STABILITY OF ELLIPTIC PDES WITH RESPECT TO PERTURBATIONS OF THE DOMAIN

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Abstract. The stability of the Pohožaev nonexistence property for stationary Schrödinger equations under perturbations of the domain is investigated in this paper. We prove that the sharp threshold for the regularity comes precisely with C^1-perturbations of the domain.

Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Let $h \in C^1(\mathbb{R}^3)$. The critical stationary Schrödinger equation in Ω, with zero Dirichlet boundary condition, is written as

$$\begin{cases} \Delta u + hu = u^5 \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\Delta u = -\sum_i \partial_i^2 u$ is the Laplace operator with nonnegative eigenvalues. By the Pohožaev identity, see [8], any solution u of (0.1) satisfies that

$$\int_\Omega hu^2 dx + \frac{1}{2}\int_\Omega (x^k \partial_k h) u^2 dx = -\frac{1}{2}\int_{\partial\Omega} (x^k \nu_k)(\partial_\nu u)^2 d\sigma \ , \tag{0.2}$$

where ν is the outward unit normal to Ω, and $x^k\partial_k h(x) = \langle x, \nabla u(x)\rangle$, $x^k \nu_k = \langle x, \nu \rangle$ by the Einstein's summation convention. In particular, it follows from (0.2) that if Ω is star-shaped with respect to the origin, and if h satisfies that

$$h(x) + \frac{1}{2} x^k \partial_k h(x) \geq 0 \tag{0.3}$$

for all $x \in \Omega$, then there are no non-trivial solutions of (0.1). The result follows from the remarks that (0.2) implies that both the left hand side in (0.3) and $\partial_\nu u$ have to be identically zero, respectively in Ω and on $\partial\Omega$, that h has to be identically zero if the left hand side in (0.3) is identically zero, and that the Hopf Lemma contradicts the equation $\partial_\nu u \equiv 0$. In Druet and Laurain [5], the stability of the Pohožaev obstruction with respect to perturbations of the potential h was investigated. It was proved that, given a star-shaped smooth domain with respect to the origin, and $\eta \in (0,1)$, $C^{0,\eta}$-perturbations of a C^1-potential h satisfying (0.3) still give rise to the Pohožaev nonexistence property, while any C^1-potential satisfying (0.3) can be approached in the L^∞-norm by potentials for which (0.1) has nontrivial solutions. The whole point of course in the stability of the nonexistence property is that (0.3) itself is not stable under $C^{0,\eta}$-perturbations, and even not stable under more regular perturbations of h. The question we ask here is the following:

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Q1: if Ω is a smooth star-shaped domain with respect to the origin, and $h \in C^1$ satisfies (0.3), down to which degree of regularity can we perturb both Ω and h in order to preserve the Pohožaev nonexistence property attached to (0.1) ?

The existence of one degree of regularity for which perturbations of both Ω and h preserve the Pohožaev nonexistence property is itself not trivial. For instance, in higher dimensions, by the Brézis and Nirenberg [2] existence result, any perturbation of $h \equiv 0$ in L^∞ which is negative somewhere would lead to a nontrivial solution of the dimensional critical analogue of (0.1), contradicting the existence of such a degree. We prove here that Q1 has a positive answer even when the domain is perturbed, and that the critical degree of regularity for the domain is C^1 by opposition to $C^{0,\eta}$-perturbations of Ω. The stability aspect in the C^1-topology for perturbations of the domain is answered by the following theorem. We let $\text{Diff}(\mathbb{R}^3)$ be the set of smooth diffeomorphisms of $\mathbb{R}^3$.

Theorem 1. *Let $\eta \in]0,1[$. Let Ω_0 be a smooth bounded domain of $\mathbb{R}^3$, star-shaped with respect to the origin, and let $h_0 \in C^1(\mathbb{R}^3)$ satisfying (0.3). Then the Pohožaev nonexistence property is stable with respect to $C^{0,\eta}$-perturbations of h_0 and C^1-perturbations of Ω_0. More precisely, there exists $\delta = \delta(h,\Omega,\eta)$, $\delta > 0$, such that for any $\tilde{h} \in C^{0,\eta}\left(\mathbb{R}^3\right)$ and any $\Phi \in C^{1,\eta} \cap \text{Diff}(\mathbb{R}^3)$ satisfying that*

$$\|h - \tilde{h}\|_{C^{0,\eta}} + \|\Phi - Id\|_{C^1} \le \delta ,$$

then (0.1) with respect to h and $\Omega = \Phi(\Omega_0)$ has no nontrivial solution.

In view of Theorem 1 it is important to note that the star-shaped property is generally not stable under C^1 or smooth perturbations of the domain as shown in the following picture, figure 1, where the star-shaped property is clearly lost under well-chosen smooth deformations of the domain.

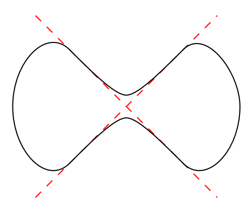

FIGURE 1. A domain with unstable star-shaped property

By Druet and Laurain [5] we know that $C^{0,\eta}$ is sharp on what concerns the perturbations of the potentials. It remains to answer the question of the sharp degree of regularity we need when perturbing the domain. We prove in the following theorem that the C^1-degree of regularity obtained in Theorem 1 is sharp.

Theorem 2. *There exists a family $(\Phi_\varepsilon)_\varepsilon$ of smooth diffeomorphims of $\mathbb{R}^3$, $0 < \varepsilon \ll 1$, such that $\Phi_\varepsilon \to Id$ in $C^{0,\eta}$ for all $0 < \eta < 1$ as $\varepsilon \to 0$, and for which the following equations*

$$\begin{cases} \Delta u = u^5 \text{ in } \Omega_\varepsilon, \\ u \ge 0 \text{ in } \Omega_\varepsilon, \\ u = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{0.4}$$

have nontrivial solutions $u = u_\varepsilon$ for all ε, where $\Omega_\varepsilon = \Phi_\varepsilon(B(0,1))$.

We prove Theorem 1 in Section 1. We prove Theorem 2 in Section 2.

