The Abel-Jacobi map of complex algebraic curves

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Université Paris 7

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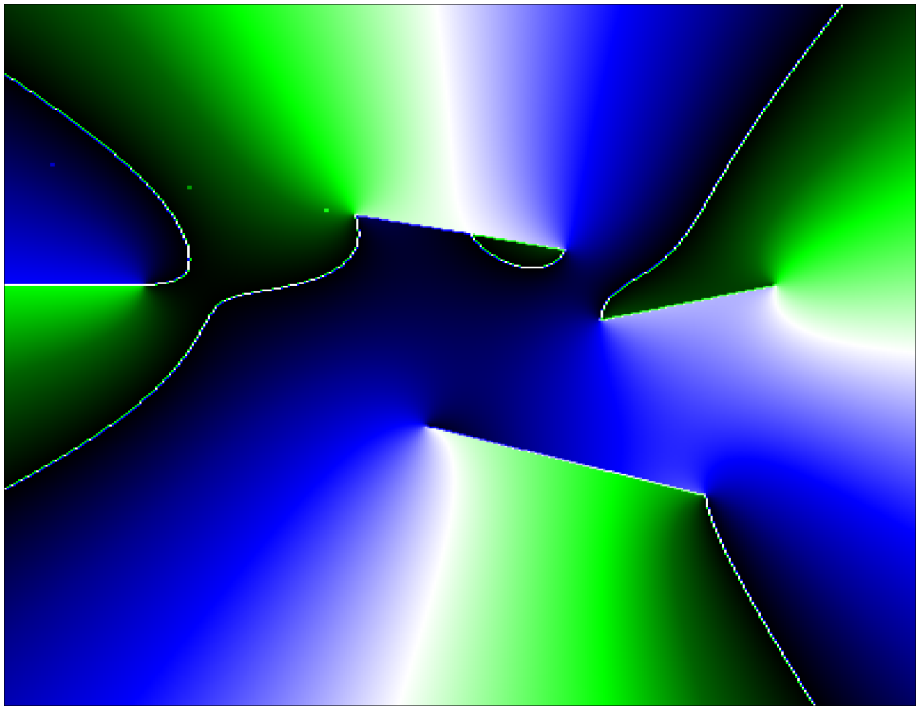
Hyperelliptic curve

$$\mathscr{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i)$$

- a_i = branch points
- one point ∞
- Riemann surface with two sheets
- genus g



color = argument of y

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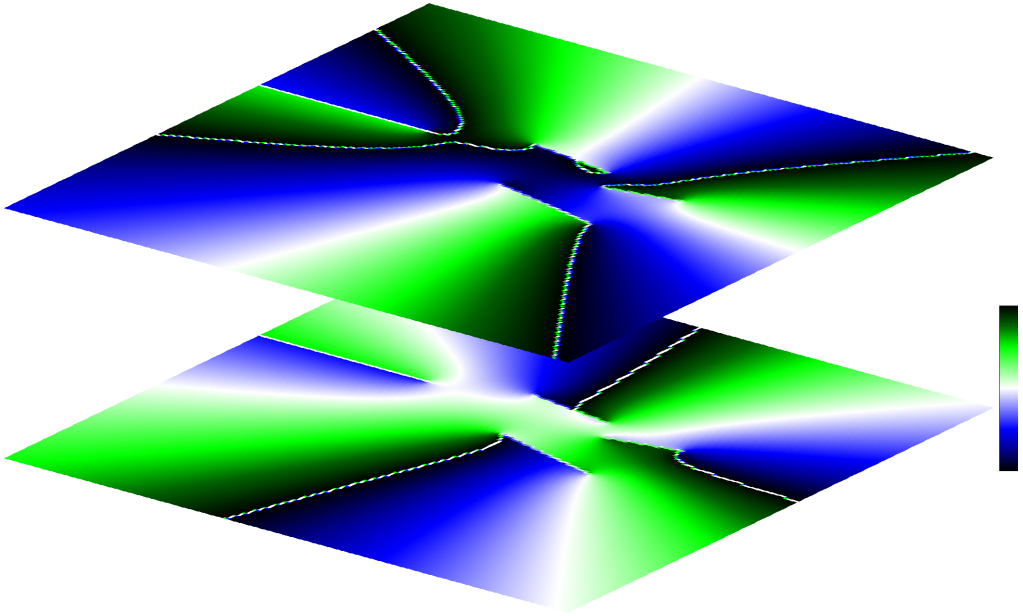

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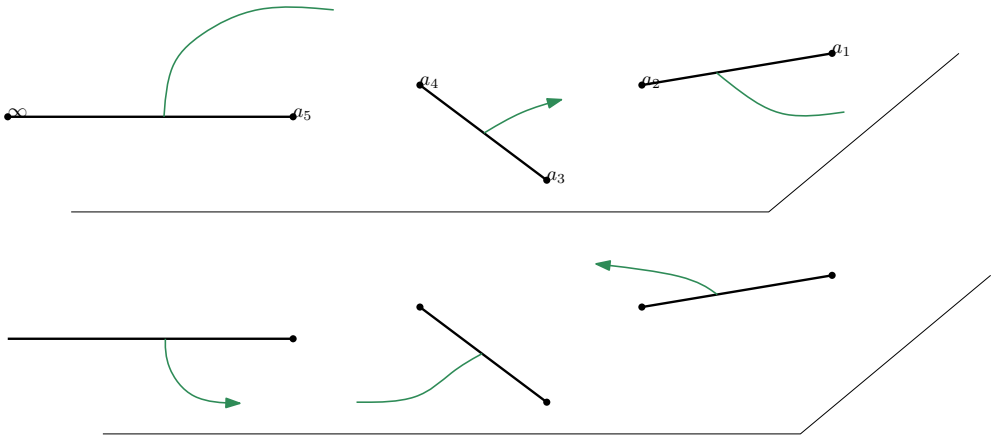

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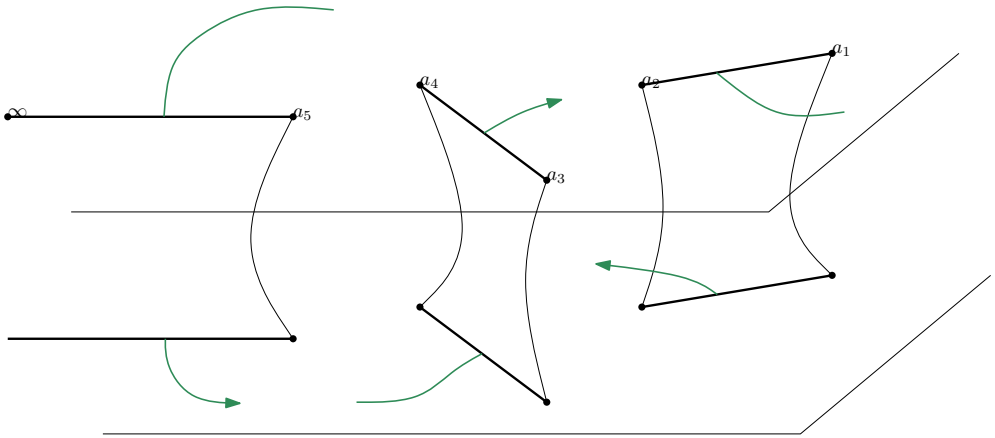

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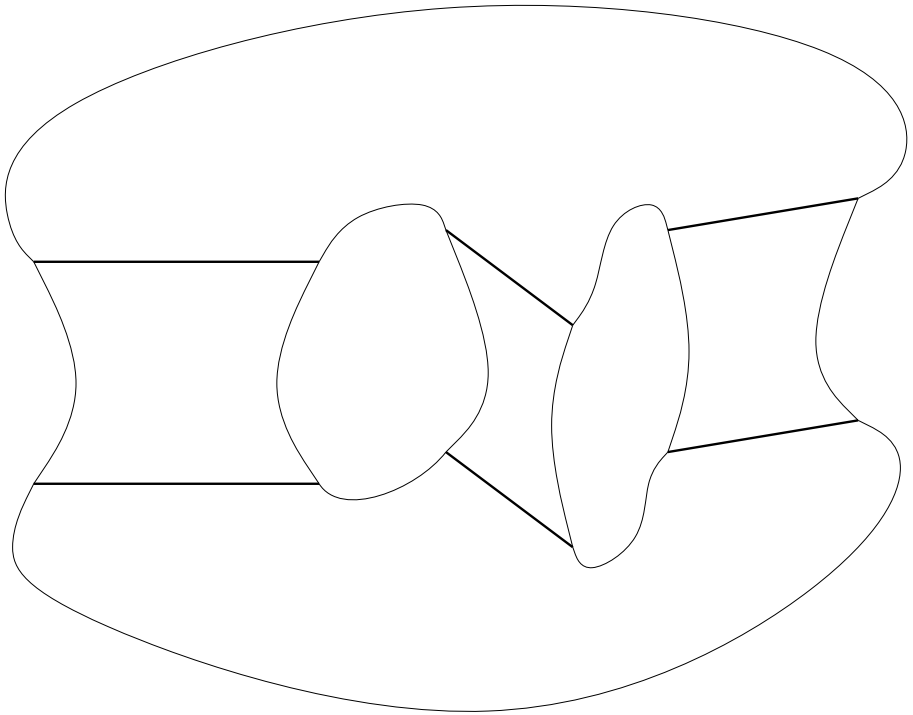

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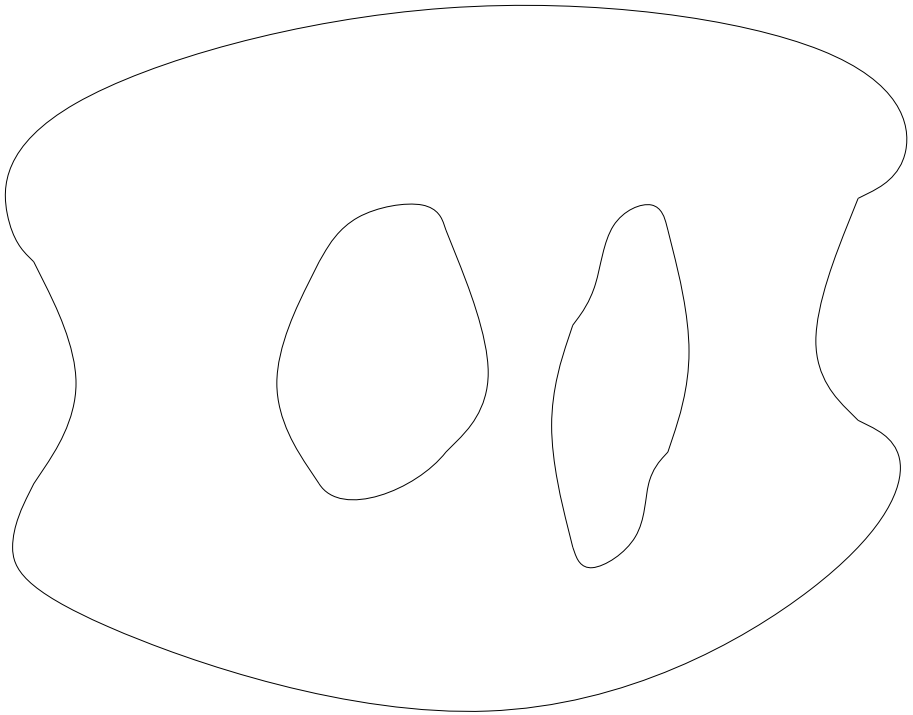

