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SuperPfaffian

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Abstract. Let $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ be a real finite dimensional supervector space provided with a non-degenerate antisymmetric even bilinear form B. Let $\mathfrak{spo}(V)$ be the Lie superalgebra of endomorphisms of V which preserve B. We consider $\mathfrak{spo}(V)$ as a supermanifold. We show that a choice of an orientation of $V_{\mathbf{1}}$ and of a square root $\boldsymbol{i}$ of -1 determines a very interesting generalized function on the supermanifold $\mathfrak{spo}(V)$, the *superPfaffian*.

When $V = V_{\mathbf{1}}$, $\mathfrak{spo}(V)$ is the orthogonal Lie algebra $\mathfrak{so}(V_{\mathbf{1}})$, the superPfaffian is the usual Pfaffian, a square root of the determinant.

When $V = V_{\mathbf{0}}$, $\mathfrak{spo}(V)$ is the symplectic Lie algebra $\mathfrak{sp}(V_{\mathbf{0}})$, the superPfaffian is a constant multiple of the Fourier transform of one the two minimal nilpotent orbits in the dual of the Lie algebra $\mathfrak{sp}(V_{\mathbf{0}})$, and it is a square root of the inverse of the determinant in the open subset of invertible elements of $\mathfrak{spo}(V)$.

In this article, we present the definition and some basic properties of the superPfaffian.

Introduction

Let V be an oriented finite dimensional real vector space provided with a non degenerate symmetric bilinear form B. The Pfaffian of $X \in \mathfrak{so}(V)$ can be defined by a suitable Berezin integral

$$\int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv)) \tag{1}$$

over the vector space V seen as an odd space (cf. [13], [2] and section 2. below). This definition still have a formal meaning for an oriented symplectic supervector space $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ ($V_{\mathbf{0}}$ is a symplectic vector space, $V_{\mathbf{1}}$ is an oriented vector space provided with a non degenerate symmetric bilinear form). This structure provides us with a well defined *Liouville integral* d_V on the supermanifold V. The integral (1) converges when X is in an open subset of $\mathfrak{spo}(V)$, and it is a square root of the inverse Berezinian. We call this function the superPfaffian. For instance, if $V = V_0$, then for $X \in \mathfrak{sp}(V)$, $B(v, Xv)$ is a quadratic form on V, and the integral (1) is convergent when it is positive definite. In this case, $\det(X)$ is strictly positive, and the superPfaffian is the positive square root of $1/\det(X)$.

Since the inverse Berezinian is not polynomial (it is only a rational function when V_0 is not 0), there is no natural extension of this superPfaffian to a function on the supermanifold $\mathfrak{spo}(V)$. The purpose of this article is to show that there is a natural extension of this superPfaffian as a *generalized function* on the supermanifold $\mathfrak{spo}(V)$. Notice that the superPfaffian is 0 when $\dim(V_{\mathbf{1}})$ is odd. Let $m = \dim(V_0)$ (which is even) and $n = \dim(V_{\mathbf{1}})$ (which we assume now to be even). Let $\boldsymbol{i} \in \mathbb{C}$ be a square root of -1. We define the superPfaffian by the formula:

$$\operatorname{Spf}(X) = \boldsymbol{i}^{(m-n)/2} \int_V d_V(v) \exp(-\frac{\boldsymbol{i}}{2} B(v, Xv)). \tag{2}$$

We prove that (2) has a well defined meaning as a generalized function of X on the supermanifold $\mathfrak{spo}(V)$.

The Berezinian is not a straightforward extension of the determinant, and the superPfaffian is not a straightforward extension of the Pfaffian. We establish several very nice properties of the superPfaffian:

a) It is an analytic function in the open set where the inverse Berezinian is defined, and, in this open set, it is a square root of the inverse Berezinian.

b) In the open set where it is convergent, it is given by (1).

c) It is a boundary value of an holomorphic function defined in a specific cone of $\mathfrak{spo}(V \otimes \mathbb{C})$, and, together with property a), this determines the superPfaffian up to sign.

d) It is harmonic. More precisely, let $SpO(V)$ be the supergroup of endomorphisms of V which preserve B. The superPfaffian is annihilated by the homogeneous constant coefficient differential operators on $\mathfrak{spo}(V)$ which are of degree > 0 and $SpO(V)$-invariant.

e) We give several formulas for $\operatorname{Spf}(X+Y)$. They generalize in an elegant manner formulas of Mathaï-Quillen for the Pfaffian, and formulas in Wick's calculus. Moreover, they are very useful to perform various computations of $\operatorname{Spf}(X)$ in matrix coordinates.

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Contents

1. Prerequisites

1.1. Notations. In this article, unless otherwise specified, all supervector spaces and superalgebras will be real. If V is a supervector space, we denote by $V_{\mathbf{0}}$ its even part and by $V_{\mathbf{1}}$ its odd part. If v is a non zero homogeneous element of V, we denote by $p(v) \in \mathbb{Z}/2\mathbb{Z}$ its parity. We put $\dim(V) = (\dim(V_{\mathbf{0}}), \dim(V_{\mathbf{1}}))$. We denote by V^* the dual supervector space $\operatorname{Hom}(V, \mathbb{R})$. If V and W are supervector spaces, $V \otimes W$ and $W \otimes V$ are supervector spaces, and they are identified by the usual rule of signs. We denote by $S(V)$ the symmetric algebra of V. Recall that it is equal to $S(V_{\mathbf{0}}) \otimes \Lambda(V_{\mathbf{1}})$, where $S(V_{\mathbf{0}})$ and $\Lambda(V_{\mathbf{1}})$ are the classical symmetric and exterior algebras of the corresponding ungraded vector spaces. We use the notation $\Lambda(U)$ only for ungraded vector spaces U. So, if V is a supervector space, $\Lambda(V)$ is the exterior algebra of the underlying vector space.

We choose a square root $\boldsymbol{i}$ of -1.

1.2. Near superalgebras. We say that a commutative superalgebra $\mathcal{P}$ is *near* if it is finite dimensional, local, and with $\mathbb{R}$ as residual field. They are the *algèbres proches* of Weil [16]. For $\alpha \in \mathcal{P}$, we denote by $\boldsymbol{b}(\alpha)$ the canonical projection of α in $\mathbb{R}$ ($\boldsymbol{b}(\alpha)$ is the *body* of α, and $\alpha - \boldsymbol{b}(\alpha)$ —a nilpotent element of $\mathcal{P}$— the *soul* of α, according to the terminology of [4]). Let $\alpha \in \mathcal{P}_{\mathbf{0}}$ be an even element. If ϕ is a smooth function defined in a neighborhood of $\boldsymbol{b}(\alpha)$ in $\mathbb{R}$ with values in some Fréchet supervector space W, we use the notation $\phi_{\mathcal{P}}(\alpha)$ (or simply $\phi(\alpha)$ if the context is clear) for the finite Taylor's series:

$$\phi_{\mathcal{P}}(\alpha) = \phi(\alpha) = \sum_{k=0}^{\infty} \frac{(\alpha - \boldsymbol{b}(\alpha))^k}{k!} \phi^{(k)}(\boldsymbol{b}(\alpha)) \in W \otimes \mathcal{P}. \tag{3}$$

In particular, if $\boldsymbol{b}(\alpha) \neq 0$, we have

$$|\alpha| = \frac{|\boldsymbol{b}(\alpha)|}{\boldsymbol{b}(\alpha)} \alpha, \tag{4}$$

and if $\boldsymbol{b}(\alpha) > 0$,

$$\sqrt{\alpha} = \sqrt{\boldsymbol{b}(\alpha)} \Big(1 + \frac{1}{2} (\frac{\alpha}{\boldsymbol{b}(\alpha)} - 1) - \frac{1}{2^2 2!} (\frac{\alpha}{\boldsymbol{b}(\alpha)} - 1)^2 + \frac{3}{2^3 3!} (\frac{\alpha}{\boldsymbol{b}(\alpha)} - 1)^3 + \dots \Big). \tag{5}$$

1.3. Supermanifolds. By a *supermanifold* we mean a smooth real supermanifold as in [3], [7], [1]. Let V be a finite dimensional supervector space. We consider V as a supermanifold. We recall some relevant definitions in this particular case. Let $\mathcal{U} \subset V_{\mathbf{0}}$ be an open set. We put

$$\mathcal{C}_V^{\infty}(\mathcal{U}) = \mathcal{C}^{\infty}(\mathcal{U}) \otimes \Lambda(V_{\mathbf{1}}^*), \tag{6}$$

where $\mathcal{C}^{\infty}(\mathcal{U})$ is the usual algebra of smooth real valued functions defined in $\mathcal{U}$, and $\Lambda(V_{\mathbf{1}}^*)$ the exterior algebra of $V_{\mathbf{1}}^*$. We say that $\mathcal{C}_V^{\infty}(\mathcal{U})$ is the superalgebra of *smooth functions on V defined in $\mathcal{U}$*. The supermanifold V is by definition the topological space $V_{\mathbf{0}}$ equipped with the sheaf of superalgebras $\mathcal{C}_V^{\infty}$. Recall that elements $\phi \in \mathcal{C}_V^{\infty}(\mathcal{U})$ are not functions in the usual sense, but we recall below how to treat them as ordinary functions.

Notice that if $\mathcal{U}$ is not empty, there is a canonical inclusion $S(V^*) \subset \mathcal{C}_V^{\infty}(\mathcal{U})$. The corresponding elements are called *polynomial functions*. One can also define rational functions. Similarly, if W is a Fréchet supervector space (for instance $W = \mathbb{C}$), we denote by $\mathcal{C}_V^{\infty}(\mathcal{U}, W) = \mathcal{C}^{\infty}(\mathcal{U}, W) \otimes \Lambda(V_1^*)$ the space of W-valued smooth functions.

We use similar notations, for instance $\mathcal{C}^{\omega}(\mathcal{U}) \subset \mathcal{C}^{\infty}(\mathcal{U})$ and $\mathcal{C}_V^{\omega}(\mathcal{U}, W) \subset \mathcal{C}_V^{\infty}(\mathcal{U}, W)$, for the real analytic functions.

If moreover V and W are complex supervector spaces, we use similar notations, for instance $\mathcal{H}(\mathcal{U}) \subset \mathcal{C}^{\infty}(\mathcal{U}, \mathbb{C})$ and $\mathcal{H}_V(\mathcal{U}, W) \subset \mathcal{C}_V^{\infty}(\mathcal{U}, W)$, for the holomorphic functions.

1.4. Evaluation of functions on supermanifolds.

Let $\mathcal{P}$ be a near superalgebra. We put

$$V_{\mathcal{P}} = (V \otimes \mathcal{P})_{\mathbf{0}}. \tag{7}$$

It is called the *set of points of* V *with values in* $\mathcal{P}$. Extending the body $\boldsymbol{b} : \mathcal{P} \to \mathbb{R}$ to a map $V \otimes \mathcal{P} \to V$, and restricting it to the even part $V_{\mathcal{P}}$, we obtain a map, still denoted by $\boldsymbol{b} : V_{\mathcal{P}} \to V_{\mathbf{0}}$. Let $\mathcal{U} \subset V_{\mathbf{0}}$ be an open set. We denote by $V_{\mathcal{P}}(\mathcal{U}) \subset V_{\mathcal{P}}$ the inverse image of $\mathcal{U}$ in $V_{\mathcal{P}}$. It is known that $V_{\mathcal{P}}(\mathcal{U})$ is canonically identified to the set of even algebra homomorphisms $\mathcal{C}^\infty_V(\mathcal{U}) \to \mathcal{P}$. Let $v \in V_{\mathcal{P}}(\mathcal{U})$. We will denote the corresponding homomorphism by $\phi \mapsto \phi(v)$, and say that $\phi(v) \in \mathcal{P}$ is the value of $\phi \in \mathcal{C}^\infty_V(\mathcal{U})$ at the point v. Note that formula (3) is a particular case of this definition, when $V = \mathbb{R}$.

For $\phi \in \mathcal{C}^\infty_V(\mathcal{U})$, we denote by $\phi_{\mathcal{P}} \in \mathcal{C}^\infty(V_{\mathcal{P}}(\mathcal{U}), \mathcal{P})$ the corresponding function. More generally, for $\phi \in \mathcal{C}^\infty_V(\mathcal{U}, W)$ we define $\phi_{\mathcal{P}} \in \mathcal{C}^\infty(V_{\mathcal{P}}(\mathcal{U}), W \otimes \mathcal{P})$. The importance of this construction is that for $\mathcal{P}$ large enough (for instance if $\mathcal{P}$ is an exterior algebra $\Lambda\mathbb{R}^N$ with $N \geq \dim(V_{\mathbf{1}})$), the map $\phi \mapsto \phi_{\mathcal{P}}$ is injective, which allows more or less to treat ϕ as an ordinary function.

Let $\mathcal{A}$ be a commutative superalgebra. We still use the notation $V_{\mathcal{A}} = (V \otimes \mathcal{A})_{\mathbf{0}}$. Polynomial functions $S(V^*)$ can be evaluated on $V_{\mathcal{A}}$, but, in general, smooth functions can be evaluated on $V_{\mathcal{A}}$ only if $\mathcal{A}$ is a near algebra. The particular case $\mathcal{A} = S(V^*)$ is important, because $V_{\mathcal{A}}$ contains a particular point, the *generic point*, corresponding to the identity in the identification of $Hom(V, V)_{\mathbf{0}}$ with $(V \otimes V^*)_{\mathbf{0}} \subset V_{\mathcal{A}}$. Let us call v the generic point. We have $f(v) = f$ for any polynomial function $f \in S(V^*)$.

