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Invariant generalized functions on $\mathfrak{sl}(2,\mathbb{R})$ with values in a $\mathfrak{sl}(2,\mathbb{R})$-module

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Abstract

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ in a finite-dimensional real vector space. Let $\mathcal{C}_V = (\mathrm{End}(V) \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ be the algebra of $\mathrm{End}(V)$-valued invariant differential operators with constant coefficients on $\mathfrak{g}$. Let $\mathcal{U}$ be an open subset of $\mathfrak{g}$. We consider the problem of determining the space of generalized functions ϕ on $\mathcal{U}$ with values in V which are locally invariant and such that $\mathcal{C}_V\phi$ is finite dimensional.

In this article we consider the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{sl}(2,\mathbb{R})$. We prove that when $\mathcal{U}$ is $SL(2,\mathbb{R})$-invariant, then ϕ is determined by its restriction to $\mathcal{U}\backslash\mathcal{N}$ where ϕ is analytic (cf. Theorem 6.1). In general this is false when $\mathcal{U}$ is not $SL(2,\mathbb{R})$-invariant and V is not trivial. Moreover, when V is not trivial, ϕ is not always locally L^1. Thus, this case is different and more complicated than the situation considered by Harish-Chandra (Amer. J. Math 86 (1964) 534; Publ. Math. 27 (1965) 5) where $\mathfrak{g}$ is reductive and V is trivial.

To solve this problem we find all the locally invariant generalized functions supported in the nilpotent cone $\mathcal{N}$. We do this locally in a neighborhood of a nilpotent element Z of $\mathfrak{g}$ (cf. Theorem 4.1) and on an $SL(2,\mathbb{R})$-invariant open subset $\mathcal{U} \subset \mathfrak{sl}(2,\mathbb{R})$ (cf. Theorem 4.2). Finally, we also give an application of our main theorem to the Superpfaffian (Superpfaffian, prepublication, e-print math.GR/0402067, 2004).

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation of $\mathfrak{g}$ in a finite-dimensional real vector space. Let $\mathcal{C}_V = (\mathrm{End}(V) \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ be the

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algebra of $\mathrm{End}(V)$-valued invariant differential operators with constant coefficients on $\mathfrak{g}$. It is the *classical family algebra* in the terminology of Kirillov (cf. [Kir00]). Let $\mathcal{U}$ be an open subset of $\mathfrak{g}$. We consider the problem of determining the space of generalized functions ϕ on $\mathcal{U}$ with values in V which are locally invariant and such that $\mathcal{C}_V\phi$ is finite dimensional.

When $V = \mathbb{R}$ is the trivial module and $\mathfrak{g}$ is reductive, the problem was solved by Harish-Chandra (cf. in particular [HC64,HC65]). Let ϕ be a locally invariant generalized function such that $S(\mathfrak{g})^{\mathfrak{g}}\phi$ is finite dimensional. He proved that ϕ is locally L^1, ϕ is determined by its restriction $\phi|_{\mathfrak{g}'}$ to the open subset $\mathfrak{g}'$ of semi-simple regular elements of $\mathfrak{g}$ and $\phi|_{\mathfrak{g}'}$ is analytic.

In this article we consider the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{sl}(2,\mathbb{R})$. In this case $\mathfrak{g}' = \mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$. Let ϕ be a locally invariant generalized function on $\mathcal{U}$ with values in V such that $\mathcal{C}_V\phi$ is finite dimensional. We prove that when $\mathcal{U}$ is $SL(2,\mathbb{R})$-invariant, then ϕ is determined by its restriction to $\mathcal{U}\backslash\mathcal{N}$ where ϕ is analytic (cf. Theorem 6.1). In general this is false when $\mathcal{U}$ is not $SL(2,\mathbb{R})$-invariant and V is not trivial. Moreover, when V is not trivial, ϕ is not always locally L^1. Finally, we also give an application of our main theorem to the Superpfaffian (cf. [Lav04]).

To solve the problem we find all the locally invariant generalized functions supported in the nilpotent cone $\mathcal{N}$. Let V_n be the $n+1$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2,\mathbb{R})$. We denote by $\mathcal{C}^{-\infty}(\mathcal{U},V_n)^{\mathfrak{sl}(2,\mathbb{R})}$ the set of locally invariant generalized functions on $\mathcal{U}$ with values in V_n. Let $\Box$ be the Casimir operator on $\mathfrak{g}$.

We denote by $\mathcal{N}^+$ (resp. $\mathcal{N}^-$) the "upper" (resp. "lower") half nilpotent cone (cf. 4.1). We put

$$\mathcal{S}_n^0(\mathcal{U}) = \{\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V_n)^{\mathfrak{sl}(2,\mathbb{R})}/\phi|_{\mathcal{U}\backslash\{0\}} = 0\}, \tag{1}$$

$$\mathcal{S}_n^{\pm}(\mathcal{U}) = \{\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V_n)^{\mathfrak{sl}(2,\mathbb{R})}/\phi|_{\mathcal{U}\backslash(\mathcal{N}^{\pm}\cup\{0\})} = 0\}, \tag{2}$$

$$\mathcal{S}_n(\mathcal{U}) = \{\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V_n)^{\mathfrak{sl}(2,\mathbb{R})}/\phi|_{\mathcal{U}\backslash\mathcal{N}} = 0\}. \tag{3}$$

Let $Z\in\mathcal{N}^+$. We assume that $\mathcal{U}$ is a suitable open neighborhood of Z (cf. Section 4.6). Let $\delta_{\mathcal{N}^{\pm}}$ be an invariant generalized function with support $\mathcal{N}^{\pm}\cup\{0\}$ (cf. Section 4.4). We construct an invariant function s_n on $\mathcal{N}\cap\mathcal{U}$ with values in V_n. We prove (cf. Theorem 4.1):

(i) When n is even, $\mathcal{S}_n(\mathcal{U})$ is an infinite-dimensional vector space with basis

$$(\Box^k(s_n\delta_{\mathcal{N}^+}))_{k\in\mathbb{N}}. \tag{4}$$

(ii) When n is odd, $\dim(\mathcal{S}_n(\mathcal{U})) = \frac{n+1}{2}$ and a basis is given by

$$(\Box^k(s_n\delta_{\mathcal{N}^+}))_{0\leqslant k\leqslant\frac{n-1}{2}}. \tag{5}$$

We assume that $\mathcal{U}$ is an $SL(2,\mathbb{R})$-invariant open subset of $\mathfrak{sl}(2,\mathbb{R})$. If $\mathcal{U}\cap\mathcal{N}\neq\emptyset$, we have $\mathcal{N}^+\subset\mathcal{U}$ or $\mathcal{N}^-\subset\mathcal{U}$. We prove (cf. Theorem 4.2)

(i)

$$\begin{cases} \mathcal{S}_n^0(\mathcal{U}) = \{0\} & \text{if } 0\notin\mathcal{U}, \\ \mathcal{S}_n^0(\mathcal{U}) \simeq (V_n\otimes S(\mathfrak{sl}(2,\mathbb{R})))^{\mathfrak{sl}(2,\mathbb{R})} & \text{if } 0\in\mathcal{U}. \end{cases} \tag{6}$$

(2) When n is even, we have

$$\mathcal{S}_n(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}) \oplus \mathrm{Vect}\{\Box^k(s_n\delta_{\mathcal{N}^+})|_{\mathcal{U}}/k\in\mathbb{N}\} \oplus \mathrm{Vect}\{\Box^k(s_n\delta_{\mathcal{N}^-})|_{\mathcal{U}}/k\in\mathbb{N}\}, \tag{7}$$

$$\mathcal{S}_n^{\pm}(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}) \oplus \mathrm{Vect}\{\Box^k(s_n\delta_{\mathcal{N}^\pm})|_{\mathcal{U}}/k\in\mathbb{N}\}. \tag{8}$$

