The Mass of an Asymptotically Flat Manifold

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Abstract

We show that the mass of an asymptotically flat n-manifold is a geometric invariant. The proof is based on harmonic coordinates and, to develop a suitable existence theory, results about elliptic operators with rough coefficients on weighted Sobolev spaces are summarised. Some relations between the mass, scalar curvature and harmonic maps are described and the positive mass theorem for n-dimensional spin manifolds is proved.

Introduction

Suppose that (M, g) is an asymptotically flat 3-manifold. In general relativity the mass of M is given by

$$\text{mass} = \frac{1}{16\pi} \oint_{S_\infty} (g_{ij,j} - g_{jj,i})\, dS^i, \tag{0.1}$$

where $g_{ij,k}$ denotes the partial derivative and dS^i is the normal surface element to S_∞, the sphere at infinity. This expression is generally attributed to [3]; for a recent review of this and other expressions for the mass and other physically interesting quantities see [4]. However, in all these works the definition depends on the choice of coordinates at infinity and it is certainly not clear whether or not this dependence is spurious. Since it is physically quite reasonable to assume that a frame at infinity ("observer") is given, this point has not received much attention in the literature (but see [15]).

It is our purpose in this paper to show that, under appropriate decay conditions on the metric, (0.1) generalises to n-dimensions for $n \geqq 3$ and gives an invariant of the metric structure (M, g). The decay conditions roughly stated (for $n = 3$) are $|g - \delta| = o(r^{-1/2})$, $|\partial g| = o(r^{-3/2})$, etc, and thus include the usual falloff conditions. We note that an example of Denisov and Solov'ev [12] shows that these conditions are optimal for the mass to be uniquely defined.

The approach we take is to construct coordinates which are harmonic near infinity and use these to show that there are no "twisted" coordinates at infinity. Harmonic coordinates have been used previously in [11] to study the mass but under more stringent conditions and with a different purpose. From the resulting uniqueness of the structure at infinity and an interpretation of the mass in terms of the scalar curvature, it is not hard to derive the uniqueness of the mass, but there is a simple but curious cancellation which deserves to be better understood.

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We remark that the interpretation of the mass in terms of scalar curvature is implicit in the Hamiltonian formulation of general relativity (e.g. [29], [36]).

In view of some future applications we have chosen to work with weak regularity assumptions on the metric. For this reason we start with a survey of the methods and results of the theory of elliptic operators on weighted Sobolev spaces; the treatment is slightly unorthodox but (we hope) clearer than the existing works, and there are some new observations. Of particular interest are the *a priori* estimates for operators transverse to the kernel (Proposition 1.12) and the description of the dimension of the kernel as the weighting is varied (Proposition 2.2 and Corollary 2.3).

Having set the stage, the uniqueness results follow readily, the main difficulty being that of deciding just what needs to be proven. Some ancillary results are also described that show that harmonic coordinates are in some sense the best possible. Section 5 emphasises this as we give an alternate interpretation of the mass in terms of harmonic maps and a quick proof of a weak version of the positive mass theorem (see [28], [31], [33]). This proof also shows that the mass estimates some quantities of partial differential equations interest. In the final section we show that Witten's proof of the positive mass theorem can be generalised to higher-dimensional spin manifolds. This result has also been announced by R. Schoen.

1. Operators on Weighted Sobolev Spaces

In this section we review those parts of the theory of weighted Sobolev spaces that will be needed later. Much of the material is well known (the treatment here is particularly indebted to [9], [25], [23] and [22]) but there are a number of technical improvements and some new observations. We have tried to emphasise the two basic ideas which underlie this subject; the use of scaling to convert local estimates into global estimates, and secondly, sharp estimates for constant coefficient operators arising from explicit expressions for the Greens function. The first is well known in the PDE literature—for example see the treatment of weighted Schauder norms in [17] and [16]—while the second has been periodically rediscovered. Some references additional to those mentioned above are [24], [5], [8] and [21], while [2], [14], [20] contain similar ideas. Most of these papers deal with elliptic systems with dominant part a scale invariant operator, not necessarily with homogeneous symbol. Since it is all that is required by our prospective applications we consider here only operators close to the Laplacian at infinity. This suffices to illustrate the main ideas.

We work initially in $\mathbf{R}^n$, $n \geqq 3$, although the physical interest is in $n = 3$, and set $r = |x|$, $\sigma = (1 + r^2)^{1/2}$. Subsets of interest are $B_R = B_R(0)$, the closed ball of radius R and centre 0, the annulus $A_R = B_{2R} \setminus B_R$ and the exterior domain $E_R = \mathbf{R}^n \setminus B_R$. The summation convention applies and partial derivatives may be denoted by subscripts, $u_i = \partial_i u = \partial u/\partial x_i$, $Du = (u_i)$. The flat metric is δ_{ij} and the standard Laplacian is $\Delta = \Sigma \partial_i \partial_i$ while Δ_g denotes a metric Laplacian,

$$\Delta_g \varphi = |g|^{-1/2} \partial_i \left(|g|^{1/2} g^{ij} \partial_j \varphi \right).$$

Constants will be denoted by C or c and their dependence on interesting parameters will be noted as appropriate.

