Computing periods of algebraic curves

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Université Paris 7

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Definition

Algebraic curve $X : f(x, y) = 0$, genus g.
Integration on X

$$P, Q \mapsto \int_P^Q \omega$$

- Holomorphic differentials $\Omega_1(X) = \langle \omega_1 \dots \omega_g \rangle$

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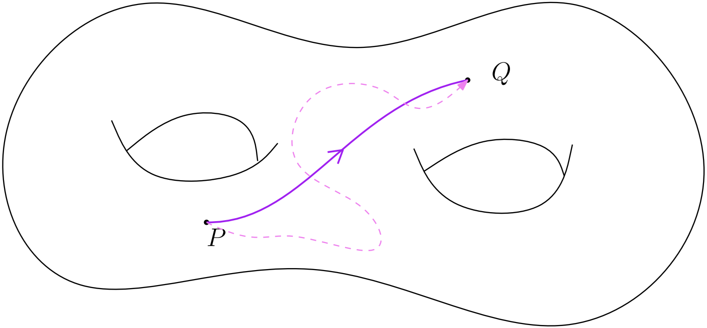
- Paths



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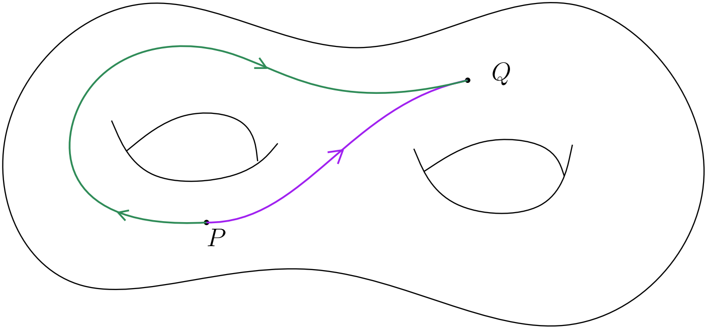
- Paths up to homology



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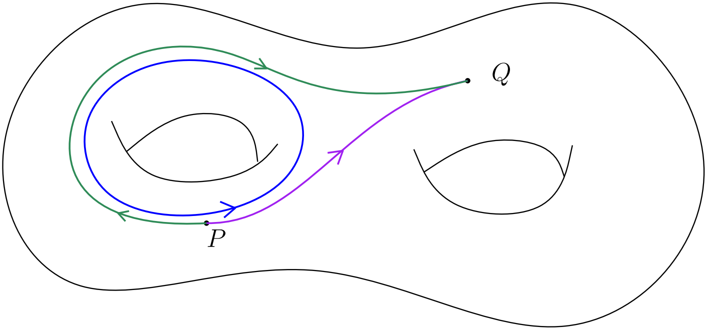
- Paths up to homology $H_1(X, \mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g}$



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- Paths up to homology $H_1(X,\mathbb{Z}) = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_{2g}$
- Integral values up to period lattice
  $\Lambda = \left\{ \int_\gamma \omega_i, \gamma \in H_1(X), 1 \leqslant i \leqslant g \right\}$.

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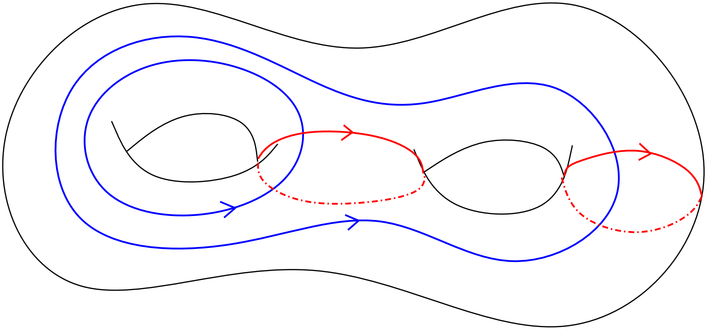
- Symplectic basis $H_1(X, \mathbb{Z}) = \langle \alpha_1, \dots \alpha_g, \beta_1, \dots \beta_g \rangle$



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- Period matrix $\Omega = \left(A = \int_{\alpha_j} \omega_i, B = \int_{\beta_j} \omega_i \right)_{1 \leqslant i,j \leqslant g}$.