(iii) When n is odd:

$$\mathcal{S}_n(\mathcal{U}) = \mathcal{S}_n^{\pm}(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}). \tag{9}$$

Finally, let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2,\mathbb{R})$. Let V be the space of a real finite-dimensional representation of $\mathfrak{g}$. Let ϕ be an invariant function defined on $\mathcal{U}$ such that $\mathcal{C}_V\phi$ is finite dimensional. This last condition is equivalent to the existence of $r\in\mathbb{N}$ and $(a_0,\ldots,a_{r-1})\in\mathbb{R}^r$ such that:

$$\left(\Box^r + \sum_{k=0}^{r-1} a_k\Box^k\right)\phi = 0.$$

Moreover, we assume that $\phi|_{\mathcal{U}\setminus\mathcal{N}} = 0$. We prove (cf. Theorem 5.3) that if $\mathcal{U}$ is $SL(2,\mathbb{R})$-invariant, then we have $\phi = 0$.

In general, when $\mathcal{U}$ is not $SL(2,\mathbb{R})$-invariant, there exist non trivial solutions of the equation $\Box^k\phi = 0$ which are supported in the nilpotent cone (cf. Theorem 5.2).

2. Notations

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. Let $\rho:\mathfrak{g}\to\mathrm{End}(V)$ be a representation of $\mathfrak{g}$ in a finite-dimensional real vector space V. Let $\mathcal{U}$ be an open subset of $\mathfrak{g}$. We denote by $\mathcal{D}_c^\infty(\mathcal{U})$ the space of compactly supported smooth densities on $\mathcal{U}$. We put

$$\mathcal{C}^{-\infty}(\mathcal{U},V) = \mathcal{L}(\mathcal{D}_c^\infty(\mathcal{U}),V), \tag{10}$$

where $\mathcal{L}$ stands for continuous homomorphisms. It is the space of generalized functions on $\mathcal{U}$ with values in V. We put $\mathcal{C}^{-\infty}(\mathcal{U}) = \mathcal{C}^{-\infty}(\mathcal{U},\mathbb{R})$. For $\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V)$ and $\mu\in\mathcal{D}_c^\infty(\mathcal{U})$, we denote by

$$\int_{\mathcal{U}} \phi(Z)\,d\mu(Z) \tag{11}$$

the image of μ by ϕ. We have

$$\mathcal{C}^{-\infty}(\mathcal{U},V) = \mathcal{C}^{-\infty}(\mathcal{U})\otimes V \tag{12}$$

(we will also write ϕv for $\phi\otimes v$).

Let $Z \in \mathfrak{g}$. We denote by ∂_Z the derivative in the direction Z. It acts on $\mathcal{C}^{-\infty}(\mathcal{U})$ and on $\mathcal{C}^{-\infty}(\mathcal{U}, V)$. We extend ∂ to a morphism of algebras from $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$. We denote by $\mathcal{L}_Z$ the differential operator defined by

$$(\mathcal{L}_Z\phi)(X) = \frac{d}{dt}\phi(X - t[Z, X])|_{t=0}. \tag{13}$$

The map $Z \mapsto \mathcal{L}_Z$ is a Lie algebra homomorphism from $\mathfrak{g}$ into the algebra of differential operators on $\mathfrak{g}$. Let $Z \in \mathfrak{g}$ and $\phi \otimes v \in \mathcal{C}^{-\infty}(\mathcal{U}) \otimes V$, we put

$$Z.(\phi \otimes v) = \phi \otimes \rho(Z)v + (\mathcal{L}_Z\phi) \otimes v. \tag{14}$$

In other words, if we extend $\mathcal{L}_Z$ (resp. $\rho(Z)$) linearly to a representation of $\mathfrak{g}$ in $\mathcal{C}^{-\infty}(\mathcal{U}, V)$, we have for $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)$:

$$Z.\phi = (\rho(Z) + \mathcal{L}_Z)\phi. \tag{15}$$

We say that $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)$ is locally invariant if for any $Z \in \mathfrak{g}$ we have $Z.\phi = 0$. We put

$$\mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{g}} = \{\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)/\forall Z \in \mathfrak{g}, Z.\phi = 0\}. \tag{16}$$

3. Support $\{0\}$ distributions

In this section we assume that $\mathfrak{g}$ is unimodular. We choose an invariant measure dZ on $\mathfrak{g}$. We define the Dirac function δ_0 on $\mathfrak{g}$ with support $\{0\}$ (which depends on the choice of dZ) by the following: Let $\mathcal{C}_c^{\infty}(\mathfrak{g})$ be the set of smooth compactly supported functions on $\mathfrak{g}$. Then:

$$\forall f \in \mathcal{C}_c^{\infty}(\mathfrak{g}), \int_{\mathfrak{g}} \delta_0(Z)f(Z)\, dZ = f(0). \tag{17}$$

We have the following well-known theorem:

Theorem 3.1. *Let $\mathfrak{g}$ be a finite-dimensional unimodular real Lie algebra and V be a finite-dimensional $\mathfrak{g}$-module. Then*

$$\{\phi \in \mathcal{C}^{-\infty}(\mathfrak{g}, V)^{\mathfrak{g}}/\phi|_{\mathfrak{g}\setminus\{0\}} = 0\} \simeq (V \otimes S(\mathfrak{g}))^{\mathfrak{g}}. \tag{18}$$

The isomorphism (which depends on the choice of dZ) sends $\sum_i v_i \otimes D_i \in (V \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ to $\sum_i (\partial_{D_i}\delta_0)v_i$.

4. Support in the nilpotent cone

From now on, we assume that $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$.

4.1. Notations

We put:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{19}$$

We denote by $(h, x, y) \in (\mathfrak{sl}(2, \mathbb{R})^*)^3$ the dual basis of (H, X, Y). Thus:

$$\begin{pmatrix} h & x \\ y & -h \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})^* \otimes \mathfrak{sl}(2, \mathbb{R}) \tag{20}$$

is the generic point of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{sl}(2, \mathbb{R})$. It is the union of three orbits:

(i) $\{0\}$.
(ii) the half cone $\mathcal{N}^+$ with equations $h^2 + xy = 0$; $x - y > 0$.
(iii) the half cone $\mathcal{N}^-$ with equations $h^2 + xy = 0$; $x - y < 0$.

We denote by $\Box$ the Casimir operator of $\mathfrak{sl}(2, \mathbb{R})$:

$$\Box = \frac{1}{2}(\partial_H)^2 + 2\partial_Y \partial_X. \tag{21}$$

It is an invariant differential operator with constant coefficients on $\mathfrak{sl}(2, \mathbb{R})$.

Let $V_1 = \mathbb{R}^2$ be the standard representation of $\mathfrak{sl}(2, \mathbb{R})$. We denote by $(e = (1, 0), f = (0, 1))$ the standard basis of $\mathbb{R}^2$. The symplectic form B such that $B(e, f) = 1$ is $\mathfrak{sl}(2, \mathbb{R})$-invariant. For $v \in V_1$, we define $\mu_1(v) \in \mathfrak{sl}(2, \mathbb{R})$ as the unique element such that:

$$\forall Z \in \mathfrak{sl}(2, \mathbb{R}), \operatorname{tr}(\mu_1(v)Z) = \tfrac{1}{2}B(v, Zv). \tag{22}$$

It defines a (moment) map:

$$\mu_1 : V_1 \to \mathfrak{sl}(2, \mathbb{R}). \tag{23}$$

We have $\mu_1(e) = \frac{1}{2}X$ and $\mu_1(f) = -\frac{1}{2}Y$. The function μ_1 is a two-fold covering of $\mathcal{N}^+$ by $V_1 \backslash \{0\}$.

