

# EQUIVARIANT COHOMOLOGY AND LOCALIZATION FORMULA IN SUPERGEOMETRY

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ABSTRACT. Let  $G$  be a compact Lie group. Let  $M$  be a smooth  $G$ -manifold and  $\mathcal{V} \rightarrow M$  be an oriented  $G$ -equivariant vector bundle. One defines the spaces of equivariant forms with generalized coefficients on  $\mathcal{V}$  and  $M$ . An equivariant Thom form  $\theta$  on  $\mathcal{V}$  is a compactly supported closed equivariant form such that its integral along the fibres is the constant function 1 on  $M$ . Such a Thom form was constructed by Mathai and Quillen [MQ86]. Its restriction to  $M$  gives a representative of the equivariant Euler class of  $\mathcal{V}$ .

In the supergeometric situation we give proper definitions of all the objects involved. But, in this case a Thom form doesn't always exist. In this article, when the action of  $G$  on  $\mathcal{V}$  is sufficiently non-trivial, we construct such a Thom form with generalized coefficients. We use it to construct an equivariant Euler form of  $\mathcal{V}$  and to generalize Berline-Vergne's localization formula ([BV83a]) to the supergeometric situation.

The aim of this article is to generalize Berline-Vergne's localization formula ([BV83a]) to the supergeometric situation.

Let  $G = (G_0, \mathfrak{g})$  be a Lie supergroup (with underlying Lie group  $G_0$  and Lie superalgebra  $\mathfrak{g}$ ) acting on the right on a globally oriented supermanifold  $M = (M_0, \mathcal{O}_M)$  (see section 2.4 for precise definitions). Let  $\alpha$  be a closed equivariant integrable form (the superanalog of closed equivariant forms with compact support). Let  $X$  be an element of  $\mathfrak{g}_0$  such that  $\overline{\exp(\mathbb{R}X)}$  is a compact subgroup of  $G_0$ . Let  $j : M(X) \hookrightarrow M$  be the subsupermanifold of zeroes of  $X$ . Under some conditions, we construct an equivariant Euler form for  $T_N M(X)$  (the normal bundle of  $M(X)$  in  $M$ ). We denote it by  $\mathcal{E}_{\mathfrak{g}}$ . Then we obtain, for  $Z$  in a neighborhood of  $X$  in  $\mathfrak{g}(X)$  (the centralizer of  $X$  in  $\mathfrak{g}$ ) the following equality:

$$(1) \quad \int_M \alpha(Z) = (2\pi)^{\frac{n+m}{2}} \int_{M(X)} \frac{j^* \alpha(Z)}{\mathcal{E}_{\mathfrak{g}}(Z)}$$

Let us describe the most remarkable condition denoted by (\*\*) (cf. Theorems 5.1 and 9.2) needed to construct an Euler form.

We assume that there is a  $G(X)$ -invariant Euclidean structure (cf. section 2.2)  $Q$  on  $T_N M(X)$ , and that  $T_N M(X)$  has an equivariant superconnection  $\mathbb{A}_{\mathfrak{g}}$  with equivariant moment  $\mu_{\mathbb{A}}$  (cf. section 4.3). We denote by  $\Gamma_{T_N M(X)}$  the sheaf of sections of  $T_N M(X)$ . The condition (\*\*) is that there is a covering of  $M_0$  by open subsets  $W$  such that:

$$(2) \quad U_+^{T_N M(X)1}(W) = \left\{ Z \in \mathfrak{g}_0(X) \middle/ \right. \\ \left. (\forall v \in \Gamma_{T_N M(X)}(W)_1 / \forall x \in W, v_{\mathbb{R}}(x) \neq 0), Q(v, \mu_{\mathbb{A}}(Z)v)_{\mathbb{R}} > 0 \right\}.$$

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contains a non empty open subset.

( $Q(v, \mu_{\mathbb{A}}(Z)v)$  is a function on  $M$  and  $Q(v, \mu_{\mathbb{A}}(Z)v)_{\mathbb{R}}$  is its restriction to its underlying manifold  $M_{\mathbf{0}}$ .)

A. S. Schwarz and O. Zaboronky found in [SZ97] a localization formula over a supermanifold in presence of symmetries given by the actions of compact groups. Here, we consider symmetries given by the actions of more general groups. In particular, we can have an action of a Lie supergroup which is not a Lie group.

One motivation for such a formula is to produce invariant generalized functions on a Lie superalgebra and to test the validity of a ‘‘Kirillov-like’’ character formula for Lie supergroups. The above condition seems very natural when applied to a coadjoint orbit of a supergroup  $G$ . When  $\mathfrak{g}$  is nilpotent, this condition allows to associate an equivalence class of unitary representations of  $G$  with the orbit as pointed out to me by M. Duflo.

In order to obtain such a formula we need to construct a representative of the equivariant Euler class of an equivariant supervector bundle  $\mathcal{V} \rightarrow M$ . It is obtained by restriction to  $M_{\mathbf{0}}$  of an equivariant Thom form on  $\mathcal{V}$  (cf. section 6).

Let us say some words about the non-equivariant case (cf. [BS84, VZ88, Vor91, Lav98]). Let  $\pi : \mathcal{V} \rightarrow M$  be a supervector bundle. We assume that  $\mathcal{V}$  and  $M$  are globally oriented. Let  $\widehat{\Omega}_f(\mathcal{V})$  (resp.  $\widehat{\Omega}_f(M)$ ) be the space integrable pseudodifferential forms on  $\mathcal{V}$  (resp.  $M$ ) (cf. sections 2.3 and 3.4). As in the purely even situation, there is a ‘‘Thom isomorphism’’ between the cohomology of  $\widehat{\Omega}_f(\mathcal{V})$  and the cohomology of  $\widehat{\Omega}_f(M)$ . But unlike the even situation, it is not given by an integration along the fibres. Indeed, there is no closed form on  $\mathcal{V}$  which is integrable along the fibres and such that  $\pi_*\alpha = 1_M$  (this is the constant function equal to 1 on  $M$  and  $\pi_*$  is the integration along the fibres of  $\mathcal{V} \rightarrow M$  cf. section 3.5).

Since we work with smooth supermanifolds (cf. [Bat79]), the supervector bundle  $\mathcal{V}$  is the direct sum of its even part  $\mathcal{V}^{\mathbf{0}}$  and its odd part  $\mathcal{V}^{\mathbf{1}}$ . Therefore we have a bundle  $\mathcal{V} \rightarrow \mathcal{V}^{\mathbf{0}} \rightarrow M$ . We denote by  $j^{\mathbf{0}} : \mathcal{V}^{\mathbf{0}} \hookrightarrow \mathcal{V}$  the embedding of  $\mathcal{V}^{\mathbf{0}}$  in  $\mathcal{V}$  by mean of the zero-section and by  $\pi^{\mathbf{0}} : \mathcal{V}^{\mathbf{0}} \rightarrow M$  the projection of  $\mathcal{V}^{\mathbf{0}}$  onto  $M$ . Let  $\alpha \in \widehat{\Omega}_f(\mathcal{V})$ . We take its restriction  $j^{\mathbf{0}*}(\alpha)$  to  $\mathcal{V}^{\mathbf{0}}$  and then we take its integral  $\pi_*^{\mathbf{0}}(j^{\mathbf{0}*}(\alpha))$  along the fibres of  $\mathcal{V}^{\mathbf{0}} \rightarrow M$ . The application  $\alpha \mapsto \pi_*^{\mathbf{0}}(j^{\mathbf{0}*}(\alpha))$  induces the ‘‘Thom isomorphism’’ between the cohomologies of  $\widehat{\Omega}_f(\mathcal{V})$  and  $\widehat{\Omega}_f(M)$ . The reverse isomorphism sends a form  $\beta \in \widehat{\Omega}_f(M)$  on  $\theta(\pi^*\beta)$  where  $\theta$  is a closed form on  $\mathcal{V}$  which is integrable along the fibres (cf. 3.5) and such that  $\pi_*^{\mathbf{0}}(j^{\mathbf{0}*}(\theta)) = 1_M$ .

Now, let  $G$  be a supergroup. Let  $\pi : \mathcal{V} \rightarrow M$  be a  $G$ -equivariant supervector bundle. Under condition (\*\*) and other technical ones (cf. Theorem 5.1), we construct a closed equivariant form with generalized coefficients  $\theta$  which is integrable along the fibres and such that  $\pi_*\theta = 1$  (cf. section 5). We call it an equivariant Thom form of  $\mathcal{V}$ . Let  $\widehat{\Omega}_{G,f}^{-\infty}(\mathcal{V})$  be the space of integrable equivariant forms with generalized coefficients on  $\mathcal{V}$ . Because of the generalized coefficients,  $\widehat{\Omega}_{G,f}^{-\infty}(\mathcal{V})$  is not an  $\widehat{\Omega}_{G,f}^{-\infty}(M)$ -module and the equivariant Thom form does not provide a Thom isomorphism between the cohomologies of  $\widehat{\Omega}_{G,f}^{-\infty}(\mathcal{V})$  and  $\widehat{\Omega}_{G,f}^{-\infty}(M)$ . Nevertheless, it provides an equivariant Euler form  $\mathcal{E}_{\mathfrak{g}}$  (cf. section 6) and

a relation between cohomology classes which is useful to obtain the localization formula (cf. section 5.3).

All notations are fixed in sections 1 to 3. These sections contain essentially well known material (cf. for example [Man88] for the most of it). They are rather detailed and as a result the article is essentially self contained.

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## 1. PREREQUISITES

We choose a square root  $\mathbf{i} \in \mathbb{C}$  of  $-1$ .

**1.1. Supervector spaces.** In this article, unless otherwise specified, all supervector spaces and superalgebras will be real. If  $V$  is a supervector space, we denote by  $V_0$  its even part and by  $V_1$  its odd part. If  $v$  is a non zero homogeneous element of  $V$ , we denote by  $p(v) \in \mathbb{Z}/2\mathbb{Z}$  its parity. We put  $\dim(V) = (\dim(V_0), \dim(V_1))$ .

Let  $(m, n) \in \mathbb{N} \times \mathbb{N}$ . We denote by  $\mathbb{R}^{(m,n)}$  the supervector space of dimension  $(m, n)$  such that  $V_0 = \mathbb{R}^m$  and  $V_1 = \mathbb{R}^n$ .

**1.2. Near superalgebras.** We say that a commutative superalgebra  $\mathcal{P}$  is *near* (they are the *algèbres proches* of Weil [Wei53]) if it is finite dimensional, local, and with  $\mathbb{R}$  as residual field. For  $\alpha \in \mathcal{P}$ , we denote by  $\mathbf{b}(\alpha)$  the canonical projection of  $\alpha$  in  $\mathbb{R}$  ( $\mathbf{b}(\alpha)$  is the *body* of  $\alpha$ , and  $\alpha - \mathbf{b}(\alpha)$  —a nilpotent element of  $\mathcal{P}$ — the *soul* of  $\alpha$ , according to the terminology of [DeW84]). Let  $\alpha \in \mathcal{P}_0$  be an even element. Let  $\phi \in \mathcal{C}^\infty(\mathbb{R}, W)$  be a smooth function defined in a neighborhood of  $\mathbf{b}(\alpha)$  in  $\mathbb{R}$ , with values in some Fréchet supervector space  $W$ . We freely use the notation (where  $\phi^{(k)}$  is the  $k$ -th derivative of  $\phi$ ):

$$(3) \quad \phi(\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha - \mathbf{b}(\alpha))^k}{k!} \phi^{(k)}(\mathbf{b}(\alpha)) \in W \otimes \mathcal{P}.$$

Since  $(\alpha - \mathbf{b}(\alpha))$  is nilpotent, the sum is finite. In particular, if  $\alpha \in \mathcal{P}_0$  is invertible, its absolute value  $|\alpha| \in \mathcal{P}_0$  is defined by the formula:

$$(4) \quad |\alpha| = \frac{|\mathbf{b}(\alpha)|}{\mathbf{b}(\alpha)} \alpha,$$

and if  $\mathbf{b}(\alpha) > 0$ , its square root is defined by the finite sum

$$\sqrt{\alpha} = \sqrt{\mathbf{b}(\alpha)} \left( 1 + \frac{1}{2} \left( \frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right) - \frac{1}{2^2 2!} \left( \frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^2 + \frac{3}{2^3 3!} \left( \frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^3 - \frac{3.5}{2^4 4!} \left( \frac{\alpha}{\mathbf{b}(\alpha)} - 1 \right)^4 + \dots \right),$$

where  $\sqrt{\lambda}$  is the unique positive square root of  $\lambda > 0$ . A notation like  $\sqrt{|\alpha|}$  (for  $\alpha \in \mathcal{P}_0$  invertible) is not ambiguous, because if  $f \in \mathcal{C}^\infty(\mathbb{R}^+)$  and  $g \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^+)$ , for  $\alpha \in \mathcal{P}_0$   $f \circ g(\alpha) = f(g(\alpha))$ .

### 1.3. Modules over a superalgebra.

1.3.1. *Definition.* We fix a commutative superalgebra  $\mathcal{A}$ . We define an  $\mathcal{A}$ -module  $V$  as an  $\mathcal{A}$ -bimodule (in particular  $V$  is a supervector space) such that the left and right  $\mathcal{A}$ -module structures satisfy the rule of signs: for  $a \in \mathcal{A}$  and  $v \in V$  non zero and homogeneous:

$$av = (-1)^{p(a)p(v)}va.$$

If  $V$  and  $W$  are  $\mathcal{A}$ -modules we denote by  $\text{Hom}_{\mathcal{A}}(V, W)$  the superalgebra of morphisms of right  $\mathcal{A}$ -modules. For  $\phi \in \text{Hom}_{\mathcal{A}}(V, W)$   $v \in V$  and  $a \in \mathcal{A}$ ,  $\phi(va) = \phi(v)a$ .

We put  $\mathfrak{gl}(V) = \text{Hom}_{\mathcal{A}}(V, V)$ .

We denote by  $V^*$  its dual module  $\text{Hom}_{\mathcal{A}}(V, \mathcal{A})$ . It is naturally a left  $\mathcal{A}$ -module. We give it a structure of  $\mathcal{A}$ -(bi)module by the rule of signs.

Let  $u \in \mathfrak{gl}(V)$  be non zero and homogeneous. We denote by  $u^*$  the endomorphism of  $V^*$  such that for any non zero homogeneous  $\phi \in V^*$  and any  $v \in V$ ,  $u^*(\phi)(v) = (-1)^{p(u)p(\phi)}\phi(u(v))$ . Let  $u, v \in \mathfrak{gl}(V)$  be non zero and homogeneous, we have  $(u \circ v)^* = (-1)^{p(u)p(v)}v^* \circ u^*$ . We extend linearly the map  $u \mapsto u^*$  to an even isomorphism super-vector spaces of  $\mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V^*)$ .

1.3.2. *Tensor product.* For two  $\mathcal{A}$ -modules  $V$  and  $W$  their tensor product  $V \otimes_{\mathcal{A}} W$  is defined by the relation:

$$\forall a \in \mathcal{A}, \forall v \in V, \forall w \in W, va \otimes w = v \otimes aw.$$

We identify  $V \otimes_{\mathcal{A}} W$  and  $W \otimes_{\mathcal{A}} V$  by the usual rule of signs (for non zero homogeneous  $v \in V$  and  $w \in W$  we identify  $v \otimes w$  and  $(-1)^{p(v)p(w)}w \otimes v$ ).

When  $\mathcal{A} = \mathbb{R}$  and  $V, W$  are Fréchet supervector spaces we denote by  $V \widehat{\otimes} W$  the completion of  $V \otimes W$ .

We denote by  $S(V)$  the symmetric  $\mathcal{A}$ -algebra of  $V$ . Let us recall its definition. We put  $V_{\mathcal{A}}^{\otimes 0} = \mathcal{A}$  and  $V_{\mathcal{A}}^{\otimes n+1} = V \otimes V_{\mathcal{A}}^{\otimes n}$ . We denote by  $T(V)$  the tensor  $\mathcal{A}$ -superalgebra of  $V$ :

$$T(V) = \bigoplus_{n \in \mathbb{N}} V_{\mathcal{A}}^{\otimes n}.$$

We denote by  $\mathcal{J}$  the ideal of  $T(V)$  generated by

$$\{v \otimes w - (-1)^{p(v)p(w)}w \otimes v / v, w \in V \text{ non zero and homogeneous}\}.$$

Then:

$$S(V) = T(V)/\mathcal{J}.$$

We denote by  $S^n(V)$  the canonical projection of  $V_{\mathcal{A}}^{\otimes n}$  in  $S(V)$ .

When  $\mathcal{A} = \mathbb{R}$  we recall that  $S(V)$  is equal to  $S(V_0) \otimes \Lambda(V_1)$ , where  $S(V_0)$  and  $\Lambda(V_1)$  are the classical symmetric and exterior algebras of the corresponding ungraded vector spaces. We use the notation  $\Lambda(U)$  only for ungraded vector spaces  $U$ . So, if  $V$  is a supervector space,  $\Lambda(V)$  is the exterior algebra of the underlying vector space.

1.3.3. *Change of parity.* For an  $\mathcal{A}$ -module  $V$ , we denote by  $\Pi V$  the  $\mathcal{A}$  supermodule with reverse parity. It is defined by  $(\Pi V)_0 = V_1$  and  $(\Pi V)_1 = V_0$ . Let  $\pi : V \rightarrow \Pi V$  the “odd identity” from  $V$  to  $\Pi V$ . Let  $a \in \mathcal{A}$ ,  $v, w \in V$  be non zero homogeneous elements. The  $\mathcal{A}$ -module structure of  $\Pi V$  is given by the relations:

$$(5) \quad \begin{aligned} \pi(va) &= \pi(v)a; \\ \pi(av) &= (-1)^{p(a)}a\pi(v). \end{aligned}$$

For  $\phi \in V^*$  non zero and homogeneous and  $v \in V$  we put:

$$(6) \quad (\Pi\phi)(\Pi v) = (-1)^{p(\phi)}\phi(v).$$

This can be extended  $\mathcal{A}$ -linearly to an odd isomorphism from  $V^*$  to  $(\Pi V)^*$ . This gives an identification between  $(\Pi V)^*$  and  $\Pi(V^*)$  (and thus we denote it by  $\Pi V^*$ ).

Similarly we put for any non zero homogeneous  $\phi \in \mathfrak{gl}(V)$  and any  $v \in V$ :

$$(7) \quad \phi(\Pi v) = (-1)^{p(\phi)}\Pi\phi(v).$$

It induces an even isomorphism  $\mathfrak{gl}(V) \rightarrow \mathfrak{gl}(\Pi V)$ . We identify this way this two algebras.

1.3.4. *Finite rank free modules.* Let  $V$  be a free  $\mathcal{A}$ -module. We say that  $V$  is of rank  $(k, l) \in \mathbb{N}^2$  if there exists a basis  $(e_1, \dots, e_k, f_1, \dots, f_l)$  of  $V$  such that  $p(e_i) = 0$  and  $p(f_j) = 1$ . We also denote by  $(e_i, f_j)$  such a standard basis. We write  $\text{rk}(V) = (k, l)$ . We put  $V^{\mathbb{R}} = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_k \oplus \mathbb{R}f_1 \oplus \dots \oplus \mathbb{R}f_l$ . Then, we have  $V \simeq V^{\mathbb{R}} \otimes_{\mathbb{R}} \mathcal{A}$ . We stress that the definition of  $V^{\mathbb{R}}$  depends on the choice of the basis  $(e_1, \dots, e_k, f_1, \dots, f_l)$  of  $V$ .

In this case  $V^* \simeq \mathcal{A} \otimes_{\mathbb{R}} (V^{\mathbb{R}})^*$  and:  $S(V) \simeq S(V^{\mathbb{R}}) \otimes_{\mathbb{R}} \mathcal{A}$ .

If  $V$  is free of rank  $(k, l)$ ,  $\Pi V$  is free of rank  $(l, k)$ .

We recall that an  $\mathcal{A}$ -module  $V$  is said to be *locally free* is for any maximal ideal  $\mathfrak{m}$  of  $\mathcal{A}$ ,  $V \otimes_{\mathcal{A}} \mathcal{A}_{\mathfrak{m}}$  is free ( $\mathcal{A}_{\mathfrak{m}}$  is the localization of  $\mathcal{A}$  at  $\mathfrak{m}$ ).

1.3.5. *Bilinear forms.* Let  $V$  be an  $\mathcal{A}$ -module. Let  $B$  be a *bilinear form* on  $V$ . It means that for  $v, w \in V$  and  $a, b \in \mathcal{A}$ :

$$(8) \quad B(va, wb) = (-1)^{p(a)p(w)}B(v, w)ab.$$

We say that  $B$  is symmetric (resp. antisymmetric) if for  $v, w$  non zero homogeneous:

$$(9) \quad B(v, w) = (-1)^{p(v)p(w)}B(w, v) \quad (\text{resp. } B(v, w) = -(-1)^{p(v)p(w)}B(w, v)).$$

In this case, we denote by  $\mathfrak{osp}(V, B)$  (resp.  $\mathfrak{spo}(V, B)$ ) the Lie subsuperalgebra of  $\mathfrak{gl}(V)$  such that for  $\mathbf{k} \in \mathbb{Z}/2\mathbb{Z}$ :

$$(10) \quad \mathfrak{osp}(V, B)_{\mathbf{k}} = (\text{resp. } \mathfrak{spo}(V, B)_{\mathbf{k}} =) \{X \in \mathfrak{gl}(V)_{\mathbf{k}} / \forall v, w \in V \setminus \{0\}, \text{ homogeneous, } B(Xv, w) + (-1)^{\mathbf{k}p(v)}B(v, Xw)\}.$$

When  $B$  is clear from the context we will denote it by  $\mathfrak{osp}(V)$  (resp.  $\mathfrak{spo}(V)$ ).

We say that  $B$  is even if for any non zero homogeneous  $v, w \in V$ ,  $p(B(v, w)) = p(v) + p(w)$ . If  $B$  is even antisymmetric and non degenerate, we say that  $(V, B)$  (or  $V$  if  $B$  is clear from the context) is a *symplectic supervector space* (cf. section 1.4 for more details on symplectic supervector spaces).

Let  $V$  be an  $\mathcal{A}$ -module and  $B$  an even bilinear form on  $V$ . We define a bilinear form  $\Pi B$  on  $\Pi V$  by

$$(11) \quad \Pi B(\Pi v, \Pi w) = (-1)^{p(v)}B(v, w).$$

If  $B$  is symmetric (resp. antisymmetric)  $\Pi B$  is antisymmetric (resp. symmetric).

When  $B$  is non degenerate it defines an isomorphism  $B^* : V \rightarrow V^*$  defined for  $v, w \in V$  by:

$$(12) \quad B^*(v)(w) = B(v, w).$$

**1.4. Symplectic oriented supervector spaces.** Since this particular case is very important in this article we pay closer attention to it. In this section  $\mathcal{A} = \mathbb{R}$  and  $V$  is finite dimensional but all the definitions makes sense if  $\mathcal{A}$  is a superalgebra and  $V$  is a free  $\mathcal{A}$ -module of finite rank.

Let  $V = V_0 \oplus V_1$  be a symplectic supervector space. It means that  $V_0$  and  $V_1$  are orthogonal, that the restriction of  $B$  to  $V_0$  is a non degenerate antisymmetric bilinear form, and that the restriction of  $B$  to  $V_1$  is a non degenerate symmetric bilinear form.

Such a space is a direct sum of  $(2, 0)$ -dimensional symplectic supervector spaces (i.e. 2-dimensional symplectic vector spaces), and of  $(0, 1)$ -dimensional symplectic supervector spaces (i.e. 1-dimensional quadratic vector spaces). We first review these building blocks.

**1.4.1. Symplectic 2-dimensional vector spaces.** Let  $V = V_0$  a purely even 2-dimensional symplectic space. A *symplectic basis* of  $V$  is a basis  $(e_1, e_2)$  such that  $B(e_1, e_2) = 1$ ,  $B(e_1, e_1) = 0$ ,  $B(e_2, e_2) = 0$ . It defines a dual *symplectic coordinate system*  $(x^1, x^2)$ , and an orientation of  $V^*$ .

**1.4.2. Symplectic 1-dimensional odd vector spaces.** Let  $V = V_1$  a purely odd 1-dimensional symplectic supervector space (i.e. a 1-dimensional quadratic space). A symplectic basis of  $V$  is a basis  $(f)$  such that  $B(f, f) = 1$ . However, such a basis does not always exists, and we allow  $f$  to be in  $V_1 \cup \mathbf{i}V_1 \subset V_1 \otimes \mathbb{C}$ . Let  $(\xi) \in V_1^* \cup \mathbf{i}V_1^*$  be the dual basis.

We will call the choice of  $(\xi)$  (the other possible choice is  $(-\xi)$ ) an *orientation* of  $V_1$ . If  $B$  is positive definite, then  $(\xi)$  is a basis of  $V_1^*$ , and so defines an orientation in the usual sense. If  $B$  is negative definite, then  $(-\mathbf{i}\xi)$  is a basis of  $V_1^*$ , and so defines an orientation in the usual sense.

**1.4.3. General case.** Let us go back to the general case.

The dimension  $m$  of  $V_0$  is even. We choose a symplectic basis  $(e_1, \dots, e_m)$  of  $V_0$ , that is  $V_0$  is the direct sum of  $m/2$  symplectic vector spaces generated by the pairs  $(e_1, e_2), (e_3, e_4), \dots$ , and  $B(e_1, e_2) = 1, B(e_3, e_4) = 1, \dots$ . The dual basis  $(x^i)$  of  $V_0^*$  is called a symplectic coordinate system.

On  $V_1$ , we define a symplectic basis  $(f_1, \dots, f_n)$  as an orthonormal basis of  $V_1 \otimes \mathbb{C}$  such that  $f_i \in V_1$  or  $f_i \in \mathbf{i}V_1$  for all  $i$ . Let  $(\xi^1, \dots, \xi^n)$  be the dual basis. The pair of functions  $\pm \xi^1 \dots \xi^n$  does not depend on the symplectic basis  $(f_1, \dots, f_n)$ . A choice of one of the two elements of  $\pm \xi^1 \dots \xi^n$  is called an *orientation* of  $V_1$ . If  $V_1$  is oriented, an oriented symplectic coordinate system on  $V_1$  is a basis for which the orientation is  $\xi^1 \dots \xi^n$ . If  $B|_{V_1}$  has signature  $(p, q)$ , then  $(-\mathbf{i})^q \xi^1 \dots \xi^n \in \Lambda^n(V_1^*)$  and so defines an orientation of  $V_1$  in the usual sense.

Let us remark that in the specially interesting case where  $B$  is positive definite, a symplectic basis is a basis of  $V_1$ , and not only of  $V_1 \otimes \mathbb{C}$ .

We define an *oriented symplectic supervector space* as a symplectic supervector space  $(V, B)$  provided with an orientation of  $V_1$ .

A symplectic oriented basis  $(e_i, f_j) = (e_1, \dots, e_m, f_1, \dots, f_n)$  is a basis of  $V \otimes \mathbb{C}$  such that  $(e_1, \dots, e_m)$  is a symplectic basis of  $V_0$ , and  $(f_1, \dots, f_n)$  an oriented symplectic basis of  $V_1 \otimes \mathbb{C}$ . We use the corresponding dual system of coordinates  $(x, \xi)$ . Then  $V$  becomes a symbol bearing a supermanifold structure, a symplectic structure, an orientation....

1.4.4. *Moment application.* A particular linear even isomorphism  $\mu$  of  $\mathfrak{spo}(V)$  to  $S^2(V^*)$ , called the moment mapping, will play an important role. Thus,  $\mu$  is an element of  $S(\mathfrak{spo}(V)^* \otimes V^*)$ . It is defined by the formula

$$(13) \quad \mu(X)(v) = -\frac{1}{2}B(v, Xv),$$

where, for any commutative superalgebra  $\mathcal{P}$  (not necessarily near),  $X \in (\mathfrak{spo}(V) \otimes \mathcal{P})_{\mathbf{0}}$  and  $v \in (V \otimes \mathcal{P})_{\mathbf{0}}$  are  $\mathcal{P}$ -valued points of  $\mathfrak{spo}(V)$  and  $V$  (cf. next section for a precise definition of  $\mathcal{P}$ -valued points), and  $B(v, Xv) \in \mathcal{P}$  is defined by the natural extension of scalars. Considering a basis  $G_k$  of  $\mathfrak{spo}(V)$ , the dual basis  $Z^k$ , the generic point  $X = G_k Z^k$ , a basis  $g_i$  of  $V$ , the dual basis  $z^i$ , and the generic point  $v = g_i z^i$ , we obtain:

$$\mu = -\frac{1}{2}B(g_i, G_k g_j) z^j Z^k z^i.$$

Let us explain the choice of the constant  $-\frac{1}{2}$  in definition of  $\mu$  and why we call  $\mu$  the moment mapping.

The symplectic form on  $V$  gives to the associated supermanifold a structure of symplectic supermanifold (cf. next section for a precise definition of a supermanifold). We define a Poisson bracket on  $S(V^*)$  by the following. Let  $f \in V^*$  we denote by  $v_f$  the element of  $V$  such that for any  $w \in V$ :

$$(14) \quad B(v_f, w) = f(w).$$

In other words  $B^*(v_f) = f$ . This gives an isomorphism from  $V^*$  onto  $V$ . For  $f, g \in V^*$ , we put:

$$(15) \quad \{f, g\} = B(v_f, v_g) \in \mathbb{R} \subset S(V^*)$$

and we extend it to a Poisson bracket on  $S(V^*)$ .

With the above definitions we have:

$$(16) \quad \{\mu(X), \mu(Y)\} = \mu([X, Y]).$$

Thus  $\mu$  is a morphism of Poisson algebras.

**1.5. Supermanifolds.** By a *supermanifold* we mean a smooth real supermanifold as in [Kos77], [Ber87] (cf. below for a precise definition).

1.5.1. *Affine supermanifolds.* Let  $V$  be a finite dimensional supervector space. We denote also by  $V$  the associated supermanifold. We say that  $V$  is an *affine supermanifold*. We recall some relevant definitions in this particular and fundamental case.

Let  $\mathcal{U} \subset V_{\mathbf{0}}$  be an open set. We put

$$\mathcal{C}_V^\infty(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda(V_{\mathbf{1}}^*),$$

where  $\mathcal{C}^\infty(\mathcal{U})$  is the usual algebra of smooth real valued functions defined in  $\mathcal{U}$ , and  $\Lambda(V_{\mathbf{1}}^*)$  is the exterior algebra of  $V_{\mathbf{1}}^*$ . We say that  $\mathcal{C}_V^\infty(\mathcal{U})$  is the superalgebra of *smooth functions on  $V$  defined in  $\mathcal{U}$* . The supermanifold  $V$  is by definition the topological space  $V_{\mathbf{0}}$  equipped with the sheaf of superalgebras  $\mathcal{C}_V^\infty$ . We denote by  $V(\mathcal{U})$  the supermanifold such that  $V(\mathcal{U})_{\mathbf{0}} = \mathcal{U}$  and sheaf of functions  $\mathcal{C}_V^\infty$  restricted to the open subsets of  $\mathcal{U}$ .

Notice that if  $\mathcal{U}$  is not empty, there is a canonical inclusion  $S(V^*) \subset \mathcal{C}^\infty(\mathcal{U})$ . The corresponding elements are called *polynomial functions*. One can also define rational functions. Similarly, if  $W$  is a Fréchet supervector space (for instance  $W = \mathbb{C}$ ), we denote by  $\mathcal{C}_V^\infty(\mathcal{U}, W) = \mathcal{C}^\infty(\mathcal{U}, W) \otimes \Lambda(V_{\mathbf{1}})^*$  the space of  $W$ -valued smooth functions.



