12. The Lie algebra $\mathring{\mathfrak{g}}$ and its loop algebra

Let R be a finite, irreducible root system with basis $\{\alpha_1, \dots, \alpha_n\}$. Let $\mathring{A}$ be its Cartan matrix and

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}^+ \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}^- = \Big(\bigoplus_{\alpha\in R^+} \mathring{\mathfrak{g}}_\alpha\Big) \oplus \mathring{\mathfrak{h}} \oplus \Big(\bigoplus_{\alpha\in R^+} \mathring{\mathfrak{g}}_{-\alpha}\Big)$$

the corresponding KM-algebra, with generators e_i, f_i for $i = 1, \dots, n$ (since $\mathring{A}$ is invertible, $\mathring{\mathfrak{h}}$ is spanned by the $\alpha_i^\vee = [e_i, f_i]$).[1]

Definition 12.1. Denote by S the Laurent polynomial ring $K[t, t^{-1}]$. Let

$$\mathcal{L}(\mathring{\mathfrak{g}}) = S \otimes_K \mathring{\mathfrak{g}} = \bigoplus_{r\in\mathbb{Z}} t^r \otimes \mathring{\mathfrak{g}}$$

be the S-Lie algebra obtained by extension of scalars.

Definition 12.2. Let $\mathcal{L}$ be a K-Lie algebra. A *central extension* of $\mathcal{L}$ by a vector space C is a Lie algebra $\widetilde{\mathcal{L}}$ such that:
(a) $\widetilde{\mathcal{L}} = C \oplus \mathcal{L}$ as vector spaces.
(b) C is central, i.e. $[C, \widetilde{\mathcal{L}}] = 0$.
(c) For every $x, y \in \mathcal{L}$, one has

$$[x, y] = \psi(x, y) + [x, y]_{\mathcal{L}}, \tag{12.1}$$

where $[\ ,\ ]_{\mathcal{L}}$ is the given bracket on $\mathcal{L}$ and ψ is a bilinear map on $\mathcal{L}$ with values in C.

The condition $[x, x] = 0$ is equivalent to $\psi(x, x) = 0$, i.e. ψ must be alternating. Further, one sees that the Jacobi identity is equivalent to the following condition, for all $x, y, z \in \mathcal{L}$:

$$\psi([x, y]_{\mathcal{L}}, z) + \psi([y, z]_{\mathcal{L}}, x) + \psi([z, x]_{\mathcal{L}}, y) = 0. \tag{12.2}$$

An alternating bilinear map $\psi : \mathcal{L} \to C$ satisfying (12.2) is called a 2-cocycle with values in C.

Definition 12.3. Let $\mathcal{L}$ be a Lie algebra over K and D a derivation of $\mathcal{L}$, i.e. a K-linear endomorphism D of $\mathcal{L}$ such that

$$D([x, y]) = [D(x), y] + [x, D(y)] \qquad \text{for all } x, y \in \mathcal{L}. \tag{12.3}$$

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[0] Version of 23/2/26.

[1] In other words, $\mathring{\mathfrak{g}}$ is a split simple Lie algebra over K, with split Cartan subalgebra $\mathring{\mathfrak{h}}$.

On the vector space $\widehat{\mathcal{L}} = \mathcal{L} \oplus KD$, there is a Lie bracket defined for all $x, y \in \mathcal{L}$ and $\lambda, \mu \in K$ by:

$$[x + \lambda D, y + \mu D] = [x, y] + \lambda D(y) - \mu D(x). \tag{12.4}$$

Note that $[D, \mathcal{L}] \subset \mathcal{L}$, hence $\mathcal{L}$ is an ideal of $\widehat{\mathcal{L}}$.

Lemma 12.4. *Let* $(\,|\,)$ *be an invariant symmetric bilinear form on* $\mathring{\mathfrak{g}}$. *On* $\mathcal{L} = \mathcal{L}(\mathring{\mathfrak{g}})$, *the bilinear form* ψ *defined by:*

$$\psi(t^r \otimes x, t^s \otimes y) = r\, \delta_{r,-s}\, (x \mid y) \tag{12.5}$$

is a 2*-cocycle.*

Proof. Firstly, if $s = r$ then $r\, \delta_{r,-r} = 0$, hence ψ is alternating. Now, let $x, y, z \in \mathring{\mathfrak{g}}$ and $r, s, u \in \mathbb{Z}$. Set $a = t^r \otimes x$, $b = t^s \otimes y$ and $c = t^u \otimes z$. Then, one has:

$$\begin{aligned}
\psi([a, b]_{\mathcal{L}}, c) &= t^{r+s+u}(r + s)\delta_{r+s+u,0}\, ([x, y] \mid z) \\
\psi([b, c]_{\mathcal{L}}, a) &= t^{r+s+u}(s + u)\delta_{r+s+u,0}\, ([y, z] \mid x) \\
\psi([c, a]_{\mathcal{L}}, b) &= t^{r+s+u}(u + r)\delta_{r+s+u,0}\, ([z, x] \mid y).
\end{aligned}$$

Further, using the symmetry and invariance of $(\,|\,)$, one obtains:

$$\begin{cases} ([y, z] \mid x) = (x \mid [y, z]) = ([x, y] \mid z), \\ ([z, x] \mid y) = (z \mid [x, y]) = ([x, y] \mid z). \end{cases} \tag{12.6}$$

