The Abel-Jacobi map of complex algebraic curves

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1. The Abel-Jacobi map

2. Numerical integration

3. Computing the periods
Hyperelliptic curve

\[ \mathcal{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i) \]

- \( a_i \) = branch points
- one point \( \infty \)
- Riemann surface with two sheets
- genus \( g \)

color = argument of \( y \)
\[
\mathcal{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i)
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Hyperelliptic curve

$\mathcal{C} : y^2 = \prod_{i=1}^{2g+1} (x - a_i)$

- $a_i =$ branch points
- one point $\infty$
- Riemann surface with two sheets
- genus $g$

color = argument of $y$
Abel-Jacobi map

Integration on the curve

\[
\text{Int} : P, Q \mapsto \int_{P}^{Q} \omega
\]

- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
Abel-Jacobi map

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- Paths
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- Paths

**Diagram:**
- Paths \( P \) to \( Q \) on the curve
- Integrals from \( P \) to \( Q \)
- Lattice points indicated
Abel-Jacobi map

Integration on the curve

\[ \text{Int} : P, Q \mapsto \int_P^Q \omega \]

- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
- Paths up to homology

\[
\Lambda = \{ \int_\gamma \omega_i, \gamma \in H_1, 1 \leq i \leq g \}\]
Abel-Jacobi map

Integration on the curve

\[ \text{Int} : P, Q \mapsto \int_{P}^{Q} \omega \]

- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
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Abel–Jacobi map

Integration on the curve

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- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
- Paths up to homology \( H_1(C) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g} \)
- Integral defined up to period lattice \( \Lambda = \left\{ \int_\gamma \omega_i, \gamma \in H_1, 1 \leq i \leq g \right\} \).
Abel-Jacobi map

Integration on the curve

\[ \text{Int} : P, Q \mapsto \int_P^Q \omega \]

- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
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- Integral defined up to period lattice
  \[ \Lambda = \left\{ \int_\gamma \omega_i, \gamma \in H_1, 1 \leq i \leq g \right\} \]

Theorem (Abel+Jacobi)

\[ 0 \to \text{Prin}(C) \to \text{Div}^0(C) \to \mathbb{C}^g / \Lambda \to 0 \]
Abel-Jacobi map

Integration on the curve

\[ \text{Int} : P, Q \mapsto \int_P^Q \omega \]

- Holomorphic differentials \( \mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g \)
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- Integral defined up to period lattice \( \Lambda = \left\{ \int_\gamma \omega_i, \gamma \in H_1, 1 \leq i \leq g \right\} \).

Theorem (Abel+Jacobi)

\[ \text{Jac}(C) \simeq \mathbb{C}^g / \Lambda \]
Standard form for the period matrix

- Homology $H_1(C) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g}$
- Holomorphic differentials $\mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g$
- $\Omega = \text{period matrix} = \left( \int_{\gamma_i} \omega_j \right) \in M_{g \times 2g}(\mathbb{C})$. 
Standard form for the period matrix

- Homology $H_1(C) = \mathbb{Z} \gamma_1 \oplus \cdots \oplus \mathbb{Z} \gamma_{2g}$

Intersection product (antisymmetric)
Symplectic basis $A_1, \ldots, A_g, B_1 \ldots B_g$ such that
$(A_i \cdot B_j = \delta_{i,j}), A_i \cdot A_j = B_i \cdot B_j = 0.$

- Holomorphic differentials $\mathcal{H}^1(C) = \mathbb{C} \omega_1 \oplus \cdots \oplus \mathbb{C} \omega_g$

- $\Omega = \text{period matrix} = \left( \int_{\gamma_i} \omega_j \right) \in M_{g \times 2g}(\mathbb{C})$. 
Standard form for the period matrix

- Homology $H_1(C) = \mathbb{Z}A_1 \oplus \cdots \oplus \mathbb{Z}B_g$
- Holomorphic differentials $\mathcal{H}^1(C) = \mathbb{C}\omega_1 \oplus \cdots \oplus \mathbb{C}\omega_g$
  take dual basis $\omega_i$ s.t. $\int_{B_j} \omega_i = \delta_{i,j}$
- $\Omega = $ period matrix $= (\tau l_g)$, $\tau = \left( \int_{A_j} \omega_i \right) \in \mathcal{H}_g$ (Siegel space).
Goals

Compute the period lattice $\Lambda$ and the Abel-Jacobi map to large precision ($> 1000$ digits).