DEFINITION 1.1. The weighted Lebesgue spaces L^p_δ, L'^p_δ, $1 \leqq p \leqq \infty$, with weight $\delta \in \mathbf{R}$ are the spaces of measureable functions in $L^p_{\mathrm{loc}}(\mathbf{R}^n)$, $L^p_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$, respectively, such that the norms $\|\cdot\|_{p,\delta}$, $\|\cdot\|'_{p,\delta}$ defined by

$$
(1.1)\qquad
\begin{aligned}
L^p_\delta:\quad \|u\|_{p,\delta} &= \begin{cases} \left(\displaystyle\int_{\mathbf{R}^n} |u|^p \sigma^{-\delta p - n}\, dx\right)^{1/p}, & p < \infty, \\ \operatorname{ess\,sup}_{\mathbf{R}^n}\left(\sigma^{-\delta}|u|\right), & p = \infty, \end{cases} \\
L'^p_\delta:\quad \|u\|'_{p,\delta} &= \begin{cases} \left(\displaystyle\int_{\mathbf{R}^n\setminus\{0\}} |u|^p r^{-\delta p - n}\, dx\right)^{1/p}, & p < \infty, \\ \operatorname{ess\,sup}_{\mathbf{R}^n\setminus\{0\}}\left(r^{-\delta}|u|\right), & p = \infty, \end{cases}
\end{aligned}
$$

are finite. The weighted Sobolev spaces are now defined in the usual way:

$$
(1.2)\qquad
\begin{aligned}
W^{k,p}_\delta:&\quad \|u\|_{k,p,\delta} = \sum_0^k \|D^j u\|_{p,\delta-j}, \\
W'^{k,p}_\delta:&\quad \|u\|'_{k,p,\delta} = \sum_0^k \|D^j u\|'_{p,\delta-j},
\end{aligned}
$$

Observe that $C^\infty_c(\mathbf{R}^n)$, $C^\infty_c(\mathbf{R}^n \setminus \{0\})$ are dense in $W^{k,p}_\delta$, $W'^{k,p}_\delta$, respectively, for $1 \leqq p < \infty$ and that $L^p_\delta = L^p(\mathbf{R}^n)$ for $\delta = -n/p$. The indexing chosen for the weights differs from that usually used but has the advantage that it directly describes the growth at infinity (see (1.10) below). Another simplification is that the rescaled function

$$
(1.3)\qquad u_R(x) = u(Rx)
$$

satisfies, by a simple change of variables,

$$
\|u_R\|'_{k,p,\delta} = R^\delta \|u\|'_{k,p,\delta}
$$

and, with an obvious notation for norms over subsets of $\mathbf{R}^n$,

$$
(1.4)\qquad \|u\|_{k,p,\delta;A_R} \approx R^{-\delta}\|u_R\|_{k,p,\delta;A_1} \quad \text{for} \quad R \geqq 1,
$$

where " $\approx$ " means "is comparable to, independent of $R \geqq 1$". The estimate (1.4) is the key to proving global weighted inequalities from local inequalities.

THEOREM 1.2. (i) *If* $1 \leqq p \leqq q \leqq \infty$, $\delta_2 < \delta_1$ *and* $u \in L^q_{\delta_2}$, *then*

$$
(1.5)\qquad \|u\|_{p,\delta_1} \leqq c\|u\|_{q,\delta_2},
$$

and hence $L^q_{\delta_2} \subset L^p_{\delta_1}$.

(ii) (Hölder inequality) *If* $u \in L^q_{\delta_1}$, $v \in L^r_{\delta_2}$ *and* $\delta = \delta_1 + \delta_2$, $1 \leqq p, q, r \leqq \infty$, $1/p = 1/q + 1/r$, *then*

$$\|uv\|_{p,\delta} \leqq \|u\|_{q,\delta_1}\|v\|_{r,\delta_2}. \tag{1.6}$$

(iii) (Interpolation inequality) *For any* $\varepsilon > 0$, *there is a* $C(\varepsilon)$ *such that, for all* $u \in W^{2,p}_\delta$, $1 \leqq p \leqq \infty$,

$$\|u\|_{1,p,\delta} \leqq \varepsilon\|u\|_{2,p,\delta} + C(\varepsilon)\|u\|_{0,p,\delta}. \tag{1.7}$$