It follows that

$$\psi([a, b]_{\mathcal{L}}, c) + \psi([b, c]_{\mathcal{L}}, a) + \psi([c, a]_{\mathcal{L}}, t^s b) = \\ 2\, t^{r+s+u}(r + s + u)\delta_{r+s+u,0}\, ([x, y] \mid z) = 0.$$

This proves that ψ it a 2-cocycle with values in K. $\square$

Definition 12.5. Let $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ be the central extension of $\mathcal{L}(\mathring{\mathfrak{g}})$ by K defined by ψ, i.e. $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}}) = Kc \oplus \mathcal{L}(\mathring{\mathfrak{g}})$, with c central, and one has:

$$[t^r \otimes x, t^s \otimes y] = t^{r+s} \otimes [x, y] + r\, \delta_{r,-s}(x \mid y)c. \tag{12.7}$$

Remark 12.6. We chose $c = 1$ as the generator of the vector space K. If one replaces the given invariant form $\varphi_0 = (\,|\,)$ be a multiple $\varphi = z\varphi_0$, for some $z \in K^\times$, the resulting central extension $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})_\varphi$ is isomorphic to $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ via the isomorphism sending c to zc.

Lemma 12.7. *Let* d *be the endomorphism of* $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$ *which kills* c *and acts by* $t\dfrac{d}{dt}$ *on* $\mathcal{L}(\mathring{\mathfrak{g}})$. *Then* d *is a derivation of* $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$.

Proof. Let us prove that $d([a,b]) = [d(a),b]+[a,d(b)]$ for all $a,b \in \widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$. Since c is central and killed by d, it suffices to treat the case where $a = t^r \otimes x$ and $b = t^s \otimes y$. Then

$$d([a,b]) = (r+s)t^{r+s} \otimes [x,y]$$

and

$$(*) \qquad [d(a),b] + [a,d(b)] = (r+s)t^{r+s} \otimes [x,y] + \psi(ra,b) + \psi(a,sb).$$

But $\psi(a,sb) = rs\,\delta_{r,-s}\,(x \mid y)$ while

$$\psi(ra,b) = r^2\,\delta_{r,-s}\,(x \mid y) = -rs\,\delta_{r,-s}\,(x \mid y).$$

Hence the RHS of $(*)$ equals $d([a,b])$. This proves that d is a derivation of $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}})$. $\square$

Definition 12.8. The Lie algebra $\widetilde{\mathcal{L}}(\mathring{\mathfrak{g}}) \oplus Kd = Kc \oplus \mathcal{L}(\mathring{\mathfrak{g}}) \oplus Kd$ is denoted by $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.

Remark 12.9. A more conceptual way to define (and understand) the cocycle ψ of (12.5) is as follows. Let $\mathrm{Res} : S \to K$ be the residue map, i.e. $\mathrm{Res}(t^r) = 0$ if $r \neq -1$ and $\mathrm{Res}(t^{-1}) = 1$. One has an exact sequence

$$0 \longrightarrow S \xrightarrow{d/dt} S \xrightarrow{\mathrm{Res}} K \longrightarrow 0.$$

For $P \in S$, denote dP/dt simply by P'. Consider the K-bilinear form on S with values in K defined for all $P,Q \in S$ by:

$$\varphi(P,Q) = \mathrm{Res}(P'Q) = -\mathrm{Res}(PQ'). \tag{12.8}$$

For all $P,Q,U \in S$, one has $(PQ)'U + (QU)'P + (UP)'Q = 2(PQU)'$ and hence:

$$\varphi(PQ,U) + \varphi(QU,P) + \varphi(UP,Q) = 0. \tag{12.9}$$

Now, for $P = t^r$ and $Q = t^s$ one has $\varphi(P,Q) = \mathrm{Res}(rt^{r-1+s}) = r\delta_{r,-s}$ and hence (12.5) can be rewritten as follows:

$$\psi(P \otimes x, Q \otimes y) = (x \mid y)\,\varphi(P,Q). \tag{12.10}$$

Then, using (12.6) and (12.9), one obtains that ψ satisfies the condition (12.2).

13. $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$ as a KM-algebra

13.1. **Generators of $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.** Firstly, we identifiy $\mathring{\mathfrak{g}}$ with the Lie subalgebra $1 \otimes \mathring{\mathfrak{g}}$ of $\mathcal{L}(\mathring{\mathfrak{g}})$.

Notation 13.1. Set Set $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus Kc \oplus Kd$ and $\mathfrak{n}^- = \mathring{\mathfrak{n}}^- \oplus \bigoplus_{r \geq 1} t^{-r} \otimes \mathring{\mathfrak{g}}$ and $\mathfrak{n}^+ = \mathring{\mathfrak{n}}^+ \oplus \bigoplus_{r \geq 1} t^{r} \otimes \mathring{\mathfrak{g}}$. Each is a Lie subalgebra of $\mathfrak{g}$.