1. Blow-up analysis and proof of theorem 1

Let Ω be a smooth bounded domain of $\mathbb{R}^3$, star-shaped with respect to the origin, and let $h \in C^1(\mathbb{R}^3)$ satisfying (0.3). Assume by contradiction that there exist some sequences $(\Phi_\varepsilon)_\varepsilon$ of $C^{1,\eta}$-diffeomorphisms of $\mathbb{R}^3$ and $(h_\varepsilon)_\varepsilon$ of C^1 functions of $\mathbb{R}^3$ such that

$$\Phi_\varepsilon \to Id \text{ in } C^1 \text{ and } h_\varepsilon \to h \text{ in } C^{0,\eta} \text{ as } \varepsilon \to 0 \tag{1.1}$$

for some $\eta > 0$ and such that there exists a sequence $(u_\varepsilon)_\varepsilon$ of C^2-solutions of

$$\begin{cases} \Delta u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^5 \text{ in } \Omega_\varepsilon \\ u_\varepsilon > 0 \text{ in } \Omega_\varepsilon \\ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{1.2}$$

for all ε, where $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$.

1.1. **Local blow-up analysis.** The local blow-up analysis we need to prove Theorem 1 is carried out in Druet and Laurain [5]. We recall in Proposition 1.1 the result we need, and refer to Druet and Laurain [5] for its proof. We let $(u_\varepsilon)_\varepsilon$ be as above and we assume that we have a sequence $(x_\varepsilon)_\varepsilon$ of points in Ω_ε and a sequence $(\rho_\varepsilon)_\varepsilon$ of positive real numbers, with $0 < 3\rho_\varepsilon \le d(x_\varepsilon, \partial\Omega_\varepsilon)$ for all ε, such that

$$\nabla u_\varepsilon(x_\varepsilon) = 0 \tag{1.3}$$

for all ε, and

$$\rho_\varepsilon \left[\sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon(x) \right]^2 \to +\infty \text{ as } \varepsilon \to 0\ . \tag{1.4}$$

Then the following proposition holds true.

Proposition 1.1. *(Druet-Laurain [5]) If there exists $C_0 > 0$ such that*

$$|x_\varepsilon - x|^{\frac{1}{2}} u_\varepsilon \le C_0 \text{ in } B(x_\varepsilon, 3\rho_\varepsilon) \tag{1.5}$$

for all ε, then there exists $C_1 > 0$ such that

$$u_\varepsilon(x_\varepsilon)u_\varepsilon(x) \le C_1 |x_\varepsilon - x|^{-1} \text{ in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\} \text{ and}$$
$$u_\varepsilon(x_\varepsilon)|\nabla u_\varepsilon(x)| \le C_1 |x_\varepsilon - x|^{-2} \text{ in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}$$

for all ε. Moreover, if $\rho_\varepsilon \to 0$, then

$$\rho_\varepsilon u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + \rho_\varepsilon x) \to \frac{1}{|x|} + b \text{ in } C^1_{loc}(B(0,2) \setminus \{0\}) \text{ as } \varepsilon \to 0$$

where b is some harmonic function in $B(0,2)$ with $b(0) = 0$ and $\nabla b(0) = 0$.

We refer to Druet and Laurain [5] for the proof of this result. Proposition 1.1 provides the local blow-up analysis we need for the proof of Theorem 1. It remains to carry over the global blow-up analysis, where the convergence $\Phi_\varepsilon \to Id$ as $\varepsilon \to 0$ plays a role. This is the subject of the following subsection.

1.2. **Global blow-up behaviour.** We prove here that the following global result holds true.

Proposition 1.2. *Assume that the sequence $(u_\varepsilon)_\varepsilon$ of solutions of (1.2) satisfies that*

$$\|u_\varepsilon\|_\infty \to +\infty \text{ as } \varepsilon \to 0 .$$

Then there exist $N \in \mathbb{N}^\star$, $x_1, \dots, x_N \in \Omega$ and $\lambda_1, \dots, \lambda_N \in \mathbb{R}_+^*$ such that, up to a subsequence,

$$\|u_\varepsilon\|_\infty u_\varepsilon(x) \to \sum_{i=1}^N \lambda_i \mathcal{G}_h(x_i, x) \text{ in } C^1_{loc}(\Omega \setminus \{x_1, \dots, x_N\}) \text{ as } \varepsilon \to 0 ,$$

where $\mathcal{G}_h$ is the Green function of the limit operator $\Delta + h$ with Dirichlet boundary condition on Ω.

In order to prove Proposition 1.2 we need two preliminary lemmas. In the first lemma we exhaust the concentration points developed by the u_ε's.

Lemma 1.1. *There exists $C > 0$ such that for all $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon})$, such that :*

$$d(x_{i,\varepsilon}, \partial\Omega_\varepsilon) u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \in [1, N_\varepsilon] ,$$

$$|x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N_\varepsilon] ,$$

and

$$\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq C$$

for all $x \in \Omega_\varepsilon$ and all $\varepsilon > 0$.

Proof of Lemma 1.1. - We let $x_\varepsilon \in \Omega_\varepsilon$ be such that $\|u_\varepsilon\|_\infty = u_\varepsilon(x_\varepsilon)$ and we set $\mu_\varepsilon = u_\varepsilon(x_\varepsilon)^{-2}$. By assumption, $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. For

$$x \in \tilde{\Omega}_\varepsilon = \left\{ x \in \mathbb{R}^3 \text{ s.t. } x_\varepsilon + \mu_\varepsilon x \in \Omega_\varepsilon \right\} ,$$

we let

$$v_\varepsilon(x) = \sqrt{\mu_\varepsilon} u_\varepsilon(x_\varepsilon + \mu_\varepsilon x)$$

which satisfies

$$\Delta v_\varepsilon + \mu_\varepsilon^2 h_\varepsilon(x_\varepsilon + \mu_\varepsilon x) v_\varepsilon = v_\varepsilon^5$$

in $\tilde{\Omega}_\varepsilon$ and $0 \leq v_\varepsilon \leq v_\varepsilon(0) = 1$. Since $\Phi_\varepsilon \to Id$ in C^1, we know that

$$\tilde{\Omega}_\varepsilon \to \Omega_0 \text{ as } \varepsilon \to 0$$

where Ω_0 is either $\mathbb{R}^3$ or a half-space. And by standard elliptic theory, we also know that $v_\varepsilon \to v_0$ in $C^1_{loc}(\Omega_0)$ as $\varepsilon \to 0$ with

$$\Delta_\xi v_0 = v_0^5 \text{ in } \Omega_0 .$$

Moreover, $v_0 = 0$ on the boundary of Ω_0 if Ω_0 is a half-space. Thanks to a result of Dancer [4], we cannot have a nonzero solution of the above equation with Dirichlet boundary condition on a half-space. Thus we know that $\Omega_0 = \mathbb{R}^3$. This proves in particular that

$$d(x_\varepsilon, \partial\Omega_\varepsilon) u_\varepsilon(x_\varepsilon)^2 \to +\infty \text{ as } \varepsilon \to 0 .$$