1.5. Coordinates and integration.

Let V be a finite dimensional supervector space. By a basis $(g_i)_{i \in I}$ of V, we mean an indexed basis consisting of homogeneous elements. The *dual basis* $(z^i)_{i \in I}$ of V^* is defined by the usual relation $z^j(g_i) = \delta^j_i$. We also say that the basis $(g_i)_{i \in I}$ is the *predual basis* of the basis $(z^i)_{i \in I}$. We call $(z^i)_{i \in I}$ *a system of coordinates on* V. The corresponding derivations of the algebra of smooth functions on V are denoted by $\frac{\partial}{\partial z^i}$. They are characterized by the rule $\frac{\partial}{\partial z^j}(z^i) = \delta^i_j$. The generic point v of V is given by the formula $v = g_i z^i \in V_{S(V^*)}$.

We use systems of coordinates of the form $(x^1, \dots, x^m, \xi^1, \dots, \xi^n)$, where $(x^1, \dots, x^m)$ is a basis of $V_{\mathbf{0}}^*$, and $(\xi^1, \dots, \xi^n)$ a basis of $V_{\mathbf{1}}^*$. They will be denoted by the symbol (x, ξ). Let $(e_1, \dots, e_m, f_1, \dots, f_n)$ be the corresponding predual basis of V: thus $(e_1, \dots, e_m)$ is a basis of $V_{\mathbf{0}}$ and $(f_1, \dots, f_n)$ is a basis of $V_{\mathbf{1}}$. These notations will be used in particular for the canonical basis of the standard (m, n)-dimensional supervector space $\mathbb{R}^{(m,n)}$. Let $I = (i_1, \dots, i_n) \in \{0, 1\}^n$. We denote by ξ^I the monomial $(\xi^1)^{i_1} \dots (\xi^n)^{i_n}$ of $S(V^*)$. Let $\mathcal{U} \subset V_{\mathbf{0}}$ be an open set. Let W be a Fréchet supervector space. Any $\phi \in \mathcal{C}^\infty_V(\mathcal{U}, W)$ can be written in a unique manner

$$\phi = \sum_I \xi^I \phi_I(x^1, \dots, x^m), \tag{8}$$

with ϕ_I is an ordinary W-valued smooth function defined in the appropriate open subset of $\mathbb{R}^m$. Note that we write ϕ_I to the right of ξ^I (recall that $\xi^I \phi_I = \pm \phi_I \xi^I$, according to the sign rule).

We denote by $\mathcal{C}^\infty_{V,c}(\mathcal{U}, W)$ the subspace of $\mathcal{C}^\infty_V(\mathcal{U}, W)$ of functions with compact support. The *distributions on* V *defined in* $\mathcal{U}$ are the elements of the (Schwartz's) dual of $\mathcal{C}^\infty_{V,c}(\mathcal{U})$. If t is a distribution, we use the notation

$$t(\phi) = \int_V t(v)\phi(v)$$

for $\phi \in \mathcal{C}^\infty_{V,c}(\mathcal{U})$. We also use complex valued distributions.

A system of coordinates (x,ξ) determines a distribution $d_{(x,\xi)}$, called the Berezin integral, by the formula

$$\int_V d_{(x,\xi)}(v)\phi(v) = (-1)^{\frac{n(n-1)}{2}} \int_{\mathbb{R}^m} \left| dx^1 \ldots dx^m \right| \phi_{(1,\ldots,1)}(x^1,\ldots,x^n) \tag{9}$$

for $\phi \in \mathcal{C}^\infty_{V,c}(V_{\mathbf{0}})$, where $\left| dx^1 \ldots dx^m \right|$ is the Lebesgue measure on $\mathbb{R}^m$.

In this article, we will be in fact interested by complex valued distributions. Then we consider basis $(e_1,\ldots,e_m,f_1,\ldots,f_n)$ of $V\otimes\mathbb{C}$, where $(e_1,\ldots,e_m)$ is a basis of $V_{\mathbf{0}}$ and $(f_1,\ldots,f_n)$ is a basis of $V_{\mathbf{1}}\otimes\mathbb{C}$. The dual basis (x,ξ) provides a coordinate system (x) on $V_{\mathbf{0}}$ and a dual basis (ξ) of $V_{\mathbf{1}}^*\otimes\mathbb{C}$. Any $f\in\mathcal{C}^\infty_{V,c}(V_{\mathbf{0}},\mathbb{C})$ can be written in the form (8), and the (complex) Berezin integral $d_{(x,\xi)}$ is again well defined by formula (9).

1.6. Generalized functions. We say that a distribution t on V defined in $\mathcal{U}$ is *smooth* (resp. *smooth compactly supported*) if there is a function $\psi\in\mathcal{C}^\infty_V(\mathcal{U})$ (resp. $\psi\in\mathcal{C}^\infty_{V,c}(\mathcal{U})$) such that $t(v)=d_{(x,\xi)}(v)\psi(v)$. It means that for any $\phi\in\mathcal{C}^\infty_{V,c}(\mathcal{U})$:

$$t(\phi) = \int_V d_{(x,\xi)}(v)\psi(v)\phi(v). \tag{10}$$

This definition does not depend on the coordinates system (x,ξ).

By definition, the *generalized functions on V defined on $\mathcal{U}$* are the elements of the (Schwartz's) dual of the space of smooth compactly supported distributions. For a generalized function ϕ and a smooth compactly supported distribution t, we write:

$$\phi(t) = (-1)^{p(t)p(\phi)} \int_V t(v)\phi(v). \tag{11}$$

We denote by $\mathcal{C}^{-\infty}(\mathcal{U})$ the space of generalized functions on $\mathcal{U}$ and by $\mathcal{C}^{-\infty}_V(\mathcal{U})$ the space of generalized functions on V defined on $\mathcal{U}$. Let us remark that, as $\mathcal{C}^\infty_V(\mathcal{U}) = \mathcal{C}^\infty(\mathcal{U})\otimes\Lambda(V_{\mathbf{1}}^*)$, we have:

$$\mathcal{C}^{-\infty}_V(\mathcal{U}) = \mathcal{C}^{-\infty}(\mathcal{U})\otimes\Lambda(V_{\mathbf{1}}^*). \tag{12}$$

Let W be a Fréchet supervector space. A W-valued generalized function is a continuous homomorphism (in sense of Schwartz) from the space of smooth compactly supported distributions to W. We denote by $\mathcal{C}^{-\infty}_V(\mathcal{U},W)$ the set of W-valued generalized functions. If W is finite dimensional, we have $\mathcal{C}^{-\infty}_V(\mathcal{U},W) = \mathcal{C}^{-\infty}_V(\mathcal{U})\otimes W$.

1.7. Berezinians. Let V be a supervector space. We denote by $\mathfrak{gl}(V)$ the Lie superalgebra of endomorphisms of V.

Let $\mathcal{A}$ be a commutative superalgebra. We write an element of $\mathfrak{gl}(V)_{\mathcal{A}}$ in the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{gl}(V)_{\mathcal{A}}. \tag{13}$$

where $A \in \mathfrak{gl}(V_0)\otimes\mathcal{A}_0$, $D\in\mathfrak{gl}(V_1)\otimes\mathcal{A}_0$, $B\in Hom(V_{\mathbf{1}},V_{\mathbf{0}})\otimes\mathcal{A}_{\mathbf{1}}$, and $C\in Hom(V_{\mathbf{0}},V_{\mathbf{1}})\otimes\mathcal{A}_{\mathbf{1}}$. Berezin introduced the following generalizations of the determinant (cf. [1, 3, 12]), called the *Berezinian* and *inverse Berezinian*.

Definition 1.1. *If D (resp. A) is invertible, we define:*

$$\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D)^{-1}, \tag{14}$$

$$(\textit{resp.}\, \mathrm{Ber}^{-}(M) = \det(A)^{-1}\det(D - CA^{-1}B)). \tag{15}$$

Definition 1.2. *Assume moreover that $\mathcal{A}$ is a near superalgebra. If D (resp. A) is invertible, we define (cf. [15]):*

$$\mathrm{Ber}_{(1,0)}(M) = \Big|\det(A - BD^{-1}C)\Big|\det(D)^{-1}, \tag{16}$$

$$(\textit{resp.}\, \mathrm{Ber}^{-}_{(1,0)}(M) = \Big|\det(A)^{-1}\Big|\det(D - CA^{-1}B),) \tag{17}$$

All these functions are multiplicative, and when both A and D are invertible, it is known that $\mathrm{Ber}^{-}(M) = \mathrm{Ber}^{-1}(M)$ and $\mathrm{Ber}^{-}_{(1,0)}(M) = \mathrm{Ber}^{-1}_{(1,0)}(M)$.

Recall that $\mathfrak{gl}(V)_{\mathbf{0}}$ consists of the matrices $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ with $A \in \mathfrak{gl}(V_{\mathbf{0}})$ and $D \in \mathfrak{gl}(V_{\mathbf{1}})$. We consider the two open sets $\mathcal{U}' = GL(V_{\mathbf{0}}) \times \mathfrak{gl}(V_{\mathbf{1}})$, and $\mathcal{U}" = \mathfrak{gl}(V_{\mathbf{0}}) \times GL(V_{\mathbf{1}})$. Formula (14) defines a rational function on the open set $\mathcal{U}"$ of the supermanifold $\mathfrak{gl}(V)$. Formula (16) defines a smooth function on the open set $\mathcal{U}"$ of the supermanifold $\mathfrak{gl}(V)$. We still denote by Ber and $\mathrm{Ber}_{(1,0)}$ the elements of $\mathcal{C}^{\infty}_{\mathfrak{gl}(V)}(\mathcal{U}")$ whose evaluation in $\mathfrak{gl}(V)_{\mathcal{A}}$ is given as above. We similarly define the elements Ber^{-} and $\mathrm{Ber}^{-}_{(1,0)}$ of $\mathcal{C}^{\infty}_{\mathfrak{gl}(V)}(\mathcal{U}')$.

1.8. Oriented symplectic supervector spaces. Let $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ be a *symplectic supervector space*, that is a supervector space provided with a *symplectic form B*: a non degenerate even skew symmetric bilinear form on V. The space $V_{\mathbf{0}}$ is a classical symplectic space. The dimension m of $V_{\mathbf{0}}$ is even. We choose a symplectic basis $(e_1, \dots, e_m)$ of $V_{\mathbf{0}}$, that is $V_{\mathbf{0}}$ is the direct sum of $m/2$ symplectic vector spaces generated by the pairs $(e_1, e_2), (e_3, e_4), \dots,$ and $B(e_1, e_2) = 1, B(e_3, e_4) = 1, \dots$. The space $V_{\mathbf{1}}$ is a classical quadratic space. We choose a symplectic basis $(f_1, \dots f_n)$ of $V_{\mathbf{1}} \otimes \mathbb{C}$, that is an orthonormal basis of $V_{\mathbf{1}} \otimes \mathbb{C}$ such that $f_j \in V_{\mathbf{1}}$ or $f_j \in \boldsymbol{i}V_{\mathbf{1}}$ for all j. The dual basis (x^i, ξ^j) of $V \otimes \mathbb{C}$ is called a *symplectic coordinate system on V*.

The pair of functions $\pm\xi^1 \dots \xi^n$ does not depend on the symplectic coordinate system. A choice of one of the two elements of $\pm\xi^1 \dots \xi^n$ is called an *orientation of V*. If V is oriented, an oriented symplectic coordinate system on V is a coordinate system for which the orientation is $\xi^1 \dots \xi^n$.

Let V be an oriented symplectic supervector space. The distribution

$$d_V = \frac{1}{(2\pi)^{m/2}} d_{(x,\xi)} \tag{18}$$

does not depend on the oriented symplectic coordinate system. We call it the *Liouville measure* of V.

1.9. Symplectic Lie superalgebras. Let $V = (V, B)$ be a symplectic supervector space. We denote by $\mathfrak{spo}(V, B)$ (or $\mathfrak{spo}(V)$) the Lie superalgebra of endomorphisms of V which leave B invariant. We have $\mathfrak{spo}(V, B)_{\mathbf{0}} = \mathfrak{sp}(V_{\mathbf{0}}) \oplus \mathfrak{so}(V_{\mathbf{1}})$.

Remark concerning the notation: Let ΠV be the space V with opposite parity. It carries a structure of orthogonal supervector space, and $\mathfrak{spo}(V)$ is isomorphic to the usual orthosymplectic Lie superalgebra $\mathfrak{osp}(\Pi V)$.

We denote by $\mu \in \mathfrak{spo}(V)^* \otimes S^2(V^*) \subset S(\mathfrak{spo}(V)^* \times V^*)$ the element such that

$$\mu(X, v) = -\frac{1}{2}B(v, Xv), \tag{19}$$

for any commutative superalgebra $\mathcal{A}$, any $X \in \mathfrak{spo}(V)_{\mathcal{A}}$ and $v \in V_{\mathcal{A}}$, where $B(v, Xv) \in \mathcal{A}$ is defined by natural extension of scalars. Considering a basis G_k of $\mathfrak{spo}(V)$, the dual basis Z^k, the generic point $X = G_k Z^k$, a basis g_i of V, the dual basis z^i, and the generic point $v = g_i z^i$, we obtain:

$$\mu = -\frac{1}{2}B(g_i, G_k g_j) z^j Z^k z^i. \tag{20}$$

By composition with the exponential map, we obtain the analytic function e^μ on the supermanifold $\mathfrak{spo}(V) \times V$, which plays an important role in what follows.

Let us explain the choice of the constant $-\frac{1}{2}$ in definition of μ and why we call μ (or its variants $\check{\mu}$ and $\hat{\mu}$ defined below) *the moment map*. The symplectic form B gives to the supermanifold V a symplectic structure, and a Poisson bracket on $S(V^*)$ by the following. Let $f \in V^*$, we denote by v_f the element of V such that for any $w \in V$: $B(v_f, w) = f(w)$. This gives an isomorphism from V^* onto V. For $f, g \in V^*$, we put: $\{f, g\} = B(v_f, v_g)$ and we extend it to a Poisson bracket on $S(V^*)$.