(iv) (Sobolev inequality) *If* $u \in W^{k,p}_\delta$, *then*

$$\|u\|_{np/(n-kp),\delta} \leqq C\|u\|_{k,q,\delta} \tag{1.8}$$

$$\text{if} \quad n - kp > 0 \text{ and } p \leqq q \leqq np/(n-kp),$$

$$\|u\|_{\infty,\delta} \leqq C\|u\|_{k,p,\delta} \quad \text{if} \quad n - kp < 0, \tag{1.9}$$

and in fact

$$|u(x)| = o(r^\delta) \quad \text{as} \quad r \to \infty. \tag{1.10}$$

(v) *If* $u \in W^{k,p}_\delta$, $0 < \alpha \leqq k - n/p \leqq 1$, *then*

$$\|u\|_{C^{0,\alpha}_\delta} \leqq C\|u\|_{k,p,\delta}, \tag{1.11}$$

where the weighted Hölder norm is defined by

$$\begin{aligned}\|u\|_{C^{0,\alpha}_\delta} = &\sup_{x\in\mathbf{R}^n}\left(\sigma^{-\delta+\alpha}(x)\sup_{4|x-y|\leqq\sigma(x)}\frac{|u(x)-u(y)|}{|x-y|^\alpha}\right)\\ &+\sup_{x\in\mathbf{R}^n}\{\sigma^{-\delta}(x)|u(x)|\}.\end{aligned} \tag{1.12}$$

Analogous to (1.10) *we have*

$$\|u\|_{C^{0,\alpha}_\delta(A_R)} = o(1) \quad \text{as} \quad R \to \infty. \tag{1.13}$$

Proof: The first two estimates follow directly from the definition and Hölders inequality. The other estimates follow from the technique of rescaling and applying local estimates, which we illustrate by proving (iv). Supposing that $p^* = np/(n-kp) < \infty$, we have

$$\begin{aligned}\|u\|_{p^*,\delta;A_R} &= \left(\int_{A_R}(\sigma^{-\delta}u)^{p^*}\sigma^{-n}\,dx\right)^{1/p^*}\\ &\leqq CR^{-\delta}\|u_R\|_{p^*,\delta;A_1}\\ &\leqq CR^{-\delta}\|u_R\|_{k,q,\delta;A_1},\end{aligned}$$

by the usual Sobolev inequality applied to $\Omega = A_1$ followed by Hölders inequality. Rescaling gives

$$\|u\|_{p^*,\delta;A_R} \leqq C\|u\|_{k,q,\delta;A_R},$$

and writing $u = \Sigma_0^\infty u_j$ with $u_0 = u|_{B_1}$, $u_j = u|_{A_2^{(j-1)}}$, $j \geqq 1$, we see that

$$\begin{aligned}\|u\|_{p^*,\delta} &= \left(\sum_0^\infty \|u_j\|_{p^*,\delta}^{p^*}\right)^{1/p^*} \\ &\leqq C\left(\sum_0^\infty \|u_j\|_{k,q,\delta}^{p^*}\right)^{1/p^*} \\ &\leqq C\left(\sum_0^\infty \|u_j\|_{k,q,\delta}^{q}\right)^{1/q},\end{aligned}$$

since $p^* \geqq q$ and $(\Sigma a_j^s)^{1/s} \leqq (\Sigma a_j^r)^{1/r}$ for $a_j \geqq 0$, $r \leqq s$. When $n - kp < 0$, the same scaling argument implies

$$\sup_{A_R} |u|\sigma^{-\delta} = \|u\|_{\infty,\delta;A_R} \leqq C\|u\|_{k,p,\delta;A_R}$$

which gives (19) and since $\|u\|_{k,p,\delta} < \infty$ we have $\|u\|_{k,p,\delta;A_R} = o(1)$ as $R \to \infty$, giving (1.10).

The counterexample $u(x) = (\log r)^{-1}$, $r \geqq 2$, shows that (i) cannot be improved to $\delta_1 = \delta_2$. The following weighted Poincaré inequality is rather more subtle than the above estimates and seems to be closely related to invertibility of the Laplacian on weighted spaces. Related ideas can be found in [32] and [1].

THEOREM 1.3. (i) *For any* $u \in W_\delta^{1,p}$ *with* $1 \leqq p < \infty$, $\delta \neq 0$, *we have*

$$\|u\|'_{p,\delta} \leqq |\delta|^{-1}\|u_r\|'_{p,\delta-1} \leqq |\delta|^{-1}\|Du\|'_{p,\delta-1}, \tag{1.14}$$

where $u_r = \partial_r u = r^{-1}x \cdot Du$.