Since R is irreducible, it has a unique maximal root

$$\gamma = \sum_{i=1}^{n} a_i \alpha_i \tag{13.1}$$

which is the highest weight of the adjoint representation. Choose generators e_γ and $e_{-\gamma}$ of $\mathring{\mathfrak{g}}_\gamma$ and $\mathring{\mathfrak{g}}_{-\gamma}$ respectively, such that $[e_\gamma, e_{-\gamma}] = \gamma^\vee$, and set $e_0 = t \otimes e_{-\gamma}$ and $f_0 = t^{-1} \otimes e_\gamma$.

Lemma 13.2. *The Lie subalgebra $\mathfrak{n}^-$ is generated by $f_0, f_1, \dots, f_n$. Similarly, $\mathfrak{n}^+$ is generated by $e_0, e_1, \dots, e_n$*

Proof. Note that $f_0, f_1, \dots, f_n$ belong to $\mathfrak{n}^-$. Let $\mathcal{N}$ be the Lie subalgebra that they generate.

Firstly, we know that $f_1, \dots, f_n$ generate $\mathring{\mathfrak{n}}^-$. Further, since e_γ is a highest weight vector of $\mathring{\mathfrak{g}}$, it generates $\mathring{\mathfrak{g}}$ as a $\mathring{\mathfrak{n}}^-$-module. Since $f_0 = t^{-1} \otimes e_\gamma$, it follows that $\mathcal{N}$ contains $t^{-1} \otimes \mathring{\mathfrak{g}}$. Next, since $\mathring{\mathfrak{g}}$ is simple one has $[\mathring{\mathfrak{g}}, \mathring{\mathfrak{g}}] = \mathring{\mathfrak{g}}$ and hence $\mathcal{N}$ contains

$$[t^{-1} \otimes \mathring{\mathfrak{g}}, t^{-1} \otimes \mathring{\mathfrak{g}}] = t^{-2} \otimes [\mathring{\mathfrak{g}}, \mathring{\mathfrak{g}}] = t^{-2} \otimes \mathring{\mathfrak{g}}.$$

By induction one obtains that $\mathcal{N}$ contains $t^{-r} \otimes \mathring{\mathfrak{g}}$ for every $r \in \mathbb{N}^*$. Thus, $\mathcal{N} = \mathfrak{n}^-$. This proves the first assertion, and the second is obtained similarly. ☐

13.2. **The subalgebra $\mathfrak{h}$ and its roots in $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.**

Notation 13.3. Recall that $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus Kc \oplus Kd$.

(a) We identify $(\mathring{\mathfrak{h}})^*$ with the orthogonal of $Kc \oplus Kd$ in $\mathfrak{h}^*$.

(b) Let δ be the element of $\mathfrak{h}^*$ defined by $\delta(\mathring{\mathfrak{h}} + Kc) = 0$ and $\delta(d) = 1$. One sets:

$$\alpha_0 = \delta - \gamma. \tag{13.2}$$

Then $\Pi = (\alpha_0, \dots, \alpha_n)$ is a K-basis of $(\mathring{\mathfrak{h}})^* \oplus K\delta$. Further, with the notation of (13.1), one has:

$$\boxed{\delta = \alpha_0 + \gamma = \alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n.} \tag{13.3}$$

Let us consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Since c is central, it suffices to consider elements of $\mathfrak{h}$ of the form $h = h_0 + zd$, with $h_0 \in \mathring{\mathfrak{h}}$ and $z \in K$. For $\alpha \in R \cup \{0\}$ and $a = t^r \otimes x$, with $r \in \mathbb{Z}$ and $x \in \mathring{\mathfrak{g}}_\alpha$, note that $[h_0, a] = t^r \otimes [h_0, x]$ since the term $r\delta_{r,0}\,(h_0 \mid x)\,c$ is 0. Hence, for $h = h_0 + zd$ one has:

$$[h, a] = zd(a) + t^r \otimes [h_0, x] = rza + \alpha(h_0)a = (r\delta + \alpha)(h)a. \tag{13.4}$$

This shows that $t^r \otimes \mathring{\mathfrak{g}}_\alpha$ has weight $r\delta + \alpha$. In particular, $t^r \otimes \mathring{\mathfrak{h}}$ has weight $r\delta$. Thus, we obtain the following:

Lemma 13.4. *For the adjoint action of $\mathfrak{h}$, the non-zero weights of $\mathfrak{g}$ are:*

(1) *$r\delta + \alpha$, with $\alpha \in R$ and $r \in \mathbb{Z}$, with multiplicity* 1*.*
(2) *$r\delta$, with $r \in \mathbb{Z} - \{0\}$, with multiplicity n.*

In particular, using definition (13.2), we see that e_0 (resp. f_0) has weight α_0 (resp. $-\alpha_0$). We set $\alpha_0^\vee = [e_0, f_0]$.

