STABILITY OF THE POHOŽAEV OBSTRUCTION IN DIMENSION 3

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Abstract. We investigate problems connected to the stability of the well-known Pohožaev obstruction. We generalize results which were obtained in the minimizing setting by Brezis and Nirenberg [2] and more recently in the radial situation by Brezis and Willem [3].

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. Let $h \in C^1(\mathbb{R}^n)$ and let us consider the equation

$$\begin{cases} \Delta u + hu = |u|^{\frac{4}{n-2}} u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{0.1}$$

where $\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. It is well-known that, if Ω is star-shaped with respect to the origin and if h satisfies

$$h(x) + \frac{1}{2}\langle x, \nabla h(x)\rangle \geq 0 \;, \tag{0.2}$$

then there are no non-trivial solutions of (0.1). This is a consequence of Pohožaev's identity (see [11] and equation (4.6) of appendix 4.3) and is referred to as the Pohožaev obstruction.

The above equation has been quite intensively studied in the past thirty years. Many existence results have been obtained if Ω is not assumed to be star-shaped or if h does not verify (0.2). It is almost impossible to give an exhaustive list of references on this equation.

In this paper, we investigate the question of non-existence of positive solutions of equation (0.1) and more precisely the stability properties of the Pohožaev obstruction.

Definition 0.1. *Let Ω be a star-shaped domain of $\mathbb{R}^n$ and let $(X, \|\,.\,\|_X)$ be some Banach space of functions on Ω (typically $X = \mathcal{C}^{k,\eta}(\Omega)$, $X = L^\infty(\Omega)$ or $X = L^p(\Omega)$). Let $h_0 \in X \cap C^1(\Omega)$ be a function which satisfies (0.2). We say that the Pohožaev obstruction is X-stable at (h_0, Ω) if the following property holds : there exists $\delta(h_0, \Omega, X) > 0$ such that for any function $h \in X$ with*

$$\|h - h_0\|_X \leq \delta(h_0, \Omega, X) \;,$$

the only non-negative C^2-solution of (0.1) is $u \equiv 0$.

We say that the Pohožaev obstruction is X-stable if it is X-stable at (h_0, Ω) for all Ω star-shaped with respect to the origin and all $h_0 \in X \cap C^1(\Omega)$ satisfying (0.2).

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Note that the property (0.2) is not stable under perturbations of the function h in any C^k-space. Since the work of Brezis and Nirenberg [2], we know that equation (0.1) behaves differently in dimension 3 and in dimensions $n \geq 4$. It is clear that, in dimensions $n \geq 4$, the Pohožaev obstruction is not X-stable for any reasonable X. Indeed, any perturbation of $h \equiv 0$ which is negative somewhere leads to a minimizing solution in dimensions $n \geq 4$ (see [2])[1]. Hence we investigate in this paper the question of the stability of the Pohožaev obstruction for various spaces X in dimension 3. We give a complete answer to this question in the following theorems.

Theorem 1. *The Pohožaev obstruction is $C^{0,\eta}$-stable for any $\eta > 0$ in dimension 3. In other words, given any $\eta > 0$, any domain Ω in $\mathbb{R}^3$, star-shaped with respect to the origin, and any function $h_0 \in C^1(\Omega)$ satisfying (0.2), there exists $\delta(\eta, \Omega, h_0) > 0$ such that, if $h \in C^{0,\eta}(\Omega)$ satisfies*

$$\|h - h_0\|_{C^{0,\eta}(\Omega)} \leq \delta(\eta, \Omega, h) ,$$

the only non-negative solution of (0.1) is $u \equiv 0$.

Note that a consequence of our theorem is the following : if Ω is a star-shaped domain in $\mathbb{R}^3$, there exists a constant $\hat{\lambda}(\Omega) > 0$ such that equation (0.1) does not possess any nontrivial positive solutions with $h \equiv \lambda$ for $\lambda > -\hat{\lambda}(\Omega)$. This is in sharp contrast with the situation for non star-shaped domains (see [1] for instance).

In the seminal paper [2], it was proved that there are no minimizing solutions of equation (0.1) in dimension 3 if the function $h \geq -\lambda^\star(\Omega)$ for some $\lambda^\star(\Omega) > 0$. Since $h \geq 0$ if h satisfies (0.2), a consequence of this result is a version of the above stability in C^0 when one considers only minimizing solutions. A necessary and sufficient condition on the function h and the domain Ω for the existence of a minimizing solution of (0.1) in dimension 3 was found in [6].

In [3], the authors studied this question in the case of the unit ball with radial functions. If we let

$$L^p_r(B) = \{u \in L^p(B) ,\ u \text{ radial } \} ,$$

then it was proved in [3] that the Pohožaev obstruction is L^∞_r-stable[2] on the unit ball of $\mathbb{R}^3$ for all functions $h \in L^\infty_r(B) \cap C^1(B)$. In [3], the question of the extension of the result to the non-radial case was explicitly asked. Our result provides an answer to this question. However, the situation is more delicate than expected in the non-radial case since, while the Pohožaev obstruction is $C^{0,\eta}$-stable for all $\eta > 0$, it is never L^∞-stable.

Theorem 2. *The Pohožaev obstruction is never L^∞-stable. In other words, given any $\varepsilon > 0$, any domain Ω in $\mathbb{R}^3$, star-shaped with respect to the origin and any function $h_0 \in C^1(\Omega)$ satisfying (0.2), we can find some function $h_\varepsilon \in L^\infty(\Omega)$ such that*

$$\|h_\varepsilon - h\|_\infty \leq \varepsilon$$

[1] Note that this remark concerns only X-stability in general. The question of X-stability at some given positive function h in dimensions $n \geq 4$ is not investigated in this paper.

[2] One should restrict oneself to radial solutions of the equation in the definition of stability.

and some positive functions $u_\varepsilon \in C^2(\Omega)$ *satisfying the equation*

$$\begin{cases} \Delta u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^5 \ \textit{in } \Omega \\ \\ u_\varepsilon = 0 \ \textit{on } \partial\Omega\,,\ u_\varepsilon > 0 \ \textit{in } \Omega \end{cases}$$

Thus the L^∞_r-stability result obtained by Brezis-Willem is really specific to the radial case. In fact, it is not really due to the symmetry of the solutions but to one of its by-product in dimension 3, precisely that sequences of solutions of equation (0.1) which are radial are either compact or develop only one concentration point. In fact, with the PDE techniques (to be compared to the ODE techniques used in [3]) we use below, we can revisit the question of the stability of the Pohožaev obstruction in dimension 3 in the radial case. We improve the result of [3] by proving that the Pohožaev obstruction is L^p_r-stable on the unit ball for all $p > 3$ but is never L^3_r-stable. For precise statements, we refer the reader to the end of section 2 and the beginning of section 3.

All these results give a complete picture of the stability of the Pohožaev obstruction in dimension 3 when the attention is restricted to non-negative solutions. The question remains widely open if one allows solutions to change sign, and is certainly more subtle due to the variety of changing-sign solutions of $\Delta u = u^5$ in $\mathbb{R}^3$.

The paper is organized as follows. Section 2 is devoted to the proofs of theorem 1 and of the corresponding result in the radial situation. The proof makes use of standard blow-up analysis in dimension 3 (see section 1) and of an extension of Pohožaev's identity to Green's functions (see Appendix 4.4). Section 3 is devoted to the proofs of theorem 2 and of the corresponding result in the radial situation. Here we have to construct examples of functions h arbitrarily close in X to some given function for which there is a positive solution of equation (0.1). It appears to be quite subtle because we need to be sharp. For instance, in order to prove theorem 2, our functions h must be close to the given function in $L^\infty(\Omega)$ but not in $C^{0,\eta}(\Omega)$ for any $\eta > 0$.

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1. Pointwise analysis around a concentration point

We consider in this section a sequence (h_ε) in $C^{0,\eta}(\mathbb{R}^n)$ for some $\eta > 0$ and a sequence (u_ε) of C^2-solutions of

$$\begin{cases} \Delta u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^5 & \text{in } \Omega \\ \\ u_\varepsilon = 0 & \text{on } \partial\Omega \\ \\ u_\varepsilon > 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is some smooth domain of $\mathbb{R}^3$ and

$$h_\varepsilon \to h \text{ in } L^p(\Omega) \text{ as } \varepsilon \to 0 \tag{1.2}$$

for some $p > 3$ where $h \in C^1(\mathbb{R}^3)$ satisfies $h \geq 0$ in Ω. Note that, as soon as h satisfies (0.2), it is non-negative.

We also assume that we have a sequence (x_ε) of points in Ω and a sequence (ρ_ε) of positive real numbers with $0 < 3\rho_\varepsilon \le d(x_\varepsilon, \partial\Omega)$ such that

$$\nabla u_\varepsilon(x_\varepsilon) = 0 \tag{1.3}$$

and

$$\rho_\varepsilon \left[\sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon(x)\right]^2 \to +\infty \text{ as } \varepsilon \to 0 . \tag{1.4}$$

We prove in this section that the following holds :

Proposition 1.1. *If there exists $C_0 > 0$ such that*

$$|x_\varepsilon - x|^{\frac{1}{2}} u_\varepsilon \le C_0 \text{ in } B(x_\varepsilon, 3\rho_\varepsilon) , \tag{1.5}$$

then there exists $C_1 > 0$ such that

$$u_\varepsilon(x_\varepsilon)u_\varepsilon(x) \le C_1|x_\varepsilon - x|^{-1} \text{ in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\} \text{ and}$$
$$u_\varepsilon(x_\varepsilon)|\nabla u_\varepsilon(x)| \le C_1|x_\varepsilon - x|^{-2} \text{ in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}.$$

Moreover, if $\rho_\varepsilon \to 0$, then

$$\rho_\varepsilon u_\varepsilon(x_\varepsilon)u_\varepsilon(x_\varepsilon + \rho_\varepsilon x) \to \frac{1}{|x|} + b \text{ in } C^1_{loc}(B(0,2) \setminus \{0\}) \text{ as } \varepsilon \to 0$$

where b is some harmonic function in $B(0,2)$ with $b(0) = 0$. At last, if the convergence in (1.2) holds in $C^{0,\eta}$, then we also get that $\nabla b(0) = 0$.

The rest of this section is dedicated to the proof of this proposition. We follow the lines of [7], section 2 (see also [8]). However, one must note that, compared to [8] and other works on this kind of blow-up analysis, some new difficulties arise since the linear term (h_ε) is only uniformly bounded in some $L^p(\Omega)$.

We divide the proof of the proposition into several claims. The first one gives the asymptotic behaviour of u_ε around x_ε at an appropriate small scale.

Claim 1.1. *After passing to a subsequence, we have that*

$$\mu_\varepsilon^{\frac{1}{2}} u_\varepsilon(x_\varepsilon + \mu_\varepsilon x) \to \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}} \text{ in } C^1_{loc}(\mathbb{R}^3), \text{ as } \varepsilon \to 0 \tag{1.6}$$

where $\mu_\varepsilon = u_\varepsilon(x_\varepsilon)^{-2}$.

Proof of Claim 1.1. Let $\tilde{x}_\varepsilon \in \overline{B(x_\varepsilon, \rho_\varepsilon)}$ and $\tilde{\mu}_\varepsilon > 0$ be such that

$$u_\varepsilon(\tilde{x}_\varepsilon) = \sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon = \tilde{\mu}_\varepsilon^{-\frac{1}{2}} . \tag{1.7}$$

Thanks to (1.4), we have that

$$\tilde{\mu}_\varepsilon \to 0 \text{ and } \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 . \tag{1.8}$$

Thanks to (1.5), we also have that

$$|x_\varepsilon - \tilde{x}_\varepsilon| = O(\tilde{\mu}_\varepsilon). \tag{1.9}$$

We set for $x \in \Omega_\varepsilon = \left\{x \in \mathbb{R}^3 \text{ s.t. } \tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x \in \Omega\right\}$,

$$\tilde{u}_\varepsilon(x) = \tilde{\mu}_\varepsilon^{\frac{1}{2}} u_\varepsilon(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x)$$

which verifies

$$\begin{aligned} &\Delta\tilde{u}_\varepsilon + \tilde{\mu}_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^5 \text{ in } \Omega_\varepsilon \ , \\ &\tilde{u}_\varepsilon(0) = \sup_{B\left(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon}\right)} \tilde{u}_\varepsilon = 1 \ , \end{aligned} \tag{1.10}$$

where $\tilde{h}_\varepsilon = h\left(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x\right)$. Thanks to (1.4), (1.7) and (1.9), we get that

$$B\left(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon}\right) \to \mathbb{R}^3 \text{ as } \varepsilon \to 0 \ . \tag{1.11}$$

Now, thanks to (1.10), (1.11), and by standard elliptic theory, we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}(\mathbb{R}^3)$ as $\varepsilon \to 0$ where U satisfies

$$\Delta U = U^5 \text{ in } \mathbb{R}^3 \text{ and } 0 \le U \le 1 = U(0) \ .$$

Thanks to the work of Caffarelli, Gidas and Spruck [4], we know that

$$U(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}} \ .$$

Moreover, thanks to (1.9), we know that, after passing to a new subsequence, $\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon} \to x_0$ as $\varepsilon \to 0$ for some $x_0 \in \mathbb{R}^3$. Hence, since x_ε is a critical point of u_ε, x_0 must be a critical point of U, namely $x_0 = 0$. We deduce that $\frac{\mu_\varepsilon}{\tilde{\mu}_\varepsilon} \to 1$ where μ_ε is as in the statement of the claim. The claim 1.1 follows. ■

For $0 \le r \le 3\rho_\varepsilon$, we set

$$\psi_\varepsilon(r) = \frac{r^{\frac{1}{2}}}{\omega_2 r^2} \int_{\partial B(x_\varepsilon, r)} u_\varepsilon d\sigma$$

where $d\sigma$ denotes the Lebesgue measure on the sphere $\partial B(x_\varepsilon, r)$ and $\omega_2 = 4\pi$ is the volume of the unit 2-sphere. We easily check, thanks to Claim 1.1, that

$$\psi_\varepsilon(\mu_\varepsilon r) = \left(\frac{r}{1 + \frac{r^2}{3}}\right)^{\frac{1}{2}} + o(1),\ \psi_\varepsilon'(\mu_\varepsilon r) = \frac{1}{2}\left(\frac{r}{1 + \frac{r^2}{3}}\right)^{\frac{3}{2}} \left(\frac{1}{r^2} - \frac{1}{3}\right) + o(1) \ . \tag{1.12}$$

We define r_ε by

$$r_\varepsilon = \max\left\{ r \in [2\sqrt{3}\mu_\varepsilon, \rho_\varepsilon] \text{ s.t. } \psi_\varepsilon'(s) \le 0 \text{ for } s \in [2\sqrt{3}\mu_\varepsilon, r] \right\} \ .$$

