Small Turing machines and generalized busy beaver competition

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Abstract

Let $TM(k, l)$ be the set of one-tape Turing machines with k states and l symbols. It is known that the halting problem is decidable for machines in $TM(2, 3)$ and $TM(3, 2)$. We prove that the decidability of machines in $TM(2, 4)$ and $TM(3, 3)$ will be difficult to settle, by giving machines in these sets for which the halting problem depends on an open problem in number theory. A machine in $TM(5, 2)$ with the same result is already known, and, moreover, this machine is the record holder for the busy beaver competitions : this is the machine in $TM(5, 2)$ which halts when starting from a blank tape, making the greatest number of steps and leaving the greatest number of non-blank symbols. We give potential winners for similar generalized busy beaver competitions in $TM(2, 3)$, $TM(2, 4)$ and $TM(3, 3)$.

Keywords : Turing machines; decidability; busy beaver competition; $3x + 1$ problem; Collatz problem.

1 Introduction

Small devices can display complex behaviors. Among the most studied such devices, one can find cellular automata, followed by Turing machines. In this

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paper, we consider one-tape Turing machines with a small number of states and symbols. Let $TM(k, l)$ denote the set of one-tape Turing machines with k states and l symbols. Precise definitions are given in the next section. For fixed k and l, the following questions have been studied, and are explained and briefly surveyed below :

1. Are all machines in $TM(k, l)$ decidable ?
2. Is there a universal machine in $TM(k, l)$?
3. Is there an undecidable machine in $TM(k, l)$?
4. Is there a machine in $TM(k, l)$ which simulates the Collatz $3x + 1$ problem ?
5. Is there a machine in $TM(k, l)$ which simulates a Collatz-like problem ?
6. What are the best machines in $TM(k, l)$ for the busy beaver competitions ?

A useful survey for questions 1, 2 and 4 can be found in Margenstern (2000) [12].

1.1 Are all machines in $TM(k, l)$ decidable ?

That is : is the halting problem decidable for all machines in $TM(k, l)$? The halting problem for a Turing machine M asks whether M stops on an input x. Formally, this is the set $K_M = \{x \in \Sigma^* : M \text{ stops on input } x\}$, where Σ is the finite input alphabet of M. The halting problem for M is decidable if the set K_M is recursive.

The halting problem is decidable for machines with only one symbol (trivial) and for machines with only one state (Hermann (1968) [5]). Minsky (1967) [17], Pavlotskaya (1973) [19], Diekert and Kudlek (1989) [4], Kudlek (1996) [7] studied machines in $TM(2, 2)$, that are decidable. Pavlotskaya proved in 1978 the decidability of machines in $TM(3, 2)$ (1978) [20] and in $TM(2, 3)$ (unpublished).

These results leave open the decidability of machines in $TM(2, 4)$, $TM(3, 3)$ and $TM(4, 2)$. In this paper, we give machines in $TM(2, 4)$ and $TM(3, 3)$ with an halting problem depending on an open problem in number theory.

Therefore, the decidability problem for these sets of machines will be difficult to settle. But it is possible that all machines in $TM(4,2)$ can be proved to be decidable.

1.2 Is there a universal machine in $TM(k,l)$?

A Turing machine is universal if it can simulate all Turing machines, or, equivalently, if its halting problem $K_M = \{x \in \Sigma^* : M \text{ stops on input } x\}$ is m-complete. The construction of universal machines in $TM(k,l)$ for small values of k and l, in the last twenty years, is mainly the work of Rogozhin (1982, 1996) [23,25].

Presently, it is known that there are universal Turing machines in the following sets :

- $TM(2,18)$: Rogozhin (1996) [25],
- $TM(3,9)$: Kudlek and Rogozhin (2002) [8],
- $TM(4,6)$: Rogozhin (1982, 1996) [23,25],
- $TM(5,5)$: Rogozhin (1982, 1996) [23,25],
- $TM(7,4)$: Minsky (1962, 1967) [16,17], Robinson (1991) [22], Rogozhin (1982, 1996) [23,25], Baiocchi (2001) [1],
- $TM(10,3)$: Rogozhin (1992, 1996) [24,25], Baiocchi (2001) [1],
- $TM(19,2)$: Baiocchi (2001) [1].