We thus have proved that

$$\left\{ x \in \Omega_\varepsilon \text{ s.t. } \nabla u_\varepsilon(x) = 0 \text{ and } d(x, \partial\Omega_\varepsilon) u_\varepsilon(x)^2 \geq 1 \right\} \neq \emptyset \tag{1.6}$$

for ε small since x_ε belongs to it. We now apply an elementary calculus lemma, see for instance lemma 4.1 of [5], to get that, for ε small enough, there exist $N_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon})$, such that :

$$\begin{aligned} & d(x_{i,\varepsilon}, \partial\Omega_\varepsilon) u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \in [1, N_\varepsilon] \ , \\ & |x_{i,\varepsilon} - x_{j,\varepsilon}|\, u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N_\varepsilon] \ , \end{aligned} \tag{1.7}$$

and

$$\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq 1 \tag{1.8}$$

for all critical point x of u_ε such that $d\left(x, \partial\Omega_\varepsilon\right) u_\varepsilon(x)^2 \geq 1$. Then, in order to end the proof of Lemma 1.1, we need to show that there exists $C > 0$ such that

$$\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \leq C \tag{1.9}$$

for all $x \in \Omega_\varepsilon$. Let us assume by contradiction that

$$\sup_{x \in \Omega_\varepsilon} \left(\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon^2(x) \right) \to +\infty \tag{1.10}$$

as $\varepsilon \to 0$. Let $y_\varepsilon \in \Omega_\varepsilon$ be such that

$$\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - y_\varepsilon| \right) u_\varepsilon\left(y_\varepsilon\right)^2 = \sup_{x \in \Omega_\varepsilon} \left(\left(\min_{i \in [1, N_\varepsilon]} |x_{i,\varepsilon} - x| \right) u_\varepsilon(x)^2 \right) . \tag{1.11}$$

We set $\nu_\varepsilon = u_\varepsilon\left(y_\varepsilon\right)^{-2}$ and $S_\varepsilon = \{x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon}\}$. Using (1.10), we have that

$$\nu_\varepsilon \to 0 \text{ as } \varepsilon \to 0 \tag{1.12}$$

and that

$$\frac{d\left(y_\varepsilon, S_\varepsilon\right)}{\nu_\varepsilon} \to +\infty \text{ as } \varepsilon \to 0 \, . \tag{1.13}$$

We let $\hat{\Omega}_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } y_\varepsilon + \nu_\varepsilon x \in \Omega_\varepsilon\}$ and we set

$$w_\varepsilon(x) = \sqrt{\nu_\varepsilon} u_\varepsilon\left(y_\varepsilon + \nu_\varepsilon x\right) \, .$$

Thanks to (1.10) and (1.11), we know that the sequence $\left(v_\varepsilon\right)_\varepsilon$ is uniformly bounded in $K \cap \hat{\Omega}_\varepsilon$ for all compact K of $\mathbb{R}^3$. As above, using standard elliptic theory, the convergence of Φ_ε to Id in C^1 and the result of Dancer [4], we can deduce that $w_\varepsilon \to w_0 \in C^1_{loc}\left(\mathbb{R}^3\right)$ with

$$\Delta w_0 = w_0^5 \text{ in } \mathbb{R}^3$$

and $0 \leq w_0 \leq 1 = w_0(0)$. Thanks to Caffarelli, Gidas and Spruck [3], we know that

$$w_0(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$$

and it follows, since w_0 has a strict maximum at 0 and since $w_\varepsilon \to w_0$ in C^1_{loc}, that u_ε had a critical point z_ε satisfying $|z_\varepsilon - y_\varepsilon| = o\left(\nu_\varepsilon\right)$ and $\nu_\varepsilon u_\varepsilon(z_\varepsilon)^2 \to 1$ as $\varepsilon \to 0$. Thanks to (1.10) and (1.11), this contradicts (1.8). This clearly proves (1.9) and ends the proof of Lemma 1.1. □

Now we define

$$d_\varepsilon = \min\left\{d(x_{i,\varepsilon}, x_{j,\varepsilon}), d(x_{i,\varepsilon}, \partial\Omega_\varepsilon) \text{ s.t. } 1 \leq i < j \leq N_\varepsilon\right\} \tag{1.14}$$

and prove that the following lemma holds true.

Lemma 1.2. *There exists $d > 0$ such that $d_\varepsilon \geq d$ for all ε.*

Proof of Lemma 1.2. - Assume that $d_\varepsilon \to 0$ as $\varepsilon \to 0$. Up to reordering the concentration points, we can assume that

$$d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon}) \text{ or } d(x_{1,\varepsilon}, \partial\Omega_\varepsilon) \ . \tag{1.15}$$

We let $\tilde{\Omega}_\varepsilon = \left\{x \in \mathbb{R}^3 \text{ s.t. } x_{1,\varepsilon} + d_\varepsilon x \in \Omega_\varepsilon\right\}$, and we set

$$v_\varepsilon(x) = d_\varepsilon^{\frac{1}{2}} u_\varepsilon\left(x_{1,\varepsilon} + d_\varepsilon x\right)$$

for $x \in \tilde{\Omega}_\varepsilon$. Then v_ε satisfies

$$\Delta v_\varepsilon + d_\varepsilon^2 h_\varepsilon\left(x_{1,\varepsilon} + d_\varepsilon\,\cdot\,\right) v_\varepsilon = v_\varepsilon^5 \text{ in } \tilde{\Omega}_\varepsilon \ . \tag{1.16}$$

After some rotation, we have, since $\Phi_\varepsilon \to Id$ in C^1 as $\varepsilon \to 0$, that

$$\lim_{\varepsilon\to 0} \tilde{\Omega}_\varepsilon = \Omega_0 = \mathbb{R}^3 \text{ or }]-\infty; d[\times\mathbb{R}^2 \text{ where } d \geq 1 \ . \tag{1.17}$$

