Using the approximate functional equation to obtain missing data

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- general approximate functional equation & avatars
- guess missing data

Approximate functional equation

3 ingredients

Cauchy formula on the critical strip

$$\frac{1}{2i\pi}\int_{\partial R} f = \sum_{\rho\in R} \operatorname{Res}_\rho f$$

Functional equation

$$f_1(s) = \epsilon f_2(1-s) \Rightarrow \int_{\downarrow} f_1 = \epsilon \int_{\uparrow} f_2$$

Dirichlet series $\times$ smooth factor

$$\frac{1}{2i\pi}\int_{\uparrow}\sum_k a_k k^{-s} h(s)\,\mathrm{d}s = \sum a_k g(k)$$

where $g = \mathscr{M}^{-1}[h]$, the inverse Mellin transform.

Approximate functional equation

Assume

- $D_1(s), D_2(s)$ are Dirichlet series
- h_1, h_2 are meromorphic and exponentially decaying in the imaginary direction
- functional equation

$$\forall s, D_1(s)h_1(s) = \varepsilon D_2(1-s)h_2(1-s)$$

then

$$\sum_{\rho} \operatorname{Res} D_1(s)h_1(s) = \sum_{k} a_1(k)g_1(k) + \epsilon a_2(k)g_2(k)$$

where h_i, g_i are Mellin transform pairs

$$g_i(x) = \frac{1}{2i\pi}\int_{(\delta)} h_i(s)x^{-s}\,\mathrm{d}s$$

L functions

- Dirichlet series

$$L(s) = \sum_{k \geqslant 1} a_k k^{-s}$$

- smooth factor

$$\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \lambda_i)$$

- functional equation

$$\Lambda(s) = L(s)\gamma(s) = \epsilon \overline{\Lambda}(1 - s)$$

- $+\ \Lambda$ meromorphic. Here assumed holomorphic.

Exact functional equation

$$L(s)\gamma(s) = \epsilon\overline{L}(1-s)\overline{\gamma}(1-s)$$

$$0 = \sum_k a_k g(k) - \epsilon\overline{a_k g}(k)$$

already not completely useless (to be continued)...

Example 1 : Theta series

For $t > 0$,

$$\forall s, L(s)\gamma(s)t^{-s} = \epsilon\overline{L}(w+1-s)\overline{\gamma}(w+1-s)\frac{t^{+(w+1-s)}}{t^{w+1}}$$

equivalent to the Theta equation

$$\forall t, t^{w+1}\Theta(t) = \overline{\Theta}(1/t)$$

where

$$\Theta(t) = \sum_k a_k g(kt)$$

is the inverse Mellin transform of Λ.

Example

h(s)	$N^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)$	$N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)$	$N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+\nu)$	general
g(x)	$e^{-\frac{\pi}{N}x^2}$	$e^{-\frac{2\pi}{\sqrt{N}}x}$	Bessel K_ν	[Dokchitser]

Example 1' : Fourier

Fourier form

$$\Lambda(s) = \epsilon\overline{\Lambda}(1-s)$$

if and only if for all $x \in \mathbb{R}+]-\frac{d\pi}{4}, \frac{d\pi}{4}[$,

$$F(x) = \epsilon\overline{F}(-x)$$

where

$$F(x) = e^{\frac{w+1}{2}x} \sum_k a_k \mathscr{M}^{-1}\left[\gamma(s); e^x\right] \quad (1)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \Lambda(\frac{w+1}{2} + it) e^{ixt} \,\mathrm{d}t \quad (2)$$

Example 2 : Lavrik's approximate functional equation

value $\Lambda(s_0)$ as a residue

$$L(s)\frac{\gamma(s)}{s-s_0}t^{-s} = \epsilon\overline{L}(1-s)\frac{\overline{\gamma}(1-s)}{1-(1-s)+s_0}\frac{t^{+(1-s)}}{t}$$

$$\Lambda(s)t^{-s} = \sum_k a_k g_{s_0}(kt) - \epsilon\overline{a_k}g_{1-s_0}(k/t)$$

Example

$h_{s_0}(s)$	$\frac{N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)}{s-s_0}$	general
$g_{s_0}(x)$	$\Gamma(s,\frac{2\pi x}{\sqrt{N}})$	[Dokchitser]

Example 3 : smoothing

- balance vertical decay (to avoid numerical cancellation)
- restore vertical decay away from s_0

$$L(s_0)\gamma(s_0)e^{i\beta s}e^{A(s-s_0)^2} = \sum_k a_k g_1(k) - \epsilon \bar{a}_k g_2(k)$$

hard part : estimates for the decay of $g_{A,\beta,s_0}(x)$...

