

# Using approximate functional equations to build L functions

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## Example : elliptic curves

Consider an elliptic curve  $E/\mathbb{Q}$  of conductor  $N$  and root number  $\varepsilon = -1$ .

The associated modular form

$$f(z) = \sum a_n q^n, \quad q = e^{2i\pi z}. \quad (1)$$

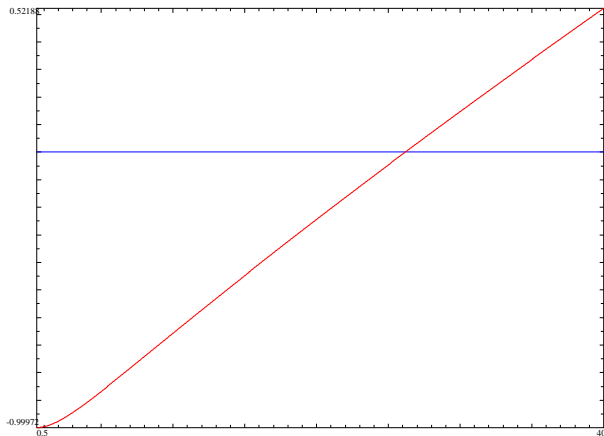
satisfies  $Wf = -f$ , so that for  $q = e^{-\frac{2\pi}{\sqrt{N}}}$ ,

$$f(q) = 0 = q + a_2 q^2 + a_3 q^3 + \dots \quad (2)$$

By Hasse,  $|a_n| \leq [\sigma_0(n)\sqrt{n}] \leq n$ , so the equality is possible only if

$$q \leq \sum_{k \geq 2} nq^n = \frac{q}{(1-q)^2} - q = \frac{q^2(2-q)}{(1-q)^2}. \quad (3)$$

```
b(q)=q*(2-q)/(1-q)^2;  
plot(N=.5,40,b(exp(-2*Pi/sqrt(N))))
```



```
b(q)=q*(2-q)/(1-q)^2;  
plot(N=.5,40,b(exp(-2*Pi/sqrt(N))))
```

```
solve(N=.5,40,b(exp(-2*Pi/sqrt(N)))-1)  
26.181852174699964975652391885916899331
```

## Theorem

*If an elliptic curve has rank  $r \geq 1$ , its conductor satisfies  $N \geq 27$ .*

## Least conductor : generalize

Degree 2 L-function  $L(s)$ , one gamma factor  $\Gamma_{\mathbb{C}}(s)$ , conductor  $N$ , weight  $k$  and sign  $\varepsilon$ . The (symmetrized) inverse Mellin transform

$$F(x) = e^{\frac{x}{2}} \sum a_n e^{-\frac{2\pi}{\sqrt{N}} e^x n} \quad (4)$$

satisfies

$$F(x) = \varepsilon \overline{F(-x)}. \quad (5)$$

In particular for all  $0 < y < \frac{\pi}{2}$ ,  $F(iy) - \varepsilon F(-iy) = 0$ .

Let  $t = e^{iy\frac{k}{2}}$ ,  $q = e^{-\frac{2\pi}{\sqrt{N}} e^{iy}}$ , one must have

$$\sum_{n \geq 1} a_n (tq^n - \varepsilon \overline{tq^n}) = 0$$

## Least conductor : generalize

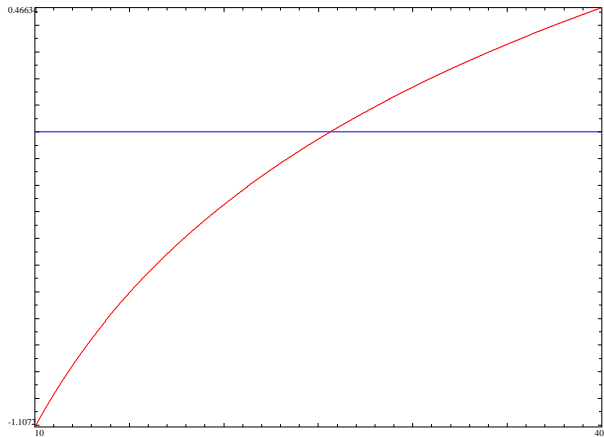
Using

- $|a_n| \leq \lfloor \sigma_0(n) n^{\frac{k-1}{2}} \rfloor \leq \sqrt{3} n^{\frac{k}{2}}$
- $\sum_{n>K} n^{\frac{k}{2}} |q|^n \leq \frac{\sqrt{K+1}^k}{1 - \frac{k}{2\alpha_y(K+1)}} |q|^K = B_K(q)$  if  $2\alpha_y(K+1) > k$

one must have

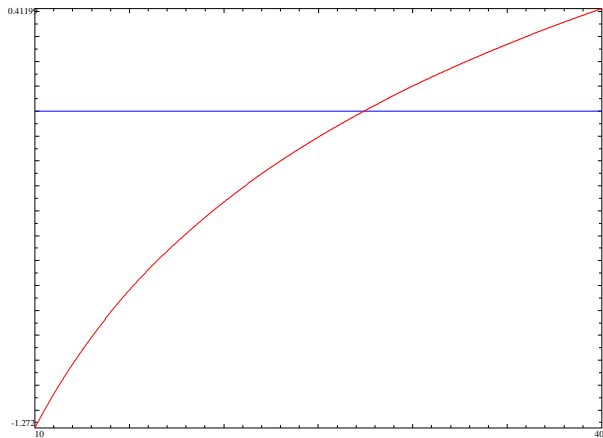
$$|t - \epsilon \bar{t}| \leq \sum_{n=2}^K \lfloor \sigma_0(n) n^{\frac{k-1}{2}} \rfloor |tq^n - \epsilon \bar{t} q^n| + 2\sqrt{3} B_K(q)$$