(ii) *If* $\delta < 0$, *there is a constant* C *such that*

$$\|u\|_{p,\delta} \leqq C\|u_r\|_{p,\delta-1} \quad \textit{for any} \quad u \in W_\delta^{1,p}. \tag{1.15}$$

Remark. By modifying σ in the interior the constant in (1.15) can be made arbitrarily close to $|\delta|^{-1}$.

Proof: By a previous remark, in order to prove (1.14) it suffices to consider $u \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$. Since $\Delta r^{(2-n)} = 0$ in $\mathbf{R}^n \setminus \{0\}$, testing with $r^{-\delta p}|u|^p$ gives

$$\int_{\mathbf{R}^n\setminus\{0\}} D(r^{2-n}) \cdot D(r^{-\delta p}|u|^p)\, dx = 0$$

which expands to

$$\int_{\mathbf{R}^n\setminus\{0\}} r^{-\delta p-n}|u|^p \leqq |\delta|^{-1}\int_{\mathbf{R}^n\setminus\{0\}} |u|^{p-1}|u_r|r^{-\delta p+1-n},$$

and then Hölders inequality gives (1.14). A similar calculation using $\Delta\sigma^{2-n} = -n(n-2)\sigma^{-2-n}$ and $\delta < 0$ with $u \in C_c^\infty(\mathbf{R}^n)$ yields

$$\int_{\mathbf{R}^n}|u|^p\lambda(r)\sigma^{-\delta p-n}\,dx \leqq \int_{\mathbf{R}^n}|u_r|^p\lambda(r)^{1-p}(r/\sigma)^p\sigma^{(-\delta+1)p-n}\,dx, \tag{1.16}$$

where $\lambda(r) = |\delta|(1-(1-n/|\delta|p)/(1+r^2/a^2))$. The estimate (1.15) now follows by noting that $\lambda(r) \to |\delta|$ as $r \to \infty$, $\lambda((r) \to n/p$ as $r \to 0$.

Next we recall the weighted Rellich-Kondrat'ev compactness theorem (see [9]) and note that counterexamples based on "travelling bumps" in x-space (for δ) and ξ-space (for k) show that the hypotheses cannot be improved. This result indicates that there is a nice duality between the k, δ indices, provided by the Fourier transform and the Paley-Wiener theorem.

LEMMA 1.4. ([9], Lemma 2.1). *For $k > j$, $\delta < \varepsilon$ and $1 \leqq p < \infty$, the inclusion $W_\delta^{k,p} \subset W_\varepsilon^{j,p}$ is compact.*

The scaling argument and standard local estimates give estimates in weighted spaces for elliptic operators whose coefficients are well behaved at infinity. Conditions appropriate for our purposes (but certainly not optimal—compare [25]) are given by

DEFINITION 1.5. The operator $u \to Pu$ defined by

$$Pu = a^{ij}(x)\,\partial_{ij}^2 u + b^i(x)\,\partial_i u + c(x)u \tag{1.17}$$

will be said to be *asymptotic to* Δ (*at rate* τ) if there exist $n < q < \infty$ and $\tau \geqq 0$ and constants C_1, λ such that

$$\begin{aligned} &\lambda|\xi|^2 \leqq a^{ij}(x)\xi_i\xi_j \leqq \lambda^{-1}|\xi|^2 \quad \text{for all} \quad x \in \mathbf{R}^n,\ \xi \in \mathbf{R}^n,\\ &\|a^{ij}-\delta_{ij}\|_{1,q,-\tau} + \|b^i\|_{0,q,-1-\tau} + \|c\|_{0,q/2,-2-\tau} \leqq C_1, \end{aligned} \tag{1.18}$$

where δ_{ij} is the usual flat metric on $\mathbf{R}^n$.

By (1.11), a^{ij} is Hölder-continuous and $|a^{ij}(x) - \delta_{ij}| = o(r^{-\tau})$ at infinity. The conditions (1.18) are also satisfied by the divergence-form operator Δ_g when g_{ij} is a uniformly elliptic metric on $\mathbf{R}^n$ and $(g_{ij} - \delta_{ij}) \in W_{-\tau}^{1,q}$ for some $\tau \geqq 0$.

It is clear from Theorem 1.2 that if P is asymptotic to Δ, then the map P: $W_\delta^{2,p} \to W_{\delta-2}^{0,p}$ is bounded for $1 \leqq p \leqq q$ and $\delta \in \mathbf{R}$. In fact the following weighted estimate holds.