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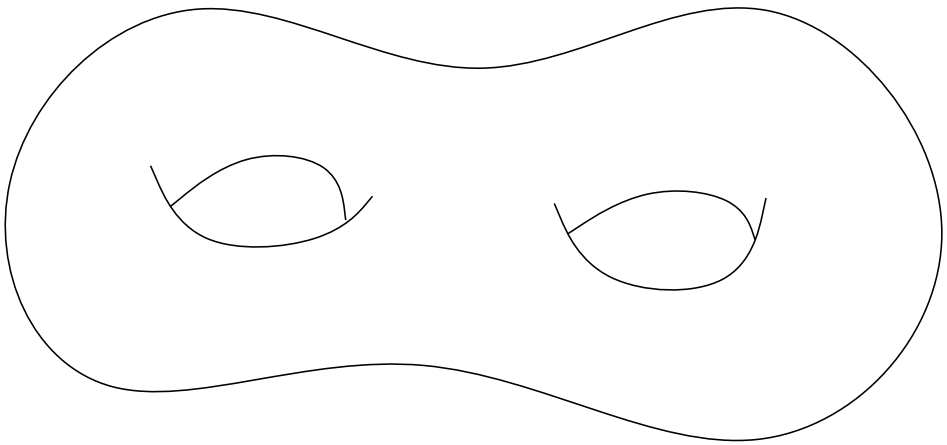

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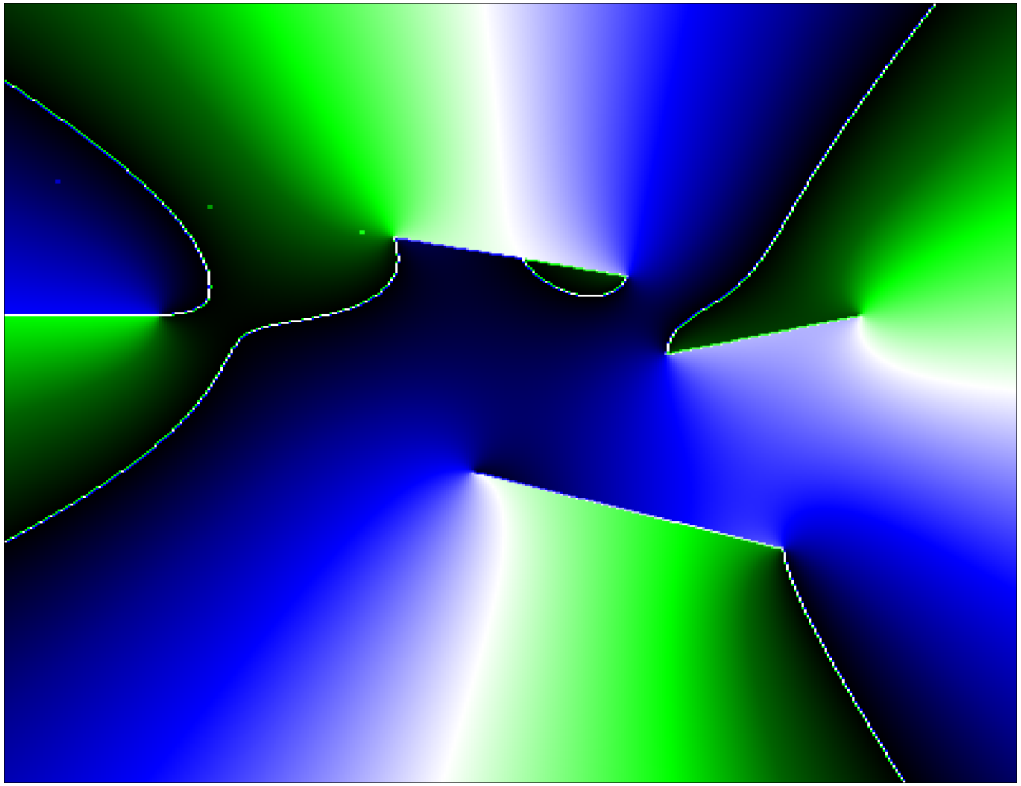

color = argument of y

Abel-Jacobi map

Integration on the curve

$$\mathrm{Int} : P, Q \mapsto \int_P^Q \omega$$

- Holomorphic differentials $\mathscr{H}^1(\mathscr{C}) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g$

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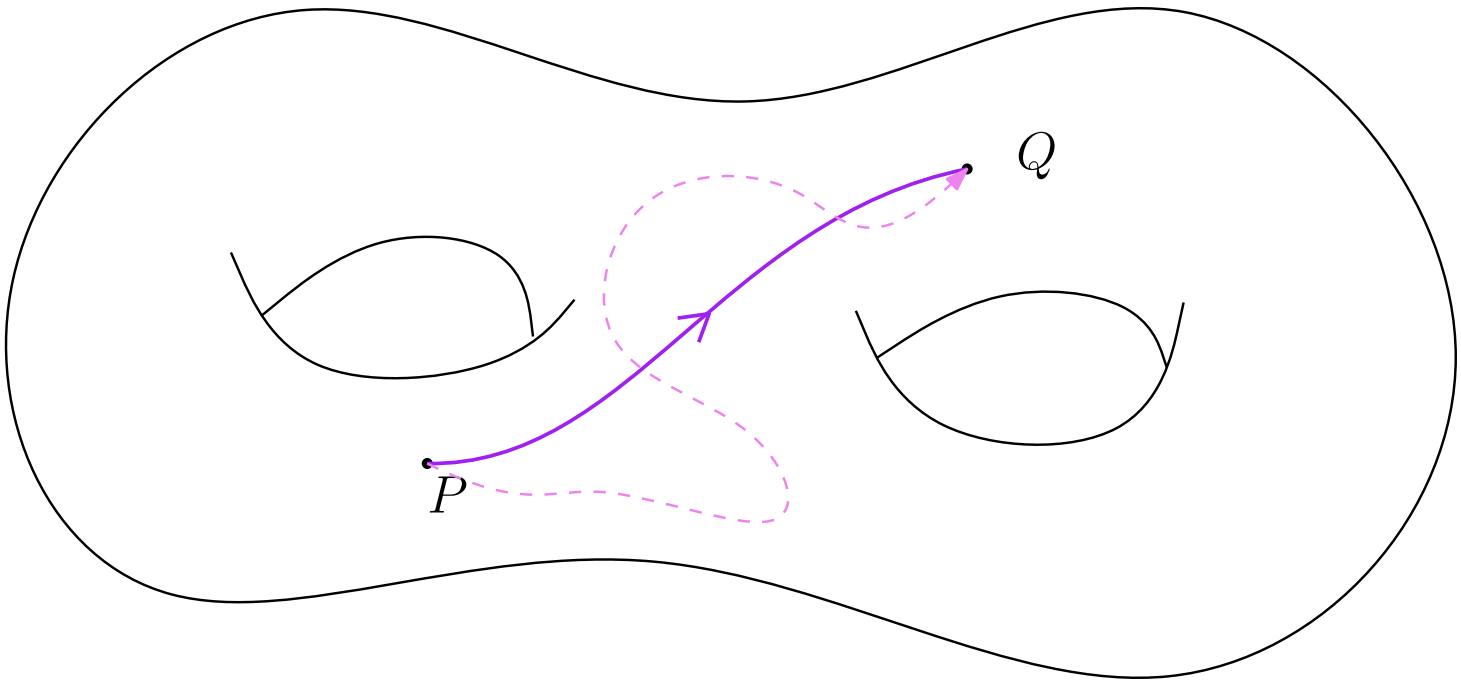

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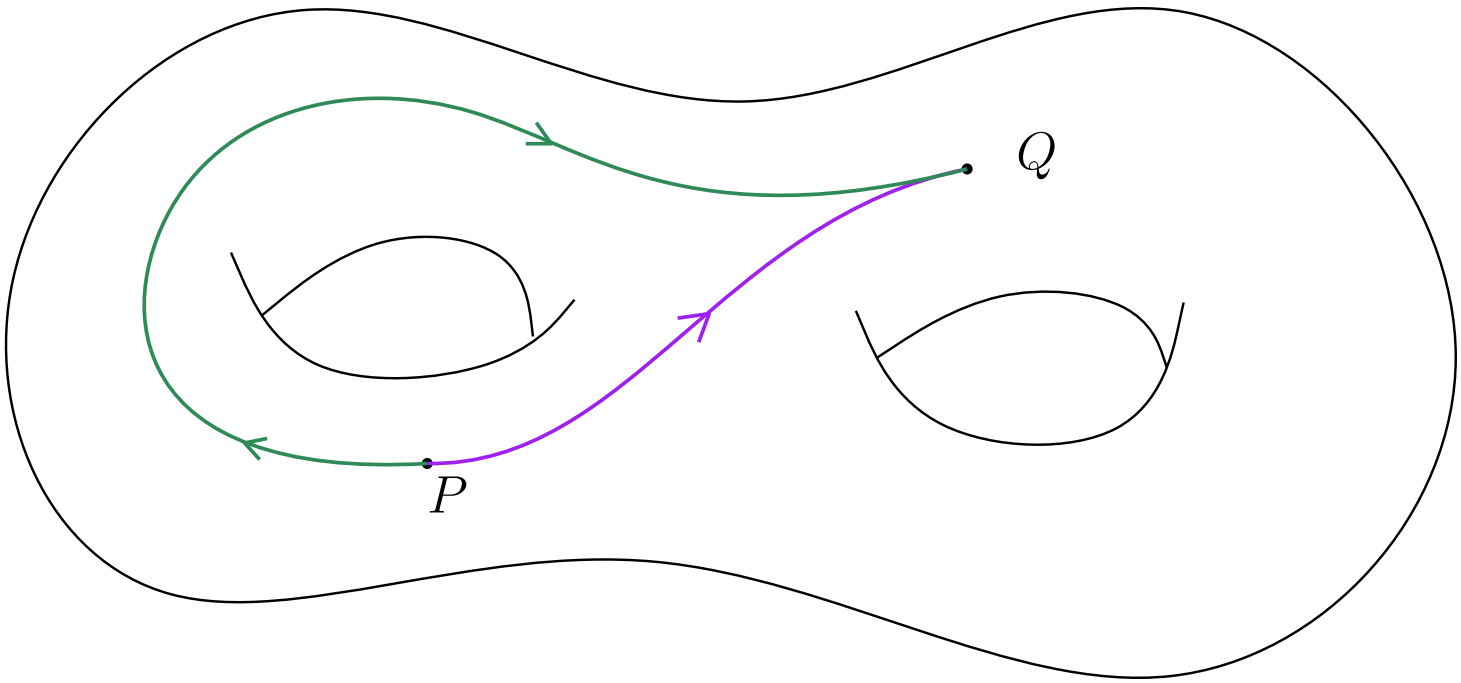

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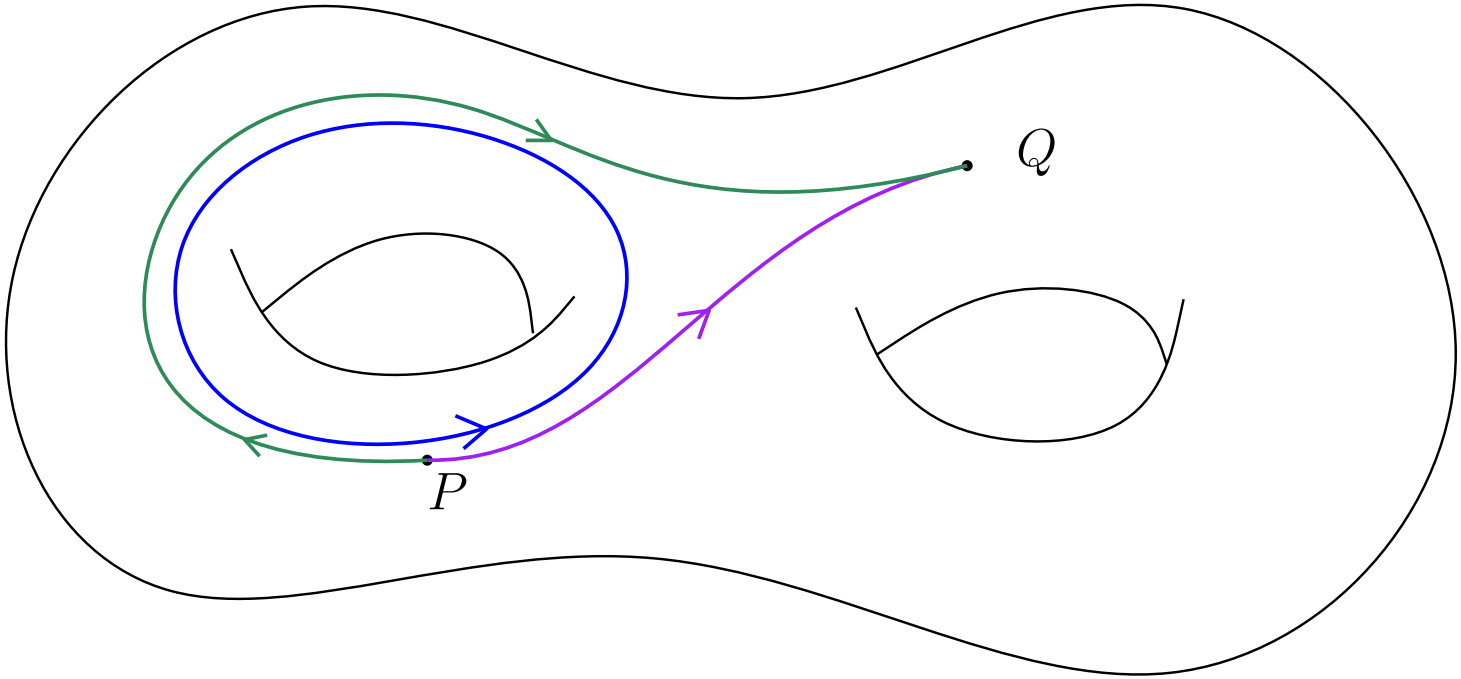