## Odd elliptic curves



$$y = 0.1 \Rightarrow N \geq 25.63$$

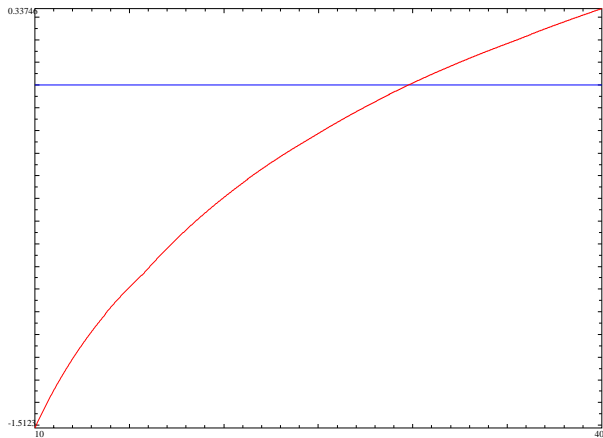
## Odd elliptic curves



$$y = 0.2 \Rightarrow N \geq 27.40$$

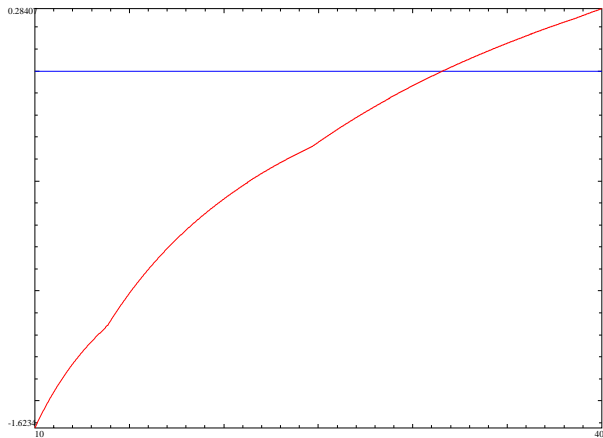


## Odd elliptic curves



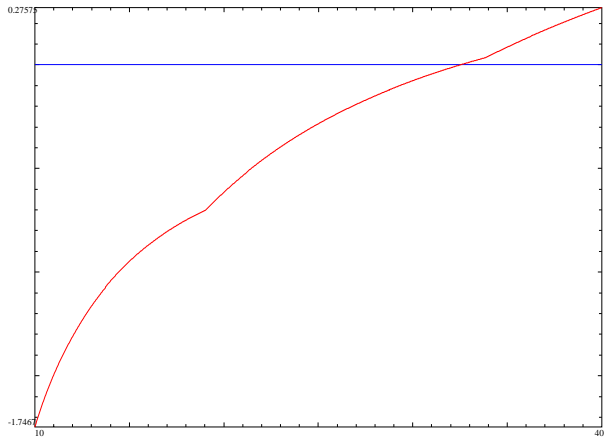
$$y = 0.3 \Rightarrow N \geq 29.77$$

## Odd elliptic curves



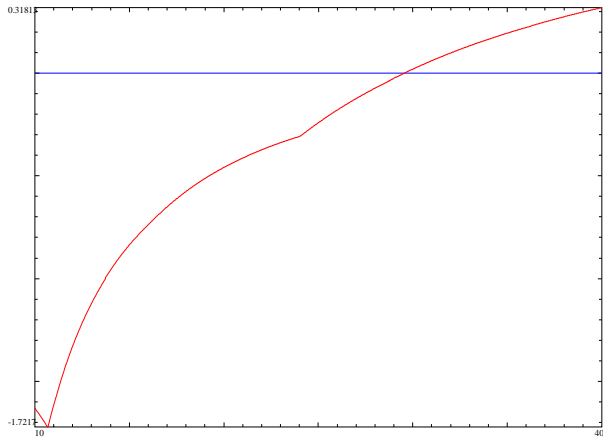
$$y = 0.4 \Rightarrow N \geq 31.58$$

## Odd elliptic curves



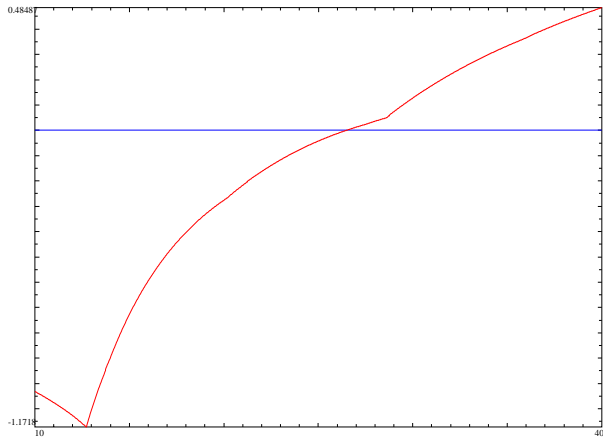
$$y = 0.5 \Rightarrow N \geq 32.53$$

## Odd elliptic curves



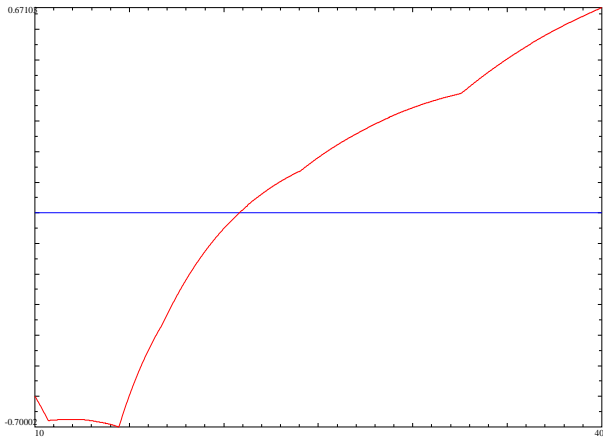
$$y = 0.6 \Rightarrow N \geq 29.57$$

## Odd elliptic curves



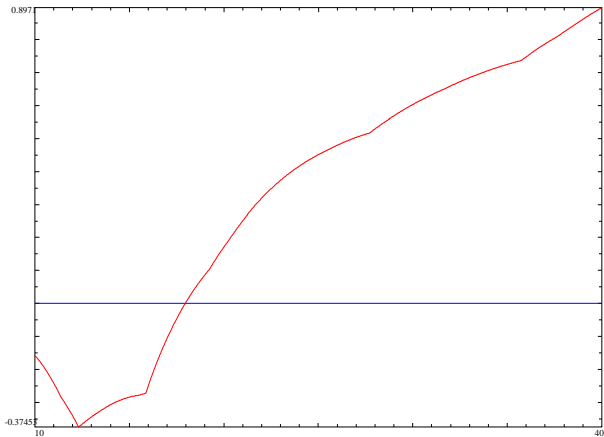
$$y = 0.7 \Rightarrow N \geq 26.48$$

## Odd elliptic curves



$$y = 0.8 \Rightarrow N \geq 20.81$$

## Odd elliptic curves



$$y = 0.9 \Rightarrow N \geq 17.95$$

- Equation  $F(0) = 0 \rightsquigarrow N \geq 27$ .
- Equation  $F(iy) + F(-iy) = 0, y = .5 \rightsquigarrow N \geq 33$