Let us set

$$\tilde{x}_{i,\varepsilon} = \frac{x_{i,\varepsilon} - x_{1,\varepsilon}}{d_\varepsilon} \ . \tag{1.18}$$

for $i = 1, \ldots, N_\varepsilon$. Given a sequence $1 \leq i_\varepsilon \leq N_\varepsilon$, there holds that

$$v_\varepsilon\left(\tilde{x}_{i_\varepsilon,\varepsilon}\right) = O(1) \Rightarrow \sup_{B(\tilde{x}_{i_\varepsilon,\varepsilon},\frac{1}{2})} v_\varepsilon = O(1) \ . \tag{1.19}$$

This is a consequence of the definition of d_ε, the last assertion of Lemma 1.1, and Harnack's inequality. One can proceed by contradiction and rescale at a point of maximum of v_ε in the above ball. Now let us define

$$\mathcal{S}_{R,\varepsilon} = \{\tilde{x}_{i,\varepsilon} | \tilde{x}_{i,\varepsilon} \in B(0,R)\}$$

for $R > 0$. After passing to a subsequence and after a diagonal extraction, we let $\mathcal{S}$ be the set

$$\mathcal{S} = \bigcup_{R>0} \lim_{\varepsilon\to 0} \mathcal{S}_{R,\varepsilon} \ .$$

We claim that

$$\forall\, 1 \leq i_\varepsilon \leq N_\varepsilon \text{ s.t. } d(x_{i_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)\,,\ v_\varepsilon\left(\tilde{x}_{i_\varepsilon,\varepsilon}\right) \to +\infty \text{ as } \varepsilon \to 0 \ . \tag{1.20}$$

First, if the assertion was false for one sequence $\left(i_\varepsilon\right)_\varepsilon$, it would be false for all. Indeed, if $v_\varepsilon\left(x_{j_\varepsilon,\varepsilon}\right) \to +\infty$ as $\varepsilon \to 0$ with $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = O(d_\varepsilon)$, then, using Proposition 1.1, we would have that $v_\varepsilon \to 0$ in a small annulus around $\lim x_{j_\varepsilon,\varepsilon}$. By Harnack's inequality, this would be the case on small annuli around any of the above $\tilde{x}_{k_\varepsilon,\varepsilon}$. If ever $v_\varepsilon\left(\tilde{x}_{i_\varepsilon,\varepsilon}\right) = O(1)$, we would then get thanks to (1.19) and the above argument that $v_\varepsilon \to 0$ uniformly in a small ball containing $\tilde{x}_{i_\varepsilon,\varepsilon}$, which is in contradiction with Lemma 1.1. Thus, if (1.20) is false, it is false for all sequences $x_{i_\varepsilon,\varepsilon}$. This means in particular thanks to (1.19) and Lemma 1.1 that $\left(v_\varepsilon\right)_\varepsilon$ is uniformly bounded in any compact subset of $\mathbb{R}^3$. After the extraction of a subsequence, it does converge to a solution U of

$$\Delta_\xi U = U^5 \text{ in } \Omega_0$$

which satisfies $U(0) \geq 1$ and $\nabla U(x) = 0$ for all $x \in \mathcal{S}$ thanks to Lemma 1.1. We then get by Dancer [4] that $\Omega_0 = \mathbb{R}^3$ and thus that $\mathcal{S}$ possesses at least two points, 0 and another. By the classification result of Caffarelli, Gidas and Spruck [3], we obtain a contradiction. Thus (1.20) is proved. Now we distinguish two cases, depending on the limit set Ω_0.

Case 1 : $\Omega_0 = \mathbb{R}^3$ - In this first case, up to a subsequence, $d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon})$ and $\mathcal{S} = \{0, \tilde{x}_2 = \lim_{\varepsilon \to 0} \tilde{x}_{2,\varepsilon}, \dots\}$ contains at least two points. Applying proposition 1.1 with $x_\varepsilon = \tilde{x}_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$, we obtain that

$$v_\varepsilon(0) v_\varepsilon(x) \to H = \frac{1}{|x|} + \frac{\lambda_2}{|x - \tilde{x}_2|} + \tilde{b}$$

in $C^1_{loc}\left(\mathbb{R}^3 \setminus \mathcal{S}\right)$ as $\varepsilon \to 0$, where $\tilde{b}$ is an harmonic function in $\Omega_0 \setminus \{S \setminus \{0, \check{x}_2\}\}$, and $\lambda_2 > 0$. Moreover $\tilde{b}(0) = -\lambda_2$. Let us, for all positive number r, rewrite H as

$$H = \sum_{\tilde{x}_i \in \mathcal{S} \cap B(0,r)} \frac{\lambda_i}{|x - \tilde{x}_i|} + \hat{b}_r,$$

where $\lambda_i > 0$. Then, taking $R > r$ big enough, we get that $\hat{b}_r > -\frac{1}{r}$ on $\partial B(0, R)$. Moreover, for any $\tilde{x}_j \in B(0,R) \setminus B(0,r)$, there exist a neighborhood $V_{j,r}$ of $\tilde{x}_j$ such that $\hat{b}_r > 0$ on $V_{j,r}$. Using the maximum principle, $\hat{b}_r > -\frac{1}{r}$ on $B(0,R)$. Since $\hat{b}_r \to \hat{b}$ on every compact set as $r \to +\infty$, we get that

$$H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|} + \hat{b}$$

with $\hat{b} \geq 0$, which proves that $\tilde{b} \geq 0$. This is in contradiction with $\tilde{b}(0) = -\lambda_2$ and finishes the proof of Lemma 1.2 in this first case.