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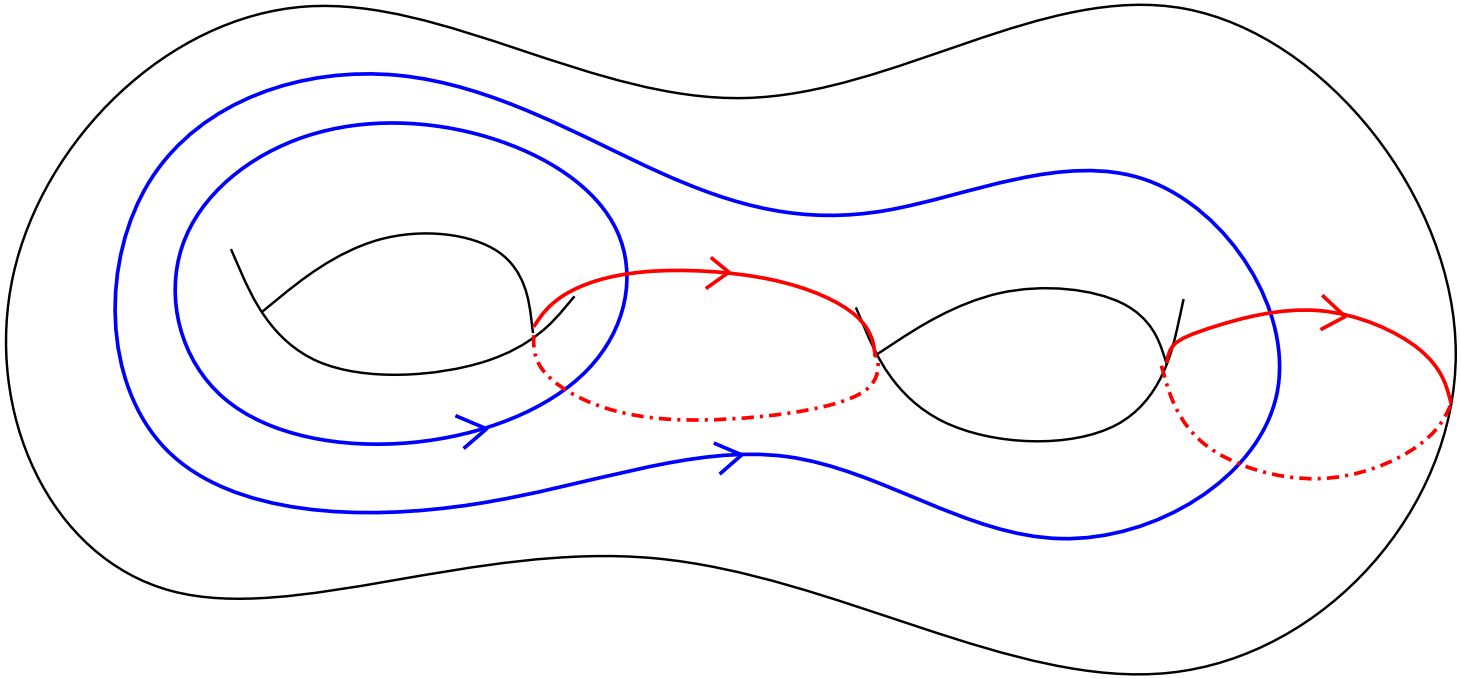

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- Integral defined up to period lattice $\Lambda = \left\{ \int_\gamma \omega_i, \gamma \in H_1, 1 \leqslant i \leqslant g \right\}$.

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Theorem (Abel+Jacobi)

$$0 \to \mathrm{Prin}(\mathscr{C}) \to \mathrm{Div}^0(\mathscr{C}) \to \mathbb{C}^g / \Lambda \to 0$$

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Theorem (Abel+Jacobi)

$$\mathrm{Jac}(\mathscr{C}) \simeq \mathbb{C}^g / \Lambda$$

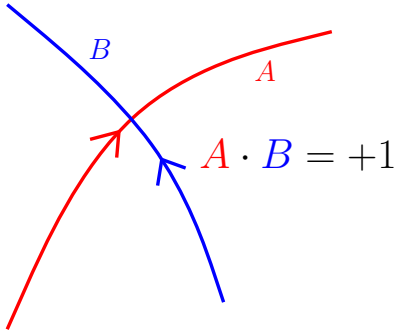
Standard form for the period matrix

- Homology $H_1(\mathscr{C}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g}$
- Holomorphic differentials $\mathscr{H}^1(\mathscr{C}) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g$
- $\Omega =$ period matrix $= \left(\int_{\gamma_i} \omega_j\right) \in M_{g\times 2g}(\mathbb{C})$.

Standard form for the period matrix

- Homology $H_1(\mathscr{C}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g}$

intersection product (antisymetric)
symplectic basis $A_1, \ldots A_g, B_1 \ldots B_g$ such that
$(A_i \cdot B_j = \delta_{i,j})$, $A_i \cdot A_j = B_i \cdot B_j = 0$.



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Standard form for the period matrix

- Homology $H_1(\mathscr{C}) = \mathbb{Z}A_1 \oplus \cdots \oplus \mathbb{Z}B_g$
- Holomorphic differentials $\mathscr{H}^1(\mathscr{C}) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g$ take dual basis ω_i s.t. $\int_{B_j} \omega_i = \delta_{i,j}$
- $\Omega =$ period matrix $= (\tau I_g)$, $\tau = \left(\int_{A_j} \omega_i\right) \in \mathfrak{H}_g$ (Siegel space).

Goals

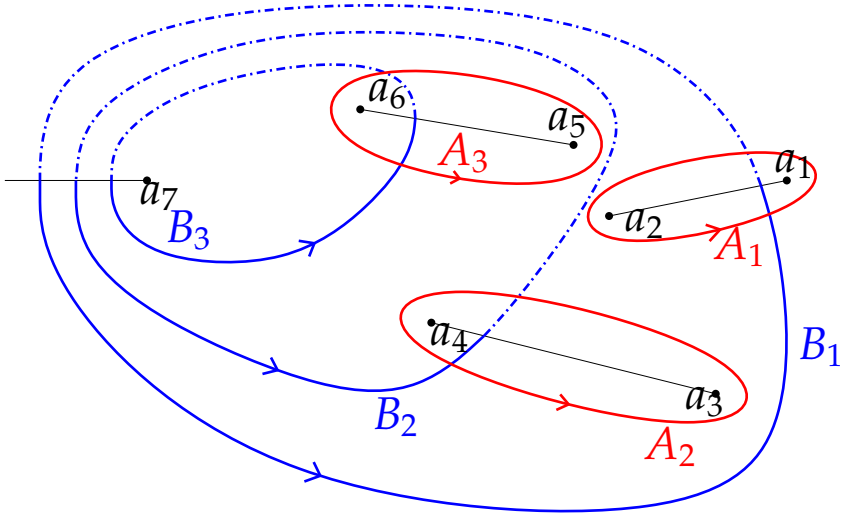
Compute the period lattice Λ and the Abel-Jacobi map to large precision (> 1000 digits).

- recognize endomorphism rings of jacobians (LLL)
- certify a divisor is non principal
- compute torsion divisors
- Riemann theta function...

Hyperelliptic setting

- standard branch cuts
- standard symplectic basis when roots are well ordered
- canonical basis of holomorphic forms

$$\frac{\mathrm{d}x^i}{y}, i = 1 \dots g$$



Need to parametrize and integrate efficiently over the loops.

Section 2

Numerical integration

Gauss-Legendre integration

Theorem

Let $f : [-1, 1] \to \mathbb{C}$ *such that* f *is holomorphic with* $|f| \leqslant M$ *on* $E_r = \{|z-1| + |z+1| < 2\cosh(r)\}$. *For all* $D > 0$ *there exists*

$$N \sim \frac{D + \log(5M)}{r \log 4} \text{ s.t. } \left| \int_{-1}^{1} f(x)\,\mathrm{d}x - \sum_{k=1}^{N} w_k f(x_k) \right| \leqslant e^{-D}$$

where x_k, w_k *are given by*

$$P_n(x_k) = 0 \text{ and } w_k = \frac{-2}{(n+1)P_n'(x_k)P_{n+1}(x_k)} \tag{1}$$

and P_n *are Legendre polynomials*

$$\begin{cases} P_0(x) = 0, P_1(x) = 1 \\ (k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x) \end{cases} \tag{2}$$

Gauss-Chebyshev integration

Theorem

Let $f : [-1, 1] \to \mathbb{C}$ *such that* f *is holomorphic with* $|f| \leqslant M$ *on* $E_r = \{|z-1| + |z+1| < 2\cosh(r)\}$. *For all* $D > 0$ *there exists*

$$N \sim \frac{D + \log(2\pi M)}{r \log 4} \text{ s.t. } \left| \int_{-1}^{1} f(x) \frac{\mathrm{d}x}{\sqrt{1-x^2}} - \sum_{k=1}^{N} w_k f(x_k) \right| \leqslant e^{-D}$$

where x_k, w_k *are given by*

$$x_k = \cos\left(\frac{(2k+1)\pi}{N}\right) \text{ and } w_k = \frac{\pi}{N} \tag{3}$$

Double-exponential integration 1

Theorem

Assume $f :]-1,1[\to \mathbb{C}$ *is holomorphic with* $|f| < M$ *on* $Z_r = \{\tanh(\sinh \mathbb{R} \pm it), t < r\}$. *For all* $D > 0$, *there exist explicit* $N, h > 0$ *such that*

$$N \sim \frac{(D + \log M)\log D}{\pi r}$$

and

$$\left| \int_{-1}^{1} f - \sum_{k=-N/2}^{N/2} f(z_k)\,\mathrm{d}z_k \right| \leqslant e^{-D}$$

where

$$\begin{cases} z_k = \tanh(\sinh(kh)) \\ \mathrm{d}z_k = \frac{h\cosh(kh)}{\cosh^2(\sinh(kh))} \end{cases} \tag{4}$$

Double-exponential integration 2

Theorem

Assume $f :]-1,1[\to \mathbb{C}$ *is holomorphic with* $|f| < M$ *on* $Z_r = \{\tanh(\sinh \mathbb{R} \pm it), t < r\}$. *For all* $D > 0$, *there exist explicit* $N, h > 0$ *such that*

$$N \sim \frac{(D + \log M) \log D}{\pi r}$$

and

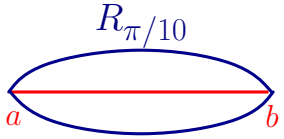
$$\left| \int_{-1}^{1} \frac{f}{\sqrt{1-x^2}} - \sum_{k=-N/2}^{N/2} f(z_k)\, \mathrm{d}z_k \right| \leqslant e^{-D}$$

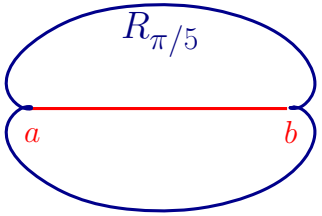
where

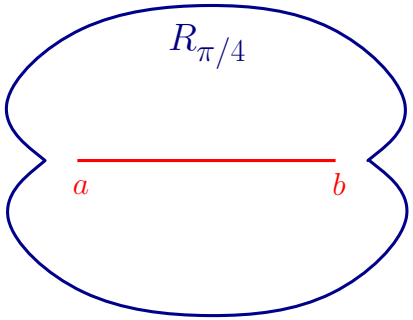
$$\begin{cases} z_k = \tanh(\sinh(kh)) \\ \mathrm{d}z_k = \frac{h \cosh(kh)}{\cosh(\sinh(kh))} \end{cases} \tag{5}$$

Convergence of DE integration

Dependency in the holomorphy domain Z_r :







$$N \sim D \log D$$

$$N \sim \frac{D \log D}{2}$$

$$N \sim \frac{2D \log D}{5}$$

Summary

Integral of to D digits.