Let $\check{\mu} : \mathfrak{spo}(V) \to S^2(V^*)$ be the linear map such that $\check{\mu}(X)(v) = \mu(X, v)$. It is an isomorphism of Lie superalgebras $\check{\mu} : \mathfrak{spo}(V) \to S^2(V^*)$. We extend $\check{\mu}$ to a morphism of commutative associative superalgebras from $S(\mathfrak{spo}(V))$ to $S(V^*)$. For $k \in \mathbb{N}$, $\check{\mu}$ induces a surjective map $\check{\mu} : S^k(\mathfrak{spo}(V)) \to S^{2k}(V^*)$. We choose a graded right inverse

$$\Xi : \bigoplus_{k \in \mathbb{N}} S^{2k}(V^*) \to S(\mathfrak{spo}(V)). \tag{21}$$

Thus, for $P \in S^{2k}(V^*)$, we have $\Xi(P) \in S^k(\mathfrak{spo}(V))$ and $\check{\mu}(\Xi(P)) = P$.

2. The superPfaffian as an holomorphic function

2.1. Definition. Let $V = V_{\mathbf{0}} \oplus V_{\mathbf{1}}$ be an oriented symplectic supervector space of dimension (m, n). For $X \in \mathfrak{sp}(V_{\mathbf{0}})$, $v \mapsto B(v, Xv)$ is a quadratic form on $V_{\mathbf{0}}$. We denote by $\mathcal{U} \subset \mathfrak{sp}(V_{\mathbf{0}})$ the open set of $X \in \mathfrak{sp}(V_{\mathbf{0}})$ for which this form is non degenerate. It is the disjoint union of the subsets $\mathcal{U}_{p,q}$ (with $p + q = m$) where (p, q) is the signature of the quadratic form. We put $\mathcal{U}^+ = \mathcal{U}_{m,0}$. It is an open convex cone in $\mathfrak{sp}(V_{\mathbf{0}})$. We denote by $\mathcal{V} = \mathcal{U} \times \mathfrak{so}(V_{\mathbf{1}})$, $\mathcal{V}_{p,q} = \mathcal{U}_{p,q} \times \mathfrak{so}(V_{\mathbf{1}})$, $\mathcal{V}^+ = \mathcal{U}^+ \times \mathfrak{so}(V_{\mathbf{1}})$ the corresponding open subsets of $\mathfrak{spo}(V)_{\mathbf{0}}$. In particular $\mathcal{V}^+$ is an open convex cone in $\mathfrak{spo}(V)_{\mathbf{0}}$.

Recall the Liouville Measure (18).

Theorem 2.1. *There exists a unique function* Spf $\in \mathcal{C}^\omega_{\mathfrak{spo}(V)}(\mathcal{V}^+, \mathbb{C})$ *such that for any near superalgebra* $\mathcal{P}$ *and any element* X *of* $\mathfrak{spo}(V)_{\mathcal{P}}$ *such that* $\boldsymbol{b}(X) \in \mathcal{V}^+$*:*

$$\mathrm{Spf}(X) = \int_V d_V(v) \exp(\mu(X, v)). \tag{22}$$

We call the function Spf defined in the preceding theorem the *superPfaffian*.

Proof. Let $\mathcal{P}$ be any near superalgebra. We denote by $\mathrm{Spf}_{\mathcal{P}}$ the function on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ such that for any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, $\mathrm{Spf}_{\mathcal{P}}(X)$ is the right hand side of (22). Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. Let $X_0 \in \mathcal{U}^+$ and $X_1 \in \mathfrak{sp}(V_{\mathbf{1}})$ such that $\boldsymbol{b}(X) = X_0 + X_1$. Thus $X = X_0 + X_1 + N$ with N nilpotent.

Let (x, ξ) be an oriented symplectic system of coordinates. Let $v = e_i x^i + f_j \xi^j$ be the generic point of V, $v_{\mathbf{0}} = e_i x^i$ be the generic point of $V_{\mathbf{0}}$ and $v_{\mathbf{1}} = f_j \xi^j$ the generic point of $V_{\mathbf{1}}$. We have $v = v_{\mathbf{0}} + v_{\mathbf{1}}$, $B(v, X_0 v) = B(v_{\mathbf{0}}, X_0 v_{\mathbf{0}})$ and $B(v, X_1 v) = B(v_{\mathbf{1}}, X_1 v_{\mathbf{1}})$.

In particular $B(v, X_1 v)$ is nilpotent. Thus $B(v, (X - X_0)v) \in (S^2(V^*) \otimes \mathcal{P})_{\mathbf{0}}$ is nilpotent. It follows that $\exp(-\frac{1}{2}B(v, (X - X_0)v)) \in (S(V^*) \otimes \mathcal{P})_{\mathbf{0}}$ is a polynomial function on V with values in $\mathcal{P}$ and that $X \mapsto \exp(-\frac{1}{2}B(v, (X - X_0)v)) \in (S(V^*) \otimes \mathcal{P})_{\mathbf{0}}$ is polynomial on $\mathfrak{spo}(V)_{\mathcal{P}}$.

Let $Z \in \mathfrak{spo}(V)_{\mathcal{P}}$ such that $\boldsymbol{b}(Z) \in \mathfrak{so}(V_{\mathbf{1}})$. Then $B(v, Zv)$ is a nilpotent element of $S^2(V^*)_{\mathcal{P}}$. We put:

$$P(Z, v_{\mathbf{0}}) = \int_{V_{\mathbf{1}}} d_{V_{\mathbf{1}}}(v_{\mathbf{1}}) \exp(-\frac{1}{2}B(v_{\mathbf{0}} + v_{\mathbf{1}}, Z(v_{\mathbf{0}} + v_{\mathbf{1}}))) \in S(V_{\mathbf{0}}^*) \otimes \mathcal{P}. \tag{23}$$

Fubini's formula gives:

$$\int_V d_V(v) \exp(-\frac{1}{2}B(v, Xv)) = \int_{V_{\mathbf{0}}} d_{V_{\mathbf{0}}}(v_{\mathbf{0}}) \exp(-\frac{1}{2}B(v_{\mathbf{0}}, X_0 v_{\mathbf{0}})) P(X - X_0, v_{\mathbf{0}}). \tag{24}$$

Since $X_0 \in \mathcal{U}^+$, $B(v_{\mathbf{0}}, X_0 v_{\mathbf{0}})$ is a positive definite quadratic form. The integral on the right hand side is a Gaussian integral on $V_{\mathbf{0}}$. Thus $\mathrm{Spf}_{\mathcal{P}}$ is an analytic function on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$.

Let $\mathcal{Q} = \Lambda(\mathfrak{spo}(V)_{\mathbf{1}}^*)$. Let $h : \mathfrak{spo}(V)_{\mathbf{0}} \hookrightarrow \big(\mathfrak{spo}(V)_{\mathbf{0}}\big)_{\mathcal{Q}}$ be the canonical embedding defined by $h(v) = v \otimes 1$. Let $H \in \big(\mathfrak{spo}(V)_{\mathbf{1}}\big)_{\mathcal{Q}}$ be the generic point of $\mathfrak{spo}(V)_{\mathbf{1}}$. We put for $X \in \mathcal{V}^+$:

$$\begin{aligned} \phi(X) &= \mathrm{Spf}_{\mathcal{Q}}(h(X) + H) \in \mathcal{Q} = \Lambda(\mathfrak{spo}(V)_{\mathbf{1}}^*) \\ &= \int_{V_{\mathbf{0}}} d_{V_{\mathbf{0}}}(v_{\mathbf{0}}) \exp(-\frac{1}{2}B(v_{\mathbf{0}}, X_0 v_{\mathbf{0}})) P(X_1 + H, v_{\mathbf{0}}). \end{aligned} \tag{25}$$

($X = X_0 + X_1$ with $X_0 \in \mathcal{U}^+$ and $X_1 \in \mathfrak{so}(V_{\mathbf{1}})$; P is defined by (23).) It defines a function

$$\phi \in \mathcal{C}^{\omega}_{\mathfrak{spo}(V)}(\mathcal{V}^+), \tag{26}$$

such that for any near superalgebra $\mathcal{P}$, any $X = Y + Z \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ with $Y \in (\mathfrak{spo}(V)_{\mathbf{0}})_{\mathcal{P}}(\mathcal{V}^+)$ and $Z \in (\mathfrak{spo}(V)_{\mathbf{1}})_{\mathcal{P}}$, $\phi(X) = \phi(Y)(Z)$.

Since ϕ is defined by a Gaussian integral on $V_{\mathbf{0}}$ all its derivatives along $\mathfrak{spo}(V)_{\mathbf{0}}$ are determined by derivation "under the integral". Moreover the above integral is $\Lambda(\mathfrak{spo}(V)_{\mathbf{1}}^*)$-linear, hence for any near superalgebra $\mathcal{P}$ and $X \in \mathfrak{spo}(V)_{\mathcal{P}}$, $\phi(X) = \mathrm{Spf}_{\mathcal{P}}(X)$.

We put $\mathrm{Spf} = \phi \in \mathcal{C}^{\omega}_{\mathfrak{spo}(V)}(\mathcal{V}^+)$ and call it the *superPfaffian.* The preceding remark shows that for any near superalgebra $\mathcal{P}$ and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ the expression $\mathrm{Spf}(X)$ is not ambiguous: the value of the function Spf at X is given by formula (22). ∎

2.2. Holomorphic extension in the appropriate subset. Formula (22) is meaningful for $X \in \mathfrak{spo}(V\otimes\mathbb{C})_{\mathcal{P}}$ with $\boldsymbol{b}(X) \in \mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)_{\mathbf{0}}$ and it defines an holomorphic function on $\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)$. Indeed, let $X \in \mathfrak{spo}(V\otimes\mathbb{C})_{\mathcal{P}}$ with $\boldsymbol{b}(X) \in \mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)_{\mathbf{0}}$. As before, let $X_0 \in \mathcal{U}^+ \times \boldsymbol{i}\,\mathfrak{sp}(V_{\mathbf{0}})$ and $X_1 \in \mathfrak{so}(V_{\mathbf{1}}\otimes\mathbb{C})$ such that $\boldsymbol{b}(X) = X_0 + X_1$. The calculations of the preceding section can be reproduced here. The right hand side of formula (24) is still a Gaussian integral and therefore defines a complex analytic function on $\mathfrak{spo}(V\otimes\mathbb{C})_{\mathcal{P}}\big(\mathcal{V}^+\times\boldsymbol{i}\,\mathfrak{spo}(V)_{\mathbf{0}}\big)$. The same arguments as in the preceding section show that formula (22) defines an holomorphic extension of Spf on $\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)_{\mathbf{0}}$ still denoted by Spf.

2.3. Invariance. Let $\mathcal{P}$ be a near superalgebra. Let $X \in \mathfrak{gl}(V)_{\mathcal{P}}$. We denote by $X^* \in \mathfrak{gl}(V)_{\mathcal{P}}$ the adjoint of X defined by:

$$\forall v, w \in V_{\mathcal{P}}, B(Xv, w) = B(v, X^*w). \tag{27}$$

We have: $(X^*)^* = X$.

Let $v \in V$, we denote by $B^{\#}(v)$ the element of V^* such that for any $w \in V$, $B^{\#}(v)(w) = B(v,w)$. This defines an isomorphism $B^{\#} : V \to V^*$. Moreover for $X \in \mathfrak{gl}(V)$ non zero and homogenous we denote by ${}^tX$ the endomorphism of V^* such that for any $\phi \in V^*$ non zero and homogenous and any $v \in V$, ${}^tX(\phi)(v) = (-1)^{p(X)p(\phi)}\phi(Xv)$. Then:

$$X^* = (B^{\#})^{-1}\,{}^tXB^{\#}. \tag{28}$$

We denote by $GL(V)_{\mathcal{P}}$ the group of invertible elements of $\mathfrak{gl}(V)_{\mathcal{P}}$. Since $GL(V)_{\mathcal{P}} \subset \mathfrak{gl}(V)_{\mathcal{P}}$, the definition of X^* is meaningful for $X = g \in GL(V)_{\mathcal{P}}$. For $g \in GL(V)_{\mathcal{P}}$ we have from (28): $\mathrm{Ber}(g^*) = \mathrm{Ber}(g)$ and $\mathrm{Ber}_{(1,0)}(g^*) = \mathrm{Ber}_{(1,0)}(g)$.

We put:

$$SpO(V)_{\mathcal{P}} = \big\{g \in GL(V)_{\mathcal{P}} \,/\, g^* = g^{-1}\big\}. \tag{29}$$

From the multiplicative property of Ber we get for $g \in SpO(V)_{\mathcal{P}}$:

$$\mathrm{Ber}(g) = \mathrm{Ber}_{(1,0)}(g) = \det(\boldsymbol{b}(g)|_{V_{\mathbf{1}}}) = \pm 1 \tag{30}$$

Proposition 2.1. *Let $\mathcal{P}$ be a near superalgebra. Let $X \in \mathfrak{spo}(V\otimes\mathbb{C})_{\mathcal{P}}\big(\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)\big)$ and $g \in GL(V)_{\mathcal{P}}$, then:*

$$\mathrm{Spf}(g^*Xg) = \mathrm{Ber}^{-1}_{(1,0)}(g)\,\mathrm{Spf}(X). \tag{31}$$

In particular, for $g \in SpO(V)_{\mathcal{P}}$, we have;

$$\mathrm{Spf}(g^{-1}Xg) = \det(\boldsymbol{b}(g)|_{V_{\mathbf{1}}})\,\mathrm{Spf}(X). \tag{32}$$

Proof. The first formula follows from the formula of change of coordinates (cf. [1]). Assume moreover that $g \in SpO(V)_{\mathcal{P}}$. Then, by definition, we have $g^* = g^{-1}$ and formula (32) follows from (30). ∎

2.4. Action of differential operators.

Let V be a symplectic finite dimensional supervector space. We assume that $\dim(V_{\mathbf{1}})$ is even.