- recognize endomorphism rings of jacobians (LLL)
- certify a divisor is non principal
- compute torsion divisors
- Riemann theta function...
Hyperelliptic setting

- standard branch cuts
- standard symplectic basis when roots are well ordered
- canonical basis of holomorphic forms

\[ \frac{d\chi^i}{y}, \, i = 1 \ldots g \]

Need to parametrize and integrate efficiently over the loops.
Section 2

Numerical integration
Gauss-Legendre integration

**Theorem**

Let $f : [-1, 1] \rightarrow \mathbb{C}$ such that $f$ is holomorphic with $|f| \leq M$ on $E_r = \{|z - 1| + |z + 1| < 2 \cosh(r)\}$. For all $D > 0$ there exists

$$N \sim \frac{D + \log(5M)}{r \log 4} \text{ s.t. } \left| \int_{-1}^{1} f(x) \, dx - \sum_{k=1}^{N} w_k f(x_k) \right| \leq e^{-D}$$

where $x_k, w_k$ are given by

$$P_n(x_k) = 0 \text{ and } w_k = \frac{-2}{(n + 1)P_n'(x_k)P_{n+1}(x_k)}$$

and $P_n$ are Legendre polynomials

$$\begin{cases}
    P_0(x) = 0, P_1(x) = 1 \\
    (k + 1)P_{k+1}(x) = (2k + 1)xP_k(x) - kP_{k-1}(x)
\end{cases}$$
Gauss-Chebyshev integration

**Theorem**

Let \( f : [-1, 1] \rightarrow \mathbb{C} \) such that \( f \) is holomorphic with \( |f| \leq M \) on \( E_r = \{|z-1| + |z+1| < 2 \cosh(r)\} \). For all \( D > 0 \) there exists

\[
N \sim \frac{D + \log(2\pi M)}{r \log 4}
\]

s.t.

\[
\left| \int_{-1}^{1} f(x) \frac{dx}{\sqrt{1-x^2}} - \sum_{k=1}^{N} w_k f(x_k) \right| \leq e^{-D}
\]

where \( x_k, w_k \) are given by

\[
x_k = \cos\left(\frac{(2k+1)\pi}{N}\right) \quad \text{and} \quad w_k = \frac{\pi}{N}
\]

(3)
Double-exponential integration 1

Theorem
Assume $f : ] -1, 1[ \to \mathbb{C}$ is holomorphic with $|f| < M$ on $Z_r = \{ \tanh(\sinh R \pm it), t < r \}$. For all $D > 0$, there exist explicit $N, h > 0$ such that

$$N \sim \frac{(D + \log M) \log D}{\pi r}$$

and

$$\left| \int_{-1}^{1} f - \sum_{k=-N/2}^{N/2} f(z_k) \, dz_k \right| \leq e^{-D}$$

where

$$\begin{cases} z_k = \tanh(\sinh(kh)) \\ dz_k = \frac{h \cosh(kh)}{\cosh^2(\sinh(kh))} \end{cases}$$

(4)
Double-exponential integration 2

Theorem
Assume $f : [-1, 1] \to \mathbb{C}$ is holomorphic with $|f| < M$ on $Z_r = \{ \tanh(\sinh R \pm it), t < r \}$. For all $D > 0$, there exist explicit $N, h > 0$ such that

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and

$$\left| \int_{-1}^{1} \frac{f}{\sqrt{1 - x^2}} - \sum_{k=-N/2}^{N/2} f(z_k) \, dz_k \right| \leq e^{-D}$$

where

$$\begin{cases} z_k = \tanh(\sinh(kh)) \\ dz_k = \frac{h \cosh(kh)}{\cosh(\sinh(kh))} \end{cases}$$ (5)
Convergence of DE integration

Dependency in the holomorphy domain $Z_r$:

\[ N \sim D \log D \]

\[ N \sim \frac{D \log D}{2} \]

\[ N \sim \frac{2D \log D}{5} \]
### Summary

Integral of to $D$ digits.