PROPOSITION 1.6. *Suppose that P is asymptotic to Δ, $1 < p \leqq q$, and $\delta \in R$. There is a constant $C = C(n, p, q, \delta, C_1, \lambda)$ such that if $u \in L^{0,p}_{\delta}$ and $Pu \in L^{0,p}_{\delta-2}$, then $u \in W^{2,p}_{\delta}$ and*

$$\|u\|_{2,p,\delta} \leqq C\big(\|Pu\|_{0,p,\delta-2} + \|u\|_{0,p,\delta}\big). \tag{1.19}$$

Proof: Elliptic regularity applies to show that $u \in W^{2,p}_{\mathrm{loc}}$, and the remaining conclusions follow from the usual interior L^p estimates (see [17], Chapter 9) and the scaling technique.

Observe that the same argument gives the estimate (1.19) with the $W'^{k,p}_{\delta}$ norms instead. We now investigate the Fredholm properties of P. The arguments which follow show that (1.19) is not sufficient to prove "Fredholmness"; a strengthened estimate with the "error term" on the right-hand side being compact with respect to $W^{2,p}_{\delta}$ is needed. To prove the scale-broken version of (1.19) which is required by the Rellich-Kondrat'ev theorem, we rely on a sharp estimate for the flat Laplacian based on an explicit expression for the integral kernel of Δ^{-1} on weighted spaces (cf. [25], [23], [24]).

The weighting parameter $\delta \in \mathbf{R}$ is said to be *nonexceptional* if $\delta \in \mathbf{R}/\{k \in \mathbf{Z}, k \neq -1, -2, \cdots, 3-n\}$, where the *exceptional values* $\{k \in \mathbf{Z},\ k \neq -1, -2, \cdots, 3-n\}$ correspond to the orders of growth of harmonic functions in $\mathbf{R}^n \setminus B_1$. It will also be useful to define

$$k^-(\delta) = \max\{k \text{ exceptional}, k < \delta\}. \tag{1.20}$$

THEOREM 1.7. *Suppose that δ is nonexceptional, $1 < p < \infty$, and s is a non-negative integer. Then the map*

$$\Delta\colon W'^{s+2,p}_{\delta} \to W'^{s,p}_{\delta-2} \tag{1.21}$$

is an isomorphism and there is a constant $C = C(n, p, \delta, s)$ such that

$$\|u\|'_{s+2,p,\delta} \leqq C\|\Delta u\|'_{s,p,\delta-2}. \tag{1.22}$$

Proof (compare [23]): It will suffice to prove (1.22) for $s = 0$. We first show that the distributional inverse of (1.21) has convolution kernel $K(x, y)$:

$$c_n K(x, y) = \begin{cases} |x-y|^{2-n} & \text{if } 2-n < \delta < 0, \\ |x-y|^{2-n} - |y|^{2-n} \sum_0^k P_j^{\lambda}(\mu)\left(\frac{|x|}{|y|}\right)^j & \text{if } k \geqq 0, \\ |x-y|^{2-n} - |x|^{2-n} \sum_0^k P_j^{\lambda}(\mu)\left(\frac{|y|}{|x|}\right)^j & \text{if } k < 2-n, \end{cases} \tag{1.23}$$

where $k = k^-(\delta)$, $\mu = (x \cdot y)/|x|\,|y|$ and P_j^{λ} are the ultraspherical functions

arising in the Taylor series expansion

$$|x-y|^{2-n} = |x|^{2-n}\sum_0^\infty P_j^\lambda(\mu)\left(\frac{|y|}{|x|}\right)^j, \qquad |y| < |x|.$$

Let us first show that (1.23) defines a bounded operator from $W_{\delta-2}^{\prime 0,p}$ to $W_\delta^{\prime 0,p}$. Since the three cases are very similar, we only consider the second in detail. Following [23], we have the estimates

$$(1.24)\quad |K(x,y)| \leqq c(n,k)|x-y|^{2-n}\begin{cases}(|x|/|y|)^{k+1} & \text{if} \quad |x| < \frac{1}{2}|y|,\\ (|x|/|y|)^{n+k-2} & \text{if} \quad |x| \geqq \frac{1}{2}|y|.\end{cases}$$

Now recall ([25], Lemma 2.1):

LEMMA 1.8. *Fix $p \in (1;\infty)$, $p' = p/(p-1)$ and let $a, b \in \mathbf{R}$ be such that $a + b > 0$. Suppose $K'(x,y)$ is the kernel*

$$K'(x,y) = |x|^{-a}|x-y|^{2-n}|y|^{-b} \quad \textit{for} \quad x \neq y,$$

and for $u \in L^p(\mathbf{R}^n)$ define

$$K'u(x) = \int K'(x,y)u(y)\,dy.$$

Then there is a constant $c = c(n,p,a,b)$ such that

$$\|K'u\|_{L^p} \leqq c\|u\|_{L^p}$$

if and only if $a < n/p$ and $b < n/p'$.