Let $X \in \mathfrak{spo}(V)$ be homogeneous. We denote by ∂_X the derivation of $\mathcal{C}^\infty_{\mathfrak{spo}(V)}\big(\mathfrak{spo}(V)_{\mathbf{0}}\big)$ such that for any homogeneous $\psi \in \mathfrak{spo}(V_{\mathbf{0}})^*$:

$$\partial_X \psi = (-1)^{p(X)p(\psi)}\psi(X). \tag{33}$$

The mapping $X \mapsto \partial_X$ extends to an isomorphism between to $S\big(\mathfrak{spo}(V)\big)$ and the superalgebra of differential operators with constant coefficients on $\mathfrak{spo}(V)$.

Let $D \in S(\mathfrak{spo}(V))$. Since $\check{\mu}$ is linear and even on $\mathfrak{spo}(V)$, we have for any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(V_{\mathbf{0}})$ and $v \in V_{\mathcal{P}}$ where $\mathcal{P}$ is a near superagebra:

$$\Big(\partial_D \exp(\mu)\Big)(X, v) = \check{\mu}(D)(v)\exp(\mu)(X, v). \tag{34}$$

Moreover we recall that since Spf is defined by a Gaussian integral all its derivatives are determined by derivation "under the integral". Thus, if $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$:

$$\partial_D \int_V d_V(v)\exp\big(\mu(X, v)\big) = \int_V d_V(v)\check{\mu}(D)(v)\exp(\mu(X, v)). \tag{35}$$

It follows:

Proposition 2.2. *Let $D \in \ker(\check{\mu}) \subset S(\mathfrak{spo}(V))$. Then, we have $\partial_D \operatorname{Spf} = 0$.*

Let $X, Y \in \mathfrak{spo}(V)$. We put $K(X, Y) = \operatorname{str}(XY)$. It defines a non degenerate symmetric even bilinear form on $\mathfrak{spo}(V)$. Let $(X_i)_{i\in I}$ be a basis of $\mathfrak{spo}(V)$ and $(X'_i)_{i\in I}$ the basis of $\mathfrak{spo}(V)$ such that $K(X_i, X'_j) = \delta_i^j$ (δ_i^j is the Dirac symbol). We put:

$$\Box_K = \sum_{i\in I} \partial_{X'_i}\partial_{X_i} \in S^2(\mathfrak{spo}(V)). \tag{36}$$

As a corollary of the above proposition we obtain:

Corollary 2.1. *Let $D \in \bigoplus_{k\in\mathbb{N}^*} S^k(\mathfrak{spo}(V))^{\mathfrak{spo}(V)}$. Thus ∂_D is an $\mathfrak{spo}(V)$-invariant differential operator on $\mathfrak{spo}(V)$ with constant coefficients and zero scalar term.*

If $dim(V_{\mathbf{0}}) > 0$ we have $\partial_D \operatorname{Spf} = 0$.

If $V = V_{\mathbf{1}}$ we have $deg(D) \neq \frac{dim(V)}{2}$ or $D \in \Big(S^{\frac{dim(V)}{2}}\big(\mathfrak{so}(V)\big)\Big)^{O(V)} \Rightarrow \partial_D \operatorname{Spf} = 0$.

In particular, in all cases we have $\Box_K \operatorname{Spf} = 0$.

Proof. We will use the following lemma:

Lemma 2.1. *For $k \geqslant 1$, if $dim(V_{\mathbf{0}}) > 0$ or $V = V_{\mathbf{1}}$ and $k \neq dim(V)$, we have:*

$$S^k(V^*)^{\mathfrak{spo}(V)} = \{0\}. \tag{37}$$

If $V = V_1$ and $k = dim(V)$, we have:

$$S^k(V^*)^{\mathfrak{so}(V)} = \Lambda^k(V^*). \tag{38}$$

Proof. Assume that $dim(V_{\mathbf{0}}) > 0$. Let $\mathcal{P}$ be a near superalgebra. Let $SpSO(V)_{\mathcal{P}}$ be the connected component of $SpO(V)_{\mathcal{P}}$. Since for $v \in V_{\mathbf{0}} \setminus \{0\}$, $SpSO(V)_{\mathcal{P}}v = V_{\mathcal{P}} \setminus \{0\}$, the invariant polynomials are constant and equality (37) follows.

In case $V = V_{\mathbf{1}}$ cf. [17].

∎

Let $D \in S^k\big(\mathfrak{spo}(V)\big)^{\mathfrak{spo}(V)}$ $(k > 0)$, then $\check{\mu}(D) \in S^{2k}(V^*)^{\mathfrak{spo}(V)}$.

Assume that $dim(V_{\mathbf{0}}) > 0$ or $V = V_{\mathbf{1}}$ and $k \neq \frac{dim(V)}{2}$, then by Lemma 2.1 we have $\check{\mu}(D) = 0$.

Assume that $V = V_{\mathbf{1}}$ and $k = \frac{dim(V)}{2}$. Then $\check{\mu}(D) \in \Lambda^{dim(V)}(V^*)$. In this case, if $\check{\mu}(D)$ is $O(V)$ invariant, then we have $\check{\mu}(D) = 0$.

The corollary follows from the proposition. ■

2.5. Taylor's formula. We still assume that V is a symplectic finite dimensional supervector space with $n = \dim(V_{\mathbf{1}})$ even. Let $(P_k)_{k\in\mathbb{N}}$ be a basis of $\bigoplus_{l\in\mathbb{N}} S^{2l}(V^*)$ made with homogenous elements with respect to parity and degree (we can replace $\mathbb{N}$ by any appropriate set of indices).

We define $c_k \in S(\mathfrak{spo}(V)^*)$ and by $\tilde{c}_k \in \mathcal{C}^{\omega}_{\mathfrak{spo}(V)}(\mathcal{V}^+)$ the formulas:

$$\exp(\mu(X, v)) = \sum_{k\in\mathbb{N}} P_k(v) c_k(X); \tag{39}$$

$$\tilde{c}_k(X) = \int_V d_V(v) P_k(v) \exp\big(\mu(X, v)\big). \tag{40}$$

We put $\partial_k = \partial_{\Xi(P_k)}$ (with Ξ defined in formula (21)). We have:

Lemma 2.2. *For any near superalgebra $\mathcal{P}$ and any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$:*

$$\tilde{c}_k(X) = \big(\partial_k \operatorname{Spf}\big)(X). \tag{41}$$

Proof. We use formula (35) to show:

$$\begin{aligned}
\partial_k \int_V d_V(v) \exp(\mu(X, v)) &= \int_V d_V(v) \check{\mu}\big(\Xi(P_k)\big)(v) \exp(\mu(X, v)) \\
&= \int_V d_V(v) P_k(v) \exp(\mu(X, v)) \\
&= \tilde{c}_k(X).
\end{aligned} \tag{42}$$

■

Theorem 2.2. *Let $\mathcal{P}$ be a near superalgebra. Let $X, Y \in \mathfrak{spo}(V)_{\mathcal{P}}$ such that $\boldsymbol{b}(Y) \in \mathfrak{so}(V_1)$, and $\boldsymbol{b}(X) \in \mathcal{U}^+$. In this case $\boldsymbol{b}(X + Y) \in \mathcal{V}^+$. Taylor's formula for* Spf *reads:*

$$\operatorname{Spf}(X + Y) = \sum_{k\in\mathbb{N}} (-1)^{p(P_k)} c_k(Y) \tilde{c}_k(X). \tag{43}$$

The sum converges as an analytic function in X.

Proof. We have

$$\exp(\mu(X + Y, v)) = \exp(\mu(Y, v)) \exp(\mu(X, v)). \tag{44}$$

Then, we expand $\exp(\mu(Y, v))$ by formula (39):

$$\exp(\mu(Y, v)) = \sum_{k\in\mathbb{N}} P_k(v) c_k(Y) = \sum_{k\in\mathbb{N}} (-1)^{p(P_k)} c_k(Y) P_k(v); \tag{45}$$

and integrate against $d_V(v)$ (since $\dim(V_{\mathbf{1}})$ is even, this operation is even and thus commute with multiplication by $c_k(Y)$ on the left).

■

2.6. Case $V = V_{\mathbf{1}}$. In the particular case where $V = V_{\mathbf{1}}$ and Spf is the ordinary pfaffian Pf, we have the following simplification which regain results of Mathaï-Quillen ([13]) and Magneron ([11]). Since here the situation is purely algebraic, we can work with $\mathbb{C}$ as ground field. We fix an oriented orthonormal basis $(f_1, \ldots, f_n)$ of V. Let $(\xi^1, \ldots, \xi^n)$ be its dual basis. Then $(\xi^J)_{J\in\{0,1\}^n}$ is a basis of $S(V^*)$. We define as above c_J, $\tilde{c}_J$ and ∂_J, for $|J| = j_1 + \cdots + j_n$ even.

For $J \in \{0,1\}^n$, we put $V_J = \mathbb{C}j_1 f_1 + \cdots + \mathbb{C}j_n f_n$. For $J = (j_1, \ldots, j_n) \in \{0,1\}^n$ we denote by $J' = (j'_1, \ldots, j'_n) \in \{0,1\}^n$ its complementary: $j_i + j'_i = 1$. We have $V = V_J \oplus V_{J'}$. We denote by $p_J : V \to V_J$ the projection of V onto V_J with $\ker(p_J) = V_{J'}$.

We consider the case $|J|$ even. Since $(f_1, \ldots, f_n)$ is an orthonormal oriented basis of V the non-degenerate symmetric bilinear form on $V_{\mathbf{1}}$ restricts to a non-degenerate symmetric bilinear form on V_J. Let $(\xi^1, \ldots, \xi^n)$ be the dual basis of $(f_1, \ldots, f_n)$. We give to V_J the orientation defined by ξ^J. Let $1 \le j_1 < \cdots < j_r \le n$ such that $\xi^J = \xi^{j_1} \ldots \xi^{j_r}$. Then $(f_{j_1}, \ldots, f_{j_r})$ is an orthonormal oriented basis of V_J.

Let $Y \in \mathfrak{so}(V_{\mathbf{1}})$. We put:

$$Y_J : V_J \to V_J; \qquad v \mapsto Y_J(v) = p_J\big(Y(v)\big). \tag{46}$$

We have $Y_J \in \mathfrak{so}(V_J)$. The matrix of Y_J in the basis $(f_{j_1}, \ldots, f_{j_r})$ is obtained from the matrix of Y in the basis $(f_1, \ldots, f_n)$ as the submatrix corresponding of rows and columns $(j_1, \ldots, j_r)$.

We define $\epsilon(J, J') \in \{-1, 1\}$ by the formula:

$$\epsilon(J, J')\xi^J \xi^{J'} = \xi^1 \ldots \xi^n; \tag{47}$$

it is the signature of the permutation $(1, \ldots, n) \mapsto (j_1, \ldots, j_r, j'_1, \ldots, j'_{n-r})$.

Using this notations we have:

Proposition 2.3. *For $V = V_{\mathbf{1}}$, formula (43) reads (cf. [13, 14, 11]):*

$$\mathrm{Pf}(X+Y) = \sum_{J\in\{0,1\}^n/|J|\,even} \epsilon(J, J')\,\mathrm{Pf}(X_J)\,\mathrm{Pf}(Y_{J'}). \tag{48}$$

Proof. We have:

$$c_J(Y) = (-1)^{\frac{|J|(|J|-1)}{2}}\,\mathrm{Pf}(Y_J). \tag{49}$$

For $J = (1, \ldots, 1)$ it is the definition of Pf, and in the other cases it follows (see [13]) by evaluating $\exp(\mu(X, v))$ at $\xi^{j'_1} = \cdots = \xi^{j'_{n-r}} = 0$.

We obtain for $Y \in \mathfrak{so}(V_{\mathbf{1}})$:

$$c_{J'}(Y) = (-1)^{\frac{n(n-1)}{2}}\epsilon(J, J')\tilde{c}_J(Y). \tag{50}$$

Since for $|J|$ even: $(-1)^{\frac{n(n-1)}{2}+\frac{|J|(|J|-1)}{2}+\frac{|J'|(|J'|-1)}{2}} = 1 = (-1)^{|J|}$, the formula follows. ■

2.7. Case $X \in (\mathfrak{spo}(V)_{\mathbf{0}})_{\mathcal{P}}$ and $Y \in (\mathfrak{spo}(V)_{\mathbf{1}})_{\mathcal{P}}$.