Lemma 13.5. *One has $\alpha_0^\vee = \dfrac{2}{(\gamma \mid \gamma)} c - \gamma^\vee$. In particular, $\Pi^\vee = (\alpha_0^\vee, \ldots, \alpha_n^\vee)$ is a K-basis of $\mathring{\mathfrak{h}} \oplus Kc$.*

Proof. Since $e_0 = t \otimes e_{-\gamma}$ and $f_0 = t^{-1} \otimes e_\gamma$, one has

$$[e_0, f_0] = [e_{-\gamma}, e_\gamma] + (e_{-\gamma} \mid e_\gamma)\, c = -\gamma^\vee + (e_{-\gamma} \mid e_\gamma)\, c.$$

Also, by assertion (2) of Th. 10.7 one has $[e_\gamma, e_{-\gamma}] = (e_\gamma \mid e_{-\gamma})\, \tau^{-1}(\gamma)$, whereas by (10.5) one has $\tau(\gamma^\vee) = 2\gamma/(\gamma \mid \gamma)$. Thus, our assumption that $[e_\gamma, e_{-\gamma}] = \gamma^\vee$ is equivalent to $(e_\gamma \mid e_{-\gamma}) = 2/(\gamma \mid \gamma)$. This proves the lemma. □

Notation 13.6. The *normalised* invariant symmetric bilinear form $(\,|\,)$ on $\mathring{\mathfrak{g}}$ is the one such that for the induced form on $(\mathring{\mathfrak{h}})^*$ one has $(\gamma \mid \gamma) = 2$. From now on, we choose this form. With this choice, the previous lemma gives:

$$c = \alpha_0^\vee + \gamma^\vee. \tag{13.5}$$

and this element c is called the *canonical central element* of $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.

Definition 13.7. Since $(\alpha_1^\vee, \ldots, \alpha_n^\vee)$ is a basis of the dual root system $R^\vee$, there exist a unique n-tuple $(c_1, \ldots, c_n) \in \mathbb{N}^n$ such that

$$\gamma^\vee = c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee. \tag{13.6}$$

Then, (13.5) can be rewritten as:

$$\boxed{c = \alpha_0^\vee + c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee.} \tag{13.7}$$

13.3. **The Cartan matrix A and the isomorphism $\mathfrak{g}(A) \simeq \widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.**

Proposition 13.8. *Set $a_{ij} = \alpha_j(\alpha_i^\vee)$ for $i, j = 0, \ldots, n$.*

(1) *The matrix $A = (a_{ij})_{i,j=0,\ldots,n}$ is a Cartan matrix of rank n. More precisely,* $\mathrm{Ker}(A)$ *and* $\mathrm{Ker}(A^t)$ *are spanned, respectively, by the transpose of the row vectors*

$$(1, a_1, \ldots, a_n) \qquad \textit{and} \qquad (1, c_1, \ldots, c_n). \tag{13.8}$$

(2) *$(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realisation of A.*

Proof. A contains the $n \times n$ submatrix $\overset{\circ}{A}$ which is invertible, hence A has rank at least n.

One has $\alpha_0 = \delta - \gamma$ and $\alpha_0^\vee = c - \gamma^\vee$. Since δ and c are orthogonal to $\overset{\circ}{\mathfrak{h}} + Kc$ and $(\overset{\circ}{\mathfrak{h}})^* + Kd$ respectively, one has $a_{00} = \gamma(\gamma^\vee) = 2$ and, for $i = 1, \ldots, n$:

$$a_{i0} = -\gamma(\alpha_i^\vee) \qquad \text{and} \qquad a_{0i} = -\alpha_i(\gamma^\vee).$$

All these are integers. Moreover, for $i = 1, \ldots, n$ one has:

$$\alpha_i(\gamma^\vee) = \frac{2(\alpha_i \mid \gamma)}{(\gamma \mid \gamma)} = \frac{(\alpha_i \mid \alpha_i)}{(\gamma \mid \gamma)}\gamma(\alpha_i^\vee). \tag{13.9}$$

Since γ is the highest weight of the adjoint representation, one has $\gamma(\alpha_i^\vee) \in \mathbb{N}$ for $i = 1, \ldots, n$. Together with (13.9), this implies that a_{i0} and a_{0i} both belong to $\mathbb{Z}^-$ and are simultaneously zero or non-zero. This shows that A is a Cartan matrix.

For every row vector $x = (x_0, \ldots, x_n)$, $A\,x^t$ is the column vector $\big((\sum_{j=0}^n x_j\alpha_j)(\alpha_i^\vee)\big)_{i=0,\ldots,n}$. Since $\alpha_0 + \sum_{i=1}^n a_i\alpha_i = \alpha_0 + \gamma = \delta$ is orthogonal to $\alpha_0^\vee, \ldots, \alpha_n^\vee$, one obtains that $A(1, a_1, \ldots, a_n)^t = 0$.

Similarly, for every row vector $y = (y_0, \ldots, y_n)$, yA is the row vector $\big(\alpha_j(\sum_{i=0}^n y_i\alpha_i^\vee)\big)_{j=0,\ldots,n}$. Since $\alpha_0^\vee + \sum_{i=1}^n c_i\alpha_i^\vee = \alpha_0^\vee + \gamma^\vee = c$ is orthogonal to $\alpha_0, \ldots, \alpha_n$, one obtains that $(1, c_1, \ldots, c_n)A = 0$, hence $A^t\,(1, c_1, \ldots, c_n)^t = 0$. This completes the proof of assertion (1).

Assertion (2) follows, because A has size $n+1$ and $\mathfrak{h}$ has dimension $2(n+1) - \operatorname{rank}(A) = n+2$, and Π and $\Pi^\vee$ are linearly independent sets in $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. $\square$

Theorem 13.9. *$\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is the KM-algebra $\mathfrak{g}(A)$.*

Proof. We have seen that $\mathfrak{h}$ is a realisation of A and that $\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^-$ and $\mathfrak{n}^+$ are the Lie subalgebras generated by $f_0, \ldots, f_n$ and $e_0, \ldots, e_n$ respectively.