In a table like Table 1, giving the properties of $TM(k,l)$ according to k and l, the sets $TM(k,l)$ presently known to contain a universal Turing machine are situated on and above a line with hyperbolic shape, which may be called the *present universality line*. Between this line and the decidable sets $TM(2,3)$ and $TM(3,2)$, there is a finite number of sets $TM(k,l)$ (presently 45), for which it is unknown whether they contain a universal Turing machine. A *true universality line* is situated somewhere between the present universality line and the decidable sets, below which there is no universal Turing machine.

symbols											
18	U										
$\vdots$	$\vdots$										
9		U									
8	T										
7											
6			U								
5		T		U							
4	O		T			U					
3	D	O		T					U		
2	D	D		O					T	$\cdots$	U
	2	3	4	5	6	7	8	9	10	$\cdots$	19	states

Table 1: Type of Turing machine in TM(states, symbols) : U = Universal, T = Three–x–plus–one, O = Open Collatz-like problem, D = all Decidable

1.3 Is there an undecidable machine in $TM(k, l)$?

It is well known that there are recursively enumerable sets that are neither m-complete, nor recursive. So, there are Turing machines that are not universal, but have an undecidable halting problem. As above, a *present undecidability line* and a *true undecidability line* can be defined, the first one being the same as the present universality line.

Presently, we can settle the halting problem for a given Turing machine either by producing an algorithm to prove it decidable, or by simulating a universal machine to prove it undecidable. When facing an instruction table for a Turing machine which is neither decidable, nor universal, we have no method available to prove it undecidable, and no more method to prove it not universal. Therefore, studying the undecidability line independently of the universality line would require a breakthrough in computability science.

1.4 Is there a machine in $TM(k, l)$ which simulates the Collatz $3x + 1$ problem ?

Let $T : \mathbb{N} - \{0\} \to \mathbb{N}$ be defined by

$$T(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ (3x+1)/2 & \text{if } x \text{ is odd.} \end{cases}$$

It is conjectured that iterating T on a positive integer always leads to the loop $T(1) = 2$, $T(2) = 1$. This is a well known open problem in number theory, known as $3x+1$ problem, Collatz problem, etc. : see Lagarias (1985) [9] for a survey.

If a machine in $TM(k,l)$ simulates the $3x+1$ problem, then we know that the decidability of machines in $TM(k,l)$ will not be settled until the $3x+1$ problem is solved. Presently, it is known that there are Turing machines which simulate the $3x+1$ problem in the following sets : $TM(2,8)$, $TM(3,5)$, $TM(4,4)$, $TM(5,3)$ and $TM(10,2)$ (results from Margenstern (2000) [12], or Baiocchi, cited in Margenstern (2000) [12]). These sets constitute a line with hyperbolic shape in Table 1, which may be called the *present* $3x+1$ *line*. This line is situated between the present universality line and the decidable sets.

1.5 Is there a machine in $TM(k,l)$ which simulates a Collatz-like problem ?

The function T defined above for the $3x+1$ problem can also be written :

$$\begin{aligned} T(2m) &= m, \\ T(2m+1) &= 3m+2. \end{aligned}$$

Given integers $d \geq 2$, $a_0, a_1, \ldots, a_{d-1}$, $b_0, b_1, \ldots, b_{d-1}$, we can define a mapping g from $\mathbb{N}$ into $\mathbb{N}$, such that, if $m \in \mathbb{N}$ and $0 \leq r \leq d-1$:

$$g(dm+r) = a_r m + b_r.$$

This definition can also be written as : if $0 \leq r \leq d-1$, $n \in \mathbb{N}$,

$$\text{if} \quad n \equiv r \pmod{d}, \quad \text{then} \quad g(n) = a_r(n-r)/d + b_r.$$

Such functions are named *one-state linear operators algorithms* (OLOA) by Kascak (1992) [6] and *periodically linear functions* by Wirsching (1998) [26]. In this paper, we need to extend such definitions to partial functions, undefined on $d\mathbb{N} + r$ for some r, and to functions of pairs of integers. We call these functions *Collatz-like functions* [15]. Iterating Collatz-like functions leads to *Collatz-like problems*. Conway (1972) [3] and Kascak (1992) [6] gave unsolvable (and m-complete) Collatz-like problems.