We fix a symplectic oriented basis (e_i, f_j) of V. Let $I = (i_1, \ldots, i_m) \in \mathbb{N}^m$ be a multiindice. We put: $x^I = (x^1)^{i_1} \ldots (x^m)^{i_m}$. Then: $\big(\xi^J x^I\big)_{(I,J)\in\mathbb{N}^m\times\{0,1\}^n}$, is a basis of $S(V^*)$. For $I=(i_1,\dots,i_m)\in\mathbb{N}^m$ we put $|I|=i_1+\dots+i_m$ and $I!=i_1!\dots i_m!$. Moreover we put:

$$\frac{\partial^{|I|}}{\partial x^I}=\frac{\partial^{|I|}}{(\partial x^1)^{i_1}\dots(\partial x^m)^{i_m}}\qquad\text{and}\qquad\frac{\partial^{|J|}}{\partial x^J}=\frac{\partial^{|J|}}{(\partial\xi^1)^{j_1}\dots(\partial\xi^n)^{j_n}}.\tag{51}$$

Let $\mathcal{P}$ be a near superalgebra. Let $v\in V\otimes V^*$ be the generic point of V. Let $X\in\mathfrak{spo}(V)_{\mathcal{P}}$. We define $c_{I,J}(X)$ as the coefficient of $\xi^J x^I$ in the Taylor formula:

$$\exp(\mu(X,v))=\sum_{(I,J)\in\mathbb{N}^m\times\{0,1\}^n}\xi^J x^I c_{I,J}(X).\tag{52}$$

(In particular, for $X\in\mathfrak{spo}(V)_{\mathcal{P}}$ such that $\boldsymbol{b}(X)\in\mathfrak{so}(V_{\mathbf{1}})$, since $\mu(X,v)$ is nilpotent, the sum is finite.) When $|I|+|J|$ is even, it defines $c_{I,J}\in S^{\frac{|I|+|J|}{2}}(\mathfrak{spo}(V)^*)$. When $|I|+|J|$ is odd, we have $c_{I,J}(X)=0$. We put:

$$\tilde{c}_{I,J}(X)=\int_V d_V(v)(\xi^J x^I)(v)\exp\big(\mu(X,v)\big).\tag{53}$$

and for $A\in\mathfrak{sp}(V_{\mathbf{0}})_{\mathcal{P}}$:

$$\tilde{c}_I(A)=\int_{V_{\mathbf{0}}} d_{V_{\mathbf{0}}}(v_{\mathbf{0}})(x^I)(v_{\mathbf{0}})\exp\big(-\frac{1}{2}B(v_{\mathbf{0}},Av_{\mathbf{0}})\big).\tag{54}$$

To avoid confusing notations, in the rest of this paragraph we denote by $\boldsymbol{B}$ the symplectic form on V. We put:

$$X=\begin{pmatrix}A&0\\0&D\end{pmatrix}\text{ and }Y=\begin{pmatrix}0&B\\C&0\end{pmatrix}.\tag{55}$$

with $A\in\mathfrak{sp}(V_{\mathbf{0}})_{\mathcal{P}}$, $D\in\mathfrak{so}(V_{\mathbf{1}})_{\mathcal{P}}$, $B\in Hom(V_{\mathbf{1}},V_{\mathbf{0}})_{\mathcal{P}}$ and $C\in Hom(V_{\mathbf{0}},V_{\mathbf{1}})_{\mathcal{P}}$. We have $C=-B^*$, where B^* is defined by:

$$\forall v\in V_{\mathbf{0}}\otimes\mathcal{P}_{\mathbf{0}},\ \forall w\in V_{\mathbf{1}}\otimes\mathcal{P}_{\mathbf{1}},\ \boldsymbol{B}(B^*v,w)=\boldsymbol{B}(v,Bw)$$

Let $v_{\mathbf{0}}$ be the generic point of $V_{\mathbf{0}}$, $v_{\mathbf{1}}$ be the generic point of $V_{\mathbf{1}}$ and $v=v_{\mathbf{0}}+v_{\mathbf{1}}$ be the generic point of V. We have:

$$\mu(X,v)=-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{0}},Av_{\mathbf{0}})-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{1}},Dv_{\mathbf{1}}).\tag{56}$$

Assume that $\boldsymbol{b}(X)\in\mathcal{V}^+$, that means $\boldsymbol{b}(A)\in\mathcal{U}^+$. Then, with the notations of the preceding subsection, for $|J|$ even, we have $(-1)^{\frac{|J'|(|J'|-1)}{2}+\frac{n(n-1)}{2}}=(-1)^{\frac{|J|(|J|-1)}{2}}$, and so:

$$\begin{aligned}\tilde{c}_{I,J}(X)=&\int_{V_{\mathbf{0}}} d_{V_{\mathbf{0}}}(v_{\mathbf{0}})x^I(v_{\mathbf{0}})\exp(-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{0}},Av_{\mathbf{0}}))\\&\qquad\int_{V_{\mathbf{1}}} d_{V_{\mathbf{1}}}(v_{\mathbf{1}})\xi^J(v_{\mathbf{1}})\exp(-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{1}},Dv_{\mathbf{1}}))\\=&(-1)^{\frac{|J|(|J|-1)}{2}}\epsilon(J,J')\tilde{c}_I(A)\,\mathrm{Pf}(D_{J'}).\end{aligned}\tag{57}$$

We compute $c_{I,J}(Y)$. Let us introduce some notations.

Since $B=-C^*$ we have: $\mu(Y,v)=-\boldsymbol{B}(v_{\mathbf{1}},Cv_{\mathbf{0}})$.

Let $(I,J)\in\mathbb{N}^m\times\{0,1\}^n$. We denote by $C_{J,I}$ the $|J|\times|I|$ matrix obtained from C by keeping j_k times the k-th line of C (in other words we keep the lines $(j_1,\dots,j_r)$) and i_k times the k-th column of C.

Example: Assume that $(m,n)=(3,4)$. Let $I=(2,0,1)$ and $J=(0,1,1,1)$ and

$$C=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\\ \beta_1&\beta_2&\beta_3\\ \gamma_1&\gamma_2&\gamma_3\\ \delta_1&\delta_2&\delta_3\end{pmatrix},\ \text{then: } C_{J,I}=\begin{pmatrix}\beta_1&\beta_1&\beta_3\\ \gamma_1&\gamma_1&\gamma_3\\ \delta_1&\delta_1&\delta_3\end{pmatrix}. \tag{58}$$

Let $r\in\mathbb{N}$. We denote by $\mathfrak{S}_r$ the group of permutations of $\{1,\dots,r\}$. We denote by ϕ_r the r-multilinear form on the $r\times r$ matrix antisymmetric in the lines, symmetric in the columns defined for $M=(a_{i,j})_{1\le i,j\le r}$ with $a_{i,j}\in\mathcal{P}_{\mathbf{1}}$ by:

$$\phi_r(M)=\sum_{\sigma\in\mathfrak{S}_r}a_{1,\sigma(1)}\dots a_{r,\sigma(r)} \tag{59}$$

In the sequel we put for $C\in Hom(V_{\mathbf{0}},V_{\mathbf{1}})\otimes\mathcal{P}_{\mathbf{1}}$ and $I,J\in\mathbb{N}^m\times\{0,1\}^n$:

$$c_{I,J}(C)=\begin{cases}0 & \text{if } |I|\neq|J|;\\ (-1)^{\frac{|J|(|J|-1)}{2}}\phi_{|J|}(C_{J,I}) & \text{if } |I|=|J|.\end{cases} \tag{60}$$

With this notations we have: $c_{I,J}(Y)=c_{I,J}(C)$.

Example: We take the preceding example with $(m,n)=(3,4)$, $Y=\begin{pmatrix}0&B\\ C&0\end{pmatrix}$ with C as above, $I=(2,0,1)$ and $J=(0,1,1,1)$. Then:

$$c_{I,J}(Y)=c_{I,J}(C)=-\phi_3(C_{J,I})=-2(\beta_1\gamma_1\delta_3+\beta_1\gamma_3\delta_1+\beta_3\gamma_1\delta_1) \tag{61}$$

Formula (43) gives:

Proposition 2.4. *Let $\mathcal{P}$ be a near superalgebra and $\begin{pmatrix}A&B\\ C&D\end{pmatrix}\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, then:*

$$\operatorname{Spf}\begin{pmatrix}A&B\\ C&D\end{pmatrix}=\sum_{(I,J)\in\mathbb{N}^m\times\mathcal{J}_n\,/\,|I|=|J|\ even}(-1)^{\frac{|J|(|J|-1)}{2}}\epsilon(J,J')c_{I,J}(C)\operatorname{Pf}(D_{J'})\tilde{c}_I(A). \tag{62}$$

2.8. $\operatorname{Spf}(-X^{-1})$ and Spf^2.

Let W be a supervector space. Let $\boldsymbol{w}\in(V\otimes W^*)_{\mathbf{0}}$. Let $\mathcal{P}$ be any near superalgebra. For $w\in W_{\mathcal{P}}$, $\boldsymbol{w}(w)$ belongs to $V_{\mathcal{P}}$ and $v\mapsto B(v,\boldsymbol{w}(w))$ is linear on V while $v\mapsto B(v,Xv)$ is quadratic. Thus, as in proof of theorem 2.1, for any near superalgebra $\mathcal{P}$ we define an analytic function $\psi_{\mathcal{P}}$ on $\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}+)\times W_{\mathcal{P}}\times\mathbb{C}$ by the formula:

$$\psi_{\mathcal{P}}(X,w,\lambda)=\int_V d_V(v)\exp\big(\mu(X,v)+\lambda B(v,\boldsymbol{w}(w))\big). \tag{63}$$

As in the proof of theorem 2.1, we prove that there is a function ψ on $\mathfrak{spo}(V)\times W\times\mathbb{C}$ defined on $\mathcal{V}^+\times W_{\mathbf{0}}\times\mathbb{C}$ such that for any near superalgebra $\mathcal{P}$ and $(X,w,\lambda)\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)\times W_{\mathcal{P}}\times\mathbb{C}$, we have $\psi(X,w,\lambda)=\psi_{\mathcal{P}}(X,w,\lambda)$.

We define:

$$\operatorname{Spf}^{\boldsymbol{w}}_\lambda(X,w)=\psi(X,w,\lambda). \tag{64}$$

Thus, for $\lambda\in\mathbb{C}$, we have $\operatorname{Spf}^{\boldsymbol{w}}_\lambda\in\mathcal{C}^\infty_{\mathfrak{spo}(V)\times W}(\mathcal{V}^+\times W_{\mathbf{0}})$.

Lemma 2.3. *(cf. [13] for the case $V = V_{\mathbf{1}}$.) Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ be invertible (since $\boldsymbol{b}(X)|_{V_{\mathbf{0}}}$ is already invertible, it means that $\boldsymbol{b}(X)|_{V_{\mathbf{1}}}$ is invertible) and $w \in W_{\mathcal{P}}$. We have:*

$$\mathrm{Spf}^{\boldsymbol{w}}_{\lambda}(X, w) = \mathrm{Spf}(X)\, \exp(\frac{-\lambda^2}{2} B(\boldsymbol{w}(w), X^{-1}\boldsymbol{w}(w))). \tag{65}$$

Proof. Since $X^* = -X$, we have:

$$\begin{aligned}\mu(X, v) + \lambda B(v, \boldsymbol{w}(w)) = &-\frac{1}{2} B(v - \lambda X^{-1}\boldsymbol{w}(w), X(v - \lambda X^{-1}\boldsymbol{w}(w))) \\ &+ \frac{\lambda^2}{2} B(X^{-1}\boldsymbol{w}(w), \boldsymbol{w}(w)).\end{aligned} \tag{66}$$

We put:

$$\phi(X, \lambda, w) = \int_V d_V(v) \exp(-\frac{1}{2} B(v - \lambda X^{-1}\boldsymbol{w}(w), X(v - \lambda X^{-1}\boldsymbol{w}(w)))). \tag{67}$$

It is an analytic function on $\mathcal{V}^+ \times \mathbb{C} \times V$. Since $d_V(v)$ is invariant by translations, we have on $\mathcal{V}^+ \times \mathbb{R} \times V$: $\phi(X, \lambda, w) = \mathrm{Spf}(X)$. By uniqueness of analytic continuation, it follows that for any $\lambda \in \mathbb{C}$, we have $\phi(X, \lambda, w) = \mathrm{Spf}(X)$. ∎

Applying this lemma for various particular values of W and $\boldsymbol{w}$ we obtain some useful formulas. First, we take W to be a supervector space isomorphic to V and $\boldsymbol{w} \in V \otimes W^*$ be an isomorphism $W \to V$. Let $D \in S(V)$, $P \in S(V^*)$ and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. We define:

$$\overline{c}_D(X) = \Big(\partial_D \exp(\mu)\Big)(X, 0); \tag{68}$$

$$\tilde{c}_P(X) = \int_V d_V(v) P(v) \exp(\mu(X, v)). \tag{69}$$

We extend the isomorphism $B^\# : V \to V^*$ to an isomorphism of algebras $B^\# : S(V) \to S(V^*)$. We put for $D \in S(V)$: $D^\# = B^\#(D)$. Let $v \in V \otimes V^*$ be the generic point of V. For $D \in S^k(V)$ we have: $\partial_D v^k = k! D$ and:

$$\partial_{\boldsymbol{w}^{-1}(D)}\big(B(v, \boldsymbol{w})^k\big) = (-1)^k k! D^\#(v). \tag{70}$$

Corollary 2.2. *Let $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ be invertible.*

1. *If $D \in S^{2k+1}(V)$, we have $\tilde{c}_{D^\#}(X) = c_D(-X^{-1}) = 0$.*
2. *If $D \in S^{2k}(V)$ we have $\tilde{c}_{D^\#}(X) = \overline{c}_D(X^{-1})\,\mathrm{Spf}(X)$.*
   For $\Re(\lambda^2) < 0$ (in this case $\lambda^2 X^{-1} \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times \boldsymbol{i}\mathfrak{spo}(V)_{\mathcal{P}}$):

$$(-\lambda^2)^{\frac{m-n-2k}{2}} (-1)^k \overline{c}_D(X) = \tilde{c}_{D^\#}(\lambda^2 X^{-1})\ \mathrm{Spf}(X).$$

Remark: We point out that in case $V = V_{\mathbf{0}}$ formula (2) is Wick's calculus (cf. for example [5, Chapter 9.1]).