Further, these generators satisfy the defining relations of the auxiliary Lie algebra $\widetilde{\mathfrak{g}}(A)$. Therefore, $\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is the quotient of $\widetilde{\mathfrak{g}}(A)$ by some ideal $\mathfrak{r}'$ such that $\mathfrak{r}' \cap \mathfrak{h} = (0)$. Hence, to prove that $\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) = \mathfrak{g}(A)$, i.e. $\mathfrak{r}' = \mathfrak{r}$, it suffices to prove that $\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ has no ideal $I \neq (0)$ such that $I \cap \mathfrak{h} = (0)$.

Assume that I is such an ideal. Since $\widehat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is the sum of its weight spaces, I must contain a weight vector $a = t^r \otimes x$, with $r \in \mathbb{Z}$ and $x \neq 0$ in $\overset{\circ}{\mathfrak{g}}_\alpha$ for some $\alpha \in R \cup \{0\}$, where r and α are not both 0 (otherwise a would be in $\overset{\circ}{\mathfrak{h}}$). Let $y \in \overset{\circ}{\mathfrak{g}}_{-\alpha}$ such that $(x \mid y) \neq 0$. Then I contains the element

$$[t^r \otimes x, t^{-r} \otimes y] = [x,y] + r(x \mid y)c = \begin{cases} (x \mid y)\big(\tau^{-1}(\alpha) + rc\big) & \text{if } \alpha \neq 0; \\ r(x \mid y)\, c & \text{else.} \end{cases}$$

In both cases, the RHS is a nonzero element of $I \cap \mathfrak{h}$, contradicting the hypothesis. This proves the theorem. □

13.4. **Invariant form on $\mathfrak{h}$ and imaginary roots.**

Proposition 13.10. *The Cartan matrix A is symmetrisable. The resulting symmetric bilinear form on $\mathfrak{h}$ satisfy $(d \mid d)_{\mathfrak{h}} = 0$ and:*

$$(\alpha_i^\vee \mid d)_{\mathfrak{h}} = \delta_{0i} \quad \textit{for } i = 0, \dots, n; \tag{13.10}$$

$$(\alpha_0^\vee \mid h)_{\mathfrak{h}} = -\gamma(h) \quad \textit{for all } h \in \mathring{\mathfrak{h}} \oplus Kc; \tag{13.11}$$

$$(\alpha_i^\vee \mid h)_{\mathfrak{h}} = \frac{2}{(\alpha_i \mid \alpha_i)}\alpha_i(h) \quad \textit{for } i = 1, \dots, n \textit{ and } h \in \mathring{\mathfrak{h}} \oplus Kc. \tag{13.12}$$

In particular, one has:

$$(c \mid \alpha_i^\vee)_{\mathfrak{h}} = 0 \quad \textit{for } i = 0, \dots, n; \tag{13.13}$$

$$(\alpha_0^\vee \mid \alpha_0^\vee)_{\mathfrak{h}} = 2; \tag{13.14}$$

and, identifying $\mathring{\mathfrak{h}}$ with its dual via $(\,|\,)$:

$$(\alpha_0^\vee \mid \alpha_i^\vee)_{\mathfrak{h}} = (-\gamma \mid \alpha_i^\vee) \quad \textit{for } i = 1, \dots, n; \tag{13.15}$$

$$(\alpha_i^\vee \mid \alpha_j^\vee)_{\mathfrak{h}} = (\alpha_i^\vee \mid \alpha_j^\vee) \quad \textit{for } i, j = 1, \dots, n. \tag{13.16}$$

Thus, the restriction of $(\,|\,)_{\mathfrak{h}}$ to $\mathring{\mathfrak{h}}$ is the normalised invariant form.

Proof. Set $\beta_0 = -\gamma$ and $\beta_i = \alpha_i$ for $i = 1, \dots, n$. Identifying $\mathring{\mathfrak{h}}$ with its dual via $(\,|\,)$, one obtains $\beta_i^\vee = 2\beta_i/(\beta_i \mid \beta_i)$ for $i = 0, \dots, n$.

Let D be the diagonal matrix whose i-th entry is $2/(\beta_i \mid \beta_i)$. Then DA is the symmetric matrix $B = \big((\beta_i \mid \beta_j)\big)_{i,j=0,\dots,n}$. Thus A symmetrisable.

Noting that $D_{00} = \dfrac{2}{(\gamma \mid \gamma)} = 1$ and using Def. 10.1 of Chap. 4, one obtains that $(\,|\,)_{\mathfrak{h}}$ is defined by $(d \mid d)_{\mathfrak{h}} = 0$ and, for all $h \in \mathfrak{h}$:

$$(\alpha_i^\vee \mid h) = D_{ii}\, \alpha_i(h) = \begin{cases} \alpha_0(h) & \text{if } i = 0; \\ \dfrac{2}{(\alpha_i \mid \alpha_i)}\alpha_i(h) & \text{if } i = 1, \dots, n. \end{cases} \tag{$*$}$$

Since $\alpha_i(d) = 0$ for $i = 1, \dots, n$ and $1 = \delta(d) = \alpha_0(d) + \sum_{i=1}^n a_i\alpha_i(d)$, one obtains (13.10). And since $\alpha_0 = \delta - \gamma$ and δ vanishes on $\mathring{\mathfrak{h}} \oplus Kc$, one obtains (13.11). And (13.12) follows immediately from $(*)$.

