L functions computations using Pari/GP

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1 Some L functions

2 A tour

Riemann zeta function

- Dirichlet series

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}$$

- completed as

$$\xi(s) = 1^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+0)\zeta(s)$$

- functional equation

$$\xi(s) = 1 \times \xi(1-s)$$

- pole at $s = 1$, residue 1.

definition

```
gp > Lzeta=[n->1,0,[0],0,1,1,0];
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```

compute values

```
gp > lfun(Lzeta, 2)
time = 2 ms.
%2 = 1.644934066848226436472415166646025
gp > Pi^ 2/6
%3 = 1.644934066848226436472415166646025
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%4 = +oo
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```

the completed xi function

```
gp > lfun(Lzeta, .5+I,1)
time = 2 ms.
%5 = -0.7772118874735735867564753709353939
+ 0.E-57*I
```

Riemann zeta function

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definition

```
gp > Lzeta=[n->1,0,[0],0,1,1,0];
```

Plot the Hardy Z-function

```
gp > lfunplot(Lzeta, 100)
time = 69 ms.
```

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definition

```
gp > Lzeta=[n->1,0,[0],0,1,1,0];
```

First zeros on the critical line

```
gp > lfunzeros(Lzeta, 50)
time = 22 ms.
%10 = [14.134725141734693790457251983562
21.022039638771554992628479593896902777,
25.010857580145688763213790992562821819,
30.424876125859513210311897530584091320,
32.935061587739189690662368964074903489,
37.586178158825671257217763480705332821,
40.918719012147495187398126914633254396,
43.327073280914999519496122165406805783,
48.005150881167159727942472749427516042,
49.773832477672302181916784678563724058]
```

Describe general L function

- Dirichlet series

$$L(s) = \sum_{n \geqslant 1} a_n n^{-s}$$

- gamma factor

$$\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \lambda_i)$$

```
gp > L=[n->a_n,0,[λ_1,..λ_d],w,N,ϵ,0]
gp > lfun(L, s)
gp > lfun(L, s, 1)
gp > lfunplot(L, maxt)
gp > lfunzeros(L, maxt)
```

- functional equation

$$\Lambda(s) = L(s)\gamma(s) = \epsilon\overline{\Lambda}(w + 1 - s)$$

- possible pole at $s = w + 1$

Dedekind zeta function

$K = \mathbb{Q}(\sqrt{5})$

Hum, is there any crossing between 33 and 34...

but there is one

the behaviour near zero indicates a pole and in fact zeta divides the L function

Dedekind zeta function

Totally real quintic

precomputations become slow

it is much more efficient to divide by zeta

Noteworthy :

always try to factor into smaller L-functions sum of complexity ¡ complexity of product

Elliptic curves

- Dirichlet series

$$L(s) = \sum_{n \geqslant 1} a_n n^{-s}$$

- gamma factor

$$\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \lambda_i)$$

- functional equation

$$\Lambda(s) = L(s)\gamma(s) = \epsilon\overline{\Lambda}(w+1-s)$$

- no pole

```
gp > e =
ellinit("11a1"); gp > L =
[n->ellak(e,n),0,[0,1],1,11,1
gp > lfun(L, 2) gp >
lfun(L, 2, 1) gp >
lfunplot(L, 20) gp >
lfunzeros(L, 20)
```

Have a look at BSD conjecture

What's behind

- Fourier transform on the critical line

$$f(t) = \Lambda(\frac{w+1}{2} + it), F(x) = \int_{\mathbb{R}} f(t) e^{-2i\pi x t} \,\mathrm{d}t$$

- Functional equation

$$F(x) = \epsilon \overline{F}(-x)$$

- Dirichlet expansion of $F(x)$, converges rapidly for $x > 0$

$$F(x) = \sum_{n \geqslant 1} a_n K(n e^x), K(t) \sim C e^{-d \pi t^{\frac{2}{d}}}$$

- Evaluation of inverse Mellin transform $K(t)$
  - for small t by generalized Taylor expansion
  - for large t by asymptotic expansion (diverges) + continued fractions evaluation (converges very well)
- Recover $f(t)$ by Poisson summation

$$f(t) = h \sum_k F(kh) e^{ikht} + O(e^{-D})$$

with explicit $h = O(1/(D+t))$, $n = O((D+t) \log D)$