Proof. The first point is clear. We consider the second point. Let $v \in V \otimes V^*$ be the generic point of V. Let $D \in S^{2k}(V)$. Then:

$$\partial_{\boldsymbol{w}^{-1}(D)} \exp(\lambda B(v, \boldsymbol{w})) = (-\lambda)^{2k} D^{\#}(v) \exp(\lambda B(v, \boldsymbol{w})). \tag{71}$$

We apply $\frac{1}{\lambda^{2k}} \partial_{\boldsymbol{w}^{-1}(D)}$ to the exponential of equality (66), we apply the distribution d_V and then, we take the value at $(X, 0)$. Since, $\overline{c}_D(\lambda^2 X^{-1}) = \lambda^{2k} \overline{c}_D(X^{-1})$, formula (2) follows.

For the second formula, we need an auxiliary result. Consider the application:

$$\phi \mapsto \mathcal{F}_\lambda(\phi) = \int_W d_W(w) \int_V d_V(v) \phi(v) \exp(\lambda B(\boldsymbol{w}(w), v)). \tag{72}$$

It is defined for $\phi \in \mathcal{C}_V^\infty(V_{\mathbf{0}})$ such that, $v \mapsto \phi(v) \exp(\lambda B(\boldsymbol{w}(w), v))$ and all its derivatives are rapidly decreasing on V for any $w \in W_{\mathcal{P}}$. It is linear and $\phi(0) = 0$ implies that $\mathcal{F}_\lambda(\phi) = 0$. Thus, there is $K_\lambda \in \mathbb{C}$ such that $\mathcal{F}_\lambda(\phi) = K_\lambda \phi(0)$. To find K_λ it is enough to consider a particular ϕ. For example $\phi(v) = \exp(\mu(X, v))$ ($\phi = \exp(\check{\mu}(X))$) for some $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ fixed. In this case formula (66) shows that:

$$\mathcal{F}_\lambda\big(\exp(\check{\mu}(X))\big) = \int_V d_V(v) \exp(\mu(X, v)) \int_W d_W(w) \exp(\frac{-\lambda^2}{2} B(\boldsymbol{w}(w), X^{-1}\boldsymbol{w}(w))).$$

hypothesis $\Re(\lambda^2) < 0$ implies that $\lambda^2 X^{-1} \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ and thus, the above integral converges. More precisely, since $\exp(\mu(X, 0)) = 1$, we have:

$$K_\lambda = \mathcal{F}_\lambda\big(\exp(\check{\mu}(X))\big) = \operatorname{Spf}(X) \operatorname{Spf}(\lambda^2 X^{-1}). \tag{73}$$

Since it does not depends on X it is sufficient to consider

$$X = \left(\begin{array}{ccc|ccc} J_2 & & 0 & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & J_2 & 0 & & 0 \\ \hline 0 & & 0 & J_2 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & 0 & & J_2 \end{array}\right) \tag{74}$$

where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We obtain: $K_\lambda = (-\lambda^2)^{\frac{n-m}{2}}$.

We multiply both sides of the exponential of equality (66) by $D^{\#}(\boldsymbol{w}(w))$, we integrate on V against $d_V(v)$ and integrate on W against $d_W(w)$. The right hand side is straightforward. For the left hand side we have (the first equality is obtained by integration by parts using equation (70)):

$$\begin{aligned} &\int_W d_W(w) D^{\#}(\boldsymbol{w}(w)) \int_V d_V(v) \exp(\mu(X, v) + \lambda B(v, \boldsymbol{w}(w))) \\ &\quad = (-\lambda)^{-2k} \int_W d_W(w) \int_V d_V(v) \Big(\partial_D \exp(\mu)\Big)(X, v) \exp(\lambda B(v, \boldsymbol{w}(w))) \\ &\quad = (-\lambda)^{-2k} \mathcal{F}_\lambda\Big(\partial_D \exp(\check{\mu}(X))\Big) = (-\lambda)^{-2k} (-\lambda^2)^{\frac{n-m}{2}} \Big(\partial_D \exp(\check{\mu})(X)\Big)(0) \\ &\quad = (-\lambda^2)^{\frac{n-m-2k}{2}} (-1)^k \overline{c}_D(X). \end{aligned} \tag{75}$$

■

In the end of this subsection, to avoid confusing notations, we denote by $\boldsymbol{B}$ the symplectic form on V; moreover, we denote by $v_{\mathbf{0}} = \sum_i e_i x^i$ (resp. $v_{\mathbf{1}} = \sum_j f_j \xi^j$) the generic point of $V_{\mathbf{0}}$ (resp. $V_{\mathbf{1}}$). Let $\mathcal{P}$ be a near superalgebra and $D \in \mathfrak{so}(V_{\mathbf{1}})_{\mathcal{P}}$. Let $J \in \{0,1\}^n$. We define $c_I(D)$ by the formula: $\exp(-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{1}}, Dv_{\mathbf{1}})) = \sum_{J \in \{0,1\}^n} \xi^J c_J(D)$. As a corollary of equation (65) we obtain:

Corollary 2.3. *Let $\mathcal{P}$ be a near superalgebra. Let $A \in \mathfrak{sp}(V_{\mathbf{0}})_{\mathcal{P}}(\mathcal{U}^+)$, $B \in Hom(V_{\mathbf{1}}, V_{\mathbf{0}}) \otimes \mathcal{P}_{\mathbf{1}}$ and $C \in Hom(V_{\mathbf{0}}, V_{\mathbf{1}}) \otimes \mathcal{P}_{\mathbf{1}}$ such that $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}$. Then for any $J \in \{0,1\}^n$ with $|J|$ even:*

$$\begin{aligned} \sum_{I \in \mathbb{N}^m \,/ |I| = |J|} (-i)^{|J|} \tilde{c}_I(A) c_{I,J}(C) &= \frac{1}{\sqrt{\det(A)}} c_J(CA^{-1}B) \\ &= \frac{(-1)^{\frac{|J|(|J|-1)}{2}}}{\sqrt{\det(A)}} \operatorname{Pf}((CA^{-1}B)_J). \end{aligned} \tag{76}$$

Proof. For $v \in V_{\mathbf{0}} \otimes \mathcal{P}_{\mathbf{0}}$ and $w \in V_{\mathbf{1}} \otimes \mathcal{P}_{\mathbf{1}}$. we have: $\boldsymbol{B}(v, Bw) = \boldsymbol{B}(w, Cv)$. The coefficient of $\xi^J x^I$ in $\exp(-\boldsymbol{B}(v_{\mathbf{0}}, Bv_{\mathbf{1}})) = \exp\left(-\frac{1}{2}(\boldsymbol{B}(v_{\mathbf{0}}, Bv_{\mathbf{1}}) + \boldsymbol{B}(v_{\mathbf{1}}, Cv_{\mathbf{0}}))\right)$ is $c_{I,J}(C)$. On the other hand, we have for $w \in V_{\mathbf{1}} \otimes \mathcal{P}_{\mathbf{1}}$:

$$\boldsymbol{B}(Bw, A^{-1}Bw) = -\boldsymbol{B}(w, CA^{-1}Bw).$$

Consider equation (65) with $V = V_{\mathbf{0}}$, $W = V_{\mathbf{1}}$, $X = A$, and: $\boldsymbol{w}(w) = Bw$. We apply $\left(\frac{\partial}{\partial \xi^n}\right)^{j_n} \ldots \left(\frac{\partial}{\partial \xi^1}\right)^{j_1}$ to (65) and take value at $(X, 0)$. Then equality (76) follows by multiplying each side by $\frac{(-\boldsymbol{i})^{|J|}}{\lambda^{|J|}}$. ■

Similarly, for $I \in \mathbb{N}^m$ and $A \in \mathfrak{sp}(V_{\mathbf{0}})_{\mathcal{P}}$, we define $c_I(A)$ by the formula $\exp(-\frac{1}{2}\boldsymbol{B}(v_{\mathbf{0}}, Av_{\mathbf{0}})) = \sum_{I \in \mathbb{N}^m} x^I c_I(A)$. We obtain:

Corollary 2.4. *Let $\mathcal{P}$ be a near superalgebra. Let $D \in \mathfrak{so}(V_{\mathbf{1}})_{\mathcal{P}}$ invertible, $B \in Hom(V_{\mathbf{1}}, V_{\mathbf{0}}) \otimes \mathcal{P}_{\mathbf{1}}$ and $C \in Hom(V_{\mathbf{0}}, V_{\mathbf{1}}) \otimes \mathcal{P}_{\mathbf{1}}$ such that $\begin{pmatrix} 0 & B \\ C & D \end{pmatrix} \in \mathfrak{spo}(V)_{\mathcal{P}}$. Then for any $I \in \mathbb{N}^m$ with $|I|$ even:*

$$\sum_{J \in \mathcal{J}_n \,/ |I| = |J|} \epsilon(J, J')(-1)^{\frac{|J|(|J|-1)}{2}} \operatorname{Pf}(D_{J'}) c_{I,J}(C) = \operatorname{Pf}(D) c_I(BD^{-1}C). \tag{77}$$

Proof. Here we apply formula (65) with $V = V_{\mathbf{1}}$, $W = V_{\mathbf{0}}$, $X = D$ and

$$\boldsymbol{w}(w) = Cw. \tag{78}$$

We apply $\frac{1}{I!}\frac{\partial^{i_1}}{\partial y_1^{i_1}} \ldots \frac{\partial^{i_m}}{\partial y_m^{i_m}}$ to (65) and take value at $(X, 0)$. Then, using (49) and (50), equality (77) follows from multiplication of each side by $\frac{(-1)^{\frac{|I|}{2}}}{\lambda^{|I|}}$. ■

Proposition 2.5. *We have in $\mathcal{C}^{\omega}_{\mathfrak{spo}(V)}(\mathcal{V}^+)$.*

$$\mathrm{Spf}^2 = \mathrm{Ber}^- . \tag{79}$$

It is equivalent to say that for any near superalgebra $\mathcal{P}$ and $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+) \times \boldsymbol{i}\,\mathfrak{spo}(V)_{\mathcal{P}}$, we have:

$$\mathrm{Spf}^2(X) = \mathrm{Ber}^-(X). \tag{80}$$

Proof. Proposition 2.1 with $g = X^{-1}$ gives:

$$\mathrm{Spf}(-X^{-1}) = \mathrm{Ber}^{-1}_{(1,0)}(X^{-1})\,\mathrm{Spf}(X). \tag{81}$$

Since on $\mathcal{V}^+$, $\mathrm{Ber}^{-1}_{(1,0)} = \mathrm{Ber}^-$, the result follows from multiplying both sides by $\mathrm{Ber}^-(X)\,\mathrm{Spf}(X)$ and using formula (73) with $\lambda = \boldsymbol{i}$.

■

2.9. Product formulas.

First we consider the case $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. Let $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We put $X^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$. Then, $X \in \mathfrak{spo}(V)_{\mathcal{P}}$ is equivalent to $X^* = -X$. Moreover, $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ implies that A is invertible. It follows that: $A^* = -A$; $D^* = -D$; $C^* = -B$. Using the formula:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}, \tag{82}$$

we obtain from (31) the formula:

$$\mathrm{Spf}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Spf}\begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} = \frac{\mathrm{Pf}(D - CA^{-1}B)}{\sqrt{\det(A)}}. \tag{83}$$

We explain the relation with formula (62). Taylor formula (48) for $\mathrm{Pf}(D - CA^{-1}B)$ gives:

$$\mathrm{Pf}(D - CA^{-1}B) = \sum_{J \in \{0,1\}^n / |J| \text{ even}} \epsilon(J, J')\,\mathrm{Pf}((-CA^{-1}B)_J)\,\mathrm{Pf}(D_{J'}). \tag{84}$$

Compatibility with formula (62) follows from formula (76) because, for $|J|$ even, we have $(-\boldsymbol{i})^{|J|} = (-1)^{\frac{|J|(|J|-1)}{2}}$.

Now, we consider the case where $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$ is invertible. As before we put $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We assume moreover that D is invertible. Using the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}^* \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix}, \tag{85}$$

We obtain:

$$\mathrm{Spf}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Spf}\begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} = \frac{\mathrm{Pf}(D)}{\sqrt{\det(A - BD^{-1}C)}}. \tag{86}$$

Compatibility with formula (62) comes from formula (43) for the function $\frac{1}{\sqrt{det(A-BD^{-1}C)}}$ and equation (77).

2.10. Homogeneity.

Proposition 2.6. *The superPfaffian is a homogeneous function of degree $\frac{n-m}{2}$ on $\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)$.*

Proof. It follows from proposition 2.1. ■

3. The SuperPfaffian as a generalized function

In this section we define Spf as a generalized function on $\mathfrak{spo}(V)$ by the formula:

$$\mathrm{Spf}(X) = \boldsymbol{i}^{\frac{m-n}{2}} \int_V d_V(v) \exp(-\frac{\boldsymbol{i}}{2} B(v, Xv)) \tag{87}$$

The meaning of this formula is the following: for any smooth compactly supported distribution t on $\mathfrak{spo}(V)$, we have:

$$\int_{\mathfrak{spo}(V)} t(X)\,\mathrm{Spf}(X) = \boldsymbol{i}^{\frac{m-n}{2}} \int_V d_V(v) \int_{\mathfrak{spo}(V)} t(X) \exp(-\frac{\boldsymbol{i}}{2} B(v, Xv)). \tag{88}$$

where the inner integral of the right hand side is a rapidly decreasing function on V (cf. below). This generalized function on the supermanifold $\mathfrak{spo}(V)$ coincides on $\mathcal{V}^+$ with the superPfaffian defined in the preceding section.