Let $Z_0 \in \mathcal{N} \backslash \{0\}$. Let $\mathcal{U}$ be a "small" neighborhood of Z_0. In this section we determine:

$$\{\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})} / \phi|_{\mathcal{U} \backslash \mathcal{N}} = 0\}. \tag{24}$$

We can assume that $Z_0 = X \in \mathcal{N}^+$.

4.2. Restriction to $X + \mathbb{R}Y$

We define a map:

$$\begin{aligned} \pi : SL(2, \mathbb{R}) \times (X + \mathbb{R}Y) &\to \mathfrak{sl}(2, \mathbb{R}) \\ (g, Z) &\mapsto Ad(g)(Z). \end{aligned} \tag{25}$$

This map is submersive. Let I_2 be the identity matrix in $SL(2, \mathbb{R})$. Let $\Delta_X \subset X + \mathbb{R}Y$ be an open interval containing X. We choose a connected open subset $\mathcal{V} \subset SL(2, \mathbb{R})$ such that $I_2 \in \mathcal{V}$. We put:

$$\mathcal{U} = \pi(\mathcal{V} \times \Delta_X). \tag{26}$$

It is an open neighborhood of X in $\mathfrak{g}$.

Lemma 4.1. *There is an injective* (*restriction*) *map*:

$$\begin{aligned} \mathfrak{I}_X : \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})} &\to \mathcal{C}^{-\infty}(\Delta_X, V) \\ \phi &\mapsto \phi_X. \end{aligned} \tag{27}$$

Proof. The map

$$\pi_{\mathcal{U}} = \pi|_{\mathcal{V} \times \Delta_X} : \mathcal{V} \times \Delta_X \to \mathcal{U} \tag{28}$$

is a submersion. Thus if $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)$, then $\pi_{\mathcal{U}}^*(\phi)$ is a well defined generalized function on $\mathcal{V} \times \Delta_X$ with values in V. Moreover,

$$\phi = 0 \iff \pi_{\mathcal{U}}^*(\phi) = 0. \tag{29}$$

Now, we assume that ϕ is locally invariant. Then, $\pi_{\mathcal{U}}^*(\phi)$ is also locally invariant and

$$\pi_{\mathcal{U}}^*(\phi) \in \mathcal{C}^{\infty}(\mathcal{V}) \widehat{\otimes} \mathcal{C}^{-\infty}(\Delta_X) \tag{30}$$

(Where $\widehat{\otimes}$ is a completed tensor product.) Thus $\pi_{\mathcal{U}}^*(\phi)$ can be restricted to $\{I_2\} \times \Delta_X \subset \mathcal{V} \times \Delta_X$ (cf. [HC64]). We identify Δ_X and $\{I_2\} \times \Delta_X$. We put:

$$\phi_X \stackrel{\text{def}}{=} \pi_{\mathcal{U}}^*(\phi)|_{\Delta_X}. \tag{31}$$

Since $\mathcal{V}$ is connected and ϕ is locally invariant, we have:

$$\pi_{\mathcal{U}}^*(\phi)(g, Z) = \rho(g)\phi_X(Z). \tag{32}$$

Thus

$$\phi_X = 0 \;\Leftrightarrow\; \pi_{\mathcal{U}}^*(\phi) = 0. \qquad \square \tag{33}$$

We have for $Z \in \mathfrak{sl}(2, \mathbb{R})$:

$$\mathcal{L}_Z = -h\partial_{[Z,H]} - x\partial_{[Z,X]} - y\partial_{[Z,Y]}. \tag{34}$$

In particular:

$$\mathcal{L}_H = -2x\partial_X + 2y\partial_Y, \tag{35}$$

$$\mathcal{L}_X = 2h\partial_X - y\partial_H, \tag{36}$$

$$\mathcal{L}_Y = x\partial_H - 2h\partial_Y. \tag{37}$$

If $\mathcal{V}$ is sufficiently small, we have $x \neq 0$ on $\mathcal{U}$. We assume that this condition is realized. It follows that on $\mathcal{U}$ we have:

$$\partial_X = -\frac{1}{2x}\mathcal{L}_H + \frac{y}{x}\partial_Y,$$

$$\partial_H = \frac{1}{x}\mathcal{L}_Y + \frac{2h}{x}\partial_Y. \tag{38}$$

We have $\Delta_X \subset \{X + yY / y \in \mathbb{R}\}$. We use the coordinate $y|_{\Delta_X}$, still denoted by y, on Δ_X. Let $\psi \in \mathcal{C}^{-\infty}(\Delta_X, V_n)$. We put $\psi(y) = \psi(X + yY)$.

Lemma 4.2. *We have*:

$$\mathfrak{I}_X(\mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}) = \{\psi \in \mathcal{C}^{-\infty}(\Delta_X, V) / (\rho(X) + y\rho(Y))\psi(y) = 0\}. \tag{39}$$

Thus

$$\mathfrak{I}_X : \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})} \to \{\psi \in \mathcal{C}^{-\infty}(\Delta_X, V) / (\rho(X) + y\rho(Y))\psi(y) = 0\} \tag{40}$$

is an isomorphism.

Proof. Since $x|_{\Delta_X} = 1$ and $h|_{\Delta_X} = 0$ we have for $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$:

$$(\mathcal{L}_X\phi)_X(y) = -y(\partial_H\phi)_X(y),$$

and

$$(\mathcal{L}_Y\phi)_X(y) = (\partial_H\phi)_X(y). \tag{41}$$

It follows that we have:

$$(\mathcal{L}_X\phi)_X(y) + y(\mathcal{L}_Y\phi)_X(y) = 0. \tag{42}$$

Let $\psi \in \mathfrak{I}_X(\mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})})$. Then, there is $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\psi = \phi_X$. We have:

$$\begin{aligned}(\rho(X) + y\rho(Y))\psi(y) &= (\rho(X) + y\rho(Y))\phi_X(y)\\ &= (\rho(X)\phi)_X(y) + y(\rho(Y)\phi)_X(y) + (\mathcal{L}_X\phi)_X(y) + y(\mathcal{L}_Y\phi)_X(y)\\ &= ((\rho(X) + \mathcal{L}_X)\phi)_X(y) + y((\rho(Y) + \mathcal{L}_Y)\phi)_X(y) = 0.\end{aligned} \tag{43}$$

Let $\psi \in \mathcal{C}^{-\infty}(\Delta_X, V)$ such that $(\rho(X) + y\rho(Y))\psi(y) = 0$. We define $\tilde{\psi} \in \mathcal{C}^{-\infty}(\mathcal{V} \times \Delta_X)$ by the formula:

$$\tilde{\psi}(g, y) = \rho(g)\psi(y). \tag{44}$$

Since ρ is a smooth function on $SL(2,\mathbb{R})$ with values in $GL(V)$, this is a well defined generalized function on $\mathcal{V} \times \Delta_X$ with values in V.