<table>
<thead>
<tr>
<th>method</th>
<th>Gauss-Legendre</th>
<th>Gauss-Chebyshev</th>
<th>DE</th>
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<tr>
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<td>$D^2 \log D$</td>
<td>$D^2 \log D$</td>
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<td>✓ ✓</td>
<td>✓</td>
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<tr>
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<td>✗</td>
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</tr>
</tbody>
</table>
Section 3

Computing the periods
Van Wamelen approach

Integrate far from branch points $a_i$
Integrate far from branch points $a_i \rightarrow$ Voronoi diagram
Van Wamelen approach

Integrate far from branch points $a_i \rightarrow$ Voronoi diagram
Integrate far from branch points $a_i \rightarrow$ Voronoi diagram

in practice $\sim 6g$ edges, so many integrals
Integrate between branch points

\[ a_1, a_2, a_3, a_4, a_5, a_6, a_7, A_1, A_2, A_3, B_1, B_2, B_3 \]
Integrate between branch points
Integrate between branch points

\[ l = \int_{B_3} \frac{dx}{y} = 2 \int_{a_7}^{a_6} \frac{dx}{y} \]

\[ = C_1 \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}} \sqrt{- \prod_{k \neq 6,7} (x(u) - a_k)} \rightarrow \text{Gauss-Chebychev} \]

\[ = C_2 \int_{\mathbb{R}} \frac{\cosh(t) \, dt}{\cosh(\sinh(t))} \sqrt{- \prod_{k \neq 6,7} (x(t) - a_k)} \rightarrow \text{DE} \]
Integrate between branch points

\[
B'_2 = B_2 - B_3
\]

\[
B'_3 = B_3
\]

Impossible for $B_1, B_2 \rightarrow$ change basis.
Integrate between branch points

\[ B_2' = B_2 - B_3 \]
\[ B_3' = B_3 \]
Integrate between branch points

\[
\begin{align*}
B_1' &= B_1 - B_2 \\
B_2' &= B_2 - B_3 \\
B_3' &= B_3
\end{align*}
\]
Integrate between branch points

\[ B'_1 = B_1 - B_2 \]
\[ B'_2 = B_2 - B_3 \]
\[ B'_3 = B_3 \]
Integrate between branch points

\[ B'_1 = B_1 - B_2 \]
\[ B'_2 = B_2 - B_3 \]
\[ B'_3 = B_3 \]

Only 2g intégrals. But here \( \int_{B'_2} \) is slow...
Best homology basis

Chain of points:

\[ r = 0.46 \]

\[ r = 0.52 \]
Best homology basis

Chain of points:

Any loop around two branch points $\rightarrow$ homology class $\neq 0$

Select $2g$ cheapest integrals.
Homology basis $\Leftrightarrow$ spanning tree.

$r = 0.46$

$r = 0.52$

$r = 0.70$
Numerical branch cuts

\[ C: y^2 = \prod_{i=1}^{2g+1} (x - a_i) \]

Evaluate \( y(x) \) on a loop? must avoid branch cuts.

\[ y = \sqrt{\prod (x - a_i)} \]
\[ y = \prod \sqrt{x - a_i} \]

Local root \( y_{a_i,a_j}(x) = \prod_k a_i \cdot a_j \sqrt{x - a_k} \) for loop \( C : a_i \rightarrow a_j \).
Numerical branch cuts

\[ C : y^2 = \prod_{i=1}^{2g+1} (x - a_i) \]

Evaluate \( y(x) \) on a loop? must avoid branch cuts.

\[ y = \sqrt{\prod (x - a_i)} \quad \quad \quad y = \prod \sqrt{x - a_i} \quad \quad \quad \text{custom } y_{a_i,a_j}(x) \]

Local root \( y_{a_i,a_j}(x) = \prod_{k}^{a_i,a_j} \sqrt{x - a_k} \) for loop \( C : a_i \rightarrow a_j \).