Case 2 : $\Omega_0 =]-\infty, d[\times \mathbb{R}^2$ - We have $\tilde{S} = \{0 = \tilde{x}_1, \tilde{x}_2, \dots\}$ and we apply Proposition 1.1 with $x_\varepsilon = x_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$ to get that

$$v_\varepsilon(0) v_\varepsilon(x) \to H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|} + \tilde{b}$$

in $C^1_{loc}(\Omega_0 \setminus S)$, where $\lambda_i > 0$, and $\tilde{b}$ is some harmonic function in Ω_0. We extend H to $\mathbb{R}^3$ by setting

$$\hat{H}(x) = \begin{cases} H(x) & \text{if } x_1 \leq d \\ -H(s(x)) & \text{otherwise} \end{cases},$$

where s is the symmetry with respect to $\{d\} \times \mathbb{R}^2$. We also extend $\tilde{b}$ by setting

$$\hat{H} = \sum_{\tilde{x}_i \in S} \left(\frac{\lambda_i}{|x - \tilde{x}_i|} - \frac{\lambda_i}{|s(x) - \tilde{x}_i|} \right) + \hat{b} \ .$$

It is clear that $\hat{b}$ is harmonic on $\mathbb{R}^3$ and satisfies $\hat{b} \geq 0$ in Ω_0 and $\hat{b} \leq 0$ in $\mathbb{R}^3 \setminus \Omega_0$. This can be proved as above. Using the Green's representation formula, one can prove that $\partial_1 \hat{b}(0) \leq 0$. We refer to Druet and Laurain [5] for details on this claim. Now we rewrite H in a neighborhood of 0 as

$$H(x) = \frac{1}{|x|} + \check{b}(x) \ ,$$

where

$$\check{b}(x) = \hat{b}(x) - \frac{1}{|s(x)|} + \sum_{\check{x}_i \in S \setminus \{0\}} \lambda_i \left(\frac{1}{|x - \check{x}_i|} - \frac{1}{|s(x) - \check{x}_i|} \right) .$$

Then it is clear with what we just said that $\partial_1 \check{b}(0) < 0$, which is in contradiction with Proposition 1.1. This ends the proof of Lemma 1.2 in this second case. □

At this point we are now in position to end the proof of proposition 1.2.

Proof of Proposition 1.2. - By Lemmas 1.1 and 1.2, there exist $C > 0, N \in \mathbb{N}^*$, and N local maxima of u_ε, $x_{1,\varepsilon}, \dots, x_{N,\varepsilon} \in \Omega_\varepsilon$, such that:

$$\begin{aligned} & d\left(x_{i,\varepsilon}, \partial\Omega_\varepsilon\right) u_\varepsilon\left(x_{i,\varepsilon}\right)^2 \geq 1 \text{ for all } i \in [1, N] \ , \\ & |x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon(x_{i,\varepsilon})^2 \geq 1 \text{ for all } i \neq j \in [1, N] \end{aligned} \tag{1.21}$$

and

$$\left(\min_{i \in [1,N]} |x_{i,\varepsilon} - x|\right) u_\varepsilon(x)^2 \leq C \tag{1.22}$$

for all $x \in \Omega_\varepsilon$. Moreover, we know that, after passing to a subsequence, $x_{i,\varepsilon} \to x_i$ as $\varepsilon \to 0$ for some $x_i \in \Omega$. We can assume that $u_\varepsilon(x_{i,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$. Indeed, otherwise, we can remove $x_{i,\varepsilon}$ from the family of concentration points, and up to changing C, the assertion remains true. Then, thanks to Harnack inequality, there exists $D > 0$ such that

$$\frac{1}{D} u_\varepsilon(x_{1,\varepsilon}) \leq u_\varepsilon(x_{i,\varepsilon}) \leq D u_\varepsilon(x_{1,\varepsilon}) \ .$$

The proposition is then a direct consequence of proposition 1.1 and of standard elliptic theory. □

Thanks to Propositions 1.1 and 1.2 we are now in position to prove Theorem 1. This is the subject of the following subsection.

1.3. **Proof of Theorem 1.** Let $i \in \{1, \dots, N\}$ and let $x_{i,\varepsilon} \to x_i \in \Omega$ be the corresponding concentration point as in the previous subsection. We apply the classical Pohožaev identity to u_ε in $B_{x_{i,\varepsilon}}(\delta)$, where $\delta > 0$ is small enough such that $B_{x_i}(2\delta) \subset \Omega$. This tells us that

$$\begin{aligned} & \int_{B_{x_{i,\varepsilon}}(\delta)} \left(h_\varepsilon u_\varepsilon^2 + 2 h_\varepsilon u_\varepsilon \langle x - x_{i,\varepsilon}, \nabla u_\varepsilon\rangle\right) dx \\ & \quad = \int_{\partial B_{x_{i,\varepsilon}}(\delta)} \left(2\delta \left(\partial_\nu u_\varepsilon\right)^2 - \delta \left|\nabla u_\varepsilon\right|^2 + u_\varepsilon \partial_\nu u_\varepsilon + \frac{\delta}{3} u_\varepsilon^6\right) d\sigma \ . \end{aligned}$$

Thanks to Proposition 1.2, we can pass to the limit in the right-hand side to obtain that

$$\begin{aligned} & \|u_\varepsilon\|_\infty^2 \int_{B_{x_{i,\varepsilon}}(\delta)} \left(h_\varepsilon u_\varepsilon^2 + 2 h_\varepsilon u_\varepsilon \langle x - x_{i,\varepsilon}, \nabla u_\varepsilon\rangle\right) dx \\ & \quad \to \int_{\partial B_{x_i}(\delta)} \left(2\delta \left(\partial_\nu G\right)^2 - \delta \left|\nabla G\right|^2 + G \partial_\nu G\right) d\sigma \end{aligned}$$

as $\varepsilon \to 0$, where

$$G = \sum_{j=1}^N \lambda_j \mathcal{G}_h\left(x_j, x\right)$$

is as in Proposition 1.2. Using the estimate of Proposition 1.1, we get by Lebesgue's dominated convergence theorem that

$$\|u_\varepsilon\|_\infty^2 \int_{B_{x_{i,\varepsilon}}(\delta)} \left(h_\varepsilon u_\varepsilon^2 + 2 h_\varepsilon u_\varepsilon \langle x - x_{i,\varepsilon}, \nabla u_\varepsilon\rangle\right) dx \to 0$$

as $\varepsilon \to 0$. Thus we have obtained that, for any $i \in \{1, \dots, N\}$,

$$\int_{\partial B_{x_i}(\delta)} \left(2\delta \left(\partial_\nu G\right)^2 - \delta \left|\nabla G\right|^2 + G\partial_\nu G\right) d\sigma = 0 \ ,$$

where G is as above. Writing for any $i = 1, \dots, N$ that

$$G(x) = \frac{\lambda_i}{\omega_2 \left|x_i - x\right|} + A_i + \frac{h\left(x_i\right)}{2\omega_2} + \gamma_i\left(x\right) \tag{1.23}$$

with γ_i a C^1-function in a neighbourhood of x_i, easy computations lead to

$$\int_{\partial B_{x_i}(\delta)} \left(2\delta \left(\partial_\nu G\right)^2 - \delta \left|\nabla G\right|^2 + G\partial_\nu G\right) d\sigma = -\lambda_i A_i$$

so that $A_i = 0$ for all $i \in \{1, \dots, N\}$. In the same spirit, we can use a modified Pohožaev identity, see Druet and Laurain [5], to obtain that

$$\int_{B_{x_{i,\varepsilon}}(\delta)} \left(h_\varepsilon u_\varepsilon - 6u_\varepsilon^5\right) \nabla u_\varepsilon \, dx = \int_{\partial B_{x_{i,\varepsilon}}(\delta)} \left(\frac{\left|\nabla u_\varepsilon\right|}{2}\nu - \partial_\nu u_\varepsilon \nabla u_\varepsilon\right) d\sigma \ .$$