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- Period matrix $\Omega = \left(A = \int_{\alpha_j} \omega_i, B = \int_{\beta_j} \omega_i \right)_{1 \leqslant i,j \leqslant g}$.
- Reduced period matrix $\tau(X) = A^{-1}B \in \mathbb{H}_g$ (Siegel space)

Two maps

X fixed, Abel-Jacobi map

$$\begin{cases} \mathrm{Pic}^0(X) \to \mathbb{C}^g/\Lambda \\ P - P_0 \mapsto \int_{P_0}^{P} \omega \end{cases}$$

X on a family, period map

$$\begin{cases} \{X_s \in S\} \to \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g \\ s \mapsto \tau(X_s) \end{cases}$$

Very useful tool...

- 1987 Chudnovsky^2
- 1988 Bost, Mestre (genus 2, real)
- 1998 Van Wamelen (hyperelliptic, Magma)
- 2001 Deconinck, Van Hoeij (general, Maple)
- 2011 Frauendiener, Klein (general, matlab)
- 2011 Molin (hyperelliptic, gp)
- 2016 Labrande (low genus)
- 2013 Mascot (modular curves)
- 2017 Costa,...Mascot (genus 2)
- 2017 Molin, Neurohr (superelliptic, Arb+Magma)
- 2018 Neurohr (general, Magma)
- 2018 Sertöz (dim > 1)
- 2018 Bruin, Sijsling, Zotine (general, Sage)

Soon available in Pari ?

Version restricted to hyperelliptic curves.

```
gp > default(realprecision, 100)
   realprecision = 115 significant digits (100 digits displayed)
gp > hyperellperiods(x^51+x+1,1);
     *** hyperellperiods: Warning: compute 41 integration points
time = 823 ms.

gp > hyperellperiods(random(1.*x^100),1);
  *** hyperellperiods: Warning: compute 97 integration points.
  *** hyperellperiods: Warning: compute 69 integration points.
  *** hyperellperiods: Warning: compute 53 integration points.
  *** hyperellperiods: Warning: compute 40 integration points.
  *** hyperellperiods: Warning: compute 29 integration points.
time = 4,600 ms.
```

The problem

X a curve, equation $f(x, y) = 0$.
We see X as a cover $y_1 \dots y_m$ of the x-plane $\mathbb{P}^1$.
We need

1. differentials
2. a basis of loops on X, lifted from loops of $\mathbb{P}^1$.
3. evaluating $y(x)$
4. numerical integration

We focus on integration (governs choices 2 and 3).

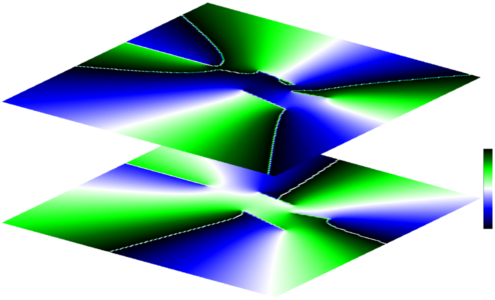
Hyperelliptic curve

$$X : y^2 = \prod_{i=1}^{2g+1} (x - a_i)$$

- $\{\infty\} \cup \{a_i\}$ = branch points
- genus g

Need

1. differentials $\Omega = \langle \frac{\mathrm{d}x}{y}, \dots \frac{\mathrm{d}x}{y} \rangle$
2. loops
3. integration



x-plane, color = $\arg(y)$

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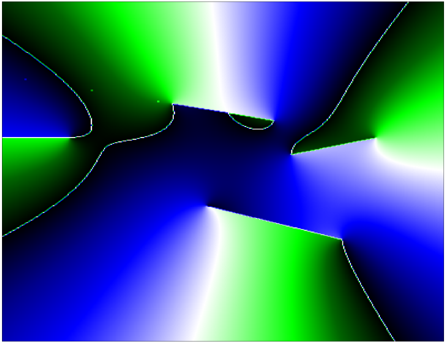

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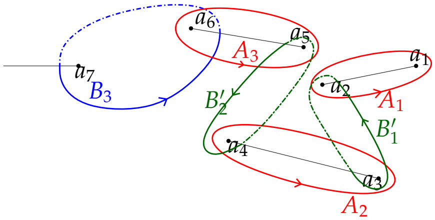

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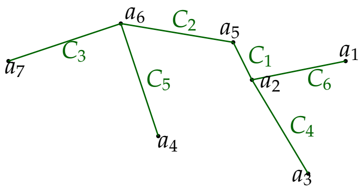