Let  $\mathcal{P}$  be a near superalgebra. We put

$$V_{\mathcal{P}} = (V \otimes \mathcal{P})_{\mathbf{0}}.$$

It is called the *set of points of  $V$  with values in  $\mathcal{P}$* . Extending the body  $\mathbf{b} : \mathcal{P} \rightarrow \mathbb{R}$  to a map  $V \otimes \mathcal{P} \rightarrow V$ , and restricting to  $V_{\mathcal{P}}$ , we obtain a map, also called the body and denoted by  $\mathbf{b}$ ,

$$\mathbf{b} : V_{\mathcal{P}} \rightarrow V_{\mathbf{0}}.$$

Let  $\mathcal{U} \subset V_{\mathbf{0}}$  be an open set. We denote by  $V_{\mathcal{P}}(\mathcal{U}) \subset V_{\mathcal{P}}$  the inverse image of  $\mathcal{U}$  in  $V_{\mathcal{P}}$ . We have  $V_{\mathcal{P}}(\mathcal{U}) = V(\mathcal{U})_{\mathcal{P}}$ . It is known that  $V_{\mathcal{P}}(\mathcal{U})$  is canonically identified to the set of even algebra homomorphisms  $\mathcal{C}_V^{\infty}(\mathcal{U}) \rightarrow \mathcal{P}$ . Let  $v \in V_{\mathcal{P}}(\mathcal{U})$ . We will denote the corresponding character by  $\phi \mapsto \phi(v)$ , and say that  $\phi(v) \in \mathcal{P}$  is the value of  $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$  at the point  $v$ .

For example, let  $v = v_i p^i \in V_{\mathcal{P}}$  (we use Einstein's summation rule, and considering tensorisation by  $\mathcal{P}$  as an extension of scalars, we write  $v_i p^i$  instead of  $v_i \otimes p^i$ ) where the  $(v_i, p^i) \in V \times \mathcal{P}$  are a finite number of pair of homogeneous elements with the same parity. Then

$$\mathbf{b}(v) = v_i \mathbf{b}(p^i),$$

which is in  $V_{\mathbf{0}}$  since  $\mathbf{b}(p^i) = 0$  if  $p^i$  is odd. Let  $\phi \in V^*$ . We denote by the same letter the corresponding element in  $\mathcal{C}_V^{\infty}(V_{\mathbf{0}})$ . Then  $\phi(v) = \phi(v_i) p^i$ , and this formula in fact completely determines the bijection between  $V_{\mathcal{P}}(\mathcal{U})$  and the set of even homomorphisms of algebras  $\mathcal{C}_V^{\infty}(\mathcal{U}) \rightarrow \mathcal{P}$  (cf. [Wei53]).

For  $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U}, W)$ , we denote by  $\phi_{\mathcal{P}}$  the corresponding function  $v \mapsto \phi(v)$  defined in  $V_{\mathcal{P}}(\mathcal{U})$ . Then  $\phi_{\mathcal{P}} \in \mathcal{C}^{\infty}(V_{\mathcal{P}}(\mathcal{U}), W \otimes \mathcal{P})$ . The importance of this construction is that for  $\mathcal{P}$  large enough (for instance if  $\mathcal{P}$  is an exterior algebra  $\Lambda \mathbb{R}^N$  with  $N \geq \dim(V_{\mathbf{1}})$ ), the map  $\phi \mapsto \phi_{\mathcal{P}}$  is injective, which allows more or less to treat  $\phi$  as an ordinary function.

We emphasize the special case  $\mathcal{P} = \mathbb{R}$ . Then  $V_{\mathbb{R}} = V_{\mathbf{0}}$ ,  $V_{\mathbb{R}}(\mathcal{U}) = \mathcal{U}$ , and  $\phi_{\mathbb{R}}$  is the projection of  $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$  to  $\mathcal{C}^{\infty}(\mathcal{U})$  which naturally extends the projection of  $\Lambda V_{\mathbf{1}}^*$  to  $\mathbb{R}$ .

To help the reader, we give two typical examples.

Let  $V = \mathbb{R}^{(1,0)}$ . Then  $V_{\mathbf{0}} = V = \mathbb{R}$ , and  $V_{\mathcal{P}} = \mathcal{P}_{\mathbf{0}}$ . Let  $\mathcal{U} \subset \mathbb{R}$  be an open subset,  $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U}, W) = \mathcal{C}^{\infty}(\mathcal{U}, W)$ , and  $\alpha \in V_{\mathcal{P}} = \mathcal{P}_{\mathbf{0}}$  such that  $\mathbf{b}(\alpha) \in \mathcal{U}$ . Then  $\phi(\alpha) \in \mathcal{P}$  is defined by formula (3).

Let  $V = \mathbb{R}^{(0,1)}$ . Then  $V_{\mathbf{0}} = \{0\}$ , and  $V_{\mathcal{P}} = \mathcal{P}_{\mathbf{1}}$ . Any  $\phi \in \mathcal{C}_V^{\infty}(\{0\}, W)$  can be written as  $\phi = c + \xi d$ , where  $c$  and  $d$  are elements of  $W$ , and  $\xi$  the standard coordinate (the identity function) on  $\mathbb{R}$ . Then, for  $\alpha \in \mathcal{P}_{\mathbf{1}}$ ,  $\phi(\alpha) \in \mathcal{P}$  is defined by formula

$$\phi(\alpha) = c + \alpha d.$$

We return now to the general case.

We define a Fréchet topology on  $\mathcal{C}_V^{\infty}(\mathcal{U}, W)$  by saying that  $(\phi_n)_{n \in \mathbb{N}} \in \mathcal{C}_V^{\infty}(\mathcal{U}, W)^{\mathbb{N}}$  converges to  $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U}, W)$  if for any normed near superalgebra  $\mathcal{P}$ ,  $((\phi_n)_{\mathcal{P}})_{n \in \mathbb{N}} \in \mathcal{C}^{\infty}(V_{\mathcal{P}}(\mathcal{U}), W \otimes \mathcal{P})^{\mathbb{N}}$  converges uniformly to  $\phi_{\mathcal{P}} \in \mathcal{C}^{\infty}(V_{\mathcal{P}}(\mathcal{U}), W \otimes \mathcal{P})$  on any compact subset  $K \subset V_{\mathcal{P}}(\mathcal{U})$ .

Let  $\mathcal{A}$  be a commutative superalgebra. We still use the notation  $V_{\mathcal{A}} = (V \otimes \mathcal{A})_{\mathbf{0}}$ . Polynomial functions  $S(V^*)$  can be evaluated on  $V_{\mathcal{A}}$ , but, in general, smooth functions can be evaluated on  $V_{\mathcal{A}}$  only if  $\mathcal{A}$  is a near algebra.

The particular case  $\mathcal{A} = S(V^*)$  is important, because  $V_{\mathcal{A}}$  contains a particular point, the *generic point*, corresponding to the identity in the identification of  $Hom(V, V)$  with

$V \otimes V^*$ . Let us call  $v$  the generic point. Then we have  $f(v) = f$  for any polynomial function  $f \in S(V^*)$ .

1.5.2. *Coordinates.* Let  $V$  be a finite dimensional supervector space. By a basis  $(g_i)_{i \in I}$  of  $V$ , we mean an indexed basis consisting of homogeneous elements. The *dual basis*  $(z^i)_{i \in I}$  of  $V^*$  is defined by the usual relation  $z^j(g_i) = \delta_i^j$  (the Dirac symbol). We will also say that the basis  $(g_i)_{i \in I}$  is the *predual basis* of the basis  $(z^i)_{i \in I}$  (the dual basis, in the canonical identification of  $V$  to the dual of  $V^*$  is  $((-1)^{p(g_i)}g_i)_{i \in I}$ ). A basis  $(z^i)_{i \in I}$  of  $V^*$  will be also called *a system of coordinates on  $V$* . The corresponding vector fields (i.e. derivations of the algebra of smooth functions on  $V$ ) are denoted by  $\frac{\partial}{\partial z^i}$ . They are characterized by the rule

$$\frac{\partial}{\partial z^j}(z^i) = \delta_j^i.$$

Notice that the generic point  $v$  of  $V$  is then given by the formula

$$(17) \quad v = g_i z^i \in V_{S(V^*)}.$$

We will mainly use standard coordinates. Let  $(m, n) = \dim(V)$ . Then they are basis of  $V^*$  of the form  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ , where  $(x^1, \dots, x^m)$  is a basis of  $V_0^*$ , and  $(\xi^1, \dots, \xi^n)$  a basis of  $V_1^*$ . Such a basis will be sometimes denoted by the symbol  $(x^i, \xi^j)$  or  $(x, \xi)$ . For the corresponding predual basis  $(e_1, \dots, e_m, f_1, \dots, f_n)$  of  $V$ , then  $(e_1, \dots, e_m)$  is a basis of  $V_0$  and  $(f_1, \dots, f_n)$  is a basis of  $V_1$ . These notations will be used in particular for the canonical basis of  $\mathbb{R}^{(m, n)}$ . Let  $I = (i_1, \dots, i_n) \in \{0, 1\}^n$ . Then we denote by  $\xi^I$  the monomial  $(\xi^1)^{i_1} \dots (\xi^n)^{i_n}$  of  $S(V^*)$ . Let  $\mathcal{U} \subset V_0$  be an open set. Let  $W$  be a Fréchet supervector space. Then any  $\phi \in \mathcal{C}_V^\infty(\mathcal{U}, W)$  is of the form

$$(18) \quad \phi = \sum_I \xi^I \phi_I(x^1, \dots, x^m),$$

with  $\phi_I$  is an ordinary  $W$ -valued smooth function defined in the appropriate open subset of  $\mathbb{R}^m$ . Notice that  $\phi_{\mathbb{R}} = \phi_{(0, \dots, 0)}(x^1, \dots, x^m)$  does not depend on the choice of the odd coordinates  $\xi^i$ . We emphasize the fact that we write  $\phi_I$  to the right of  $\xi^I$  (recall that  $\xi^I \phi_I = \pm \phi_I \xi^I$ , according to the sign rule).

1.5.3. *General supermanifolds.* We define *supermanifold*  $M$  of dimension  $(m, n)$  as a pair  $(M_0, \mathcal{O}_M)$  where  $M_0$  is a smooth manifold of dimension  $m$  and  $\mathcal{O}_M$  is a sheaf of commutative superalgebras on  $M_0$  such that locally on a sufficiently small open subset  $\mathcal{U}$  of  $M_0$ :

$$(19) \quad \mathcal{O}_M(\mathcal{U}) \simeq \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda((\mathbb{R}^n)^*)$$

where  $\mathcal{C}^\infty(\mathcal{U})$  denotes the algebra of smooth functions on  $\mathcal{U}$ . Sometimes, when  $\mathcal{U} = M_0$  we will use the abuse of notations  $\mathcal{O}(M) = \mathcal{O}_M(M_0)$ . If moreover  $\mathcal{U}$  is a coordinates set of  $M_0$ , we say that it is a coordinates set of  $M$ . In this case, let  $(x^1, \dots, x^m)$  be coordinates on  $\mathcal{U}$  and  $(\xi^1, \dots, \xi^n)$  be the standard basis of  $(\mathbb{R}^n)^*$ , we say that  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  is a coordinates system on  $\mathcal{U}$ . We will also use the notation  $(x^i, \xi^j)$  or  $(x, \xi)$  to denote such a coordinates system.

We stress that if  $\mathcal{U}$  is a coordinates set of  $M$ , the supermanifold  $M(\mathcal{U}) = (\mathcal{U}, \mathcal{O}_M|_{\mathcal{U}})$  isomorphic to the affine supermanifold  $\mathbb{R}^{(m, n)}$ .

Let  $\mathcal{P}$  be a near superalgebra. As for affine supermanifolds we can define the set  $M_{\mathcal{P}}$  of points of  $M$  with values in  $\mathcal{P}$ . It is the set of even algebra homomorphisms  $\mathcal{O}_M(M_0) \rightarrow \mathcal{P}$ .

Let  $v \in M_{\mathcal{P}}$ . We still denote by  $\phi \mapsto \phi(v)$  the corresponding character.

For  $\mathcal{P} = \mathbb{R}$ , we have  $M_{\mathbb{R}} = M_0$ .

The canonical projection  $\mathbf{b} : \mathcal{P} \rightarrow \mathbb{R}$  induces a projection  $\mathbf{b} : M_{\mathcal{P}} \rightarrow M_{\mathbb{R}} = M_0$ . Let  $v \in M_{\mathcal{P}}$  and  $\phi \in \mathcal{O}_M(M_0)$ , then by definition  $\phi(\mathbf{b}(v)) = \mathbf{b}(\phi(v))$ . This definition coincides in case of an affine supermanifold  $V$  with the body map  $\mathbf{b} : V_{\mathcal{P}} \rightarrow V_0$ . Let  $\mathcal{U}$  be an open subset of  $M_0$ . As in the affine case we denote by  $M_{\mathcal{P}}(\mathcal{U})$  the inverse image of  $\mathcal{U}$  in  $M_{\mathcal{P}}$ . It is canonically isomorphic to the set of even algebra homomorphisms  $\mathcal{O}_M(\mathcal{U}) \rightarrow \mathcal{P}$ .

The set  $M_{\mathcal{P}}(\mathcal{U})$  is canonically provided with a structure of a smooth manifold (cf. [Wei53]). As in the affine case, for  $\phi \in \mathcal{O}_M(\mathcal{U})$ , we denote by  $\phi_{\mathcal{P}} : v \in M_{\mathcal{P}}(\mathcal{U}) \mapsto \phi(v)$  the corresponding function. It is a smooth function on  $M_{\mathcal{P}}(\mathcal{U})$ .

We stress that on a coordinates set  $\mathcal{U}$  of  $M$  the map  $\phi \mapsto \phi_{\mathbb{R}}$  is the projection  $\mathcal{O}_M(\mathcal{U}) \rightarrow \mathcal{C}^{\infty}(\mathcal{U})$  which extends the canonical projection from  $\Lambda((\mathbb{R}^m)^*)$  to  $\mathbb{R}$ .

Let  $\mathcal{U}$  be an open subset of  $M_0$ . As in the affine case, we provide  $\mathcal{O}_M(\mathcal{U})$  with a Fréchet space topology.

**1.5.4. Morphisms of supermanifolds.** Let  $M, N$  be two supermanifolds. A morphism  $\pi : M \rightarrow N$  of supermanifolds is a morphism  $(\pi_0, \pi^*)$  topological spaces with sheafs of commutative superalgebras. Then  $\pi_0 : M_0 \rightarrow N_0$  is a morphism of smooth manifolds and for an open subset  $\mathcal{U} \in N_0$ ,  $\pi^* : \mathcal{O}_{N_0}(\mathcal{U}) \rightarrow \mathcal{O}_{M_0}(\pi_0^{-1}(\mathcal{U}))$  is an even morphism of superalgebras.

Let  $\mathcal{P}$  be a near superalgebra. Then  $\pi$  induces a morphism  $\pi_{\mathcal{P}} : M_{\mathcal{P}} \rightarrow N_{\mathcal{P}}$  of smooth manifolds. In particular we have  $\pi_{\mathbb{R}} = \pi_0$ .

Let  $\phi \in \mathcal{O}(N)$ . We will also use the notation  $v \mapsto \phi(\pi(v))$  to denote the function  $v \mapsto (\pi^*\phi)(v)$ . Such a notation is not confusing because for  $v \in M_{\mathcal{P}}$ ,  $(\pi^*\phi)(v) = \phi(\pi_{\mathcal{P}}(v))$ .

## 1.6. Supergroups.

**1.6.1. Definition.** We recall (cf. [Kos77]) that a *supergroup*  $G$  can be defined a a couple  $(G_0, \mathfrak{g})$  where  $G_0$  is a Lie group,  $\mathfrak{g}$  is a Lie superalgebra such that  $\mathfrak{g}_0$  is the Lie algebra of  $G_0$  and there is a representation  $\text{Ad} : G_0 \rightarrow GL(\mathfrak{g}_0) \times GL(\mathfrak{g}_1)$  such that its differential is the restriction to  $\mathfrak{g}_0$  of the adjoint representation of  $\mathfrak{g}$ .

Such a supergroup defines a supermanifold  $G = G_0 \times \mathfrak{g}_1$  (where  $\mathfrak{g}_1$  is the purely odd affine supermanifold associated to the odd supervector space  $\mathfrak{g}_1$ ).

Let  $\mathcal{P}$  be any near superalgebra then  $G_{\mathcal{P}}$  is canonically a Lie group with Lie algebra  $\mathfrak{g}_{\mathcal{P}}$ .

*Example 1:  $GL(V)$ .* Let  $V$  be a supervector space. We denote by  $GL(V)$  the supergroup with underlying Lie group  $GL(V)_0 = GL(V_0) \times GL(V_1)$  (where  $V_0$  and  $V_1$  are considered as ungraded vector spaces) and Lie superalgebra  $\mathfrak{gl}(V)$ . When  $V = \mathbb{R}^{(m,n)}$  we denote it by  $GL(m, n)$ .

In this case  $GL(V)_{\mathcal{P}} = GL(V_{\mathcal{P}})$  is the group of invertibles elements of  $\mathfrak{gl}(V)_{\mathcal{P}} = \mathfrak{gl}(V_{\mathcal{P}})$ .

*Example 2:  $SpO$  and  $SpSO$ .* Let  $V$  a symplectic supervector space We denote by  $SpO(V)$  the subsupergroup of  $GL(V)$  such that  $SpO(V)_0 = Sp(V_0) \times O(V_1)$  and Lie superalgebra  $\mathfrak{sp}o(V)$ . We denote by  $SpSO(V)$  its connected component:  $SpSO(V)_0 = Sp(V_0) \times SO(V_1)$  and the same Lie superalgebra.

When  $V$  is identified to  $\mathbb{R}^{(m,n)}$  by the choice of a symplectic basis we denote them by  $SpO(m, n)$  and  $SpSO(m, n)$  respectively.

1.6.2. *Representation.* Let  $G = (G_0, \mathfrak{g})$  be a Lie supergroup. Let  $V$  be a Fréchet supervector space. A representation  $\rho$  of  $G$  in  $V$  is a differentiable representation  $\rho_0 : G_0 \rightarrow GL(V_0) \times GL(V_1)$  of  $G_0$  in  $V$  and a representation also denoted by  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  of  $\mathfrak{g}$  in  $V$  such that  $\rho|_{\mathfrak{g}_0}$  is the differential  $d\rho_0$  of  $\rho_0$ .

Let  $\mathcal{P}$  be any near superalgebra. The representation  $\rho$  induces a representation  $\rho_{\mathcal{P}}$  of  $G_{\mathcal{P}}$  in  $V_{\mathcal{P}}$ . It is defined by a morphism  $\rho_{\mathcal{P}} : G_{\mathcal{P}} \rightarrow GL(V_{\mathcal{P}}) = GL(V)_{\mathcal{P}}$ .

1.6.3. *Action on a supermanifold.* Let  $G = (G_0, \mathfrak{g})$  be a supergroup and  $M$  be a supermanifold. We say that  $M$  is a  $G$ -supermanifold or that  $G$  acts (on the right) on  $M$  if  $G_0$  acts on the right of  $(M_0, \mathcal{O}_M(M_0))$  and there is a morphism of Lie superalgebras  $X \mapsto X_M$  of  $\mathfrak{g}$  into the derivations of  $\mathcal{O}_M(M_0)$ . We denote by  $x \in M_0 \mapsto xg$  the action of  $g \in G_0$  on  $M_0$  and  $\phi \in \mathcal{O}_M(M_0) \mapsto g\phi$  its action on  $\mathcal{O}_M(M_0)$ . For  $g \in G_0$ ,  $x \in M_0$  and  $\phi \in \mathcal{O}_M(M_0)$ , we have  $(g\phi)(x) = \phi(xg)$ . Moreover, we assume the following compatibility between the actions of  $G_0$  and  $\mathfrak{g}$ . For any  $X \in \mathfrak{g}_0$  and  $\phi \in \mathcal{O}_M(M_0)$ ,

$$(20) \quad X_M \phi = \frac{d}{dt} \exp(tX) \phi \Big|_{t=0}$$

(where  $\exp$  is the exponential map from  $\mathfrak{g}_0$  into  $G_0$ ).

Let  $\mathcal{P}$  be a near superalgebra. Such an action induces a Lie group right action of  $G_{\mathcal{P}}$  on  $M_{\mathcal{P}}$ .

Let  $\mathcal{U}$  be a  $G_0$ -invariant subset of  $M_0$ . Let  $\phi \in \mathcal{O}_M(\mathcal{U})$ . We say that  $\phi$  is  $G$ -invariant if  $\phi$  is  $G_0$ -invariant and for any  $X \in \mathfrak{g}$ ,  $X_M \phi = 0$ . We denote by  $\mathcal{O}_M(\mathcal{U})^G$  the set of  $G$ -invariant functions on  $M$  defined on  $\mathcal{U}$ .

Equivalently this means that for any near superalgebra  $\mathcal{P}$ ,  $\phi_{\mathcal{P}}$  is a  $G_{\mathcal{P}}$ -invariant function.

1.7. **Rapidly decreasing functions.** We say (cf. for example [Hör83, Chapter 7]) that  $\phi \in \mathcal{C}^\infty(\mathbb{R}^m)$  is rapidly decreasing if for any  $(\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$  and any  $(\beta_1, \dots, \beta_m) \in \mathbb{N}^m$ ,

$$(21) \quad \text{Sup} \left| (x^1)^{\beta_1} \dots (x^m)^{\beta_m} \frac{\partial^{\alpha_1}}{\partial (x^1)^{\alpha_1}} \dots \frac{\partial^{\alpha_m}}{\partial (x^m)^{\alpha_m}} \phi(x^1, \dots, x^m) \right| < +\infty.$$

where  $(x^1, \dots, x^m)$  are the canonical coordinates on  $\mathbb{R}^m$ .

Let  $V$  be a supervector space. Let  $\phi \in \mathcal{C}_V^\infty(V_0)$  be a smooth function on  $V$ . Let  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  be a basis of  $V^*$ . We put  $\phi = \sum_I \xi^I \phi_I(x^1, \dots, x^m)$  where  $\phi_I \in \mathcal{C}^\infty(\mathbb{R}^m)$ .

We say that  $\phi$  is *rapidly decreasing* if for any  $I$ ,  $\phi_I$  is a rapidly decreasing function on  $\mathbb{R}^m$ .

This definition does not depend on the choice of the basis  $(x^i, \xi^j)$  of  $V^*$ . We denote by  $\mathcal{S}_V(V_0)$  or  $\mathcal{S}(V)$  the set of rapidly decreasing functions on  $V$ .

Equivalently,  $\phi \in \mathcal{C}_V^\infty(V_0)$  is rapidly decreasing if for any near superalgebra  $\mathcal{P}$ ,  $\phi_{\mathcal{P}}$  is rapidly decreasing on  $V_{\mathcal{P}}$ .

Let  $W$  be a Fréchet supervector space. Similarly we define the set  $\mathcal{S}_V(V_0, W)$  of rapidly decreasing functions with values in  $W$ . In this case, condition (21) must be

satisfied when the absolute value  $|\cdot|$  is replaced by any seminorm defining the topology of  $W$ .

**1.8. Supertrace and Berezinians.** Let  $V$  be a supervector space.

Let  $\mathcal{A}$  be a commutative superalgebra. We write an element of  $\mathfrak{gl}(V)_{\mathcal{A}}$  in the form

$$(22) \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V)_{\mathcal{A}}.$$

where  $A \in \mathfrak{gl}(V_0) \otimes \mathcal{A}_0$ ,  $D \in \mathfrak{gl}(V_1) \otimes \mathcal{A}_0$ ,  $B \in \text{Hom}(V_1, V_0) \otimes \mathcal{A}_1$ , and  $C \in \text{Hom}(V_0, V_1) \otimes \mathcal{A}_1$

We recall the definition of the supertrace:

**Definition 1.1.** *The supertrace of  $M \in \mathfrak{gl}(V)_{\mathcal{A}}$  is defined by*

$$(23) \quad \text{str}(M) = \text{tr}(A) - \text{tr}(D),$$

where  $\text{tr}$  is the ordinary trace.

Berezin introduced the following generalization of the determinant (cf. [Ber87, BL75, Man88]), called the *Berezinian*.

If  $D$  is invertible, we define:

$$(24) \quad \text{Ber}(M) = \det(A - BD^{-1}C) \det(D)^{-1}.$$

Assume moreover that  $\mathcal{A}$  is a near superalgebra. If  $D$  is invertible, we define (cf. [Vor91]):

$$(25) \quad \text{Ber}_{(1,0)}(M) = \left| \det(A - BD^{-1}C) \right| \det(D)^{-1}.$$

All these functions are multiplicative.

Recall that  $\mathfrak{gl}(V)_{\mathbf{0}}$  consists of the matrices  $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$  with  $A \in \mathfrak{gl}(V_0)$  and  $D \in \mathfrak{gl}(V_1)$ . We consider the open set  $\mathcal{U}' = \mathfrak{gl}(V_0) \times GL(V_1)$ . Formula (24) defines a rational function on the open set  $\mathcal{U}'$  of the supermanifold  $\mathfrak{gl}(V)$ . We still denote by  $\text{Ber}$  and  $\text{Ber}_{(1,0)}$  the elements of  $\mathcal{C}_{\mathfrak{gl}(V)}^{\infty}(\mathcal{U}')$  whose evaluation in  $\mathfrak{gl}(V)_{\mathcal{A}}$  is given as above.

Since  $GL(V)_{\mathbf{0}} \subset \mathcal{U}'$  these functions are well defined on  $GL(V)$ .

We keep the preceding notations. We assume moreover that  $V$  is a symplectic supervector space, then if  $M \in SpO(V)_{\mathcal{A}}$ :

$$(26) \quad \text{Ber}(M) = \text{Ber}_{(1,0)}(M) = \det(D - CA^{-1}B) \in \{\pm 1\};$$

and if  $M \in SpSO(V)_{\mathcal{A}}$ :

$$(27) \quad \text{Ber}(M) = \text{Ber}_{(1,0)}(M) = 1.$$

1.9. **Berezinian modules.** (cf. [Man88]) Let  $\mathcal{A}$  be a commutative superalgebra. Let  $V$  be an locally free  $\mathcal{A}$ -module of finite type. We put  $K(V) = S(\Pi V) \otimes S(V^*)$ . Let  $(e_i)$  and  $(x^i)$  be families of vectors in  $V$  and  $V^*$  respectively such that for any  $v \in V$  we have  $v = \sum_i e_i x^i(v)$ . We put:

$$d_V = \sum_i \pi e_i x^i.$$

Then, by definition:

$$(28) \quad \mathcal{B}er(V) = H(K(V), d_V),$$

where  $H$  denotes the homology of  $(K(V), d_V)$ .

Let  $\omega \in K(V)$ , we denote by  $\mathcal{B}er(\omega)$  its canonical image in  $\mathcal{B}er(V)$ .

In particular, assume that  $V$  is free. Let  $(e_1, \dots, e_m, f_1, \dots, f_n)$  be a standard basis of  $V$ , then

$$(29) \quad \mathcal{B}er(V) = \mathcal{A} \mathcal{B}er(\pi e_1 \dots \pi e_m \xi^1 \dots \xi^n).$$

Let  $V, V'$  be two free  $\mathcal{A}$ -modules of finite type. Let  $\phi : V \rightarrow V'$  be an isomorphism. Then we define:

$$(30) \quad \begin{aligned} \mathcal{B}er(\phi) : \mathcal{B}er(V) &\rightarrow \mathcal{B}er(V') \\ \mathcal{B}er(\pi e_1 \dots \pi e_n \xi_1 \dots \xi_m) &\mapsto \mathcal{B}er(\pi \phi(e_1) \dots \pi \phi(e_n) (\phi^{-1})^*(\xi_1) \dots, (\phi^{-1})^*(\xi_m)). \end{aligned}$$

In the particular case where  $V' = V$ , the map  $\mathcal{B}er(\phi)$  coincides with the multiplication by  $\mathcal{B}er(\phi) \in \mathcal{A}$  defined in the preceding section. We put  $V^{\mathbb{R}} = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_m \oplus \mathbb{R}f_1 \oplus \dots \oplus \mathbb{R}f_n$  thus  $V = V^{\mathbb{R}} \otimes \mathcal{A}$ . Let  $\phi \in GL(V) \simeq GL(V^{\mathbb{R}})_{\mathcal{A}}$ . We have:

$$(31) \quad \mathcal{B}er(\pi \phi(e_1) \dots \pi \phi(e_m) (\phi^{-1})^*(\xi^1) \dots (\phi^{-1})^*(\xi^n)) = \mathcal{B}er(\phi) \mathcal{B}er(\pi e_1 \dots \pi e_m \xi^1 \dots \xi^n).$$

Now, for later use, we stress:

$$(32) \quad K(\Pi V^*) = S(\Pi(\Pi V^*)) \otimes S((\Pi V^*)^*) = S(V^*) \otimes S(\Pi V) = K(V)$$

Now,  $(\pi x^i)$  is a family of  $\Pi V^*$  and  $(\pi e_i)$  is a family of  $(\Pi V^*)^* = \Pi V$ . On the other hand, we have  $\pi(\pi x^i) = x^i$ . Thus, by rule of signs:

$$(33) \quad d_{\Pi V^*} = \sum_i x^i \pi e_i = d_V.$$

It follows that  $(K(V), d_V)$  and  $(K(\Pi V^*), d_{\Pi V^*})$  are canonically isomorphic. Therefore:

$$(34) \quad \mathcal{B}er(V) = H(K(V), d_V) = H(K(\Pi V^*), d_{\Pi V^*}) = \mathcal{B}er(\Pi V^*).$$

## 2. SOME SUPERGEOMETRY

### 2.1. Supervector bundles.

2.1.1. *Definition.* Let  $M$  be a supermanifold. We recall that a *supervector bundle*  $\mathcal{V}$  on  $M$  is defined by the locally free sheaf  $\Gamma_{\mathcal{V}}$  of  $\mathcal{O}_M$ -modules of its sections. More precisely,  $\Gamma_{\mathcal{V}}$  is a sheaf of  $\mathcal{O}_M$  modules such that locally,  $\Gamma_{\mathcal{V}}(\mathcal{U})$  is a free  $\mathcal{O}_M(\mathcal{U})$  module.

In this article, unless otherwise specified, the supervector bundle will be of finite rank denoted by  $(k, l)$ . This means that locally,  $\Gamma_{\mathcal{V}}(\mathcal{U})$  is a free  $\mathcal{O}_M(\mathcal{U})$  module of rank  $(k, l)$ . We put  $\text{rk}(\mathcal{V}) = (k, l)$ .

In particular the *tangent bundle*  $TM$  of  $M$  is the supervector bundle which sheaf of sections is  $\Gamma_{TM} = \mathcal{D}er_M$ , the sheaf of derivations of  $\mathcal{O}_M$ . We stress that since by derivation we mean left derivation  $\mathcal{D}er_M$  is naturally a sheaf of left  $\mathcal{O}_M$ -modules. As usual we provide it with a structure of right  $\mathcal{O}_M$ -module by the rule of signs.