Let K_1, K_2 be the operator kernels $|x-y|^{2-n}(|x|/|y|)^\alpha$ with $\alpha = k+1$, $\alpha = n+k-2$, respectively. The above lemma shows that the kernel

$$K_1'(x,y) = |x|^{-\delta-n/p}K_1(x,y)|y|^{\delta-2+n/p}$$

defines a bounded operator $L^p \to L^p$ when $\delta + n/p - k - 1 < n/p$ and $-\delta - n/p + k + 3 < n/p'$, i.e., when $3 - n + k < \delta < k+1$, and then

$$\|K_1u\|_{p,\delta}' \leqq c\|r^{2-\delta-n/p}u\|_{L^p} \leqq c\|u\|_{p,\delta-2}'.$$

Similarly, $K_2'(x,y) = |x|^{-\delta-n/p}K_2(x,y)|y|^{\delta-2+n/p}$ is bounded $L^p \to L^p$ when $k < \delta < n+k-2$ and then K_2: $W_{\delta-2}^{\prime 0,p} \to W_\delta^{\prime 0,p}$ is likewise bounded. These two estimates and (1.24) show that K: $W_{\delta-2}^{\prime 0,p} \to W_\delta^{\prime 0,p}$ is bounded when $k < \delta < k+1$ as claimed and the boundedness for the other two cases in (1.23) follows

similarly. The distributional identities

$$(1.25) \qquad \Delta_x K(x, y) = \Delta_y K(x, y) = \delta(x - y) \quad \text{in} \quad D'(\mathbf{R}^n \setminus \{0\}).$$

imply that $K(\Delta u) = u$ for all $u \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$, so the boundedness of K gives

$$\|u\|'_{0,p,\delta} \leqq C\|\Delta u\|'_{0,p,\delta-2} \quad \text{for all} \quad u \in W_\delta'^{k,p},$$

since $C_c^\infty(\mathbf{R}^n \setminus \{0\})$ is dense. This and the scaling estimate (1.19) yield (1.22) for $s = 0$. Now suppose $\{u_i\} \subset W_\delta'^{2,p}$, $\{f_i\} \subset W_{\delta-2}'^{0,p}$ are sequences such that $f_i = \Delta u_i$ and $f_i \to f$. By (1.22), $\{u_i\}$ is a Cauchy sequence and hence is convergent to $u \in W_\delta'^{2,p}$ with $\Delta u = f$, since Δ: $W_\delta'^{2,p} \to W_{\delta-2}'^{0,p}$ is bounded. This map thus has closed range and by (1.25) we have $\Delta(Kf)(x) = f(x)$ for $f \in C_c^\infty(\mathbf{R}^n \setminus \{0\})$ and $Kf \in W_\delta'^{0,p}$, so it is also surjective. Finally, (1.22) shows that the kernel is trivial and this establishes the isomorphism.

It is clear that the above arguments require only the scale-invariant estimate (1.19) and explicit estimates for the Greens function such as (1.24). Thus the generalization to operators with homogeneous symbol is straightforward, and a change of variables $\mathbf{R}^n \setminus \{0\} \approx S^{n-1} \times \mathbf{R}$ permits the generalization to dilation-invariant operators (see [5], [22], [2]).

The estimate (1.22) implies the following well-known Liouville theorem.

COROLLARY 1.9. *Suppose that* $\Delta u = 0$ *and* $u \in L_\delta^p$, $1 < p < \infty$, *and let* $k = k^-(\delta)$. *If* $k < 0$, *then* $u \equiv 0$; *while if* $k \geqq 0$, *then* u *is a harmonic polynomial of degree less than or equal to* k.

Proof: Elliptic regularity shows that $u \in C^\infty(\mathbf{R}^n)$. Let $h_k(x)$ be the Taylor series expansion of u about $x = 0$,

$$u(x) = h_k(x) + O(|x|^{k+1}) \quad \text{as} \quad |x| \to 0.$$

Since $\Delta u(0) = 0$, $h_k(x)$ is a harmonic polynomial so that $(u - h_k) \in L_\delta'^p$, and the estimate (1.22) applied to $u - h_k$ now shows that $u \equiv h_k$.

The scale-broken estimate (1.26) which follows is the key to proving Fredholm properties.

THEOREM 1.10. *Suppose that* P *is asymptotic to* Δ *and* $\delta \in \mathbf{R}$ *is nonexceptional. For* $1 < p \leqq q$ *the map* P: $W_\delta^{2,p} \to W_{\delta-2}^{0,p}$ *has finite-dimensional kernel and closed range and, for any* $u \in W_\delta^{2,p}$,

$$(1.26) \qquad \|u\|_{2,p,\delta} \leqq C\left(\|Pu\|_{0,p,\delta-2} + \|u\|_{L^p(B_R)}\right),$$

where C, R *are (computable) constants depending on* P, δ, n, p, q.