Homology basis ⇔ spanning tree

Computing periods

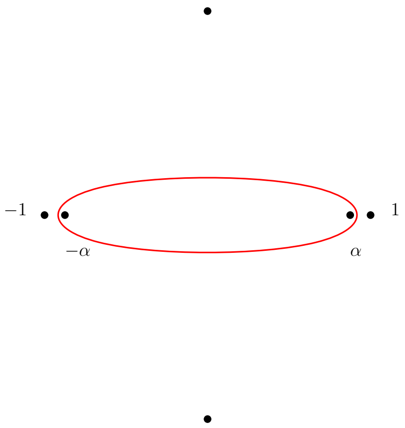
We use as an example

$$X : y^2 = x^6 - 2x^2 + 1$$

[lmfdb :800.a.409600.1]
Compute the period

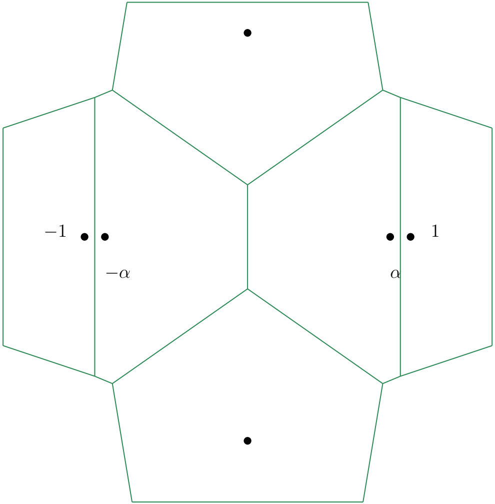
$$\int_\gamma \frac{\mathrm{d}x}{y}$$

where $\alpha = \sqrt{\phi} < 1$



Method 1 - Voronoi

Combine paths far from branch points a_i +Gauss-Legendre integration



Van Wamelen (magma), Bruin & Zotine (sage)

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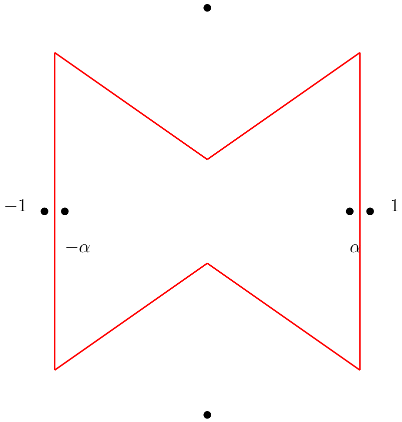
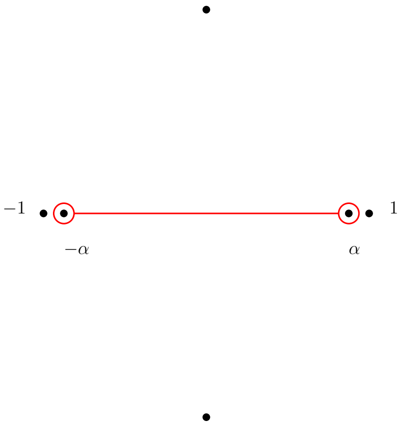


Van Wamelen (magma), Bruin & Zotine (sage)

Method 2 - Singular integrals

$$\int_\gamma \frac{\mathrm{d}x}{y} = \int_a^b \frac{\mathrm{d}x}{y} + \int_b^a \frac{\mathrm{d}x}{-y}$$
$$= 2\int_a^b \frac{\mathrm{d}x}{y}$$
$$= 2\int_a^b g(u) \frac{\mathrm{d}x}{\sqrt{x-a}\sqrt{x-b}}$$



Gauss-Chebychev integration.

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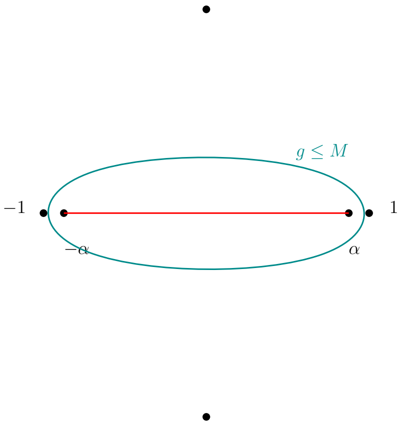


Gauss-Chebychev integration. Explicit error bound $\frac{2\pi M}{e^{2rn}-1}$.