In this paper, we give machines in $TM(2,4)$ and $TM(3,3)$ with halting problems depending on Collatz-like problems which seem to be presently open. Such a machine is known to exist in $TM(5,2)$ [15]. So the sets $TM(2,4)$, $TM(3,3)$ and $TM(5,2)$ constitute a line with hyperbolic shape in Table 1, which may be called the *present Collatz-like line*. This line is situated between the present $3x+1$ line and the decidable sets. It is unknown whether there is a machine simulating a Collatz-like problem in $TM(4,2)$.

1.6 What are the best machines in $TM(k,l)$ for the busy beaver competitions ?

Let $HTM(k,l)$ be the set of Turing machines in $TM(k,l)$ which stop when starting from a blank tape. For $M \in HTM(k,l)$, let $s(M)$ be the number of computation steps made by Turing machine M, and let $\sigma(M)$ be the number of symbols distinct from the blank symbol left by M when it stops. The greatest values of $s(M)$ and $\sigma(M)$ lead to the definition of the following functions of k and l :

$$\begin{aligned} S(k,l) &= \max\{s(M) : M \in HTM(k,l)\}, \\ \Sigma(k,l) &= \max\{\sigma(M) : M \in HTM(k,l)\}. \end{aligned}$$

For $l = 2$ symbols, we get the classical busy beaver competition defined by Rado (1962) [21]. It is known that :

- $S(2,2) = 6$ and $\Sigma(2,2) = 4$: Rado (1962) [21],
- $S(3,2) = 21$ and $\Sigma(3,2) = 6$: Lin and Rado (1965) [10],
- $S(4,2) = 107$ and $\Sigma(4,2) = 13$: Brady (1983) [2] and Kopp (cited by Machlin and Stout (1990) [11]),
- $S(5,2) \geq 47176870$ and $\Sigma(5,2) \geq 4098$: Marxen and Buntrock (1990) [13],
- $S(6,2) \geq 3 \times 10^{1730}$ and $\Sigma(6,2) \geq 1.29 \times 10^{865}$: Marxen and Buntrock in 2001 [14].

For $l \geq 3$, we get two generalized busy beaver competitions between machines in $HTM(k,l)$. In this paper, we give machines showing that :

- $S(2,3) \geq 38$ and $\Sigma(2,3) \geq 9$,

- $S(2,4) \geq 7195$ and $\Sigma(2,4) \geq 90$,
- $S(3,3) \geq 40737$ and $\Sigma(3,3) \geq 208$.

We conjecture that the lower bounds for $(k,l) = (2,3)$ and $(2,4)$ are the best ones, but that the lower bounds for $(k,l) = (3,3)$ can be improved.

The machine in $HTM(2,4)$ giving the lower bounds is the machine considered in subsection 1.5, with an open Collatz-like halting problem. Similarly, the machine in $HTM(5,2)$ giving the lower bounds was previously shown in [15] to have an open Collatz-like halting problem.

2 Definitions and notations

The Turing machines we consider are the standard ones used in papers on small universal Turing machines or busy beaver competition. They have a unique one-dimensional tape infinite in both directions, and a unique two-way read–write head. There is a blank symbol denoted by 0. Initially, a finite word, the input, is written on the tape, other cells contain the blank symbol, the head reads the leftmost symbol of the input, and the state is the initial state. At each step, according to the current state of the machine and the symbol read by the head, the symbol is modified, the head moves left or right (and cannot stay reading the same cell), and the state is modified. The computation stops when a special halting state is reached. We can suppose that, when a machine halts, it writes a 1, moves right, and enters state H.

Formally, a Turing machine is $M = (Q \cup \{H\}, \Sigma, \delta)$, where Q is the finite set of non-halting states, Σ is the finite set of symbols (including the blank symbol 0), and δ is the next move function :

$$\delta : Q \times \Sigma \to (\Sigma \times \{L, R\} \times Q) \cup \{(1, R, H)\}$$

If $\delta(q,a) = (b, D, q')$, then, when the state is q and the head reads symbol a, Turing machine M replaces symbol a by symbol b, moves in direction $D \in \{L, R\}$ (L for left and R for right), and enters state q'. We denote non-halting states by $A, B, C, \ldots$, and symbols by $0, 1, 2, \ldots$.