Thanks to Proposition 1.2, we can pass to the limit in the right-hand side to obtain that

$$\begin{aligned} &\left\|u_\varepsilon\right\|_\infty^2 \int_{\partial B_{x_{i,\varepsilon}}(\delta)} \left(\frac{\left|\nabla u_\varepsilon\right|}{2}\nu - \partial_\nu u_\varepsilon \nabla u_\varepsilon\right) d\sigma \\ &\quad \to \int_{\partial B_{x_i}(\delta)} \left(\frac{\left|\nabla G\right|}{2}\nu - \partial_\nu G \nabla G\right) d\sigma \end{aligned}$$

as $\varepsilon \to 0$. Using Proposition 1.1 and integration by parts, we get by Lebesgue's dominated convergence theorem, using the fact that $h_\varepsilon \to h_0$ in $C^{0,\eta}$, that

$$\left\|u_\varepsilon\right\|_\infty^2 \int_{B_{x_{i,\varepsilon}}(\delta)} \left(h_\varepsilon u_\varepsilon - 6u_\varepsilon^5\right) \nabla u_\varepsilon \, dx \to \frac{1}{2} h\left(x_i\right) \int_{\partial B_{x_i}(\delta)} G^2 \, d\sigma$$

as $\varepsilon \to 0$. Thus we arrive to

$$\int_{\partial B_{x_i}(\delta)} \left(\frac{\left|\nabla G\right|}{2}\nu - \partial_\nu G \nabla G\right) d\sigma = \frac{1}{2} h\left(x_i\right) \int_{\partial B_{x_i}(\delta)} G^2 \, d\sigma \ .$$

Letting δ go to 0 in the above equality easily leads to $\nabla \gamma_i\left(x_i\right) = 0$. Since Ω is starshaped with respect to 0, there holds that

$$\sum_{i=1}^N \lambda_i \left(A_i + 2\langle x_i, \nabla \gamma_i\left(x_i\right)\rangle\right) < 0 \ ,$$

where the A_i's and γ_i's are as in (1.23). We refer to lemma 4.4 of Druet-Laurain [5] for the proof of this Green-Pohožaev identity. This last inequality is clearly in contradiction with the fact that $A_i = 0$ for all i and that $\nabla \gamma_i\left(x_i\right) = 0$. This ends the proof of Theorem 1.

2. A counter-example to better stability results. Proof of theorem 2

In this section, we construct a non star-shaped domain, as close as we want to the unit ball in any $C^{0,\eta}$-topology, for which problem (1.2) admits a positive solution. The idea is to perturb the unit ball, adding a small domain to it, controling the $W^{1,\infty}$-norm of the diffeomorphism which sends this domain to the unit ball (see picture 2 below). Our construction will clearly prove Theorem 2 and is optimal in view of Theorem 1. We mainly follow ideas of Passaseo [7] and start by defining the domain that we will glue to $B(0,1)$. For simplicity of notations, we prefer to glue a fixed domain to $B\left(0,\frac{1}{\varepsilon}\right)$ and then rescale the ambient space. Let us set

$$P_{\varepsilon,r_1}=\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}-\varepsilon\le x\le\frac{1}{\varepsilon}+1\text{ and }y^2+z^2\le\frac{1}{4}\right\}\setminus(B_\varepsilon\cup C_{\varepsilon,r_1}),$$

where B_ε is the ball of center $\left(\frac{1}{\varepsilon}+\frac{1}{2},0,0\right)$ and radius $\frac{1}{4}$, and

$$C_{\varepsilon,r_1}=\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}+\frac{1}{2}\le x\le\frac{1}{\varepsilon}+1;\ y^2+z^2\le\frac{r_1^2}{4}\right\}$$

with $r_1<\frac{1}{8}$ to be chosen later. Smoothing

$$U_{\varepsilon,r_1}=B\left(0,\frac{1}{\varepsilon}\right)\cup P_{\varepsilon,r_1}$$

at the vertices, we obtain a smooth, non star-shaped domain, $\tilde\Omega_{\varepsilon,r_1}$, as shown in the following picture, figure 2.

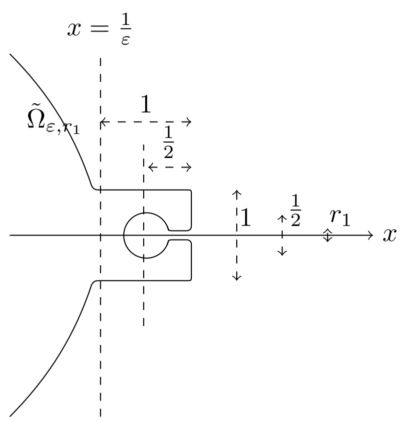

Figure 2. $\tilde\Omega_{\varepsilon,r_1}$

Moreover, there exists a diffeomorphism Φ_ε from $B\left(0,\frac{1}{\varepsilon}\right)$ onto $\tilde\Omega^\varepsilon_{r_1}$ with gradient's norm is bounded independently of ε. Indeed, it is clear that there exists a diffeomorphism from $K_\varepsilon=\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}-\varepsilon\le x\le\frac{1}{\varepsilon}+1\text{ and }y^2+z^2\le\frac{1}{4}\right\}$ to P_{ε,r_1} which is identity on $\left\{(x,y,z)\text{ s.t. }\frac{1}{\varepsilon}-\varepsilon\le x\le\frac{1}{\varepsilon}\text{ and }y^2+z^2\le\frac{1}{4}\right\}$ and depends only on r_1. From the one hand, we can extend this diffeomorphism to a diffeomorphism from $B\left(0,\frac{1}{\varepsilon}\right)\cup K_\varepsilon$ to $\tilde\Omega^\varepsilon_{r_1}$. From the other hand, using the fact that $B\left(0,\frac{1}{\varepsilon}\right)\cup K_\varepsilon$ is star shaped with respect to 0 with easily find a diffeomorphism from $B\left(0,\frac{1}{\varepsilon}\right)$ to $B\left(0,\frac{1}{\varepsilon}\right)\cup K_\varepsilon$ whose gradient is bounded by 2. Compiling all this information we obtain the desired diffeomorphism Φ_ε whose norm depends only on r_1, finally we omit the subscript r_1 since this quantity will be soon fixed. We set