Since $\alpha_1, \dots, \alpha_n$ and δ vanish on c, one obtains (13.13). Further, let $\overset{\circ}{\tau}$ be the isomorphism $\overset{\circ}{\mathfrak{h}} \xrightarrow{\sim} (\overset{\circ}{\mathfrak{h}})^*$ induced by $(\,|\,)$. Then for all $\lambda \in (\overset{\circ}{\mathfrak{h}})^*$ and $h \in \overset{\circ}{\mathfrak{h}}$, one has $(\lambda \mid \overset{\circ}{\tau}(h)) = \lambda(h)$. Since, for $i = 1, \dots, n$, one has:

$$\overset{\circ}{\tau}(\alpha_i^\vee) = \frac{2}{(\alpha_i \mid \alpha_i)}\alpha_i,$$

one obtains $\big(-\gamma \mid \overset{\circ}{\tau}(\alpha_i^\vee)\big) = -\gamma(\alpha_i^\vee) = (\alpha_0 \mid \alpha_i^\vee)_{\mathfrak{h}}$. This gives (13.15) where in the RHS we have written, by abuse of notation, $\alpha_i^\vee$ instead of $\overset{\circ}{\tau}(\alpha_i^\vee)$. Similarly, for $i, j = 1, \dots, n$ one has:

$$\frac{2}{(\alpha_i \mid \alpha_i)}\alpha_i(\alpha_j^\vee) = \Big(\frac{2}{(\alpha_i \mid \alpha_i)}\alpha_i \mid \overset{\circ}{\tau}(\alpha_j^\vee)\Big) = \big(\overset{\circ}{\tau}(\alpha_i^\vee) \mid \overset{\circ}{\tau}(\alpha_j^\vee)\big).$$

This gives (13.16), with the same abuse of notation as above. □

Let ω_0 be the element of $\mathfrak{h}^*$ defined by $\omega_0(\overset{\circ}{\mathfrak{h}} + Kd) = 0$ and $\omega_0(c) = 1$. Then $(\alpha_0, \dots, \alpha_n, \omega_0)$ is a K-basis of $\mathfrak{h}^*$.

Corollary 13.11. *The isomorphism $\tau : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ induced by $(\,|\,)_{\mathfrak{h}}$ satisfies $\tau(d) = \omega_0$, $\tau(\alpha_i^\vee) = 2\alpha_i/(\alpha_i \mid \alpha_i)$ for $i = 1, \dots, n$ and $\tau(c) = \delta$. Further, one has $\tau(\alpha_0^\vee) = \alpha_0$ and*

$$\text{(13.17)} \qquad \begin{cases} (\delta \mid \alpha_i)_{\mathfrak{h}^*} = 0 \text{ for } i = 0, \dots, n, \text{ hence } (\delta \mid \delta) = 0; \\ (\alpha_i \mid \alpha_j)_{\mathfrak{h}^*} = (\alpha_i \mid \alpha_j) \quad \text{for } i, j = 1, \dots, n. \end{cases}$$

In particular, the restriction of $(\,|\,)_{\mathfrak{h}^*}$ to $(\overset{\circ}{\mathfrak{h}})^*$ is the normalised invariant form.

Proof. For $h \in \mathfrak{h}$ one has $\tau(d)(h) = (d \mid h) = 0$ if $h \in \overset{\circ}{\mathfrak{h}} \oplus Kd$, and $= 1$ if $h = \alpha_0^\vee$. Thus $\tau(d) = \omega_0$. Note also that since $c = \alpha_0^\vee + \sum_{i=1}^n c_i\alpha_i^\vee$, one has $(d \mid c) = 1$.

Next, one has $\tau(c)(\alpha_i^\vee) = (c \mid \alpha_i^\vee) = 0$ for $i = 0, \dots, n$, hence $\tau(c) = z\delta$ for some $z \in K$. Since $\tau(c)(d) = (c \mid d) = 1$, one obtains $z = 1$. Thus $\tau(c) = \delta$. We have seen in the proof of Prop. 13.10 that $\tau(\alpha_i^\vee) = 2\alpha_i/(\alpha_i \mid \alpha_i)$ for $i = 1, \dots, n$. This gives (13.17).