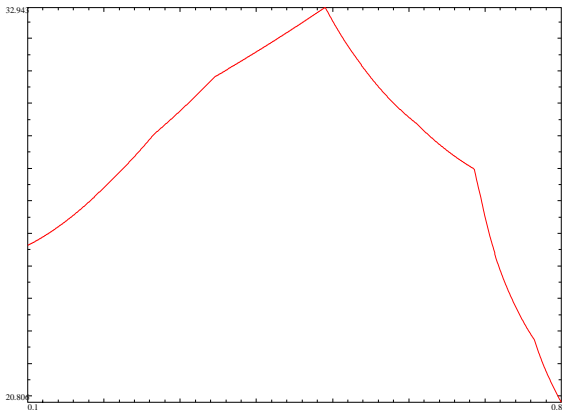
### Theorem

*If an elliptic curve has rank  $r \geq 1$ , its level satisfies  $N \geq 33$ .*



## Automatic results

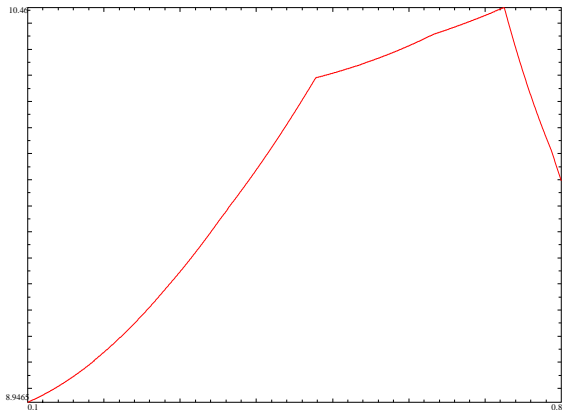
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 2, \varepsilon = -1$$

## Automatic results

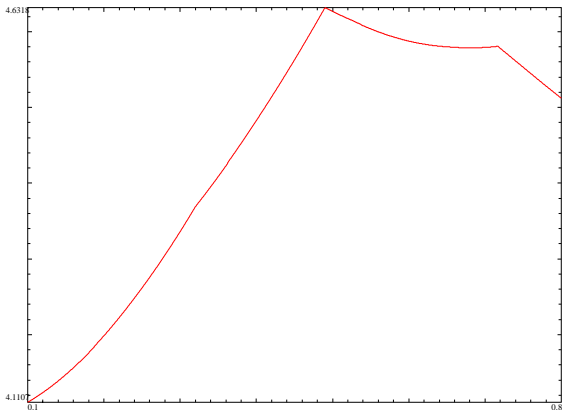
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 2, \varepsilon = +1$$

## Automatic results

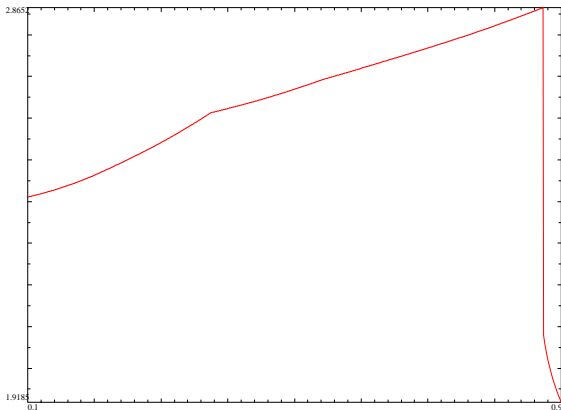
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 4, \varepsilon = +1$$

## Automatic results

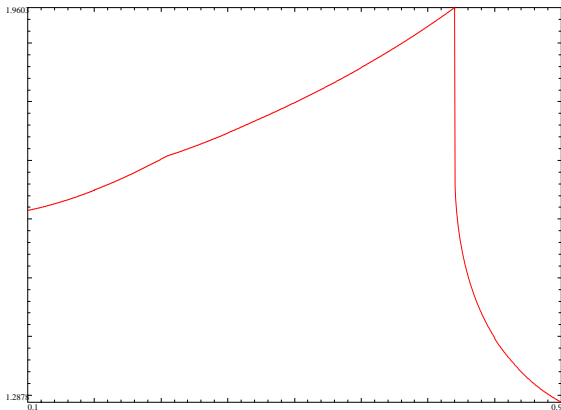
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 6, \varepsilon = +1$$

## Automatic results

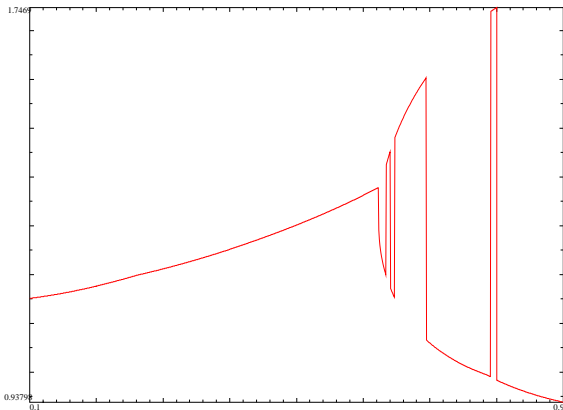
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 8, \varepsilon = +1$$