Thanks to (1.12), the set on which the maximum is taken is not empty for ε small enough, and moreover

$$\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{1.13}$$

We prove now the following :

Claim 1.2. *There exists $C > 0$, independent of ε, such that*

$$\begin{aligned} &u_\varepsilon(x) \le C\mu_\varepsilon^{\frac{1}{2}} |x_\varepsilon - x|^{-1} \text{ in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\} \text{ and} \\ &|\nabla u_\varepsilon(x)| \le C\mu_\varepsilon^{\frac{1}{2}} |x_\varepsilon - x|^{-2} \text{ in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\} \ . \end{aligned}$$

Proof of Claim 1.2. We follow the proof of Lemma 1.5 and 1.6 of [8]. However, there is an extra-difficulty due to the fact that we do not assume any pointwise convergence of h_ε to h. We first prove that for any given $0<\nu<\frac{1}{2}$, there exists $C_\nu>0$ such that

$$u_\varepsilon(x)\le C_\nu\left(\mu_\varepsilon^{\frac{1}{2}(1-2\nu)}|x-x_\varepsilon|^{-(1-\nu)}+\alpha_\varepsilon\left(\frac{r_\varepsilon}{|x-x_\varepsilon|}\right)^\nu\right) \tag{1.14}$$

for all $x\in B\left(x_\varepsilon,2r_\varepsilon\right)$ and ε small enough, where

$$\alpha_\varepsilon=\left(\sup_{\partial B(x_\varepsilon,r_\varepsilon)}u_\varepsilon\right)\ . \tag{1.15}$$

First of all, we can use (1.5) and apply the Harnack inequality, see for instance theorem 4.17 of [10], to get the existence of some $C>0$ such that

$$\frac{1}{C}\max_{\partial B(x_\varepsilon,r)}\left(u_\varepsilon+r\left|\nabla u_\varepsilon\right|\right)\le\frac{1}{\omega_2 r^2}\int_{\partial B(x_\varepsilon,r)}u_\varepsilon d\sigma\le C\min_{\partial B(x_\varepsilon,r)}u_\varepsilon \tag{1.16}$$

for all $0<r<\frac{5}{2}\rho_\varepsilon$ and all $\varepsilon>0$. The details of the proof of such an assertion may be found in [8], lemma 1.3. Hence, thanks to (1.12) and (1.13), we have that

$$|x-x_\varepsilon|^{\frac{1}{2}}u_\varepsilon(x)\le C\psi_\varepsilon(r)\le C\psi_\varepsilon(R\mu_\varepsilon)=C\left(\frac{R}{1+\frac{R^2}{3}}\right)^{\frac{1}{2}}+o(1)$$

for all $R\ge 2\sqrt{3}$, all $r\in[R\mu_\varepsilon,r_\varepsilon]$, all ε small enough and all $x\in\partial B\left(x_\varepsilon,r\right)$. Thus we get that

$$\sup_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,R\mu_\varepsilon)}|x-x_\varepsilon|^{\frac{1}{2}}u_\varepsilon(x)=e(R)+o(1) \tag{1.17}$$

where $e(R)\to 0$ as $R\to+\infty$. Let $0<\sigma\le 1$ and $\mathcal{G}_{\varepsilon,\sigma}$ be the Green function of the operator $\Delta+\frac{h_\varepsilon}{\sigma}$ in Ω with Dirichlet boundary condition. Thanks to the fact that h is non-negative (this is an assumption in this section), we can use lemma 4.2 to get the existence of some $C_\sigma>0$ such that

$$\left||x-y|\mathcal{G}_{\varepsilon,\sigma}(x,y)-\frac{1}{\omega_2}\right|\le C_\sigma|x-y|, \tag{1.18}$$

and that

$$\left||x-y|^2|\nabla\mathcal{G}_{\varepsilon,\sigma}(x,y)|-\frac{1}{\omega_2}\right|\le C_\sigma|x-y|, \tag{1.19}$$

for all $x\neq y\in\Omega$. We fix $0<\nu<\frac{1}{2}$ and we set

$$\Phi_{\varepsilon,\nu}=\mu_\varepsilon^{\frac{1}{2}(1-2\nu)}\mathcal{G}_{\varepsilon,1-\nu}(x_\varepsilon,x)^{1-\nu}+\alpha_\varepsilon\big(r_\varepsilon\mathcal{G}_{\varepsilon,\nu}(x_\varepsilon,x)\big)^\nu.$$

Thanks to (1.18), (1.14) reduces to prove that

$$\sup_{B(x_\varepsilon,2r_\varepsilon)}\frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}}=O(1)\ .$$

We let $y_\varepsilon\in\overline{B(x_\varepsilon,2r_\varepsilon)\setminus\{x_\varepsilon\}}$ be such that

$$\sup_{B(x_\varepsilon,2r_\varepsilon)}\frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}}=\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)}.$$

We are going to consider the several possible beahviour of the sequence (y_ε).

First of all, assume that

$$\frac{|x_\varepsilon-y_\varepsilon|}{\mu_\varepsilon}\to R\text{ as }\varepsilon\to 0\ .$$

Thanks to Claim 1.1, we have in this case that

$$\mu_\varepsilon^{\frac{1}{2}} u_\varepsilon(y_\varepsilon) \to (1+R^2)^{-\frac{1}{2}} \text{ as } \varepsilon \to 0.$$

On the other hand, thanks to (1.17), we can write that

$$\begin{aligned}\mu_\varepsilon^{\frac{1}{2}} \Phi_{\varepsilon,\nu}(y_\varepsilon) &= \left(\frac{\mu_\varepsilon}{\omega_2 |x_\varepsilon - y_\varepsilon|}\right)^{1-\nu} + O\left(\alpha_\varepsilon \mu_\varepsilon^{\frac{1}{2}} \left(\frac{r_\varepsilon}{|x_\varepsilon - y_\varepsilon|}\right)^{\nu}\right) + o(1) \\ &= (R\omega_2)^{\nu-1} + O\left((r_\varepsilon^{\frac{1}{2}} \alpha_\varepsilon) \mu_\varepsilon^{\frac{1}{2}(1-2\nu)} r_\varepsilon^{\frac{1}{2}(2\nu-1)}\right) + o(1) \\ &= (R\omega_2)^{\nu-1} + o(1),\end{aligned}$$

if $R > 0$, and $\mu_\varepsilon^{\frac{1}{2}} \Phi_{\varepsilon,\nu}(y_\varepsilon) \to +\infty$ as $\varepsilon \to 0$ if $R = 0$. In any case, $\left(\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu(y_\varepsilon)}}\right)$ is bounded.

Assume now that there exists $\delta > 0$ such that $y_\varepsilon \in B(y_\varepsilon, r_\varepsilon) \setminus B(y_\varepsilon, \delta r_\varepsilon)$. Thanks to Harnack's inequality (1.16), we get that $u_\varepsilon(y_\varepsilon) = O(\alpha_\varepsilon)$ which, thanks to (1.18), easily gives that $\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu(y_\varepsilon)}} = O(1)$.

Hence, we are left with the following situation :

$$\frac{|y_\varepsilon - x_\varepsilon|}{r_\varepsilon} \to 0 \text{ and } \frac{|x_\varepsilon - y_\varepsilon|}{\mu_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \,. \tag{1.20}$$

Thanks to the definition of y_ε, we can then write that

$$\frac{\Delta u_\varepsilon(y_\varepsilon)}{u_\varepsilon(y_\varepsilon)} \geq \frac{\Delta \Phi_{\varepsilon,\nu}(y_\varepsilon)}{\Phi_{\varepsilon,\eta}(y_\varepsilon)}$$

which gives, thanks to the definition of $\Phi_{\varepsilon,\nu}$ and multiplying by $|x_\varepsilon - y_\varepsilon|^2$, that

$$\begin{aligned}|x_\varepsilon - y_\varepsilon|^2 u_\varepsilon(y_\varepsilon)^4 \geq\ & \nu(1-\nu) \frac{|x_\varepsilon - y_\varepsilon|^2}{\Phi_{\varepsilon,\eta}(y_\varepsilon)} \left(\alpha_\varepsilon r_\varepsilon^{\nu} \frac{|\nabla G_{\varepsilon,\nu}(x_\varepsilon, y_\varepsilon)|^2}{G_{\varepsilon,\nu}(x_\varepsilon, y_\varepsilon)^2} G_{\varepsilon,\nu}(x_\varepsilon, y_\varepsilon)^{\nu}\right. \\ & \left. + \mu_\varepsilon^{\frac{1}{2}(1-2\eta)} \frac{|\nabla G_{\varepsilon,1-\nu}(x_\varepsilon, y_\varepsilon)|^2}{G_{\varepsilon,1-\nu}(x_\varepsilon, y_\varepsilon)^2} G_{\varepsilon,1-\nu}(x_\varepsilon, y_\varepsilon)^{1-\nu}\right) .\end{aligned}$$

Here is the main difference with [8]. Thanks to our choice of $\Phi_{\varepsilon,\nu}$, the terms involving h_ε disappear, which is necessary since we did not assume any pointwise convergence of h_ε. Thanks to (1.17), the left-hand side goes to 0 as $\varepsilon \to 0$. Then, thanks to (1.18), (1.19) and (1.20), we get that

$$o(1) \geq \nu(1-\nu) + o(1)$$

which is a contradiction, and shows that this last case can not occur. This ends the proof of (1.14).

We now claim that there exists $C > 0$, independent of ε, such that

$$u_\varepsilon(x) \leq C\left(\mu_\varepsilon^{\frac{1}{2}} |x - x_\varepsilon|^{-1} + \alpha_\varepsilon\right) \text{ in } B(x_\varepsilon, r_\varepsilon) \,. \tag{1.21}$$

Thanks to Claim 1.1 and (1.16), this holds for all sequences $y_\varepsilon \in B(x_\varepsilon, r_\varepsilon) \setminus \{x_\varepsilon\}$ such that $|y_\varepsilon - x_\varepsilon| = O(\mu_\varepsilon)$ or $\frac{|y_\varepsilon - x_\varepsilon|}{r_\varepsilon} \not\to 0$. Thus we may assume from now that

$$\frac{|y_\varepsilon - x_\varepsilon|}{\mu_\varepsilon} \to +\infty \text{ and } \frac{|y_\varepsilon - x_\varepsilon|}{r_\varepsilon} \to 0 \text{ as } \varepsilon \to 0 \,.$$

Thanks to the Green representation formula, we write with (1.18) and (1.19) that

$$\begin{aligned} u_\varepsilon(y_\varepsilon) &= \int_{B(x_\varepsilon,r_\varepsilon)} \mathcal{G}_{\varepsilon,1}(\Delta u_\varepsilon + h_\varepsilon u_\varepsilon)\,dx \\ &\quad + O\left(r_\varepsilon^{-1}\int_{\partial B(x_\varepsilon,r_\varepsilon)} |\partial_\nu u_\varepsilon|\ d\sigma\right) \\ &\quad + O\left(r_\varepsilon^{-2}\int_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\,d\sigma\right)\ . \end{aligned}$$

This gives with (1.15), (1.16) and (1.18) that

$$u_\varepsilon(y_\varepsilon) = O\left(\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1}|\Delta u_\varepsilon + h_\varepsilon u_\varepsilon|dx\right) + O\left(\alpha_\varepsilon\right)\ . \tag{1.22}$$

Using (1.14) with $\nu = \frac{1}{5}$, we can write that

$$\begin{aligned} &\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1}|\Delta u_\varepsilon + h_\varepsilon u_\varepsilon|dx \\ &= \int_{B(x_\varepsilon,\mu_\varepsilon)} \frac{u_\varepsilon^5}{|x-y_\varepsilon|}dx + \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1}u_\varepsilon^5 dx \\ &= O\left(\mu_\varepsilon^{\frac{1}{2}}|y_\varepsilon - x_\varepsilon|^{-1}\right) + \alpha_\varepsilon^5 r_\varepsilon \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1}|x-x_\varepsilon|^{-1}\,dx \\ &\quad + \mu_\varepsilon^{\frac{3}{2}} \int_{B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} |x-y_\varepsilon|^{-1}|x-x_\varepsilon|^{-4}dx \\ &= O\left(\mu_\varepsilon^{\frac{1}{2}}|y_\varepsilon - x_\varepsilon|^{-1}\right) + O(\alpha_\varepsilon^5 r_\varepsilon^2)\ . \end{aligned}$$

Thanks to (1.13) and to (1.17), this leads to

$$\int_{B(x_\varepsilon,r_\varepsilon)} |x-y_\varepsilon|^{-1}|\Delta u_\varepsilon|dx \le O(\mu_\varepsilon^{\frac{1}{2}}|y_\varepsilon - x_\varepsilon|^{-1}) + o(\alpha_\varepsilon),$$

which, thanks to (1.22), proves (1.21).

In order to end the proof of the first part of the claim, we just have to prove that

$$\alpha_\varepsilon = \sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon = O\left(\mu_\varepsilon^{\frac{1}{2}} r_\varepsilon^{-1}\right)\ . \tag{1.23}$$

For that purpose, we use the definition of r_ε to write that

$$(\beta r_\varepsilon)^{\frac{1}{2}}\psi_\varepsilon(\beta r_\varepsilon) \ge (r_\varepsilon)^{\frac{1}{2}}\psi_\varepsilon(r_\varepsilon)$$

for all $0 < \beta < 1$. Using (1.16), this leads to

$$r_\varepsilon^{\frac{1}{2}}\left(\sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right) \le C(\beta r_\varepsilon)^{\frac{1}{2}}\left(\sup_{\partial B(x_\varepsilon,\beta r_\varepsilon)} u_\varepsilon\right)\ .$$

Thanks to (1.21), we obtain that

$$\left(\sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right) \le C\beta^{\frac{1}{2}}\left(\mu_\varepsilon^{\frac{1}{2}}(\beta r_\varepsilon)^{-1} + \sup_{\partial B(x_\varepsilon,r_\varepsilon)} u_\varepsilon\right)\ .$$

Choosing β small enough clearly gives (1.23) and thus the pointwise estimate on u_ε of the claim. The estimate on ∇u_ε then follows from standard elliptic theory.■

We now prove the following :

Claim 1.3. *If $r_\varepsilon \to 0$ as $\varepsilon \to 0$, up to passing to a subsequence,*

$$r_\varepsilon u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + r_\varepsilon x) \to \frac{1}{|x|} + b \text{ in } C^1_{loc}\left(B(0,2) \setminus \{0\}\right) \text{ as } \varepsilon \to 0$$

where b is some harmonic function in $B(0,2)$. Moreover, if $r_\varepsilon < \rho_\varepsilon$, then $b(0) = 1$.