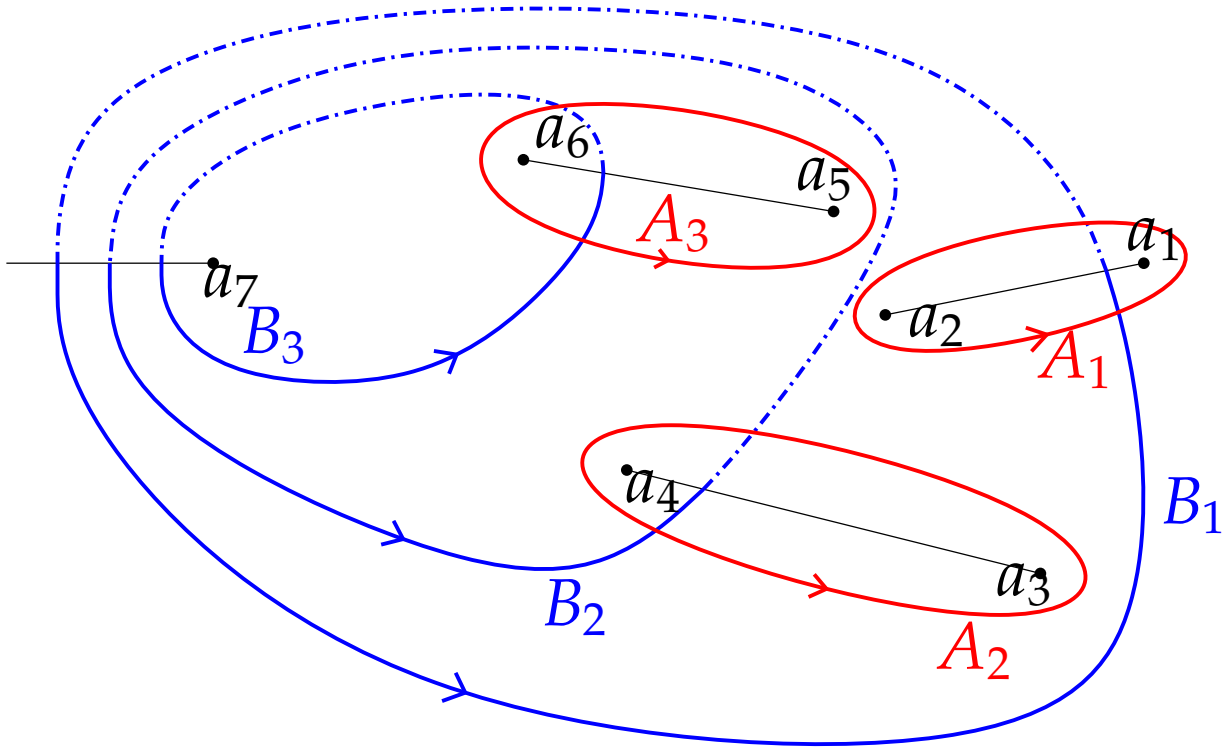
method	Gauss-Legendre	Gauss-Chebyshev	DE
# points	D	D	$D \log D$
cost points	$D^2 \log^3 D$	$D^2 \log D$	$D^2 \log D$
cost integral	$D^2 \log^2 D$	$D^2 \log D$	$D^2 \log^3 D$
holomorphic	✓✓	✗	✓
holomorphic/$\sqrt{1-x^2}$	✗	✓✓	✓
whatever	✗	✗	✓

Section 3

Computing the periods

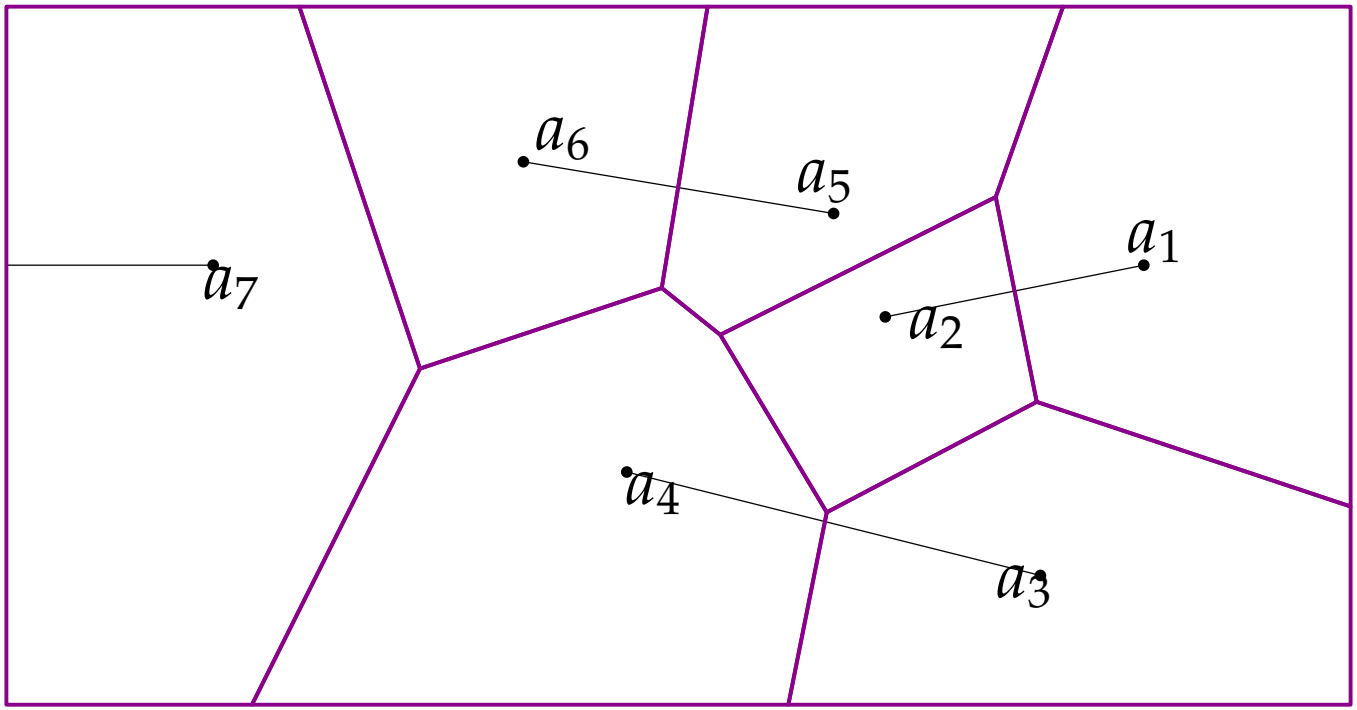
Van Wamelen approach

Integrate far from branch points a_i



Van Wamelen approach

Integrate far from branch points $a_i \to$ Voronoi diagram



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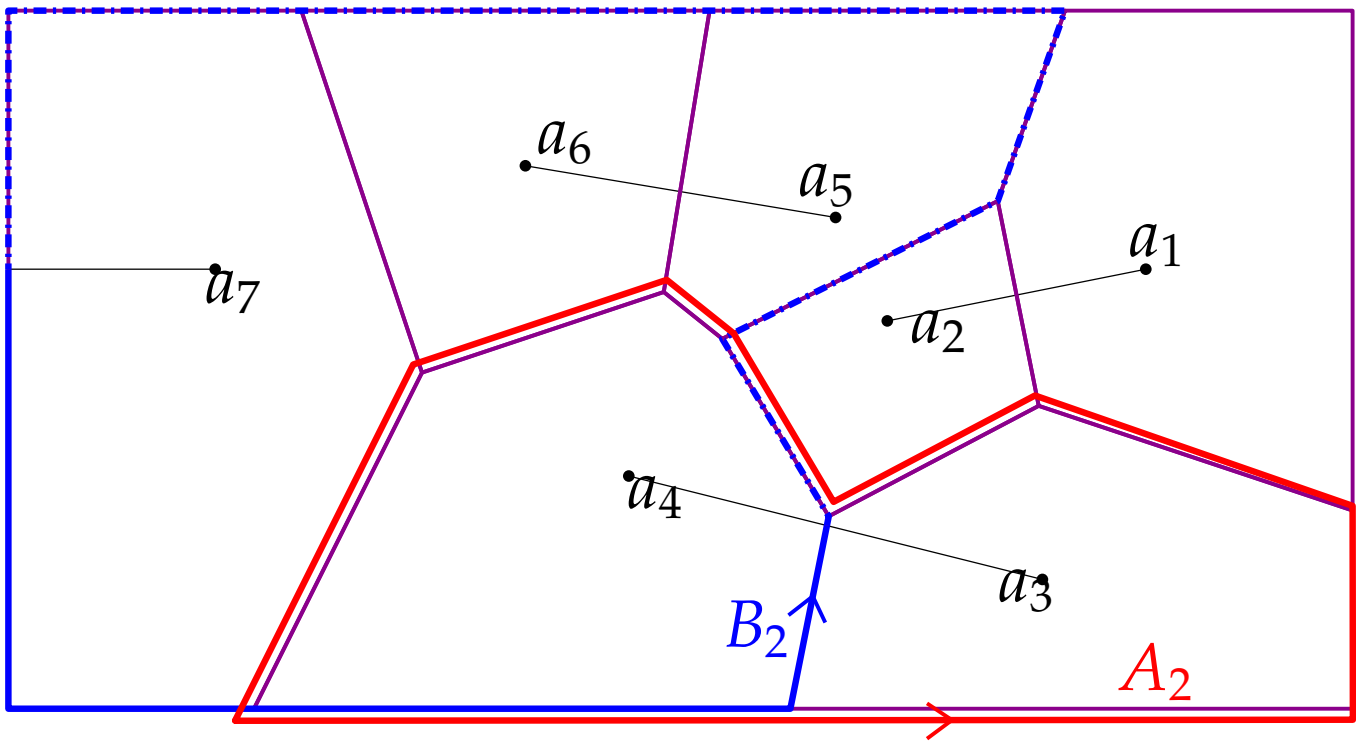