## Automatic results

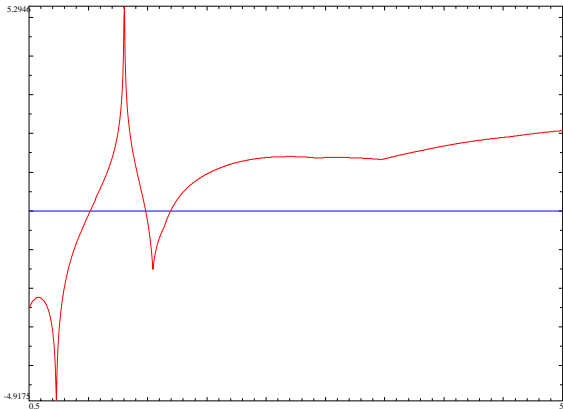
Plot on  $y$  for the least value of  $N$  satisfying inequality.



$$k = 10, \varepsilon = +1$$

## Bifurcation

Bifurcation diagram : the ratio is not monotonous and crosses the 1-axis several times.



Still the result  $N > 1.7$  is correct, combining several plots.

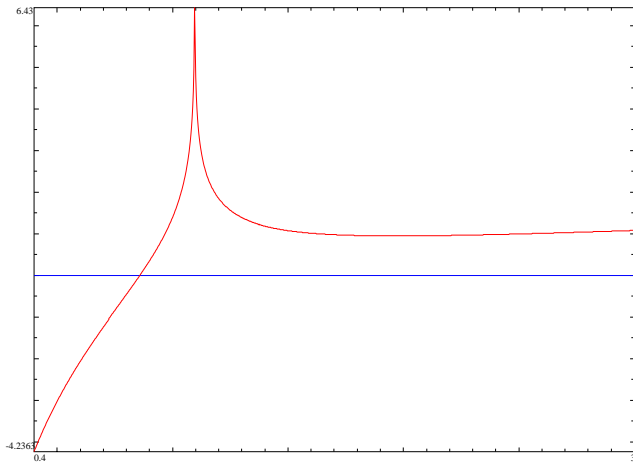
## Results for degree 2

weight  $k$  and level  $N$ , root number  $\epsilon$ .

$k$	$\epsilon$	bound	LMFDB object
2	1	$N > 10.45$	11a
	-1	$N > 32.95$	37a
4	1	$N > 4.63$	5.4.1a
	-1	$N > 12.24$	13.4.1a
6	1	$N > 2.85$	3.6.1a
	-1	$N > 6.55$	7.6.1b
8	1	$N > 1.95$	2.8.1a
	-1	$N > 4.09$	(server error)
10	1	$N > 1.37$	2.10.1a
	-1	$N > 2.75$	3.10.1b
12	1	*	$\Delta_\tau$
	-1	$N > 1.97$	4.12.1a