Method 2

```
f = x^6-2*x^2+1;

period_intnum() =
  x3=(sqrt((sqrt(5)-1)/2));
  2*intnum(x=[-x3,-1/2],[x3,-1/2],subst(f,'x,x)^(-1/2));


period_gc(p=precision(1.)) =
    my(r,n,Q,x3,xk);
    x3 = sqrt((sqrt(5)-1)/2);
    r = acosh((abs(1/x3+1)+abs(1/x3-1))/2);
    n = ceil((p*log(10)+log(2*Pi)+1) / (2*r));
    /* x=x3*u, -1<u<1 */
    my(Q=((x3*'x)^2-1)*((x3*'x)^2+(1+sqrt(5))/2)/x3^4);
    2/x3^2 * Pi / n * sum(k=1,n,
      xk=cos((2*k-1)*Pi/(2*n));
      1/sqrt(-subst(Q,'x,xk))
    );
```

Timings

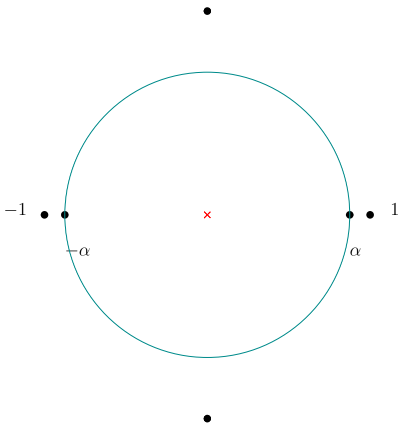
Single integral $\int_\gamma \frac{\mathrm{d}x}{y}$ on $X : y^2 = x^6 - 2x^2 + 1$.

prec	intnum	gc	PARI
100	9	2	2
200	29	7	7
500	283	64	41
1000	2.7s	519	261
2000	$\star$	5.7s	1.7s
5000	$\star$	84.9s	19.4s
10000	$\star$	$\star$	107s

Method 3 - Power series

Taylor expansion at 0,

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}x}{\sqrt{1-2x^2+x^6}} = \sum a_k x^k$$



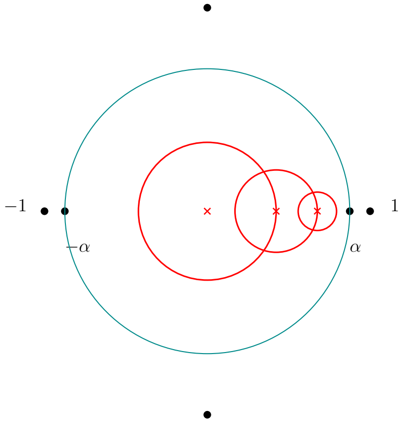
Method 3 - Power series

Taylor expansion at 0, x_1, x_2,

$$\frac{\mathrm{d}x}{y} = \sum a_k x^k$$

$$\int \frac{\mathrm{d}x}{y} = \sum a_k \frac{x^{k+1}}{k+1}$$

series converges for $|x| < \alpha$ (and $a_k \sim \alpha^{-k}$)



Evaluate at $x_1 = t\alpha$ for $t < 1$, x_2, $x_3 = \alpha(1-(1-t)^3)$
each $\sim p/\log(1/t)$ terms for precision p

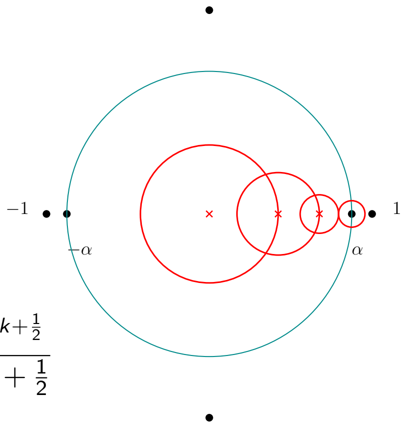
Method 3 bis - Puiseux series

Puiseux expansion near root α

$$f(\alpha + t) = tg(t)$$

$$\frac{\mathrm{d}x}{y} = g(t)^{-\frac{1}{2}} \frac{\mathrm{d}t}{\sqrt{t}} = \sum b_k t^{k-\frac{1}{2}}$$

$$\int \frac{\mathrm{d}x}{y} = \int g(t)^{-\frac{1}{2}} \frac{\mathrm{d}t}{\sqrt{t}} = \sum b_k \frac{t^{k+\frac{1}{2}}}{k+\frac{1}{2}}$$



Chain $0 \to x_1 \to x_2 \to x_3 \leftarrow \alpha$.