Let $(g, Z) \in \mathcal{V} \times \Delta_X$. Let $(g', Z') \in \mathcal{V} \times \Delta_X$ such that $\mathrm{Ad}(g)(Z) = \mathrm{Ad}(g')(Z')$. Then, $\mathrm{Ad}((g')^{-1}g)Z = Z'$. We put $G^Z = \{g'' \in SL(2,\mathbb{R})/\mathrm{Ad}(g'')(Z) = Z\}$. For $g'' \in SL(2,\mathbb{R})$, we have $\mathrm{Ad}(g'')(Z) \in \Delta_X \Leftrightarrow g'' \in G^Z$. Then, the fiber of $\pi_{\mathcal{U}}$ at (g, Z) is included in $\{(g', Z)/g^{-1}g' \in G^Z\}$. Moreover, for $Z' \in \mathfrak{sl}(2,\mathbb{R})$, $[Z, Z'] = 0 \Leftrightarrow Z' \in \mathbb{R}Z$. Thus, since $\mathcal{V}$ is connected, the condition $(\rho(X) + y\rho(Y))\psi(y) = 0$ on Δ_X ensures that $\tilde{\psi}$ is constant along the fibers of $\pi_{\mathcal{U}}$. Thus there is a well defined generalized function $\bar{\psi}$ on $\mathcal{U}$ such that:

$$\pi^*_{\mathcal{U}}(\bar{\psi}) = \tilde{\psi}. \tag{45}$$

It follows from the construction that $(\bar{\psi})_X = \psi$. $\square$

The hypothesis $\phi|_{\mathcal{U}\setminus\mathcal{N}} = 0$ means that ϕ_X is supported in $\{X\} \subset \Delta_X$.

4.3. Radial part of $\square$

In the neighborhood $\mathcal{U}$ of X defined in Section 4.2:

$$\begin{aligned}\square &= \frac{1}{2}(\partial_H)^2 + 2\partial_Y\partial_X\\ &= \frac{1}{2}\left(\frac{1}{x}\mathcal{L}_Y + \frac{2h}{x}\partial_Y\right)^2 + 2\partial_Y\left(\frac{-1}{2x}\mathcal{L}_H + \frac{y}{x}\partial_Y\right).\end{aligned} \tag{46}$$

We define the radial part of $\Box$ as the differential operator $\Box_X$ on $\mathcal{C}^{-\infty}(\Delta_X, V)$:

$$\Box_X = \left(3 + \rho(H) + 2y\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} + \frac{1}{2}\rho(Y)^2. \tag{47}$$

This definition is justified by the following lemma:

Lemma 4.3. *Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$, then we have*:

$$(\Box\phi)_X = \Box_X\phi_X. \tag{48}$$

Proof. Since $x|_{\Delta_X} = 1$ and $h|_{\Delta_X} = 0$, we have:

$$\begin{aligned}
(\Box\phi)_X &= \frac{1}{2}\left(\left(\left(\mathcal{L}_Y^2 + 2\mathcal{L}_Y\frac{h}{x}\frac{\partial}{\partial y}\right)\phi\right)_X + 2\left(\frac{-1}{2}\left(\frac{\partial}{\partial y}\mathcal{L}_H\phi\right)_X + \left(\frac{\partial}{\partial y}y\frac{\partial}{\partial y}\phi\right)_X\right)\right)\\
&= \frac{1}{2}\left(\left(\rho(Y)^2 + 2(x\partial_H - 2h\partial_Y)\frac{h}{x}\frac{\partial}{\partial y}\right)\phi\right)_X\\
&\quad + 2\left(\frac{-1}{2}\left(-\rho(H)\frac{\partial}{\partial y}\phi_X\right) + \frac{\partial}{\partial y}\phi_X + y\left(\frac{\partial}{\partial y}\right)^2\phi_X\right)\\
&= \frac{1}{2}\left(\rho(Y)^2 + 2\frac{\partial}{\partial y}\right)\phi_X + \left(\rho(H) + 2 + 2y\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}\phi_X\\
&= \left(3 + \rho(H) + 2y\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y}\phi_X + \frac{1}{2}\rho(Y)^2\phi_X = \Box_X\phi_X. \qquad \Box
\end{aligned} \tag{49}$$

4.4. The Dirac function $\delta_{\mathcal{N}^+}$ (resp. $\delta_{\mathcal{N}^-}$)

Let $dZ = dx\,dy\,dh$ be the Lebesgue measure on $\mathfrak{sl}(2,\mathbb{R})$. Let $(e^*, f^*) \in (V_1^*)^2$ be the dual basis of (e, f). The Lebesgue measure $dv = -2de^*\,df^*$ on V_1 is $\mathfrak{sl}(2,\mathbb{R})$-invariant. We define an invariant generalized function $\delta_{\mathcal{N}^+}$ (resp. $\delta_{\mathcal{N}^-}$) on $\mathfrak{sl}(2,\mathbb{R})$ and supported in $\mathcal{N}^+ \cup \{0\}$ (resp. $\mathcal{N}^- \cup \{0\}$) by

$$\forall g \in \mathcal{C}_c^\infty(\mathfrak{sl}(2,\mathbb{R})), \quad \int_{\mathfrak{sl}(2,\mathbb{R})} \delta_{\mathcal{N}^+}(Z)g(Z)\,dZ \stackrel{\text{def}}{=} \int_{V_1} g\circ\mu_1(v)\,dv,$$

$$\left(\text{resp. } \forall g \in \mathcal{C}_c^\infty(\mathfrak{sl}(2,\mathbb{R})), \quad \int_{\mathfrak{sl}(2,\mathbb{R})} \delta_{\mathcal{N}^-}(Z)g(Z)\,dZ \stackrel{\text{def}}{=} \int_{V_1} g\circ(-\mu_1)(v)\,dv\right). \tag{50}$$

We put

$$\delta_X = (\delta_{\mathcal{N}^+})_X \in \mathcal{C}^{-\infty}(\Delta_X). \tag{51}$$

We still denote by dy the Lebesgue measure on Δ_X. It is invariant. Let $g \in \mathcal{C}_c^\infty(\Delta_X)$.

Then we have:

$$\int_{\Delta_X} \delta_X(y)g(y)\,dy = g(0). \tag{52}$$

4.5. Irreducible representations

If $V = V^1 \oplus \cdots \oplus V^n$ where V^i is an irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$, then we have:

$$\mathcal{C}^{-\infty}(\mathcal{U}, V) = \bigoplus_{i=1}^{n} \mathcal{C}^{-\infty}(\mathcal{U}, V^i), \tag{53}$$

every subspace being stable for $\mathfrak{sl}(2,\mathbb{R})$. Thus we can assume from now on that the representation of $\mathfrak{sl}(2,\mathbb{R})$ in V is irreducible.

We fix the Cartan subalgebra $\mathfrak{h} = \mathbb{R}H$ and the positive root $2h$ (we still denote by h its restriction to $\mathfrak{h}$). Let $n \in \mathbb{N}$. We denote by V_n the irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ with highest weight nh. We have $\dim(V_n) = n+1$. We decompose V_n under the action of $\mathbb{R}H$. We fix $v_0 \in V_n \backslash \{0\}$ a vector of weight $-nh$:

$$\rho(H)v_0 = -nv_0. \tag{54}$$

We put for $0 \leqslant i \leqslant n$: $v_i = \rho(X)^i v_0$. We have $\rho(X)v_n = 0$ and $\rho(H)v_i = (-n+2i)v_i$. On the other hand, $\rho(Y)v_0 = 0$ and for $1 \leqslant i \leqslant n$: $\rho(Y)v_i = (n-i+1)iv_{i-1}$.