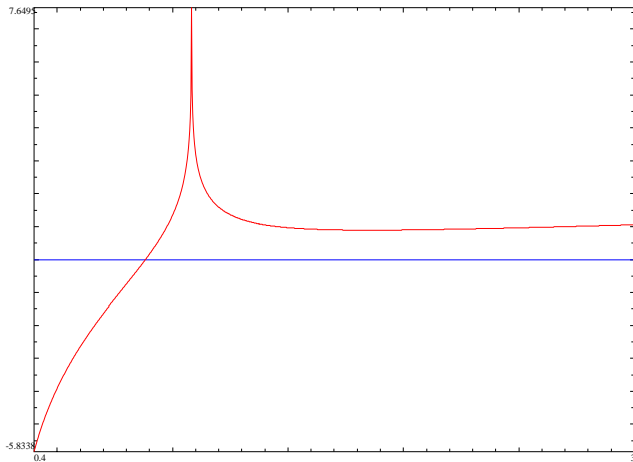


## $k=12$ : Ramanujan $\Delta$ function



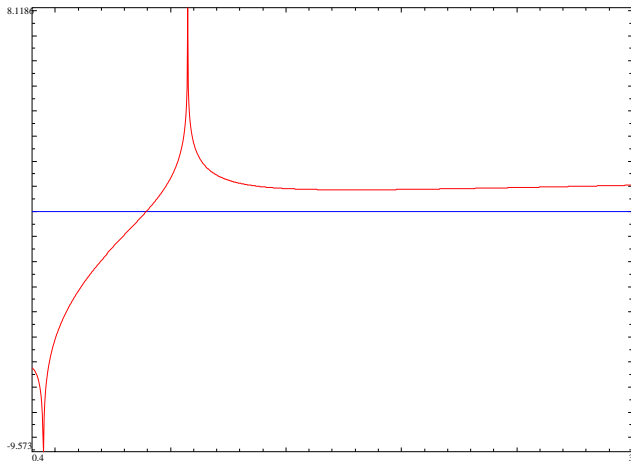
$$x = 0.05i, N \geq 0.86$$

## $k=12$ : Ramanujan $\Delta$ function



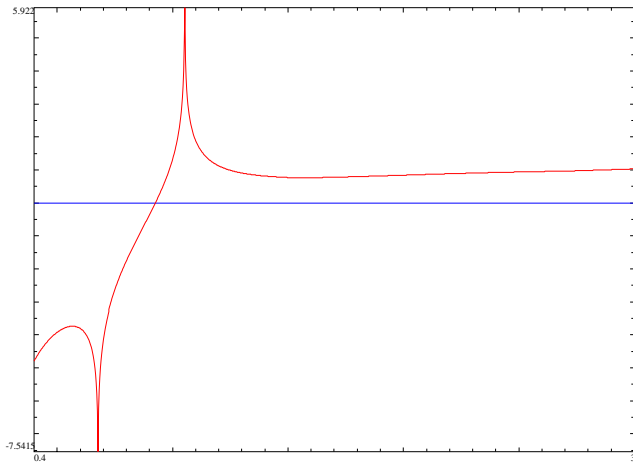
$$x = 0.20i, N \geq 0.88$$

## $k=12$ : Ramanujan $\Delta$ function



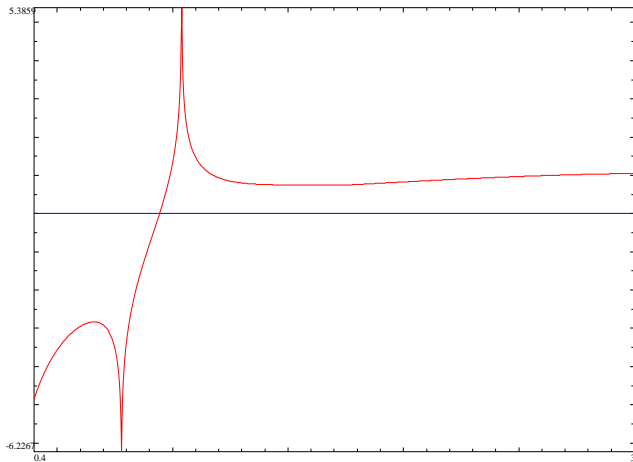
$$x = 0.25i, N \geq 0.90$$

## $k=12$ : Ramanujan $\Delta$ function



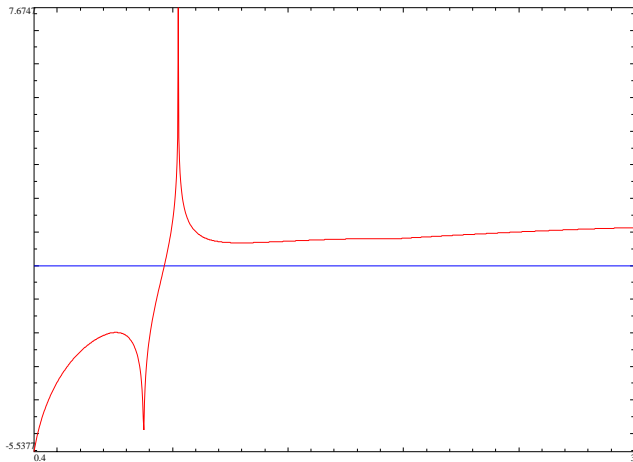
$$x = 0.35i, N \geq 0.93$$

## $k=12$ : Ramanujan $\Delta$ function



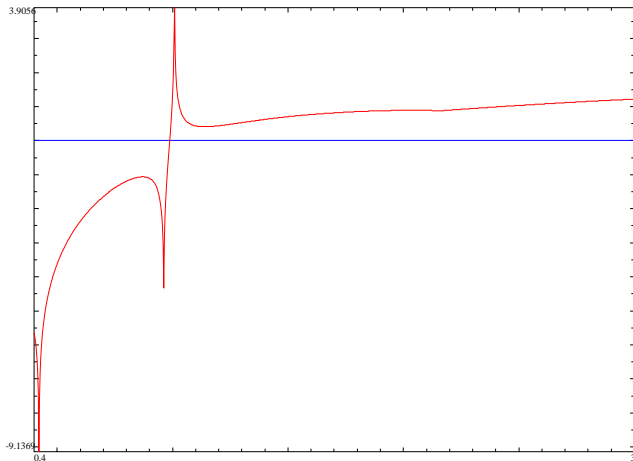
$$x = 0.40i, N \geq 0.94$$

## $k=12$ : Ramanujan $\Delta$ function



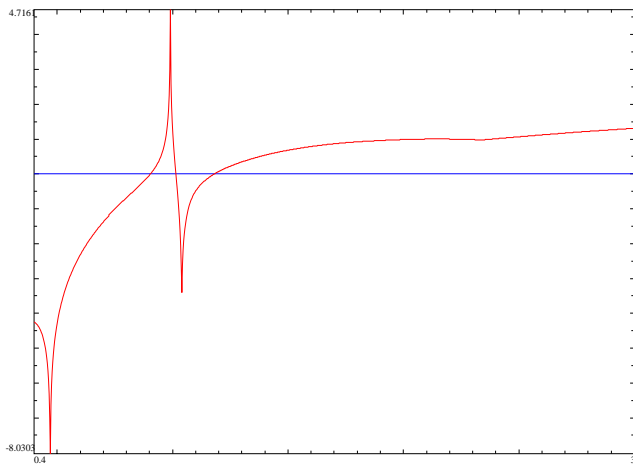
$$x = 0.45i, N \geq 0.96$$

## $k=12$ : Ramanujan $\Delta$ function



$$x = 0.50i, N \geq 0.99$$

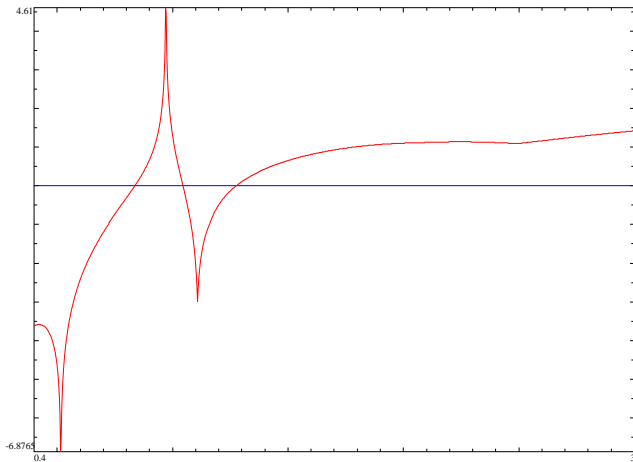
## $k=12$ : Ramanujan $\Delta$ function



$x = 0.55i, 0.90 \leq N \leq 1.01$  or  $N \geq 1.18$

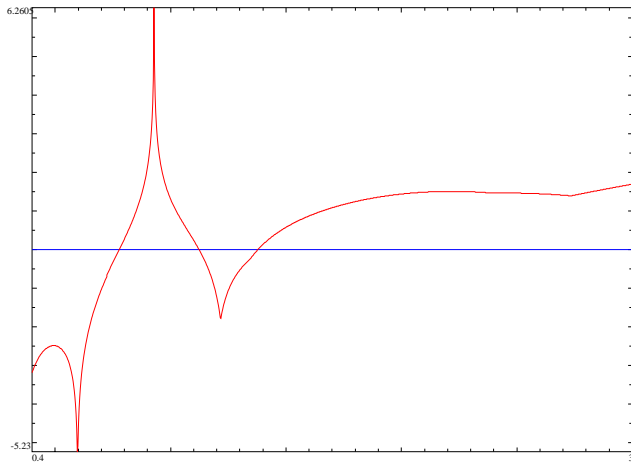


## k=12 : Ramanujan $\Delta$ function



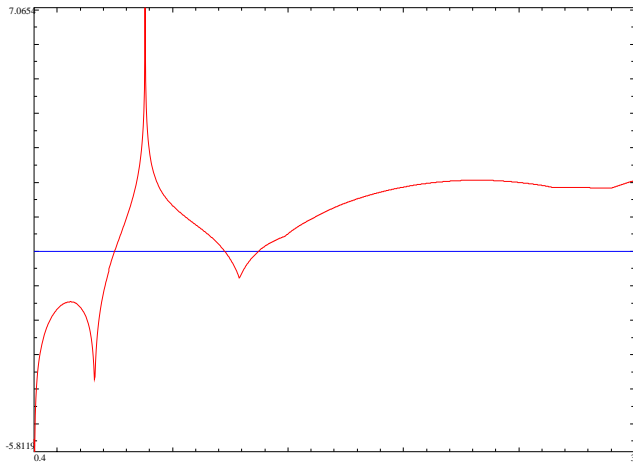
$$x = 0.60j, 0.84 \leq N \leq 1.04 \text{ or } N \geq 1.27$$

## $k=12$ : Ramanujan $\Delta$ function



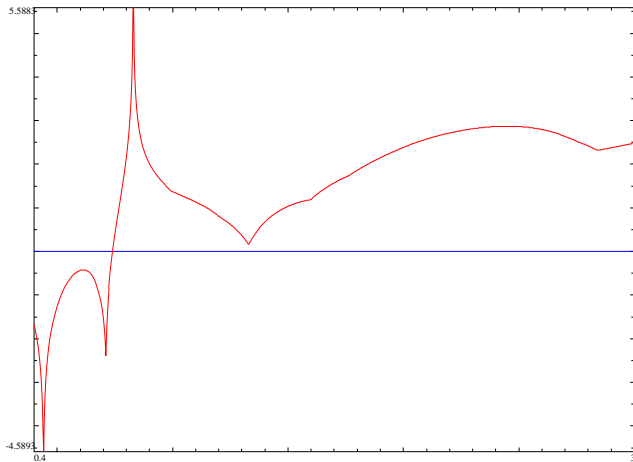
$$x = 0.70i, 0.78 \leq N \leq 1.12 \text{ or } N \geq 1.37$$

## $k=12$ : Ramanujan $\Delta$ function



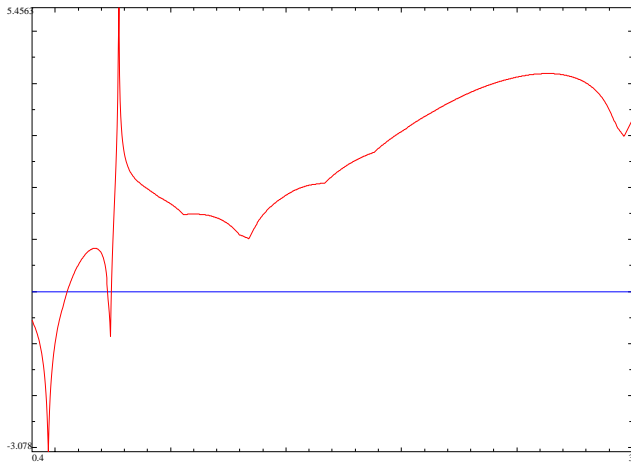
$$x = 0.80i, N \geq 0.75$$

## $k=12$ : Ramanujan $\Delta$ function



$$x = 0.90i, N \geq 0.74$$

## $k=12$ : Ramanujan $\Delta$ function



$$x = 1.00i, N \geq 0.55$$

## $k=12$ : Ramanujan $\Delta$ function

Sounds like this function *should not* exist.

Cannot for  $N < .9999$ , nor for  $1.00001 < N < 1.37$ .