Proof of Claim 1.3. We set, for $x \in B(0,2)$,

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{-\frac{1}{2}} r_\varepsilon u_\varepsilon(x_\varepsilon + r_\varepsilon x)$$

which verifies

$$\Delta \tilde{u}_\varepsilon + r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 \text{ in } B(0,2) \tag{1.24}$$

where $\tilde{h}_\varepsilon = h(x_\varepsilon + r_\varepsilon x)$. Thanks to Claim 1.2, there exists $C > 0$ such that

$$\tilde{u}_\varepsilon(x) \leq \frac{C}{|x|} \text{ in } B(0,2) \setminus \{0\}. \tag{1.25}$$

Then, thanks to standard elliptic theory, we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}\left(B(0,2) \setminus \{0\}\right)$ as $\varepsilon \to 0$ where U is a non-negative solution of

$$\Delta U = 0 \text{ in } B(0,2) \setminus \{0\} .$$

Then, thanks to the Bôcher theorem on singularities of harmonic functions, we get that

$$U(x) = \frac{\lambda}{|x|} + b(x)$$

where b is some harmonic function in $B(0,2)$ and $\lambda \geq 0$. Now, integrating (1.24) on $B(0,1)$, we get that

$$\int_{\partial B(0,1)} \partial_\nu \tilde{u}_\varepsilon d\sigma = \int_{B(0,1)} \left(r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon - \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 \right) dx$$

Thanks to Claim 1.2,

$$\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon dx \to 0 \text{ as } \varepsilon \to 0$$

and, thanks to Claim 1.1,

$$\int_{B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \tilde{u}_\varepsilon^5 dx \to \int_{\mathbb{R}^3} \left(1 + \frac{|x|^2}{3}\right)^{-\frac{5}{2}} dx = \omega_2 \text{ as } \varepsilon \to 0 .$$

On the other hand, we have that

$$\int_{\partial B(0,1)} \partial_\nu \tilde{u}_\varepsilon d\sigma \to -\omega_2 \lambda \text{ as } \varepsilon \to 0 .$$

We deduce that $\lambda = 1$, which proves the first part of the Claim.

Now, if $r_\varepsilon < \rho_\varepsilon$, we have thanks to the definition of r_ε that

$$\psi_\varepsilon'(r_\varepsilon) = 0 .$$

Setting $\tilde{\psi}_\varepsilon(r) = \left(\frac{r_\varepsilon}{\mu_\varepsilon}\right)^{\frac{1}{2}} \psi_\varepsilon(r_\varepsilon r)$ for $0 < r < 2$, we see that

$$\tilde{\psi}_\varepsilon(r) \to \frac{r^{\frac{1}{2}}}{\omega_2 r^2} \int_{\partial B(0,r)} U d\sigma = r^{-\frac{1}{2}} + r^{\frac{1}{2}} b(0) \ .$$

We deduce that $b(0) = 1$, which ends the proof of the Claim. ■

We prove at last the following :

Claim 1.4. *Using the notations of Claim 1.3, we have that $b(0) = 0$, and if the convergence in (1.2) holds in $C^{0,\eta}$, then $\nabla b(0) = 0$.*

Proof of Claim 1.4. We use the notation of the proof of Claim 1.3. Let us apply the Pohožaev identity (4.4) of appendix 4.3 to $\tilde{u}_\varepsilon$ in $B(0,1)$. We obtain that

$$\frac{1}{2} \int_{B(0,1)} r_\varepsilon^2 \left(\tilde{h}_\varepsilon \tilde{u}_\varepsilon^2 + \tilde{h}_\varepsilon < x, \nabla \tilde{u}_\varepsilon^2 > \right) dx = \tilde{B}_1^\varepsilon + \tilde{B}_2^\varepsilon$$

where

$$\tilde{B}_1^\varepsilon = \int_{\partial B(0,1)} (\partial_\nu \tilde{u}_\varepsilon)^2 + \frac{1}{2} \tilde{u}_\varepsilon \partial_\nu \tilde{u}_\varepsilon - \frac{|\nabla \tilde{u}_\varepsilon|^2}{2} d\sigma \text{ and}$$

$$\tilde{B}_2^\varepsilon = \int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} d\sigma \ .$$

Thanks to Claim 1.2 and to Lebesgue dominated convergence theorem, we can pass to the limit to obtain that

$$\int_{\partial B(0,1)} (\partial_\nu U)^2 + \frac{1}{2} U \partial_\nu U - \frac{|\nabla U|^2}{2} d\sigma = 0 \ .$$

Since b is harmonic, it is easily checked that the left-hand side is just $-\frac{\omega_2 b(0)}{2}$. This proves that $b(0) = 0$.

In order to prove the second part of the Claim, we apply the Pohožaev identity (4.7) of appendix 4.3 to $\tilde{u}_\varepsilon$ in $B(0,1)$. We obtain that

$$\begin{aligned} &\int_{\partial B(0,1)} \left(\frac{|\nabla \tilde{u}_\varepsilon|^2}{2} \nu - \partial_\nu \tilde{u}_\varepsilon \nabla \tilde{u}_\varepsilon \right) d\sigma \\ &= - \int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \frac{\nabla \tilde{u}_\varepsilon^2}{2} \, dx - \int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} \nu \, d\sigma \ . \end{aligned} \tag{1.26}$$

It is clear that

$$\int_{\partial B(0,1)} \left(\frac{|\nabla \tilde{u}_\varepsilon|^2}{2} \nu - \partial_\nu \tilde{u}_\varepsilon \nabla \tilde{u}_\varepsilon \right) d\sigma \to \int_{\partial B(0,1)} \left(\frac{|\nabla U|^2}{2} \nu - \partial_\nu U \nabla U \right) d\sigma \text{ as } \varepsilon \to 0 \ .$$

Moreover, thanks to the fact that b is harmonic, we easily get that

$$\int_{\partial B(0,1)} \left(\frac{|\nabla U|^2}{2} \nu - \nabla U \partial_\nu U \right) d\sigma = \omega_2 \nabla b(0) \ .$$

It remains to deals with the right-hand side of (1.26). It is clear that

$$\int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon}\right)^2 \frac{\tilde{u}_\varepsilon^6}{6} \nu d\sigma \to 0 \text{ as } \varepsilon \to 0 \ .$$

Then we rewrite the first term of the right-hand side of (1.26) as

$$\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \frac{\nabla \tilde{u}_\varepsilon^2}{2}\, dx = \int_{B(0,1)} r_\varepsilon^2 (\tilde{h}_\varepsilon - \tilde{h}_\varepsilon(0)) \frac{\nabla \tilde{u}_\varepsilon^2}{2}\, dx + \tilde{h}_\varepsilon(0) \int_{B(0,1)} r_\varepsilon^2 \frac{\nabla \tilde{u}_\varepsilon^2}{2}\, dx \ .$$

If we assume that the convergence of (h_ε) holds in $C^{0,\eta}$, we can use Lebesgue dominated convergence theorem to obtain that the first term of the right-hand side goes to 0 as $\varepsilon \to 0$. Then, integrating by parts the second term, we get

$$\tilde{h}_\varepsilon(0) \int_{B(0,1)} r_\varepsilon^2 \frac{\nabla \tilde{u}_\varepsilon^2}{2} dx = \tilde{h}_\varepsilon(0) \int_{\partial B(0,1)} r_\varepsilon^2 \frac{\tilde{u}_\varepsilon^2}{2} \nu d\sigma$$

which clearly goes to 0 as $\varepsilon \to 0$. Finally, collecting the above informations, and passing to the limit $\varepsilon \to 0$ in (1.26), we get that $\nabla b(0) = 0$ if the convergence of (h_ε) holds in $C^{0,\eta}$, which achieves the proof of the Claim. ■

We are now in position to end the proof of proposition 1.1. If $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$ then we deduce the proposition from claims 1.3 and 1.4. If $\rho_\varepsilon \not\to 0$ as $\varepsilon \to 0$, then claims 1.3 and 1.4 give that $r_\varepsilon \not\to 0$ as $\varepsilon \to 0$. Then, using the Harnack inequality (1.16), one can extend the result of Claim 1.2 to $B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}$, which proves the first part of Proposition 1.1 when $\rho_\varepsilon \not\to 0$, and ends the proof of the whole proposition.

2. Stability of the Pohožaev obstruction

We prove theorem 1 and give some stability result for radial solutions on the unit ball (see the end of the section). We assume by contradiction that there exists a sequence (h_ε) of functions in $C^{0,\eta}\left(\mathbb{R}^3\right)$ for some $\eta > 0$ and a sequence (u_ε) of C^2-solutions of (1.1) where Ω is some smooth domain of $\mathbb{R}^3$ star-shaped with respect to the origin and $h_\varepsilon \to h$ in $L^p(\Omega)$ as $\varepsilon \to 0$ for some $p > 3$ where $h \in C^1(\mathbb{R}^3)$ satisfies (0.2). Sometimes we will assume that $h_\varepsilon \to h$ in $C^{0,\eta}$ as $\varepsilon \to 0$.

We claim first that

$$\|u_\varepsilon\|_\infty \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{2.1}$$

Indeed, if (u_ε) is uniformly bounded in $L^\infty(\Omega)$, then it is clear that $\left(\frac{u_\varepsilon}{\|u_\varepsilon\|_\infty}\right)$ is uniformly bounded in $W^{2,p}(\Omega)$ for some $p > 3$, and thus, after passing to a subsequence, $\frac{u_\varepsilon}{\|u_\varepsilon\|_\infty} \to u$ in $C^1_{loc}(\Omega)$ where u is a positive solution of

$$\Delta u + hu = \left(\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty^4\right) u^5 \text{ in } \Omega$$

with $u = 0$ on $\partial\Omega$. Since $h \geq 0$, it is clear that $\|u_\varepsilon\|_\infty \not\to 0$ as $\varepsilon \to 0$. Then $\tilde{u} = (\lim_{\varepsilon\to 0}\|u_\varepsilon\|_\infty)\, u$ is a non-trivial solution of (0.1), which is a contradiction since (0.2) holds. Thus (2.1) is proved.

Then the sequence (u_ε) develops some concentration phenomena. We prove that this leads to a contradiction as follows : in Claim 2.1, mimicking [8], we exhaust a family of critical points of u_ε, $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon})$, such that each sequence $(x_{i_\varepsilon,\varepsilon})$ satisfies the assumptions of Section 1 with

$$\rho_\varepsilon = \min_{1\leq i\leq N_\varepsilon, i\neq i_\varepsilon} \{|x_{i,\varepsilon} - x_{i_\varepsilon,\varepsilon}|, d(x_{i_\varepsilon,\varepsilon}, \partial\Omega)\} \ .$$

In Claim 2.2, we prove that these concentration points are in fact isolated. In other words, we prove that (u_ε) develops only finitely many concentration points. We prove that such a configuration of concentrations points must satisfy two relations involving the Green function of $\Delta + h$ at these points. And it is impossible to find such a configuration thanks to some Pohožaev identity on Green functions we prove in Appendix 4.4. Claim 2.1 is rather classical. The core of the proof lies in Claim 2.2. Avoiding bubble accumulation in the interior Ω in dimension 3 is by now classical. The main difficulty here is to avoid boundary bubble accumulation. The rest of the section is devoted to the details of the proof we just sketched.

Claim 2.1. *There exists $D > 0$ such that for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon})$ such that :*

$$d(x_{i,\varepsilon}, \partial\Omega) u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \in [1, N_\varepsilon] \,,$$
$$|x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N_\varepsilon] \,,$$

and

$$\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq D$$

for all $x \in \Omega$ and all $\varepsilon > 0$.

Proof of Claim 2.1. First of all, we claim that

$$\{x \in \Omega \text{ s.t. } \nabla u_\varepsilon(x) = 0 \text{ and } d(x, \partial\Omega) u_\varepsilon(x)^2 \geq 1\} \neq \emptyset \tag{2.2}$$

for ε small enough. Let us prove (2.2). Let $y_\varepsilon \in \Omega$ be a point where u_ε achieves his maximum. We set $\mu_\varepsilon = u_\varepsilon(y_\varepsilon)^{-2} \to 0$ as $\varepsilon \to 0$. We set also for all $x \in \Omega_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } y_\varepsilon + \mu_\varepsilon x \in \Omega\}$,

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{1}{2}} u_\varepsilon(y_\varepsilon + \mu_\varepsilon x)$$

which verifies

$$\Delta \tilde{u}_\varepsilon + \mu_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^5 \text{ in } \Omega_\varepsilon,$$

where $\tilde{h}_\varepsilon = h(y_\varepsilon + \mu_\varepsilon x)$. Note that $0 \leq \tilde{u}_\varepsilon \leq \tilde{u}_\varepsilon(0) = 1$. Thanks to standard elliptic theory, we get that $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}(\Omega_0)$ where U satisfies

$$\Delta U = U^5 \text{ in } \Omega_0 \text{ and } 0 \leq U \leq 1 = U(0),$$

and where $\Omega_0 = \lim_{\varepsilon\to 0} \Omega_\varepsilon$. Thanks to [5], we have $\Omega_0 = \mathbb{R}^3$, which proves that $d(y_\varepsilon, \partial\Omega) u_\varepsilon(y_\varepsilon)^2 \to +\infty$ as $\varepsilon \to 0$. This ends the proof of (2.2).