4.6. A basic function on $\mathcal{N}^+$

We construct a function $s_n : \mathcal{U} \cap \mathcal{N}^+ \to V_n$ which is the basic tool to generate all the generalized functions we are looking for.

4.6.1. Case n even

In this case V_n is isomorphic to the irreducible component of $S^{\frac{n}{2}}(\mathfrak{sl}(2,\mathbb{R}))$ (under adjoint action of $\mathfrak{sl}(2,\mathbb{R})$) generated by $X^{\frac{n}{2}}$. From now on we will identify V_n with this component. We denote by $s_n : \mathcal{N} \to V_n$ the invariant map defined by:

$$s_n(Z) = Z^{\frac{n}{2}}. \tag{55}$$

4.6.2. Case $n = 1$

We recall that $\mu_1 : V_1 \backslash \{0\} \to \mathcal{N}^+$ is a two-fold covering with $\mu_1(e) = \frac{1}{2}X$. If $\mathcal{U}$ is a sufficiently small connected neighborhood of X, there exists a unique continuous section s_1 of μ_1 in $\mathcal{U} \cap \mathcal{N}^+$ such that $s_1(\frac{1}{2}X) = e$. We have $s_1 : \mathcal{U} \cap \mathcal{N}^+ \to V_1$. It satisfies:

$$\forall Z \in \mathcal{U} \cap \mathcal{N}^+,\ \ \mu_1(s_1(Z)) = Z. \tag{56}$$

4.6.3. Case n odd

More generally, when n is odd, V_n is isomorphic to the irreducible component of $V_1 \otimes S^{\frac{n-1}{2}}(\mathfrak{sl}(2,\mathbb{R}))$ generated by $e \otimes X^{\frac{n-1}{2}}$. From now on we will identify V_n with this component. Let $\mathcal{U}$ be the above neighborhood of X. We define a function $s_n : \mathcal{U} \cap \mathcal{N}^+ \to V_n$ by:

$$\forall Z \in \mathcal{U} \cap \mathcal{N}^+, \; s_n(Z) = s_1(Z) \otimes Z^{\frac{n-1}{2}} \in V_n. \tag{57}$$

4.7. Basic theorem

Let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2,\mathbb{R})$. We put:

$$\mathcal{S}_n(\mathcal{U}) = \{\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2,\mathbb{R})} / \phi|_{\mathcal{U} \setminus \mathcal{N}} = 0\}. \tag{58}$$

Theorem 4.1. *Let $n \in \mathbb{N}$. Let $\mathcal{U}$ be an open connected neighborhood of X such that the function s_n is well defined on $\mathcal{U} \cap \mathcal{N}$ (cf. Section* 4.6) *and $\mathfrak{I}_X$ is bijective (cf. Section* 4.2). *Then*:

(i) *When n is even, $\mathcal{S}_n(\mathcal{U})$ is an infinite-dimensional vector space with basis*:

$$(\square^k (s_n \delta_{\mathcal{N}^+}))_{k \in \mathbb{N}}. \tag{59}$$

(ii) *When n is odd*, $\dim(\mathcal{S}_n(\mathcal{U})) = \frac{n+1}{2}$ *and a basis is given by*:

$$(\square^k (s_n \delta_{\mathcal{N}^+}))_{0 \leqslant k \leqslant \frac{n-1}{2}}. \tag{60}$$

Remark. Since $\delta_{\mathcal{N}^+}(Z)dZ$ is a measure on $\mathfrak{sl}(2,\mathbb{R})$ with support $\mathcal{N}^+ \cup \{0\}$ and s_n is a smooth function on $\mathcal{U} \cap \mathcal{N}$ with values in V_n, $s_n \delta_{\mathcal{N}^+}$ is a well defined generalized function on $\mathcal{U}$ with values in V_n.

Proof. Thanks to the isomorphism $\mathfrak{I}_X$ we have to determine the space:

$$\{\psi \in \mathcal{C}^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \text{ and } (\rho(X) + y\rho(Y))\psi(y) = 0\}. \tag{61}$$

Let $\psi \in \mathcal{C}^{-\infty}(\Delta_X, V_n)$. We write:

$$\psi(y) = \sum_{i=0}^{n} \psi_i(y) v_i, \tag{62}$$

where $\psi_i \in \mathcal{C}^{-\infty}(\Delta_X)$ and $(v_i)_{0 \leqslant i \leqslant n}$ is the basis defined in Section 4.5. We put:

$$\delta^k(y) = \left(\frac{\partial}{\partial y}\right)^k \delta_X(y). \tag{63}$$

Since ψ is supported in $\mathcal{N}$ and $\Delta_X \cap \mathcal{N} = \{X\}$, there exists $a_{i,k} \in \mathbb{R}$, all equal to zero but for finite number, such that:

$$\psi_i(y) = \sum_{k \in \mathbb{N}} a_{i,k} \delta^k(y). \tag{64}$$

For $n = 0$, we have $\rho = 0$ and the condition $(\rho(X) + y\rho(Y))\psi(y) = 0$ is automatically satisfied.

For $n \geqslant 1$, we put $\alpha_i = (n - i + 1)i$. We have $y\delta^0(y) = 0$ and for $k \geqslant 1$, $y\delta^k(y) = -k\delta^{k-1}(y)$. Thus:

$$\sum_{0 \leqslant i \leqslant n-1,\, k \in \mathbb{N}} a_{i,k}\delta^k(y)v_{i+1} - \sum_{1 \leqslant i \leqslant n,\, k \geqslant 1} \alpha_i a_{i,k} k \delta^{k-1}(y) v_{i-1} = 0. \tag{65}$$

It follows:

$$\begin{cases} a_{n-1,k} = 0 & \text{for } k \geqslant 0, \\ a_{1,k} = 0 & \text{for } k \geqslant 1, \\ a_{i-1,k} = (k+1)(i+1)(n-i)a_{i+1,k+1} & \text{for } n \geqslant 2,\ 1 \leqslant i \leqslant n-1 \text{ and } k \geqslant 0. \end{cases} \tag{66}$$

It follows in particular

(i) from the first and the last relations that $\forall i, k \geqslant 0$ with $2i + 1 \leqslant n$: $a_{n-(2i+1),k} = 0$,
(ii) from the last relation that $\forall i \geqslant 0$ with $2i \leqslant n$, $(a_{n-2i,k})_{k \geqslant 0}$ is completely determined by $(a_{n,k})_{k \geqslant 0}$.

We distinguish between the two cases according to the parity of n.

n even: In this case, for $n \geqslant 2$, the second relation follows from (i). Hence the map:

$$\{\psi \in \mathcal{C}^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \quad \text{and} \quad (\rho(X) + y\rho(Y))\psi(y) = 0\} \to \mathbb{R}^{\mathbb{N}}$$

$$\psi(y) = \sum_{0 \leqslant i \leqslant n,\ k \in \mathbb{N}} a_{i,k}\delta^k(y)v_i \mapsto (a_{n,k})_{k \in \mathbb{N}} \tag{67}$$

is bijective. This is also true for $n = 0$.

n odd: It follows from the two last relations that for $k \geqslant i \geqslant 1$ $a_{2i-1,k} = 0$. In particular, the map:

$$\{\psi \in \mathcal{C}^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \quad \text{and} \quad (\rho(X) + y\rho(Y))\psi(y) = 0\} \to \mathbb{R}^{\frac{n+1}{2}}$$

$$\psi(y) = \sum_{0 \leqslant i \leqslant n,\ k \in \mathbb{N}} a_{i,k}\delta^k(y)v_i \mapsto (a_{n,0}, \ldots, a_{n,\frac{n-1}{2}}) \tag{68}$$

is bijective.