(still the min ratio for  $N = 1$  is 1.3, so the coefficients of  $\Delta$  are not forced to reach their bounds.)

## General L functions

- Dirichlet series  $L(s) = \sum_{n \geq 1} a_n n^{-s}$
- gamma factor of level  $N$  and degree  $d$

$$\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^d \Gamma_{\mathbb{R}}(s + \lambda_j)$$

- functional equation of weight  $k$

$$\Lambda(s) = L(s)\gamma(s) = \epsilon \overline{\Lambda}(k - s)$$

- Ramanujan bound  $a_n \leq \sigma_0(n)^{d-1} n^{\frac{k-1}{2}}$
- $\Lambda$  meromorphic. Here assume holomorphic.

## Theta equations

### Fourier form

$$\Lambda(s) = \epsilon \bar{\Lambda}(k - s)$$

if and only if for all  $x \in \mathbb{R}_+ ] - \frac{d\pi}{4}, \frac{d\pi}{4} [$ ,

$$F(x) = \epsilon \bar{F}(-x)$$

where

$$F(x) = e^{\frac{k}{2}x} \sum_n a_n \mathcal{M}^{-1} [\gamma(s); e^x n] \quad (6)$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \Lambda\left(\frac{k}{2} + it\right) e^{ixt} dt \quad (7)$$



## Inverse Mellin transforms

$$\mathcal{M}^{-1}[\gamma; x] \left| \begin{array}{l} \gamma(s) \\ N^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s) \\ e^{-\frac{\pi}{N}x^2} \end{array} \right| \left| \begin{array}{l} N^{\frac{s}{2}} \Gamma_{\mathbb{C}}(s) \\ e^{-\frac{2\pi}{\sqrt{N}}x} \end{array} \right| \left| \begin{array}{l} N^{\frac{s}{2}} \Gamma_{\mathbb{C}}(s) \Gamma_{\mathbb{C}}(s + \nu) \\ \text{Bessel } K_{\nu} \end{array} \right|$$

More gamma factors (real shifts) now available in Pari/gp

```
g=gammamellininvinit([0,0,1,1,1])  
gammamellininv(g,x)
```

**Conclusion** For any L function, easy to produce equations  $\sum_n a_n x_n$  satisfied by the Dirichlet series, with  $x_n \rightarrow 0$  exponentially.