Further, one has $\tau(\alpha_0^\vee)(h) = (\alpha_0^\vee \mid h) = -\gamma(h)$ for all $h \in \overset{\circ}{\mathfrak{h}} \oplus Kc$. Thus, $\tau(\alpha_0^\vee) = -\gamma + z\delta$, for some $z \in K$. Since $\tau(\alpha_0^\vee)(d) = (d \mid \alpha_0^\vee) = 1$, one has $z = 1$ and hence $\tau(\alpha_0^\vee) = \delta - \gamma = \alpha_0$. □

Remark 13.12. The equalities $\tau(c) = \delta$ and $\tau(\alpha_0^\vee) = \alpha_0$ give the following equality:

$$\alpha_0 + \sum_{i=1}^n a_i\alpha_i = \delta = \tau(\alpha_0^\vee) + \sum_{i=1}^n c_i\tau(\alpha_i^\vee) = \alpha_0 + \sum_{i=1}^n \frac{2c_i}{(\alpha_i \mid \alpha_i)}\alpha_i.$$

Therefore, one has $a_i = \dfrac{2c_i}{(\alpha_i \mid \alpha_i)}$ for $i = 1, \dots, n$. This can be seen directly, as follows. Since $(\gamma \mid \gamma) = 2$ one has, on the one hand, $\gamma^\vee = \dfrac{2}{(\gamma \mid \gamma)}\gamma = \gamma = \sum_{i=1}^n a_i\alpha_i$. And one has $\gamma^\vee = \sum_{i=1}^n c_i\alpha_i^\vee = \sum_{i=1}^n \dfrac{2c_i}{(\alpha_i \mid \alpha_i)}\alpha_i$ on the other hand.

Remark 13.13. If α is a real root, there exists $w \in W$ such that $w\alpha$ is a simple root α_i, and hence one has $(\alpha \mid \alpha) = (\alpha_i \mid \alpha_i) > 0$. Thus, since $(\delta \mid \delta) = 0$, the roots $k\delta$, with $k \in \mathbb{Z}^\times$, are imaginary. By the proposition below, these are the only imaginary roots. For this reason, δ is called the *fundamental* imaginary root.

Note that, since $(\delta \mid \alpha_i^\vee) = 0$ one has $s_i\delta = \delta$ for $i = 0, \dots, n$, hence δ is invariant by the Weyl group W.

Lemma 13.14. (1) *As a vector space, $(\mathring{\mathfrak{h}})^*$ is spanned by the short roots, and also by the long roots.*

(2) *There exists a positive short root α such that $(\alpha \mid \gamma^\vee) = 1$.*

Proof. We know that $E = (\mathring{\mathfrak{h}})^*$ is spanned by the set of simple roots. Thus, if all simple roots have the same length, they are both short and long, and (1) follows.

So, assume that β, α are adjacent simple roots, with β long and α short. Then $(\alpha \mid \beta^\vee) = -1$ and $(\beta \mid \alpha^\vee) = -p$ with $p = 2$ or 3, hence $s_\beta(\alpha) = \alpha + \beta$ and $s_\alpha(\beta) = \beta + p\alpha$.

We know that all roots of the same length are conjugate under $\mathring{W}$, the Weyl group of R. Let R_s, resp. R_ℓ, be the subset of short, resp. long, roots of R and let E_s, resp. E_ℓ, be the subspace of E spanned by R_s, resp. R_ℓ. Both E_s and E_ℓ are stable by $\mathring{W}$. Since $\beta = s_\beta(\alpha) - \alpha$ and $\alpha = (1/p)(s_\alpha(\beta) - \beta)$, we see that E_s contains β, hence all long roots, and E_ℓ contains α, hence all short roots. Therefore, $E_s = E = E_\ell$.

(2) By (1), γ cannot be orthogonal to all short roots, hence there exists a positive short root α such that $(\gamma \mid \alpha)$ is non-zero; then it must be positive since γ is a dominant weight. Finally, since α is a short root, one must have $(\alpha \mid \gamma^\vee) = 1$. ☐

Proposition 13.15. *For every $\alpha \in R$ and $k \in \mathbb{Z}$, the root $k\delta + \alpha$ is real.*

Proof. Note first that Φ^+_{re} contains R^+ and $\alpha_0 = \delta - \gamma$. Hence it contains also $\delta - w\gamma$ for each $w \in \mathring{W}$. Hence Φ^+_{re} contains all roots $\delta - \beta$, with $\beta \in R_\ell$. Let $k \geq 2$ and suppose that Φ^+_{re} contains $r\delta + R_\ell$ for $r = 1, \dots, k-1$. Then it contains also $s_0((k-2)\delta + \gamma)$, which equals:

$$(k-2)\delta + \gamma - \langle (k-2)\delta + \gamma, \alpha_0^\vee\rangle\alpha_0 = (k-2)\delta + \gamma + \langle \gamma, \gamma^\vee\rangle(\delta - \gamma) = k\delta - \gamma,$$

and hence Φ^+_{re} contains $k\delta + R_\ell$. By induction, one obtains that it contains $k\delta + R_\ell$ for all $k \in \mathbb{N}^*$.

Suppose now that R contains roots of different lengths. By Lemma 13.14, there exists $\alpha \in R_s^+$ such that $(\alpha \mid \gamma^\vee) = 1$. Suppose that Φ_{re}^+ contains $k\delta + \alpha$, which is true for $k = 0$. Then it contains $s_0(k\delta + \alpha)$, which equals:

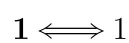
$$k\delta{+}\alpha - \langle k\delta{+}\alpha, \alpha_0^\vee\rangle\alpha_0 = k\delta{+}\alpha + \langle\alpha, \gamma^\vee\rangle(\delta - \gamma) = (k+1)\delta{+}s_\gamma\alpha.$$

Hence Φ_{re}^+ contains $(k+1)\delta{+}ws_\gamma\alpha$, for all $w \in \mathring{W}$, i.e. it contains $(k+1)\delta{+}R_s$. By induction, it follows that Φ_{re}^+ contains $k\delta + R_s$, for all $k \in \mathbb{N}^*$. This proves the proposition. $\square$