Now, applying Lemma 4.1, see Appendix 4.1, for ε small enough, there exist $N_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon})$, such that :

$$d(x_{i,\varepsilon}, \partial\Omega) u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \in [1, N_\varepsilon] \,,$$
$$|x_{i,\varepsilon} - x_{j,|} u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N_\varepsilon] \,,$$

and

$$\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq 1 \tag{2.3}$$

for all critical point x of u_ε such that $d(x, \partial\Omega) u_\varepsilon(x)^2 \geq 1$. It remains to show that there exists $D > 0$ such that

$$\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq D$$

for all $x \in \Omega$. We proceed by contradiction, assuming that

$$\sup_{x\in\Omega} \left(\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon^2(x) \right) \to +\infty \tag{2.4}$$

as $\varepsilon \to 0$. Let $z_\varepsilon \in \Omega$ be such that

$$\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - z_\varepsilon|\right) u_\varepsilon(z_\varepsilon)^2 = \sup_{x\in\Omega}\left(\left(\min_{i\in[1,N_\varepsilon]} |x_{i,\varepsilon} - x|\right) u_\varepsilon(x)^2\right) .$$

We set $\hat{\mu}_\varepsilon = u_\varepsilon(z_\varepsilon)^{-2}$ and $S_\varepsilon = \{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\}$. Thanks to (2.4), we check that

$$\hat{\mu}_\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

and that

$$\frac{d(S_\varepsilon, z_\varepsilon)}{\hat{\mu}_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 . \tag{2.5}$$

Then we set, for all $x \in \hat{\Omega}_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } z_\varepsilon + \hat{\mu}_\varepsilon x \in \Omega\}$,

$$\hat{u}_\varepsilon(x) = \hat{\mu}_\varepsilon^{\frac{1}{2}}\, \hat{u}_\varepsilon(z_\varepsilon + \hat{\mu}_\varepsilon x)$$

which verifies

$$\Delta \hat{u}_\varepsilon + \hat{\mu}_\varepsilon^2 \hat{h}_\varepsilon \hat{u}_\varepsilon = \hat{u}_\varepsilon^5 \text{ in } \Omega_\varepsilon$$

where $\hat{h}_\varepsilon = h(z_\varepsilon + \hat{\mu}_\varepsilon x)$. Note that $\hat{u}_\varepsilon(0) = 1$ and also that

$$\lim_{\varepsilon\to 0} \sup_{B(0,R)\cap\Omega_\varepsilon} \hat{u}_\varepsilon = 1$$

for all $R > 0$ thanks to (2.4) and (2.5). Standard elliptic theory gives then that $\hat{u}_\varepsilon \to \hat{U}$ in $C^1_{loc}(\hat{\Omega}_0)$ where U satisfies

$$\Delta \hat{U} = \hat{U}^5 \text{ in } \hat{\Omega}_0 \text{ and } 0 \le \hat{U} \le 1 = \hat{U}(0)$$

with $\hat{\Omega}_0 = \lim_{\varepsilon\to 0} \hat{\Omega}_\varepsilon$. As above, we deduce that $\hat{\Omega}_0 = \mathbb{R}^3$, which gives that

$$\lim_{\varepsilon\to 0} d(z_\varepsilon, \partial\Omega) u_\varepsilon^2(z_\varepsilon) \to +\infty . \tag{2.6}$$

Moreover, thanks to [4], we know that

$$\hat{U}(x) = \frac{1}{\left(1 + \frac{|x|^2}{3}\right)^{\frac{1}{2}}} .$$

Since $\hat{U}$ has a strict local maximum at 0, there exists $\hat{x}_\varepsilon$, a critical point of u_ε, such that $|z_\varepsilon - \hat{x}_\varepsilon| = o(\hat{\mu}_\varepsilon)$ and $\hat{\mu}_\varepsilon u_\varepsilon(\hat{x}_\varepsilon)^2 \to 1$ as $\varepsilon \to 0$. Thanks to (2.5) and (2.6), this contradicts (2.3) and proves the Claim. ■

We define

$$d_\varepsilon = \min\{d(x_{i,\varepsilon}, x_{j,\varepsilon}), d(x_{i,\varepsilon}, \partial\Omega) \text{ s.t. } 1 \le i < j \le N_\varepsilon\}$$

and prove :

Claim 2.2. *If the convergence of h_ε to h holds in $C^{0,\eta}$, then there exists $d > 0$ such that $d_\varepsilon \ge d$.*

Proof of Claim 2.2. Assume that $d_\varepsilon \to 0$ as $\varepsilon \to 0$. There are two cases to consider : either the distance between two critical points goes to 0, or one of them goes to the boundary. In the first case, the arguments which lead to a contradiction follow closely [7], but in the second case we have to be more precise looking at the "artificial" singularities created by the boundary.

Up to reordering the concentration points, we can assume that

$$d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon}) \text{ or } d(x_{1,\varepsilon}, \partial\Omega) .$$

For $x \in \Omega_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } x_{1,\varepsilon} + d_\varepsilon x \in \Omega\}$, we set

$$\tilde{u}_\varepsilon(x) = d_\varepsilon^{\frac{1}{2}} u_\varepsilon(x_{1,\varepsilon} + d_\varepsilon x)$$

which verifies

$$\Delta \tilde{u}_\varepsilon + d_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = \tilde{u}_\varepsilon^5 \text{ in } \Omega_\varepsilon,$$

where $\tilde{h}_\varepsilon = h(x_{1,\varepsilon} + d_\varepsilon x)$. We have, up to a harmless rotation,

$$\lim_{\varepsilon \to 0} \Omega_\varepsilon = \Omega_0 = \mathbb{R}^3 \text{ or }]-\infty; d[\times \mathbb{R}^2 \text{ where } d \geq 1 .$$

We also set

$$\tilde{x}_{i,\varepsilon} = \frac{x_{i,\varepsilon} - x_{1,\varepsilon}}{d_\varepsilon} .$$

We claim that, for any sequence $i_\varepsilon \in [1, N_\varepsilon]$ such that

$$\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) = O(1) , \tag{2.7}$$

we have that

$$\sup_{B(\tilde{x}_{i_\varepsilon,\varepsilon},\frac{1}{2})} \tilde{u}_\varepsilon = O(1) . \tag{2.8}$$

Indeed, let $y_\varepsilon \in \overline{B(\tilde{x}_{i_\varepsilon,\varepsilon}, \frac{1}{2})}$ be such that $\sup_{B(\tilde{x}_{i_\varepsilon,\varepsilon},\frac{1}{2})} \tilde{u}_\varepsilon = \tilde{u}_\varepsilon(y_\varepsilon)$ and assume by contradiction that

$$\tilde{u}_\varepsilon(y_\varepsilon)^2 \to +\infty \text{ as } \varepsilon \to 0 . \tag{2.9}$$

Thanks to the definitions of d_ε, y_ε and the last assertion of Claim 2.1, we can write that

$$|d_\varepsilon(y_\varepsilon - \tilde{x}_{i_\varepsilon,\varepsilon})| u_\varepsilon(x_{1,\varepsilon} + d_\varepsilon y_\varepsilon)^2 \leq D$$

so that

$$|y_\varepsilon - \tilde{x}_{i_\varepsilon,\varepsilon}| = o(1) . \tag{2.10}$$

For $x \in B(0, \frac{1}{3\hat{\mu}_\varepsilon})$ and ε small enough, we set

$$\hat{u}_\varepsilon(x) = \hat{\mu}_\varepsilon^{\frac{1}{2}} \tilde{u}_\varepsilon(y_\varepsilon + \hat{\mu}_\varepsilon x)$$

where $\hat{\mu}_\varepsilon = u_\varepsilon(y_\varepsilon)^{-2}$. It satisfies

$$\Delta \hat{u}_\varepsilon + (\hat{\mu}_\varepsilon d_\varepsilon)^2 \hat{h}_\varepsilon \hat{u}_\varepsilon = \hat{u}_\varepsilon^5 \text{ in } B(0, \frac{1}{3\hat{\mu}_\varepsilon}) \text{ and } \hat{u}_\varepsilon(0) = \sup_{B(0,\frac{1}{3\hat{\mu}_\varepsilon})} \hat{u}_\varepsilon = 1$$

where $\hat{h}_\varepsilon = \tilde{h}_\varepsilon(y_\varepsilon + \hat{\mu}_\varepsilon x)$. Thanks to (2.9), $B(0, \frac{1}{3\hat{\mu}_\varepsilon}) \to \mathbb{R}^3$ as $\varepsilon \to +\infty$. Then $(\hat{u}_\varepsilon)$ is uniformly locally bounded and, by standard elliptic theory, $\hat{u}_\varepsilon$ converges to $\hat{U}$ in $C^1_{loc}(\mathbb{R}^3)$ where $\hat{U}$ satisfies

$$\Delta \hat{U} = \hat{U}^5 \text{ in } \mathbb{R}^3 \text{ and } 0 \leq \hat{U} \leq 1 = \hat{U}(0) .$$

Thanks to [4] and to the fact that $\frac{\tilde{x}_{i_\varepsilon,\varepsilon} - y_\varepsilon}{\hat{\mu}_\varepsilon}$ is bounded, we can write that

$$\liminf_{\varepsilon \to 0} \frac{\tilde{u}_\varepsilon(x_{i_\varepsilon,\varepsilon})}{\tilde{u}_\varepsilon(y_\varepsilon)} > 0$$

which is a contradiction with (2.7) and (2.9), and achieves the proof of (2.8).

For $R > 0$, we set $S_{R,\varepsilon} = \{\tilde{x}_{i,\varepsilon} | \tilde{x}_{i,\varepsilon} \in B(0,R)\}$. Thanks to definition of d_ε, up to a subsequence, $S_{R,\varepsilon} \to S_R$ as $\varepsilon \to 0$, where S_R is a not empty finite set, then up to performing a diagonal extraction, we can define the countable set

$$S = \bigcup_{R>0} S_R \ .$$

Thanks to the previous definition, we are ready to prove the following assertion :

$$\forall\, i_\varepsilon \in [1, N_\varepsilon] \text{ s.t. } d(x_{i_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)\,,\ \tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) \to +\infty \text{ as } \varepsilon \to 0\ . \tag{2.11}$$

Assume that there exists i_ε such that $d(x_{i_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)$ with $\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon})$ bounded, then for all sequences j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)$, $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon})$ is bounded. Indeed, if there exists a sequence j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)$ and $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$, thanks to Claim 2.1, we can apply Proposition 1.1 with $x_\varepsilon = \tilde{x}_{j_\varepsilon,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$. We obtain that up to a subsequence $\tilde{u}_\varepsilon \to 0$ in $C^1_{loc}(B(\tilde{x}, \frac{2}{3})) \setminus \{\tilde{x}\}$, where $\tilde{x} = \lim_{\varepsilon\to 0} \tilde{x}_{j_\varepsilon,\varepsilon}$. But $(\tilde{u}_\varepsilon)$ is uniformly bounded in $B(\tilde{y}, \frac{1}{2})$, where $\tilde{y} = \lim_{\varepsilon\to 0} \tilde{x}_{i_\varepsilon,\varepsilon}$. We thus obtain thanks to Harnack's inequality that $\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) \to 0$ as $\varepsilon \to 0$, which is a contradiction with the first or the second assertion of Claim 2.1.

Thus we have proved that for all sequence j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)$, $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon})$ is bounded, which proves that $(\tilde{u}_\varepsilon)$ is uniformly bounded in a neighborhood of any finite subset of S. But thanks to Claim 2.1, $\tilde{u}_\varepsilon$ is bounded in any compact subset of $\Omega_0 \setminus S$. This clearly proves that $\tilde{u}_\varepsilon$ is uniformly bounded on any compact of Ω_0. Then, by standard elliptic theory, $\tilde{u}_\varepsilon \to U$ in $C^1_{loc}(\Omega_0)$ as $\varepsilon \to 0$, where U is a nonnegative solution of

$$\Delta U = U^5 \text{ in } \Omega_0\ .$$

But, thanks to the first or second assertion of Claim 2.1, we know that $U(0) \geq 1$, hence we have necessarily that $\Omega_0 = \mathbb{R}^3$, and thus U possesses at least two critical points, namely 0 and $\tilde{x}_2 = \lim_{\varepsilon\to 0} \tilde{x}_{2,\varepsilon}$. Thanks to [4], this is impossible. This ends the proof of (2.11).

We are now going to consider two cases, depending on Ω_0.

Case 1 : $\Omega_0 = \mathbb{R}^3$ - In this case, up to a subsequence, $d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon})$ and $S = \{0, \tilde{x}_2 = \lim_{\varepsilon\to 0} \tilde{x}_{2,\varepsilon}, \dots\}$ contains at least two points. Applying Proposition 1.1 with $x_\varepsilon = \tilde{x}_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$, we obtain that

$$\tilde{u}_\varepsilon(0)\tilde{u}_\varepsilon(x) \to H = \frac{1}{|x|} + \frac{\lambda_2}{|x - \tilde{x}_2|} + \tilde{b} \text{ in } C^1_{loc}(\mathbb{R}^3 \setminus S) \text{ as } \varepsilon \to 0$$

where $\tilde{b}$ is an harmonic function in $\Omega_0 \setminus \{S \setminus \{0, \tilde{x}_2\}\}$, and $\lambda_2 > 0$. Moreover $\tilde{b}(0) = -\lambda_2$. We prove in the following that $\tilde{b}$ is nonnegative, which will give a contradiction and end the study of this case. To check that $\tilde{b}$ is nonnegative, for all positive number r, we rewrite H as

$$H = \sum_{\tilde{x}_i \in S \cap B(0,r)} \frac{\lambda_i}{|x - \tilde{x}_i|} + \hat{b}_r,$$

where $\lambda_i > 0$. Then, taking $R > r$ big enough, we get that $\hat{b}_r > \frac{-1}{r}$ on $\partial B(0,R)$. Moreover, for any $\tilde{x}_j \in B(0,R) \setminus B(0,r)$, there exist a neighborhood $V_{j,r}$ of $\tilde{x}_j$ such that $\hat{b}_r > 0$ on $V_{j,r}$. Thanks to the maximum principle, $\hat{b}_r > \frac{-1}{r}$ on $B(0,R)$. Since

$\hat{b}_r \to \hat{b}$ on every compact set as $r \to +\infty$, we get that $H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|} + \hat{b}$ with $\hat{b} \geq 0$, which proves that $\tilde{b} \geq 0$. This is the contradiction we were looking for, and this ends the proof of the claim in this first case.