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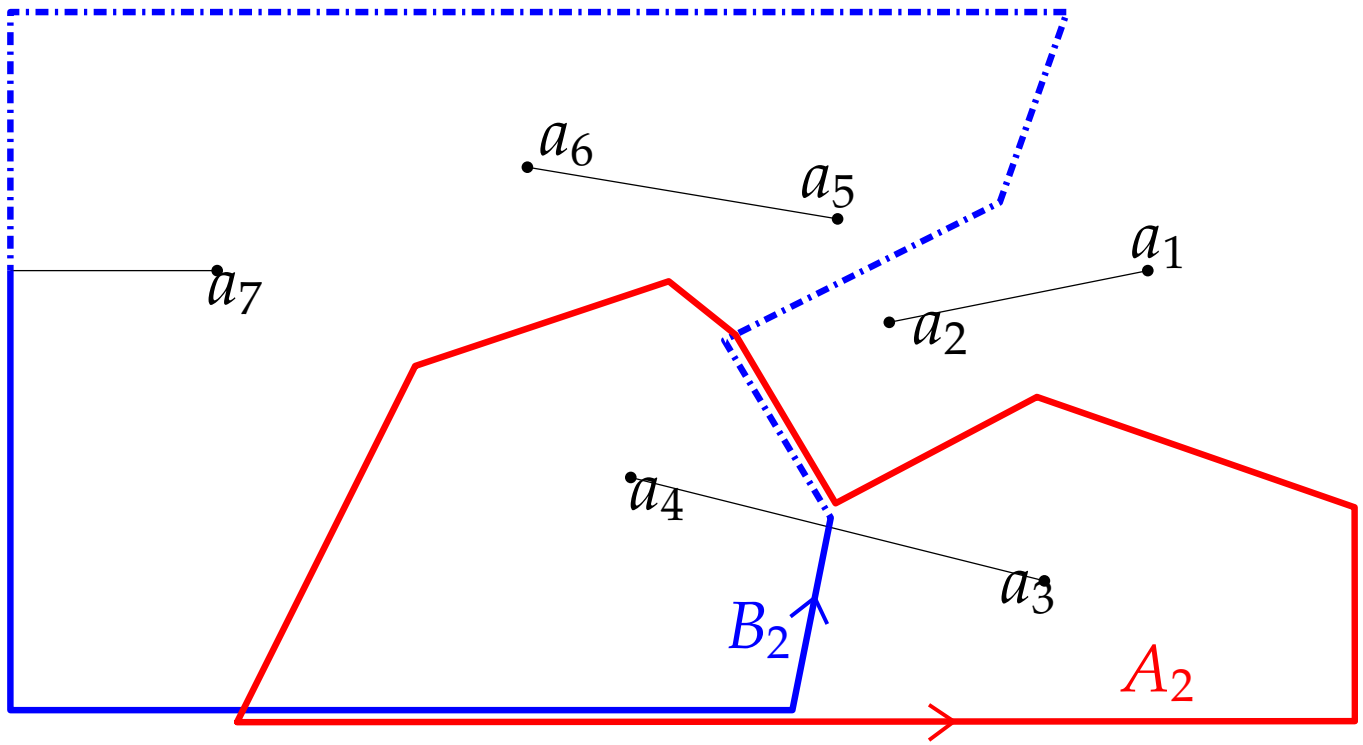
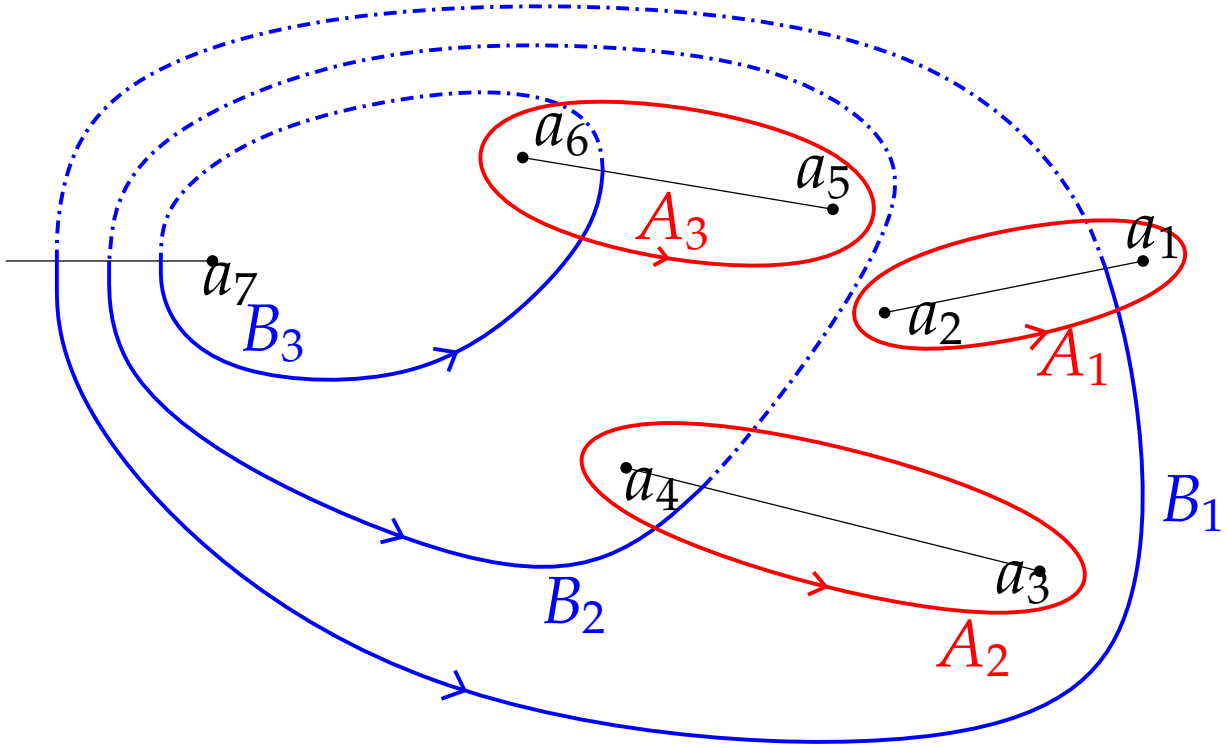


in practice $\sim 6g$ edges, so many integrals

Integrate between branch points



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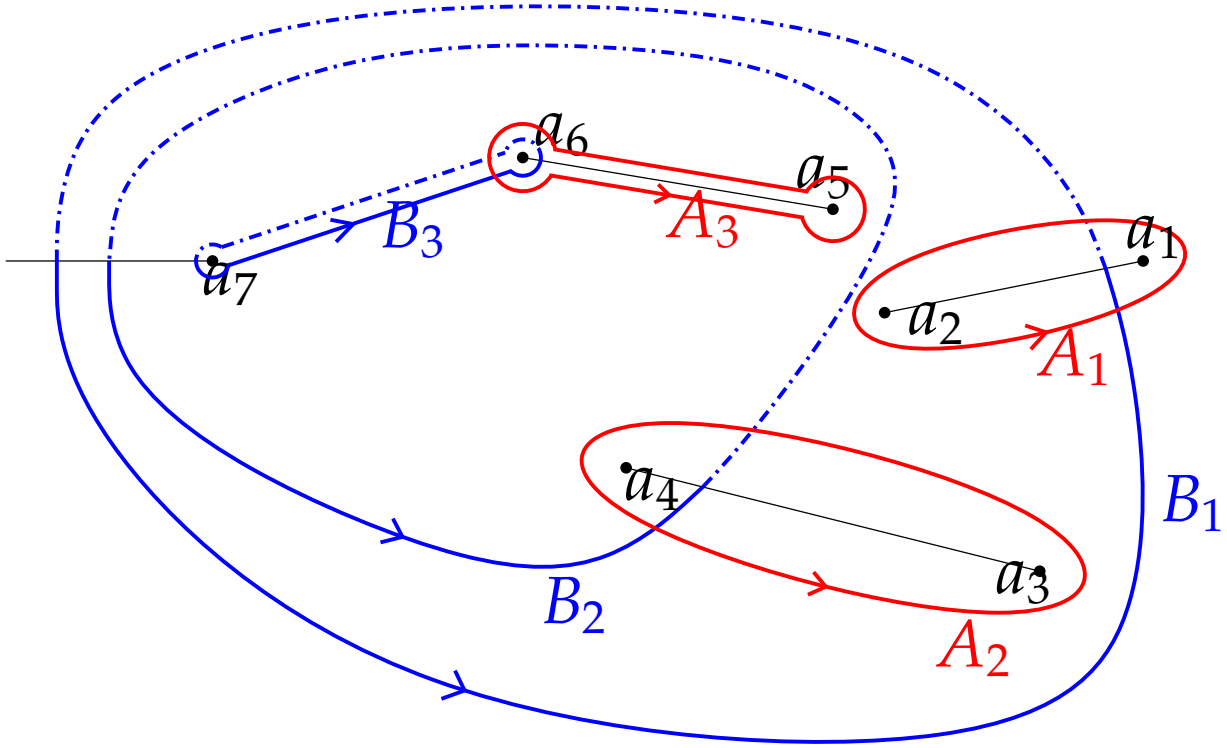

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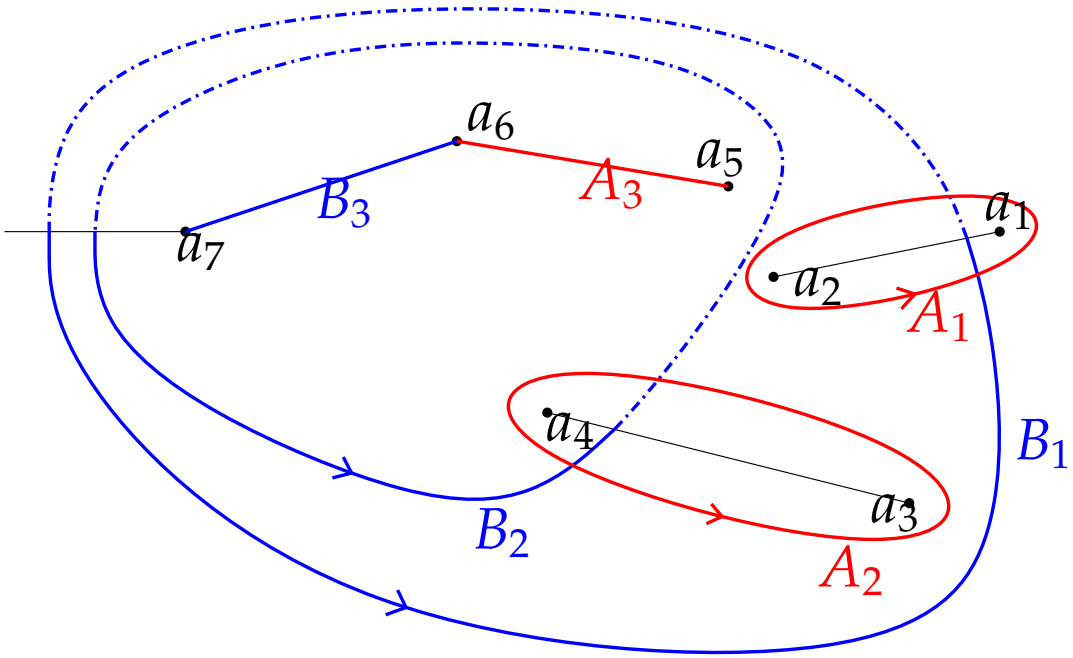
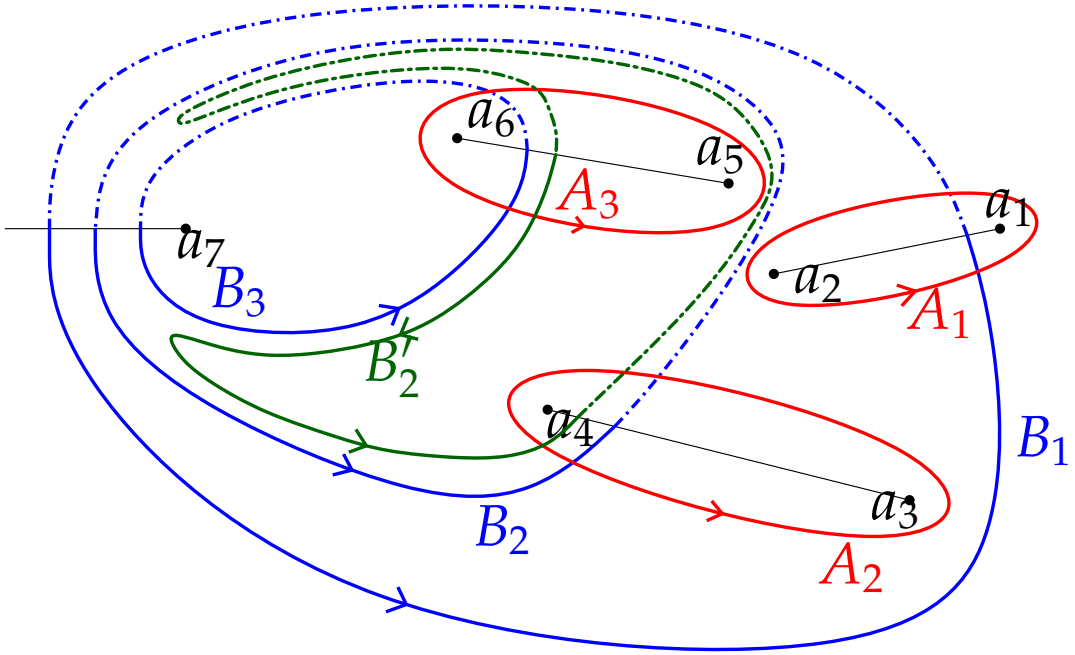


$$I = \int_{B_3} \frac{\mathrm{d}x}{y} = 2\int_{a_7}^{a_6} \frac{\mathrm{d}x}{y}$$

$$= C_1 \int_{-1}^{1} \frac{\mathrm{d}u}{\sqrt{1-u^2}\sqrt{-\prod_{k\neq 6,7}(x(u)-a_k)}} \rightarrow \text{ Gauss-Chebychev}$$

$$= C_2 \int_{\mathbb{R}} \frac{\cosh(t)\,\mathrm{d}t}{\cosh(\sinh(t))\sqrt{-\prod_{k\neq 6,7}(x(t)-a_k)}} \rightarrow \text{ DE}$$