Example 4 : explicit formulas

Generalize a bit :

$$(D(s) + \Psi(s))h(s) = (D(1-s) + \Psi(1-s))h(1-s)$$

then

$$\sum_\rho \mathrm{Res}(D(s) + \psi(s))h(s) = \int_{\partial R} \psi(s)h(s) + \sum_k a_k g(k)$$

Use with

$$\frac{\Lambda'(s)}{\Lambda(s)} = \frac{\gamma'(s)}{\gamma(s)} + \sum_p \log p \frac{F_p'(p^{-s})}{F_p(p^{-s})}$$

to obtain

$$\sum_\rho h(\rho) = \int_{\partial R} \Psi(s)h(s) + \sum_p \log(p) \sum_l c_{p,l} g(p^l)$$

with $\Psi(s) = \frac{\gamma'(s)}{\gamma(s)} = \log(N) + \sum \psi_{\mathbb{R}}(s + \lambda_j)$.

The game

- goal : one functional equation

$$\Lambda(s) = \epsilon\overline{\Lambda}(1-s)$$

- tools : many approximate functional equations

$$\sum_{\rho} \mathrm{Res}_{\rho}\, L(s)h_1(s) = \sum_{k} a_k g_1(k) - \epsilon\overline{a_k} g_2(k)$$

$\Theta(t)$, $F(x)$, pole at s_0, gaussian smooth factor...

CheckFunctionalEquation

$$\forall s, \Lambda(s) = \epsilon\overline{\Lambda}(1-s)$$

if and only if

$$\forall t, t^{w+1}\Theta(t) = \epsilon\overline{\Theta}(1/t)$$

if and only if

$$\forall x, F(x) = \epsilon\overline{F}(-x)$$

In case $L = \overline{L}$

$\epsilon = 1$ if and only if F is even, and $\epsilon = -1$ if and only if F is odd.

Guess/compute root number

$$F(x) = \epsilon \overline{F}(-x)$$

In particular, if $F(0) \neq 0$,

$$\epsilon = e^{2 \arg F(0)}$$

(if $F(0)$ vanishes, take quotient at $x \neq 0$)

Example

χ Dirichlet character modulo N

$$\epsilon = \frac{\tau(\chi)}{\sqrt{N}}$$

numerical computation of Gauss sums in $O(\sqrt{N})$

Level

N must be an integer.

$$\text{plot}\left|\frac{t\Theta(t/\sqrt{N})}{\Theta(\frac{1}{t\sqrt{N}})}\right|$$

Structure of the Dirichlet series

- Euler product

$$L(s) = \prod F_p(p^{-s})^{-1}$$

- Euler factor

$$F_p(T) = 1 + c_{p,1}T + \dots T^d$$

- local functional equation

$$c_{p,d-j} = \chi(p) p^{\frac{w}{2}(d-2j)} c_{p,j}$$

with χ Dirichlet character modulo N

- Ramanujan bounds

$$c_{p,j} \leqslant \binom{d}{j} p^{j\frac{w}{2}} \quad (3)$$

$$a_k \leqslant \sigma_0(k)^{d-1} k^{\frac{w}{2}} \quad (4)$$

Some Euler factors

unknown factor

$$F_p(T) = 1 + c_{p,1}T + \dots c_{p,d}T^d$$

expand formally

$$a_{p^e} \in \mathbb{Q}(c_{p,1}, \dots c_{p,d})$$

any approximate functional equation gives a power series

$$P(c_{p,1}, \dots c_{p,d})$$

vanishing at $c_{p,1}, \dots c_{p,d}$.

Ramanujan bounds

$$c_{p,j} \leqslant \binom{d}{j} p^{j\frac{w}{2}}$$

Newton method from random points

$$z \leftarrow z - J(z)^{-1}P(z)$$

Least conductor

Consider a modular form of weight $k = w + 1$ and level N such that the root number is $\epsilon = -1$.