Case 2 : $\Omega_0 =]-\infty, d[\times \mathbb{R}^2$ - We still denote $S = \{0 = \tilde{x}_1, \tilde{x}_2, \dots\}$ and we apply Proposition 1.1 with $x_\varepsilon = x_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$ to get that

$$\tilde{u}_\varepsilon(0)\tilde{u}_\varepsilon(x) \to H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|} + \tilde{b} \text{ in } C^1_{loc}(\Omega_0 \setminus S)$$

where $\lambda_i > 0$, and $\tilde{b}$ is some harmonic function in Ω_0. We extend H to $\mathbb{R}^3$ by setting

$$\hat{H}(x) = \begin{cases} H(x) & \text{if } x_1 \leq d \\ -H(s(x)) & \text{otherwise} \end{cases}$$

where s is the symmetry with respect to $\{d\} \times \mathbb{R}^2$. We also extend $\tilde{b}$ by setting

$$\hat{H} = \sum_{\tilde{x}_i \in S} \left(\frac{\lambda_i}{|x - \tilde{x}_i|} - \frac{\lambda_i}{|s(x) - \tilde{x}_i|} \right) + \hat{b} \; .$$

It is clear that $\hat{b}$ is harmonic on $\mathbb{R}^3$ and satisfies $\hat{b} \geq 0$ in Ω_0 and $\hat{b} \leq 0$ in $\mathbb{R}^3 \setminus \Omega_0$. This can be proved as in Case 1. Let $\mathcal{G}_R$ the Green function of the laplacian on the ball centered in 0 with radius R, we get thanks to the Green representation formula that

$$\hat{b}(x) = \int_{\partial B(0,R)} \partial_\nu \mathcal{G}_R(x,y)\hat{b}(y)d\sigma$$

which gives since

$$\partial_\nu \mathcal{G}_R(x,y) = \frac{R^2 - |x|^2}{\omega_2 R \, |x-y|^3}$$

on $\partial B(0,R)$ that

$$\partial_1 \hat{b}(0) = \frac{3}{\omega_2 R^4} \int_{\partial B(0,R)} y_1 \hat{b}(y) d\sigma \; .$$

Now we decompose $\partial B(0,R)$ into three sets, namely

$$\begin{aligned} A &= \{y \in \partial B(0,R) \text{ s.t. } y_1 \geq d\} \; , \\ B &= \{y \in \partial B(0,R) \text{ s.t. } 0 \leq y_1 \leq d\} \; , \\ C &= \{y \in \partial B(0,R) \text{ s.t. } y_1 \leq 0\} \; . \end{aligned}$$

In A and B, we have that $y_1 \hat{b}(y) \leq d\hat{b}(y)$, and in C, we have that $y_1 \hat{b}(y) \leq 0$. Since $\hat{b} \geq 0$ in C, we arrive to

$$\partial_1 \hat{b}(0) \leq \frac{3d}{\omega_2 R^4} \int_{A \cup B} \hat{b}(y) d\sigma \leq \frac{3d}{\omega_2 R^4} \int_{\partial B(0,R)} \hat{b}(y) d\sigma = \frac{3d\hat{b}(0)}{R^2} \; .$$

Passing to the limit $R \to +\infty$ gives that $\partial_1 \hat{b}(0) \leq 0$. In order to obtain a contradiction, we rewrite H in a neighborhood of 0 as

$$H(x) = \frac{1}{|x|} + \check{b}(x)$$

where

$$\check{b}(x) = \hat{b}(x) - \frac{1}{|s(x)|} + \sum_{\check{x}_i \in S \setminus \{0\}} \lambda_i \left(\frac{1}{|x - \check{x}_i|} - \frac{1}{|s(x) - \check{x}_i|} \right) .$$

As is easily checked, $\partial_1 \check{b}(0) < 0$, which is a contradiction with Proposition 1.1. This ends the proof of Claim 2.2 in this second case.

We are now ready to prove theorem 1. Thanks to Claim 2.1, there exist $D > 0, N \in \mathbb{N}^*$ and N local maxima of u_ε, $x_{1,\varepsilon}, \dots, x_N$, such that:

$$d(x_{i,\varepsilon}, \partial\Omega) u_\varepsilon (x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \in [1, N] ,$$
$$|x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon (x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N]$$

and

$$\left(\min_{i \in [1,N]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq D$$

for all $x \in \Omega$. We can assume that $u_\varepsilon(x_{i,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$. Indeed, otherwise we can remove $x_{i,\varepsilon}$ from the family of concentration points, and up to changing D, the assertion remains true. Then, thanks to Harnack inequality, there exists $C > 0$ such that

$$\frac{1}{C} u_\varepsilon(x_{1,\varepsilon}) \leq u_\varepsilon(x_{i,\varepsilon}) \leq C u_\varepsilon(x_{1,\varepsilon}) . \tag{2.12}$$

Now, thanks to the results of section 1 and by standard elliptic theory, we have that, after passing to a subsequence,

$$u_\varepsilon(x_{1,\varepsilon}) u_\varepsilon(x) \to G \text{ in } C^2_{loc}(\Omega \setminus \{x_1, \dots, x_N\}) \text{as } \varepsilon \to 0$$

where

$$G(x) = \sum_{i=1}^{N} \lambda_i \mathcal{G}_h(x_i, x)$$

with $\mathcal{G}_h$ the Green function of the limit operator $\Delta + h$ with Dirichlet boundary condition on Ω. Thanks to (2.12), we know that $\lambda_i > 0$ for $1 \leq i \leq N$. This can be rewritten as

$$G(x) = \frac{\lambda_i}{\omega_2 |x - x_i|} + G_i(x) \tag{2.13}$$

where G_i is a continuous function on $\Omega \setminus \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N\}$. Thanks to lemma 4.3, we can write

$$G_i(x) = G_i(x_i) + \frac{h(x_i)}{2\omega_2} |x - x_i| + \gamma_i(x) \tag{2.14}$$

where $\gamma_i \in C^1(\Omega)$ and $\gamma_i(0) = 0$. We claim that

$$G_i(x_i) = 0 \text{ for all } 1 \leq i \leq N . \tag{2.15}$$

In order to prove this, we apply the Pohožaev identity (4.4) to u_ε on the ball $B(x_{i,\varepsilon}, \delta)$ for some $\delta > 0$ small enough. This gives

$$\begin{aligned} &\frac{1}{2} \int_{B(x_{i,\varepsilon},\delta)} \left(h_\varepsilon u_\varepsilon^2 + h_\varepsilon < x - x_{i,\varepsilon}, \nabla u_\varepsilon^2 > \right) dx \\ &= \int_{\partial B(x_{i,\varepsilon},\delta)} \left(\delta \left(\partial_\nu u_\varepsilon\right)^2 - \delta \frac{|\nabla u_\varepsilon|^2}{2} + \frac{1}{2} u_\varepsilon \partial_\nu u_\varepsilon + \frac{\delta}{6} u_\varepsilon^6 \right) d\sigma . \end{aligned} \tag{2.16}$$

Thanks to the fact that h_ε is bounded in $L^p(\mathbb{R}^3)$ for some $p>3$ and Proposition 1.1., we get the uniform estimate

$$u_\varepsilon\left(x_{i,\varepsilon}\right)^2\left|\frac{1}{2}\int_{B(x_{i,\varepsilon},\delta)}\left(h_\varepsilon u_\varepsilon^2+h_\varepsilon < x-x_{i,\varepsilon},\nabla u_\varepsilon^2 >\right)dx\right|\le e(\delta)$$

where $e\in C^0(\mathbb{R})$ with $e(0)=0$. Using (2.13), we get that

$$\begin{aligned}&\int_{\partial B(x_{i,\varepsilon},\delta)}\left(\delta\left(\partial_\nu u_\varepsilon\right)^2-\delta\frac{|\nabla u_\varepsilon|^2}{2}+\frac{1}{2}u_\varepsilon\partial_\nu u_\varepsilon\right)d\sigma+\int_{\partial B(x_{i,\varepsilon},\delta)}\frac{\delta}{6}u_\varepsilon^6\,d\sigma\\&\quad=u_\varepsilon\left(x_{i,\varepsilon}\right)^{-2}\int_{\partial B(x_i,\delta)}\left(\delta\left(\partial_\nu G\right)^2-\delta\frac{|\nabla G|^2}{2}+\frac{1}{2}G\partial_\nu G\right)d\sigma+o\left(u_\varepsilon\left(x_{i,\varepsilon}\right)^{-2}\right)\ .\end{aligned}$$

Using (2.14), we easily get that

$$\int_{\partial B(x_i,\delta)}\left(\delta\left(\partial_\nu G\right)^2-\delta\frac{|\nabla G|^2}{2}+\frac{1}{2}G\partial_\nu G\right)d\sigma=-\frac{1}{2}\lambda_i G_i\left(x_i\right)+o(1)\text{ as }\delta\to 0\ .$$

Collecting the above informations proves (2.15).

We are going to prove now that $\nabla\gamma_i\left(x_i\right)=0$ where γ_i is as in (2.14). This will contradict lemma 4.4 of Appendix 4.4 and will achieve the proof of the theorem. For that purpose, we apply the Pohožaev identity (4.7) to u_ε on the ball $B\left(x_{i,\varepsilon},\delta\right)$ for some $\delta>0$ small enough. We obtain that

$$\begin{aligned}&u_\varepsilon\left(x_{i,\varepsilon}\right)^2\int_{\partial B(x_{i,\varepsilon},\delta)}\left(\frac{|\nabla u_\varepsilon|}{2}\nu-\nabla u_\varepsilon\partial_\nu u_\varepsilon\right)d\sigma\\&\quad=u_\varepsilon\left(x_{i,\varepsilon}\right)^2\int_{B(x_{i,\varepsilon},\delta)}h_\varepsilon\frac{\nabla u_\varepsilon^2}{2}dx-u_\varepsilon\left(x_{i,\varepsilon}\right)^2\int_{\partial B(x_{i,\varepsilon},\delta)}\nabla u_\varepsilon^6\,d\sigma\ .\end{aligned}\tag{2.17}$$

It is clear that we can pass to the limit in the left-hand side. Moreover, thanks to (2.15) and (2.14), we have that

$$\int_{\partial B(x_i,\delta)}\left(\frac{|\nabla G|}{2}\nu-\nabla G\partial_\nu G\right)d\sigma\to\nabla\gamma_i(x_i)\text{ as }\delta\to 0.$$

Now we look at the right-hand side of (2.17). It is clear that

$$u_\varepsilon\left(x_{i,\varepsilon}\right)^2\int_{\partial B(x_{i,\varepsilon},\delta)}\nabla u_\varepsilon^6\,d\sigma\to 0\text{ as }\varepsilon\to 0\ .$$

Then we write that

$$\int_{B(x_{i,\varepsilon},\delta)}h_\varepsilon\frac{\nabla u_\varepsilon^2}{2}\,dx=\int_{B(x_{i,\varepsilon},\delta)}\left(h_\varepsilon-h_\varepsilon(x_{i,\varepsilon})\right)\frac{\nabla u_\varepsilon^2}{2}\,dx+h_\varepsilon(x_{i,\varepsilon})\int_{B(x_{i,\varepsilon},\delta)}\frac{\nabla u_\varepsilon^2}{2}\,dx\ .$$

Assuming that the convergence of h_ε to h holds in $C^{0,\eta}$, it is clear that the first term of the right-hand side goes to 0 as $\varepsilon\to 0$. Integrating by parts the second term, we get that

$$h_\varepsilon(x_{i,\varepsilon})\int_{B(x_{i,\varepsilon},\delta)}\frac{\nabla u_\varepsilon^2}{2}dx=h_\varepsilon(x_{i,\varepsilon})\int_{\partial B(x_{i,\varepsilon},\delta)}\frac{u_\varepsilon^2}{2}\nu\,d\sigma\to h\left(x_i\right)\int_{\partial B(x_i,\delta)}\frac{G^2}{2}\nu d\sigma$$

as $\varepsilon\to 0$. It is easily checked that the above goes to 0 as $\delta\to 0$.

Finally, collecting the above informations, and passing consecutively to the limit $\varepsilon\to 0$ and $\delta\to 0$ in (2.17), we get that $\nabla\gamma_i(x_i)=0$ for all i, which achieves the proof of theorem 1 thanks to lemma 4.4. ■

Let us now give a precise statement of what we meant by stability of the Pohožaev obstruction in the radial situation in the introduction. We will prove the following :

Theorem 3. *Let B be the unit ball of $\mathbb{R}^3$. Let h_0 be a C^1-radial function which satisfies (0.2). Then for any $p > 3$, there exists $\delta > 0$ (depending on h_0 and p) such that if $h \in C^{0,\eta}(B)$ for some $\eta > 0$ with $\|h - h_0\|_{L^p(B)} \leq \delta$, then there exists no positive radial solution of equation (0.1) in the unit ball.*

Proof of theorem 3. We proceed as for the proof of theorem 1. Note that, since u_ε is radial, there can be only one concentration point, namely 0. Thanks to claim 2.1, the result of section 1 and standard elliptic theory, we have that

$$u_\varepsilon(0)u_\varepsilon(x) \to \omega_2 \mathcal{G}_h(x,0) \text{ in } C^1_{loc}(\Omega \setminus \{0\}) \text{as } \varepsilon \to 0$$

where $\mathcal{G}_h$ is the Green function of the limit operator $\Delta + h$. We have that

$$\mathcal{G}_h(x,0) = \frac{1}{\omega_2 |x|} + g(x)$$

where g is a continuous function on Ω which satisfies

$$\Delta g + hg = -\frac{h}{\omega_2 |x|} \text{ in } \Omega \text{ and } g = -\omega_2 \text{ on } \partial\Omega \ .$$

By the maximum principle, we see that g is negative so that $g(0) < 0$. Now we can proceed as in the proof of (2.15) to get a contradiction. Note that the proof of (2.15) did not require the $C^{0,\eta}$ convergence of h_ε. In the above proof, this $C^{0,\eta}$ convergence had been used only in the proof of claim 2.2 (which is given for free in the radial situation) and in the last part of the proof to deal with the case of several concentration points (this can not happen in the radial situation). ■

We shall prove in the next section that the above theorem is sharp in the radial situation.

3. Construction of blowing-up examples and instability of the Pohožaev obstruction

In this section, we prove theorem 2. In fact, we will first prove the corresponding result in the radial situation (thus showing that our theorem 3 is sharp) since it contains the main ideas and the computations are a little bit less involved.

We first need some results on Green's functions of coercive operators $\Delta + h$ with Dirichlet boundary condition on domains of $\mathbb{R}^3$. We let Ω be a smooth domain of $\mathbb{R}^3$ and $h \in C^1(\Omega)$ be such that the operator $\Delta + h$, with Dirichlet boundary conditions, is coercive. Then there exists a unique function $\mathcal{G} : \Omega \times \Omega \setminus \{(x,x), x \in \Omega\} \mapsto \mathbb{R}$, symmetric, positive, such that

$$\Delta_y \mathcal{G}(x,y) + h(y)\mathcal{G}(x,y) = \omega_2 \delta_x$$

in Ω and $\mathcal{G}(x,y) = 0$ for $y \in \partial\Omega$ for all $x \in \Omega$. It is easily checked that $\mathcal{G}(x,y)$ has the following expansion in the neighbourhood of the diagonal

$$\mathcal{G}(x,y) = \frac{1}{|x-y|} + \frac{1}{2}h(x)|x-y| + \gamma_x(y) \tag{3.1}$$

where $\gamma_x \in C^1(\Omega)$ satisfies

$$\Delta_y \gamma_x(y) + h(y)\gamma_x(y) = \frac{h(x) - h(y)}{|x-y|} - \frac{1}{2}h(x)h(y)|x-y|$$

in Ω with

$$\gamma_x(y) = -\frac{1}{|x-y|} - \frac{1}{2}h(x)\,|x-y|$$

for all $y \in \partial\Omega$.