This proves the first part of the theorem on the dimension of $\mathcal{S}_n(\mathcal{U})$. It remains to prove that the functions $\Box^k(s_n\delta_{\mathcal{N}})$ form a basis of $\mathcal{S}_n(\mathcal{U})$. We have for $\psi(y) = \sum_{i=0}^n \sum_{k\in\mathbb{N}} a_{i,k}\delta^k(y)v_i \in \mathcal{C}^{-\infty}(\Delta_X, V_n)$ such that $\rho(X+yY)\psi(y) = 0$

$$\begin{aligned}\Box_X\psi(y) &= (3+\rho(H)+2y\partial_Y)\sum_{k\in\mathbb{N}} a_{n,k}\delta^{k+1}(y)v_n + \sum_{i=0}^{n-1}\ldots v_i\\ &= \sum_{k\in\mathbb{N}}(n-2k-1)a_{n,k}\delta^{k+1}(y)v_n + \sum_{i=0}^{n-1}\ \ldots\ v_i,\end{aligned} \tag{69}$$

where … are elements of $\mathcal{C}^{-\infty}(\Delta_X)$.

n even: Since $v_n = X^{\frac{n}{2}}$, we have $(s_n\delta_{\mathcal{N}})_X(y) = \delta_X(y)X^{\frac{n}{2}}$. By induction on k, it follows:

$$\begin{aligned}(\Box^k(s_n\delta_{\mathcal{N}}))_X(y) &= (n-2k+1)\ldots(n-1)\delta^k(y)X^{\frac{n}{2}}\\ &\quad + \text{terms with } X^{\frac{n}{2}-i} \text{ for } i\geqslant 1.\end{aligned} \tag{70}$$

Since n is even $n-2k+1\neq 0$. The result follows.

n odd: Since $v_n = e\otimes X^{\frac{n-1}{2}}$, we have $(s_n\delta_{\mathcal{N}})_X(y) = \delta_X(y)(e\otimes X^{\frac{n-1}{2}})$. By induction on k, it follows:

$$\begin{aligned}(\Box^k(s_n\delta_{\mathcal{N}}))_X(y) &= (n-2k+1)\ldots(n-1)\delta^k(y)(e\otimes X^{\frac{n-1}{2}})\\ &\quad + \text{terms with } e\otimes X^{\frac{n-1}{2}-i} \text{ for } i\geqslant 1.\end{aligned} \tag{71}$$

In this case for $k = \frac{n+1}{2}$, $n-2k+1 = 0$. Thus, since $\Box^k(s_n\delta_{\mathcal{N}})$ is invariant, it follows from the isomorphism (68) that for $k\geqslant\frac{n+1}{2}$: $\Box^k(s_n\delta_{\mathcal{N}}) = 0$. The result follows. $\Box$

4.8. Global version

Let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2,\mathbb{R})$. We put:

$$\mathcal{S}_n^0(\mathcal{U}) = \{\phi\in\mathcal{C}^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2,\mathbb{R})}/\phi|_{\mathcal{U}\setminus\{0\}} = 0\}, \tag{72}$$

$$\mathcal{S}_n^\pm(\mathcal{U}) = \{\phi\in\mathcal{C}^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2,\mathbb{R})}/\phi|_{\mathcal{U}\setminus(\mathcal{N}^\pm\cup\{0\})} = 0\}. \tag{73}$$

Theorem 4.2. *Let $\mathcal{U}$ be an $SL(2,\mathbb{R})$-invariant open subset of $\mathfrak{sl}(2,\mathbb{R})$. Then we have*:

(i)

$$\begin{cases}\mathcal{S}_n^0(\mathcal{U}) = \{0\} & \text{if } 0\notin\mathcal{U},\\ \mathcal{S}_n^0(\mathcal{U}) \simeq (V_n\otimes S(\mathfrak{sl}(2,\mathbb{R})))^{\mathfrak{sl}(2,\mathbb{R})} & \text{if } 0\in\mathcal{U}.\end{cases} \tag{74}$$

(ii) *When n is even, we have*:

$$\mathcal{S}_n(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}) \oplus \mathrm{Vect}\{\square^k(s_n\delta_{\mathcal{N}^+})|_{\mathcal{U}}/k\in\mathbb{N}\} \oplus \mathrm{Vect}\{\square^k(s_n\delta_{\mathcal{N}^-})|_{\mathcal{U}}/k\in\mathbb{N}\}, \tag{75}$$

$$\mathcal{S}_n^{\pm}(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}) \oplus \mathrm{Vect}\{\square^k(s_n\delta_{\mathcal{N}^\pm})|_{\mathcal{U}}/k\in\mathbb{N}\}. \tag{76}$$

(iii) *When n is odd*:

$$\mathcal{S}_n(\mathcal{U}) = \mathcal{S}_n^{\pm}(\mathcal{U}) = \mathcal{S}_n^0(\mathcal{U}). \tag{77}$$

Proof. (i) It follows from Theorem 3.1.

(ii) When n is even, the function $\delta^{\pm}_{\mathcal{N}}$ is defined on $\mathfrak{sl}(2,\mathbb{R})$, the function s_n is defined on $\mathcal{N}$ and the product $s_n\delta_{\mathcal{N}^\pm}$ is well defined (cf. Remark of Theorem 4.1). Then the result follows from Theorem 4.1.

(iii) Let n be odd. We assume that $\mathcal{U}\cap\mathcal{N}\neq\emptyset$. Since $\mathcal{U}$ is $SL(2,\mathbb{R})$-invariant, we have $\mathcal{N}^+\subset\mathcal{U}$ or $\mathcal{N}^-\subset\mathcal{U}$. We assume that $\mathcal{N}^+\subset\mathcal{U}$ (the case $\mathcal{U}\subset\mathcal{N}^-$ is similar).

Let $\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V)^{\mathfrak{sl}(2,\mathbb{R})}$. Let $\mathcal{U}_0\subset\mathcal{U}$ be a suitable neighborhood of X where s_1 (and thus s_n) is defined (cf. Section 4.6). There exists $(a_0,\dots,a_{\frac{n-1}{2}})\in\mathbb{R}^{\frac{n+1}{2}}$ such that on $\mathcal{U}_0$ (cf. Theorem 4.1):

$$\phi(Z) = \sum_{k=0}^{\frac{n+1}{2}} a_k\square^k(s_n(Z)\delta_{\mathcal{N}^+}(Z)) = \sum_{k=0}^{\frac{n+1}{2}} a_k\square^k((s_1(Z)\otimes Z^{\frac{n-1}{2}})\delta_{\mathcal{N}^+}(Z)). \tag{78}$$

Since $\mu_1 : V_1\backslash\{0\}\to\mathcal{N}^+$ is a non trivial two-fold covering, there is not any continuous section. In other words there is not any continuous $SL(2,\mathbb{R})$-invariant map $s:\mathcal{N}^+\to V_1$ such that for any $Z\in\mathcal{U}_0$, $s(Z) = s_1(Z)$. Thus $a_0=\cdots=a_{\frac{n-1}{2}}=0$. The result follows. $\square$

5. Invariant solutions of differential equations

5.1. Introduction

Let $\mathcal{C}_V = (\mathrm{End}(V)\otimes S(\mathfrak{sl}(2,\mathbb{R})))^{\mathfrak{sl}(2,\mathbb{R})}$ be the algebra of $\mathrm{End}(V)$-valued invariant differential operators with constant coefficients on $\mathfrak{g}$. It is the *classical family algebra* in the terminology of Kirillov (cf. [Kir00]). When $V=V_n$ is the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$, we put $\mathcal{C}_n = \mathcal{C}_{V_n}$.