Proof: Define the operator norm

$$\|P - \Delta\|_{\mathrm{op}} = \sup\left\{\|(P - \Delta)u\|_{0,p,\delta-2}\colon u \in W_\delta^{2,p},\ \|u\|_{2,p,\delta} = 1\right\}$$

and let $\|\cdot\|_{\mathrm{op},R}$ denote the same norm restricted to functions with support in $E_R = \mathbf{R}^n \setminus B_R$. Then if $\operatorname{supp}(u) \subset E_R$, since $q > n$,

$$\|(P - \Delta)u\|_{0,p,\delta-2} \leqq \sup_{|x|>R}\left\{|a^{ij}(x) - \delta_{ij}|\right\}\|D^2u\|_{0,p,\delta-2}$$

$$+ C\|b\|_{0,q,1;E_R}\|Du\|_{1,p,\delta-1} + C\|c\|_{0,q/2,2;E_R}\|u\|_{2,p,\delta},$$

using (1.6) and (1.8). Since P is asymptotic to Δ this shows that

$$\|P - \Delta\|_{\mathrm{op},R} = o(1) \quad \text{as} \quad R \to \infty. \tag{1.27}$$

Let $\chi \in C_c^\infty(B_2)$ be a patch function, $0 \leqq \chi \leqq 1$, $\chi \equiv 1$ in B_1, and set $\chi_R(x) = \chi(x/R)$. Writing $u = u_0 + u_\infty$, $u_0 = \chi_R u$, $u_\infty = (1 - \chi_R)u$ with R a constant to be determined, the sharp estimate (1.22) yields

$$\|u_\infty\|_{2,p,\delta} \leqq C\|\Delta u_\infty\|_{0,p,\delta-2}$$

$$\leqq C\left(\|Pu_\infty\|_{0,p,\delta-2} + \|P - \Delta\|_{\mathrm{op},R}\|u_\infty\|_{2,p,\delta}\right)$$

and proceeding as before we estimate

$$\|Pu_\infty\|_{0,p,\delta} \leqq \|Pu\|_{0,p,\delta-2} + \|2a^{ij}u_i\,\partial_j\chi_R + \left(a^{ij}\,\partial_{ij}^2\chi_R + b^i\,\partial_i\chi_R\right)u\|_{0,p,\delta-2;A_R}$$

$$\leqq \|Pu\|_{0,p,\delta-2} + C\|u\|_{1,p,\delta;A_R}.$$

Since $\|P - \Delta\|_{\mathrm{op},R} = o(1)$, for R sufficiently large

$$\|u_\infty\|_{0,p,\delta} \leqq C\left(\|Pu\|_{0,p,\delta-2} + \|u\|_{1,p,\delta;A_R}\right), \tag{1.28}$$

and using (1.19) and the interpolation inequality (1.7) gives (1.26). Now suppose that $\{u_k\}$ is a sequence in $\ker P$ satisfying $\|u_k\|_{2,p,\delta} = 1$, so that by the Rellich lemma we may assume that $\{u_k\}$ converges strongly in $L^p(B_R)$. Estimate (1.26) now shows that $\|u_j - u_k\|_{2,p,\delta} \to 0$ as $\min(j,k) \to \infty$, thus $\{u_k\}$ is Cauchy and hence convergent in $W_\delta^{2,p}$ which implies that $\ker P$ is finite-dimensional. To show that P has closed range we follow [9], Theorem 6.3. Since $\dim\ker P < \infty$, there is a closed subspace Z such that $W_\delta^{2,p} = Z + \ker P$ and we claim there is a constant C such that

$$\|u\|_{2,p,\delta} \leqq C\|Pu\|_{0,p,\delta-2} \quad \text{for all} \quad u \in Z.$$

For if this were not the case, there would be a sequence $\{u_k\} \subset Z$ such that $\|u_k\|_{2,p,\delta} = 1$ and $\|Pu_k\|_{0,p,\delta-2} \to 0$. The usual Rellich lemma applied to (1.26) shows that $\{u_k\}$ has a subsequence which is Cauchy in Z and whose limit is a non-zero element of $\ker P \cap Z$, which is a contradiction. The argument in Theorem 1.7 now implies that P has closed range.