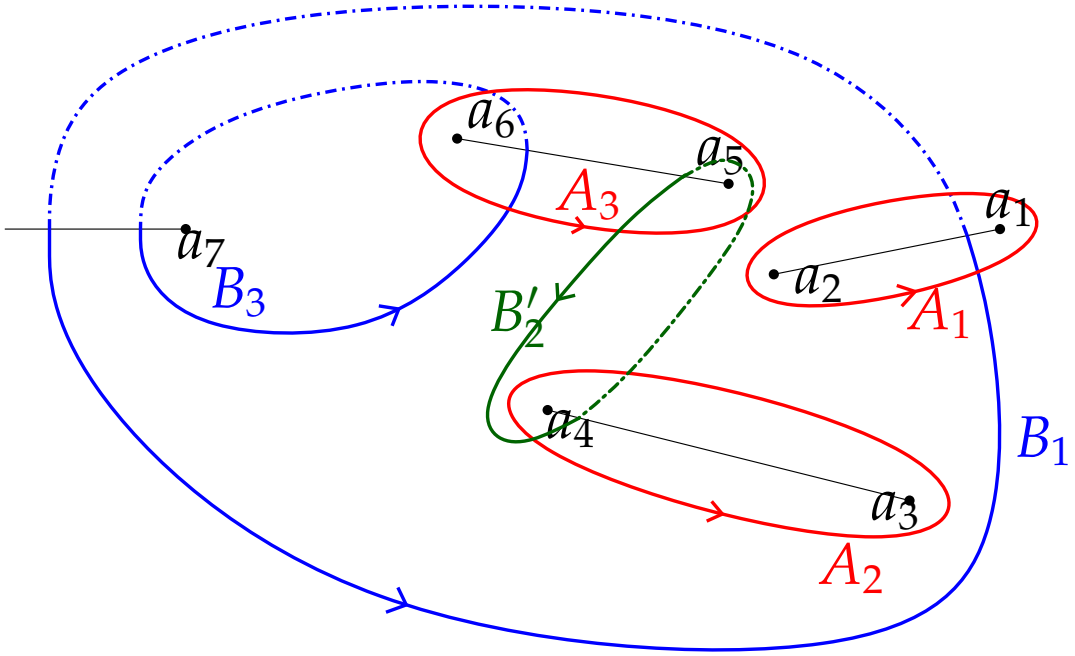
Integrate between branch points



$$B_2' = B_2 - B_3$$
$$B_3' = B_3$$

Impossible for $B_1, B_2 \rightarrow$ change basis.

Integrate between branch points



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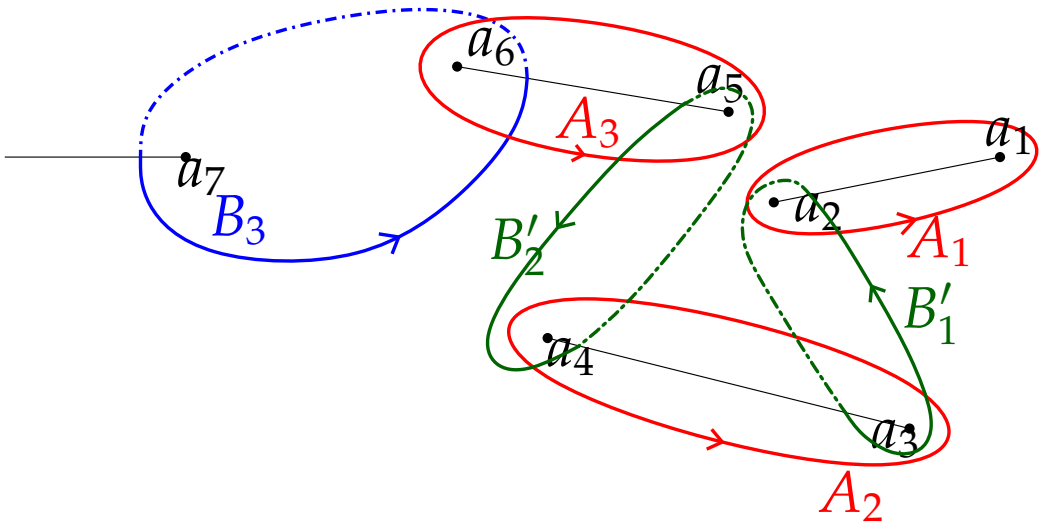
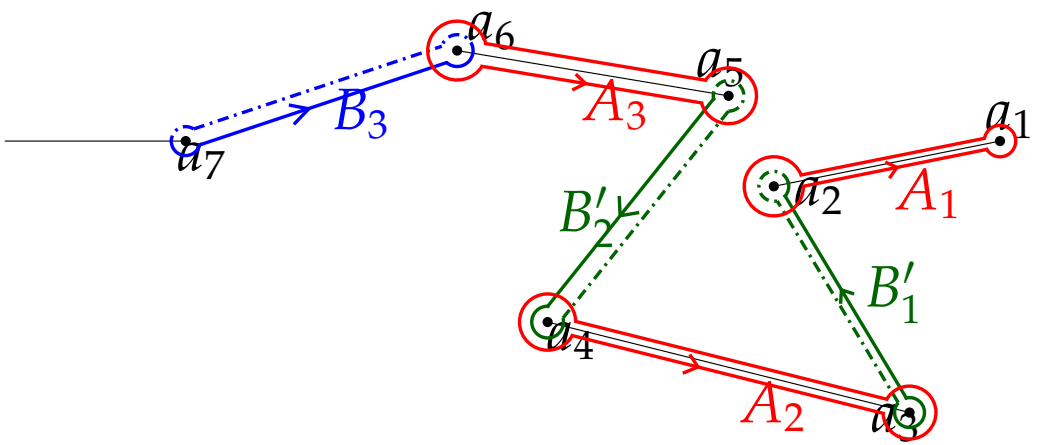
Integrate between branch points



$$
\begin{aligned}
B_1' &= B_1 - B_2 \\
B_2' &= B_2 - B_3 \\
B_3' &= B_3
\end{aligned}
$$

Integrate between branch points



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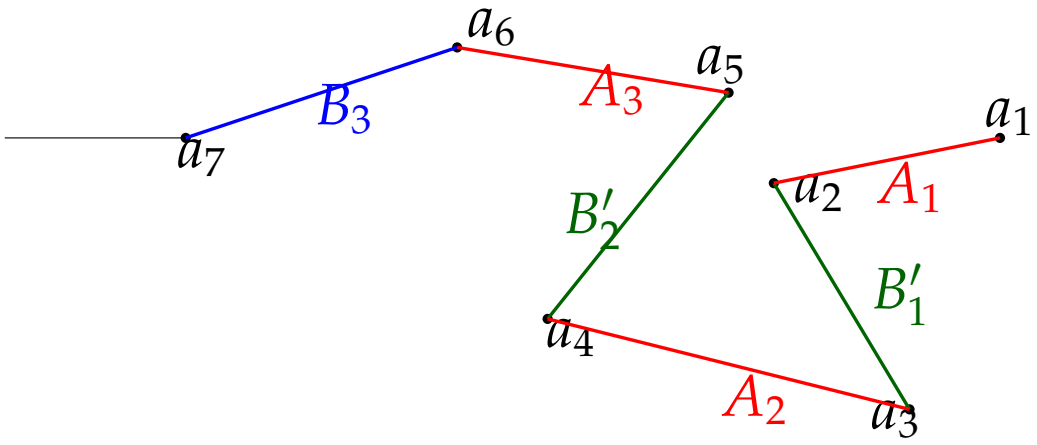

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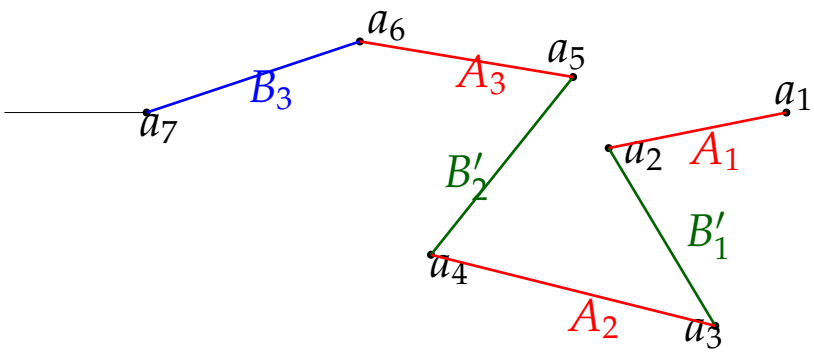
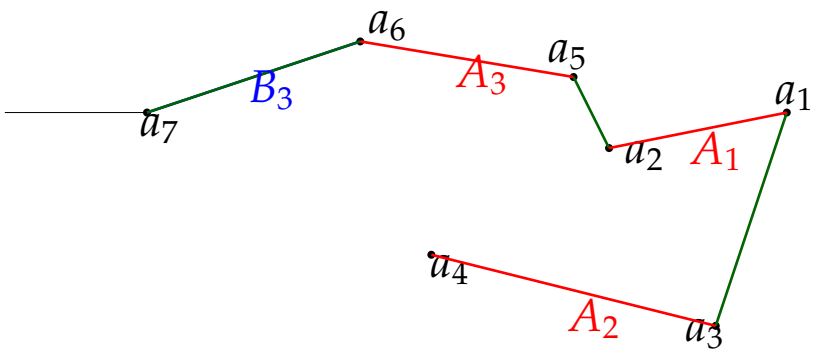
Only $2g$ intégrals. But here $\int_{B_2'}$ is slow...