Then the Theta series must vanish at 1,

$$0 = \Theta(1) = \sum_k a_k q^k, q = e^{-\frac{2i\pi}{\sqrt{N}}}$$

and by Ramanujan this is not possible unless

$$q \leqslant \sum_{k \geqslant 2} \lfloor \sigma_0(k)\sqrt{k} \rfloor q^k, q = e^{-\frac{2\pi}{\sqrt{N}}}$$

Least conductor

- Equation $F(0) = 0 \rightsquigarrow N \geqslant 26$.
- Equation $F(iy) + F(-iy) = 0, y > 0 \rightsquigarrow N \geqslant 33$

For other signs/weights, many bounds are optimal

w	$\epsilon = 1$	$\epsilon = -1$
0	$N > 14.61$	$N > 44.62$
1	$N > 10.45$	$N > 32.83$
2	$N > 6.21$	$N > 16.86$
3	$N > 4.63$	$N > 12.24$
5	$N > 2.85$	$N > 6.55$
7	$N > 1.95$	$N > 4.09$
9	$N > 1.37$	$N > 2.75$
11	$\star$	$N > 1.97$

Enumerate Dirichlet series

Find inductively all Dirichlet series numerically compatible with some approximate functional equation $\sum a_k x_k = 0$.

- $a_1 = 1$
- $a_2 \in [-b_2, b_2]$ s.t.

$$\left|(a_1 x_1) + a_2 x_2\right| \leqslant b_3 \left|x_3\right| + b_4 \left|x_4\right| + b_5 \left|x_5\right| + b_6 \left|x_6\right| + b_7 \left|x_7\right| + \dots$$

- $a_3 \in [-b_3, b_3]$ s.t.

$$\left|(a_1 x_1 + a_2 x_2) + a_3(x_3 + a_2 x_6)\right| \leqslant +b_4 \left|x_4\right| + b_5 \left|x_5\right| + \cdots + b_7 \left|x_7\right| +$$

Framework

Write $k \prec p^e$ if k is p-smooth and $p^e \nmid k$.
Assume $\{a_k\}$ known for $k \prec p^e$. Equation for coefficient a_{p^e} :

$$\left(\sum_{k \prec p^e} a_k x_k\right) + a_{p^e}\left(\sum_{m \prec p} a_m x_{p^e m}\right) = \sum_{k \succ p^e} a_k x_k$$

If $e \geqslant \frac{d}{2}$, by reciprocity of $F_p(T)$ all a_{p^ℓ} in terms of a_{p^j}, $j \leqslant e$

$$\left(\sum_{k \prec p^e} a_k x_k\right) + \sum_{\ell \geqslant e} a_{p^\ell}\left(\sum_{m \prec p} a_m x_{p^\ell m}\right) = \sum_{k \succ p^\infty} a_k x_k$$

Ramanujan bound $|a_k| \leqslant b_k = \left|\sigma_0(k)^{d-1} k^{\frac{w}{2}}\right|$ on the tail $\rightsquigarrow$ polynomial equation

$$|P(a_{p^e})| \leqslant r$$

$\rightsquigarrow$ solve in integers in $]-b_{p^e}, b_{p^e}[$.

Good and bad tails

Case of polynomial equation of degree 1 :

$$|v + a_p w| \leqslant r = b_{p_+} |x_{p_+}| + b_{p_{++}} |x_{p_{++}}| + \dots$$

- $|w| \geqslant r \Rightarrow$ at most one solution a_p.
- usually nice at big prime gaps
- usually horrible for twin primes

the ratio $\frac{r}{w}$ can be studied a priori

Build equations

Combine approximate functional equations :

$$X_k = (x_{k,1}, \dots x_{k,n})$$

define V, W to be the sum of X_k and $R = (b_k X_k)_{k \succ p}$.
For all $\lambda \in \mathbb{R}^n$,

$$|V.\lambda + a_p W.\lambda| \leqslant \|R.\lambda\|_1$$

Choose λ to minimize $\frac{\| \ R.\lambda\|}{| \ W.\lambda|}$: Least absolute deviations problem. Can be solved with iterated + weighted least squares.

Examples

Degree 2, one complex gamma factor $\gamma(s) = N^{\frac{s}{2}}\Gamma_{\mathbb{C}}(s)$

- first weights give

w	level N
0	15, 20, 23, 24, 31, 32, 35, 36, 39(2), 40, 44, 48,
1	11, 14, 15, 17, 19, 20, 21, 24, 26(2), 27, 30, 32,
2	16, 27, 32, 64,
3	5, 6, 7, 8, 9, 10, 12, 14(2), 15(2)

- Modular forms expansions 805b and 805c start to differ at primes 11 and 13, with values exchanged
- For $N = 66$, $w = 1$ and central character $\chi = (\frac{-66}{\cdot})$, a match exists for the 71 first primes, then disapears.

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- For $N = 66$, $w = 1$ and central character $\chi = \left(\frac{-66}{\cdot}\right)$, a match exists for the 71 first primes, then disapears. χ is trivial up to $p = 19$, and $\pi(19^2) = 72$