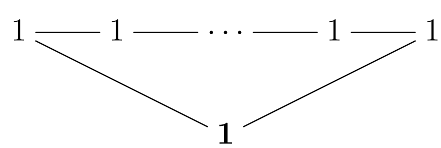
13.5. **The Dynkin diagram of $\widehat{\mathcal{L}}(\mathring{\mathfrak{g}})$.** We give below the Dynkin diagram of A, using Carter's notation $\widetilde{X}_n$, which is a bit shorter than Kač's notation $X_n^{(1)}$. Each diagram $\widetilde{X}_n$ is obtained from the finite diagram X_n by adding a vertex indicated in bold, which corresponds to the simple root α_0. Further, we replace the vertices $\circ$ by the integers $(1, a_1, \ldots, a_n)$ of Prop. 13.8.

When $\mathring{\mathfrak{g}}$ is of type A_n, D_n, E_n, where all roots have the same length, the Cartan matrix A is symmetric. Thus, the vector $(1, c_1, \ldots, c_n)$ of Prop. 13.8 is the same as $(1, a_1, \ldots, a_n)$ and we do not repeat it.

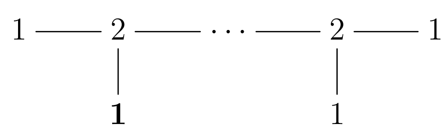
$\widetilde{A}_1$



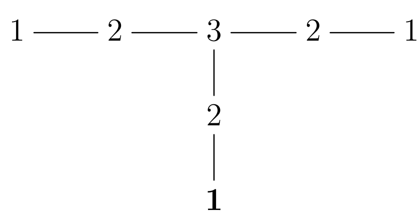
$\widetilde{A}_n,\ n \geq 2$



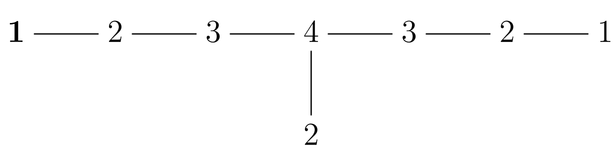
$\widetilde{D}_n,\ n \geq 4$



$\widetilde{E}_6$



$\widetilde{E}_7$



$\widetilde{E}_8$

$$\begin{array}{ccccccccccccccc} 2 & \text{---} & 4 & \text{---} & 6 & \text{---} & 5 & \text{---} & 4 & \text{---} & 3 & \text{---} & 2 & \text{---} & \mathbf{1} \\ & & & & | & & & & & & & & & & \\ & & & & 3 & & & & & & & & & & \end{array}$$

When $\mathring{\mathfrak{g}}$ is of type B_n, C_n or F_4 or G_2, we obtain the following Dynkin diagrams, where the additional coordinates $c_0, c_1, \ldots, c_n$ are indicated as subscripts or supscripts.

In type C_n $(n \geq 2)$, the highest root is $\gamma = \alpha_n + 2\sum_{i=1}^{n-1} \alpha_i$ and $\gamma^\vee$ is the highest short root in type B_n, which is $\alpha_1^\vee + \cdots + \alpha_n^\vee$. So we get:

$\widetilde{C}_n,\ n \geq 2$

$$\underset{1}{\mathbf{1}} \Longrightarrow \underset{1}{2} \text{---} \underset{1}{2} \text{---} \cdots \text{---} \underset{1}{2} \Longleftarrow \underset{1}{1}$$

In type B_n $(n \geq 3)$, the highest root is $\gamma = \alpha_1 + 2\sum_{i=2}^{n} \alpha_i$ and $\gamma^\vee$ is the highest short root in type C_n, which is $\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$. So we get:

$\widetilde{B}_n,\ n \geq 3$

$$\begin{array}{ccccccccc} \overset{1}{1} & \text{---} & \overset{2}{2} & \text{---} & \cdots & \text{---} & \overset{2}{2} & \Longrightarrow & \overset{1}{2} \\ & & | & & & & & & \\ & & \underset{1}{\mathbf{1}} & & & & & & \end{array}$$

In type F_4, the highest root is $\gamma = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, while $\gamma^\vee$ is the highest short root in type $F_4^\vee \simeq F_4$, which is $2\alpha_1^\vee + 3\alpha_2^\vee + 2\alpha_3^\vee + \alpha_4^\vee$. So we get:

$\widetilde{F}_4$

$$\underset{1}{\mathbf{1}} \text{---} \underset{2}{2} \text{---} \underset{3}{3} \Longrightarrow \underset{2}{4} \text{---} \underset{1}{2}$$

In type G_2, the highest root is $\gamma = 2\alpha_1 + 3\alpha_2$, while $\gamma^\vee$ is the highest short root in type $G_2^\vee \simeq G_2$, which is $2\alpha_1^\vee + \alpha_2^\vee$. So we get:

$\widetilde{G}_2$

$$\underset{1}{\mathbf{1}} \text{---} \underset{2}{2} \Rrightarrow \underset{1}{3}$$

References for this chapter

They are [Ca05], Chap. 17-18, and [Ka90], Chap. 6-7.