Let  $x \in M_0$ . We put  $\mathfrak{m}_x = \{\phi \in \mathcal{O}_M(M_0) / \phi(x) = 0\}$ . We recall that  $\mathfrak{m}_x$  is a maximal ideal of  $\mathcal{O}_M(M_0)$  and that  $\mathcal{O}_M(M_0)/\mathfrak{m}_x \simeq \mathbb{R}$ .

Then we put  $\mathcal{V}_x = \Gamma_{\mathcal{V}}(M_0)/\mathfrak{m}_x = \Gamma_{\mathcal{V}}(M_0) \otimes_{\mathcal{O}_M(M_0)} \mathcal{O}_M(M_0)/\mathfrak{m}_x$ . It is the fibre of  $\mathcal{V}$  at  $x$ . If  $\mathcal{V} = TM$  we denote it by  $T_x M$ . It is a supervector space with  $\dim(\mathcal{V}_x) = \text{rk}(\mathcal{V})$ . For  $v \in \Gamma_{\mathcal{V}}(M_0)$  we denote by  $v(x)$  its image by the canonical projection on  $\mathcal{V}_x$ .

*Example:* assume that  $\mathcal{V}$  is trivial. In this case there is a supervector space  $V$  such that for any open subset  $\mathcal{U} \subset M_0$ , we have  $\Gamma_M(\mathcal{U}) = V \otimes \mathcal{O}_M(\mathcal{U})$ . Then for any  $x \in M_0$ ,  $\mathcal{V}_x = V$ . Moreover, let  $(e_i)$  be a basis of  $V$  and  $v = \sum_i e_i \phi_i \in \Gamma_{\mathcal{V}}(M_0)$  ( $\phi_i \in \mathcal{O}_M(M_0)$ ), we have  $v(x) = \sum_i e_i \phi_i(x)$ .

In general  $\mathcal{V}$  is not trivial but locally trivial: there exists a supervector space  $V$  such that for  $\mathcal{U}$  sufficiently small,  $\Gamma_{\mathcal{V}}(\mathcal{U}) \simeq V \otimes \mathcal{O}_M(\mathcal{U})$ . We call such an open subset  $\mathcal{U}$  a *trivialization subset* of  $\mathcal{V}$  and such a supervector space  $V$  the *generic fibre* of  $M$ . If  $\mathcal{U}$  is any open subset of  $M$  we denote by  $\mathcal{V}(\mathcal{U})$  the restriction of  $\mathcal{V}$  to  $M(\mathcal{U})$ .

Let  $M$  be a supermanifold. Let  $\mathcal{V}$  be a supervector bundle on  $M$ .

We denote by  $\mathcal{V}^*$  the dual supervector bundle of  $\mathcal{V}$ . It is the supervector bundle whose sheaf of sections is the dual sheaf of  $\Gamma_{\mathcal{V}}$ :  $\Gamma_{\mathcal{V}^*}(\mathcal{U}) = (\Gamma_{\mathcal{V}}(\mathcal{U}))^* = \text{Hom}(\Gamma_{\mathcal{V}}(\mathcal{U}), \mathcal{O}_M(\mathcal{U}))$ .

We denote by  $\Pi\mathcal{V}$  the supervector bundle whose sheaf of sections is  $\Pi\Gamma_{\mathcal{V}}$ :  $\Gamma_{\Pi\mathcal{V}}(\mathcal{U}) = \Pi\Gamma_{\mathcal{V}}(\mathcal{U})$ .

We denote by  $S(\mathcal{V})$  the supervector bundle whose sheaf of sections is  $S(\Gamma_{\mathcal{V}})$ :  $\Gamma_{S(\mathcal{V})}(\mathcal{U}) = S(\Gamma_{\mathcal{V}}(\mathcal{U}))$ .

Let  $\mathcal{V} \rightarrow M$  and  $\mathcal{W} \rightarrow M$  be two supervector bundles. We denote by  $\mathcal{V} \otimes_{\mathcal{O}_M} \mathcal{W}$  the supervector bundle on  $M$  whose sheaf of sections is  $\Gamma_{\mathcal{V}} \otimes_{\mathcal{O}_M} \Gamma_{\mathcal{W}}$ .

We denote by  $\mathcal{B}er(\mathcal{V})$  the *Berezinian bundle* of  $\mathcal{V}$ . It is the vector bundle on  $M$  whose sheaf of sections is

$$(35) \quad \mathcal{U} \mapsto \Gamma_{\mathcal{B}er(\mathcal{V})}(\mathcal{U}) = \mathcal{B}er(\Gamma_{\mathcal{V}}(\mathcal{U})).$$

Since  $\mathcal{B}er(\Gamma_{\mathcal{V}}) = \mathcal{B}er(\Pi\Gamma_{\mathcal{V}}^*)$  (cf. (34)) we canonically identify  $\mathcal{B}er(\mathcal{V})$  and  $\mathcal{B}er(\Pi\mathcal{V}^*)$ .

2.1.2. *Supermanifold structure.* Let  $M$  be a supermanifold. Let  $\mathcal{V}$  be a supervector bundle on  $M$ . As in the purely even case  $\Gamma_{\mathcal{V}}$  determines a supermanifold also denoted by  $\mathcal{V}$  with a canonical projection  $\pi : \mathcal{V} \rightarrow M$ .

Let  $\mathcal{U}$  be an open subset of  $M_0$ . Thus  $\pi_0^{-1}(\mathcal{U})$  is an open subset of  $\mathcal{V}_0$ . We have a canonical injection  $S(\Gamma_{\mathcal{V}^*}(\mathcal{U})) \hookrightarrow \mathcal{O}_{\mathcal{V}}(\pi_0^{-1}(\mathcal{U}))$ . We call the elements of  $S(\Gamma_{\mathcal{V}^*}(\mathcal{U}))$  the functions on  $\mathcal{V}$  polynomial in the fibres.

*Example:* If  $\mathcal{V}$  is trivial,  $\mathcal{V} = M \times V$  and

$$S(\Gamma_{\mathcal{V}^*}(\mathcal{U})) = S(V^*) \otimes \mathcal{O}_M(\mathcal{U}) \subset \mathcal{C}_V^\infty(V) \widehat{\otimes} \mathcal{O}_M(\mathcal{U}) \simeq \mathcal{O}_{\mathcal{V}}(\pi_0^{-1}(\mathcal{U})).$$

Let  $\mathcal{P}$  be a near superalgebra. Then  $\pi_{\mathcal{P}} : \mathcal{V}_{\mathcal{P}} \rightarrow M_{\mathcal{P}}$  is a vector bundle on  $M_{\mathcal{P}}$ . Let  $x \in M_{\mathcal{P}}$ . We denote by  $(\mathcal{V}_{\mathcal{P}})_x$  the fibre of  $\mathcal{V}_{\mathcal{P}}$  at  $x$ . We have  $\mathbf{b}(x) \in M_0$  and:

$$(\mathcal{V}_{\mathbf{b}(x)})_{\mathcal{P}} = (\mathcal{V}_{\mathcal{P}})_x = \{v \in \mathcal{V}_{\mathcal{P}} / \pi_{\mathcal{P}}(v) = x\}.$$

2.1.3. *Bilinear forms.* A bilinear form  $B$  on  $\mathcal{V}$  is a bilinear form on the sheaf  $\mathcal{O}_M$ -modules  $\Gamma_{\mathcal{V}}$ . Let  $x \in M_0$  and  $v_x, w_x \in \mathcal{V}_x$ . We choose  $v, w \in \Gamma_{\mathcal{V}}(M_0)$  such that  $v(x) = v_x$  and  $w(x) = w_x$ . We put

$$(36) \quad B_x(v_x, w_x) = B(v, w)_{\mathbb{R}}(x).$$

This definition does not depend on the choice of  $v$  and  $w$ . This defines a bilinear form  $B_x$  on the supervector space  $\mathcal{V}_x$ .

If  $B$  is antisymmetric, we denote by  $\mathfrak{spo}(\mathcal{V})$  the supervector bundle such that for any open subset  $\mathcal{U}$  of  $M_0$ :  $\Gamma_{\mathfrak{spo}(\mathcal{V})}(\mathcal{U}) = \mathfrak{spo}(\Gamma_{\mathcal{V}}(\mathcal{U}), B)$ . If  $B$  is symmetric we define  $\mathfrak{osp}(\mathcal{V})$  in the same way.

The form  $B$  determines a form  $\mathfrak{p}B$  on  $\Pi\mathcal{V}$  in the usual way.

2.1.4. *Rapidly decreasing functions along the fibres.* Let  $\pi : \mathcal{V} \rightarrow M$  be a supervector bundle on  $M$ . Let  $W$  be a Fréchet supervector space. Let  $\phi \in \mathcal{O}_{\mathcal{V}}(\mathcal{V}_0, W)$ . We say that  $\phi$  is *rapidly decreasing along the fibres* if for any trivialization subset  $\mathcal{U}$ , such that  $\Gamma_{\mathcal{V}}(\mathcal{U}) \simeq V \otimes \mathcal{O}_M(\mathcal{U})$  ( $V$  is the generic fibre of  $\mathcal{V}$ ),  $\phi \in \mathcal{S}_V(V_0) \widehat{\otimes}_{\mathcal{O}_M(\mathcal{U})} \mathcal{O}_M(\mathcal{U}, W)$ . We denote by  $\mathcal{S}_{\mathcal{V}}(\mathcal{V}_0, W)$  the set of rapidly decreasing functions along the fibres of  $\mathcal{V}$ .

Equivalently this means that for any near superalgebra  $\mathcal{P}$  and any  $x \in M_{\mathcal{P}}$ ,  $\phi_{\mathcal{P}}|_{(\mathcal{V}_{\mathcal{P}})_x} \in \mathcal{C}_{(\mathcal{V}_{\mathcal{P}})_x}^\infty((\mathcal{V}_{\mathcal{P}})_x, W)$  is rapidly decreasing.

2.1.5. *G-equivariant vector bundle.* Let  $G = (G_0, \mathfrak{g})$  be a supergroup. Let  $M$  be a  $G$ -supermanifold. Let  $\mathcal{V} \rightarrow M$  be a vector bundle. We say that  $\mathcal{V}$  is an equivariant vector bundle if there is a representation of  $G$  in the Fréchet supervector space  $\Gamma_{\mathcal{V}}(M_0)$  satisfying the following conditions. Let  $g \in G_0$ , we denote by  $v \in \Gamma_{\mathcal{V}}(M_0) \mapsto gv$  its action on  $\Gamma_{\mathcal{V}}(M_0)$  and we denote by  $\mathcal{L}^{\mathcal{V}} : \mathfrak{g} \mapsto \mathfrak{gl}(\Gamma_{\mathcal{V}}(M_0))$  the representation of  $\mathfrak{g}$ . For any non zero and homogeneous  $v, w \in \Gamma_{\mathcal{V}}(M_0)$ , any  $\phi \in \mathcal{O}_M(M_0)$  any  $g \in G_0$  and any non zero and homogeneous  $X \in \mathfrak{g}$  we require that:

$$(37) \quad \begin{aligned} g(v\phi) &= (gv)(g\phi) \\ \mathcal{L}^{\mathcal{V}}(X)(v\phi + w) &= (\mathcal{L}^{\mathcal{V}}(X)v)\phi + (-1)^{p(X)p(v)}v(X_M\phi). \end{aligned}$$



**2.2. Euclidean superstructure.** Let  $\mathcal{V} \rightarrow M$  be a supervector bundle. An *Euclidean structure* on  $\mathcal{V}$  is an even non degenerate symmetric bilinear form  $Q$  on  $\Gamma_{\mathcal{V}}$  such that for any  $x \in M_0$  the symmetric bilinear form  $Q_x$  on  $\mathcal{V}_x$  is positive definite on the even part  $(\mathcal{V}_x)_0$ . We stress that since  $Q$  is non degenerate,  $Q_x$  is non degenerate and thus if we forget the graduation,  $((\mathcal{V}_x)_1, Q_x|_{(\mathcal{V}_x)_1})$  is a symplectic vector space.

If we assume only that for any  $x$  in  $M_0$  the restriction of the form  $Q_x$  to  $(\mathcal{V}_x)_0$  is positive definite (we no longer assume that it is non degenerate, and thus  $(\mathcal{V}_x)_1$  is no longer symplectic), we say that  $Q$  is a *weak Euclidean structure*.

By definition a (weak) Euclidean structure on a supermanifold  $M$  is a (weak) Euclidean structure on  $TM$ .

We stress that if  $M$  has a (weak) Euclidean structure, then  $M_0$  is a Riemannian manifold.

### 2.3. Pseudodifferential forms.

2.3.1. *Definition.* Let  $M$  be a supermanifold and  $TM$  its tangent bundle. We put:

$$(38) \quad \widehat{M} = \Pi TM.$$

If  $\dim(M) = (n, m)$ , then  $\dim(\widehat{M}) = (n + m, n + m)$ . The pseudodifferential forms on  $M$  are the functions on  $\widehat{M}$  (cf. [BL77a, BL77b, Vor91]).

Let  $\widehat{\pi} : \widehat{M} \rightarrow M$  be the canonical projection of  $\widehat{M}$  onto  $M$ . Let  $\mathcal{U}$  be an open subset of  $M_0$ . We have  $\widehat{M}(\mathcal{U})_0 = \widehat{\pi}_0^{-1}(\mathcal{U})$ . We put

$$(39) \quad \widehat{\Omega}_M(\mathcal{U}) = \mathcal{O}_{\widehat{M}}(\widehat{M}(\mathcal{U})_0).$$

It is by definition the algebra of *pseudodifferential forms* on  $\mathcal{U}$ . We put  $\widehat{\Omega}(M) = \widehat{\Omega}_M(M_0)$ . Let  $W$  be a Fréchet supervector space we put similarly:

$$(40) \quad \widehat{\Omega}_M(\mathcal{U}, W) = \mathcal{O}_{\widehat{M}}(\widehat{M}(\mathcal{U})_0, W);$$

and  $\widehat{\Omega}(M, W) = \widehat{\Omega}_M(M_0, W)$ .

In particular, when  $W = \Gamma_{\mathcal{V}}(M_0)$  is the  $\mathcal{O}_M(M_0)$ -module of sections of a supervector bundle (possibly not of finite rank) on  $M$ , we put:

$$(41) \quad \widehat{\Omega}(M, \mathcal{V}) = \widehat{\Omega}(M, \Gamma_{\mathcal{V}}(M_0)) = \Gamma_{\mathcal{V}}(M_0) \underset{\mathcal{O}_M(M_0)}{\widehat{\otimes}} \widehat{\Omega}(M).$$

Let us look at this pseudodifferential forms in coordinates.

Let  $\mathcal{U}$  be a coordinates set of  $M$ . Let  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  be a local coordinates system on  $\mathcal{U}$  (with respective parities  $p(x^i) = 0$  and  $p(\xi^j) = 1$ ).

We denote by  $(dx^1, \dots, dx^m, d\xi^1, \dots, d\xi^n)$  the associated coordinates on the fibres of  $\widehat{M}$  ( $p(dx^i) = 1$  and  $p(d\xi^j) = 0$ ). They are defined as follows. Let  $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \xi^1}, \dots, \frac{\partial}{\partial \xi^n})$  be the local basis of sections of  $TM$  such that  $\frac{\partial}{\partial x^i} x^j = \delta_i^j$  (the Dirac symbol) and  $\frac{\partial}{\partial \xi^i} \xi^j = \delta_i^j$ . Then  $(\Pi(\frac{\partial}{\partial x^1}, \dots, \Pi \frac{\partial}{\partial x^m}, \Pi \frac{\partial}{\partial \xi^1}, \dots, \Pi \frac{\partial}{\partial \xi^n}))$  is a local basis of sections of  $\Pi TM$ . Let  $((\Pi \frac{\partial}{\partial x^1})^*, \dots, (\Pi \frac{\partial}{\partial x^m})^*, (\Pi \frac{\partial}{\partial \xi^1})^*, \dots, (\Pi \frac{\partial}{\partial \xi^n})^*)$  be its dual basis. We have

$$dx^i = (\Pi \frac{\partial}{\partial x^i})^* \text{ and } d\xi^j = (\Pi \frac{\partial}{\partial \xi^j})^*;$$

these are elements of  $\Gamma_{(\Pi TM)^*}(\mathcal{U})$  and thus functions on  $\widehat{M} = \Pi TM$  that are linear along the fibres.

*Case where  $M$  is an affine manifold  $V$ .* (In fact the case of a coordinates set is isomorphic to it) Let  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  be a basis of  $V^*$  (coordinates on  $V$ ). We have  $TV \simeq V \oplus V$  and  $\widehat{V} \simeq V \oplus \Pi V$ . Then  $(\pi x^1, \dots, \pi x^m, \pi \xi^1, \dots, \pi \xi^n)$  is a basis of  $(\Pi V)^*$ . We put

$$dx^i = \pi x^i \text{ and } d\xi^j = \pi \xi^j.$$

A pseudodifferential form  $\omega$  can be written locally as:

$$(42) \quad \omega = \sum_{I,J} dx^I \xi^J \omega_{I,J}(x, d\xi)$$

where the  $\omega_{I,J}$  are smooth functions of the variables  $x^i$  and  $d\xi^j$ .

Let  $M, N$  be two supermanifolds. Let  $\pi : M \rightarrow N$  be a morphisms of supermanifolds. Let  $\mathcal{U}$  be an open subset of  $N_{\mathbf{0}}$ . Then  $\pi$  induces as usual a morphism  $\pi^* : \widehat{\Omega}_N(\mathcal{U}) \rightarrow \widehat{\Omega}_M(\pi_0^{-1}(\mathcal{U}))$  (cf. [Man88]).

**2.3.2. Exterior differential...** Let  $\mathcal{U}$  be a coordinates set of  $M$ . On  $\widehat{\Omega}_M(\mathcal{U})$  the exterior differential is given by the odd vector field on  $\widehat{M}$ :

$$(43) \quad d = \sum_i dx^i \frac{\partial}{\partial x^i} + \sum_j d\xi^j \frac{\partial}{\partial \xi^j}.$$

Let  $\zeta = \sum_i f_i \frac{\partial}{\partial x^i} + \sum_j g_j \frac{\partial}{\partial \xi^j} \in \Gamma_{TM}(\mathcal{U})$  ( $f_i, g_j \in \mathcal{O}_M(\mathcal{U})$ ) be an homogeneous vector field on  $M$  (that is a derivation of  $\mathcal{O}_M(\mathcal{U})$ ). The inner product  $\iota(\zeta)$  and the Lie derivative  $\mathcal{L}(\zeta)$  of  $\zeta$  are defined by the following vector fields on  $\widehat{M}$ :

$$(44) \quad \iota(\zeta) = (-1)^{p(\zeta)} \left( \sum_i f_i \frac{\partial}{\partial x^i} + \sum_j g_j \frac{\partial}{\partial \xi^j} \right),$$

$$(45) \quad \mathcal{L}(\zeta) = [d, \iota(\zeta)] = d\iota(\zeta) + (-1)^{p(\zeta)} \iota(\zeta)d.$$

These definitions does not depend on the choice of coordinates and thus can be extended to any open subset  $\mathcal{U}$  of  $M_{\mathbf{0}}$ .

Thus  $\iota$  (resp.  $\mathcal{L}$ ) is an odd (resp. even) morphism of  $\mathcal{O}_M(\mathcal{U})$ -modules from  $\Gamma_{TM}(\mathcal{U})$  to  $\Gamma_{T\widehat{M}}(\widehat{M}(\mathcal{U}))$ . We recall the Cartan relations:

$$(46) \quad \mathcal{L}(\zeta)f = \zeta f.$$

$$(47) \quad \iota(\zeta)(df) = (-1)^{p(\zeta)} \zeta f,$$

$$(48) \quad [\mathcal{L}(\zeta), d] = 0,$$

$$(49) \quad [\iota(\gamma), \mathcal{L}(\zeta)] = \iota([\gamma, \zeta]),$$

$$(50) \quad [\iota(\gamma), \iota(\zeta)] = 0,$$

$$(51) \quad [\mathcal{L}(\gamma), \mathcal{L}(\zeta)] = \mathcal{L}([\gamma, \zeta]).$$

In particular  $\mathcal{L}$  is a morphism of sheaf of Lie superalgebras.

**2.4. Orientation.** (cf. [Vor91]) We say that a supermanifold  $M$  is *oriented* if  $M_{\mathbf{0}}$  is oriented.

We say that  $M$  is *globally oriented* if  $\widehat{M}$  is oriented.

If  $V$  is a supervector space, we say that it is oriented (resp. globally oriented) if  $V_{\mathbf{0}}$  (resp.  $V$  as a non graded vector space) is oriented.

We stress that in this case the definitions orientation and global orientation for the supervector space  $V$  or for its associated affine supermanifold coincides.

*Example:* Let  $V$  be an oriented symplectic supervector space. Then  $V$  is oriented in the above sense and also globally oriented. Thus the terminology is not too confusing.

For a supervector bundle  $\mathcal{V} \rightarrow M$  we say that  $\mathcal{V}$  is an oriented (resp. globally oriented) supervector bundle if  $\mathcal{V}_{\mathbf{0}} \rightarrow M_{\mathbf{0}}$  (resp.  $(\widehat{\mathcal{V}})_{\mathbf{0}} \rightarrow (\widehat{M})_{\mathbf{0}}$ ) is an oriented vector bundle.

*Example:* If  $\mathcal{V} = M \times V$  is a trivial supervector bundle. it is oriented (resp. globally oriented) if  $V$  is oriented (resp. globally oriented).

**2.5. Oriented symplectic supervector bundle.** Let  $\mathcal{V} \rightarrow M$  be a supervector bundle. We say that  $\mathcal{V}$  is symplectic if there is an even non degenerate antisymmetric bilinear form  $B$  on  $\mathcal{V}$ . If moreover  $\mathcal{V}$  is a globally oriented supervector bundle, we say that  $\mathcal{V}$  is an oriented symplectic supervector bundle. This is equivalent to say that its fibres are oriented symplectic supervector spaces or that  $\Gamma_{\mathcal{V}}$  is a sheaf of locally free oriented symplectic  $\mathcal{O}_M$ -modules.

We stress that if  $\mathcal{V}$  as an Euclidean structure  $Q$ , then  $(\Pi\mathcal{V}, \Pi Q)$  is a symplectic supervector bundle. Moreover, if  $\mathcal{V}$  is an oriented supervector bundle,  $\Pi\mathcal{V}$  is an oriented symplectic supervector bundle.

We stress that  $M$  is globally oriented if  $\mathcal{B}er(TM)$  is a trivial vector bundle on  $M$ . In this case, up to a multiplication by a function  $\phi \in \mathcal{O}_M(M_{\mathbf{0}})$  such that  $\phi_{\mathbb{R}} > 0$  there are exactly two different basis of  $\Gamma_{\mathcal{B}er(TM)}(M_{\mathbf{0}})$ . The choice of a global orientation of  $M$  corresponds to the choice of such a basis.

Let  $(x^i, \xi^j)$  be a local coordinates system of  $M$ . Let  $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}, \frac{\partial}{\partial \xi^1}, \dots, \frac{\partial}{\partial \xi^n})$  be the corresponding local basis of sections of  $TM$ . We say that  $(x^i, \xi^j)$  is globally oriented local system of coordinates if locally the orientation of  $M$  is defined by the basis:

$$\mathcal{B}er\left(\Pi \frac{\partial}{\partial x^1} \dots \Pi \frac{\partial}{\partial x^m} \xi^1 \dots \xi^n\right)$$

of  $\Gamma_{\mathcal{B}er(TM)}(M_{\mathbf{0}})$ .

For a supervector bundle it is globally oriented (resp. oriented) if  $\mathcal{B}er(\mathcal{V}) \rightarrow M$  (resp.  $\mathcal{B}er(\mathcal{V}_{\mathbf{0}}) \rightarrow M_{\mathbf{0}}$ ) is trivial. In this case, the choice of an orientation corresponds to a choice of a basis of  $\mathcal{B}er(\mathcal{V})$  (up to multiplication by  $\phi \in \mathcal{O}_M(M_{\mathbf{0}})$  with  $\phi_{\mathbb{R}} > 0$ ). We say that a local basis of section  $(e_i, f_j)$  with dual basis  $(x^i, \xi^j)$  is globally oriented if the orientation is defined by the basis:

$$\mathcal{B}er\left(\Pi e_1 \dots \Pi e_m \xi^1 \dots \xi^n\right)$$

of  $\Gamma_{\mathcal{B}er(\mathcal{V})}(M_{\mathbf{0}})$ .

**2.6. Superconnections.**

2.6.1. *Definitions.* Let  $\mathcal{V} \rightarrow M$  be a supervector bundle (possibly infinite dimensional). Let  $\Gamma_{\mathcal{V}}$  be its sheaf of sections. Let  $\Gamma_{\mathcal{V}} \otimes_{\mathcal{O}_M} \widehat{\Omega}_M$  be the sheaf of pseudodifferential forms on  $M$  with values in  $\mathcal{V}$ . For an open subset  $\mathcal{U}$  of  $M_0$  we put

$$\widehat{\Omega}_M(\mathcal{U}, \mathcal{V}) = \Gamma_{\mathcal{V}}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \widehat{\Omega}_M(\mathcal{U})$$

According to Quillen (cf. [MQ86, BGV92]), a *superconnection* on  $\mathcal{V}$  is an odd endomorphism of the sheaf of supervector spaces  $\Gamma_{\mathcal{V}} \otimes_{\mathcal{O}_M} \widehat{\Omega}_M$  such that, for  $\alpha \in \widehat{\Omega}_M(\mathcal{U})$  and  $\omega \in \widehat{\Omega}_M(\mathcal{U}, \mathcal{V})$  non zero and homogeneous:

$$(52) \quad \mathbb{A}(\omega\alpha) = (\mathbb{A}\omega)\alpha + (-1)^{p(\omega)}\omega(d\alpha).$$

Locally, on a trivialization subset  $\mathcal{U}$  of  $\mathcal{V}$  there is  $\omega \in \widehat{\Omega}_M(\mathcal{U}, \mathfrak{gl}(\mathcal{V}))_1$  such that:

$$(53) \quad \mathbb{A} = d + \omega.$$

The superconnection  $\mathbb{A}$  acts on  $\mathfrak{gl}(\Gamma_{\mathcal{V}}) \otimes_{\mathcal{O}_M} \widehat{\Omega}_M$  by means of the supercommutation bracket of endomorphisms of  $\Gamma_{\mathcal{V}} \otimes_{\mathcal{O}_M} \widehat{\Omega}_M$ .

The curvature of a superconnection is the operator  $F = \mathbb{A}^2 \in \mathfrak{gl}(\mathcal{V}) \otimes \widehat{\Omega}_M(\mathcal{U})$ . Locally,  $F = d\omega + \omega^2$ . It satisfies the Bianchi identity:

$$(54) \quad \mathbb{A}(F) = 0.$$

Let  $B$  be an even bilinear form on  $\mathcal{V}$ . We extend it  $\widehat{\Omega}_M$ -linearly to a bilinear form on  $\Gamma_{\mathcal{V}} \otimes_{\mathcal{O}_M} \widehat{\Omega}_M$ .

We say that  $\mathbb{A}$  preserves  $B$  (or leave  $B$  invariant) if for any non zero homogeneous  $\omega, \omega' \in \widehat{\Omega}_M(\mathcal{U}, \mathcal{V})$  we have:

$$(55) \quad B(\mathbb{A}\omega, \omega') + (-1)^{p(\omega)}B(\omega, \mathbb{A}\omega') = dB(\omega, \omega').$$

In this case when  $B$  is symmetric (resp. antisymmetric) we have:

$$(56) \quad F = \mathbb{A}^2 \in \mathfrak{osp}(\mathcal{V}) \otimes \widehat{\Omega}_M(\mathcal{U}) \text{ (resp. } \in \mathfrak{spo}(\mathcal{V}) \otimes \widehat{\Omega}_M(\mathcal{U}) \text{)}.$$

2.6.2. *“Induced” superconnections.* Let  $\mathcal{V} \rightarrow M$  be a supervector bundle and  $\mathbb{A}$  be a superconnection on  $\mathcal{V}$ . We denote by  $\mathbb{A}^{\Pi}$  the superconnection on  $\Pi\mathcal{V}$  defined for  $\omega \in \widehat{\Omega}(\mathcal{U}, \mathcal{V})$  by:

$$(57) \quad \mathbb{A}^{\Pi}(\Pi\omega) = -\Pi(\mathbb{A}\omega).$$

Recall that  $\widehat{\Omega}(\mathcal{U}, \Pi\mathcal{V})$  is the  $\widehat{\Omega}_M(\mathcal{U})$ -module  $\Gamma_{\Pi\mathcal{V}}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \widehat{\Omega}_M(\mathcal{U})$ . Now,  $\mathbb{A}$  determines also a superconnection  $\mathbb{A}^{\Pi*}$  (resp.  $\mathbb{A}^*$ ) on  $\Pi\mathcal{V}^*$  (resp.  $\mathcal{V}^*$ ) by the following formula. Let  $\alpha \in \widehat{\Omega}(\mathcal{U}, \Pi\mathcal{V}^*)$  (resp.  $\alpha \in \widehat{\Omega}(\mathcal{U}, \mathcal{V}^*)$ ) non zero and homogeneous and  $\beta \in \widehat{\Omega}(\mathcal{U}, \Pi\mathcal{V})$  (resp.  $\beta \in \widehat{\Omega}(\mathcal{U}, \mathcal{V})$ ):

$$(58) \quad \mathbb{A}^{\Pi*}(\alpha)(\beta) = (-1)^{p(\alpha)}\alpha(\mathbb{A}^{\Pi}\beta) \quad \text{(resp. } \mathbb{A}^*(\alpha)(\beta) = (-1)^{p(\alpha)}\alpha(\mathbb{A}\beta)\text{)}.$$

We extend  $\mathbb{A}^{\Pi*}$  (resp.  $\mathbb{A}^*$ ) to a derivation of the algebra  $\widehat{\Omega}(\mathcal{U}, S(\Pi\mathcal{V}^*)) = S(\widehat{\Omega}(\mathcal{U}, \Pi\mathcal{V}^*))$  (resp.  $\widehat{\Omega}(\mathcal{U}, S(\mathcal{V}^*)) = S(\widehat{\Omega}(\mathcal{U}, \mathcal{V}^*))$ ). Thus  $\mathbb{A}^{\Pi*}$  (resp.  $\mathbb{A}^*$ ) is a superconnection on

$S(\Pi\mathcal{V}^*)$  (resp.  $S(\mathcal{V}^*)$ ). It is defined for  $\alpha, \beta \in \widehat{\Omega}(\mathcal{U}, S(\Pi\mathcal{V}^*))$  (resp.  $\alpha, \beta \in \widehat{\Omega}(\mathcal{U}, S(\mathcal{V}^*))$ ) non zero and homogeneous,

$$(59) \quad \begin{aligned} \mathbb{A}^{\Pi*}(\alpha\beta) &= (\mathbb{A}^{\Pi*}\alpha)\beta + (-1)^{p(\alpha)}\alpha(\mathbb{A}^{\Pi*}\beta) \\ (\text{resp. } \mathbb{A}^{\Pi*}(\alpha\beta) &= (\mathbb{A}^{\Pi*}\alpha)\beta + (-1)^{p(\alpha)}\alpha(\mathbb{A}^{\Pi*}\beta). \end{aligned}$$

### 3. INTEGRATION

#### 3.1. Integration on a supervector space.

3.1.1. *Definition.* Let  $V$  be a supervector space,  $W$  be Fréchet supervector space and  $\mathcal{U}$  be an open subset of  $V_0$ .