3.1. A well defined generalized function. In this section we prove that formula (87) defines a generalized function on $\mathfrak{sp}(V_{\mathbf{0}})$, with values in a finite dimensional subspace of $S\big((\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}})^*\big)$. This ensures that formula (87) defines generalized function on $\mathfrak{spo}(V)$. The point is to show that

$$\int_{\mathfrak{spo}(V)} t(X) \exp(-\frac{\boldsymbol{i}}{2} B(v, Xv))$$

is rapidly decreasing on V.

Let us fix some notations. Let $v \in V \otimes V^*$ be the generic point of V. We denote by $\widetilde{\mu} \in \mathfrak{spo}(V)^* \otimes S^2(V^*)$ the polynomial of degree 2 on V with values in $\mathfrak{spo}(V)^*$ such that for any near superalgebra $\mathcal{P}$ and any $X \in \mathfrak{spo}(V)_{\mathcal{P}}$:

$$\widetilde{\mu}(v)(X) = \mu(X, v). \tag{89}$$

In particular for $u \in V_{\mathcal{P}}$, $\widetilde{\mu}(u) \in \mathfrak{spo}(V)^*_{\mathcal{P}}$.

Let ρ be a smooth compactly supported distribution on $\mathfrak{sp}(V_{\mathbf{0}})$. We denote by $\widehat{\rho}$ its Fourier transform. It is the smooth rapidly decreasing function on $\mathfrak{sp}(V_{\mathbf{0}})^*$ (in sense of Schwartz) which is defined for $f \in \mathfrak{sp}(V_{\mathbf{0}})^*$ by the formula:

$$\widehat{\rho}(f) = \int_{\mathfrak{sp}(V_{\mathbf{0}})} \rho(X) \exp(-\boldsymbol{i} f(X)), \tag{90}$$

We fix $J \in \mathcal{U}^+$. Then $B(v, Jv)$ is a positive definite quadratic form on $V_{\mathbf{0}}$. For $u \in V_{\mathbf{0}}$, we put: $\|u\| = \sqrt{\frac{1}{2} B(u, Ju)}$.

We fix a norm N' on $\mathfrak{spo}(V)_{\mathbf{0}}$ and denote by N the associated norm on $\mathfrak{spo}(V)^*_{\mathbf{0}}$. For $f \in \mathfrak{spo}(V)^*_{\mathbf{0}}$ we have: $N(f) = \underset{Y \in \mathfrak{spo}(V)_{\mathbf{0}} \setminus \{0\}}{Sup} \frac{|f(Y)|}{N'(Y)}$. Thus we have for $u \in V_{\mathbf{0}}$:

$$N(\widetilde{\mu}(u)|_{\mathfrak{sp}(V_{\mathbf{0}})}) \geqslant \frac{|\widetilde{\mu}(u)(J)|}{N'(J)} = \frac{\|u\|^2}{N'(J)}; \tag{91}$$

where $\widetilde{\mu}(u)|_{\mathfrak{sp}(V_{\mathbf{0}})}$ is the restriction of $\widetilde{\mu}(v_{\mathbf{0}})$ to $\mathfrak{sp}(V_{\mathbf{0}})$.

For $X'' \in \big(\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}}\big)_{\mathcal{P}}$, we put:

$$\phi(X'', v) = \int_{\mathfrak{sp}(V_{\mathbf{0}})} \rho(X') \exp\big(-\frac{\boldsymbol{i}}{2} B(v, (X' + X'')v)\big) \tag{92}$$

Lemma 3.1. *For any $X'' \in \big(\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}}\big)_{\mathcal{P}}$, $\phi(X'', v)$ is a well defined rapidly decreasing function on V. Moreover, ϕ is polynomial in X''.*

Proof. Let $v_{\mathbf{0}} \in V_{\mathbf{0}} \otimes V_{\mathbf{0}}^*$ be the generic point of $V_{\mathbf{0}}$ and $v_{\mathbf{1}} \in V_{\mathbf{1}} \otimes V_{\mathbf{1}}^*$ be the generic point of $V_{\mathbf{1}}$. We have $v = v_{\mathbf{0}} + v_{\mathbf{1}}$. Then: $B(v, (X' + X'')v) = -2\widetilde{\mu}(v_{\mathbf{0}})(X') + B(v, X''v)$. It follows

$$\phi(X'', v) = \widehat{\rho}\big(-\widetilde{\mu}(v_{\mathbf{0}})|_{\mathfrak{sp}(V_{\mathbf{0}})}\big) \exp\big(-\frac{\boldsymbol{i}}{2} B(v, X''v)\big). \tag{93}$$

Since $\boldsymbol{b}(X'') \in \mathfrak{so}(V_{\mathbf{1}})$, we have:

$$B(v, X''v) = B(v_{\mathbf{1}}, \boldsymbol{b}(X'')v_{\mathbf{1}}) + B(v, (X'' - \boldsymbol{b}(X''))v). \tag{94}$$

Hence, $B(v, X''v)$ is nilpotent and $X'' \mapsto \exp(-\frac{\boldsymbol{i}}{2} B(v, X''v))$ defines a polynomial function on $\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}}$ with values in $S(V^*)$. In particular ϕ is polynomial in X''. Since $\widehat{\rho}$ is rapidly decreasing on $\mathfrak{sp}(V_{\mathbf{0}})^*$, formula (91) ensures that the function $\widehat{\rho}(-\widetilde{\mu}(v_{\mathbf{0}})|_{\mathfrak{sp}(V_{\mathbf{0}})})$ is rapidly decreasing on $V_{\mathbf{0}}$. Thus, for any $X'' \in \big(\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}}\big)_{\mathcal{P}}$, $\phi(X'', v)$ is a rapidly decreasing function on V. ■

The integral:

$$h(X'') = \int_V d_V(v)\phi(X'', v) = \int_V d_V(v) \int_{\mathfrak{sp}(V_{\mathbf{0}})} \rho(X') \exp\big(-\frac{\boldsymbol{i}}{2} B(v, (X' + X'')v)\big) \tag{95}$$

converges and it defines a polynomial function on $\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}}$. This means that Spf is a generalized function on $\mathfrak{sp}(V_{\mathbf{0}})$ with values in $S\big((\mathfrak{so}(V_{\mathbf{1}}) \oplus \mathfrak{spo}(V)_{\mathbf{1}})^*\big)$.

3.2. Comparison with the analytic version of section 2.. In this section we denote by Spf_{an} the analytic superPfaffian defined on $\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)$ by formula (22) and by Spf_{gene} the generalized superPfaffian defined on $\mathfrak{spo}(V)$ by formula (87).

The function

$$(X, \boldsymbol{i}Y) \mapsto \boldsymbol{i}^{\frac{m-n}{2}} \mathrm{Spf}_{an}(\boldsymbol{i}(X + \boldsymbol{i}Y)) \tag{96}$$

is analytic on $\mathfrak{spo}(V) \times \boldsymbol{i}\mathcal{V}^-$. We consider it as an analytic function on the open cone $\mathfrak{spo}(V)_{\mathbf{0}} \times \boldsymbol{i}\mathcal{V}^-$ of $\mathfrak{spo}(V \otimes \mathbb{C})_{\mathbf{0}}$ with values in $\Lambda(\mathfrak{spo}(V)_{\mathbf{1}}^*)$.

We fix a relatively compact open neighborhood $\mathcal{X}$ of 0 in $\mathfrak{spo}(V)_{\mathbf{0}}$. Since Spf is homogeneous of degree $\frac{n-m}{2}$ it follows that for any relatively compact open subset $\mathcal{W} \subset \mathfrak{spo}(V)_{\mathbf{0}}$, there exists a constant $K_{\mathcal{W}}$ such that for any $(X, \boldsymbol{i}Y) \in \mathcal{W} \times \boldsymbol{i}(\mathcal{V}^- \cap \mathcal{X})$ and any homogeneous differential operator $\mathcal{D} \in \Lambda(\mathfrak{spo}(V)_{\mathbf{1}})$ we have for some $k \in \mathbb{N}$:

$$\Big|\big(\mathcal{D}\,\mathrm{Spf}_{an}\big)(\boldsymbol{i}(X + \boldsymbol{i}Y))\Big| = \Big|\big(\mathcal{D}\,\mathrm{Spf}_{an}\big)(-Y + \boldsymbol{i}X))\Big| \leqslant K_{\mathcal{W}} N'(Y)^{-k}. \tag{97}$$

(N' is a norm on $\mathfrak{spo}(V)_{\mathbf{0}}$.)

Then [6, Theorem 3.1.15] shows that its limit when Y goes to 0 in $\mathcal{V}^-$ exists as a generalized function on $\mathfrak{spo}(V)_{\mathbf{0}}$ with values in $\Lambda(\mathfrak{spo}(V)^*_{\mathbf{1}})$. We have:

$$\mathrm{Spf}_{gene}(X) = \lim_{Y\to 0, Y\in\mathcal{V}^-} \boldsymbol{i}^{\frac{m-n}{2}} \mathrm{Spf}_{an}(\boldsymbol{i}(X+\boldsymbol{i}Y)). \tag{98}$$

Since Spf_{an} is the holomorphic extension of $\mathrm{Spf}_{an}|_{\mathcal{V}^+}$. It is entirely determined by $\mathrm{Spf}_{an}|_{\mathcal{V}^+}$. On the other hand, since $\mathrm{Spf}_{gene}(X)$ is the limit of $\boldsymbol{i}^{\frac{n-m}{2}} \mathrm{Spf}_{an}(\boldsymbol{i}(X+\boldsymbol{i}Y))$, Spf_{gene} is determined by Spf_{an} and thus by $\mathrm{Spf}_{an}|_{\mathcal{V}^+}$. In particular it follows that Spf_{gene} possesses the properties of Spf_{an}.

Let $\mathcal{P}$ be a near superalgebra and $X\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, let $\epsilon>0$, we have:

$$\mathrm{Spf}_{an}((\epsilon+\boldsymbol{i})X) = (\boldsymbol{i}+\epsilon)^{\frac{n-m}{2}} \mathrm{Spf}_{an}(X) \tag{99}$$

It follows, taking the limit of (99) when ϵ goes to zero that:

$$\mathrm{Spf}_{gene}|_{\mathcal{V}^+} = \mathrm{Spf}_{an}|_{\mathcal{V}^+}. \tag{100}$$

($\phi|_{\mathcal{V}^+}$ denotes the restriction of the (generalized) function ϕ to the open set $\mathcal{V}^+$.) In particular, Spf_{gene} is analytic on $\mathcal{V}^+$. From now on Spf stands for Spf_{gene}, and for $(X,Y)\in\mathfrak{spo}(V\otimes\mathbb{C})_{\mathcal{P}}\big(\mathcal{V}^+\times\boldsymbol{i}\mathfrak{spo}(V)_{\mathbf{0}}\big)$ $\mathrm{Spf}(X+\boldsymbol{i}Y)$ stands for $\mathrm{Spf}_{an}(X+\boldsymbol{i}Y)$.

3.3. Evaluation of Spf on $\mathcal{V}_{p,q}$. We recall notations $\mathcal{U}_{p,q}$ and $\mathcal{V}_{p,q}$ from section 2..

Proposition 3.1. *Let $\mathcal{P}$ be a near superalgebra. Let $(p,q)\in\mathbb{N}^2$ such that $p+q=m$. Let $X\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}_{p,q})$. It means that X is represented in a symplectic basis $(e_1,\dots,e_m,f_1,\dots,f_n)$ by $X=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathfrak{spo}(V)_{\mathcal{P}}$, with $\boldsymbol{b}(A)\in\mathcal{U}_{p,q}$. We have:*

$$\mathrm{Spf}\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \boldsymbol{i}^q \frac{\mathrm{Pf}(D-CA^{-1}B)}{\sqrt{|\det(A)|}}. \tag{101}$$

where we recall that Pf *is the ordinary Pfaffian.*

Proof. The first equality in formula (83) implies that it is enough to prove the formula for $X\in\big(\mathfrak{spo}(\mathcal{V}_{p,q})_{\mathbf{0}}\big)_{\mathcal{P}}$.

First, consider the particular case where $X\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$. In this case $(p,q)=(m,0)$, thus the coefficient is $\boldsymbol{i}^0=1$ and for $X\in\mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V}^+)$, $\boldsymbol{b}(A)\in\mathcal{U}^+$ and thus $\det(\boldsymbol{b}(A))>0$. The proposition reduces in this case to formula (83). Since $\mathcal{V}_{p,q}$ is a purely even real manifold, it is enough to consider $\mathcal{P}=\mathbb{R}$ and $X\in\mathcal{V}_{p,q}$. We put $X=\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}$, with $A\in\mathcal{U}_{p,q}$ and $D\in\mathfrak{so}(V_{\mathbf{0}})$. Then:

$$\mathrm{Spf}(X) = \boldsymbol{i}^{\frac{m-n}{2}} \int_{V_{\mathbf{0}}} d_{V_{\mathbf{0}}}(v_{\mathbf{0}}) \exp(-\frac{\boldsymbol{i}}{2}B(v_{\mathbf{0}},Av_{\mathbf{0}})) \int_{V_{\mathbf{1}}} d_{V_{\mathbf{1}}}(v_{\mathbf{1}}) \exp(-\frac{\boldsymbol{i}}{2}B(v_{\mathbf{1}},Dv_{\mathbf{1}})). \tag{102}$$

On one hand:

$$\int_{V_{\mathbf{1}}} d_{V_{\mathbf{1}}}(v_{\mathbf{1}}) \exp(-\frac{\boldsymbol{i}}{2}B(v_{\mathbf{1}},Dv_{\mathbf{1}})) = \boldsymbol{i}^{\frac{n}{2}} \mathrm{Pf}(D). \tag{103}$$

On the other hand, it is well known (cf. for example [6, formula 3.4.6]) that for $A \in \mathcal{U}_{p,q}$:

$$\int_{V_{\mathbf{0}}} |dx^1 \dots dx^m| \exp\big(-\frac{\boldsymbol{i}}{2} B(v_{\mathbf{0}}, Av_{\mathbf{0}})\big) = \frac{(2\pi)^{\frac{m}{2}}}{\exp(\boldsymbol{i}\frac{p-q}{4}\pi)\sqrt{|\det(A)|}}. \tag{104}$$

Since $p+q=m$, $\frac{\boldsymbol{i}^{\frac{m}{2}}}{\exp(\boldsymbol{i}\frac{p-q}{4}\pi)} = \boldsymbol{i}^q$ and the formula follows. ∎

In particular, it implies that Spf is smooth on $\mathcal{V}$ (in fact it is analytic) and that for any $X \in \mathfrak{spo}(V)_{\mathcal{P}}(\mathcal{V})$, we have: $\mathrm{Spf}(X)^2 = \mathrm{Ber}^-(X)$.