Let $TM(k,l)$ be the set of Turing machines with card$(Q) = k$ states and card$(\Sigma) = l$ symbols. Then

$$\text{card } TM(k,l) = (2kl+1)^{kl}$$

Let Σ^* be the set of finite words from alphabet Σ, λ the empty word, $|x|$ the length of $x \in \Sigma^*$, and Σ^n the set of words with length n. If $x \in \Sigma^*$, then we define $x^0 = \lambda$, $x^1 = x$, and, for any $n \geq 1$, $x^{n+1} = x^n x$. An infinite to the right string of 0's is denoted by 0^ω, and an infinite to the left string of 0's, by ${}^\omega 0$.

A *configuration* of machine M is a two-side infinite string ${}^\omega 0 x (Za) y 0^\omega$, where $Z \in Q \cup \{H\}$, $a \in \Sigma$, $x, y \in \Sigma^*$. Then, the word $xay \in \Sigma^*$ is written on the tape, between two infinite strings of 0's, the state is Z and the head scans symbol a.

The initial configuration of M on an input $x_1 x_2 \dots x_n \in \Sigma^*$ is

$$ {}^\omega 0 (A x_1) x_2 \dots x_n 0^\omega . $$

On a blank tape, M starts from ${}^\omega 0 (A0) 0^\omega$. Note that, if $x \in \Sigma^*$, M has the same behavior on $x0^n$, for all $n \in \mathbb{N}$.

If C_1 and C_2 are two configurations of M, and $p \in \mathbb{N}$, then we write $C_1 \vdash (p)\ C_2$ if the next move function δ leads from C_1 to C_2 in p steps. We write $C \vdash (p)\ END$ if configuration C leads in p steps to a final configuration, that is a configuration with final state H.

3 Turing machines with 2 states and 3 symbols

The machine M_0 defined below is the record holder for the generalized busy beaver competitions in $TM(2,3)$.

Definition 3.1 Instruction table for $M_0 \in TM(2,3)$ *:*

M_0	0	1	2
A	$1RB$	$2LB$	$1RH$
B	$2LA$	$2RB$	$1LB$

Proposition 3.2 *(i) Machine M_0 halts on a blank tape in 38 steps, leaving 9 non-blank letters : $s(M_0) = 38$ and $\sigma(M_0) = 9$.*

(ii) $S(2,3) \geq 38$ and $\Sigma(2,3) \geq 9$.

Proof : it can be checked that ${}^\omega 0 (A0) 0^\omega \vdash (38)\ {}^\omega 0 2^7 1 (H2) 0^\omega$. □

We conjecture that M_0 is the winner in the generalized busy beaver competition in $TM(2,3)$, so $S(2,3) = 38$ and $\Sigma(2,3) = 9$.

4 Turing machines with 2 states and 4 symbols

The machine M_1 defined below is the current record holder for the generalized busy beaver competitions in $TM(2,4)$.

Definition 4.1 Instruction table for $M_1 \in TM(2,4)$ *:*

M_1	0	1	2	3
A	$1RB$	$2LA$	$1RA$	$1LA$
B	$3LA$	$1RH$	$2RB$	$2RA$

Proposition 4.2 *(i) Machine M_1 halts on a blank tape in 7195 steps, leaving 90 non-blank letters :* $s(M_1) = 7195$ *and* $\sigma(M_1) = 90$.

(ii) $S(2,4) \geq 7195$ *and* $\Sigma(2,4) \geq 90$.

Proof : it can be checked that $^{\omega}0(A0)0^{\omega} \vdash (7195)\ ^{\omega}012^{88}1(H0)0^{\omega}$. □

We conjecture that M_1 is the winner in the generalized busy beaver competition in $TM(2,4)$, so $S(2,4) = 7195$ and $\Sigma(2,4) = 90$.

The halting problem for machine M_1 depends on a Collatz-like problem, as shown by the following proposition.