We will denote by  $\mathcal{C}_{V,c}^\infty(\mathcal{U}, W)$  the subspace of  $\mathcal{C}_V^\infty(\mathcal{U}, W)$  of function with compact support.

The *distributions on  $V$  defined in  $\mathcal{U}$*  are the elements of the (Schwartz's) dual of  $\mathcal{C}_{V,c}^\infty(\mathcal{U})$ . If  $t$  is a distribution, we will use the notation

$$t(\phi) = \int_V t(v)\phi(v)$$

for  $\phi \in \mathcal{C}_{V,c}^\infty(\mathcal{U})$ . We will also use complex valued distributions, defined in an obvious way.

A *Berezin integral* (or *Haar*, or *Lebesgue*) is by definition a distribution on  $V$  which is invariant by translations (i.e. which vanishes on functions of the form  $\partial_X\phi$  where  $\phi \in \mathcal{C}_{V,c}^\infty(V)$  and  $\partial_X$  is the vector field on  $V$  with constant coefficients corresponding to  $X \in V$ : for  $f \in V^*$ ,  $\partial_X f = (-1)^{p(X)p(f)}f(X)$ ). Up to a multiplicative constant, there is exactly one Berezin integral, and it is an important matter in this article to choose a particular one for the symplectic oriented supervector spaces (see below and [Lav03]).

A choice of a standard system of coordinates determines a specific choice  $d_{(x,\xi)}$  of a Berezin integral by the formula

$$(60) \quad \begin{aligned} \int_V d_{(x,\xi)}(v)\phi(v) &= (-1)^{\frac{n(n-1)}{2}} \int_{\mathbb{R}^m} |dx^1 \dots dx^m| \phi_{(1,\dots,1)}(x^1, \dots, x^n) \\ &= \int_{\mathbb{R}^m} |dx^1 \dots dx^m| \left( \frac{\partial}{\partial \xi^1} \dots \frac{\partial}{\partial \xi^n} \phi \right)_{\mathbb{R}}(x^1, \dots, x^m), \end{aligned}$$

for  $\phi \in \mathcal{C}_{V,c}^\infty(V_0)$ , where  $|dx^1 \dots dx^m|$  is the Lebesgue measure on  $\mathbb{R}^m$ .

Note that this formula can also be applied to any  $\phi \in \mathcal{C}_{V,c}^\infty(\mathcal{U}, W)$  with a result in  $W$  and also to any rapidly decreasing

$$\phi \in \mathcal{C}_V^\infty(\mathcal{U}, W).$$

The choice of sign is such that Fubini's formula holds. More precisely, let  $V, W$  be two supervector spaces of dimensions  $(m, n)$  and  $(p, q)$ . Let  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$  be standard coordinates on  $V$  and  $(y^1, \dots, y^p, \eta^1, \dots, \eta^q)$  be standard coordinates on  $W$ . Then  $(x, y, \xi, \eta) = (x^1, \dots, x^m, y^1, \dots, y^p, \xi^1, \dots, \xi^n, \eta^1, \dots, \eta^q)$  defines standard coordinates on  $V \times W$ . Let  $\phi(v, w)$  is a smooth compactly supported function on  $V \times W$ . Then:

$$(61) \quad \int_{V \times W} d_{(x,y,\xi,\eta)}(v, w)\phi(v, w) = \int_V d_{(x,\xi)}(v) \left( \int_W d_{(y,\eta)}(w)\phi(v, w) \right).$$

We write:

$$(62) \quad d_{(x,y,\xi,\eta)}(v, w) = d_{(x,\xi)}(v)d_{(y,\eta)}(w).$$

In particular, since  $V = V_0 \oplus V_1$ ,  $d_{x,\xi} = d_x d_\xi$  and formula (60) is a particular case of formula (61).

Let us stress that if  $V_1$  is not  $\{0\}$ , in the setting of supermanifolds there is no natural notion of measure on  $V$  and no natural notion of positive distribution on  $V$ . Thus we use these notions only for (ungraded) vector spaces, or for the even part  $V_0$  of a supervector space, which is then regarded as an ungraded vector space. Otherwise, we use the terms *distribution* or *integral*.

In this article, we will be in fact interested by complex valued distributions. Then we allow standard basis  $(e_1, \dots, e_m, f_1, \dots, f_n)$  of  $V \otimes \mathbb{C}$ , where  $(e_1, \dots, e_m)$  is a basis of  $V_0$  and  $(f_1, \dots, f_n)$  is a basis of  $V_1 \otimes \mathbb{C}$ . Then the dual basis  $(x, \xi)$  provides a coordinate system  $(x)$  on  $V_0$  and a dual basis  $(\xi)$  of  $V_1^* \otimes \mathbb{C}$ . Any  $f \in C_{V,c}^\infty(V_0, \mathbb{C})$  can be written in the form (18), and the (complex) Berezin integral  $d_{(x,\xi)}$  is again well defined by formula (60).

3.1.2. *Change of variables.* Now, we recall the formula for change of variable in integration (cf. [Ber87] for example).

Let  $V$  be a supervector space. Let  $(x^i, \xi^j)$  be a standard basis of  $V^*$ . Let  $h : V \rightarrow V$  be an isomorphism of supermanifolds. We put  $y^i = \phi^* x^i$  and  $\eta^j = \phi^* \xi^j$ .

Let  $J(h)$  be the jacobian matrix of  $h$  defined by:

$$(63) \quad J(h) = \begin{pmatrix} \frac{\partial y_j}{\partial x_i} & \frac{\partial y_j}{\partial \xi_i} \\ \frac{\partial \eta_j}{\partial x_i} & \frac{\partial \eta_j}{\partial \xi_i} \end{pmatrix} \in C_V^\infty(V_0, \mathfrak{gl}(V))_0 = \mathfrak{gl}(V)_{C_V^\infty(V_0)}.$$

Moreover since  $h$  is an isomorphism,  $J(h)$  is invertible, and  $\text{Ber}_{(1,0)}(J(h))$  is the function on  $V$  such that for any near superalgebra  $\mathcal{P}$  and any  $v \in V_{\mathcal{P}}$ ,  $\text{Ber}_{(1,0)}(J(h))(v) = \text{Ber}_{(1,0)}(J(h)(v))$ .

Now the formula for change of variables is:

$$(64) \quad \int_V d_{(x,\xi)}(v) \text{Ber}_{(1,0)}(J(h))(v) \phi(h(v)) = \int_V d_{(x,\xi)}(v) \phi(v).$$

3.1.3. *Link with  $\mathcal{B}er(V)$ .* Assume that  $V$  is oriented. Let  $h : V \rightarrow V$  be an isomorphism of supermanifolds. Then if  $h$  preserves orientation, that is if  $(x^1, \dots, x^m)$  and  $(h^* x^1, \dots, h^* x^m)$  define the same orientation, we have:

$$(65) \quad \text{Ber}(J(h)) = \text{Ber}_{(1,0)}(J(h)).$$

Thus map:

$$d_{(x,\xi)} \mapsto \mathcal{B}er(\pi e_1 \dots \pi e_m \xi^1 \dots \xi^n)$$

induces an isomorphism from the  $(1, 0)$  or  $(0, 1)$ -dimensional (depending on the parity of  $n$ ) vector space of Berezin integrals on  $V$  and  $\mathcal{B}er(V)$ . This isomorphism is even or odd depending on the parity of  $m$ . When  $V$  is oriented we will identify  $d_{(x,\xi)}$  and  $\mathcal{B}er(\pi e_1 \dots \pi e_m \xi^1 \dots \xi^n)$ .

3.2. **Integration in symplectic oriented supervector spaces.** Let  $V = V_0 \oplus V_1$  be an oriented symplectic supervector space (cf. subsection 1.4).

Since such a space is a direct sum of  $(2, 0)$ -dimensional symplectic supervector spaces, and of  $(0, 1)$ -dimensional symplectic supervector spaces (i.e. 1-dimensional quadratic vector spaces). We first review these building blocks.

3.2.1. *Symplectic 2-dimensional vector spaces.* Let  $V = V_0$  a purely even 2-dimensional symplectic space. Let  $(x^1, x^2)$  be a symplectic coordinate system. It defines a *Liouville integral* (a particular normalization of the Berezin integral):

$$\phi \in \mathcal{C}_c^\infty(V) = \mathcal{C}_c^\infty(\mathbb{R}^2) \mapsto \int_V d_V(v)\phi(v) = \frac{1}{2\pi} \int |dx^1 dx^2| \phi(x^1, x^2).$$

3.2.2. *Symplectic 1-dimensional odd vector spaces.* Let  $V = V_1$  a purely odd 1-dimensional oriented symplectic supervector space. Let  $(\xi) \in V_1^* \cup \mathbf{i}V_1^*$  be a symplectic oriented coordinates system. It defines a *Liouville integral* (which is complex valued if  $B$  is negative definite)  $d_V$ :

$$\phi = a + \xi b \in \Lambda(V^* \otimes \mathbb{C}) \mapsto \int_V d_V(v)\phi(v) = b.$$

3.2.3. *General case.* Let us go back to the general case. Since  $V_0$  is a classical symplectic space, there is a canonical normalization of Lebesgue integral on  $V_0$ , the Liouville integral, which we recall now.

The dimension  $m$  of  $V_0$  is even. Let  $(x^i)$  be a symplectic coordinate system. The Liouville integral on  $V_0$  is

$$\frac{1}{(2\pi)^{m/2}} |dx^1 \dots dx^m|.$$

The Liouville integral does not depend on the choice of the symplectic basis of  $V_0$ .

Let  $(\xi^1, \dots, \xi^n)$  be an oriented symplectic coordinate system on  $V_1$ .

We call the corresponding Berezin integral  $d_\xi$  the Liouville integral of the oriented symplectic space  $V_1$ .

Let  $(x, \xi)$  be an oriented symplectic coordinates system on  $V$ . The associated Berezin integral  $\frac{1}{(2\pi)^{\frac{n}{2}}} d_{(x,\xi)}$  will be denoted by  $d_V$ .

**3.3. Generalized functions.** Let  $V$  be a finite dimensional supervector space and  $\mathcal{U} \subset V_0$  be an open set. Let  $(x, \xi)$  be a standard coordinates system on  $V$ . We will say that a distribution  $t$  on  $V$  defined in  $\mathcal{U}$  is *smooth* (resp. *smooth compactly supported*) if there is a function  $\psi \in \mathcal{C}_V^\infty(\mathcal{U})$  (resp.  $\psi \in \mathcal{C}_{V,c}^\infty(\mathcal{U})$ ) such that  $t(v) = d_{x,\xi}(v)\psi(v)$ . In particular, for any  $\phi \in \mathcal{C}_{V,c}^\infty(\mathcal{U})$  (resp.  $\phi \in \mathcal{C}_V^\infty(\mathcal{U})$ ):

$$(66) \quad t(\phi) = \int_V d_{x,\xi}(v)\psi(v)\phi(v).$$

This definition does not depend on the standard coordinates system  $(x, \xi)$ .

By definition, the *generalized functions on  $V$  defined on  $\mathcal{U}$*  are the elements of the (Schwartz's) dual of the space of smooth compactly supported distributions. For a generalized function  $\phi$  and a smooth compactly supported distribution  $t$ , we write:

$$(67) \quad \phi(t) = (-1)^{p(t)p(\phi)} \int_V t(v)\phi(v).$$

(The spaces of distributions and thus of generalized functions are naturally  $\mathbb{Z}/2\mathbb{Z}$ -graded.)

We denote by  $\mathcal{C}_V^{-\infty}(\mathcal{U})$  the set of generalized functions on  $V$  defined on  $\mathcal{U}$ .

Let us remark that, as  $\mathcal{C}_V^\infty(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U}) \otimes \Lambda(V_1^*)$ , we have:

$$(68) \quad \mathcal{C}_V^{-\infty}(\mathcal{U}) = \mathcal{C}^{-\infty}(\mathcal{U}) \otimes \Lambda(V_1^*).$$

Let  $W$  be a Fréchet supervector space. A  $W$ -valued generalized function is a continuous homomorphism (in sense of Schwartz) from the space of smooth compactly supported distributions on  $V$  to  $W$ . We denote by  $\mathcal{C}_V^{-\infty}(\mathcal{U}, W)$  the set of  $W$ -valued generalized functions. If  $W$  is finite dimensional, we have  $\mathcal{C}_V^{-\infty}(\mathcal{U}, W) = \mathcal{C}_V^{-\infty}(\mathcal{U}) \otimes W$ . We will be in particular concerned with the cases  $W = \mathbb{C}$  and  $W = \widehat{\Omega}(M)$  for some supermanifold  $M$ .

Let  $G$  be a Lie supergroup acting on  $V$ . Then, we have a representation of  $G$  on  $\mathcal{C}_{V,c}^{\infty}(V_0)$  and thus a representation of  $G$  on the  $\mathcal{C}_{V,c}^{\infty}(V_0)$ -module of compactly supported distributions and finally on  $\mathcal{C}_V^{-\infty}(V_0)$ .

We denote by  $\mathcal{C}_V^{-\infty}(V_0)^G$  the set of the  $G$ -invariant generalized functions on  $V$ .

**3.4. Integration of pseudodifferential forms.** Let  $M$  be a supermanifold. Let  $W$  be a Fréchet supervector space. Let  $\omega \in \widehat{\Omega}(M, W)$  be a pseudodifferential form on  $M$  with values in  $W$ .

We say that  $\omega$  is *integrable* if it is compactly supported on  $M_0$  and if rapidly decreasing along the fibres as a function on the supervector bundle  $\widehat{M} \rightarrow M$ .

We denote by  $\widehat{\Omega}_f(M, W)$  (resp.  $\widehat{\Omega}_f(M)$  when  $W = \mathbb{R}$ ) the set of integrable pseudodifferential forms on  $M$ .

*Example:* Assume that  $M$  is an affine supermanifold. Let  $(x^i, \xi^j)$  be a coordinates system on  $M$ . Let  $\omega \in \widehat{\Omega}(M)$ . We put:

$$\omega = \sum_{I,J} dx^I \xi^J \omega_{I,J}(x^i, d\xi^j).$$

Then  $\omega$  is integrable if it is a compactly supported function in the  $x^i$  and rapidly decreasing in the  $d\xi^j$ .

Let  $M$  be a globally oriented supermanifold of dimension  $(m, n)$ . Let  $\omega$  be an integrable  $W$ -valued pseudodifferential form which support is included in a coordinates domain  $\mathcal{U}$  with coordinates  $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$ . We put (cf. BL77a, BL77b, Vor91):

$$(69) \quad \int_M \omega = \int_{\widehat{M}(\mathcal{U})} d_{(x,d\xi,dx,\xi)} \omega.$$

It does not depend on the globally oriented coordinate system.

We extend this definition to a general integrable form on  $M$  by means of a partition of unity.

**3.5. Direct image of pseudodifferential forms.** Let  $M$  be a supermanifold of dimension  $(m, n)$  and  $\pi : \mathcal{V} \rightarrow M$  be a vector bundle of rank  $(k, l)$ .

Let  $W$  be a Fréchet supervector space. We say that  $\omega \in \widehat{\Omega}_{\mathcal{V}}(\mathcal{V}_0, W)$  is *integrable along the fibres* if  $\omega$  is rapidly decreasing along the fibres as a function on the supervector bundle  $\widehat{\mathcal{V}} \rightarrow \widehat{M}$ .

We denote by  $\widehat{\Omega}_{\pi_*}(\mathcal{V}, W)$  (resp.  $\widehat{\Omega}_{\pi_*}(\mathcal{V})$  when  $W = \mathbb{R}$ ) the set of pseudodifferential forms on  $\mathcal{V}$  that are integrable along the fibres.

When  $M$  and  $\mathcal{V}$  are globally oriented, and  $\omega \in \widehat{\Omega}_{\mathcal{V}}(\mathcal{V}, W)$  is integrable along the fibres, we define the direct image  $\pi_*\omega$  of  $\omega$  as the unique pseudodifferential form on  $M$  such



that for any pseudodifferential form  $\alpha$  on  $M$  which is compactly supported in  $(\widehat{M})_{\mathbf{0}}$  we have (cf. [BGV92] with an other normalization):

$$(70) \quad \int_M (\pi_* \omega) \alpha = \frac{1}{(2\pi)^{\frac{k+l}{2}}} \int_{\mathcal{V}} \omega(\pi^* \alpha).$$

Let  $\alpha \in \widehat{\Omega}_{\mathcal{V}}(\mathcal{V}, W)$  be integrable along the fibres and  $\beta \in \widehat{\Omega}_M(M)$ , then:

$$(71) \quad \pi_*(\alpha \pi^*(\beta)) = (\pi_* \alpha) \beta.$$

Indeed, since  $\pi^* : \widehat{\Omega}_M(M) \rightarrow \widehat{\Omega}_{\mathcal{V}}(\mathcal{V})$  is a morphism of superalgebras, if  $\alpha \in \widehat{\Omega}_M(M)$  is compactly supported on  $(\widehat{M})_{\mathbf{0}}$ , we have:

$$(72) \quad \int_{\mathcal{V}} \alpha(\pi^* \beta)(\pi^* \gamma) = \int_{\mathcal{V}} \alpha \pi^*(\beta \gamma) = (2\pi)^{\frac{k+l}{2}} \int_M (\pi_* \alpha) \beta \gamma.$$

The application  $\pi_* : \widehat{\Omega}_{\mathcal{V}}(\mathcal{V}, \mathcal{A}) \rightarrow \widehat{\Omega}_M(M, \mathcal{A})$  is a morphism of  $\widehat{\Omega}_M(M)$ -modules with parity  $k + l \pmod{\mathbb{Z}/2\mathbb{Z}}$ . Indeed, the integration on  $\mathcal{V}$  is an operator of parity  $k + l + m + n \pmod{\mathbb{Z}/2\mathbb{Z}}$  and the integration on  $M$  has parity  $m + n \pmod{\mathbb{Z}/2\mathbb{Z}}$ .

We denote by  $d_{\mathcal{V}}$  (resp.  $d_M$ ) the exterior differential on  $\mathcal{V}$  (resp.  $M$ ). Let  $\omega \in \widehat{\Omega}_{\mathcal{V}}(\mathcal{V}, \mathcal{A})$  be integrable along the fibres, then we have:

$$(73) \quad d_M \pi_* \omega = (-1)^{k+l} \pi_*(d_{\mathcal{V}} \omega).$$

### 3.6. Integration along the fibres.

3.6.1. *Definition.* Let  $\pi : \mathcal{V} \rightarrow M$  be a supervector bundle of rank  $(k, l)$ .

We call *volume form on the fibres*  $\mathcal{V}$  the sections of the bundle

$$\mathcal{V}ol(\mathcal{V}) = \mathcal{B}er(\mathcal{V}) \times_M \mathcal{V} \rightarrow \mathcal{V}.$$

We have for an open subset  $\mathcal{U} \subset M_{\mathbf{0}}$ :

$$\Gamma_{\mathcal{V}ol(\mathcal{V})}(\pi_{\mathbf{0}}^{-1}(\mathcal{U})) = \Gamma_{\mathcal{B}er(\mathcal{V})}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \mathcal{O}_{\mathcal{V}}(\pi_{\mathbf{0}}^{-1}(\mathcal{U})).$$

The canonical inclusion  $M \hookrightarrow \mathcal{V}$  by means of the zero section (it corresponds to the canonical projection  $S(\Gamma_{\mathcal{V}^*}) \rightarrow \mathcal{O}_M$ ) induces a canonical inclusion

$$\mathcal{B}er(\mathcal{V}) = \mathcal{B}er(\mathcal{V}) \times_M M \hookrightarrow \mathcal{V}ol(\mathcal{V}).$$

We call the sections of  $\mathcal{B}er(\mathcal{V})$  the *Berezinians volume forms along the fibres of*  $\mathcal{V}$ .

Let  $\mathcal{U} \subset M$  be an open subset. We say that a volume form  $\mathcal{D} \in \Gamma_{\mathcal{V}ol(\mathcal{V})}(\pi_{\mathbf{0}}^{-1}(\mathcal{U}))$  is rapidly decreasing along the fibres if

$$\mathcal{D} \in \Gamma_{\mathcal{B}er(\mathcal{V})}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \mathcal{S}_{\mathcal{V}}(\pi_{\mathbf{0}}^{-1}(\mathcal{U})).$$

Now we assume that  $\mathcal{V} \rightarrow M$  is an oriented supervector bundle.

Let  $\mathcal{U} \subset M$  be an open subset. To any volume form  $\mathcal{D} \in \Gamma_{\mathcal{V}ol(\mathcal{V})}(\pi_{\mathbf{0}}^{-1}(\mathcal{U}))$  on the fibres of  $\mathcal{V}$  we associate canonically a distribution on  $\mathcal{V}$  defined on  $\pi_{\mathbf{0}}^{-1}(\mathcal{U})$  with values in  $\mathcal{O}_M(\mathcal{U})$ .

Assume that  $\mathcal{U}$  is a trivialization subset of  $\mathcal{V}$ . Let  $(e_i, f_j)$  be a standard basis of  $\Gamma_{\mathcal{V}}(\mathcal{U})$  (as  $\mathcal{O}_M(\mathcal{U})$ -module). Let  $(x^i, \xi^j)$  be its dual basis. Let  $V = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k \oplus \mathbb{R}f_1 \oplus \cdots \oplus \mathbb{R}f_l$  be the generic fibre of  $\mathcal{V}$ .

We recall that we identify the Berezin integral  $d_{(x,\xi)}$  on  $V$  with a section of  $\mathcal{B}er(V) \subset \mathcal{B}er(\mathcal{V}) \simeq \mathcal{B}er(V) \otimes \mathcal{O}_M(\mathcal{U})$ . Then there exists  $h_{\mathcal{U}} \in \mathcal{O}_{\mathcal{V}}(\pi_0^{-1}(\mathcal{U}))$  such that  $\mathcal{D}|_{\mathcal{U}} = d_{(x,\xi)}h_{\mathcal{U}}$ .

On the other hand we have  $\mathcal{O}_{\mathcal{V}}(\pi_0^{-1}(\mathcal{U})) = \mathcal{C}_V^{\infty}(V_0, \mathcal{O}_M(\mathcal{U}))$ .

Let  $W$  be a Fréchet supervector space. Let

$$\phi \in \mathcal{C}_{V,c}(V_0, \mathcal{O}_M(\mathcal{U}) \otimes W) \subset \mathcal{O}_{\mathcal{V}}(\pi_0^{-1}(\mathcal{U}), W)$$

be a function on  $\mathcal{V}$  with values in  $W$  that is rapidly decreasing along the fibres. We assume moreover that  $\phi$  is supported in  $\mathcal{U}$ . Let  $\mathcal{P}$  be a near superalgebra. We have  $\mathcal{V}_{\mathcal{P}}(\pi_0^{-1}(\mathcal{U})) = V_{\mathcal{P}} \times M_{\mathcal{P}}(\mathcal{U})$ . For  $v \in V_{\mathcal{P}}$ , we have  $\phi(v) \in \mathcal{C}_{M_{\mathcal{P}}}^{\infty}(M_{\mathcal{P}}(\mathcal{U}), W_{\mathcal{P}})$  and we write  $\phi(v, m)$  for its value at  $m \in M_{\mathcal{P}}(\mathcal{U})$ .

We put for any near superalgebra  $\mathcal{P}$  and any  $m \in M_{\mathcal{P}}(\mathcal{U})$ :

$$(74) \quad \left( \int_{\mathcal{V}/M} \mathcal{D}\phi \right)(m) = \int_{\mathcal{V}/M} \mathcal{D}_m(v)\phi(v, m) = \int_V d_{(x,\xi)}(v) h_{\mathcal{U}}(v, m)\phi(v, m) \in \mathcal{O}_M(\mathcal{U}).$$

This formula defines  $\int_{\mathcal{V}/M} \mathcal{D}\phi$  as a function in  $\mathcal{O}_M(\mathcal{U})$ . The formula for change of variables shows that it does not depends on the choice of the basis  $(e_i, f_j)$ .

We extend it to any  $\phi \in \mathcal{O}_{\mathcal{V}}(\mathcal{V}_0)$  that is rapidly decreasing along the fibres by means of a partition of unity on  $M_0$  associated with trivialization subsets. It is called the *integral along the fibres* of  $\phi$  against the volume form  $\mathcal{D}$ .

In the same way we define the integral along the fibres of a rapidly decreasing volume form  $\mathcal{D}$  and we denote it by  $\int_{\mathcal{V}/M} \mathcal{D} \in \mathcal{O}(M)$ .

*Example 1:* Let  $M$  be a point and  $\mathcal{V} = \widehat{V}$  for some supervector spaces  $V$ . Assume that  $\mathcal{V}$  is oriented (this means that  $V$  is globally oriented). Let  $(x, \xi)$  be standard globally oriented coordinates on  $V$ . Then  $d_{(x,d\xi,dx,\xi)}$  defines a canonical volume form on  $\widehat{V}$ . It is canonical in the sense that it does not depends on the choice of standard globally oriented coordinates  $(x, \xi)$ . Then, integration of integrable pseudodifferential forms on  $V$  is integration of functions on  $\mathcal{V} = \widehat{V}$  against this canonical volume form.

*Example 2:* More generally, we can do the same construction when  $M$  is not trivial. We consider  $\mathcal{V} \rightarrow M$  a globally oriented supervector bundle and  $\widehat{\mathcal{V}} \rightarrow \widehat{M}$  be defined as usual (it is an oriented supervector bundle). We replace  $(x, \xi)$  by a standard globally oriented basis of sections of  $\mathcal{V}^* \rightarrow M$ . We obtain a canonical volume form on the fibres of  $\widehat{\mathcal{V}} \rightarrow \widehat{M}$  (since  $\mathcal{V}$  is a globally oriented supervector bundle,  $\mathcal{B}er(\widehat{\mathcal{V}}) \rightarrow \widehat{M}$  is trivial). We denote it  $d_{(x,d\xi,dx,\xi)}$ . Let  $\pi : \mathcal{V} \rightarrow M$  be the projection of  $\mathcal{V}$  onto  $M$ . Then, the direct image  $\pi_*\omega$  of a pseudodifferential form integrable along the fibres is the integration of  $\omega$  along the fibres of  $\widehat{\mathcal{V}} \rightarrow \widehat{M}$  against  $\frac{1}{(2\pi)^{\frac{k+l}{2}}}d_{(x,d\xi,dx,\xi)}$ .

**3.6.2. Symplectic oriented case.** Assume that  $\mathcal{V}$  is a symplectic oriented supervector bundle.

Let  $\mathcal{U} \subset M_0$  be a trivialization subset and  $V$  be the generic fibre. Then  $V$  is an oriented symplectic supervector space. In formula (74) we can assume that  $(e_i, f_j)$  is a symplectic oriented basis. In this case we take the Liouville integral  $d_V = \frac{1}{(2\pi)^{\frac{l}{2}}}d_{(x,\xi)}$  on  $V$  (instead of  $d_{(x,\xi)}$ ) which does not depend on the choice of the symplectic oriented symplectic basis  $(e_i, f_j)$ . We can take  $h_{\mathcal{U}}$  to be constant equal to 1. Thus we obtain a

canonical oriented symplectic volume form on the fibres of  $\mathcal{V}$  called the *Liouville volume form* and denoted by  $\mathcal{D}_{\mathcal{V}} \in \Gamma_{\mathcal{B}er(\mathcal{V})}(M_{\mathbf{0}})$ .

**3.6.3. Volume forms and superconnection.** Let  $\mathbb{A}$  be a superconnection on  $\mathcal{V}$ . We recall that  $\mathbb{A}$  determines a superconnection  $\mathbb{A}^*$  on  $S(\mathcal{V}^*)$  by formulas (58) and (59). Then, we extend  $\mathbb{A}^*$  by continuity to a derivation of  $\widehat{\Omega}_M(\mathcal{U}, \mathcal{O}(\mathcal{V}))$ .

There is a superconnection  $\mathbb{A}^{\mathcal{B}er}$  on  $\mathcal{B}er(\mathcal{V})$  defined locally as follows. Let  $\mathcal{U}$  be a trivialization set of  $\mathcal{V}$ . Let  $V$  be a generic fibre of  $\mathcal{V}$ . On  $\mathcal{U}$ , we have  $\mathbb{A} = d + \omega$  where  $d$  is the exterior differential and  $\omega$  is an odd pseudodifferential form on  $U$  with values in  $\mathfrak{gl}(V)$ . Then we put  $\mathbb{A}^{\mathcal{B}er} = d + str(\omega)$ . This does not depend on the trivialization and thus defines a superconnection on  $\mathcal{B}er(\mathcal{V})$ .

We extend it to a superconnection  $\mathbb{A}^{\mathcal{V}ol}$  on  $\mathcal{V}ol(\mathcal{V})$  by the following.

Let  $\phi \in \widehat{\Omega}(M, \mathcal{O}(\mathcal{V})) = \mathcal{O}(\mathcal{V}) \widehat{\otimes}_{\mathcal{O}(M)} \widehat{\Omega}(M)$  and  $\mathcal{D}$  be a volume form on the fibres. Then if  $\text{rk}(\mathcal{V}) = (k, l)$ :

$$(75) \quad \mathbb{A}^{\mathcal{V}ol}(\mathcal{D}\phi) = (\mathbb{A}^{\mathcal{V}ol}\mathcal{D})\phi + (-1)^k \mathcal{D}(\mathbb{A}^*\phi).$$

Since, locally,  $\mathbb{A}^*$  is the sum of  $d$  (the differential of  $\widehat{\Omega}(M)$ ) and of a derivation of  $\mathcal{O}(\mathcal{V})$  with coefficients in  $\widehat{\Omega}(M)$  (in particular, they are constant in the fibres), we have:

$$(76) \quad \int_{\mathcal{V}/M} \mathbb{A}^{\mathcal{V}ol}(\mathcal{D}\phi) = d \int_{\mathcal{V}/M} \mathcal{D}\phi.$$

Moreover, if we suppose that  $\mathbb{A}^{\mathcal{V}ol}\mathcal{D} = 0$ , then we have:

$$(77) \quad (-1)^k \int_{\mathcal{V}/M} \mathcal{D}(\mathbb{A}\phi) = d \int_{\mathcal{V}/M} \phi \mathcal{D}.$$

Now, assume that  $\mathcal{V}$  is an oriented symplectic supervector space. Let  $\mathbb{A}$  be a superconnection on  $\mathcal{V}$  which leaves the symplectic structure invariant. Let  $\mathcal{U} \subset M_{\mathbf{0}}$  be a trivialization subset of  $\mathcal{V}$ . On  $\mathcal{U}$ ,  $\mathbb{A} = d + \omega$  with  $\omega \in \widehat{\Omega}(\mathcal{U}, \mathfrak{spo}(\mathcal{V}))$ . Let  $\mathcal{D}_{\mathcal{V}}$  be the Liouville volume form on the fibres of  $\mathcal{V}$ . Since on  $\mathfrak{spo}(\Gamma_{\mathcal{V}}(\mathcal{U}))$  the supertrace is zero, we have:

$$(78) \quad \mathbb{A}\mathcal{D}_{\mathcal{V}} = str(\omega) \mathcal{D}_{\mathcal{V}} = 0.$$

It follows, that for any  $\phi \in \widehat{\Omega}(M, \mathcal{O}(\mathcal{V})) = \mathcal{O}(\mathcal{V}) \widehat{\otimes}_{\mathcal{O}(M)} \widehat{\Omega}(M)$  such that  $\phi$  and  $\mathbb{A}^*\phi$  are rapidly decreasing along the fibres:

$$(79) \quad \int_{\mathcal{V}/M} \mathcal{D}_{\mathcal{V}}(\mathbb{A}^*\phi) = d \int_{\mathcal{V}/M} \mathcal{D}_{\mathcal{V}}\phi.$$

This result extends naturally to the equivariant situation defined in the next section.