Method 3

```
intseries(x0,x1,n) =
  subst(Pol(
    intformal((subst(f,'x,'x+x0)+O('x^n))^(-1/2))),'x,x1-x0);

intsing(x0,x1,n) =
  h = x1-x0;
  f0 = real(Pol(Vec(subst(f,'x,x0+'x),poldegree(f))));
  a  = Vec((subst(f0,'x,'x)+O('x^n))^(-1/2));
  a  = Polrev(vector(n,k,a[k]/(k-1/2)));
  sqrt(h)*subst(a,'x,h);

period_ser(p=precision(1.)) =
    x3 = sqrt((sqrt(5)-1)/2);
    t = solve(t=0,1,(1-t)^3-(1-x3)*t);
    n = ceil(-(p*log(10)-log(1-t))/log(t));
    z = vector(4,i,1-(1-t)^(i-1))*x3;
    4*(sum(k=1,3,intseries(z[k],z[k+1],n))+intsing(x3,z[4],n));
```

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Single integral $\int_\gamma \frac{\mathrm{d}x}{y}$ on $X : y^2 = x^6 - 2x^2 + 1$.

prec	intnum	gc	PARI	series
100	9	2	2	4
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500	283	64	41	65
1000	2.7s	519	261	410
2000	$\star$	5.7s	1.7s	3.0s
5000	$\star$	84.9s	19.4s	32s
10000	$\star$	$\star$	107s	152s

Method 4 - Binary splitting

$$z(x) = \int_0^x \frac{\mathrm{d}x}{y} = \sum a_k x^k$$

satisfies the linear recursion

$$\forall n, d_6(n) a_{n+6} + d_4(n) a_{n+4} + d_0(n) a_n = 0$$

with degree 2 polynomials d_0, d_4, d_6 coming from differential equation

$$2f(x)z'' + f'(x)z' = 0$$

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Fast evaluation of a_n

$$(a_{n+1}, \dots a_{n+6}) = (a_n, \dots a_{n+6-1}) M(n)$$

where $M(n)$ is the companion matrix of the recurrence.

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Fast evaluation of a_n with a product tree

$$(a_{n+1}, \dots a_{n+6}) = (a_0, \dots a_5) M(0) M(1) \dots M(n)$$

Same for $S_n = \sum_{k<n} a_k (\frac{p}{q})^k$.

Method 4

```
d6 = 2*'n^2+22*'n+60
d4 = -4*'n^2-32*'n-64
d0 = 2*'n^2+4*'n
delta = [d0,0,0,0,d4,0,d6];

/* get rapidly a[n] */
A(n) =
  v0=[0, 1, 0, 1/3, 0, 3/10];
  vm = vector(n-5,k,matcompanion(Polrev(subst(delta,'n,k-1))));
  v0*vecprod(vm);


A(1000)
time = 24 ms

Vecrev(intseries(0,'x,1000));
time = 1,032 ms.
```

Ore Algebra Analytic

Involved process to get things fast and rigourous.

Sage package developped by Marc Mezzarobba (analytic continuation, very general).

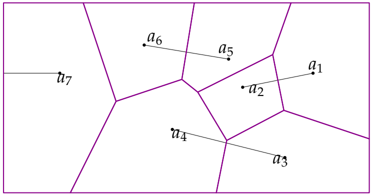
```
from ore_algebra import OreAlgebra
A.<x> = ZZ['x']
Dops.<Dx> = OreAlgebra(A, 'Dx')

f=x^6-2*x^2+1
D=f*Dx^2+2*f.diff()*Dx

X=f.roots(QQbar)
time D.numerical_transition_matrix([0,X[2][0]],1e-500)[0,1]
Wall time: 1.35 s
```

Timings

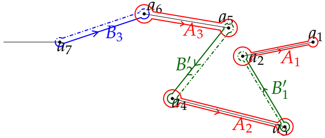
Single integral $\int_\gamma \frac{\mathrm{d}x}{y}$ on $X : y^2 = x^6 - 2x^2 + 1$.

prec	intnum	gc	PARI	series	OreAlg
100	9	2	2	4	365
200	29	7	7	10	562
500	283	64	41	65	1.2s
1000	2.7s	519	261	410	1.8s
2000	$\star$	5.7s	1.7s	3.0s	3.5s
5000	$\star$	84.9s	19.4s	32s	8.3s
10000	$\star$	$\star$	107s	152s	20.7s

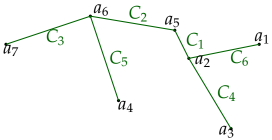
General curves



Bruin&al 2018 : Voronoi cell's approach generalized (Sage)



Neurohr 2018 : mixed tree + loops (Magma)



Tree + differential equations (work in progress with Marc)