Best homology basis

Chain of points :



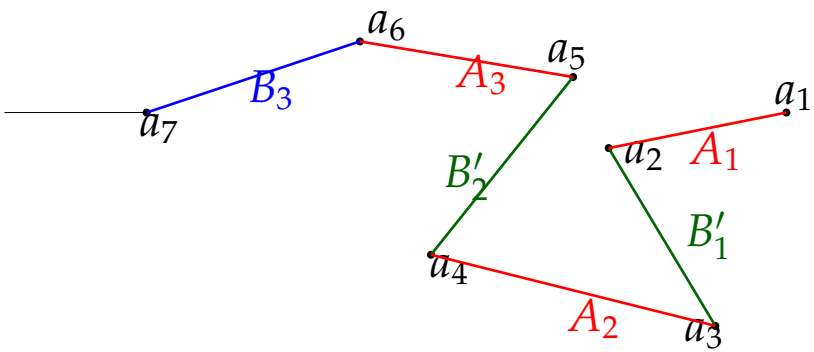
$r = 0.46$



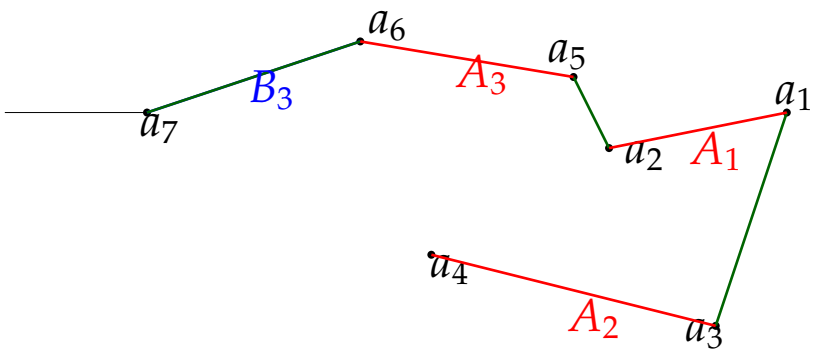
$r = 0.52$

Best homology basis

Chain of points :

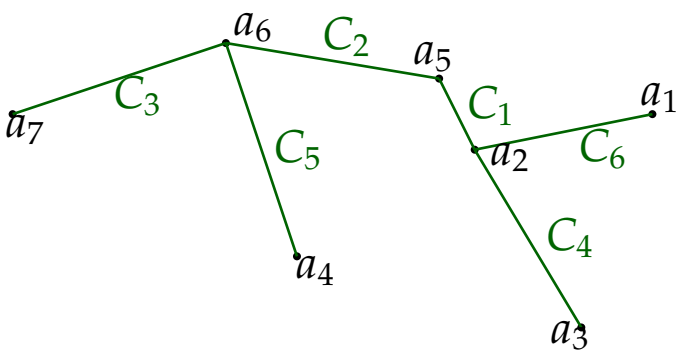


$r = 0.46$



$r = 0.52$

Any loop around two branch points $\rightarrow$ homology class $\neq 0$



$r = 0.70$

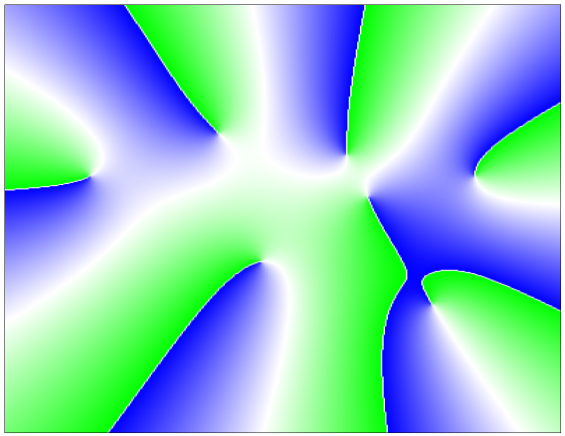
Select $2g$ cheapest integrals.
Homology basis $\Leftrightarrow$ spanning tree.

Numerical branch cuts

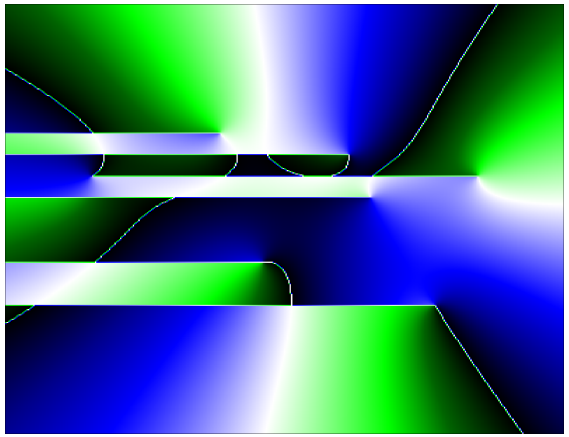
$$\mathscr{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i)$$

Evaluate $y(x)$ on a loop ? must avoid branch cuts.

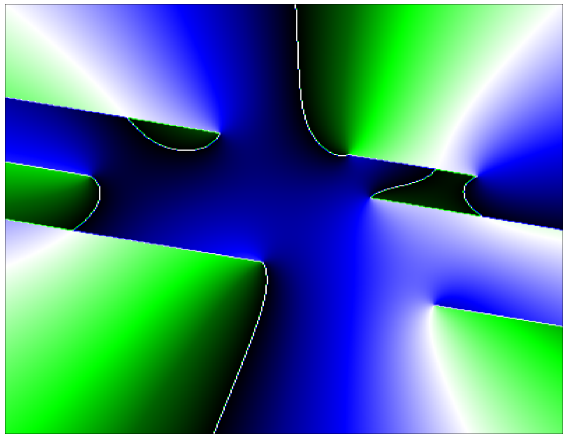
$y = \sqrt{\prod(x - a_i)}$



$y = \prod \sqrt{x - a_i}$



custom $y_{a_i,a_j}(x)$



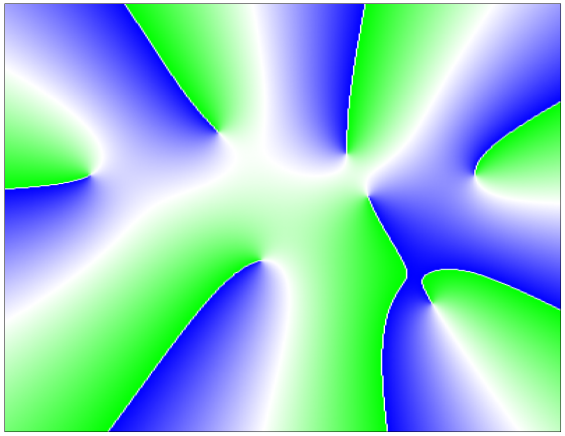
Local root $y_{a_i,a_j}(x) = \prod_k \sqrt[a_i,a_j]{x - a_k}$ for loop $C : a_i \to a_j$.

Numerical branch cuts

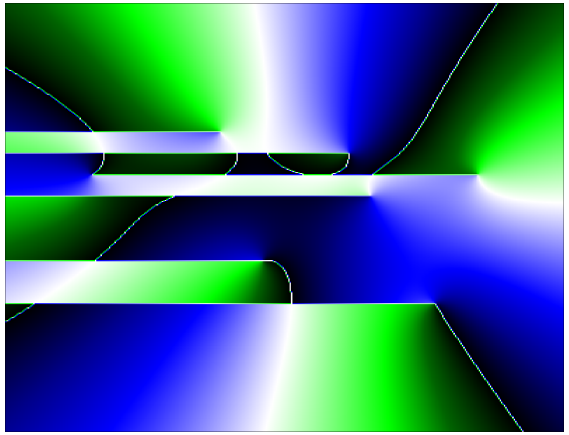
$$\mathscr{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i)$$

Evaluate $y(x)$ on a loop ? must avoid branch cuts.

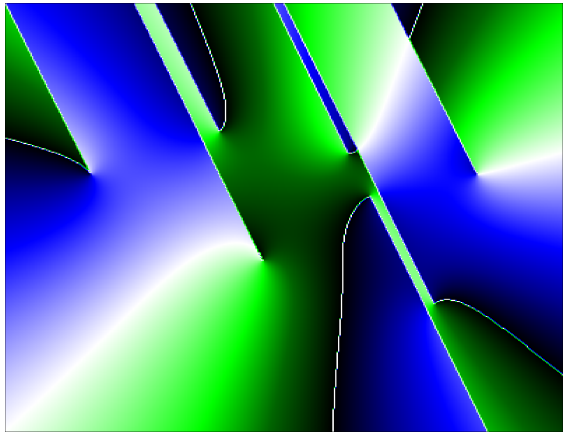
$y = \sqrt{\prod(x - a_i)}$



$y = \prod \sqrt{x - a_i}$



custom $y_{a_i,a_j}(x)$

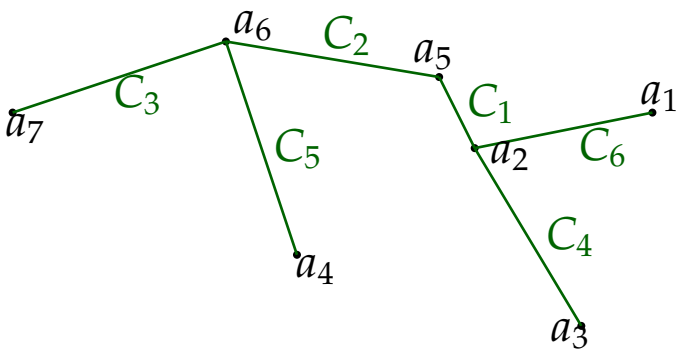


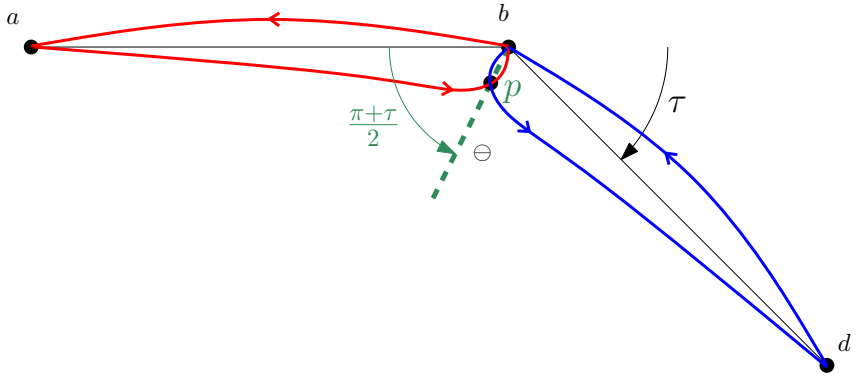
Local root $y_{a_i,a_j}(x) = \prod_k \sqrt[a_i,a_j]{x - a_k}$ for loop $C : a_i \to a_j$.

Lifting problem : $\int_{C_j} \omega_i$ on local sheet, period up to sign.

Retrieve symplecticity

Compute intersection $C_i \cdot C_j$ using local square root at end point.





Lemma

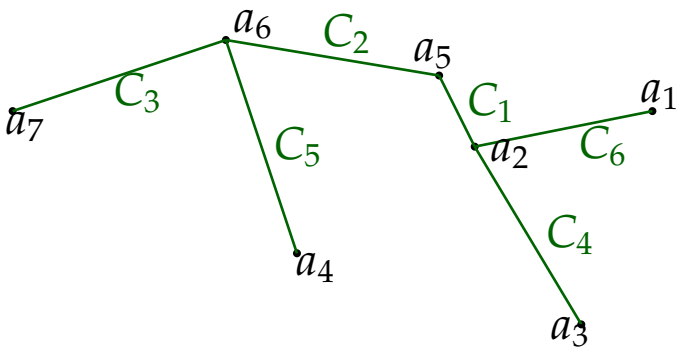
Let C_i and C_j such that

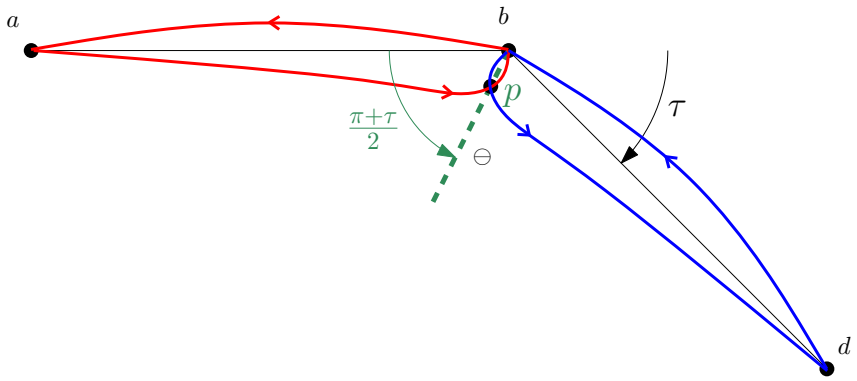
- $\int_{C_i} \frac{\mathrm{d}x}{y} = 2\int_a^b \frac{\mathrm{d}x}{y_{a,b}(x)}$
- $\int_{C_j} \frac{\mathrm{d}x}{y} = 2\int_b^d \frac{\mathrm{d}x}{y_{b,d}(x)}$.