### 3.7. Fourier transform.

3.7.1. *Fourier transform of smooth rapidly decreasing distributions.* Let  $V$  be a supervector space of dimension  $(n, m)$ . Let  $t$  be a smooth rapidly decreasing distribution on  $V$ . We define the *Fourier transform of  $t$*  as the function  $\widehat{t}$  on  $V^*$  defined by:

$$(80) \quad \widehat{t}(h) = \int_V t(v) \exp(-\mathbf{i}h(v)).$$

This means that for any near superalgebra  $\mathcal{P}$  and any  $h \in V_{\mathcal{P}}^*$ ,  $\widehat{t}(h)$  is given by the integral on the right hand side.

We stress that, this implies that  $\widehat{t}$  is a rapidly decreasing function on  $V^*$ .

3.7.2. *Fourier transform of rapidly decreasing functions.* Let  $V$  be a supervector space of dimension  $(n, m)$ . Let  $\phi$  be a rapidly decreasing function on  $V$ . We define the *Fourier transform of  $\phi$*  as the distribution  $\widehat{\phi}$  on  $V^*$  defined by the following. Let  $(e_i, f_j)$  be a standard basis of  $V$ . Let  $(x^i, \xi^j)$  be its dual basis. We put:

$$(81) \quad \widehat{\phi}(h) = \frac{(-1)^{\frac{n(n-1)}{2}} \mathbf{i}^n}{(2\pi)^m} d_{(e_i, f_j)}(h) \int_V d_{(x, \xi)}(v) \phi(v) \exp(\mathbf{i}h(v)).$$

The formula for change of variables (64) implies that this formula does not depends on the choice of  $(e_i, f_j)$ .

Fourier inversion formula reads:

$$(82) \quad \widehat{\widehat{\phi}}(v) = \phi(v).$$

(Cf. below the case of a volume forms on the fibres of a supervector bundle for an other justification of the normalization)

*Example:*  $\dim(V) = (0, 1)$ . Since the purely even case is well known we examine the purely odd case. The simplest one is a one dimensional purely odd vector space  $V$ . Let  $(f)$  be a base of  $V$ . Let  $(\xi)$  be its dual basis. Let  $\phi = a + \xi b \in \mathcal{C}_V^\infty(V_{\mathbf{0}})$  be any function on  $V$ . Then,  $\phi$  is rapidly decreasing.

Let  $h = -\xi f \in V^* \otimes V$  be the generic point of  $V^*$ . Then

$$(83) \quad \begin{aligned} \widehat{\phi} &= \widehat{\phi}(h) = \mathbf{i} d_{(f)}(h) \int_V d_{(\xi)}(v) \phi(v) \exp(\mathbf{i}h(v)) \\ &= \mathbf{i} d_{(f)}(h) \int_V d_{(\xi)}(v) ((a + \xi b)(1 - \mathbf{i}\xi f))(v) \\ &= \mathbf{i} d_{(f)}(h)(b - \mathbf{i}a f)(h) \end{aligned}$$

Let  $v = f\xi \in V \otimes V^*$  be the generic point of  $V$ . Then we have:

$$(84) \quad \begin{aligned} \widehat{\widehat{\phi}} &= \widehat{\widehat{\phi}}(v) = \mathbf{i} \int_{V^*} d_{(f)}(h)(b - \mathbf{i}a f)(h) \exp(-\mathbf{i}v(h)) \\ &= \mathbf{i} \int_{V^*} d_{(f)}(h) ((b - \mathbf{i}a f)(1 - \mathbf{i}f\xi))(h) \\ &= (a + b\xi)(v) = \phi(v) \end{aligned}$$

3.7.3. *Fourier transform on the fibres of a supervector bundle.* Let  $\mathcal{V} \rightarrow M$  be a supervector bundle of rank  $(k, l)$ . Let  $\mathcal{D}$  be a rapidly decreasing volume form along the fibres of  $\mathcal{V}$ . We define the Fourier transform of  $\mathcal{D}$  as the function  $\widehat{\mathcal{D}}$  on  $\mathcal{V}^*$  defined by:

$$(85) \quad \widehat{\mathcal{D}}(h) = \int_{\mathcal{V}/M} \mathcal{D}(v) \exp(-ih(v)).$$

We stress that this definition coincides with the precedent one if  $M$  is a point and thus  $\mathcal{V}$  is a supervector space.

The supervector space  $\widetilde{\mathcal{V}} = \mathcal{V}^* \oplus \mathcal{V}$  has a standard symplectic structure  $B$  defined as follows. Let  $\mathcal{U} \subset M_{\mathbf{0}}$  be a trivialization subset of  $\mathcal{V}$ . Let  $(g_i)$  be an homogeneous basis of sections of  $\mathcal{V}$  on  $\mathcal{U}$ . Let  $(z^i)$  be its dual basis. We put  $B(\Gamma_{\mathcal{V}}, \Gamma_{\mathcal{V}}) = B(\Gamma_{\mathcal{V}^*}, \Gamma_{\mathcal{V}^*}) = 0$  and  $B(z^i, g_j) = \delta_i^j$ . This does not depend on the choice of  $(g_i)$  and thus defines  $B$  globally.

Moreover we give to  $\widetilde{\mathcal{V}}$  the orientation defined by the basis  $(z^i, g_i)$  (it is independent to the choice of the basis  $(g_i)$ ). We recall that its dual basis is  $((-1)^{p(g_i)} g_i, z^i)$ .

Let  $\mathcal{D}_{\widetilde{\mathcal{V}}}$  be the canonical Liouville volume form along the fibres of  $\widetilde{\mathcal{V}}$ . Let  $\mathcal{D}$  be a Berezin volume form on  $\mathcal{V}$ . Since  $\mathcal{B}er(\widetilde{\mathcal{V}}) = \mathcal{B}er(\mathcal{V}^*) \otimes_M \mathcal{B}er(\mathcal{V})$ , there is an unique Berezin volume form  $\mathcal{D}^*$  on  $\mathcal{V}^*$  such that  $\mathcal{D}^* \otimes \mathcal{D} = \mathcal{D}_{\widetilde{\mathcal{V}}}$ .

Let  $\phi \in \mathcal{S}_{\mathcal{V}}(\mathcal{V}_{\mathbf{0}})$ . We define the Fourier transform of  $\phi$  as the volume form  $\widehat{\phi}$  along the fibres of  $\mathcal{V}^*$  defined by:

$$(86) \quad \widehat{\phi}(h, m) = \mathcal{D}_m^*(h) \int_{\mathcal{V}/M} \mathcal{D}_m(v) \phi(v, m) \exp(ih(v))$$

Once more this definition does not depends on  $\mathcal{D}$  and coincides with the preceding one when  $M$  is a point.

We check this assertion in the fundamental cases of a  $(1, 0)$ ,  $(0, 1)$  and  $(0, n)$  dimensional supervector spaces.

*Case of a  $(1, 0)$  dimensional vector space.* Let  $(e)$  be a basis of  $\mathcal{V}$ . Let  $(x)$  be its dual basis. Then  $(x, e)$  is an oriented symplectic basis of  $\widetilde{\mathcal{V}} = \mathcal{V}^* \oplus \mathcal{V}$ . Its dual basis is  $(e, x)$ , and  $\mathcal{D}_{\widetilde{\mathcal{V}}} = \frac{1}{2\pi} d_{(e,x)} = \frac{1}{2\pi} d_{(e)} d_{(x)}$ . Thus  $d_{(x)}^* = \frac{1}{2\pi} d_{(e)}$ .

*Case of a  $(0, 1)$  dimensional vector space.* Let  $(f)$  be a basis of  $\mathcal{V}$ . Let  $(\xi)$  be its dual basis. Then  $(\frac{f+\xi}{\sqrt{2}}, \mathbf{i}\frac{\xi-f}{\sqrt{2}})$  is an oriented symplectic basis of  $\widetilde{\mathcal{V}} = \mathcal{V}^* \oplus \mathcal{V}$ . In fact it is a basis of  $\widetilde{\mathcal{V}} \otimes \mathbb{C}$ . Its dual basis is  $(\frac{\xi-f}{\sqrt{2}}, \mathbf{i}\frac{\xi+f}{\sqrt{2}})$  and  $\mathcal{D}_{\widetilde{\mathcal{V}}} = d_{(\frac{\xi-f}{\sqrt{2}}, \mathbf{i}\frac{\xi+f}{\sqrt{2}})} = \mathbf{i} d_{(f)} d_{(\xi)}$ . Thus  $d_{(\xi)}^* = \mathbf{i} d_{(f)}$ .

*Case of a  $(0, n)$  dimensional vector space.* We have to be very carrefull on orientation. Let  $(f_i)$  be a basis of  $\mathcal{V}$  with dual basis  $(\xi^j)$ . The symplectic form  $B$  on the odd supervector space  $\widetilde{\mathcal{V}}$  is a quadratic form of signature  $(n, n)$ . The orientation of the real supervector space  $\widetilde{\mathcal{V}}$  corresponding to the symplectic basis  $(\frac{f+\xi}{\sqrt{2}}, \mathbf{i}\frac{\xi-f}{\sqrt{2}}, \dots, \frac{f+\xi}{\sqrt{2}}, \mathbf{i}\frac{\xi-f}{\sqrt{2}})$  is given by:

$$(87) \quad (-\mathbf{i})^n (-\mathbf{i} f_1 \xi^1) \dots (-\mathbf{i} f_n \xi^n) = (-1)^n (-1)^{\frac{n(n-1)}{2}} f_1 \dots f_n \xi^1 \dots \xi^n.$$

On the other hand, the orientation of  $\widetilde{\mathcal{V}}$  corresponding to  $(\xi^1, \dots, \xi^n, f_1, \dots, f_n)$  is given by:

$$(88) \quad (-1)^n f_1 \dots f_n \xi^1 \dots \xi^n.$$

Thus we obtain:

$$(89) \quad d_{(\xi^j)^*} = (-1)^{\frac{n(n-1)}{2}} \mathbf{i}^n d_{(f_j)}.$$

Now the Fourier inversion formula reads:

$$(90) \quad \widehat{\widehat{\phi}} = \phi.$$

#### 4. EQUIVARIANT COHOMOLOGY

4.1. **Definitions.** Let  $M$  be a supermanifold and  $G = (G_0, \mathfrak{g})$  be a supergroup acting on  $M$  on the right. Let  $\mathcal{U} \subset \mathfrak{g}_0$  be a  $G_0$  invariant open subset. Let  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C}))$  be the superalgebra of the functions on  $\mathcal{U}$  with values in the complex pseudodifferential forms on  $M$ :

$$(91) \quad \mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C})) = \mathcal{O}_{\mathfrak{g} \times \widehat{M}}(\mathcal{U} \times (\widehat{M})_0, \mathbb{C}) = \mathcal{O}_{\mathfrak{g}}(\mathcal{U}) \widehat{\otimes} \widehat{\Omega}_M(M_0) \otimes \mathbb{C}.$$

We consider the subalgebra of the  $G$ -invariant elements of  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}_M(M)_0, \mathbb{C})$ . We denote it by:

$$(92) \quad \widehat{\Omega}_G(\mathcal{U}, M) = (\mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C})))^G.$$

We call the elements of this algebra the *equivariant forms*, on  $M$ .

We extend by  $\mathbb{C} \otimes \mathcal{O}_{\mathfrak{g}}(\mathcal{U})$ -linearity and continuity the exterior differential on  $\widehat{\Omega}(M)$  to  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C}))$ . We recall that  $d$  is an odd vector field on  $\widehat{M}$ . We consider it now as a vector field on  $\mathfrak{g} \times \widehat{M}$ . Let  $(G_i)$  be an homogeneous basis of  $\mathfrak{g}$ . We denote by  $(g^i)$  its dual basis. We recall definition (20) for  $G_{iM} = (G_i)_M$ . We put:

$$(93) \quad \iota = \sum_i \iota(G_{iM}) g^i,$$

(cf. (44)). Then  $\iota$  is a derivation of  $\mathcal{O}_{\mathfrak{g} \times \widehat{M}}(\mathcal{U} \times (\widehat{M})_0, \mathbb{C})$ , called operator of contraction. It is an odd vector field on  $\mathfrak{g} \times \widehat{M}$  which satisfies  $\iota^2 = 0$ .

We define the equivariant differential on  $\mathcal{O}_{\mathfrak{g} \times \widehat{M}}(\mathcal{U} \times (\widehat{M})_0, \mathbb{C})$  by the odd vector field:

$$(94) \quad d_{\mathfrak{g}} = d - \mathbf{i}\iota.$$

We have:

$$(95) \quad d_{\mathfrak{g}}^2 = -\mathbf{i}\mathcal{L},$$

with

$$(96) \quad \mathcal{L} = \sum_i \mathcal{L}(G_{iM}) g^i.$$

The differentials  $d$  and  $\iota$  commute with the action of  $G$  and so leave  $\widehat{\Omega}_G(\mathcal{U}, M)$  stable. Therefore,  $d_{\mathfrak{g}}$  induces a derivation of  $\widehat{\Omega}_G(U, M)$ . Moreover, on  $\widehat{\Omega}_G(U, M)$ , we have  $d_{\mathfrak{g}}^2 = 0$ .

We denote by  $\widehat{H}_G(\mathcal{U}, M)$  the cohomology of  $(\widehat{\Omega}_G(\mathcal{U}, M), d_{\mathfrak{g}})$  and we call it the equivariant cohomology of  $M$  (with coefficients in  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U})^G$ ).

We need variants of this definition. We define:

4.1.1. The space  $\widehat{\Omega}_{G,f}(U, M)$  of *integrable equivariant forms*:

$$(97) \quad \widehat{\Omega}_{G,f}(U, M) = \mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}_f(M, \mathbb{C}))^G$$

We have  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}_f(M, \mathbb{C})) \subset \mathcal{O}_{\mathfrak{g}}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C})) = \mathcal{O}_{\mathfrak{g} \times \widehat{M}}(\mathcal{U} \times (\widehat{M})_{\mathbf{0}}, \mathbb{C})$ . The integrable equivariant forms are  $G$ -invariant functions on the supervector bundle  $\mathfrak{g} \times \widehat{M} \rightarrow \mathfrak{g} \times M$  which are rapidly decreasing in the fibres and compactly supported on  $M_{\mathbf{0}}$ .

4.1.2. The space  $\widehat{\Omega}_G^{-\infty}(U, M)$  of *equivariant forms with generalized coefficients*: More precisely:

$$(98) \quad \widehat{\Omega}_G^{-\infty}(U, M) = \mathcal{C}_{\mathfrak{g}}^{-\infty}(U, \widehat{\Omega}(M, \mathbb{C}))^G.$$

This is the space of  $G$ -invariant continuous homomorphisms from the space of smooth compactly supported distributions on  $\mathfrak{g}(\mathcal{U})$  with values in  $\widehat{\Omega}(M, \mathbb{C})$ . Let  $\alpha \in \widehat{\Omega}_G^{-\infty}(U, M)$ . Let  $t$  be any smooth compactly supported distribution  $t$  on  $\mathfrak{g}$  with support in  $\mathcal{U}$ , then:

$$(99) \quad \alpha(t) = (-1)^{p(t)p(\alpha)} \int_{\mathfrak{g}} t(X)\alpha(X) \in \widehat{\Omega}(M, \mathbb{C}).$$

We recall that  $\mathcal{C}^{-\infty}(U) = \mathcal{C}_{\mathfrak{g}_0}^{-\infty}(U) \otimes \Lambda(\mathfrak{g}_1^*)$ . We put for any smooth compactly supported distribution  $t$  on  $\mathcal{U}$ :

$$(100) \quad \alpha(t) = (-1)^{p(t)p(\alpha)} \int_{\mathfrak{g}_0} t(X)\alpha(X) \in \Lambda(\mathfrak{g}_1^*) \otimes \widehat{\Omega}(M, \mathbb{C}).$$

*Example:* Let  $M$  be a point, then:

$$(101) \quad \Omega_G^{-\infty}(U, M) = \mathcal{C}_{\mathfrak{g}}^{-\infty}(U)^G = \left( \mathcal{C}^{-\infty}(U) \otimes \Lambda(\mathfrak{g}_1^*) \right)^G,$$

4.1.3. The space  $\widehat{\Omega}_{G,f}^{-\infty}(U, M)$  of *integrable equivariant forms with generalized coefficients*:

More precisely:

$$(102) \quad \widehat{\Omega}_{G,f}^{-\infty}(U, M) = \mathcal{C}_{\mathfrak{g}}^{-\infty}(U, \widehat{\Omega}_f(M, \mathbb{C}))^G.$$

This is the space of  $G$ -invariant continuous homomorphisms from the space of smooth compactly supported distributions on  $\mathfrak{g}(\mathcal{U})$  with values in  $\widehat{\Omega}_f(M, \mathbb{C})$ .

Let  $\alpha \in \widehat{\Omega}_{G,f}^{-\infty}(U, M)$ . Let  $t$  be any smooth compactly supported distribution  $t$  on  $\mathfrak{g}$  with support in  $\mathcal{U}$ , then:

$$(103) \quad \alpha(t) = (-1)^{p(t)p(\alpha)} \int_{\mathfrak{g}} t(X)\alpha(X) \in \widehat{\Omega}_f(M, \mathbb{C}).$$

As before, for any smooth compactly supported distribution  $t$  on  $\mathcal{U}$ :

$$(104) \quad \alpha(t) = (-1)^{p(t)p(\alpha)} \int_{\mathfrak{g}_0} t(X)\alpha(X) \in \Lambda(\mathfrak{g}_1^*) \otimes \widehat{\Omega}_f(M, \mathbb{C}).$$

Let  $\alpha \in \widehat{\Omega}_{G,f}^{-\infty}(\mathcal{U}, M)$ . Then  $\int_M \alpha \in \mathcal{C}_{\mathfrak{g}}^{-\infty}(\mathcal{U}, \mathbb{C})$  is defined for any smooth compactly supported distribution  $t$  on  $\mathcal{U}$  by the formula:

$$(105) \quad \left( \int_M \alpha \right)(t) = \int_M (\alpha(t)) = (-1)^{p(\alpha)p(t)} \int_M \int_{\mathfrak{g}} t(X)\alpha(X).$$

4.1.4. Let  $\pi : \mathcal{V} \rightarrow M$  be a  $G$  equivariant supervector bundle. We will consider also the space  $\widehat{\Omega}_{G,\pi_*}^{-\infty}(U, \mathcal{V})$  of *equivariant forms on  $\mathcal{V}$  with generalized coefficients which are integrable along the fibres*:

More precisely:

$$(106) \quad \widehat{\Omega}_{G,\pi_*}^{-\infty}(\mathcal{U}, \mathcal{V}) = \mathcal{C}_{\mathfrak{g}}^{-\infty}(\mathcal{U}, \widehat{\Omega}_{\pi_*}(\mathcal{V}, \mathbb{C}))^G$$

If  $\alpha \in \widehat{\Omega}_{G,\pi_*}^{-\infty}(\mathcal{U}, \mathcal{V})$ , for any compactly supported smooth distribution  $t$  on  $\mathfrak{g}_0$  which support is included in  $\mathcal{U}$ :

$$(107) \quad (\pi_*\alpha)(t) = \pi_*(\alpha(t)) \in \widehat{\Omega}(M).$$

This defines  $\pi_*\alpha$  as an element of  $\widehat{\Omega}_G^{-\infty}(\mathcal{U}, M)$ .

The differential  $d_{\mathfrak{g}}$  is defined on these spaces as well as the corresponding cohomology spaces. If  $M$  has a global orientation, we have (cf. [Lav98]):

$$(108) \quad \int_M d_{\mathfrak{g}}\alpha = 0, \text{ for } \alpha \in \widehat{\Omega}_{G,f}^{-\infty}(\mathcal{U}, M).$$

4.2. **Some proprieties.** Let  $G$  be a supergroup and  $M, N$  be two  $G$ -supermanifolds. Let  $\pi : N \rightarrow M$  be an equivariant morphism of  $G$ -supermanifolds. Let  $\alpha \in \widehat{\Omega}_G(M)$ . We have  $\pi^*\alpha \in \widehat{\Omega}_G(N)$  and:

$$(109) \quad \pi^*d_{\mathfrak{g}}\alpha = d_{\mathfrak{g}}\pi^*\alpha.$$

Therefore  $\pi^*$  induces an application in equivariant cohomology.

In particular, if  $M$  is a point  $H_G(\mathcal{U}, M) = \mathcal{O}_{\mathfrak{g}}(\mathcal{U})^G$  and  $\widehat{H}_G(\mathcal{U}, N)$  is an  $\mathcal{O}_{\mathfrak{g}}(\mathcal{U})^G$ -algebra (cf. for example [BGV92] in the classical situation).

Moreover, if  $\pi$  defines an equivariant supervector bundle and if  $M$  and  $N$  have a  $G$ -invariant global orientation, if the fibres of  $N$  are of dimension  $(k, l)$ , the following equation for  $\alpha \in \widehat{\Omega}_{G,\pi_*}(\mathcal{V})$ :

$$(110) \quad d_{\mathfrak{g}}\pi_*\alpha = (-1)^{(k+l)}\pi_*d_{\mathfrak{g}}\alpha.$$



**4.3. Equivariant superconnection.** (cf [BGV92]) Let  $G = (G_0, \mathfrak{g})$  be a supergroup,  $M$  a  $G$ -supermanifold and  $\mathcal{V} \rightarrow M$  an equivariant supervector bundle. There is a natural representation of  $G$  in the space of pseudodifferential forms on  $M$  and also on the space  $\widehat{\Omega}(M, \mathcal{V})$  of pseudodifferential forms with value in  $\mathcal{V}$  (that is sections of  $\mathcal{V} \times_M \widehat{M} \rightarrow \widehat{M}$ ).

We denote by  $\mathcal{L}^\mathcal{V}$  the associated representation of  $\mathfrak{g}$  in  $\widehat{\Omega}(M, \mathcal{V})$ . This is a morphism of Lie superalgebras, from  $\mathfrak{g}$  into first order differential operators on  $\widehat{\Omega}(M, \mathcal{V})$ .

Let us consider equivariant forms on  $M$  with values in  $\mathcal{V}$ . For an open  $G_0$ -invariant subset  $\mathcal{U} \subset \mathfrak{g}_0$  we put:

$$(111) \quad \widehat{\Omega}_G(\mathcal{U}, M, \mathcal{V}) = \mathcal{O}_\mathfrak{g}(\mathcal{U}, \widehat{\Omega}(M, \mathcal{V} \otimes \mathbb{C}))^G.$$

Let  $\mathbb{A}$  be a  $G$ -invariant superconnection on  $\mathcal{V}$ . We extend it  $\mathcal{O}_\mathfrak{g}(\mathcal{U})$ -linearly to an endomorphism of  $\mathcal{O}_\mathfrak{g}(\mathcal{U}, \widehat{\Omega}(M, \mathcal{V} \otimes \mathbb{C}))$ . We extend linearly the contraction operator  $\iota$  on  $\mathcal{O}_\mathfrak{g}(\mathcal{U}, \widehat{\Omega}(M, \mathbb{C}))$  to  $\mathcal{O}_\mathfrak{g}(\mathcal{U}, \widehat{\Omega}(M, \mathcal{V} \otimes \mathbb{C}))$ . We put:

$$(112) \quad \mathbb{A}_\mathfrak{g} = \mathbb{A} - \mathbf{i}\iota.$$

It is a differential operator on  $\mathcal{O}_\mathfrak{g}(\mathcal{U}, \widehat{\Omega}(M, \mathcal{V} \otimes \mathbb{C}))$  which leaves  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathcal{V})$  invariant. We call it the *equivariant superconnection* associated with  $\mathbb{A}$ .

For  $\alpha \in \widehat{\Omega}_G(\mathfrak{g}, M)$  and  $\omega \in \widehat{\Omega}_G(\mathfrak{g}, M, \mathcal{V})$  non zero and homogeneous we have:

$$(113) \quad \mathbb{A}_\mathfrak{g}(\omega\alpha) = (\mathbb{A}_\mathfrak{g}\omega)\alpha + (-1)^{p(\omega)}\omega(d_\mathfrak{g}\alpha).$$

The superconnection  $\mathbb{A}_\mathfrak{g}$  acts on  $\widehat{\Omega}_G(\mathcal{U}, M, \mathfrak{gl}(\mathcal{V}))$  by means of the supercommutation bracket of the algebra of endomorphisms of  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathcal{V})$ .

Let  $\eta$  be the application which sends a form  $\theta \in \widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{gl}(\mathcal{V}))$  on the multiplication on the left by  $\theta$  in  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{gl}(\mathcal{V}))$ . The *equivariant curvature*  $F_\mathfrak{g}$  of  $\mathbb{A}_\mathfrak{g}$  is the element of  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{gl}(\mathcal{V}))$  that satisfies:

$$(114) \quad \eta(F_\mathfrak{g}) = \mathbb{A}_\mathfrak{g}^2 + \mathbf{i}\mathcal{L}^\mathcal{V}.$$

The Bianchi identity is proved as in the classical case(cf [BGV92] proposition 7.4 p.210):

$$(115) \quad \mathbb{A}_\mathfrak{g}F_\mathfrak{g} = 0.$$

We define the *equivariant momentum* (for the  $G$ -invariant superconnection  $\mathbb{A}$ ) as:

$$(116) \quad \mu_\mathbb{A} = \mathcal{L}^\mathcal{V} - [\mathbb{A}, \iota].$$

It is an element of  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{gl}(\mathcal{V}))$  linear on  $\mathfrak{g}$  which satisfies:

$$(117) \quad F_\mathfrak{g} = \mathbb{A}^2 + \mathbf{i}\mu_\mathbb{A} = F + \mathbf{i}\mu_\mathbb{A}.$$

## 5. EQUIVARIANT THOM FORM

Let  $G$  be a supergroup and  $\pi : \mathcal{V} \rightarrow M$  be a  $G$ -equivariant supervector bundle. When  $\mathcal{V}$  is not purely even, there is no smooth equivariant closed pseudodifferential form  $\theta$ , integrable along the fibres and such that  $\pi_*\theta = 1_M$  (cf. [Lav98]). Nevertheless, if the action of  $G$  is sufficiently non-trivial, we shall show that there is a closed equivariant form with generalized coefficients satisfying this propriety.

Here is an example where such a form does not exist. Let  $M$  be a point and  $V = \mathbb{R}^{(0,2)}$ . We take  $G = \{e\}$ . A smooth equivariant form is a function:

$$(118) \quad \omega(\xi^1, \xi^2, d\xi^1, d\xi^2) = \omega_{(0,0)}(d\xi^1, d\xi^2) \\ + \xi^1 \omega_{(1,0)}(d\xi^1, d\xi^2) + \xi^2 \omega_{(0,1)}(d\xi^1, d\xi^2) + \xi^1 \xi^2 \omega_{(1,1)}(d\xi^1, d\xi^2).$$

The condition  $\omega$  integrable means that  $\omega_{(0,0)}$ ,  $\omega_{(1,0)}$ ,  $\omega_{(0,1)}$  and  $\omega_{(1,1)}$  are rapidly decreasing in  $d\xi^1$  and  $d\xi^2$ . and the condition  $d\omega = 0$  implies that  $\omega_{(1,1)} = 0$ . Thus  $\pi\omega = \int_V \omega = 0$ .

## 5.1. Preliminaries.

### 5.1.1. Definitions.

**Definition 5.1.** *Let  $G = (G_0, \mathfrak{g})$  be a supergroup. Let  $M$  be a  $G$ -supermanifold. Let  $\pi : \mathcal{V} \rightarrow M$  be an equivariant supervector bundle. We assume that  $\mathcal{V}$  and  $M$  are globally oriented.*

*An equivariant Thom form on  $\mathcal{V}$  is an equivariantly closed form  $\theta \in \widehat{\Omega}_{G, \pi_*}^{-\infty}(\mathfrak{g}, \mathcal{V})$  which is integrable along the fibres and such that  $\pi_*\theta = 1$ .*

This last equality means that if  $t$  is a smooth compactly supported distribution  $\mathfrak{g}$ , then

$$(119) \quad (\pi_*\theta)(t) = \int_{\mathfrak{g}} t(X) \in \widehat{\Omega}(M, \mathbb{C}).$$

Following Mathai-Quillen (cf. [MQ86], see also [BGV92] and [DV88]) we will construct such a Thom form for an Euclidean equivariant supervector bundle  $\mathcal{V} \rightarrow M$  with a sufficiently non trivial action of  $G$ .

Let  $Q$  be a  $G$ -invariant Euclidean structure on  $\mathcal{V}$ .

We recall that  $\pi Q$  is symplectic form on  $\Pi\mathcal{V}$  (cf. equation (11)).

We recall that we denote by  $\mu$  the moment application from  $\mathfrak{spo}(\Pi\mathcal{V})$  to  $S^2(\Pi\mathcal{V}^*) \subset \mathcal{O}(\Pi\mathcal{V})$  (cf. section 1.4.4).

We suppose that  $\mathcal{V}$  has a  $G$ -invariant superconnection  $\mathbb{A}$  which leaves the Euclidean structure invariant. Therefore, the curvature  $\mathbb{A}^2$  is a pseudodifferential form on  $M$  with values in  $\mathfrak{osp}(\mathcal{V})$ :  $\mathbb{A}^2 \in \widehat{\Omega}(M, \mathfrak{osp}(\mathcal{V}))$  and  $\mathbb{A}^{\Pi^2} \in \widehat{\Omega}(M, \mathfrak{spo}(\Pi\mathcal{V}))$  (cf. section 2.6.1).

5.1.2. *First condition.* We describe the first condition we need to construct a Thom form.

We put  $\mathbb{A}^{\Pi^2} = (\mathbb{A}^{\Pi})^2$ .

The application  $\mu$  (cf. formula (13)) sends  $\mathfrak{spo}(\Pi\mathcal{V})$  on  $S^2(\Pi\mathcal{V}^*)$ . Therefore,  $\mu(\mathbb{A}^{\Pi^2})$  is a function on  $\widehat{M}$  with values in  $S^2(\Pi\mathcal{V}^*)$ . In other words,  $\mu(\mathbb{A}^{\Pi^2})$  is a function on the bundle  $\widehat{M} \times_{\Pi\mathcal{V}} \Pi\mathcal{V} \rightarrow \widehat{M}$  which is polynomial and homogeneous of degree 2 in the fibres.

So  $\mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}}$  is a function on  $(\widehat{M} \times_{\Pi\mathcal{V}} \Pi\mathcal{V})_{\mathbf{0}} = (\widehat{M})_{\mathbf{0}} \times_{M_{\mathbf{0}}} (\Pi\mathcal{V})_{\mathbf{0}}$ . The condition is:

$$(*) \quad \mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}} \leq 0.$$

Let  $v \in \Gamma_{\mathcal{V}}(M_{\mathbf{0}})_{\mathbf{1}}$  be an odd section of  $\mathcal{V}$ . Then  $\pi v$  is an even section of  $\Pi\mathcal{V}$ . We have:

$$(120) \quad \mu(\mathbb{A}^{\Pi^2})(\pi v) = -\frac{1}{2} \pi Q(\pi v, \mathbb{A}^{\Pi^2} \pi v) = \frac{1}{2} Q(v, \mathbb{A}^2 v) \in \widehat{\Omega}(M) = \mathcal{O}_{\widehat{M}}((\widehat{M})_{\mathbf{0}}).$$

Condition (\*) means that for any  $v \in \Gamma_{\mathcal{V}}(M_{\mathbf{0}})_{\mathbf{1}}$ , the restriction of  $\frac{1}{2}Q(v, \mathbb{A}^2 v)$  to  $(\widehat{M})_{\mathbf{0}}$ , is non positive.