3.1. **The radial case.** We start by proving that the Pohožaev identity is not L^3_r-stable in the unit ball of $\mathbb{R}^3$. More precisely, we prove the following result :

Theorem 4. *Let $h \in C^1(B)$ be a non-negative radial function on the unit ball B of $\mathbb{R}^3$. For any $\varepsilon > 0$, there exists a radial function $\tilde{h} \in C^{0,\eta}(B)$ with $\left\|\tilde{h} - h\right\|_{L^3(B)} \leq \varepsilon$ such that the equation*

$$\Delta \tilde{u} + \tilde{h}\tilde{u} = \tilde{u}^5 \text{ in } B \;\; \tilde{u} = 0 \text{ on } \partial B$$

admits a positive radial solution.

Note that a function h which satisfies (0.2) is necessarily non-negative. Let us prove the theorem in the rest of this subsection. We let $h \in C^1(B)$ be a non-negative radial function where B is the unit ball of $\mathbb{R}^3$. We let $\mathcal{G}$ be the Green function of $\Delta + h$ and let $G(x) = \mathcal{G}(0, x)$. We set

$$u_\varepsilon(x) = U_\varepsilon(x) + \eta_\varepsilon(x) V_\varepsilon(x) \tag{3.2}$$

where

$$\begin{aligned} U_\varepsilon(x) &= \varepsilon^{\frac{1}{2}}\left(\varepsilon^2 + G(x)^{-2}\right)^{-\frac{1}{2}} \\ V_\varepsilon(x) &= -\gamma_0(0)\varepsilon^{\frac{1}{2}} G(x)^{-3}\left(\varepsilon^2 + G(x)^{-2}\right)^{-\frac{3}{2}} \\ \eta_\varepsilon(x) &= \eta(x)\frac{\ln\left(\varepsilon^2 + |x|^2\right)}{\ln \varepsilon^2} \end{aligned} \tag{3.3}$$

where η is a smooth positive function such that $\eta = 1$ on the ball of radius $\frac{1}{4}$ and $\eta = 0$ outside of the ball of radius $\frac{1}{2}$. Here, γ_0 comes from the asymptotic expansion (3.1). It is easily checked that u_ε is a $C^{2,\eta}$-positive function in B and that $u_\varepsilon = 0$ on ∂B. Moreover, we have that

$$\frac{\eta_\varepsilon V_\varepsilon}{U_\varepsilon} \to 0 \text{ in } L^\infty(B) \text{ as } \varepsilon \to 0\,. \tag{3.4}$$

We claim that

$$\frac{3u_\varepsilon^5 - \Delta u_\varepsilon}{u_\varepsilon} \to h \text{ in } L^3(B) \text{ as } \varepsilon \to 0\,, \tag{3.5}$$

which clearly implies the theorem. Straightforward computations give that

$$\Delta U_\varepsilon + hU_\varepsilon = 3U_\varepsilon^5\left|\nabla G^{-1}\right|^2 + h(x)\varepsilon U_\varepsilon^3 \tag{3.6}$$

and that

$$\begin{aligned} \Delta V_\varepsilon + hV_\varepsilon =& 15U_\varepsilon^4 V_\varepsilon + 12\gamma_0(0)\varepsilon^{\frac{5}{2}} G^4\left(1+\varepsilon^2G^2\right)^{-\frac{5}{2}} \\ &- \varepsilon^{\frac{1}{2}}\gamma_0(0)h\left(1+\varepsilon^2G^2\right)^{-\frac{5}{2}}\left(1+4\varepsilon^2G^2\right) \\ &- 3\gamma_0(0)\varepsilon^{\frac{5}{2}}G^4\left(1+\varepsilon^2G^2\right)^{-\frac{7}{2}}\left(1-4\varepsilon^2G^2\right)\left(\left|\nabla G^{-1}\right|^2 - 1\right)\,. \end{aligned} \tag{3.7}$$

It is easily checked that this implies that

$$\Delta u_\varepsilon + hu_\varepsilon - 3u_\varepsilon^5 = o\left(u_\varepsilon\right) \tag{3.8}$$

in $B_0(1)\setminus B_0\left(\frac{1}{2}\right)$. Using the expansion of G and its consequence

$$\left|\nabla G^{-1}\right|^2 = 1 - 4\gamma_0(0)G^{-1} + O\left(G^{-2}\right) ,$$

we can then write thanks to (3.4) that

$$\begin{aligned}\frac{\Delta u_\varepsilon + hu_\varepsilon - 3u_\varepsilon^5}{u_\varepsilon} =& O\left(|x|^2 U_\varepsilon^4\right) + O\left(\varepsilon U_\varepsilon^2\right) + O\left(|x| U_\varepsilon^4 \left|1-\eta_\varepsilon\right|\right) \\ &+ O\left(U_\varepsilon^{-1}\left|\nabla V_\varepsilon\right|\left|\nabla \eta_\varepsilon\right|\right) + O\left(U_\varepsilon^{-1}\left|V_\varepsilon\right|\left|\Delta \eta_\varepsilon\right|\right)\end{aligned} \tag{3.9}$$

in $B_0\left(\frac{1}{2}\right)$. It is easily checked that

$$|x|^2 U_\varepsilon^4 \to 0 \text{ and } \varepsilon U_\varepsilon^2 \to 0 \text{ in } L^p(B) \text{ as } \varepsilon \to 0 \tag{3.10}$$

for all $1 \le p < +\infty$. Let us write now that

$$\begin{aligned}\int_B |x|^3 U_\varepsilon^{12} \left|1-\eta_\varepsilon\right|^3 \, dx &= O\left(\varepsilon^6 \int_0^1 r^5 \left(\varepsilon^2+r^2\right)^{-6} \left|1 - \frac{\ln\left(\varepsilon^2+r^2\right)}{\ln \varepsilon^2}\right|^3 dr\right) \\ &= O\left(\int_0^{\varepsilon^{-1}} r^5 \left(1+r^2\right)^{-6} \left|\frac{\ln\left(1+r^2\right)}{\ln \varepsilon^2}\right|^3 dr\right) \\ &= O\left(\left|\ln \varepsilon^2\right|^{-3}\right) = o(1)\end{aligned}$$

thanks to the dominated convergence theorem. We can also write that

$$\begin{aligned}\int_B U_\varepsilon^{-3} \left|\nabla V_\varepsilon\right|^3 \left|\nabla \eta_\varepsilon\right|^3 \, dx &= O\left(\left|\ln \varepsilon^2\right|^{-3} \int_0^1 r^{11}\left(\varepsilon^2+r^2\right)^{-6} dr\right) \\ &\quad + O\left(\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\frac{\ln\left(\varepsilon^2+r^2\right)}{\ln \varepsilon^2}\right|^3 dr\right) \\ &= O\left(\left|\ln \varepsilon^2\right|^{-3}\right) = o(1)\end{aligned}$$

and that

$$\begin{aligned}\int_B U_\varepsilon^{-3} \left|V_\varepsilon\right|^3 \left|\Delta \eta_\varepsilon\right|^3 &= O\left(\left|\ln \varepsilon^2\right|^{-3} \int_0^1 r^5 \left(\varepsilon^2+r^2\right)^{-1} dr\right) + O\left(\left|\ln \varepsilon^2\right|^{-3}\right) \\ &= O\left(\left|\ln \varepsilon^2\right|^{-3}\right) = o(1) .\end{aligned}$$

Coming back to (3.9) with these last estimates, we get (3.5). This ends the proof of theorem 4. ■

3.2. **The general case.** Here we prove that the Pohožaev identity is never L^∞-stable. In fact we will even prove a stronger result :

Theorem 5. *Let Ω be a smooth domain of $\mathbb{R}^3$ and let $h \in C^1(\Omega)$ be such that the operator $\Delta + h$ is coercive. For any $\varepsilon > 0$, there exists $\tilde{h} \in C^{0,\eta}(\Omega)$ with $\left\|\tilde{h} - h\right\|_\infty \le \varepsilon$ such that the equation*

$$\begin{cases} \Delta \tilde{u} + \tilde{h}\tilde{u} = \tilde{u}^5 \text{ in } \Omega \\ \tilde{u} = 0 \text{ on } \partial\Omega ,\ \tilde{u} > 0 \text{ in } \Omega \end{cases}$$

admits a solution.

It is clear that this result implies theorem 2. It is sufficient to remember that a function h which satisfies (0.2) is necessarily non-negative and that a non-negative h leads to a coercive operator $\Delta + h$. The rest of this subsection is devoted to the proof of this theorem.

We will construct a sequence of functions $u_\varepsilon \in C^\infty(\Omega)$, positive in Ω, null on the boundary of Ω, such that

$$\frac{\Delta u_\varepsilon - 3u_\varepsilon^5}{u_\varepsilon} \to h \text{ in } L^\infty(\Omega) \text{ as } \varepsilon \to 0 . \tag{3.11}$$

This will clearly prove the theorem.

We let $\mathcal{G}$ be the Green function of the operator $\Delta+h$ in Ω with Dirichlet boundary conditions. Note first that $\gamma_x(x) \to -\infty$ as x approaches $\partial\Omega$. In particular, there exists a point $x_1 \in \Omega$ such that $\gamma_{x_1}(x_1) < 0$. For $x \in \Omega \setminus \{x_1\}$, we set

$$\lambda(x) = \left(-\frac{\gamma_{x_1}(x_1)}{\mathcal{G}(x_1,x)}\right)^2$$

and

$$F(x) = \mathcal{G}(x_1,x)^2 - \gamma_{x_1}(x_1)\gamma_x(x) .$$

Since $F(x) \to +\infty$ as $x \to x_1$ and $F(x) \to -\infty$ as x approaches $\partial\Omega$ and since F is continous, there exists x_2 such that $F(x_2) = 0$. We let then $\lambda = \lambda(x_2)$ and we have

$$\sqrt{\lambda} G_2(x_1) + \gamma_1(x_1) = G_1(x_2) + \sqrt{\lambda}\gamma_2(x_2) = 0 \tag{3.12}$$

where

$$G_1(x) = \mathcal{G}(x_1,x) , \; G_2(x) = \mathcal{G}(x_2,x) , \; \gamma_1(x) = \gamma_{x_1}(x) , \; \gamma_2(x) = \gamma_{x_2}(x) . \tag{3.13}$$

We let $\delta > 0$ be such that $\delta \le 10 d(x_1,\partial\Omega)$ and $\delta \le 10 d(x_2,\partial\Omega)$. We fix $\eta \in C^\infty(\mathbb{R})$ such that $\eta(r) = 1$ for $|r| \le \delta$ and $\eta(r) = 0$ for $|r| \ge 2\delta$. We set in the following

$$\begin{aligned}
u_\varepsilon = {} & \varepsilon^{-\frac{1}{2}} U(\varepsilon G_1) + (\lambda\varepsilon)^{-\frac{1}{2}} U(\lambda\varepsilon G_2) \\
& + \eta(|x-x_1|)\gamma_1(x_1)\varepsilon^{\frac{1}{2}} V(\varepsilon G_1) + \eta(|x-x_2|)\gamma_2(x_2)(\lambda\varepsilon)^{\frac{1}{2}} V(\lambda\varepsilon G_2) \\
& - \eta(|x-x_1|)\psi_\varepsilon(\varepsilon G_1)\varepsilon^{\frac{1}{2}}(x-x_1)^i \\
& \quad . \left(\left(1+\varepsilon^2 G_1^2\right)^{-\frac{3}{2}} \partial_i\gamma_1(x_1) + \lambda^{\frac{1}{2}}\partial_i G_2(x_1)\right) \\
& - \eta(|x-x_2|)\psi_\varepsilon(\lambda\varepsilon G_2)(\lambda\varepsilon)^{\frac{1}{2}}(x-x_2)^i \\
& \quad . \left(\left(1+\lambda^2\varepsilon^2 G_2^2\right)^{-\frac{3}{2}} \partial_i\gamma_2(x_2) + \lambda^{-\frac{1}{2}}\partial_i G_1(x_2)\right) \\
& + \eta(|x-x_1|)\varepsilon^{\frac{3}{2}}\psi_\varepsilon(\varepsilon G_1)\left(h(x_1)W(\varepsilon G_1) - \frac{3}{2}\gamma_1(x_1)^2 U(\varepsilon G_1)^5\right) \\
& + \eta(|x-x_2|)(\lambda\varepsilon)^{\frac{3}{2}}\psi_\varepsilon(\lambda\varepsilon G_2)\left(h(x_2)W(\lambda\varepsilon G_2) - \frac{3}{2}\gamma_2(x_2)^2 U(\lambda\varepsilon G_2)^5\right)
\end{aligned} \tag{3.14}$$

where we adopt Einstein summation conventions and U, V, W and ψ_ε are given by

$$U(r) = r\left(1+r^2\right)^{-\frac{1}{2}},\ V(r) = 1-\left(1+r^2\right)^{-\frac{3}{2}},\ \psi_\varepsilon(r) = 1+\frac{\ln\left(1+r^{-2}\right)}{\ln\varepsilon^2} \text{ and}$$
$$W(r) = -\frac{13}{4}U + 8\left(2U^3-U\right)\ln U - 2\left(U^{-1}-8U+8U^3\right) r\arctan\left(\frac{1}{r}\right) . \tag{3.15}$$

It is easily checked that u_ε is $C^{2,\eta}$ in Ω and that $u_\varepsilon = 0$ on $\partial\Omega$. We claim now that (3.11) holds for this specific u_ε and that u_ε is positive in Ω. We shall prove this claim in three steps. First, we can prove it rather easily in $\Omega\setminus B_{x_1}(2\delta)\bigcup B_{x_2}(2\delta)$ because, in this region, u_ε is simply

$$u_\varepsilon = \varepsilon^{-\frac{1}{2}}U\left(\varepsilon G_1\right) + (\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right) .$$

Now, noticing that $U' = r^{-3}U^3$ and that $U'' = -3r^{-4}U^5$, simple computations lead to

$$\begin{aligned}
&\Delta\left(\varepsilon^{-\frac{1}{2}}U\left(\varepsilon G_1\right) + (\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right)\right) + h\left(\varepsilon^{-\frac{1}{2}}U\left(\varepsilon G_1\right) + (\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right)\right)\\
&\quad= 3\left(\varepsilon^{-\frac{1}{2}}U\left(\varepsilon G_1\right)\right)^5\left|\nabla G_1^{-1}\right|^2 + 3\left((\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right)\right)^5\left|\nabla G_2^{-1}\right|^2\\
&\qquad+ h\varepsilon^{-\frac{1}{2}}U\left(\varepsilon G_1\right)\left(1-\left(1+\varepsilon^2G_1^2\right)^{-1}\right)\\
&\qquad+ h(\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right)\left(1-\left(1+\lambda^2\varepsilon^2G_2^2\right)^{-1}\right)
\end{aligned} \tag{3.16}$$

in Ω. In the region we are interested in, this clearly leads to

$$\Delta u_\varepsilon + hu_\varepsilon - 3u_\varepsilon^5 = o\left(u_\varepsilon\right)$$

which proves that (3.11) holds in this region while u_ε is clearly positive in this region.