Proposition 4.3 *Let denote the following configurations of M_1 : for every $n \geq 0$, $C_1(n,0) = {}^{\omega}0(A0)2^n0^{\omega}$, and $C_1(n,1) = {}^{\omega}0(A0)2^n30^{\omega}$. Then, for every $k \geq 0$,*

$$\begin{array}{rll}
C_1(3k,0) & \vdash (15k^2+7k+3) & C_1(5k+1,1) \\
C_1(3k+1,0) & \vdash (15k^2+22k+11) & END \\
C_1(3k+2,0) & \vdash (15k^2+27k+13) & C_1(5k+4,0) \\
C_1(3k,1) & \vdash (15k^2+28k+16) & END \\
C_1(3k+1,1) & \vdash (15k^2+33k+19) & C_1(5k+5,0) \\
C_1(3k+2,1) & \vdash (15k^2+43k+33) & C_1(5k+7,1)
\end{array}$$

Proof : The result is given by a tedious analysis of the behavior of machine M_1. □

So the halting problem for M_1 involves the study of the function $g_1 : \mathbb{N} \times \{0,1\} \to \mathbb{N} \times \{0,1\}$ defined by

$$\begin{array}{rcl} g_1(3k,0) & = & (5k+1,1), \\ g_1(3k+1,0) & & \text{undefined}, \\ g_1(3k+2,0) & = & (5k+4,0), \\ g_1(3k,1) & & \text{undefined}, \\ g_1(3k+1,1) & = & (5k+5,0), \\ g_1(3k+2,1) & = & (5k+7,1). \end{array}$$

The behavior of iterating g_1 on an element of $\mathbb{N} \times \{0,1\}$ is an open problem. We can conjecture that iterating g_1 always leads to an undefined value, but no method is known to prove this result. Note that no less than 23 iterations of g_1 on $(81,0)$ lead to an undefined value, and so, that machine M_1 stops on ${}^{\omega}0(A0)2^{81}0^{\omega}$ in more than 10^{14} computation steps.

5 Turing machines with 3 states and 3 symbols

We define below three machines M_2, M_3, $M_4 \in TM(3,3)$. Machine M_2 is the current record holder for the generalized busy beaver competition in $TM(3,3)$ according to the number of steps taken by the computation. Machine M_3 is the current record holder according to the number of non-blank letters left on the tape. Machine M_4 has a halting problem that depends on an open Collatz-like problem.

Definition 5.1 Instruction tables for M_2, M_3, $M_4 \in TM(3,3)$ *:*

M_2	0	1	2
A	$1RB$	$0LA$	$1LA$
B	$2RC$	$1RB$	$1RA$
C	$2LA$	$0RB$	$1RH$

M_3	0	1	2
A	$1RB$	$2RA$	$1LA$
B	$1LC$	$1RC$	$0LA$
C	$2LA$	$2RB$	$1RH$

M_4	0	1	2
A	$1LB$	$0LB$	$2RB$
B	$2LC$	$2LB$	$1RB$
C	$2RA$	$2LA$	$1RH$

Proposition 5.2 *(i) Machine M_2 halts on a blank tape in 40737 steps, leaving 200 non-blank letters : $s(M_2) = 40737$ and $\sigma(M_2) = 200$.*

(ii) $S(3,3) \geq 40737$.

Proof : it can be checked that
${}^\omega 0(A0)0^\omega \vdash (40737)\ {}^\omega 01(21111)^{36}2112111121(H1)012012120120^\omega$. □

We conjecture that a better machine for function s can be found in $TM(3,3)$, so that $S(3,3) > 40737$.

Proposition 5.3 *(i) Machine M_3 halts on a blank tape in 11082 steps, leaving 208 non-blank letters : $s(M_3) = 11082$ and $\sigma(M_3) = 208$.*

(ii) $\Sigma(3,3) \geq 208$.

Proof : it can be checked that ${}^\omega 0(A0)0^\omega \vdash (11082)\ {}^\omega 011(21)^{102}1(H1)0^\omega$.
□

We conjecture that a better machine for function σ can be found in $TM(3,3)$, so that $\Sigma(3,3) > 208$.