This condition is trivially satisfied when  $\mathbb{A}^2 = 0$ . Moreover, this condition depends of the choice of the particular superconnection we choose.

5.1.3. *More notations.* We denote by  $\mathbb{A}_{\mathfrak{g}}$  the equivariant superconnection associated with  $\mathbb{A}$ . We denote by  $\pi^*S(\Pi\mathcal{V}^*)$  the bundle with  $\mathcal{V}$  as base space which is the pullback of  $S(\Pi\mathcal{V}^*)$  by  $\pi$ :  $\pi^*S(\Pi\mathcal{V}^*) = S(\Pi\mathcal{V}^*) \times_M \mathcal{V}$ .

Since  $\Pi\mathcal{V}^* = S^1(\Pi\mathcal{V}^*)$ , there is a natural injection of  $\pi^*\Pi\mathcal{V}^*$  in  $\pi^*S(\Pi\mathcal{V}^*)$ .

To help the reader we fix (abuse of) notations in the following commutative diagram:

$$(121) \quad \begin{array}{ccc} p^*\pi^*(\Pi\mathcal{V}) = \Pi\mathcal{V} \times_M \widehat{\mathcal{V}} & \longrightarrow & \widehat{\mathcal{V}} \\ \downarrow & & \downarrow p \\ \pi^*(\Pi\mathcal{V}) = \Pi\mathcal{V} \times_M \mathcal{V} & \longrightarrow & \mathcal{V} \\ \downarrow \pi & & \downarrow \pi \\ \Pi\mathcal{V} & \longrightarrow & M \end{array}$$

The left vertical projections correspond to the right vertical ones with the same letter and thus “act only on the second factor”. In the preceding diagram we can change  $\Pi\mathcal{V}$  into  $\Pi\mathcal{V}^*$  without any other changes.

We denote by  $\mathbf{v}$  be the tautological section with change of parity of  $\Pi\mathcal{V} \times_M \mathcal{V} \rightarrow \mathcal{V}$ . Let  $\mathcal{U} \subset M_{\mathbf{0}}$  be a trivialization subset of  $\mathcal{V}$ . Let  $(e_i)_{i \in I}$  be an homogeneous basis of sections of  $\mathcal{V}$ . Let  $(x^i)$  be its dual basis. Then,  $(\pi e_i)$  the basis of sections of  $\Pi\mathcal{V}$ . By abuse of notations we also denote by  $(\pi e_i)$  the basis of sections  $(\pi^*\pi e_i)$  of  $\Pi\mathcal{V} \times_M \mathcal{V}$ . We have:

$$(122) \quad \mathbf{v} = \sum_i (\pi e_i) x^i \in \Gamma_{\Pi\mathcal{V}}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \Gamma_{\mathcal{V}^*}(\mathcal{U}) \subset \Gamma_{\pi^*\Pi\mathcal{V}}(\pi_0^{-1}(\mathcal{U})).$$

Since the above formula does not depend on the choice of  $(e_i)_{i \in I}$ , this defines a global section  $\mathbf{v} \in \Gamma_{\pi^*\Pi\mathcal{V}}(\mathcal{V}_{\mathbf{0}})$ .

We recall (cf. section 1.3.5 for the definition of  $\pi Q^*$ ) that:

$$(123) \quad \pi Q^*(\mathbf{v}) \in \Gamma_{\pi^*\Pi\mathcal{V}^*}(\mathcal{V}_{\mathbf{0}}).$$

Let  $\mathcal{P}$  be a near superalgebra. We explicit  $\pi Q^*(\mathbf{v})$  for  $\mathcal{P}$ -points. For  $w \in (\pi^*\Pi\mathcal{V})_{\mathcal{P}}$ , we recall that  $\pi(w)$  is its projection on  $(\Pi\mathcal{V})_{\mathcal{P}}$ . For any  $w \in (\pi^*\Pi\mathcal{V})_{\mathcal{P}}$  and any  $v \in \mathcal{V}_{\mathcal{P}}$  we have:

$$(124) \quad \pi Q^*(\mathbf{v}(v))(w) = \pi Q(\pi v, \pi(w)).$$

In particular  $\pi Q^*(\mathbf{v})(\mathbf{v}) = \pi Q(\mathbf{v}, \mathbf{v})$ .

We recall (cf. section 2.6.2) that  $\mathbb{A}$  determines a superconnection  $\mathbb{A}^{\Pi}$  on  $\Pi\mathcal{V}$  and  $\mathbb{A}^{\Pi^*}$  on  $S(\Pi\mathcal{V}^*)$ . This determines equivariant superconnections  $\mathbb{A}_{\mathfrak{g}}^{\Pi}$  on  $\widehat{\Omega}_G(\mathfrak{g}_{\mathbf{0}}, M, \Pi\mathcal{V})$  and  $\mathbb{A}_{\mathfrak{g}}^{\Pi^*}$  on the algebra  $\widehat{\Omega}_G(\mathfrak{g}_{\mathbf{0}}, M, S(\Pi\mathcal{V}^*))$ .

The differential operator  $\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi}$  on  $\widehat{\Omega}_G(\mathfrak{g}, \mathcal{V}, \pi^* \Pi \mathcal{V})$  is an equivariant superconnection defined for  $\omega \in \widehat{\Omega}_G(\mathfrak{g}, M, \Pi \mathcal{V})$  by:

$$(125) \quad \pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi}(\pi^* \omega) = \pi^*(\mathbb{A}_{\mathfrak{g}}^{\Pi} \omega).$$

We have:

$$(126) \quad \widehat{\Omega}_G(\mathfrak{g}_0, \mathcal{V}, \pi^* \Pi \mathcal{V}) = \widehat{\Omega}_G(\mathfrak{g}_0, M, \Pi \mathcal{V}) \otimes_{\widehat{\Omega}_G(M)} \widehat{\Omega}_G(\mathcal{V}).$$

With respect to this decomposition we have:

$$(127) \quad \pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi} = \mathbb{A}_{\mathfrak{g}}^{\Pi} \otimes 1 + 1 \otimes d_{\mathfrak{g}},$$

where  $(1 \otimes d_{\mathfrak{g}})(\alpha \otimes \beta) = (-1)^{p(\alpha)} \alpha \otimes d_{\mathfrak{g}} \beta$  and  $(\mathbb{A}_{\mathfrak{g}}^{\Pi} \otimes 1)(\alpha \otimes \beta) = \mathbb{A}_{\mathfrak{g}}^{\Pi} \alpha \otimes \beta$ .

We define similarly the equivariant connection  $\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*}$  on  $\widehat{\Omega}_G(\mathfrak{g}, \mathcal{V}, \pi^* S(\Pi \mathcal{V}^*))$ .

Now,

$$\widehat{\Omega}_G(\mathfrak{g}_0, \mathcal{V}, \pi^* S(\Pi \mathcal{V}^*)) \subset \mathcal{O}_{\mathfrak{g}} \left( \mathfrak{g}_0, \mathcal{O}(\Pi \mathcal{V} \times_M \widehat{\mathcal{V}}) \right)^G$$

and the inclusion is dense. The operator  $\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*}$  can be extended by continuity to this last space.

We stress that, if  $\omega \in \widehat{\Omega}_G(\mathfrak{g}_0, \mathcal{V}, \pi^* S(\Pi \mathcal{V}^*))_0$  and  $f$  is an entire function of the complex variable, the function  $f(\omega)$  is well defined in  $\mathcal{O}_{\mathfrak{g}} \left( \mathfrak{g}_0, \mathcal{O}(\Pi \mathcal{V} \times_M \widehat{\mathcal{V}}) \right)^G$ .

Finally, let  $v \in \Gamma_{\pi^* \Pi \mathcal{V}}(\mathcal{V}_0)$  be non zero and homogeneous. We denote by  $\partial_v$  the derivation of  $\Gamma_{\pi^* S(\Pi \mathcal{V}^*)}(\mathcal{V}_0)$  such that for  $\phi \in \Gamma_{\pi^* \Pi \mathcal{V}^*}(\mathcal{V}_0)$  non zero and homogeneous:

$$(128) \quad \partial_v \phi = (-1)^{p(v)p(\phi)} \phi(v).$$

We extend it  $\widehat{\Omega}_G(\mathfrak{g}_0, \mathcal{V})$ -linearly to a derivation of  $\widehat{\Omega}_G(\mathfrak{g}_0, M, \pi^* S(\Pi \mathcal{V}^*))$  and then by continuity to a derivation of  $\mathcal{O}_{\mathfrak{g}} \left( \mathfrak{g}_0, \mathcal{O}(\Pi \mathcal{V} \times_M \widehat{\mathcal{V}}) \right)^G$ .

We denote by  $v^{\Pi}$  the generic point of the  $\mathcal{O}(\mathcal{V})$ -module  $\Gamma_{\Pi \mathcal{V} \times_M \mathcal{V}}(\mathcal{V}_0)$ . Let  $((\pi e_i)^*)$  be the dual basis of  $(\pi e_i)$ . We have:

$$(129) \quad v^{\Pi} = \sum_i \pi e_i (\pi e_i)^*$$

We have:

$$(130) \quad \partial_v v^{\Pi} = v.$$

5.1.4. *The form  $\omega_{\mathcal{V}}$ .* The operator

$$(131) \quad \mathbb{B} = \pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*} + \mathbf{i} \partial_{\mathbf{v}}$$

defines an equivariant superconnection on  $\pi^* S(\Pi\mathcal{V}^*)$ .

We denote by  $F_{\mathfrak{g}}^{\Pi}$  the equivariant curvature of  $\mathbb{A}_{\mathfrak{g}}^{\Pi}$ .

Let  $\mu_{\mathbb{A}}$  be the equivariant moment of  $\mathbb{A}_{\mathfrak{g}}$ :

$$\mu_{\mathbb{A}} \in \widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{osp}(\mathcal{V})).$$

Let  $\mu_{\mathbb{A}\Pi}$  be the equivariant moment of  $\mathbb{A}^{\Pi}$ . We recall that we identified  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{osp}(\mathcal{V}))$  and  $\widehat{\Omega}_G(\mathfrak{g}, M, \mathfrak{spo}(\Pi\mathcal{V}))$ . Under this identification we have  $\mu_{\mathbb{A}\Pi} = \mu_{\mathbb{A}}$ .

We put:

$$(132) \quad \begin{aligned} \omega_{\mathcal{V}} &= \frac{1}{2} \pi Q(\mathbb{B}v^{\Pi}, \mathbb{B}v^{\Pi}) + \mathbf{i} \mu(\mu_{\mathbb{A}}(X)) \\ &= -\frac{1}{2} \pi Q(\mathbf{v}, \mathbf{v}) + \mathbf{i} \pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*} \pi Q^*(\mathbf{v}) + \mu(F_{\mathfrak{g}}^{\Pi}) \in \widehat{\Omega}_G(\mathfrak{g}, \mathcal{V}, \pi^* S(\Pi\mathcal{V}^*)). \end{aligned}$$

**Proposition 5.1.**

$$(133) \quad (\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*} + \mathbf{i} \partial_{\mathbf{v}}) \omega_{\mathcal{V}} = 0,$$

If  $f$  is an entire function of one complex variable, we have:

$$(134) \quad (\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*} + \mathbf{i} \partial_{\mathbf{v}}) f(\omega_{\mathcal{V}}) = 0.$$

*Proof.* Since  $\pi Q$  is antisymmetric we have:

$$(135) \quad \mathbb{B} \pi Q(\mathbb{B}v^{\Pi}, \mathbb{B}v^{\Pi}) = \pi Q(\mathbb{B}^2 v^{\Pi}, \mathbb{B}v^{\Pi}) + \pi Q(\mathbb{B}v^{\Pi}, \mathbb{B}^2 v^{\Pi}) = 0.$$

On the other hand, since  $Q$  is  $\mathbb{A}$ -invariant:

$$(136) \quad \pi^* \mathbb{A}^{\Pi*} \mu(\mu_{\mathbb{A}}(X)) = -\frac{1}{2} \pi^* \mathbb{A}^{\Pi*} \pi Q(v^{\Pi}, \mu_{\mathbb{A}}(X) v^{\Pi}) = 0.$$

Finally, since  $\pi Q$  is antisymmetric and  $\mu_{\mathbb{A}}(X) \in \Gamma_{\mathfrak{spo}(\Pi\mathcal{V})}(M_0)$ :

$$(137) \quad \partial_{\mathbf{v}} \mu(\mu_{\mathbb{A}}(X)) = -\frac{1}{2} \pi Q(\mathbf{v}, \mu_{\mathbb{A}}(X) v^{\Pi}) - \frac{1}{2} \pi Q(v^{\Pi}, \mu_{\mathbb{A}}(X) \mathbf{v}) = 0.$$

Hence  $\mathbb{B} \mu(\mu_{\mathbb{A}}(X)) = 0$  and equality (133) follows.

The last equality follows immediately, since  $\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi*} + \mathbf{i} \partial_{\mathbf{v}}$  is a derivation of the superalgebra  $\mathcal{O}_{\mathfrak{g}}(\mathfrak{g}_0, \mathcal{O}(\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}))^G$ .  $\square$

5.1.5. *An equivariantly closed form.* Since  $Q$  is an euclidean structure on  $\mathcal{V}$ ,  $\pi Q$  is a symplectic structure on  $\Pi\mathcal{V}$ . Moreover, since  $\mathcal{V}$  is globally oriented,  $\Pi\mathcal{V}$  is an oriented symplectic supervector bundle in sense of section 2.5. Let  $\mathcal{D}_{\Pi\mathcal{V}}$  be the Liouville volume form along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ . We denote by

$$\phi \mapsto T(\phi) = \int_{\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}/\widehat{\mathcal{V}}} \mathcal{D}_{\Pi\mathcal{V}} \phi$$

the corresponding distribution (cf. section 3.6).

Recall that  $\omega_{\mathcal{V}} \in \widehat{\Omega}_G(\mathfrak{g}, \mathcal{V}, \pi^*S(\Pi\mathcal{V}^*)) \subset \mathcal{O}(\mathfrak{g}, \Pi\mathcal{V} \times_M \widehat{\mathcal{V}})$ . Thus, if all the functions involved are integrable along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}$ , we obtain:

$$(138) \quad (-1)^k T\left((\pi^*\mathbb{A}_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})f(\omega_{\mathcal{V}})\right) = d_{\mathfrak{g}}T((f(\sigma^*(\omega_{\mathcal{V}}))),$$

where  $(k, l)$  is the dimension of the fibres of  $\mathcal{V}$ . This follows from formula (110) and from the fact that since  $\mathbf{v}$  is a section of  $\pi^*\Pi V$ ,  $\partial_{\mathbf{v}}$  is a derivation along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$  with constant coefficients in the direction of the fibres and thus  $T(\partial_{\mathbf{v}}f(\omega_{\mathcal{V}})) = 0$ .

Since  $(\pi^*\mathbb{A}_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})f(\omega_{\mathcal{V}}) = 0$ , it follows that for any entire function  $f$  of one complex variable such that  $f(\omega_{\mathcal{V}})$  is rapidly decreasing along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ :

$$(139) \quad d_{\mathfrak{g}}T\left(f(\omega_{\mathcal{V}})\right) = 0.$$

Therefore the form  $T(f(\omega_{\mathcal{V}}))$  is an equivariantly closed form on  $\mathcal{V}$ . To find a Thom form we just have to find a “good” function  $f$ . The choice  $f(z) = \exp(z)$  gives, up to a multiplicative constant, a Thom form on  $\mathcal{V}$ .

Moreover, the equivariant cohomology class of this form in  $\widehat{H}_{G_f}^{\infty}(\mathcal{V})$  does not depend on the choice of the  $G$ -invariant superconnection  $\mathbb{A}$ . Let  $\omega' \in \widehat{\Omega}(M, \mathfrak{osp}(\mathcal{V}))_1^G$ . We put  $\mathbb{A}(t) = \mathbb{A} + t\omega'$  and denote its equivariant curvature by  $F(t)_{\mathfrak{g}}$ . We put

$$(140) \quad \omega_{\mathcal{V}}(t) = -\frac{1}{2}\Pi Q(\mathbf{v}, \mathbf{v}) + \mathbf{i}\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*}\Pi Q^*(\mathbf{v}) + \mu(F(t)_{\mathfrak{g}}).$$

We have:

$$(141) \quad \begin{aligned} \frac{d}{dt}\omega_{\mathcal{V}}(t) &= \mathbf{i}(\pi^*\omega'^*)\Pi Q^*(\mathbf{v}) + \tau([\mathbb{A}(t)_{\mathfrak{g}}, \omega']) \\ &= (\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})\mu(\omega'). \end{aligned}$$

and since  $(\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \partial_{\mathbf{v}})\omega_{\mathcal{V}}(t) = 0$ :

$$(142) \quad \frac{d}{dt}\exp(\omega_{\mathcal{V}}(t)) = (\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})\left(\mu(\omega')\exp(\omega_{\mathcal{V}}(t))\right)$$

Thus, since  $\omega_{\mathcal{V}}(0)$  and  $\omega_{\mathcal{V}}(1)$  corresponds to the superconnections  $\mathbb{A}$  and  $\mathbb{A} + \omega'$ :

$$(143) \quad \exp(\omega_{\mathcal{V}}(1)) - \exp(\omega_{\mathcal{V}}(0)) = \int_0^1 (\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})\left(\mu(\omega')\exp(\omega_{\mathcal{V}}(t))\right)dt,$$

and:

$$(144) \quad \begin{aligned} T\left(\exp(\omega_{\mathcal{V}}(1)) - \exp(\omega'_{\mathcal{V}}(0))\right) \\ &= T\left(\int_0^1 (\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})\left(\mu(\omega')\exp(\omega_{\mathcal{V}}(t))\right)dt\right) \\ &= \int_0^1 T\left((\pi^*\mathbb{A}(t)_{\mathfrak{g}}^{\Pi^*} + \mathbf{i}\partial_{\mathbf{v}})\left(\mu(\omega')\exp(\omega_{\mathcal{V}}(t))\right)\right)dt \\ &= d_{\mathfrak{g}}\int_0^1 T(\mu(\omega')\exp(\omega_{\mathcal{V}}(t)))dt. \end{aligned}$$

5.1.6. *Two additional notations.* Let  $\mathcal{U} \subset M_{\mathbf{0}}$  be an open subset. We put

$$(145) \quad U_+^{\mathcal{V}_1}(\mathcal{U}) = \left\{ X \in \mathfrak{g}_{\mathbf{0}} \mid (\forall v \in \Gamma_{\mathcal{V}}(\mathcal{U})_{\mathbf{1}} / \forall x \in \mathcal{U}, v_{\mathbb{R}}(x) \neq 0), Q(v, \mu_{\mathbb{A}}(X)v)_{\mathbb{R}} > 0 \right\}.$$

For  $X \in \mathfrak{g}_{\mathbf{0}}$ , and  $v \in \Gamma_{\mathcal{V}}(\mathcal{U})$ , we have  $Q(v, \mu_{\mathbb{A}}(X)v) \in \widehat{\Omega}_M(\mathcal{U}) = \mathcal{O}_{\widehat{M}}((\widehat{M}(\mathcal{U}))_{\mathbf{0}})$ . Thus  $Q(v, \mu_{\mathbb{A}}(X)v)_{\mathbb{R}}$  is a function on  $(\widehat{M}(\mathcal{U}))_{\mathbf{0}}$ . Thus  $Q(v, \mu_{\mathbb{A}}(X)v) > 0$  means

$$(146) \quad \forall \mathbf{m} \in (\widehat{M}(\mathcal{U}))_{\mathbf{0}}, Q(v, \mu_{\mathbb{A}}(X)v)_{\mathbb{R}}(\mathbf{m}) = Q_{x_{\mathbf{m}}}(v_{\mathbb{R}}(x_{\mathbf{m}})v, \mu_{\mathbb{A}}(X)(\mathbf{m})v_{\mathbb{R}}(x_{\mathbf{m}})) > 0.$$

where  $x_{\mathbf{m}}$  is the projection of  $\mathbf{m} \in (\widehat{M}(\mathcal{U}))_{\mathbf{0}}$  on  $\mathcal{U}$ .

Since  $v$  is supposed to be odd,  $v_{\mathbb{R}}(x_{\mathbf{m}}) \in (\mathcal{V}_{x_{\mathbf{m}}})_{\mathbf{1}}$ . On the other hand,  $Q_{x_{\mathbf{m}}}|_{(\mathcal{V}_{x_{\mathbf{m}}})_{\mathbf{1}}}$  is symplectic, thus  $w \mapsto Q_{x_{\mathbf{m}}}(w, \mu_{\mathbb{A}}(X)(\mathbf{m})w)$  is a quadratic form on  $(\mathcal{V}_{\mathbf{m}})_{\mathbf{1}}$  and the condition says that this quadratic form is positive definite for any  $\mathbf{m} \in (\widehat{M}(\mathcal{U}))_{\mathbf{0}}$ .

On the other hand we put:

$$(147) \quad U^{\mathcal{V}_1}(\mathcal{U}) = \text{Interior of } \left\{ X \in \mathfrak{g}_{\mathbf{0}} \mid \forall \mathbf{m} \in (\widehat{M}(\mathcal{U}))_{\mathbf{0}}, \mu_{\mathbb{A}}(X)(\mathbf{m})|_{(\mathcal{V}_{x_{\mathbf{m}}})_{\mathbf{1}}} \text{ is invertible} \right\}.$$

We describe these subsets in the particular case when  $M$  is a point and  $\mathbb{A}$  is the trivial superconnection  $d$ .

In this case  $\mathcal{V}$  is a Euclidean supervector space with a representation  $\rho = (\rho_{\mathbf{0}}, \rho)$  of  $G = (G_{\mathbf{0}}, \mathfrak{g})$ . In particular  $\rho : \mathfrak{g} \rightarrow \mathfrak{osp}(\mathcal{V}, Q)$ . Then for  $X \in \mathfrak{g}_{\mathbf{0}}$ ,  $\mu_{\mathbb{A}}(X) = \rho(X)$ . We have (since  $M$  is a point we omit  $\mathcal{U}$ ):

$$(148) \quad U^{\mathcal{V}_1} = \{ X \in \mathfrak{g}_{\mathbf{0}} / \rho(X)|_{\mathcal{V}_1} \text{ is invertible.} \}$$

It is an open subset of  $\mathfrak{g}_{\mathbf{0}}$ .

Since  $(\mathcal{V}_1, Q|_{\mathcal{V}_1})$  is, as an ungraded vector space, a symplectic vector space,  $w \mapsto B(w, \rho(X)w)$  is a quadratic form on (the ungraded vector space)  $\mathcal{V}_1$ . Thus  $U_+^{\mathcal{V}_1}$  is the open subset of  $X \in \mathfrak{g}_{\mathbf{0}}$  such that this quadratic form is positive definite; and  $U^{\mathcal{V}_1}$  is the open subset of  $X \in \mathfrak{g}_{\mathbf{0}}$  such that this quadratic form is non degenerate.

5.2. **Construction of an equivariant Thom form.** We put:

$$(149) \quad \theta = T\left(\exp(\omega_{\mathcal{V}})\right).$$

**Theorem 5.1.** *Let  $G = (G_{\mathbf{0}}, \mathfrak{g})$  be a supergroup. Let  $M$  be a  $G$ -supermanifold. Let  $\pi : \mathcal{V} \rightarrow M$  be an equivariant supervector bundle on  $M$  with rank  $(k, l)$ . We assume that  $\mathcal{V}$  and  $M$  are globally oriented, and that  $\mathcal{V}$  is endowed with an Euclidean structure denoted by  $Q(\cdot, \cdot)$ , all these structures being  $G$ -invariants.*

*Finally we assume that there is a  $G$ -invariant superconnection which leaves the Euclidean structure invariant. We assume:*

$$(*) \quad \mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}} \leq 0 \text{ on } (\widehat{M} \times_M \Pi\mathcal{V})_{\mathbf{0}}.$$

(\*\*) *There is a covering of  $M_{\mathbf{0}}$  by open subsets  $\mathcal{U}$  such that  $U_+^{\mathcal{V}_1}(\mathcal{U})$  contains a non empty open subset (cf. formula (145)).*

Then, the equivariant form  $\theta$  defined above is a Thom form. It does not depend on the choice of the superconnection  $\mathbb{A}$ .

Moreover, let  $\mathcal{U}$  be an open subset of  $M_0$ . Then the restriction of  $\theta$  to  $U^{\mathcal{V}_1}(\mathcal{U}) \times \mathcal{V}(\mathcal{U})$  is a smooth equivariant form.

*Proof.* Up to the multiplicative constant we have to check that the form  $T(\exp(\omega_{\mathcal{V}}))$  has the required properties. We already know that it is an equivariantly closed form, so we have to check that the form  $T(\exp(\omega_{\mathcal{V}}))$  is a well defined equivariant form with generalized coefficients, integrable along the fibres and to evaluate its integral.

5.2.1. *A form with generalized coefficients.* First we have to show that  $T(\exp(\omega_{\mathcal{V}}))$  is a well defined generalized function on  $\mathfrak{g}$ .

It means that for any smooth compactly supported distribution  $t$  on  $\mathfrak{g}$  the integral

$$(150) \quad \int_{\mathfrak{g}} t(X) \exp(\omega_{\mathcal{V}}(X)) \in \mathcal{O}(\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}).$$

is a rapidly decreasing function along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ .

In this case, by definition, we put:

$$(151) \quad \langle \theta, t \rangle = T\left(\int_{\mathfrak{g}} t(X) \exp(\omega_{\mathcal{V}}(X))\right).$$

Since  $F_{\mathfrak{g}}^{\Pi}(X) = \mathbb{A}^{\Pi^2} + \mathbf{i}\mu_{\mathbb{A}}(X)$  and  $\pi^* \mathbb{A}_{\mathfrak{g}}^{\Pi^*}(X) \Pi Q^*(\mathbf{v}) = \pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v})$ , we have:

$$(152) \quad \omega_{\mathcal{V}}(X) = -\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) + \mathbf{i}\pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v}) + \mu(\mathbb{A}^{\Pi^2}) + \mathbf{i}\mu(\mu_{\mathbb{A}}(X)).$$

Thus the preceding integral (150) is equal to:

$$(153) \quad \exp\left(-\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) + \mathbf{i}\pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v}) + \mu(\mathbb{A}^{\Pi^2})\right) \int_{\mathfrak{g}} t(X) \exp(\mathbf{i}\mu(\mu_{\mathbb{A}}(X))).$$

Since the problem is entirely local on  $M$ , we restrict us to a trivialization subset  $\mathcal{U}$  of  $\mathcal{V}$ . Moreover hypotheses (\*\*\*) allows us to assume that  $U_+^{\mathcal{V}_1}(\mathcal{U})$  contains a non empty open subset. To avoid boring notations we assume that  $\mathcal{V}$  is trivial and  $\mathcal{U} = M_0$ .

We have to show that  $\int_{\mathfrak{g}} t(X) \exp(\mathbf{i}\mu(\mu_{\mathbb{A}}(X)))$  is rapidly decreasing along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ . We stress that this problem is completely similar to the one of sections 3.1 and 3.7 of [Lav03]. We reproduce here the argument.

Let  $(e_i, f_j)$  be a standard basis of sections of  $\mathcal{V}$  on  $M$ . Then  $(\Pi f_j, \Pi e_i)$  is a standard basis of sections of  $\Pi\mathcal{V}$ . By abuse of notations we also denote by  $(\Pi f_j, \Pi e_i)$  the basis of sections  $(\pi^* \Pi f_j, \pi^* \Pi e_i)$  of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}$ . Let  $((\Pi f_j)^*, (\Pi e_i)^*)$  be its dual basis. We have

$$(154) \quad v^{\Pi} = \sum_j \Pi f_j (\Pi f_j)^* + \sum_i \Pi e_i (\Pi e_i)^*.$$

We put  $v_0 = \sum_j \Pi f_j (\Pi f_j)^*$  and  $v_1 = \sum_i \Pi e_i (\Pi e_i)^*$ .



We have

$$(155) \quad \begin{aligned} \mu(\mu_{\mathbb{A}}(X)) &= -\frac{1}{2}\pi Q(v^{\Pi}, \mu_{\mathbb{A}}(X)v^{\Pi}) \\ &= -\frac{1}{2}\left(\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{0}}) + \pi Q(v_{\mathbf{1}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}}) + 2\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}})\right). \end{aligned}$$

Thus,

$$(156) \quad \begin{aligned} \exp(\mu(\mu_{\mathbb{A}}(X))) &= \exp\left(-\frac{1}{2}\left(\pi Q(v_{\mathbf{1}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}}) + 2\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}})\right)\right) \\ &\quad \exp\left(-\frac{1}{2}\left(\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{0}})\right)\right). \end{aligned}$$

Since  $v_{\mathbf{1}}$  is a linear combination of nilpotent elements,

$$\exp\left(-\mathbf{i}\frac{1}{2}\left(\pi Q(v_{\mathbf{1}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}}) + 2\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}})\right)\right)$$

is a polynomial function on  $\mathfrak{g}$  with values in  $\Gamma_{S(\Pi\nu^*) \times_M \widehat{\mathcal{V}}_0}$ .

Now we put

$$\rho(X) = t(X) \exp\left(-\mathbf{i}\frac{1}{2}\left(\pi Q(v_{\mathbf{1}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}}) + 2\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{1}})\right)\right);$$

It is a smooth compactly supported distribution on  $\mathfrak{g}$  with values in  $\Gamma_{S(\Pi\nu^*) \times_M \widehat{\mathcal{V}}_0}$ .

Thus:

$$(157) \quad \int_{\mathfrak{g}} t(X) \exp(\mathbf{i}\mu(\mu_{\mathbb{A}}(X))) = \int_{\mathfrak{g}} \rho(X) \exp(-\frac{\mathbf{i}}{2}\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{0}}))$$

Since  $U_+^{\mathcal{V}_1}(M_0)$  contains a non empty subset, there exists a basis  $(G_i)$  of  $\mathfrak{g}_0$  such that  $G_i \in U_+^{\mathcal{V}_1}(M_0)$ . Let  $(g^i)$  be its dual basis. Let  $(H_j)$  be a basis of  $\mathfrak{g}_1$  and  $(h^j)$  be its dual basis. We put  $X_0 = \sum_i G_i g^i$  and  $X_1 = \sum_j H_j h^j$ . We recall that a distribution can be integrated on  $\mathfrak{g}_1$  and then on  $\mathfrak{g}_0$ . We put:

$$(158) \quad \sigma(X_0) = \int_{\mathfrak{g}_1} \rho(X_0 + X_1) \exp(-\frac{\mathbf{i}}{2}\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X_1)v_{\mathbf{0}})).$$

It is a smooth compactly supported distribution on  $\mathfrak{g}_0$  with values in  $\Gamma_{S(\Pi\nu^*) \times_M \widehat{\mathcal{V}}_0}$ .