$$\Omega_{\varepsilon,r_1} = \varepsilon\tilde{\Omega}_{\varepsilon,r_1}.$$

Then Ω_{ε,r_1} is diffeomorphic to $B(0,1)$ by $\Psi_\varepsilon \equiv \varepsilon\Phi_\varepsilon\left(\frac{\cdot}{\varepsilon}\right)$, which is bounded, independently of ε, in $W^{1,\infty}(B(0,1))$. The end of this section is devoted to the proof that for r_1 sufficiently small, $\Delta u = u^5$ admits a positive solution with Dirichlet boundary data on Ω_{ε,r_1} for all $\varepsilon \in (0,\varepsilon_0)$, $0 < \varepsilon_0 \ll 1$. In what follows we consider the minimisation problem

$$S^\star = \inf_{u\in H^\star} E(u) \ ,$$

where

$$E(u) = \int_{\Omega_{\varepsilon,r_1}} |\nabla u|^2 \, dv \ ,$$

and

$$H^* = \left\{ u \in H^1_{s,0}(\Omega_{\varepsilon,r_1}) \text{ s.t. } \int_{\Omega_{\varepsilon,r_1}} u^6 \, dv = 1 \text{ and } g(u) \geq 1 + \frac{\varepsilon}{2} \right\} \ ,$$

with

$$H^1_{s,0}(\Omega_{\varepsilon,r_1}) = \left\{ u \in H^1_0(\Omega_{\varepsilon,r_1}) \text{ s.t. } u(x,,T(y,z)) = u(x,y,z) \text{ a.e. for all } T \in O(2) \right\}$$

and

$$g(u) = \int_{\Omega_{\varepsilon,r_1}} x|u|^6 \, dv \ .$$

Roughly speaking, H^* is the space of H^1_0-functions which are radial with respect to the x-axis and whose first coordinate of the "center of mass" is greater than or equal to $1 + \frac{\varepsilon}{2}$. In particular their center of mass does not belong to Ω_{ε,r_1}. It will also be convenient to consider the subspace of functions of H^* whose first coordinate of the "center of mass" is exactly $1 + \frac{\varepsilon}{2}$:

$$H^\star_0 = \left\{ u \in H^1_{s,0}(\Omega_{\varepsilon,r_1}) \text{ s.t. } \int_{\Omega_{\varepsilon,r_1}} u^6 \, dv = 1 \text{ and } g(u) = 1 + \frac{\varepsilon}{2} \right\} \ .$$

We prove in the following that the above minimisation problem admits a solution (Lemmas 2.1 to 2.3 below) which does not belong to $H^\star_0$ (the choice of r_1 is crucial here). The proof consists in showing that this solution is also a solution of our equation.

Lemma 2.1. *For any $r_1 > 0$ we have that*

$$S^* = \inf\{E(u) \ s.t. \ u \in H^*\} > S$$

and that

$$S^\star_0 = \inf\{E(u) \ s.t. \ u \in H^\star_0\} \geq \tilde{S} > S \ ,$$

where

$$S = \inf_{u\in C^\infty_c(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^3} u^6 \, dx\right)^{\frac{1}{3}}} \ ,$$

$\tilde{S} = \inf_{u\in\tilde{H}} E(u)$, $\tilde{H} = \left\{ u \in H^1_{s,0}(\Omega_\varepsilon) \ s.t. \ \int_{\Omega_{\varepsilon,r_1}} u^6 \, dv = 1 \ and \ g(u) = 1 + \frac{\varepsilon}{2} \right\}$, *and* $\Omega_\varepsilon = \Omega_{\varepsilon,0} \bigcup \left\{ (t,0,0),\ 1 + \frac{3\varepsilon}{4} < t < 1 + \varepsilon \right\}$.

Proof of Lemma 2.1. It is well known that S is achieved by the family of functions

$$U_{y,\lambda}(x) = C\frac{\lambda^{\frac{1}{2}}}{\left(\lambda^2 + \frac{|x-y|^2}{3}\right)^{\frac{1}{2}}} \,,$$

where $y \in \mathbb{R}^3$ and $\lambda > 0$ (see Aubin [1] or Talenti [10]). In the rest of this section, we choose C, independent of λ, such that $\|U_{\lambda,y}\|_6 = 1$. Then we just need to prove that $S^\star > S$ and that $\tilde{S} > S$. The proof relies on the concentration-compactness principle of Lions, see [6] or chapter 1 of Struwe [9]. We assume by contradiction that $S^\star$, or $\tilde{S}$, is equal to S. It is clear in this case that $S^\star$, or $\tilde{S}$, cannot be achieved since minimizers for S are known and of noncompact support. Let $(u_n)_n$ be a minimizing sequence for E in $H^\star$, or $\tilde{H}$. Thanks to the concentration-compactness principle, we know that there exist two sequences $x_n \in \Omega$ and $\lambda_n \to 0$ such that

$$\|u_n - U_{x_n,\lambda_n}\|_6 \to 0 \,.$$

It is clear that $x_n \to x_0$ as $n \to +\infty$, after passing to a subsequence. For symmetry reason, x_0 lies on the x-axis and $|x_0| \geq 1 + \frac{\varepsilon}{2}$ (in the case of $S^\star$) or $|x_0| = 1 + \frac{\varepsilon}{2}$ (in the case of $\tilde{S}$). In particular, $x_0 \notin \overline{\Omega_{\varepsilon,r_1}}$ (in the case of $S^\star$) and $x_0 \notin \overline{\Omega_\varepsilon}$ (in the case of $\tilde{S}$). Since $u_n^6\,dx \rightharpoonup \delta_{x_0}$ as $n \to +\infty$, we get a contradiction with the fact that $\int_{\Omega_{\varepsilon,r_1}} u_n^6\,dx = 1$. This ends the proof of Lemma 2.1. □

The second result we need is as follows.