Note that, for machine M_3, we have

$$\begin{array}{lll} {}^\omega 0(A0)1^{2k+1}0^\omega & \vdash (5k^2+25k+21) & {}^\omega 0(A0)1^{5k+6}0^\omega, \\ {}^\omega 0(A0)1^{2k+2}0^\omega & \vdash (2k+5) & \text{END} \end{array}$$

Let $g_3 : \mathbb{N} \to \mathbb{N}$ be defined by :

$$\begin{array}{rcl} g_3(2k+1) & = & 5k+6 \\ g_3(2k) & & \text{undefined} \end{array}$$

Then iterating g_3 on a positive integer always leads to an undefined value, so the halting problem for machine M_3 does not depend on a true Collatz-like problem, but on a 'pseudo-Collatz-like' problem which is not an open problem. The integers leading to many iterations of g_3 are given by integer approximations of the solution of the equation $x = 2k + 1 = 5k + 6$ in the ring of 2-adic integers (that is $k = -5/3$, $x = -7/3 = 1 + 2 + \sum_{n=2}^{\infty} 2^{2n}$).

The halting problem for machine M_4 depends on a Collatz-like problem, as shown by the following proposition.

Proposition 5.4 *Let, for every $n \geq 0$, $C_4(n)$ denote the following configuration of M_4 : $C_4(n) = {}^{\omega}01^n(B0)220^{\omega}$.*
Then ${}^{\omega}0(A0)2^n0^{\omega} \vdash (3n+11)\ C_4(n+1)$, *and, for every $k \geq 0$,*

$$\begin{array}{lll} C_4(4k+4) & \vdash (12k^2+46k+41) & C_4(6k+7) \\ C_4(4k+1) & \vdash (12k^2+28k+25) & C_4(6k+5) \\ C_4(4k+2) & \vdash (12k^2+16k+8) & END \\ C_4(8k+3) & \vdash (156k^2+242k+86) & C_4(18k+9) \\ C_4(8k+7) & \vdash (156k^2+344k+191) & END \end{array}$$

Proof : The result is given by a tedious analysis of the behavior of M_4. □

Note that, for M_4, ${}^{\omega}0(A0)0^{\omega} \vdash (13044)\ {}^{\omega}021(H2)2^{144}0^{\omega}$, so $s(M_4) = 13044$, and $\sigma(M_4) = 147$.

Let $g_4 : \mathbb{N} - \{0\} \to \mathbb{N} - \{0\}$ be defined by :

$$\begin{array}{lcl} g_4(4k+4) & = & 6k+7 \\ g_4(4k+1) & = & 6k+5 \\ g_4(4k+2) & & \text{undefined} \\ g_4(8k+3) & = & 18k+9 \\ g_4(8k+7) & & \text{undefined} \end{array}$$

As for function g_1 defined above, we can conjecture that iterating g_4 on $\mathbb{N} - \{0\}$ always leads to an undefined value, but no method is known to solve such a problem.

6 Conclusion

It is clear from Table 1 that the present universality line and the present $3x + 1$ line could be lowered by some later works. The present Collatz-like line is already on its lowest possible level, with the possible exception of $TM(4, 2)$, but we conjecture that all machines in this set can be proved to be decidable.

Secondly, note that $S(k,l) > S(l,k)$ and $\Sigma(k,l) > \Sigma(l,k)$ for $(k,l) = (2,3)$ and $(2,4)$. We conjecture that this is true for any $k < l$, $k \geq 2$. This lack of symmetry between the number of states k and the number of symbols l can be found again in the following facts :

- there are universal machines :
  - in $TM(19,2)$ and $TM(2,18)$,
  - in $TM(10,3)$ and $TM(3,9)$,
  - in $TM(7,4)$ and $TM(4,6)$,
- there are $3x+1$ machines in $TM(10,2)$ and $TM(2,8)$,
- there are open Collatz-like machines in $TM(5,2)$ and $TM(2,4)$.

Finally, note that Oberschelp et al. (1988) [18] consider Turing machines that cannot print and move in one computation step, and are defined by quadruples instead of quintuples. A parallel study in this context is still to be done.

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symbols												
18	U											
$\vdots$	$\vdots$											
9		U										
8	T											
7												
6			U									
5		T		U								
4	O		T			U						
3	D	O		T					U			
2	D	D		O					T	$\cdots$	U	
	2	3	4	5	6	7	8	9	10	$\cdots$	19	states

Table 2: Type of Turing machine in TM(states, symbols) : U = Universal, T = Three–x–plus–one, O = Open Collatz-like problem, D = all Decidable