We will now prove that (3.11) holds in $B_{x_1}(2\delta)$ and that u_ε is positive in this ball. By symmetry, it is clear that the proof of the fact that (3.11) holds in $B_{x_2}(2\delta)$ is exactly the same[3]. In order to simplify the notations, we will assume that $x_1 = 0$, which we can always do by translating Ω. We will denote G_1 by G and γ_1 by γ. We also set

$$\begin{aligned}
&U_\varepsilon = \varepsilon^{-\frac{1}{2}}U\left(\varepsilon G\right),\ V_\varepsilon = \varepsilon^{\frac{1}{2}}V\left(\varepsilon G\right),\ W_\varepsilon = \varepsilon^{\frac{3}{2}}W\left(\varepsilon G\right),\\
&\varphi_\varepsilon = \psi_\varepsilon\left(\varepsilon G\right),\ Y_\varepsilon = -\frac{3}{2}\varepsilon^{\frac{3}{2}}U\left(\varepsilon G\right)^5,\ \tilde U_\varepsilon = (\lambda\varepsilon)^{-\frac{1}{2}}U\left(\lambda\varepsilon G_2\right) \text{ and}\\
&Z_\varepsilon = -\varepsilon^{\frac{1}{2}}x^i\left(\left(1+\varepsilon^2G^2\right)^{-\frac{3}{2}}\partial_i\gamma(0) + \lambda^{\frac{1}{2}}\partial_iG_2(0)\right) .
\end{aligned} \tag{3.17}$$

With these notations, we have that, in $B_0(2\delta)$,

$$u_\varepsilon = U_\varepsilon + \tilde U_\varepsilon + \eta\left(|x|\right)\gamma(0)V_\varepsilon + \eta\left(|x|\right)\varphi_\varepsilon\left(h(0)W_\varepsilon + \gamma(0)^2Y_\varepsilon + Z_\varepsilon\right) . \tag{3.18}$$

Let us write thanks to (3.1) that

$$\begin{aligned}
\left|\nabla G^{-1}\right|^2 = 1 - 4\gamma(0)G^{-1} + 3\left(2\gamma(0)^2 - h(0)\right)G^{-2}\\
- 6G^{-1}x^i\partial_i\gamma(0) + o\left(G^{-2}\right)
\end{aligned} \tag{3.19}$$

[3]The symmetry is precisely the following : if we see u_ε as a function of x_1, x_2, ε and λ, namely $u_\varepsilon\left(x_1,x_2,\varepsilon,\lambda\right)$, one has that $u_\varepsilon\left(x_2,x_1,\lambda\varepsilon,\lambda^{-1}\right) = u_\varepsilon\left(x_1,x_2,\varepsilon,\lambda\right)$.

and let us also remark that

$$G^{-2}=\varepsilon U_\varepsilon^{-2}-\varepsilon^2\ . \tag{3.20}$$

Let us write thanks to (3.12) that

$$\begin{aligned}&\tilde{U}_\varepsilon=-\varepsilon^{\frac{1}{2}}\gamma(0)+(\lambda\varepsilon)^{\frac{1}{2}}\,x^i\partial_iG_2(0)+O\left(\varepsilon^{\frac{3}{2}}U_\varepsilon^{-2}\right),\\&V_\varepsilon=O\left(\varepsilon^{\frac{3}{2}}U_\varepsilon^2\right),\,W_\varepsilon=O\left(\varepsilon^{\frac{3}{2}}\right),\,Y_\varepsilon=O\left(\varepsilon^{\frac{3}{2}}\right)\text{ and }Z_\varepsilon=O\left(\varepsilon U_\varepsilon^{-1}\right)\ .\end{aligned} \tag{3.21}$$

Thanks to (3.18) and to (3.21), it is easily checked that u_ε is positive in $B_0\left(2\delta\right)$. Lengthy but straightforward computations lead then to

$$\begin{aligned}&\left|\nabla\varphi_\varepsilon\right|=O\left(\frac{1}{\varepsilon^{\frac{1}{2}}\ln\frac{1}{\varepsilon}}U_\varepsilon\right)\ ,\ \left|\nabla V_\varepsilon\right|=O\left(\varepsilon U_\varepsilon^3\right)\ ,\ \left|\nabla W_\varepsilon\right|=O\left(\varepsilon U_\varepsilon\right)\ ,\\&\left|\nabla Y_\varepsilon\right|=O\left(\varepsilon U_\varepsilon\right)\text{ and }\left|\nabla Z_\varepsilon\right|=O\left(\varepsilon^{\frac{1}{2}}\right)\end{aligned} \tag{3.22}$$

in $B_0\left(2\delta\right)$ and to

$$\begin{aligned}\Delta U_\varepsilon+hU_\varepsilon&=3U_\varepsilon^5-12\gamma(0)G^{-1}U_\varepsilon^5-18G^{-1}x^i\partial_i\gamma(0)U_\varepsilon^5\\&\quad+18\gamma(0)^2G^{-2}U_\varepsilon^5+h(0)\left(\varepsilon^2-8G^{-2}\right)U_\varepsilon^5+o\left(U_\varepsilon\right)\ ,\\\Delta\tilde{U}_\varepsilon+h\tilde{U}_\varepsilon&=3\tilde{U}_\varepsilon^5\left|\nabla G_2^{-1}\right|^2+\lambda\varepsilon h\tilde{U}_\varepsilon^3=O\left(\varepsilon^{\frac{5}{2}}\right)\ ,\\\Delta V_\varepsilon+hV_\varepsilon&=15U_\varepsilon^4V_\varepsilon-15\varepsilon^{\frac{1}{2}}U_\varepsilon^4+12G^{-1}U_\varepsilon^5\\&\quad+12\gamma(0)\left(5\varepsilon^{-1}G^{-4}U_\varepsilon^7-4G^{-2}U_\varepsilon^5\right)+o\left(U_\varepsilon\right)\ ,\\\Delta W_\varepsilon+hW_\varepsilon&=15U_\varepsilon^4W_\varepsilon+8\varepsilon U_\varepsilon^3-9\varepsilon^2U_\varepsilon^5+o\left(U_\varepsilon\right)\ ,\\\Delta Y_\varepsilon+hY_\varepsilon&=15U_\varepsilon^4Y_\varepsilon+30\varepsilon^3U_\varepsilon^7-30\varepsilon^4U_\varepsilon^9+o\left(U_\varepsilon\right)\ ,\\\Delta\varphi_\varepsilon&=O\left(\frac{1}{\varepsilon\ln\frac{1}{\varepsilon}}U_\varepsilon^2\right)\ ,\\\Delta Z_\varepsilon+hZ_\varepsilon&=15U_\varepsilon^4Z_\varepsilon+18U_\varepsilon^5G^{-1}\partial_i\gamma(0)x^i\\&\quad+15\left(\lambda\varepsilon\right)^{\frac{1}{2}}U_\varepsilon^4x^i\partial_iG_2(0)+O\left(\varepsilon U_\varepsilon^{-1}\right)+o\left(U_\varepsilon\right)\end{aligned} \tag{3.23}$$

in $B_0\left(2\delta\right)$. It follows easily from the above equations that

$$\frac{\Delta u_\varepsilon+hu_\varepsilon-3u_\varepsilon^5}{u_\varepsilon}\to0\text{ in }L^\infty\left(B_0\left(2\delta\right)\setminus B_0\left(\delta\right)\right)\text{ as }\varepsilon\to0\ .$$

It remains to prove the result in $B_0\left(\delta\right)$. Thanks to (3.21), one can easily check that

$$\frac{u_\varepsilon}{U_\varepsilon}\to1+\sqrt{\lambda}\frac{G_2}{G_1}\text{ in }L^\infty\big(B_0\left(\delta\right)\big)\text{ as }\varepsilon\to0 \tag{3.24}$$

so that we can write that

$$3u_\varepsilon^5=3U_\varepsilon^5+15U_\varepsilon^4\left(u_\varepsilon-U_\varepsilon\right)+30U_\varepsilon^3\left(u_\varepsilon-U_\varepsilon\right)^2+O\left(U_\varepsilon^2\left|u_\varepsilon-U_\varepsilon\right|^3\right)\ .$$

Using again (3.22), we deduce that

$$\begin{aligned}3u_\varepsilon^5\ =\ &3U_\varepsilon^5-15\varepsilon^{\frac{1}{2}}\gamma(0)U_\varepsilon^4+15\left(\lambda\varepsilon\right)^{\frac{1}{2}}U_\varepsilon^4x^i\partial_iG_2(0)\\&+15\gamma(0)U_\varepsilon^4V_\varepsilon+15U_\varepsilon^4\varphi_\varepsilon\left(h(0)W_\varepsilon+\gamma(0)^2Y_\varepsilon+Z_\varepsilon\right)\\&+30\gamma(0)^2U_\varepsilon^3\left(V_\varepsilon-\varepsilon^{\frac{1}{2}}\right)^2+o\left(U_\varepsilon\right)\end{aligned}$$

in $B_0(\delta)$. Thanks to (3.21), to (3.22) and to (3.23), we can also write that

$$\begin{aligned}\Delta u_\varepsilon + hu_\varepsilon &= 3U_\varepsilon^5 + 15\gamma(0)U_\varepsilon^4 V_\varepsilon + 15(\lambda\varepsilon)^{\frac{1}{2}}\varphi_\varepsilon U_\varepsilon^4 x^i\partial_i G_2(0)\\ &\quad -15\gamma(0)\varepsilon^{\frac{1}{2}}U_\varepsilon^4 + 15U_\varepsilon^4\varphi_\varepsilon\left(h(0)W_\varepsilon + \gamma(0)^2 Y_\varepsilon + Z_\varepsilon\right)\\ &\quad +30\gamma(0)^2\left(2\varepsilon^{-1}G^{-4}U_\varepsilon^7 - G^{-2}U_\varepsilon^5\right) + h(0)\left(\varepsilon^2 - 8G^{-2}\right)U_\varepsilon^5\\ &\quad +h(0)\varphi_\varepsilon\left(8\varepsilon U_\varepsilon^3 - 9\varepsilon^2 U_\varepsilon^5\right) + 30\gamma(0)^2\varphi_\varepsilon\left(\varepsilon^3 U_\varepsilon^7 - \varepsilon^4 U_\varepsilon^9\right)\\ &\quad +18(\varphi_\varepsilon - 1)U_\varepsilon^5|x|\partial_i\gamma(0)x^i + o(U_\varepsilon)\ .\end{aligned}$$

Combining these two last equations, we get that

$$\begin{aligned}\Delta u_\varepsilon + hu_\varepsilon - 3u_\varepsilon^5 &= -30\gamma(0)^2\Big(U_\varepsilon^3\left(V_\varepsilon - \varepsilon^{\frac{1}{2}}\right)^2 - \varphi_\varepsilon\varepsilon^3 U_\varepsilon^7\\ &\qquad +\varphi_\varepsilon\varepsilon^4 U_\varepsilon^9 - 2\varepsilon^{-1}G^{-4}U_\varepsilon^7 + G^{-2}U_\varepsilon^5\Big)\\ &\quad +h(0)\left(\varepsilon^2 U_\varepsilon^5 - 8G^{-2}U_\varepsilon^5 + 8\varphi_\varepsilon\varepsilon U_\varepsilon^3 - 9\varphi_\varepsilon\varepsilon^2 U_\varepsilon^5\right) + o(U_\varepsilon)\\ &\quad +18(\varphi_\varepsilon - 1)|x|x^i\partial_i\gamma(0)U_\varepsilon^5\end{aligned}$$

It remains to remark using (3.20) that

$$\begin{aligned}&U_\varepsilon^3\left(V_\varepsilon - \varepsilon^{\frac{1}{2}}\right)^2 - \varphi_\varepsilon\varepsilon^3 U_\varepsilon^7 + \varphi_\varepsilon\varepsilon^4 U_\varepsilon^9 - 2\varepsilon^{-1}G^{-4}U_\varepsilon^7 + G^{-2}U_\varepsilon^5\\ &\quad = \varepsilon^2 G^{-2}U_\varepsilon^9(1 - \varphi_\varepsilon)\\ &\quad = -\frac{\varepsilon^2}{\ln\varepsilon^2}\ln\left(1 + \varepsilon^{-2}G^{-2}\right)G^{-2}U_\varepsilon^9\\ &\quad = -\frac{U_\varepsilon}{\ln\varepsilon^2}\varepsilon^6\ln\left(1 + \varepsilon^{-2}G^{-2}\right)G^{-2}\left(\varepsilon^2 + G^{-2}\right)^{-4}\\ &\quad = O\left(\frac{U_\varepsilon}{\ln\frac{1}{\varepsilon}}\right) = o(U_\varepsilon)\ ,\end{aligned}$$

that

$$\begin{aligned}&\varepsilon^2 U_\varepsilon^5 - 8G^{-2}U_\varepsilon^5 + 8\varphi_\varepsilon\varepsilon U_\varepsilon^3 - 9\varphi_\varepsilon\varepsilon^2 U_\varepsilon^5\\ &\quad = \left(-9\varepsilon^2 U_\varepsilon^5 + 8\varepsilon U_\varepsilon^3\right)(\varphi_\varepsilon - 1)\\ &\quad = \frac{U_\varepsilon}{\ln\varepsilon^2}\ln\left(1 + \varepsilon^{-2}G^{-2}\right)\left(-9\left(1 + \varepsilon^{-2}G^{-2}\right)^{-2} + 8\left(1 + \varepsilon^2 G^{-2}\right)^{-1}\right)\\ &\quad = O\left(\frac{U_\varepsilon}{\ln\frac{1}{\varepsilon}}\right) = o(U_\varepsilon)\end{aligned}$$

and that

$$\begin{aligned}(\varphi_\varepsilon - 1)G^{-1}x^i\partial_i\gamma(0)U_\varepsilon^5 &= O\left(\frac{U_\varepsilon}{\ln\varepsilon^2}\ln\left(1 + \varepsilon^{-2}G^{-2}\right)\varepsilon^{-2}G^{-2}\left(1 + \varepsilon^{-2}G^{-2}\right)^{-2}\right)\\ &= O\left(\frac{U_\varepsilon}{\ln\frac{1}{\varepsilon}}\right) = o(U_\varepsilon)\end{aligned}$$

to conclude thanks to (3.24) that (3.11) holds in $B_0(\delta)$ for this choice of u_ε. As already said, this proves that (3.11) holds for u_ε given by (3.14) and this ends the proof of the theorem. ■

As already said, this result implies theorem 2.

4. Appendix

4.1. **A general simple lemma on functions.**

We prove a new version of the simple Lemma 1.1 of [8], replacing the compact manifold M by a domain Ω in $\mathbb{R}^n$.