## Results for weight $k=1$

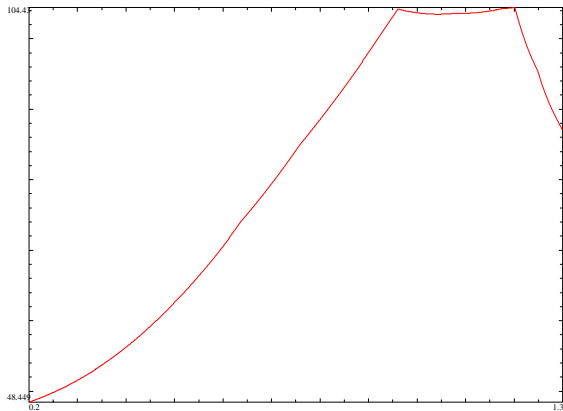
### Theorem (Smallest discriminants of number fields)

Let  $K/\mathbb{Q}$  be a number field of signature  $r, s$ .

- if  $r, s = 3, 0$ , then  $|\Delta| \geq 25$  ;
- if  $r, s = 1, 1$ , then  $|\Delta| \geq 15$  ;
- if  $r, s = 4, 0$ , then  $|\Delta| \geq 105$  ;
- if  $r, s = 2, 1$ , then  $|\Delta| \geq 64$  ;
- if  $r, s = 0, 2$ , then  $|\Delta| \geq 40$  ;
- if  $r, s = 5, 0$ , then  $|\Delta| \geq 356$ .

## "Proof" (case 4,0)

$\zeta_K^*(s) = \zeta_K(s)/\zeta(s)$  is degree 4, weight 1, holomorphic L function, with  $\gamma(s) = N^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^4$ .



## Example : weight k=1

### First weight 1 L functions

gamma	N	formulas
[0]	5, 8, 12, ...	$\left(\frac{5}{\cdot}\right), \left(\frac{8}{\cdot}\right), \left(\frac{12}{\cdot}\right), \dots$
[1]	3, 4, 7, ...	$\left(\frac{-3}{\cdot}\right), \left(\frac{-4}{\cdot}\right), \left(\frac{-7}{\cdot}\right), \dots$
[0, 0]	25, 40, 49, ...	$\left(\frac{5^2}{\cdot}\right), \left(\frac{5 \times 8}{\cdot}\right), \zeta_{x^3-x^2-2*x+1}^*$ , ...
[0, 1]	15, 20, 23, ...	$\left(\frac{-3 \times 5}{\cdot}\right), \left(\frac{-4 \times 5}{\cdot}\right), \zeta_{x^3-x^2+1}^*$ , ...
[1, 1]	9, 12, 16, ...	$\left(\frac{(-3)^2}{\cdot}\right), \left(\frac{-3 \times -4}{\cdot}\right), \left(\frac{-4 \times -4}{\cdot}\right), \dots$
[0, 0, 0]	125, 200, 245, ...	$\left(\frac{5^3}{\cdot}\right), \left(\frac{5^2 \times 8}{\cdot}\right), \left(\frac{5}{\cdot}\right) \zeta_{x^3-x^2-2*x+1}^*$ , ...
[0, 0, 1]	75, 100, 115, ...	$\left(\frac{-3 \times 5^2}{\cdot}\right), \left(\frac{-4 \times 5^2}{\cdot}\right), \left(\frac{5}{\cdot}\right) \zeta_{x^3-x^2+1}^*$ , ...
[0, 1, 1]	45, 60, 69, ...	$\left(\frac{(-3)^2 \times 5}{\cdot}\right), \left(\frac{-3 \times -4 \times 5}{\cdot}\right), \left(\frac{-3}{\cdot}\right) \zeta_{x^3-x^2+1}^*$ , ...
[1, 1, 1]	27, 36, 48, ...	$\left(\frac{(-3)^3}{\cdot}\right), \left(\frac{(-3)^2 \times -4}{\cdot}\right), \left(\frac{-3 \times (-4)^2}{\cdot}\right), \dots$

## Enumerate Dirichlet series

Could play same game testing values of  $a_2$ ...

Or even find inductively all Dirichlet series numerically compatible with some approximate functional equation

$\sum_n a_n x_n = 0$  and bounds  $|a_n| \leq b_n$ .

- $a_1 = 1$
- $a_2 \in [-b_2, b_2]$  s.t.

$$|(a_1 x_1) + a_2 x_2| \leq B_2 = b_3 |x_3| + b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + \dots$$

- $a_3 \in [-b_3, b_3]$  s.t.

$$|(a_1 x_1 + a_2 x_2) + a_3 x_3| \leq B_3 = b_4 |x_4| + b_5 |x_5| + b_6 |x_6| + b_7 |x_7| + \dots$$

## Enumerate Dirichlet series

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- $a_3 \in [-b_3, b_3]$  s.t.

$$|(a_1 x_1 + a_2 x_2) + a_3 (x_3 + a_2 x_6)| \leq B_3 = b_4 |x_4| + b_5 |x_5| + \dots + b_7 |x_7| + \dots$$

## More structure on Dirichlet series

- Euler product  $L(s) = \prod F_p(p^{-s})^{-1}$
- Euler factor  $F_p(T) = 1 + c_{p,1}T + \dots T^d$
- local functional equation

$$c_{p,d-j} = \chi(p)p^{\frac{w}{2}(d-2j)}c_{p,j}$$

with  $\chi$  Dirichlet character modulo  $N$

- Ramanujan bounds : using  $|\text{roots}| \leq p^{\frac{w}{2}}$

$$c_{p,j} \leq \binom{d}{j} p^{j\frac{w}{2}}$$

$$a_k \leq k^{\frac{w}{2}} \prod_{p^e \parallel k} \binom{e+d-1}{d-1}$$

## Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients  $c_{p,e}$  for  $e \leq \frac{d}{2}$  (or  $e \leq d - 1$  if  $p \mid N$ ).
- Depth-first search (constant in memory)