Now, we have:

$$(159) \quad \begin{aligned} \int_{\mathfrak{g}} t(X) \exp(\mathbf{i}\mu(\mu_{\mathbb{A}}(X))) &= \int_{\mathfrak{g}} \rho(X) \exp(-\frac{\mathbf{i}}{2}\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X)v_{\mathbf{0}})) \\ &= \int_{\mathfrak{g}_0} \sigma(X_0) \exp(-\frac{\mathbf{i}}{2}\pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(X_0)v_{\mathbf{0}})) \\ &= \widehat{\sigma}\left(\frac{1}{2}\sum_i \pi Q(v_{\mathbf{0}}, \mu_{\mathbb{A}}(G_i)v_{\mathbf{0}})g^i\right) \in \mathcal{O}(\Pi\nu \times_M \widehat{\mathcal{V}}). \end{aligned}$$

Since  $\sigma$  is smooth and compactly supported,  $\widehat{\sigma}$  is rapidly decreasing on  $\mathfrak{g}_0^*$ . Since  $G_i \in U_+^{\mathcal{V}_1}(M_0)$ , for any  $\mathbf{m} \in \widehat{\mathcal{V}}_0$ ,

$$\Pi Q(v_0, \mu_{\mathbb{A}}(G_i)v_0)(\mathbf{m}) = - \left( \sum_{r,s} Q(f_r, \mu_{\mathbb{A}}(G_i)f_s)(\Pi f_s)^*(\Pi f_r)^* \right)(\mathbf{m})$$

is a negative definite form on  $(\mathcal{V}_{x_m})_1 = ((\Pi\mathcal{V} \times_M \widehat{\mathcal{V}})_{\mathbf{m}})_0$ . Thus, thanks to the Taylor formula (3), the above function is rapidly decreasing along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}$ .

Now,  $-\frac{1}{2}\Pi Q(\mathbf{v}, \mathbf{v})$  is constant along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ , and  $\exp(i\pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v}))$  has (also by Taylor formula (3)) almost polynomial growth. Finally, hypothesis (\*) ensures that  $\exp(\mu(\mathbb{A}^{\Pi^2}))$  has also almost polynomial growth.

It follows that

$$(160) \quad \int_{\mathfrak{g}} t(X) \exp(\omega_{\mathcal{V}}(X)) = \exp \left( -\frac{1}{2}\Pi Q(\mathbf{v}, \mathbf{v}) + i\pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v}) + \mu(\mathbb{A}^{\Pi^2}) \right) \int_{\mathfrak{g}} t(X) \exp(i\mu(\mu_{\mathbb{A}}(X))).$$

is a rapidly decreasing function along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ .

5.2.2.  $\theta$  is integrable along the fibres of  $\mathcal{V}$ . This means that for any smooth compactly supported distribution  $t$  on  $\mathfrak{g}$ ,  $\langle \theta, t \rangle$  is an integrable form along the fibres of  $\mathcal{V}$ . The problem being local on  $M$  we still assume that  $\mathcal{V}$  is trivial.

Since  $-\frac{1}{2}\Pi Q(\mathbf{v}, \mathbf{v})$  is constant along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{V}}$ ,

$$(161) \quad \langle \theta, t \rangle = \exp(-\frac{1}{2}\Pi Q(\mathbf{v}, \mathbf{v})) T \left( \exp(i\pi^* \mathbb{A}^{\Pi^*} \Pi Q^*(\mathbf{v}) + \mu(\mathbb{A}^{\Pi^2})) \int_{\mathfrak{g}} t(X) \exp(i\mu(\mu_{\mathbb{A}}(X))) \right).$$

We recall that  $v^{\Pi}$  is the generic point of  $\Gamma_{\Pi\mathcal{V} \times_M \widehat{M}}(\widehat{M}_0)$  (cf. formula (154)) and that  $\mathcal{D}_{\Pi\mathcal{V}}$  is the Liouville volume form along the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}$ . We put:

$$(162) \quad \begin{aligned} \psi(v^{\Pi}) &= \mathcal{D}_{\Pi\mathcal{V}}(v^{\Pi}) \exp(\mu(\mathbb{A}^{\Pi^2})(v^{\Pi})) \int_{\mathfrak{g}} t(X) \exp(i\mu(\mu_{\mathbb{A}}(X)))(v^{\Pi}) \\ &= \mathcal{D}_{\Pi\mathcal{V}}(v^{\Pi}) \int_{\mathfrak{g}} t(X) \exp(\mu(F_{\mathfrak{g}}(X)))(v^{\Pi}) \in \Gamma_{\mathcal{V} \circ \Pi\mathcal{V} \times_M \widehat{\mathcal{V}}}(\widehat{\mathcal{V}}_0). \end{aligned}$$

The preceding section shows that  $\psi$  is a rapidly decreasing volume form on the fibres of  $\Pi\mathcal{V} \times_M \widehat{\mathcal{V}}$ .

Before going further, we have to consider the objects involved under another point of view.

First, locally, we consider  $\mathbf{v} \in \Gamma_{\Pi\mathcal{V}}(\mathcal{U}) \otimes_{\mathcal{O}_M(\mathcal{U})} \Gamma_{\mathcal{V}^*}(\mathcal{U}) \simeq \text{Hom}_{\mathcal{O}_M(\mathcal{U})}(\Gamma_{\mathcal{V}}(\mathcal{U}), \Gamma_{\Pi\mathcal{V}}(\mathcal{U}))$ .

Thus  $\mathbf{v}$  is identified to odd canonical isomorphism of sheafs of  $\mathcal{O}_M$ -modules  $\Pi : \Gamma_{\mathcal{V}} \rightarrow \Gamma_{\Pi\mathcal{V}}$ .

Now, since  $\mathbb{A} = d + \omega$  with  $\omega \in \widehat{\Omega}_M(M_0, \mathfrak{sp}\mathfrak{o}(\mathcal{V}))_1$ ,  $\pi^* \mathbb{A}^{\Pi} \mathbf{v}$  realizes an even isomorphism of sheafs of  $\mathcal{O}_{\mathcal{V} \times_M \widehat{M}}$ -modules between  $\Gamma_{\widehat{\mathcal{V}}}$  and  $\Gamma_{\Pi\mathcal{V} \times_M (\mathcal{V} \times_M \widehat{M})}$ .

Finally  $\pi^* \mathbb{A}^{\Pi*} \Pi Q^*(\mathbf{v}) = \pi^* \Pi Q^*(\mathbb{A}^{\Pi} \mathbf{v})$  realizes an even linear isomorphism of sheafs of  $\mathcal{O}_{\mathcal{V} \times_M \widehat{M}}$ -modules between  $\Gamma_{\widehat{\mathcal{V}}}$  and  $\Gamma_{\Pi \mathcal{V} \times_M (\mathcal{V} \times_M \widehat{M})}$ .

It follows that:

$$(163) \quad T \left( \exp \left( \mathbf{i} \pi^* \mathbb{A}^{\Pi*} \Pi Q^*(\mathbf{v}) + \mu(\mathbb{A}^{\Pi^2}) \int_{\mathfrak{g}} t(X) \exp(\mathbf{i} \mu(\mu_{\mathbb{A}}(X))) \right) \right) = \widehat{\psi}(\pi^* \mathbb{A}^{\Pi*} \Pi Q^*(\mathbf{v}))$$

is rapidly decreasing along the fibres of  $\widehat{\mathcal{V}} \rightarrow \mathcal{V} \times_M \widehat{M}$ .

Since  $\exp \left( -\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) \right)$  is constant along these fibres and rapidly decreasing along the fibres of  $\mathcal{V} \times_M \widehat{M} \rightarrow \widehat{M}$ , it follows that

$$\langle \theta, t \rangle = \exp \left( -\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) \right) \widehat{\psi}(\pi^* \mathbb{A}^* \Pi Q^*(\mathbf{v}))$$

is a pseudodifferential form on  $\mathcal{V}$  that is integrable along the fibres.

5.2.3. *Evaluation of  $\pi_* \theta$ .* Now, we have to check that  $\pi_* \theta$  is a constant generalized function on  $\mathfrak{g}$ .

We stress that in  $\Pi \mathcal{V} \times_M \widehat{\mathcal{V}}$  there are two copies of  $\Pi \mathcal{V}$ , we denote by  $\widehat{\Pi} \mathcal{V}$  the copy of  $\Pi \mathcal{V}$  in  $\widehat{\mathcal{V}}$ :  $\widehat{\mathcal{V}} = \mathcal{V} \times_M \widehat{\Pi} \mathcal{V} \times_M \widehat{M}$ .

We denote by  $v'$  the generic point of  $\Gamma_{\mathcal{V} \times_M \widehat{M}}(\widehat{M}_0)$  and by  $v^{\widehat{\Pi}}$  the generic point of  $\widehat{\Pi} \mathcal{V} \times_M \widehat{M}$ .

We recall that  $\mathbf{v} \in \Gamma_{\Pi \mathcal{V} \times_M \mathcal{V}}(\mathcal{V}_0) \subset \widehat{\Omega}(\mathcal{V}, \Pi \mathcal{V} \times_M \mathcal{V})$ . It follows from formula (127) with  $\mathbb{A}$  in place of  $\mathbb{A}_{\mathfrak{g}}$  that:

$$(164) \quad \pi^* \mathbb{A}^{\Pi*} \Pi Q^*(\mathbf{v}) = \Pi Q^*(\pi^* \mathbb{A}^{\Pi} \mathbf{v}) = \Pi Q^*((1 \otimes d)\mathbf{v}) + \Pi Q^*((\mathbb{A}^{\Pi} \otimes 1)\mathbf{v}).$$

We put  $\mathbf{d} = 1 \otimes d$ . Since  $\Pi Q^*((\mathbb{A}^{\Pi} \otimes 1)\mathbf{v})$  is constant along the fibres of  $\widehat{\mathcal{V}} \rightarrow \mathcal{V} \times_M \widehat{M}$  and  $d_{(d\xi, dx)}$  is invariant by translation, we obtain:

$$(165) \quad \begin{aligned} \langle \pi_* \theta, t \rangle &= \pi_* (\langle \theta, t \rangle) \in \widehat{\Omega}_M(M_0) \\ &= \frac{1}{(-1)^{\frac{k+l}{2}}} \int_{\widehat{\mathcal{V}}/\widehat{M}} d_{(x, d\xi, dx, \xi)}(v', v^{\widehat{\Pi}}) \\ &\quad \exp \left( -\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) \right) (v') \widehat{\psi}(\pi^* \mathbb{A}^*(\Pi Q^*(\mathbf{v}))) (v', v^{\widehat{\Pi}}) \\ &= \frac{1}{(2\pi)^{\frac{k+l}{2}}} \int_{\mathcal{V} \times_M \widehat{M}/\widehat{M}} d_{(x, \xi)}(v') \exp \left( -\frac{1}{2} \Pi Q(\mathbf{v}, \mathbf{v}) \right) (v') \\ &\quad \int_{\widehat{\Pi} \mathcal{V} \times_M \widehat{M}/\widehat{M}} d_{(d\xi^j, dx^i)}(v^{\widehat{\Pi}}) \widehat{\psi}(\Pi Q^*(\mathbf{d}\mathbf{v})) (v^{\widehat{\Pi}}). \end{aligned}$$

Now, as above for  $\pi^* \mathbb{A}^{\Pi*} \Pi Q^*(\mathbf{v})$ ,  $\mathbf{d}\mathbf{v} : \widehat{\mathcal{V}} = \widehat{\Pi} \mathcal{V} \times_M \mathcal{V} \times_M \widehat{M} \rightarrow \Pi \mathcal{V} \times_M \mathcal{V} \times_M \widehat{M}$  is an even isomorphism of supervector bundles. It induces an isomorphism  $\mathcal{B}er(\mathbf{d}\mathbf{v}) : \mathcal{B}er(\widehat{\Pi} \mathcal{V}) \rightarrow \mathcal{B}er(\Pi \mathcal{V})$ .

We recall that  $(e_i, f_j)$  is a standard basis of  $\Gamma_{\mathcal{V}}(\mathcal{U})$  and  $(x^i, \xi^j)$  is its dual basis. Then  $(\Pi f_j, \Pi e_i)$  is a standard basis of  $\Gamma_{\Pi\mathcal{V}}(\mathcal{U})$ . Its dual basis is  $(-\Pi\xi^j, \Pi x^i)$ . Thus we have:

$$(166) \quad \mathcal{B}er(\mathbf{d}\mathbf{v})(d_{(d\xi^j, dx^i)}) = (-1)^l (d_{(\Pi\xi^j, \Pi x^i)}).$$

Thus:

$$(167) \quad \int_{\widehat{\Pi\mathcal{V}} \times_M \widehat{M}/\widehat{M}} d_{(d\xi^j, dx^i)}(v^{\widehat{\Pi}}) \widehat{\psi}(\Pi Q^*(\mathbf{d}\mathbf{v}))(v^{\widehat{\Pi}}) = (-1)^l \int_{\Pi\mathcal{V} \times_M \widehat{M}/\widehat{M}} d_{(\Pi\xi^j, \Pi x^i)}(v^{\Pi}) \widehat{\psi}(\Pi Q^*(v^{\Pi})).$$

On the other hand,  $d_{(x^i, \xi^j)} \in \mathcal{B}er(\mathcal{V})$ . We recall that we canonically identified  $\mathcal{B}er(\mathcal{V})$  and  $\mathcal{B}er(\Pi\mathcal{V}^*)$ . We assume that  $(-\Pi f_j, \Pi e_i)$  is an oriented symplectic basis of  $\Pi\mathcal{V}$ . Under this identification:

$$(168) \quad d_{(x^i, \xi^j)} = d_{(-\Pi f_j, \Pi e_i)} = \frac{(2\pi)^l}{(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k} (d_{(\Pi\xi^j, \Pi x^i)})^*;$$

and

$$(169) \quad \mathcal{D}_{\Pi\mathcal{V}} = \frac{1}{(2\pi)^{\frac{l}{2}}} d_{(-\Pi f_j, \Pi e_i)} = \frac{1}{(2\pi)^{\frac{l}{2}}} d_{(x^i, \xi^j)}$$

It follows from Fourier inversion formula (since  $l$  is even  $(-1)^l = 1$ ):

$$(170) \quad \begin{aligned} d_{(x, \xi)} \int_{\widehat{\Pi\mathcal{V}} \times_M \widehat{M}/\widehat{M}} d_{(d\xi^j, dx^i)}(v^{\widehat{\Pi}}) \widehat{\psi}(\Pi Q^*(\mathbf{d}\mathbf{v}))(v^{\widehat{\Pi}}) &= \frac{(2\pi)^l}{(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k} \mathcal{D}_{\Pi\mathcal{V}} \int_{\mathfrak{g}} t(X) \\ &= \frac{(2\pi)^{\frac{l}{2}}}{(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k} d_{(x, \xi)} \int_{\mathfrak{g}} t(X). \end{aligned}$$

Since  $k$  is even,  $(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k = 1$ . Moreover,  $\Pi Q(\mathbf{v}, \mathbf{v})(v') = Q(v', v')$ . Thus:

$$(171) \quad \begin{aligned} \langle \pi_* \theta, t \rangle &= \frac{1}{(2\pi)^{\frac{k+l}{2}}} \frac{(2\pi)^{\frac{l}{2}}}{(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k} \int_{\mathcal{V} \times_M \widehat{M}/\widehat{M}} d_{x, \xi}(v') \exp\left(-\frac{1}{2} Q(v', v')\right) \int_{\mathfrak{g}} t(X) \\ &= \frac{1}{(2\pi)^{\frac{k+l}{2}}} \frac{(2\pi)^{\frac{l}{2}}}{(-1)^{\frac{k(k-1)}{2}} \mathbf{i}^k} (2\pi)^{\frac{k}{2}} \int_{\mathfrak{g}} t(X) \\ &= \int_{\mathfrak{g}} t(X) \end{aligned}$$

Consequently  $\theta$  is a Thom form on  $\mathfrak{g}$ . Moreover its restriction to  $\pi^{-1}(W)$  is  $\mathcal{C}^\infty$  on  $U^{\mathcal{V}_1}(W)$ . □

**Remark:** If  $V = V_1$ , the hypotheses imply that the moment map  $\mu_{\mathbb{A}}$  is proper. Let  $\phi$  be smooth compactly supported function on  $\mathfrak{g}_0$ . The fact that  $\int_{\mathfrak{g}_0} dX \exp \mathbf{i} \mu(\mu_{\mathbb{A}}(X)) \phi(X)$  defines an integrable function on the vector space  $V_1$  is a particular case of Lemma 12 in [Ver97].

*Example:* Let us consider the case where  $M$  is a point,  $\mathcal{V} = \mathbb{R}^{(0,2)}$  ( $\Gamma_{\mathcal{V}}(M) = \mathbb{R}^{(0,2)}$ ) and  $\mathfrak{g} = \mathbb{R}X$ . The action of  $X$  is defined by the field of vectors:

$$(172) \quad X_{\mathcal{V}} = \xi \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial \xi}.$$

In the canonical basis  $(f_1, f_2)$  of  $\mathbb{R}^{(0,2)}$ ,  $\mathcal{L}^{\mathcal{V}}(X)$  is represented by

$$(173) \quad X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Euclidean structure  $Q$  is given by  $Q(f_1, f_2) = 1$ . We denote by  $z$  the element of  $\mathfrak{g}^*$  such that  $z(X) = 1$ . We denote by  $(\xi, \eta)$  the dual basis of  $(f_1, f_2)$ . Thus  $(\pi\xi, \pi\eta)$  is a basis of  $\Pi(\mathbb{R}^{(0,2)})^* = (\mathbb{R}^2)^*$ . We have:

$$(174) \quad \omega_{\mathcal{V}} = \xi\eta - \mathbf{i}(d\xi\pi\eta - d\eta\pi\xi) - \frac{\mathbf{i}}{2}z((\pi\xi)^2 + (\pi\eta)^2).$$

On the open set  $U = (\mathbb{R} \setminus \{0\})X$ , we have

$$(175) \quad \begin{aligned} \theta &= \int_{\Pi\mathbb{R}^{(0,2)}} |d(\pi\xi)d(\pi\eta)| \exp(\omega_{\mathcal{V}}) \\ &= (1 + \xi\eta) \int_{\mathbb{R}^2} |d(\pi\xi)d(\pi\eta)| \exp\left(-\frac{\mathbf{i}}{2}z\left((\pi\xi - \frac{d\eta}{z})^2 + (\pi\eta + \frac{d\xi}{z})^2\right)\right. \\ &\quad \left. + \frac{\mathbf{i}}{2}\left(\frac{d\xi^2}{z} + \frac{d\eta^2}{z}\right)\right) \\ &= \frac{2\mathbf{i}\pi}{z} \exp\left(\xi\eta - \frac{\mathbf{i}}{2z}(d\xi^2 + d\eta^2)\right). \end{aligned}$$

It appears that  $\theta$  is rapidly decreasing on  $\widehat{\mathbb{R}^{(0,2)}}$  but only as a generalized function on  $\mathfrak{g}$ . More precisely, for any compactly supported function  $\phi \in \mathcal{C}^\infty(\mathbb{R})$  which vanishes at 0, we have:

$$(176) \quad \int_{\mathbb{R}} |dz| \theta(z) \phi(z) = -2\mathbf{i}\pi \exp(\xi\eta) \left(\frac{du}{u} \widehat{\phi}\left(\frac{1}{u}\right)\right) \left(-\frac{1}{2}(d\xi^2 + d\eta^2)\right),$$

where  $\left(\frac{du}{u} \widehat{\phi}\left(\frac{1}{u}\right)\right)$  denotes the Fourier transform of the smooth distribution  $\frac{du}{u} \phi\left(\frac{1}{u}\right)$  on  $\mathbb{R} \setminus \{0\}$ . Since  $\phi$  is compactly supported, this integral is rapidly decreasing in  $d\xi$  and  $d\eta$ .

**5.3. A relation between cohomology classes.** Let  $G = (G_0, \mathfrak{g})$  be a supergroup. Let  $\pi : \mathcal{V} \mapsto M$  be an oriented  $G$ -equivariant supervector bundle. We assume that  $\mathcal{V}$  and  $M$  are globally oriented.

**Lemma 5.1.** *Let  $\mathcal{U}$  be a  $G$ -invariant open subset of  $\mathfrak{g}_0$ . Let  $\theta$  be a Thom form on  $\mathcal{V}$ . Let  $\alpha \in \widehat{\Omega}_{G,f}^\infty(\mathcal{U}, M)$ , then*

$$(177) \quad \theta(\pi^*\alpha) \in \widehat{\Omega}_{G,f}^{-\infty}(\mathcal{U}, \mathcal{V}).$$

**Remark:** This product is well defined because  $\alpha$  has smooth coefficients.

*Proof.* We have to show that this product defines a generalized function in the integrable forms. It is enough to check locally that we can integrate along the fibres of  $\mathcal{V} \rightarrow M$ , and then to check that the resulting form is integrable on  $M$ . But since  $\pi^*(\alpha)$  is constant along the fibres, the integrability along the fibres comes from that of  $\theta$ . Thus we have

$\pi_*(\theta \pi^*(\alpha)) = \pi_*(\theta)\alpha = \alpha$ . Hence, the resulting form is  $\alpha$ , which is integrable by hypothesis.  $\square$

**Theorem 5.2.** *Let  $\mathcal{U}$  be a  $G_0$ -invariant open subset of  $\mathfrak{g}_0$ . Let  $\theta$  be a Thom form on  $\mathcal{V}$ . Let  $\alpha$  be an equivariantly closed form with smooth coefficients ( $\alpha \in \widehat{\Omega}_{G,f}^\infty(\mathcal{U}, \mathcal{V})$ ). We have the following equality between cohomology classes in  $\widehat{H}_{G,f}^{-\infty}(\mathcal{U}, \mathcal{V})$ :*

$$(178) \quad \alpha \equiv \theta \pi^*(\pi_*\alpha).$$

*Proof.* The proof in the even situation (cf. [KV93]) can be repeated here. We consider the supervector bundle  $\mathcal{V} \times_M \mathcal{V}$  over  $M$ . For  $t \in \mathbb{R}$ , we denote by  $\sigma_t$  the linear transformation in the fibres of  $\mathcal{V} \times_M \mathcal{V}$  which is defined for any section  $x$  and  $y$  of  $\mathcal{V}$  by:

$$(179) \quad \sigma_t(x, y) = \left( (\cos t)x + (\sin t)y, -(\sin t)x + (\cos t)y \right).$$

We have  $\sigma_0 = Id$ , and  $\sigma_{\frac{\pi}{2}} = \sigma$  where, for any sections  $x$  and  $y$  of  $\mathcal{V}$ ,  $\sigma(x, y) = (y, -x)$ . We put:

$$(180) \quad S = \frac{d}{dt} \sigma_t \in \Gamma_{T(\mathcal{V} \times_M \mathcal{V})}((\mathcal{V} \times_M \mathcal{V})_0).$$

Since for all  $t$ ,  $\sigma_t$  commutes with the action of  $G$ ,  $\mathcal{L}(S)$  and  $\iota(S)$  leave  $\widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$  invariant. Thus we have the following relation (between derivations of  $\widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$ ):

$$(181) \quad \mathcal{L}(S) = d_{\mathfrak{g}}\iota(S) + \iota(S)d_{\mathfrak{g}},$$

( $S$  is an even field of vectors and thus for  $X \in \mathfrak{g}$   $\iota(X)\iota(S) + \iota(S)\iota(X) = [\iota(X), \iota(S)] = 0$ ).

We put for  $\nu \in \widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$ :

$$(182) \quad H\nu(X) = \int_0^{\frac{\pi}{2}} \sigma_t^*(\iota(S)\nu(X)) dt.$$

It defines an application  $H : \widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V}) \rightarrow \widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$ .

For any  $\nu \in \widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$  we have:

$$(183) \quad \sigma^*\nu - \nu = (d_{\mathfrak{g}}H - Hd_{\mathfrak{g}})\nu.$$

We denote by  $p_i : \mathcal{V} \times_M \mathcal{V} \rightarrow \mathcal{V}$ ,  $i = 1$  (resp. 2), the projection on the first (resp. the second) factor.

Let  $\alpha$  be an integrable equivariantly closed form on  $\mathcal{V}$ . Then (for similar reasons as those we used in the preceding lemma)  $(p_1^*\theta)(p_2^*\alpha)$  is a well defined in  $\widehat{\Omega}_{G,f}^{-\infty}(\mathfrak{g}, \mathcal{V} \times_M \mathcal{V})$ .

It is a closed form because  $\alpha$  and  $\theta$  are closed. The previous relation shows that  $(p_1^*\theta)(p_2^*\alpha)$  is in the same cohomology class as  $\sigma^*((p_1^*\theta)(p_2^*\alpha))$ .

For any form  $\beta$  on  $\mathcal{V}$ , we denote by  $\overline{\beta}$  the form  $\tau^*\beta$  where  $\tau : \mathcal{V} \rightarrow \mathcal{V}$ ,  $v \mapsto -v$ . So we have  $\sigma^*((p_1^*\theta)(p_2^*\alpha)) = (p_2^*\theta)(\overline{p_1^*\alpha})$ . We integrate these two cohomologically equivalent forms along the fibres of  $p_1$ . We obtain  $(\pi_* \text{ and } \theta \text{ have same parity } p(\pi_*) = p(\theta) =$

$\frac{n+m}{2}(\text{mod } 2)$ ):

$$(184) \quad (p_1)_*(p_1^*\theta p_2^*\alpha) = \pi^*(\pi_*\theta)\alpha,$$

$$(185) \quad \begin{aligned} (p_1)_*(p_2^*\theta p_1^*\bar{\alpha}) &= (-1)^{(n+m)p(\theta)}\theta\pi^*(\pi_*\bar{\alpha}), \\ &= (-1)^{(n+m)(p(\theta)+1)}\theta\pi^*(\pi_*\alpha), \\ &= \theta\pi^*(\pi_*\alpha). \end{aligned}$$

Since  $\pi_*\theta = 1$ , we obtain the requested equality in cohomology:

$$(186) \quad \theta\pi^*(\pi_*\alpha) \equiv \pi^*(\pi_*\theta)\alpha = \alpha.$$

□

## 6. EQUIVARIANT EULER FORM

**Definition 6.1.** *Let  $G = (G_0, \mathfrak{g})$  be a supergroup. Let  $M$  be a  $G$ -supermanifold. Let  $\mathcal{V} \rightarrow M$  be an equivariant supervector bundle. We assume that  $\mathcal{V}$  and  $M$  are globally oriented. Let  $j$  be the injection of  $M$  into  $\mathcal{V}$  by means of the zero section. We assume that there is an equivariant Thom form  $\theta$  on  $\mathcal{V}$ . We put:*

$$(187) \quad \mathcal{E}_{\mathfrak{g}} = j^*\theta \in \widehat{\Omega}_G(M).$$

We say that  $\mathcal{E}_{\mathfrak{g}}$  is an equivariant Euler form on  $\mathcal{V}$ .

Now, we assume that there is a  $G$ -invariant Euclidean structure denoted by  $Q$  on the fibres and an equivariant superconnection denoted by  $\mathbb{A}_{\mathfrak{g}}$  on  $\mathcal{V}$ . We denote by  $F_{\mathfrak{g}} = \mathbb{A}_{\mathfrak{g}}^2 + \mathbf{i}\mathcal{L}^{\mathcal{V}}$  its equivariant curvature. It is an  $\mathfrak{osp}(\mathcal{V}, Q)$ -valued equivariant form.

We recall that we identified  $\mathfrak{osp}(\mathcal{V}, Q)$  and  $\mathfrak{spo}(\Pi\mathcal{V}, \Pi Q)$ . This identifies  $F_{\mathfrak{g}}$  and  $F_{\mathfrak{g}}^{\Pi}$ . Thus, we can consider  $\mu(F_{\mathfrak{g}})$  instead of  $\mu(F_{\mathfrak{g}}^{\Pi})$ .

We assume that the hypothesis in Theorem 5.1 are satisfied (in particular  $(*)$  and  $(**)$ ).

Let  $\mathcal{D}_{\Pi\mathcal{V}}$  be Liouville volume form on the oriented symplectic bundle  $(\Pi\mathcal{V} \times_M \widehat{M}, \Pi Q)$ .

We put (compare with [Lav03]):

$$(188) \quad \text{Spf}(-\mathbf{i}F_{\mathfrak{g}}(X)) = \mathbf{i}^{\frac{k-l}{2}} \int_{\Pi\mathcal{V} \times_M \widehat{M}/\widehat{M}} \mathcal{D}_{\Pi\mathcal{V}}(v^{\Pi}) \exp(\mu(F_{\mathfrak{g}}(X)(v^{\Pi}))).$$

The hypotheses  $(*)$  and  $(**)$  ensure that this definition makes sense as a generalized function on  $\mathfrak{g}$  with values in the pseudodifferential forms on  $M$ . It means that for any smooth compactly supported distribution  $t$  on  $\mathfrak{g}$ , the integral

$$(189) \quad \int_{\mathfrak{g}} t(X) \exp(\mu(F_{\mathfrak{g}}(X))) \in \mathcal{O}(\Pi\mathcal{V} \times_M \widehat{M})$$

is rapidly decreasing along the fibres of  $\Pi\mathcal{V} \times_M \widehat{M} \rightarrow \widehat{M}$  (cf. subsection 5.2.1 in demonstration of Theorem 5.1). Then by definition:

$$(190) \quad \langle \text{Spf}(-\mathbf{i}F_{\mathfrak{g}}(X)), t(X) \rangle = \mathbf{i}^{\frac{k-l}{2}} \int_{\Pi\mathcal{V} \times_M \widehat{M}/\widehat{M}} \mathcal{D}_{\Pi\mathcal{V}}(v^{\Pi}) \int_{\mathfrak{g}} t(X) \exp(\mu(F_{\mathfrak{g}}(X)(v^{\Pi}))).$$

**Proposition 6.1.** *We have the following equality between cohomology classes:*

$$(191) \quad \mathcal{E}_{\mathfrak{g}} \equiv \frac{1}{\mathbf{i}^{\frac{k-l}{2}}} \text{Spf}(-\mathbf{i}F_{\mathfrak{g}}),$$

where  $(k, l)$  is the dimension of the fibres of  $\mathcal{V}$ .

**Remark:** In [BGV92] the equivariant Euler form is defined by the preceding formula (in the purely even case).

*Proof.* It is enough to show that this equality is true for the particular equivariant Euler form obtained with the Thom form constructed in Theorem 5.1.

When  $k$  is odd,  $\text{Spf}(-\mathbf{i}F_{\mathfrak{g}})$  is zero.

When  $k$  is even, we have (the existence of an Euclidean structure ensures that  $l$  is even):

$$(192) \quad \begin{aligned} j^*\theta &= \int_{\Pi V \times_{\widehat{M}/\widehat{M}} \widehat{M}} \mathcal{D}_{\Pi V}(v^{\Pi}) \exp(\mu(F_{\mathfrak{g}})(v^{\Pi})) \\ &= \frac{1}{\mathbf{i}^{\frac{k-l}{2}}} \text{Spf}(-\mathbf{i}F_{\mathfrak{g}}). \end{aligned}$$

□

## 7. FIRST LOCALIZATION FORMULAS

**7.1. The linear situation.** Let  $G = (G_0, \mathfrak{g})$  be a supergroup and  $V$  be a supervector space of dimension  $(m, n)$ . We assume that we have a representation of  $G$  in  $V$ . We denote by  $\rho$  the associated representation of  $\mathfrak{g}$ . We assume that  $V$  is  $G$ -invariantly globally oriented and has a  $G$ -invariant Euclidean structure  $Q$ . In particular  $m$  is even.