3.4. Example: $\mathfrak{spo}(2,2)$.

Let us consider as an example the case $\mathfrak{g} = \mathfrak{spo}(2,2)$. Let $\mathcal{P}$ be a near superalgebra. The algebra $\mathfrak{spo}(2,2)_{\mathcal{P}}$ is the set of matrices:

$$X = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|cc} a & b & \beta & \delta \\ c & -a & -\alpha & -\gamma \\ \hline -\alpha & -\beta & 0 & -d \\ -\gamma & -\delta & d & 0 \end{array}\right) \tag{105}$$

where $a, b, c, d \in \mathcal{P}_{\mathbf{0}}$ and $\alpha, \beta, \gamma, \delta \in \mathcal{P}_{\mathbf{1}}$. Here $V = \mathbb{R}^{(2,2)}$ is endowed with the symplectic form $\boldsymbol{B}$ given in the canonical base (e_1, e_2, f_1, f_2) ($|e_i| = 0$ and $|f_i| = 1$) by $\boldsymbol{B}(f_i, f_j) = \delta_i^j$, $\boldsymbol{B}(e_1, e_2) = -\boldsymbol{B}(e_2, e_1) = 1$, $\boldsymbol{B}(e_1, e_1) = \boldsymbol{B}(e_2, e_2) = 0$ and $\boldsymbol{B}(e_i, f_j) = \boldsymbol{B}(f_j, e_i) = 0$, and the orientation of $V_{\mathbf{1}}$ defined by the basis (f_1, f_2).

We have:

$$CA^{-1}B = \frac{1}{a^2+bc}\begin{pmatrix} 0 & -(a\alpha\delta - b\alpha\gamma + c\beta\delta + a\beta\gamma) \\ a\alpha\delta - b\alpha\gamma + c\beta\delta + a\beta\gamma & 0 \end{pmatrix}. \tag{106}$$

We denote by Spf_0 the superPfaffian on $\mathfrak{sp}(\mathbb{R}^2) = \mathfrak{sl}(2,\mathbb{R})$. For $X \in \mathfrak{spo}(2,2)_{\mathcal{P}}(\mathcal{V})$ ($\mathcal{V} = \mathcal{U} \times \mathfrak{so}(2)$), we get from (101):

$$\mathrm{Spf}(X) = \Big(d - \frac{a(\alpha\delta + \beta\gamma) - b\alpha\gamma + c\beta\delta}{a^2+bc}\Big)\,\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{107}$$

with (cf. formula (101)):

$$\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{1}{\sqrt{-(a^2+bc)}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{U}_{2,0} \tag{108}$$

$$\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{-1}{\sqrt{-(a^2+bc)}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{U}_{0,2} \tag{109}$$

$$\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{\boldsymbol{i}}{\sqrt{a^2+bc}} \quad \text{if } \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathcal{U}_{1,1}. \tag{110}$$

We denote by (x, y, ξ, η) the system of coordinates on $\mathbb{R}^{(2,2)}$ dual of (e_1, e_2, f_1, f_2). Let $v = e_1x + e_2y + f_1\xi + f_2\eta$ be the generic point of $\mathbb{R}^{(2,2)}$. We have:

$$\mu(X, v) = d\eta\xi + \alpha\xi x + \beta\xi y + \gamma\eta x + \delta\eta y - \frac{c}{2}x^2 + axy + \frac{b}{2}y^2. \tag{111}$$

Since $d_V(v) = \frac{1}{2\pi}|dx\,dy|\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}$, we have:

$$\int_{\mathbb{R}^{(2,2)}} d_V(v)\exp(\boldsymbol{i}\mu(X,v)) = \\ \frac{1}{2\pi}\int_{\mathbb{R}^2}|dx\,dy|\,\Big(\boldsymbol{i}d-(\alpha x+\beta y)(\gamma x+\delta y)\Big)\Big)\exp\boldsymbol{i}(-\frac{c}{2}x^2+axy+\frac{b}{2}y^2), \quad (112)$$

We compute the integral on the right hand side. We put

$$\mathcal{H} = \begin{pmatrix} \frac{\alpha\delta+\beta\gamma}{2} & \beta\delta \\ -\alpha\gamma & -\frac{\alpha\delta+\beta\gamma}{2} \end{pmatrix}. \quad (113)$$

If $v_{\mathbf{0}} = e_1x+e_2y$ is the generic point of $\mathbb{R}^2$, we have:

$$-(\alpha x+\beta y)(\gamma x+\delta y) = \boldsymbol{B}(v_{\mathbf{0}},\mathcal{H}v_{\mathbf{0}}). \quad (114)$$

Thus

$$-\int_{\mathbb{R}^2}|dx\,dy|\,(\alpha x+\beta y)(\gamma x+\delta y)\exp\boldsymbol{i}(-\frac{c}{2}x^2+axy+\frac{b}{2}y^2) \\ = 2\boldsymbol{i}\partial_{\mathcal{H}}\int_{\mathbb{R}^2}|dx\,dy|\,\exp-\frac{\boldsymbol{i}}{2}\boldsymbol{B}(v_{\mathbf{0}},Av_{\mathbf{0}}). \quad (115)$$

Since

$$\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \frac{\boldsymbol{i}}{2\pi}\int_{\mathbb{R}^2}|dx\,dy|\exp-\frac{\boldsymbol{i}}{2}\boldsymbol{B}(v_{\mathbf{0}},Av_{\mathbf{0}}),$$

we have in $\mathcal{C}^{-\infty}_{\mathfrak{spo}(V)}(\mathfrak{spo}(V)_{\mathbf{0}})$:

$$\mathrm{Spf}(X) = \Big(d+2\Big(\frac{\alpha\delta+\beta\gamma}{2}\frac{\partial}{\partial a}+\beta\delta\frac{\partial}{\partial b}-\alpha\gamma\frac{\partial}{\partial c}\Big)\Big)\,\mathrm{Spf}_0\begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \quad (116)$$

From this equation we deduce again (107).

3.5. The superPfaffian as the Fourier transform of a coadjoint orbit. In this section we assume that $V = V_{\mathbf{0}}$. Let $\mathrm{Sp}(V)$ be the corresponding symplectic group, and $\mathfrak{sp}(V)$ its Lie algebra. Recall, formula (89), the moment map $\widetilde{\mu}: V \to \mathfrak{sp}(V)^*$. The image $\widetilde{\mu}(V)$ is the disjoint union of a nilpotent $\mathrm{Sp}(V)$-orbit Ω and of $\{0\}$. The coadjoint orbit Ω is a symplectic manifold. The Fourier transform of its Liouville measure (defined as in [2], section 7.5) is a generalized function $\mathcal{F}_\Omega$ on $\mathfrak{sp}(V)$. The restriction of $\widetilde{\mu}$ to $V\setminus\{0\}$ is a double cover of Ω. It follows easily that we have :

$$Spf = 2\mathcal{F}_\Omega. \quad (117)$$

3.6. Singularities and wave front set.

We recall that a generalized function ϕ on $\mathfrak{spo}(V)$ is a generalized function on $\mathfrak{spo}(V)_{\mathbf{0}}$ with values in $\Lambda(\mathfrak{spo}(V)^*_{\mathbf{1}})$. The support of singularities of ϕ, denoted by singsupp(ϕ), and for any $X \in$ singsupp(ϕ) its wave front set at X, denoted by $WF_X(\phi)$ is defined as usual (cf. for example [6, chapter 8]). For the superPfaffian the Taylor formula (62) shows that the support of singularities and the wave front set of Spf are those of its restriction to $\mathfrak{sp}(V_{\mathbf{0}})$. The preceding section implies:

Proposition 3.2. *We have* $\mathrm{singsupp}(\mathrm{Spf}) = \mathfrak{spo}(V)_{\mathbf{0}} \setminus \mathcal{V}$ *and for any* $X \in \mathrm{singsupp}(\phi)$ $WF_X(\mathrm{Spf}) \subset \widetilde{\mu}(V_{\mathbf{0}}) \setminus \{0\}$.

3.7. Uniqueness results. We put:

$$(\mathcal{V}^-)^0 = \left\{ f \in \mathfrak{spo}(V)_{\mathbf{0}}^* \,/\, \forall X \in \mathcal{V}^-, f(X) \geqslant 0 \right\}. \tag{118}$$

We have $\widetilde{\mu}(V_{\mathbf{0}}) \subset (\mathcal{V}^-)^0$. As a direct application of [6, Theorem 8.4.15] we obtain.

Theorem 3.1. *Let* V *be a symplectic supervector space. Let* $\mathcal{V}^+$, $\mathcal{V}^-$, $(\mathcal{V}^-)^0$, $\mathrm{singsupp}(\mathrm{Spf})$, *be defined as above. Let* $\phi \in \mathcal{C}^{-\infty}_{\mathfrak{spo}(V)}(\mathfrak{spo}(V)_{\mathbf{0}})$ *be a generalized function on* $\mathfrak{spo}(V)$, *such that:*

1. ϕ *is smooth on* $\mathcal{V}^+$ *and* $(\phi|_{\mathcal{V}^+})^2 = Ber^-$
2. $WF(\phi) \subset \mathfrak{spo}(V)_{\mathbf{0}} \times (\mathcal{V}^-)^0$.

Then, we have $\phi = \mathrm{Spf}$ *or* $-\mathrm{Spf}$. *More precisely, an orientation of* $V_{\mathbf{1}}$ *chooses between* Spf *and* $-\mathrm{Spf}$.

Proof. Let $\phi \in \mathcal{C}^{-\infty}_{\mathfrak{spo}(V)}(\mathfrak{spo}(V)_{\mathbf{0}})$ satisfying the above conditions. Condition (2) and [6, Theorem 8.4.15] imply that for any open convex cone Γ with closure included in $\mathcal{V}^- \bigcup \{0\}$, there is an analytic function F on $\mathfrak{spo}(V) \times \boldsymbol{i}\Gamma$ such that on $\mathfrak{spo}(V)$:

$$\phi(X) = \lim_{Y \to 0,\, Y \in \Gamma} F(X + \boldsymbol{i}Y). \tag{119}$$

We choose an orientation of $V_{\mathbf{1}}$. Condition (1) implies that on $\mathcal{V}^+$, $\phi(X) = \pm \mathrm{Spf}(X)$. Thus, by [6, Theorem 3.1.15 and Remark], we have on $\mathfrak{spo}(V) \times \boldsymbol{i}\Gamma$

$$F(X + \boldsymbol{i}Y) = \pm \boldsymbol{i}^{\frac{m-n}{2}} \mathrm{Spf}(\boldsymbol{i}(X + \boldsymbol{i}Y)). \tag{120}$$

where we recall that $\mathrm{Spf}(\boldsymbol{i}(X + \boldsymbol{i}Y))$ is the analytic function on $\mathcal{V}^+ \times \boldsymbol{i}\,\mathfrak{spo}(V)$ defined by formula (22). Thus equation (98) implies that $\phi = \pm \mathrm{Spf}$.

Since changing the orientation of $V_{\mathbf{1}}$ changes Spf to $-\mathrm{Spf}$, the last remark follows.

■

Remark: In the definition of Spf besides the orientation of $V_{\mathbf{1}}$, we chose a square root $\boldsymbol{i}$ of -1. Changing $\boldsymbol{i}$ into $-\boldsymbol{i}$ changes $\mathrm{Spf}(X)$ into $\overline{\mathrm{Spf}}(X)$ which is the limit of $(-\boldsymbol{i})^{\frac{m-n}{2}} \mathrm{Spf}(-\boldsymbol{i}(X + \boldsymbol{i}Y))$ when Y goes to 0 in $\mathcal{V}^+ = -\mathcal{V}^-$. In particular:

$$WF(\overline{\mathrm{Spf}}) = -WF(\mathrm{Spf}). \tag{121}$$

In [10] we proved an other unicity theorem for the superPfaffian in the case of $\mathfrak{sl}(2, \mathbb{R})$. In this article it is proved that the superPfaffian is the unique square root of $1/\det$ which is harmonic and $Sp(2, \mathbb{R})$-invariant up to sign and complex conjugation.

3.8. Concluding Remarks. One motivation for this study comes from differential supergeometry. The equivariant Euler form of an equivariant real Euclidean oriented fiber bundle is equal to the Pfaffian of the equivariant curvature of an equivariant connection. With complications partly due to the fact that the superPfaffian is only a generalized function, this is still true in supergeometry (cf

[8, 9]). Formula (2) may be considered as particular typical case of the localization formula in supergeometry.

A second motivation is the close relationship between the superPfaffian and the distribution character of the metaplectic representation of the simply connected Lie supergroup with Lie superalgebra $\mathfrak{spo}(V)$. This will be studied in another paper.

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