Let $\mathcal{U}\subset\mathfrak{sl}(2,\mathbb{R})$ be an open subset. It is a natural and interesting problem to determine the generalized functions $\phi\in\mathcal{C}^{-\infty}(\mathcal{U},V)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\mathcal{C}_V\phi$ is finite dimensional.

We recall that $S(\mathfrak{sl}(2,\mathbb{R}))^{\mathfrak{sl}(2,\mathbb{R})} = \mathbb{R}[\square]$. It is a subalgebra of $\mathcal{C}_V$. An other subalgebra of $\mathcal{C}_V$ is $\mathrm{End}(V)^{\mathfrak{sl}(2,\mathbb{R})}$. When $V = V_n$, we put:

$$M_n = \rho_n(X)Y + \rho_n(Y)X + \frac{1}{2}\rho_n(H)H \in \mathcal{C}_n. \tag{79}$$

According Rozhkovskaya (cf. [Roz03]), $\mathcal{C}_n$ is a free $S(\mathfrak{sl}(2,\mathbb{R}))^{\mathfrak{sl}(2,\mathbb{R})}$-module with basis $\mathcal{B}_n = (1, M_n, \dots, (M_n)^n)$.

Lemma 5.1. *Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$. Then we have*

$$\dim_{\mathbb{R}}(\mathcal{C}_V\phi) < \infty \iff \dim_{\mathbb{R}}(\mathbb{R}[\square]\phi) < \infty. \tag{80}$$

Proof. We argue as in [Roz03]. Let H be the set of harmonic polynomials in $S(\mathfrak{sl}(2,\mathbb{R}))$. Then, $S(\mathfrak{sl}(2,\mathbb{R})) = \mathbb{R}[\square] \otimes H$ (cf. [Kos63]), and:

$$\mathcal{C}_V = \mathbb{R}[\square] \otimes (H \otimes \mathrm{End}(V))^{\mathfrak{sl}(2,\mathbb{R})}. \tag{81}$$

Since $\dim_{\mathbb{R}}(H \otimes \mathrm{End}(V))^{\mathfrak{sl}(2,\mathbb{R})} < \infty$, the result follows:

Remark. Since $\mathbb{R}[\square] \subset \mathbb{R}[\square] \otimes \mathrm{End}(V)^{\mathfrak{sl}(2,\mathbb{R})} \subset \mathcal{C}_V$, the condition $\dim(\mathcal{C}_V\phi) < \infty$ is also equivalent to the existence of $r \in \mathbb{N}$ and $(A_0, \dots, A_{r-1}) \in \left(\mathrm{End}(V)^{\mathfrak{sl}(2,\mathbb{R})}\right)^r$ such that:

$$(\square^r + A_{r-1}\square^{r-1} + \cdots A_1\square + A_0)\phi = 0. \tag{82}$$

Useful examples of (82) are $(\square - \lambda)^k\phi = 0$ for $\lambda \in \mathbb{C}$ and generalized functions with values in a complex representation. We give such an example below.

Definition 5.1. Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$. We say that ϕ is $\square$-finite if $\dim_{\mathbb{R}}(\mathbb{R}[\square]\phi) < \infty$.

In other words, ϕ is $\square$-finite if there exists $r \in \mathbb{N}$ and $(a_0, \dots, a_{r-1}) \in \mathbb{R}^r$ such that

$$(\square^r + a_{r-1}\square^{r-1} + \cdots a_1\square + a_0)\phi = 0. \tag{83}$$

Example (This was our original motivation to study this problem). Let $\mathfrak{g} = \mathfrak{g}_{\mathbf{0}} \oplus \mathfrak{g}_{\mathbf{1}}$ be a Lie superalgebra. We define the generalized functions on $\mathfrak{g}$ as the generalized functions on $\mathfrak{g}_{\mathbf{0}}$ with values in the exterior algebra $\Lambda(\mathfrak{g}_{\mathbf{1}}^*)$ of $\mathfrak{g}_1^*$

$$\mathcal{C}^{-\infty}(\mathfrak{g}) \stackrel{\mathrm{def}}{=} \mathcal{C}^{-\infty}(\mathfrak{g}_{\mathbf{0}}) \otimes \Lambda(\mathfrak{g}_{\mathbf{1}}^*) = \mathcal{C}^{-\infty}(\mathfrak{g}_{\mathbf{0}}, \Lambda(\mathfrak{g}_1^*)). \tag{84}$$

We assume that $\mathfrak{g}$ has a non degenerate invariant symmetric even bilinear form B. Let $\Omega \in S^2(\mathfrak{g})$ be the Casimir operator associated with B. We have $\Omega = \Omega_{\mathbf{0}} + \Omega_{\mathbf{1}}$ with

$\Omega_0 \in S^2(\mathfrak{g}_0)$ and $\Omega_1 \in \Lambda^2(\mathfrak{g}_1)$. We consider Ω_1 as an element of $\mathrm{End}(\Lambda(\mathfrak{g}_1^*))$ acting by interior product. When they can be evaluated (cf. for example [Lav98, Chapitre III.5]), the Fourier transforms of the coadjoint orbits in $\mathfrak{g}^*$ are invariant generalized functions ϕ on $\mathfrak{g}$ subject to equations of the form $(\Omega - \lambda)\phi = 0$ with $\lambda \in \mathbb{C}$. It can be written $(\Omega_0 + (\Omega_1 - \lambda))\phi = 0$ (for $\mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R})$ it is of the form (82) with $\Omega_0 = \Box$ and $A_0 = \Omega_1 - \lambda$). We have:

$$(\Omega_0 - \lambda)^k = \sum_{i=0}^{k} \binom{k}{i} (\Omega - \lambda)^i (-\Omega_1)^{k-i}. \tag{85}$$

For $k > \frac{\dim(\mathfrak{g}_1)}{2}$, we have $\Omega_1^k = 0$. It follows that for $k > 1 + \frac{\dim(\mathfrak{g}_1)}{2}$ we have:

$$(\Omega_0 - \lambda)^k \phi = 0. \tag{86}$$

this equation is of the form of (82).

5.2. Generalized functions with support $\{0\}$

We immediately obtain from Theorem 3.1

Theorem 5.1. *Let V be a representation of $\mathfrak{sl}(2, \mathbb{R})$. Let $\phi \in \mathcal{C}^{-\infty}(\mathfrak{sl}(2, \mathbb{R}), V)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\phi|_{\mathfrak{sl}(2,\mathbb{R})\setminus\{0\}} = 0$ and ϕ is $\Box$-finite. Then, we have $\phi = 0$.*

5.3. Support in the nilpotent cone: local version

Theorem 5.2. *Let $n \in \mathbb{N}$. Let V_n be the irreducible $n+1$-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$. Let W be a finite-dimensional vector space with trivial action of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{U}$ be an open connected neighborhood of X such that the function s_n is well defined on $\mathcal{U} \cap \mathcal{N}$ (cf. Section 4.6) and $\mathfrak{I}_X$ is bijective (cf. Section 4.2). Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, W \otimes V_n)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\phi|_{\mathcal{U} \setminus \mathcal{N}} = 0$. Let $r \in \mathbb{N}$ and $(a_0, \ldots, a_{r-1}) \in \mathbb{R}^r$ such that: $(\Box^r + \sum_{k=0}^{r-1} a_k \Box^k)\phi = 0$.*

Then, we have $\phi = 0$ when at least one of the following conditions is satisfied:

(i) *n is even;*
(ii) *n is odd and $a_0 \neq 0$.*

Proof. Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, W \otimes V_n)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\phi|_{\mathcal{U} \setminus \mathcal{N}} = 0$. From Theorem 4.1 we obtain that there exist $p \in \mathbb{N}$, with $p = \frac{n-1}{2}$ if n is odd and $(w_0, \ldots, w_p) \in W^{p+1}$, such that:

$$\phi = \sum_{i=0}^{p} w_i \otimes \Box^i (s_n \delta_{\mathcal{N}^+}). \tag{87}$$

Then:

(i) When n is even, for $0 \leqslant j \leqslant p + r$, we have $\sum_{k+i=j} a_k w_i = 0$.