Lemma 2.2. *For r_1 small enough, there exists $V_\varepsilon \in H^*$ such that*

$$E(V_\varepsilon) < \min\left(2^{\frac{2}{3}}S, \tilde{S}\right)$$

for all ε.

Proof of Lemma 2.2. Let $y_\varepsilon = \left(1 + \frac{7\varepsilon}{8}, 0, 0\right)$ and $U_\lambda = U_{y_\varepsilon,\lambda}$. We let

$$\eta_\lambda\left(x_1, x_2, x_3\right) = \psi\left(|x - y_\varepsilon|\right) \times \begin{cases} 0 & \text{if } r \leq r_\lambda \\ 2\dfrac{\ln \frac{r}{r_\lambda}}{\ln \frac{1}{r_\lambda}} & \text{if } r_\lambda \leq r \leq \sqrt{r_\lambda} \\ 1 & \text{if } r \geq \sqrt{r_\lambda} \end{cases}$$

where $0 \leq \psi \leq 1$ is a smooth function with value 1 on $\left[0, \frac{\varepsilon}{32}\right]$ and 0 on $\left[\frac{\varepsilon}{16}, +\infty\right)$, $r^2 = x_2^2 + x_3^2$, and

$$r_\lambda = e^{-\frac{1}{\lambda^2}} \,.$$

We set

$$V_\lambda = \frac{\eta_\lambda U_\lambda}{\|\eta_\lambda U_\lambda\|_6} \,.$$

It is clear that $V_\lambda \in H^\star$. We have that

$$\int_{\Omega_{\varepsilon,r_\lambda}} \left|\nabla\left(\eta_\lambda U_\lambda\right)\right|^2 dx \quad \leq \quad \int_{\Omega_{\varepsilon,r_\lambda}} \left|\nabla U_\lambda\right|^2 dx + o(1)$$

since

$$\int_{\Omega_{\varepsilon,r_\lambda}} \left|\nabla \eta_\lambda\right|^2 U_\lambda^2\,dx = O\left(\lambda\right) \,.$$

Moreover, it is easily checked that

$$\int_{\Omega_{\varepsilon,r_\lambda}} (\eta_\lambda U_\lambda)^6 \ dx \quad = \quad \int_{\Omega_{\varepsilon,r_\lambda}} U_\lambda^6 \, dx + o(1) \ .$$

We can deduce that

$$E\left(V_\lambda\right) \to S \text{ as } \lambda \to 0 \ .$$

Using Lemma 2.1, we obtain some λ_0 small enough such that

$$E\left(V_{\lambda_0}\right) < \min\left(2^{\frac{2}{3}} S, \tilde{S}\right) \ .$$

Thus the statement of Lemma 2.2 holds for $r_1 = r_{\lambda_0}$. This ends the proof of Lemma 2.2. □

The third lemma we need is as follows.

Lemma 2.3. *S^* is achieved by $U_\star \in H^*$ with $U_\star \geq 0$.*

Proof of Lemma 2.3. Let $(u_n)_n$ be a (non-negative) minimizing sequence in $H^\star$ for $S^\star$. It is clear that $g\left(u_n\right) \not\to 1 + \frac{\varepsilon}{2}$ as $n \to +\infty$. Indeed, otherwise, taking φ some smooth radial function centered at 0 with compact support in $B_0\left(\frac{1}{2}\right)$, we would have that

$$\frac{u_n + \eta_n \varphi}{\left\| u_n + \eta_n \varphi \right\|_6} \in H_0^\star$$

for some $\eta_n \to 0$. Since the energy of this sequence would also converge to $S^\star$, we would get a contradiction with the fact that $S_0^\star > S^\star$, as proved in Lemmas 2.1 and 2.2. Looking at $u_n + \eta \varphi_n$ for any sequence $(\varphi_n)_n$ in H_0^1 of norm 1, radial with respect to the x-axis, we can write that

$$\frac{u_n + \eta \varphi_n}{\left\| u_n + \eta \varphi_n \right\|_6} \in H^\star$$

for $\eta > 0$ small enough. Writing that the energy of this function is greater than or equal to $S^\star$ leads to the fact that

$$\int_{\Omega_{\varepsilon,r_1}} \left(\langle \nabla u_n, \nabla \varphi_n \rangle + S^\star u_n^5 \varphi_n \right) \, dx = o\left(\left\| \nabla \varphi_n \right\|_2 \right)$$

for all sequences $(\varphi_n)_n$. We set $v_n = \left(S^\star\right)^{\frac{1}{4}} u_n$. It is not difficult to check that it is a Palais-Smale sequence for the free energy

$$F(u) = \frac{1}{2} \int_{\Omega_{\varepsilon,r_1}} \left|\nabla u\right|^2 \, dx - \frac{1}{6} \int_{\Omega_{\varepsilon,r_1}} u^6 \, dx$$

with

$$F\left(v_n\right) \to \frac{1}{3} \left(S^\star\right)^{\frac{3}{2}} \in \left(\frac{1}{3} S^{\frac{3}{2}}, \frac{2}{3} S^{\frac{3}{2}} \right)$$

thanks to Lemmas 2.1 and 2.2. Since the functional F satisfies the Palais-Smale condition in this interval (see Struwe [9], section III.3.1), we obtain that $v_n \to v_0$ strongly as $n \to +\infty$ and thus that $u_n \to U_\star$ which is the desired solution of our minimization problem. This ends the proof of Lemma 2.3. □

At this point wa can prove Theorem 2.

Proof of Theorem 2. It suffices to prove that $U_\star$ is a solution to our problem. In order to prove this we remark that the constraint on $g(u)$ does not play any role in the Lagrange multiplier since $S^\star < \tilde{S} < S_0^\star$ by Lemma 2.2. Moreover,

$$\int \langle \nabla\phi, \nabla U_\star \rangle \, dx = \int \phi U_\star^5 \, dx$$

for all $\phi \in H^*$. Then we have that

$$\Delta U_\star = U_\star^5$$

using the fact that $\Delta^{-1}(u) \in H^*$ for any $u \in H^*$. Finally, we check that $U_\star$ is positive by the maximum principle. This proves Theorem 2. □

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