Lemma 4.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Let $u \in C^1(\Omega)$ be a function positive in the interior and null on the boundary. Assume that*

$$\{x \in \Omega \text{ s.t. } \nabla u(x) = 0 \text{ and } d(x, \partial\Omega)u^2(x) \geq 1\} \neq \emptyset \, .$$

Then there exist $N \in \mathbb{N}^*$ and N critical points of u, denoted by $(x_1, \dots, x_N)$, such that

$$d(x_i, \partial\Omega)u(x_i)^2 \geq 1 \text{ for all } i \in [1, N],$$
$$|x_i - x_j|u(x_i)^2 \geq 1 \text{ for all } i \neq j \in [1, N]$$

and

$$\left(\min_{i\in[1,N]} |x_i - x|\right) u(x)^2 \leq 1$$

for all critical points x of u such that $d(x, \partial\Omega)u(x)^2 \geq 1$.

Proof of Lemma 4.1. Let $\mathcal{C}_u$ be the set of critical points of u. Thanks to the Hopf Lemma, it is clear that $\mathcal{C}_u$ is a compact set of Ω. We let

$$K_0 = \{x \in \mathcal{C}_u \text{ s.t. } d(x, \partial\Omega)u^2(x) \geq 1\}$$

and we assume that $K_0 \neq \emptyset$. We let $x_1 \in K_0$ and $K_1 \subset K_0$ be such that

$$u(x_1) = \max_{K_0} u$$

and

$$K_1 = \left\{x \in K_0 \text{ s.t. } |x_1 - x|u(x)^2 \geq 1\right\} \, .$$

Then we proceed by induction. Assuming we have constructed $K_0 \supset \cdots \supset K_p$ and $x_1, \dots, x_p$ such that $x_i \in K_{i-1}$ for all $i \in [1, p]$, we let $x_{p+1} \in K_p$ and $K_{p+1} \subset K_p$ be such that

$$u(x_{p+1}) = \max_{K_p} u$$

and

$$K_{p+1} = \left\{x \in K_p \text{ s.t. } |x_{p+1} - x|u(x_{p+1})^2 \geq 1 \text{ and } \min_{i\in[1,p]} |x - x_i|u(x)^2 \geq 1\right\} \, .$$

We claim that, at some step in the process, $K_p = \emptyset$. In order to prove it, we remark that at each step of the construction,

$$|x_i - x_j|u(x_i)^2 \geq 1 \text{ for all } i \neq j \in [1, p], \tag{4.1}$$

which will prove the claim, since Ω is bounded. We prove (4.1) by induction. Let $p \geq 1$. By definition, for all $x \in K_p$, we have

$$|x_i - x|u(x)^2 \geq 1 \text{ for all } i \in [1, p] \, .$$

This holds in particular for $x = x_{p+1}$. Then, for all $x \in K_p$, we also easily check that

$$|x_i - x|u(x_i)^2 \geq 1 \text{ for all } i \in [1, p] \, ,$$

which is also true for x_{p+1}, and proves (4.1). Let $N \in \mathbb{N}^*$ be such that $K_N = \emptyset$. We claim that

$$\left(\min_{i\in[1,N]} |x_i - x|\right) u^2(x) \leq 1 \tag{4.2}$$

for all $x \in K_0$ which, together with (4.1), will end the proof of the lemma. Let $x \in K_0$. Since $K_N = \emptyset$, there exists p such that $x \in K_{p-1}$ and $x \notin K_p$. Then we have either

$$|x_p - x|u^2(x_p) < 1$$

or

$$\min_{i \in [1,p]} |x - x_i|u(x) < 1 \ .$$

In the second case, (4.2) is clearly true while in the first, using the definition of x_p, we have that

$$|x_p - x|u^2(x) \le |x_p - x|u^2(x_p) < 1 \ ,$$

which proves that (4.2) also holds. As already sais, this proves the lemma. ■

4.2. **Green function of $\Delta + h$.** We prove here some basic estimates on Green's functions of operators $\Delta + h$ where h is of low regularity.

Lemma 4.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^3$. Let $h \in L^p(\Omega)$ for some $p > 3$. Then there exists $\delta > 0$ such that if*

$$\|h_-\|_{\frac{3}{2}} < \delta \ , \tag{4.3}$$

then the operator $\Delta + h$ admits a positive Green function $\mathcal{G}_h$ which verifies the following estimates :

$$\left| |x-y|\mathcal{G}_h(x,y) - \frac{1}{\omega_2} \right| \le C|x-y|$$

and

$$\left| |x-y|^2 |\nabla \mathcal{G}_h(x,y)| - \frac{1}{\omega_2} \right| \le C|x-y|$$

for all $x \neq y \in \Omega$, where C is a positive constant depending only on Ω, $\|h\|_p$ and δ.

Proof of lemma 4.2. We divide the proof into three steps.

Step 1 : $\Delta + h$ is coercive if $\|h_-\|_{\frac{3}{2}}$ is small enough - Let $u \in H_0^1(\Omega)$. We write that

$$\int_\Omega \left(|\nabla u|^2 + hu^2\right) \, dx \ge \int_\Omega \left(|\nabla u|^2 - h_- u^2\right) \, dx \ge \|\nabla u\|_2^2 - \|h_-\|_{\frac{3}{2}} \|u\|_6$$

thanks to Hölder's inequalities. One can then use Sobolev's embeddings and the fact that $\|h_-\|_{\frac{3}{2}}$ is small to conclude this first step.

Step 2 : Existence and a priori estimate - Let $\mathcal{G}(x,y)$ be the Green function of the Laplacian. Then solving

$$\Delta_y \mathcal{G}_h(x,y) + h\mathcal{G}_h(x,y) = \delta_x \text{ in } \Omega,$$
$$\mathcal{G}_h(x,y) = 0 \text{ on } \partial\Omega,$$

is equivalent to solving

$$\Delta_y \beta(x,y) + h\beta(x,y) = -h\mathcal{G}(x,y),$$
$$\beta(x,y) = 0 \text{ on } \partial\Omega.$$

Since $h \in L^p(\Omega)$ for some $p > 3$, there exists $q > \frac{3}{2}$ such that $h\mathcal{G}(x,.) \in L^q(\Omega)$. The existence of β follows from the coercivity of $\Delta + h$ and Lax-Milgram theorem.

Moreover, using again the coercivity of $\Delta + h$ and Sobolev's embeddings, we get that

$$\frac{1}{C}\|\nabla\beta\|_2^2 \le \int_\Omega (|\nabla\beta|^2 + h\beta^2)dx = \int_\Omega -h\mathcal{G}\beta dx \le \|h\mathcal{G}\|_{\frac{3}{2}}\|\beta\|_3 \le C\|\nabla\beta\|_2,$$

for some $C > 1$ depending only on $\|h\|_p$, $\|h_-\|_{\frac{3}{2}}$ and Ω. This gives an a priori bound on $\|\nabla\beta\|_2$.

Step 3 : estimates and positivity - Thanks to the previous Step, there exists $C > 0$ which depends only on $\|h\|_p$ and $\|h_-\|_{\frac{3}{2}}$, and $q > \frac{3}{2}$ such that

$$\|h(\beta + G(x,.))\|_q \le C \ .$$

Now, thanks to standard elliptic theory (see for instance theorem 9.13 of [9]), we see that $\beta \in L^\infty$ and

$$\|\beta\|_\infty \le C$$

where C is a positive constant which depends only on $\|h\|_p$ and $\|h_-\|_{\frac{3}{2}}$. This proves the first estimate of the lemma. The second follows by standard elliptic theory. Positivity of the Green function is an easy consequence of the coercivity of the operator $\Delta + h$. ∎

4.3. **General Pohožaev's identities.** For the sake of completeness, we derive here several forms of the classical Pohožaev identity [11] we used in this paper. Assume that u is a C^2- solution of

$$\Delta u = u^5 - hu \text{ in } \Omega \ .$$

Multiplying this equation by $< x, \nabla u >$ and integrating by parts, one easily gets that

$$\frac{1}{2}\int_\Omega \left(hu^2 + h < x, \nabla u^2 >\right) dx = B_1 + B_2, \tag{4.4}$$

where

$$B_1 = \int_{\partial\Omega} \left(< x, \nabla u > \partial_\nu u + \frac{1}{2}u\partial_\nu u - < x, \nu > \frac{|\nabla u|^2}{2}\right) d\sigma \text{ and}$$

$$B_2 = \int_{\partial\Omega} < x, \nu > \frac{u^6}{6} d\sigma \ .$$

Hence, if $u = 0$ on $\partial\Omega$, we get that

$$\frac{1}{2}\int_\Omega h\left(u^2 + < x, \nabla u^2 >\right) dx = \int_{\partial\Omega} < x, \nu > (\partial_\nu u)^2 d\sigma \ . \tag{4.5}$$

Integrating by parts again, we get the Pohožaev identity in its usual form :

$$\int_\Omega \left(h + \frac{< x, \nabla h >}{2}\right) u^2 dx = -\int_{\partial\Omega} < x, \nu > (\partial_\nu u)^2 d\sigma \ . \tag{4.6}$$

In a similar way, multiplying the equation by ∇u and integrating by parts, one can derive the following Pohožaev's identity :

$$\int_{\partial\Omega} \left(\frac{|\nabla u|^2}{2}\nu - \partial_\nu u \nabla u + \frac{u^6}{6}\nu\right) d\sigma = \int_\Omega h \frac{\nabla u^2}{2} dx \ . \tag{4.7}$$

4.4. **Pohožaev's identity for Green functions.** In this section, we prove a useful Pohožaev identity for a sum of Green's functions. First of all, we easily derive the following Lemma from standard elliptic theory :

Lemma 4.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Let $y \in \Omega$ and let g be a weak solution in $H^1(\Omega)$ of*

$$\Delta g + hg = -\frac{h}{\omega_2 |x-y|} \text{ in } \Omega .$$

Then g is continuous and can be written as

$$g(x) = g(y) + \frac{h(y)}{2}|x-y| + \gamma_y(x) \text{ in } \Omega \tag{4.8}$$

where $\gamma_y \in C^1(\Omega)$ satisfies $\gamma_y(y) = 0$.

Applying the previous decomposition lemma to Green's functions, we get the following Pohožaev identity on the regular parts of them.

Lemma 4.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$, star-shaped with respect to 0 and let $h \in C^1(\Omega)$ which satisfies (0.2). Let $\mathcal{G}_h$ be the Green function of $\Delta + h$. Let also $N \in \mathbb{N}^*$, $x_1, \dots, x_N \in \Omega$, $\lambda_1, \dots, \lambda_N$ some positive real numbers and*

$$G(x) = \sum_{i=1}^N \lambda_i \mathcal{G}_h(x, x_i) .$$

Then, using lemma 4.3, we write G in a neighbourhood of x_i as

$$G(x) = \frac{\lambda_i}{\omega_2 |x-x_i|} + m_i + \lambda_i \frac{h(x_i)}{2}|x-x_i| + \gamma_i(x)$$

where $m_i \in \mathbb{R}$ and $\gamma_i \in C^1(\Omega)$ satisfies $\gamma_i(0) = 0$. Then we have that

$$\sum_{i=1}^N \lambda_i \left(m_i + 2 < x_i, \nabla\gamma_{x_i}(x_i) >\right) < 0 .$$

Proof of lemma 4.4. We let $\delta > 0$ be such that the $B(x_i,\delta)$ are disjoint and do not intersect the boundary of Ω and we set

$$\Omega_\delta = \Omega \setminus \left\{ \bigcup_{i=1}^N B(x_i,\delta) \right\} .$$

Multiplying the equation satisfied by G by $< x, \nabla G >$ and after some integrations by parts, we obtain that

$$\begin{aligned}
&\int_{\Omega_\delta} \left(\frac{1}{2} < x, \nabla h > + h\right) G^2 \, dx \\
&= \int_{\partial\Omega} \left(\frac{1}{2} < x, \nu > \left(|\nabla G|^2 + hG^2\right) - \left((x, \nabla G) + \frac{1}{2}G\right)\partial_\nu G\right) d\sigma \\
&\quad - \sum_{i=1}^N \int_{\partial B(x_i,\delta)} \left(\frac{1}{2} < x, \nu > \left(|\nabla G|^2 + hG^2\right) - \left((x, \nabla G) + \frac{1}{2}G\right)\partial_\nu G\right) d\sigma
\end{aligned}$$

where ν denotes the outer normal to $\partial\Omega$ and to $\partial B(x_i, \delta)$ respectively. Noting that $G = 0$ on $\partial\Omega$, we have that

$$\int_{\partial\Omega}\left(\frac{1}{2} < x,\nu > \left(|\nabla G|^2 + hG^2\right) - \left((x,\nabla G) + \frac{1}{2}G\right)\partial_\nu G\right)\,d\sigma$$
$$= -\frac{1}{2}\int_{\partial\Omega} < x,\nu > |\nabla G|^2\,d\sigma < 0$$

since Ω is star-shaped. Since h satisfies (0.2), we arrive to

$$\sum_{i=1}^N \int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x,\nu > \left(|\nabla G|^2 + hG^2\right) - \left((x,\nabla G) + \frac{1}{2}G\right)\partial_\nu G\right)\,d\sigma \le -C_0$$

where C_0 is independent of δ. It is easily checked that

$$\int_{\partial B(x_i,\delta)} < x,\nu > hG^2\,d\sigma \to 0 \text{ as } \delta \to 0\ .$$

In order to estimate the remaining terms, we write that

$$\int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x,\nu > |\nabla G|^2 - \left((x,\nabla G) + \frac{1}{2}G\right)\partial_\nu G\right)\,d\sigma$$
$$= \int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x - x_i,\nu > |\nabla G|^2 - \left((x - x_i,\nabla G) + \frac{1}{2}G\right)\partial_\nu G\right)\,d\sigma$$
$$+ \int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x_i,\nu > |\nabla G|^2 - (x_i,\nabla G)\,\partial_\nu G\right)\,d\sigma\ .$$

Then, thanks to the expansion of G in a neighbourhood of x_i, one can easily check that

$$\int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x - x_i,\nu > |\nabla G|^2 - \left((x - x_i,\nabla G) + \frac{1}{2}G\right)\partial_\nu G\right)\,d\sigma \to \frac{\lambda_i}{2}m_i$$

and

$$\int_{\partial B(x_i,\delta)}\left(\frac{1}{2} < x_i,\nu > |\nabla G|^2 - (x_i,\nabla G)\,\partial_\nu G\right)\,d\sigma \to \lambda_i < x_i, \nabla\gamma_{x_i}(x_i) >$$

as $\delta \to 0$. Combining the above results gives the desired inequality. ■

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