(ii) When n is odd, for $0 \leqslant j \leqslant \frac{n-1}{2}$, we have $\sum_{k+i=j} a_k w_i = 0$.

The result follows. $\square$

Remark. When n is odd, in contrast with the classical case ($V = V_0$ is the trivial representation) there exist (in a neighborhood of X) non trivial locally invariant solutions of the equation $\square^k \phi = 0$ supported in the nilpotent cone! For example, if $k \geqslant \frac{n+1}{2}$ the functions $\phi = \square^i (s_n \delta_{\mathcal{N}^+})$ for $0 \leqslant i \leqslant \frac{n-1}{2}$ are not trivial, supported in the nilpotent cone and satisfy the equation $\square^k \phi = 0$.

When we consider the equation $(\square - \lambda)^k \phi = 0$ for $\lambda \in C \backslash \{0\}$, then the trivial solution is again the only one supported in the nilpotent cone.

5.4. Support in the nilpotent cone: global version

Theorem 5.3. *Let V be a real finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathcal{U}$ be an $SL(2,\mathbb{R})$-invariant open subset of $\mathfrak{sl}(2,\mathbb{R})$. Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$ such that $\phi|_{\mathcal{U} \backslash \mathcal{N}} = 0$ and ϕ is $\square$-finite. Then we have $\phi = 0$.*

Proof. It is enough to prove the theorem for V irreducible. Then, the result follows from Theorems 4.2, 5.2 and 5.1. $\square$

6. General invariant generalized functions

6.1. Main theorem

Theorem 6.1. *Let V be a real finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$. Let $\mathcal{U}$ be an $SL(2,\mathbb{R})$-invariant open subset of $\mathfrak{sl}(2,\mathbb{R})$. Let $\phi \in \mathcal{C}^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})}$ such that ϕ is $\square$-finite. Then ϕ is determined by $\phi|_{\mathcal{U} \backslash \mathcal{N}}$ and $\phi|_{\mathcal{U} \backslash \mathcal{N}}$ is an analytic function.*

Proof. The fact that ϕ is determined by $\phi|_{\mathcal{U} \backslash \mathcal{N}}$ follows from Theorem 5.3. The fact that $\phi|_{\mathcal{U} \backslash \mathcal{N}}$ is analytic can be proved exactly as in [HC65]. $\square$

Remark. In general ϕ will not be locally L^1. Indeed, let $\phi_0 \in \mathcal{C}^{-\infty}(\mathfrak{sl}(2,\mathbb{R}))^{\mathfrak{sl}(2,\mathbb{R})}$ a non zero $\square$-finite generalized function. Then ϕ_0 is locally L^1, but for $k \in \mathbb{N}^*$:

$$M_n^k \phi_0 \in \mathcal{C}^{-\infty}(\mathfrak{sl}(2,\mathbb{R}), \mathrm{End}(V_n))^{\mathfrak{sl}(2,\mathbb{R})} \tag{88}$$

is usually not locally L^1.

6.2. Application to the superpfaffian

Let us consider the Lie superalgebra $\mathfrak{g} = \mathfrak{spo}(2,2n)$. Its even part is $\mathfrak{g}_{\mathbf{0}} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(2n,\mathbb{R})$. Its odd part is $\mathfrak{g}_{\mathbf{1}} = V_1 \otimes W$ where W is the standard $2n$-dimensional representation of $\mathfrak{so}(2n,\mathbb{R})$.

In [Lav04] we constructed a particular invariant generalized function Spf on $\mathfrak{spo}(2,2n)$ called Superpfaffian. It generalizes the Pfaffian on $\mathfrak{so}(2n,\mathbb{R})$ and the inverse square root of the determinant on $\mathfrak{sl}(2,\mathbb{R})$. As it is a polynomial of degree n on $\mathfrak{so}(2n,\mathbb{R})$, we may consider that we have:

$$\mathrm{Spf} \in \mathcal{C}^{-\infty}\left(\mathfrak{sl}(2,\mathbb{R}), \bigoplus_{k=0}^{n} S^k(\mathfrak{so}(2n,\mathbb{R})^*) \otimes \Lambda(\mathfrak{g}_1^*)\right)^{\mathfrak{sl}(2,\mathbb{R})}. \tag{89}$$

Let Ω (resp. $\square$, $\Omega'_{\mathbf{0}}$, $\Omega_{\mathbf{1}}$) be the Casimir operator on $\mathfrak{spo}(2,2n)$ (resp. on $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{so}(2n,\mathbb{R})$, $\mathfrak{g}_1$). Then $\Omega = \square + \Omega'_{\mathbf{0}} + \Omega_{\mathbf{1}}$ and

$$\Omega'_{\mathbf{0}} + \Omega_{\mathbf{1}} \in \mathrm{End}\left(\bigoplus_{k=0}^{n} S^k(\mathfrak{so}(2n,\mathbb{R})^*) \otimes \Lambda(\mathfrak{g}_1^*)\right)^{\mathfrak{sl}(2,\mathbb{R})} \tag{90}$$

is a nilpotent endomorphism. The superpfaffian satisfies:

$$(\square + (\Omega'_{\mathbf{0}} + \Omega_{\mathbf{1}}))\mathrm{Spf} = \Omega\,\mathrm{Spf} = 0. \tag{91}$$

The function Spf is analytic on $\mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$ and in [Lav04] an explicit formula is given for $\mathrm{Spf}(X) \in \oplus_{k=0}^{n} S^k(\mathfrak{so}(2n,\mathbb{R})^*) \otimes \Lambda(\mathfrak{g}_1^*)$ with $X \in \mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$. However, since Spf is not locally L^1 (cf. [Lav04]), it is not clear whether Spf is determined by its restriction to $\mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$ or not. In [Lav04] we proved that Spf is characterized, as an invariant generalized function on $\mathfrak{sl}(2,\mathbb{R})$, by its restriction to $\mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$ and its wave front set.

From the preceding results we obtain this new characterization of Spf:

Theorem 6.2. *Let* $\phi \in \mathcal{C}^{-\infty}(\mathfrak{sl}(2,\mathbb{R}), \oplus_{k=0}^{n} S^k(\mathfrak{so}(2n,\mathbb{R})^*) \otimes \Lambda(\mathfrak{g}_1^*))^{\mathfrak{sl}(2,\mathbb{R})}$ *such that*:

(i) *for* $X \in \mathfrak{sl}(2,\mathbb{R})\backslash\mathcal{N}$, $\phi(X) = \mathrm{Spf}(X) \in \oplus_{k=0}^{n} S^k(\mathfrak{so}(2n,\mathbb{R})^*) \otimes \Lambda(\mathfrak{g}_1^*)$;
(ii) $\Omega\phi = 0$.

Then we have $\phi = \mathrm{Spf}$.

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