Lifting problem: \( \int_{C_j} \omega_i \) on local sheet, period up to sign.
Retrieve symplecticity

Compute intersection $C_i \cdot C_j$ using local square root at end point.

Lemma

Let $C_i$ and $C_j$ such that

\begin{align*}
\int_{C_i} \frac{dx}{y} &= 2 \int_{a}^{b} \frac{dx}{y_{a,b}(x)} \\
\int_{C_j} \frac{dx}{y} &= 2 \int_{b}^{d} \frac{dx}{y_{b,d}(x)}.
\end{align*}

Then

\[
\arg\left(\frac{\prod_{a \neq a,b}^{a,b} \sqrt{b-a_k}}{\prod_{a \neq b,d}^{b,d} \sqrt{b-a_k}}\right) = (C_i \cdot C_j) \frac{\pi + \tau}{2}
\]
Retrieve symplecticity

Compute intersection $C_i \cdot C_j$ using local square root at end point.

Symplectic reduction gives a standard basis

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$
Algorithm

1. improve the set of branch points via Moebius transform
2. compute integration cost for every edge \((a_i, a_j)\)
3. find minimum spanning tree \((C_1, \ldots, C_{2g})\)
4. compute periods \(\Omega_C = \left(\int_{C_j} \frac{dx^i}{y}\right)_{i,j}\) (Gauss / DE)
5. compute intersection matrix \(X = (C_i \cdot C_j)\)
6. perform symplectic reduction \(^tPXP = J_g\)
7. big period matrix \(\Omega = \Omega_C P = (\Omega_0\Omega_1)\)
8. small period matrix \(\tau = \Omega_0^{-1}\Omega_1 \in \mathbb{H}_g\)

Complexity: \(O\left(g^2D^2 \log(D)\right)\) bit operations (+trivially parallel).
Superelliptic curves

\[ \mathcal{C}: y^m = \prod_{i=1}^{d} (x - a_i) \]

- \(a_i = \) branch points
- \(\gcd(m, d) = \) points at infinity
- Riemann surface with \(m\) sheets
- genus \(g = \frac{(m-1)(d-1)-\gcd(m,d)+1}{2}\)
- the \(\frac{dx^i}{y^j}\) with \(jd \geq mi + \gcd(m, d)\) form a basis of differentials.

Everything generalises, \(-1 = \zeta_2\) replaced by \(\zeta_m\).

(joint work with C. Neurohr, 2017)
Code and timings

- Arb + Pari/GP + Magma code (Neurohr) (github/pascalmolin/hcperiods)
- Comparison for period matrix

<table>
<thead>
<tr>
<th>genus</th>
<th>curve</th>
<th>digits</th>
<th>38</th>
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<th>600</th>
<th>1200</th>
<th>3000</th>
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Code and timings

- Arb + Pari/GP + Magma code (Neurohr)  
(github/pascalmolin/hcperiods)
- Comparison for period matrix

![Graph 1](image1.png)
![Graph 2](image2.png)
![Graph 3](image3.png)
![Graph 4](image4.png)
General curves

\[ \mathcal{C} : f(x, y) = p_m(x)y^m + \ldots + p_0(x) = 0 \]

- branch points = \{x, \text{disc}_y(f)(x) = 0\}
- Riemann surface with \( m \) sheets
- compute \( y_1(x), \ldots, y_m(x) \) by Newton
- basis of differentials \( \frac{h(x,y) \, dx}{\partial_y f(x,y)} \)
Bruin&al 2018: Voronoi cell’s approach generalized.

Neurohr 2017: mixed tree + loops.

Tree only? need (at least) Puiseux expansions at branch points.

General curves.
General curves

Bruin & al 2018: Voronoi cell’s approach generalized

Neurohr 2017: mixed tree + loops

Tree only? need (at least) Puiseux expansions at branch points

use differential equation! may be already available through Marc’s code