Then $\arg\left(\frac{\prod_{a_k \neq a,b} \sqrt[a,b]{b-a_k}}{\prod_{a_k \neq b,d} \sqrt[b,d]{b-a_k}}\right) = (C_i \cdot C_j)\frac{\pi+\tau}{2}$

Retrieve symplecticity

Compute intersection $C_i \cdot C_j$ using local square root at end point.





symplectic reduction gives a standard basis

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

Algorithm

1. improve the set of branch points via Moebius transform
2. compute integration cost for every edge (a_i, a_j)
3. find minimum spanning tree $(C_1, \dots C_{2g})$
4. compute periods $\Omega_C = \left(\int_{C_j} \frac{\mathrm{d}x^i}{y}\right)_{i,j}$ (Gauss / DE)
5. compute intersection matrix $X = (C_i \cdot C_j)$
6. perform symplectic reduction ${}^tPXP = J_g$
7. big period matrix $\Omega = \Omega_C P = (\Omega_0 \Omega_1)$
8. small period matrix $\tau = \Omega_0^{-1}\Omega_1 \in \mathbb{H}_g$

Complexity : $O\left(g^2 D^2 \log(D)\right)$ bit operations (+trivially parallel).

Superelliptic curves

$$\mathscr{C} : y^m = \prod_{i=1}^{d} (x - a_i)$$

- a_i = branch points
- $\gcd(m, d)$ points at infinity
- Riemann surface with m sheets
- genus $g = \frac{(m-1)(d-1)-\gcd(m,d)+1}{2}$
- the $\frac{\mathrm{d}x^i}{y^j}$ with $jd \geqslant mi + \gcd(m, d)$ form a basis of differentials.

Everything generalises, $-1 = \zeta_2$ replaced by ζ_m.
(joint work with C. Neurohr, 2017)

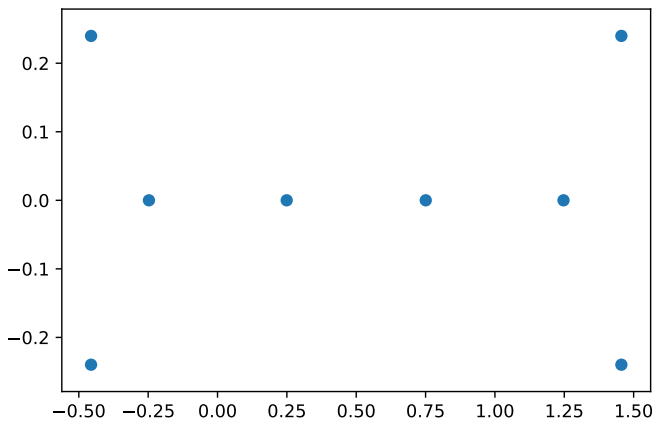
Code and timings

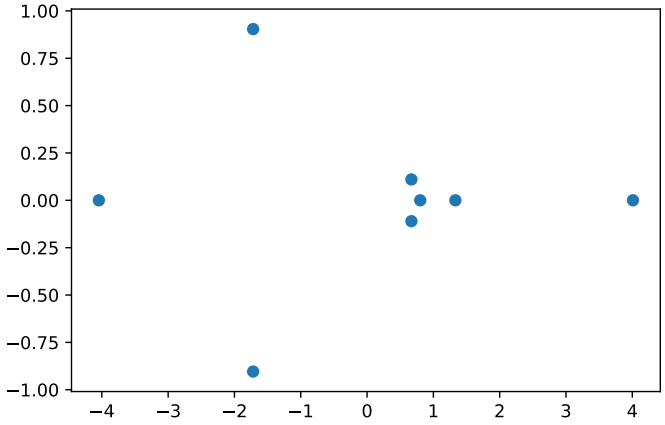
- Arb + Pari/GP + Magma code (Neurohr) (`github/pascalmolin/hcperiods`)
- Comparison for period matrix

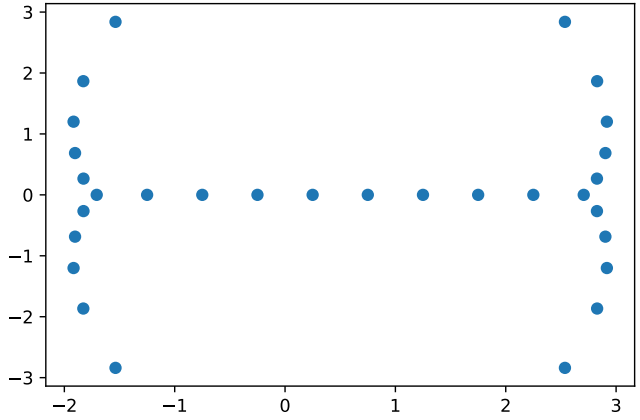
genus	curve	digits	38	154	600	1200	3000
3	$\mathcal{B}_{2,8}$	Arb	3.2e-3	1.5e-2	2.2e-1	1.03	9.94
		Magma	0.78	1.66	22.3	1208	—
	$\widetilde{\mathcal{B}}_{2,8}$	Arb	3.8e-3	1.2e-2	1.5e-1	6.9e-1	6.79
		Magma	0.81	1.35	22.2	1285	—
14	$\mathcal{B}_{2,30}$	Arb	5.1e-2	2.4e-1	4.30	20.3	193
		Magma	11.6	11.6	259	21312	—
	$\widetilde{\mathcal{B}}_{2,30}$	Arb	5.3e-2	2.5e-1	4.59	21.9	208
		Magma	246	412	1102	26578	—

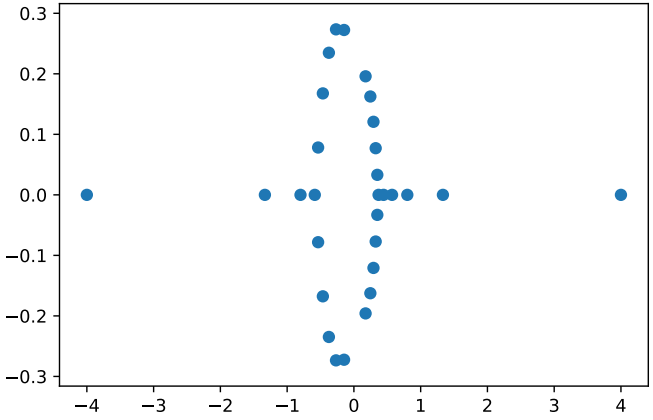
Code and timings

- Arb + Pari/GP + Magma code (Neurohr) (`github/pascalmolin/hcperiods`)
- Comparison for period matrix







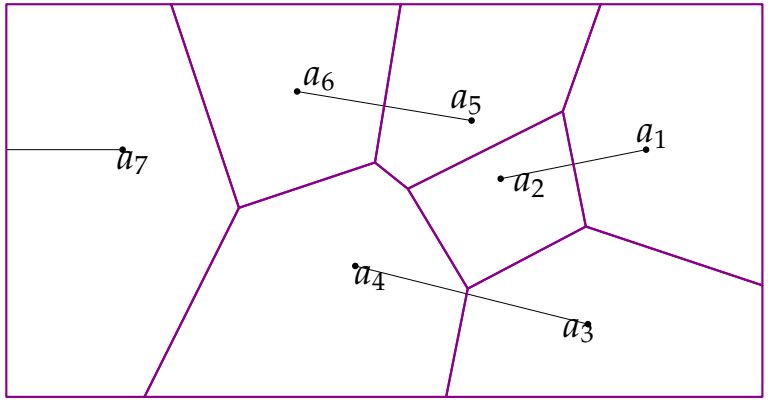


General curves

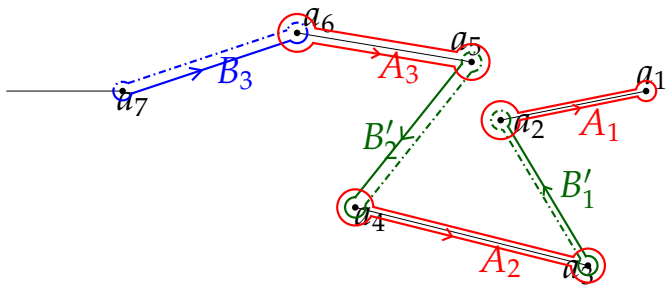
$$\mathscr{C} : f(x, y) = p_m(x)y^m + \dots p_0(x) = 0$$

- branch points $= \{x, \mathrm{disc}_y(f)(x) = 0\}$
- Riemann surface with m sheets
- compute $y_1(x), \dots y_m(x)$ by Newton
- basis of differentials $\frac{h(x,y)\,\mathrm{d}x}{\partial_y f(x,y)}$

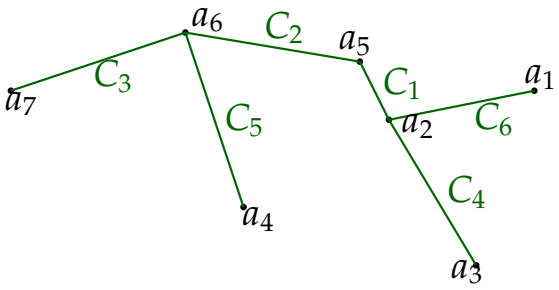
General curves



Bruin&al 2018 : Voronoi cell's approach generalized

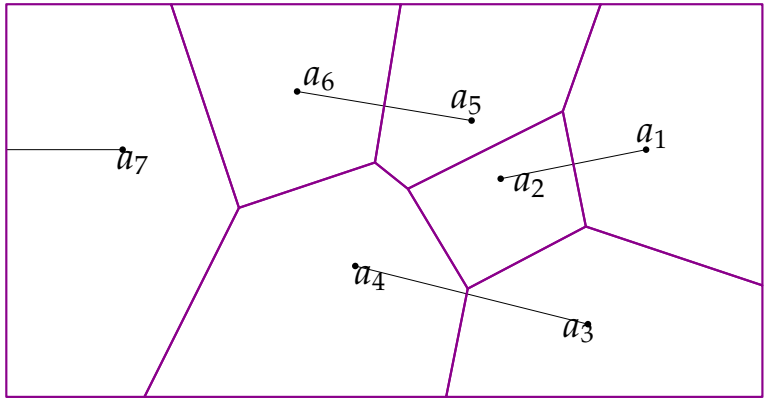


Neurohr 2017 : mixed tree + loops



Tree only ? need (at least) Puiseux expansions at branch points

General curves



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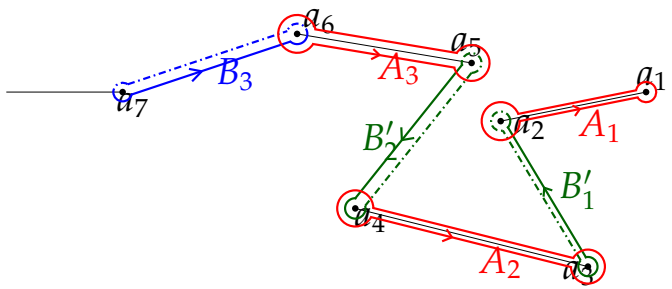

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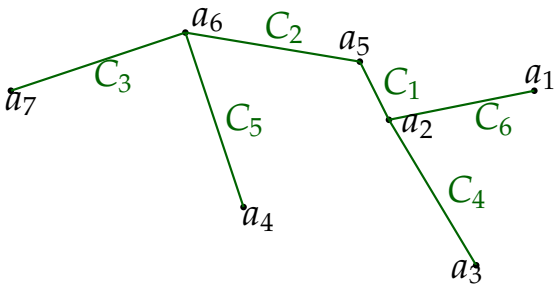

Tree only ? need (at least) Puiseux expansions at branch points

use differential equation ! may be already available through Marc's code