## Examples

```
gp > lfunbuild([[[]],1,[0,1],2,11,1],45,[2])
time = 11 ms.
%1 = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
gp > lfunbuild([[[]],1,[0,1],2,12,1],45,[2])
time = 11 ms.
%2 = [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```



## Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients  $c_{p,e}$  for  $e \leq \frac{d}{2}$  (or  $e \leq d - 1$  if  $p \mid N$ ).
- Depth-first search (constant in memory)

## Examples

```
gp> lfunbuild([[[]],1,[0,1],2,26,1],59,[1])
```

```
time = 7 ms.
```

```
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]
```

## Idea

- Explore a tree of Dirichlet series (or local Euler factors)
- Search on Euler coefficients  $c_{p,e}$  for  $e \leq \frac{d}{2}$  (or  $e \leq d - 1$  if  $p \mid N$ ).
- Depth-first search (constant in memory)

## Examples

```
gp> lfunbuild([[1, [0,1], 2, 26, 1], 59, [1])  
time = 7 ms.  
%3 = [1, 3, 3, 3, 2, 26, 4, 15, 5, 2, 2, 2, 2, 12, 2, 2, 2]
```

the choice of equation is still important !

```
gp> lfunbuild([[1, [0,1], 2, 26, 1], 59, [[.42]])  
time = 7 ms.  
%4 = [1, 3, 3, 2, 3, 2, 4, 2, 3, 2, 2, 2, 2, 2, 4, 2, 2]
```

# Program lfunbuild

## Two independant functions

- prepare seach tree
  - compute a family of equations
  - identify search variables, ranges, search levels in the tree
  - craft nice equation for each level
  - compute tails  $B_p$
- prune tree using depth first search (constant memory)
  - start at  $p = 2$
  - solve  $|\text{polynomial}(a_p)| \leq B_p$
  - for each possible value  $a_p$ 
    - propagate value in Dirichlet series
    - recursively descend next level

```
gp > for(N=10,100,print(N," : ",Vec(lfunbuild([[[]],1,[0,1],2,N,1],31,[2])))
```

## Framework

Write  $k \prec p^e$  if  $k$  is  $p$ -smooth and  $p^e \nmid k$ .

Assume  $\{a_k\}$  known for  $k \prec p^e$ . Equation for coefficient  $a_{p^e}$  :

$$\left( \sum_{k \prec p^e} a_k x_k \right) + a_{p^e} \left( \sum_{m \prec p} a_m x_{p^e m} \right) = \sum_{k \succ p^e} a_k x_k$$

Ramanujan bound  $|a_n| \leq b_n$  on the tail

$\rightsquigarrow$  polynomial equation  $|P(a_{p^e})| \leq r$

$\rightsquigarrow$  solve in integers in  $[-b_{p^e}, b_{p^e}]$ .

## Framework

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If  $e \geq \frac{d}{2}$ , by reciprocity of  $F_p(T)$  all  $a_{p^\ell}$  in terms of  $a_{p^j}$ ,  $j \leq e$

$$\left( \sum_{n \prec p^e} a_n x_n \right) + \sum_{\ell \geq e} a_{p^\ell} \left( \sum_{m \prec p} a_m x_{p^\ell m} \right) = \sum_{n \succ p^\infty} a_n x_n$$

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## Good and bad tails

Case of polynomial equation of degree 1 :

$$|S_0 + a_p S_1| \leq B_p = b_{p+} |x_{p+}| + b_{p++} |x_{p++}| + \dots$$

- $|S_1| \geq B_p \Rightarrow$  at most one solution  $a_p$ .
- the ratio  $\frac{B_p}{x_p}$  can be studied a priori
- usually nice at big prime gaps
- can be horrible for twin primes

## Combine equations

Combine approximate functional equations :

$$X_n = (x_{n,1}, \dots, x_{n,r})$$

define  $S = \sum_{n \prec p^e} a_n X_n$ ,  $T = \sum_{m \prec p} a_m X_{p^e m}$ , and  $R = (b_n X_n)_{n \succ p}$ .  
For all  $\lambda \in \mathbb{R}^n$ ,

$$|S \cdot \lambda + a_p T \cdot \lambda| \leq \|R \cdot \lambda\|_1$$

Choose  $\lambda$  to minimize  $\frac{\|R \cdot \lambda\|}{|W \cdot \lambda|}$  : "least absolute deviations".  
Can be solved with iterated + weighted least squares.





## Observations

- Ramanujan  $\Delta$  very easy to build (despite  $k = 12$ ).
- Need many equations to cancel tail

```
gp > lfunbuild([[[]],1,[0,0,1],3,121,1],41,[vector(10,k,k/18)])  
time = 450 ms.  
%4 = [1, 13, 193, 216, 105, 67, 41, 45, 16, 3, 3, 3, 3]
```

otherwise the tree explodes

```
gp > lfunbuild([[[]],1,[0,0,1],3,121,1],41,[4])  
time = 766 ms.  
%5 = [1, 12, 159, 236, 234, 2003, 1933, 1803, 108, 58, 321, 1042, 15121]
```

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- For  $N = 66$ ,  $k = 2$  and central character  $\chi = \left(\frac{-66}{\cdot}\right)$ , a match exists for the 71 first primes, then disappears.  $\left(\frac{-66}{\cdot}\right)$  is trivial up to  $p = 19$ , and  $\pi(19^2) = 72$

# Conclusion

## ToDo

- work still in progress... [bugs, precision]
- optimize equations
- try other sources of equations
- write tree search in ball arithmetic (Arb)

## Goals

- compute interesting examples. Challenge :  $\Gamma_{\mathbb{C}}(s-6)\Gamma_{\mathbb{C}}(s)$ ,  
 $k = 20$ ,  $N = 1$ ,  $a_2 = 0...$
- prove nothing exists outside what is expected
- fill missing bad Euler factors rigorously (e.g. symmetric powers)