Let  $U$  be the open subset of  $\mathfrak{g}_0$  defined by:

$$(193) \quad U = \{X \in \mathfrak{g}_0 / \rho(X) \text{ is invertible}\}.$$

Let  $U_+$  be the open subset of  $\mathfrak{g}_0$  such that:

$$(194) \quad U_+ = \{X \in \mathfrak{g}_0 / v \mapsto Q(v, \rho(X)v) \text{ is a scalar product on } V_1.\}.$$

We have  $U_+ \subset U$ . In order to satisfy condition (\*\*), we assume that  $U_+$  is not empty. In particular, it implies that  $m$  is even.

**Proposition 7.1.** *Let  $\alpha \in \widehat{\Omega}_{G,f}^{\infty}(U, V)$  be an equivariantly closed form on  $V$ . We denote by  $j$  the canonical injection of  $\{0\}$  in  $V$ . Then we have in  $\mathcal{C}^{\infty}(U)$ :*

$$(195) \quad \int_V \alpha(X) = \mathbf{i}^{\frac{m-n}{2}} (2\pi)^{\frac{n+m}{2}} \frac{(j^*\alpha)(X)}{\text{Spf}(\rho(X))}.$$

*Proof.* Following section 5.3, we know that the cohomology class of  $\alpha$  in  $\widehat{H}_{G,f}^{\infty}(U, V)$  is equal to that of  $\theta(\pi_*\alpha) = \frac{1}{(2\pi)^{\frac{m+n}{2}}} \theta(\int_V \alpha)$ , where  $\theta$  is an equivariant Thom form on  $V$  ( $\theta$  is smooth on  $U$ ).

We use the notation of the preceding section, just replacing  $\mathcal{V}$  by  $V$  and paying attention to the fact that, in this case, we can take  $\mathbb{A} = 0$  ( $M$  is a point). The equivariant moment  $\mu_{\mathbb{A}}$  is the representation  $\rho$  of  $\mathfrak{g}$  in  $V$  ( $\rho : \mathfrak{g} \rightarrow \mathfrak{osp}(V)$ ) and  $F_{\mathfrak{g}} = \mathbf{i}\rho$ .

We just have to evaluate  $j^*\theta$ . In this case condition (\*) is trivial and condition (\*\*) follows from the assumption that  $U_+$  is not empty.



We obtain:

$$(196) \quad j^*\theta(X) = \frac{1}{\mathbf{i}^{\frac{m-n}{2}}} \text{Spf}(\rho(X)).$$

Since on  $U$ , the function  $X \mapsto \text{Spf}(\rho(X))$  is smooth and invertible, the formula follows.  $\square$

**7.2. The fibered situation.** We can generalize the preceding formula to the case of a  $G$ -equivariant oriented Euclidean supervector bundle.

We put:

$$(197) \quad U^\mathcal{V}(\mathcal{U}) = \text{Interior of } \left\{ X \in \mathfrak{g}_0 / \forall \mathbf{m} \in (\widehat{M}(\mathcal{U}))_0, \mu_{\mathbb{A}}(X)(\mathbf{m}) \text{ is invertible} \right\}.$$

We have  $U^\mathcal{V}(\mathcal{U}) \subset U^{\mathcal{V}_1}(\mathcal{U})$ .

**Proposition 7.2.** *Let  $G = (G_0, \mathfrak{g})$  be a supergroup and  $M$  be a  $G$ -supermanifold. Let  $\mathcal{V} \rightarrow M$  be a  $G$ -equivariant oriented Euclidean supervector bundle.*

*We assume that there is a superconnection  $\mathbb{A}$  satisfying condition (\*). We denote by  $\mu_{\mathbb{A}}$  the equivariant moment of  $\mathbb{A}$ .*

*We assume that  $U_+^{\mathcal{V}_1}(M)_0 \subset \mathfrak{g}_0$  (cf formula (145)) contains a non empty open subset, which implies condition (\*\*) (cf. Theorem 5.1 for conditions (\*) and (\*\*)).*

*Let  $\alpha \in \widehat{\Omega}_{G,f}^\infty(U^\mathcal{V}(M), \mathcal{V})$  be an equivariant integrable form with smooth coefficients ( $U^\mathcal{V}(M)$  defined by formula (197)).*

*We denote by  $j$  the injection of  $M$  into  $\mathcal{V}$  by means of the zero section and by  $\mathcal{E}_{\mathfrak{g}}$  an equivariant Euler form on  $M$  associated to  $\mathcal{V}$ . We have the following equality of functions in  $\mathcal{C}^\infty(U^\mathcal{V}(M))$ :*

$$(198) \quad \int_{\mathcal{V}} \alpha(X) = (2\pi)^{\frac{m+n}{2}} \int_M \frac{j^*\alpha(X)}{\mathcal{E}_{\mathfrak{g}}(X)}$$

*(On  $U^\mathcal{V}(M)$ ,  $\mathcal{E}_{\mathfrak{g}}$  is smooth and invertible.)*

*Proof.* Exactly the same as in the case where  $M$  is just a point.  $\square$

## 8. PRELIMINARIES TO THE LOCALIZATION FORMULA

**8.1. Introduction.** Let  $G = (G_0, \mathfrak{g})$  be a connected supergroup. Let  $M$  be a globally oriented  $G$ -supermanifold. We also assume that  $M$  has a  $G$ -invariant weak Euclidean structure denoted by  $Q$ . Let  $\alpha$  be an equivariantly closed form with smooth coefficients defined on  $\mathfrak{g}_0$ . Let  $X \in \mathfrak{g}_0$  such that  $\overline{\exp(\mathbb{R}X)} \subset G_0$  is compact. Let  $M(X)$  be the supermanifold of zeroes of  $X$  in  $M$  (it will be defined in the next subsection). We express  $\int_M \alpha$  on a neighborhood of  $X$  in  $\mathfrak{g}(X)_0$  in terms of an integral on  $M(X)$ . This method comes from [BV83a],[BV83b] and [Bis86].

To achieve this we have to assume that the normal bundle  $T_N M(X)$  of  $M(X)$  in  $M$  (cf. section 9.1 for a definition) has a  $G(X)$ -invariant Euclidean structure, and has a  $G(X)$ -invariant superconnection,  $\mathbb{A}$ , which preserves the Euclidean structure of  $T_N M(X)$ .

Moreover, we assume:

$$(*) \quad \mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}} \leq 0 \text{ on } (\Pi T_N M(X))_0;$$

(\*\*) there is a covering of  $M_0$  by open subsets  $W$  such that  $U_+^{T_N M(X)_1}(W)$  contains a non empty open subset (cf. formula (145) for  $U_+^{T_N M(X)_1}(W)$ ).

First, we define  $M(X)$ , the manifold of zeroes of  $X$  in  $M$ .

Then, we construct:

- (i) an open neighborhood  $U$  of  $X$  in  $\mathfrak{g}_0$ ,
- (ii) an equivariant form  $\beta \in \widehat{\Omega}_G^\infty(U, M)$ ,
- (iii) an open neighborhood  $V$  of  $M(X)_0$  in  $M$ ,

such that  $d_{\mathfrak{g}}\beta$  is invertible as equivariant form on  $M$  defined on the complement of  $\overline{V}$  in  $M_0$  (where  $\overline{V}$  is the closure of  $V$ ).

Then, we prove the localization formula in the case of isolated zeroes and in the general case.

**8.2. Supermanifold of zeroes.** Let  $X \in \mathfrak{g}_0$  be such that  $\overline{\exp(\mathbb{R}X)} \subset G_0$  is compact. Let  $\mathcal{I}$  be the sheaf of ideals of  $\mathcal{O}_M$  such that for any open subset  $\mathcal{U}$  of  $M$ ,  $\mathcal{I}(\mathcal{U})$  is the ideal generated by

$$\{X_M f, f \in \mathcal{O}(\mathcal{U})\}.$$

We recall that  $(X_M)_{\mathbb{R}}$  is the canonical projection of  $X_M$  in  $\Gamma_{TM}(M_0) \otimes_{\mathcal{O}_M(M_0)} \mathcal{C}^\infty(M_0)$ .

Since  $\overline{\exp(\mathbb{R}X)}$  is compact, the set of zeroes of  $(X_M)_{\mathbb{R}}$  in  $M_0$  is a manifold denoted by  $M(X)_0$  (cf. [BGV92]).

**Proposition 8.1.** *The sheaf of ideals  $\mathcal{I}$  defines a subsupermanifold of  $M$  denoted by  $M(X)$  which underlying manifold is  $M(X)_0$  and such that for any open subset  $\mathcal{U} \subset M_0$ :*

$$(199) \quad \mathcal{O}_{M(X)}(\mathcal{U} \cap M(X)_0) = \mathcal{O}_M(\mathcal{U})/\mathcal{I}(\mathcal{U}).$$

Then we put:

**Definition 8.1.** *We call supermanifold of zeroes of  $X \in \mathfrak{g}_0$  in  $M$  the subsupermanifold  $M(X)$  of  $M$  defined in the preceding proposition.*

*Example:* Let  $M$  be a supervector space with a representation  $(\rho_0, \rho)$  of  $G = (G_0, \mathfrak{g})$ . Let  $X \in \mathfrak{g}_0$ . Then  $M(X) = \ker(\rho(X))$ .

Let  $G_0(X)$  (resp.  $\mathfrak{g}(X)$ ) be the centralizer of  $X$  in  $G_0$  (resp.  $\mathfrak{g}$ ). We put  $G(X) = (G_0(X), \mathfrak{g}(X))$ . It is a subsupergroup of  $G$  called the centralizer of  $X$  in  $G$ .

The supermanifold  $M(X)$  is stable under the action of the centralizer  $G(X)$  of  $X$  in  $G$ .

### 8.3. An equivariant form with invertible equivariant differential.

**8.3.1. Construction of the form.** Let  $X \in \mathfrak{g}_0$  be a central element in  $\mathfrak{g}$  ( $\mathfrak{g}(X) = \mathfrak{g}$ ). Let  $M(X)$  be the set of zeroes of  $X$ .

Let  $\alpha$  be an equivariantly closed form on  $M$  with smooth coefficients defined on  $\mathfrak{g}_0$ . We denote by  $Supp(\alpha)$  its support. It is the smallest closed subset  $F$  of  $M_0$  such that  $\alpha$  vanishes on  $M_0 \setminus F$ . We assume that  $Supp(\alpha)$  is compact.

We will prove that if  $Supp(\alpha) \cap M(X)_0 = \emptyset$ , there is a  $G_0$ -invariant open neighborhood  $U'$  of  $X$  such that the restriction of  $\alpha$  to  $U'$  is  $d_{\mathfrak{g}}$ -exact.

For any  $Y$  in  $\mathfrak{g}$ , we put:

$$(200) \quad \beta(Y) = \pi Q^*(\pi Y_M) \in \Gamma_{\pi T^*M}(M_0) \subset \widehat{\Omega}_M(M_0).$$

This defines  $\beta \in \mathfrak{g}^* \otimes \widehat{\Omega}(M) \subset \mathcal{O}_{\mathfrak{g}}(\mathfrak{g}_0, \widehat{\Omega}(M))$ . Since  $Q$  is  $G$ -invariant, this form is equivariant.

We have  $d_{\mathfrak{g}}\beta(Y) = -iQ(Y_M, Y_M) + d\beta(Y)$ . To ensure that  $d_{\mathfrak{g}}\beta(Y)$  is invertible it is sufficient to show that  $Q(Y_M, Y_M)_{\mathbb{R}}$  is not zero. First we remark that the function

$$(m, Y) \mapsto Q_m(Y_M(m), Y_M(m))$$

is continuous and positive on  $M_0 \times \mathfrak{g}_0$ .

8.3.2. *Construction of  $U_a(\alpha, X, V)$ .* We use the following lemma:

**Lemma 8.1.** *Let  $W$  and  $V$  be two topological spaces where  $V$  is compact, and  $\phi$  be a continuous function on  $W \times V$  into  $\mathbb{R}$ . Then the set*

$$(201) \quad \{m \in W, \forall v \in V \phi(m, v) > 0\}$$

*is open in  $W$ .*

Let  $V$  be a closed neighborhood of  $M(X)_0$ . Let  $a > 0$ . Then by the preceding lemma:

$$(202) \quad U_a(\alpha, X, V) = \{Y \in \mathfrak{g}_0 / \forall m \in \text{Supp}(\alpha) \cap \overline{M_0 \setminus V}, Q_m(Y_M(m), Y_M(m)) > a\}$$

is open (we recall that  $\overline{M_0 \setminus V}$  designs the closure of  $M_0 \setminus V$  in  $M_0$ ).

Moreover  $Q(X_M, X_M)_{\mathbb{R}} > 0$  on  $M_0 \setminus M(X)_0$ . As  $\overline{\text{Supp}(\alpha)}$  is compact,  $Q(X_M, X_M)_{\mathbb{R}}$  has a strictly positive lower bound  $A$  on  $\text{Supp}(\alpha) \cap \overline{M_0 \setminus V}$ . Then  $U_a(\alpha, X, V)$  is not empty if  $a \leq A$ .

8.3.3. *Exactitude of  $\alpha$ .*

**Lemma 8.2.** *Let  $X \in \mathfrak{g}_0$ , central in  $\mathfrak{g}$ .*

*Let  $\alpha \in \widehat{\Omega}_G^\infty(\mathfrak{g}_0, M)$  such that  $d_{\mathfrak{g}}\alpha = 0$ ,  $\text{Supp}(\alpha)$  is compact and  $\text{Supp}(\alpha) \cap M(X)_0 = \emptyset$ .*

*Then, there exists an open neighborhood  $U$  of  $X$  in  $\mathfrak{g}_0$  such that  $\alpha$  is  $d_{\mathfrak{g}}$ -exact in  $\widehat{\Omega}_G^\infty(U, M)$ .*

*Moreover, if  $\alpha$  is integrable,  $\alpha$  is  $d_{\mathfrak{g}}$ -exact in  $\widehat{\Omega}_{G,f}^\infty(U, M)$ .*

*Proof.* Since  $\text{Supp}(\alpha) \cap M(X)_0 = \emptyset$  and  $X$  is central in  $\mathfrak{g}$  there exists a  $G$ -invariant closed neighborhood  $V$  of  $M(X)_0$  such that  $V \cap \text{Supp}(\alpha) = \emptyset$ . Then, it is enough to take  $U = U_a(\alpha, X, V)$  for a small enough  $a$  and to check that  $\alpha = d_{\mathfrak{g}}\left(\frac{\beta\alpha}{d_{\mathfrak{g}}\beta}\right)$ .

If  $\alpha$  is integrable,  $\frac{\beta\alpha}{d_{\mathfrak{g}}\beta}$  is integrable because  $\frac{\beta}{d_{\mathfrak{g}}\beta}$  is a smooth bounded function on  $U_a(\alpha, X, V) \times \widehat{M}(M_0 \setminus V)$ .  $\square$

**Remark:** Since  $\beta$  is smooth, the lemma is still true for an equivariant form  $\alpha$  with generalized coefficients.

## 9. LOCALIZATION FORMULA

9.1. **The localization.** As in the preceding paragraph, we consider a connected supergroup  $G = (G_0, \mathfrak{g})$  and a globally oriented  $G$ -supermanifold  $M$  with a weak  $G$ -invariant superstructure.

Let  $\alpha$  be an integrable equivariant form with smooth coefficients. In particular this implies that  $\text{Supp}(\alpha)$  is compact.

9.1.1.  $X$  and  $T_N M(X)$ . We fix  $X \in \mathfrak{g}_0$  such that  $\overline{\exp(\mathbb{R}X)} \subset G_0$  is compact. We assume that  $X$  is central in  $\mathfrak{g}$ . Otherwise, we replace  $\mathfrak{g}$  by  $\mathfrak{g}(X)$ , the centralizer of  $X$  in  $\mathfrak{g}$ , and  $G$  by the connected component of identity of the centralizer  $G(X)$  of  $X$  in  $G$ .

Let  $T_N M(X) = (TM|_{M(X)})/TM(X) \rightarrow M(X)$  be the normal bundle of  $M(X)$  in  $M$ . Its sheaf of sections is given for any open subset  $\mathcal{U} \subset M_0$  by:

$$(203) \quad \Gamma_{T_N M(X)}(\mathcal{U} \cap M(X)_0) = \left( \Gamma_{TM}(\mathcal{U}) \otimes_{\mathcal{O}_{M(\mathcal{U})}} \mathcal{O}_{M(X)}(\mathcal{U} \cap M(X)_0) \right) / \Gamma_{TM(X)}(\mathcal{U} \cap M(X)_0).$$

Then  $X$  determines a supervector bundle orientation of  $T_N M(X)$  by the following (cf. for example [BGV92] in the purely even case).

Since  $Q$  is  $G$ -invariant,  $Q$  determines by restriction a weak Euclidean structure on  $TM(X)$  and then on  $T_N M(X)$ . We still denote the quotient weak Euclidean structure by  $Q$ . Then  $\mathcal{L}^\vee(X) \in \Gamma_{\mathfrak{osp}(T_N M(X))}(M(X)_0)$  is invertible. Thus  $(v, w) \mapsto Q(v, \mathcal{L}^\vee(X)w)$  defines a symplectic structure on  $T_N M(X)$ . This gives as usual an orientation of  $T_N M(X) \rightarrow M(X)$ .

We recall that since  $M$  is oriented,  $TM \rightarrow M$  is oriented. We give to  $M(X)$  the orientation such that the resulting the quotient supervector bundle orientation on  $T_N M(X) = (TM|_{M(X)})/TM(X) \rightarrow M(X)$  is the one determined by  $X$ .

In order to ensure the existence of a Thom form on  $T_N M(X)$ , we assume that  $T_N M(X)$  has an Euclidean structure (not just a weak Euclidean structure). For simplicity we assume and a  $G$ -invariant superconnection  $\mathbb{A}$  which preserves the Euclidean structure of  $T_N M(X)$ . Moreover, we assume that  $\mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}} \leq 0$  on  $\left( \widehat{M(X)} \times_{M(X)} \Pi T_N M(X) \right)_0$  (condition  $(*)$  with  $\mu$  defined by formula (13) with  $\mathcal{A} = \mathcal{O}(M(X))$ ,  $V = \Gamma_{\Pi T_N(M(X))}(M(X)_0)$  and  $B = \pi Q$ ).

9.1.2. *A neighborhood of  $M(X)$ .* Let  $V$  be a  $G_0$ -invariant open neighborhood of  $M(X)_0$  in  $M_0$  such that  $M(V)$  is isomorphic to  $T_N M(X)(V')$  where  $V'$  is an open subset of  $T_N M(X)_0$ . Since  $M_0$  has a  $G_0$ -invariant weak Euclidean structure, such an open neighborhood exists. The action of  $G(X)$  on  $V$  can be transferred to  $V'$  by means of the isomorphism between  $M(V)$  and  $M(X)(V')$ .

Moreover, we choose a closed neighborhood  $V''$  of  $M(X)_0$  in  $M_0$  such that  $V'' \subset V$ . We denote by  $U_a(\alpha, X, V'')$  the  $G$ -invariant open neighborhood of  $X$  in  $\mathfrak{g}$  defined in (202) for  $a > 0$  small enough. Since  $\alpha$  is integrable,  $\alpha \in \widehat{\Omega}_{G,f}^\infty(U_a(\alpha, X, V''), M)$ .

9.1.3. *A limit formula.* We put for  $t \in \mathbb{R}$ ,  $\phi_t(x) = \exp(itx)$ . Let  $\beta$  be the form constructed in the preceding section (cf. formula (200)). Then  $d_{\mathfrak{g}}\beta$  is even and we can define  $\phi_t(d_{\mathfrak{g}}\beta)$  by means of the Taylor formula (3).

Let  $\psi_t$  be the analytic function such that  $\psi_t(x) = \frac{(1-\phi_t(x))}{x}$  for  $x \neq 0$  and  $\psi_t(0) = it \exp'(0) = it$ . We put  $\gamma_t = \beta\psi_t(d_{\mathfrak{g}}\beta)$ . We have  $(1 - \phi_t(d_{\mathfrak{g}}\beta))\alpha = d_{\mathfrak{g}}(\gamma_t\alpha)$ . Thus we have as functions in  $\mathcal{C}^\infty(\mathfrak{g})$ :

$$(204) \quad \forall t \in \mathbb{R}, \quad \int_M \alpha(Z) = \int_M \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z).$$

(This is equivalent to say that this equality holds for any  $Z \in \mathfrak{g}_{\mathcal{P}}$  where  $\mathcal{P}$  is any near superalgebra.) In particular:

$$(205) \quad \int_M \alpha(Z) = \lim_{t \rightarrow +\infty} \int_M \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z).$$

Since  $\phi_t(d_{\mathfrak{g}}\beta)$  is smooth and bounded and  $\alpha$  is integrable, these integrals make sense.

9.1.4. *A partition of unity.* Let  $\chi_1 + \chi_2 = 1$  be a partition of unity on  $M$  such that  $\chi_1$  is equal to 1 on  $V''$  and vanishes on the complementary of  $V$ . Then we have:

$$(206) \quad \int_M \phi_t(d_{\mathfrak{g}}\beta)\alpha = \int_{M(M_0 \setminus \overline{V''})} \chi_2 \phi_t(d_{\mathfrak{g}}\beta)\alpha + \int_{M(V)} \chi_1 \phi_t(d_{\mathfrak{g}}\beta)\alpha.$$

By construction of  $U_a(\alpha, V'', X)$ , the function  $\Im m((\iota\beta)_{\mathbb{R}})$  ( $\Im m$  denoted the imaginary part of the complex) has a strictly positive lower bound on  $U_a(\alpha, V'', X) \times \text{Supp}(\alpha) \cap \overline{M_0 \setminus V''}$ . Therefore

$$(207) \quad \lim_{t \rightarrow \infty} \int_{M(M_0 \setminus \overline{V''})} \chi_2 \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z) = 0$$

9.2. **The case of isolated zeroes and smooth coefficients.** Now we assume that the zeroes of  $X$  are isolated.

Let  $p$  be such a zero. We can assume that  $M(X) = \{p\}$  and that  $V$  is a neighborhood of  $p$ . Let  $V' \subset (T_p M)_0$  open such that  $M(V)$  is isomorphic to  $\mathcal{V}' = T_p M(V') = V' \times (T_p(M))_1$ . We denote by  $\tau_p : \mathcal{V}' \rightarrow M(V)$  this isomorphism. We transform the action of  $G$  on  $M(V)$  to an action on  $\mathcal{V}'$  by means of  $\tau_p$ .

We have to evaluate the limit of the integral of  $\tau_p^*(\chi_1 \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z))$  on  $\mathcal{V}'$ . We denote by  $\delta_t$  the contraction of  $T_p M$  by a factor of  $\frac{1}{\sqrt{t}}$ . We transform the action of  $G$  on  $\mathcal{V}'$  into an action on  $\delta_t(\mathcal{V}')$  by means of  $\delta_t$ . We have  $\lim_{t \rightarrow \infty} \delta_t(\mathcal{V}') = T_p M$  and the action of  $G$  which is obtained in the limit is equal to the tangent action of  $G$  on  $T_p M$ .

Then the limit we are looking for is:

$$(208) \quad \lim_{t \rightarrow \infty} \int_{\delta_t(\mathcal{V}')} \delta_t^* \tau_p^*(\chi_1 \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z)).$$

We denote by  $j_p$  the injection of the origin into  $T_p(M)$ . We have:

$$(209) \quad \begin{aligned} \lim_{t \rightarrow \infty} \delta_t^* \tau_p^*(\chi_1) &= 1, \\ \lim_{t \rightarrow \infty} \delta_t^* \tau_p^*(\alpha)(Z) &= j_p^*(\alpha)(Z). \end{aligned}$$

In the last equality,  $j_p^*(\alpha)(Z)$  is seen as a constant function on  $T_p M$ . Finally  $\lim_{t \rightarrow \infty} t \delta_t^* \tau_p^*(Q)$  is the weak Euclidean structure  $Q_p$  on  $T_p M$ . It is  $G$ -invariant.

For  $Z$  in  $\mathfrak{g}$ , we recall that  $Z_{T_p M}$  denotes the vector field generated by the linear action of  $Z$  in  $T_p M$ . Let  $\beta_p$  be the equivariant form on  $T_p M$  defined by for  $Z$  in  $\mathfrak{g}$  by  $\beta_p(Z) = (\pi Q_p)^*(\pi Z_{T_p M})$ . We have:

$$(210) \quad \lim_{t \rightarrow \infty} \delta_t^* \tau_p^*(\phi_t(d_{\mathfrak{g}}\beta(Z))) = \phi_1(d_{\mathfrak{g}}\beta_p(Z)).$$

Therefore,

$$(211) \quad \lim_{t \rightarrow \infty} \int_{\delta_t(\mathcal{V}')} \delta_t^* \tau_p^*(\chi_1 \phi_t(d_{\mathfrak{g}}\beta(Z))\alpha(Z)) = j_p^*(\alpha)(Z) \int_{T_p M} \exp(-id_{\mathfrak{g}}\beta_p(Z)).$$

But the form under the integral on the right hand side is equivariantly closed and integrable; so the evaluation of this integral has already been done in Proposition 7.1. We obtained:

**Theorem 9.1.** *Let  $G = (G_0, \mathfrak{g})$  be a supergroup and  $M$  be a globally oriented real  $G$ -supermanifold which has a  $G$ -invariant Euclidean structure denoted by  $Q$ .*

*Let  $\alpha \in \widehat{\Omega}_{G,f}^\infty(\mathfrak{g}_0, M)$  such that  $d_{\mathfrak{g}}\alpha = 0$ .*

*Let  $X \in \mathfrak{g}_0$  such that  $\overline{\exp(\mathbb{R}X)} \subset G_0$  is compact. We assume that the zeroes of  $X$  in  $M$  are isolated.*

*Let  $p \in M(X)$ . We denote by  $\tau_p$  the representation of  $\mathfrak{g}(X)$  in  $T_pM$ .*

*We assume that  $T_pM$  has a  $G$ -invariant weak Euclidean structure denoted by  $Q'_p$*

*We assume that  $\forall p \in M(X)$ :*

$$(212) \quad \{Z \in \mathfrak{g}_0(X) / v \mapsto Q'_p(v, \tau_p(Y)v) \text{ is positive definite on } (T_pM)_1.\} \neq \emptyset$$

*Then, there exists an open subset  $O$  of  $\mathfrak{g}_0(X)$  such that  $X \in O$  and as functions in  $\mathcal{C}_\mathfrak{g}^\infty(O)$ :*

$$(213) \quad \int_M \alpha(Z) = i^{\frac{m-n}{2}} (2\pi)^{\frac{n+m}{2}} \sum_{p \in M(X)} \frac{j_p^*(\alpha)(Z)}{Spf(\tau_p(Z))}.$$

*Proof.* As the fixed points are isolated,  $\tau_p(Z)$  is invertible in  $\mathfrak{gl}(T_pM)$  on a neighborhood of  $X$ . We denote by  $\mathcal{O}(p)$  the open subset of  $\mathfrak{g}_0(X)$  defined by:

$$(214) \quad \mathcal{O}(p) = \{Z \in \mathfrak{g}_0(X) / \tau_p(Z) \text{ is invertible in } \mathfrak{gl}(T_pM)\}.$$

Let us consider  $O_p = U_\alpha(\alpha, X, V^n) \cap \mathcal{O}(p)$ . On  $O_p$ ,  $Spf \circ \tau_p$  is smooth and invertible. As  $j_p^*(\exp(-id_{\mathfrak{g}}\beta_p(Z))) = 1$  and as the hypotheses imply that  $m$  is even, we obtain the formula considering

$$(215) \quad O = \bigcap_{p \in M(X)_0 \cap Supp(\alpha)} O(p)$$

$(M(X)_0 \cap Supp(\alpha))$  is finite. □

**9.3. Localization formula for non-isolated zeroes.** In section 8 we restricted the problem to an integration in a  $G$ -invariant neighborhood of  $M(X)$  in  $M$ . But such a neighborhood is isomorphic to an open neighborhood of  $M(X)$  in  $T_N M(X)$  on which we can transfer the action of  $G$ . As in the case of isolated zeroes we can restrict the problem to an integration on  $T_N M(X)$ . Here, instead of applying Proposition 7.1 we apply Proposition 7.2.

Let  $\alpha$  be an integrable equivariant form with smooth coefficients. Let  $C$  be a relatively compact  $G_0$ -invariant open neighborhood of the support of  $\alpha$  in  $M_0$ . As the support of  $\alpha$  is  $G_0$ -invariant and as  $M_0$  has a  $G_0$ -invariant Rimanian structure, such a neighborhood exists.

In order to apply Proposition 7.2 we have to check that  $X$  is in the open subsupermanifold  $U^{T_N M(X)}(M(X)_0 \cap C)$  (cf. formula (147) for a definition). Indeed, since  $M(X)$  is the manifold of zeroes of  $X$ , the action of  $X$  on the fibres of  $T_N M(X)$  is invertible.

Finally, we have:

**Theorem 9.2.** *Let  $G = (G_0, \mathfrak{g})$  be a supergroup and  $M$  be a globally oriented real  $G$ -supermanifold with a weak Euclidean structure.*

*Let  $\alpha \in \widehat{\Omega}_{G,f}^\infty(\mathfrak{g}_0, M)$  such that  $d_{\mathfrak{g}}\alpha = 0$ .*

Let  $X \in \mathfrak{g}_0$  such that  $\overline{\exp(\mathbb{R}X)} \subset G_0$  is compact.

We denote by  $T_N M(X)$  the normal bundle to the manifold of zeroes of  $X$  in  $M$ . We assume  $T_N M(X)$ , has a  $G(X)$ -invariant Euclidean structure  $Q'$  and a  $G(X)$ -invariant superconnection,  $\mathbb{A}$  which preserves the Euclidean structure of  $T_N M(X)$ .

Moreover we assume:

(\*)  $\mu(\mathbb{A}^{\Pi^2})_{\mathbb{R}} \leq 0$  on  $\left( \prod_{M(X)} T_N M(X) \times \widehat{M}(X) \right)_0$  (cf. formula (13) for the definition of  $\mu$  with  $\mathcal{A} = \mathcal{O}(\widehat{M}(X))$ ,  $V = \Gamma_{\prod_{M(X)} T_N M(X) \times \widehat{M}(X)}(\widehat{M}(X)_0)$  and  $B = \Pi Q'$ );

(\*\*) there is a covering of  $M$  by open supersubmanifolds  $W$  such that  $U_+^{T_N M(X)_1}(W)$  contains a non empty open subset (cf. formula (145) for the definition of  $U_+^{T_N M(X)_1}(W)$  with  $\mathcal{U} = W$  and  $\mathcal{V} = T_N M(X)$ ).

We denote by  $j$  the canonical injection of  $M(X)$  into  $M$  and by  $\mathcal{E}_{\mathfrak{g}}$  a representative of the  $G(X)$ -equivariant Euler class of the normal bundle  $T_N M(X)$ .

Then, there exists an open subset  $O$  of  $\mathfrak{g}(X)_0$  such that  $X \in O$  and as a function on  $\mathfrak{g}(X)(O)$  we have:

$$(216) \quad \int_M \alpha(Z) = (2\pi)^{\frac{n+m}{2}} \int_{M(X)} \frac{j^* \alpha(Z)}{\mathcal{E}_{\mathfrak{g}}(Z)}.$$

(This is equivalent to say that this equality holds for any  $Z \in \mathfrak{g}(X)_{\mathcal{P}}(O)$  where  $\mathcal{P}$  is any near superalgebra.)

*Proof.* As in the case of isolated zeroes the proof is similar to that of the non-super case (cf. for example [BGV92]).  $\square$

We can note that the integral on the right hand side is a sum of integrals on the connected components of  $M(X)$ . As the support of  $\alpha$  is compact the number of such integrals that are not equal to zero is finite.

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