We are interested in the dimension of the kernel of P, which will be denoted by

$$N(P,\delta) = \dim\ker\left(P\colon W_\delta^{2,p} \to W_{\delta-2}^{0,p}\right) \tag{1.29}$$

with $1 < p \leqq q$ (from Proposition 1.6 it is clear that $N(P,\delta)$ does not depend on p). Purely function-theoretic arguments give some information and yield as a by-product some estimates which do not seem to be derivable by direct PDE methods. The first result is the upper-semicontinuity of $N(P,\delta)$ and is well known (e.g. [25], [19]). To simplify notation we write $X = W_\delta^{2,p}$, $Y = W_{\delta-2}^{0,p}$ for δ nonexceptional and $1 < p \leqq q$, so that Theorem 1.10 shows that $P\colon X \to Y$ is *semi-Fredholm* (see [19]), i.e., has finite-dimensional kernel and closed range.

PROPOSITION 1.11. *Suppose that $P\colon X \to Y$ is a semi-Fredholm map between Banach spaces X, Y. Then there are constants $C, \varepsilon > 0$ depending only on P such that if $P'\colon X \to Y$ is any semi-Fredholm map satisfying $\|P - P'\|_{\mathrm{op}} < \varepsilon$, then*

$$\dim\ker P' \leqq \dim\ker P, \tag{1.30}$$

and for any $u \in X$ we have the estimate

$$\|u - \ker P\|_X \leqq C\|Pu\|_Y, \tag{1.31}$$

where

$$\|u - \ker P\|_X = \inf\{\|u - w\|_X\colon w \in \ker P\}$$

is the quotient norm on $X/(\ker P)$.

Proof: The argument in Theorem 1.10 gives (1.31). Now let $\varepsilon = (2C)^{-1}$ and suppose (1.30) fails, so there is $u \in \ker P'$ such that $\|u\|_X = 1$ and $\|u - \ker P\|_X > \frac{1}{2}$. Then

$$\begin{aligned}\tfrac{1}{2} < \|u - \ker P\|_X &\leqq C\|Pu\|_Y \leqq C\|P - P'\|_{\mathrm{op}}\|u\|_X \\ &< C\varepsilon,\end{aligned}$$

which is a contradiction.

Intuitively $C(P)^{-1}$ measures the distance (in the operator norm) of P to a semi-Fredholm operator with larger kernel. This picture is reinforced by the following partial converse.

PROPOSITION 1.12. *Let X, Y be as above and set*

$$\mathbf{F} = \{P\colon X \to Y,\ P \text{ satisfies the conditions } (1.18)\}.$$

Suppose that $\mathbf{U} \subset \mathbf{F}$ has the property that there is a constant N such that

$$\dim(\ker P) = N \quad \text{for all} \quad P \in \mathbf{U}.$$

Then given $P \in \mathbf{U}$, there are constants $\varepsilon > 0, C < \infty$, such that

$$\|u - \ker P'\|_X \leqq C\|P'u\|_Y \tag{1.32}$$

for all $u \in X$ and all $P' \in \mathbf{U}$ with $\|P - P'\|_{\mathrm{op}} \leqq \varepsilon$.

Proof: Suppose this is not the case, so that there are sequences $P_k \to P$ in $\mathbf{U}$ and $\{u_k\} \subset X$ such that $\|u_k\|_X = 1$, $\|u_k - \ker P_k\|_X > \frac{1}{2}$ and

$$\|u_k - \ker P_k\|_X \geqq k\|P_k u_k\|_Y.$$

This shows that $P_k u_k \to 0$; thus using (1.26) and the Rellich lemma again and passing to a subsequence we see that $u_k \to u$ in X, $Pu = 0$, $\|u\|_X = 1$ and $\|u - \ker P_k\|_X \geqq \frac{1}{4}$ for k sufficiently large. A similar argument shows that $\ker P_k$ converges to an N-dimensional subspace of $\ker P$, which however cannot contain u. This contradicts $\dim\ker P = N$.

Finally we note the following result which indicates that the set of operators for which $\dim\ker$ is bigger than usual is "small".

THEOREM 1.13 ([19], Theorem IV.5.31). *Suppose that $P\colon X \to Y$ is semi-Fredholm and $A\colon X \to Y$ is P-bounded (i.e., $\|Au\|_Y \leqq C(\|Pu\|_Y + \|u\|_X)$ for all $u \in X$). Then there is $\lambda_0 > 0$ such that $P + \lambda A$ is semi-Fredholm for $|\lambda| < \lambda_0$ and*

$$\dim\ker(P + \lambda A) = \text{constant} \quad \text{for} \quad 0 < |\lambda| < \lambda_0.$$

(Note that $\lambda = 0$ is not included in the last conclusion.)

If the formal adjoint

$$P^*\colon W^{0,p'}_{2-n-\delta} \to W^{-2,p'}_{-n-\delta} \tag{1.33}$$

of P also satisfies the conditions (1.18), then more information can be obtained by using the Fredholm index. Here $W^{-2,p'}_{-n-\delta}$ is the subspace of $D'(\mathbf{R}^n)$ consisting of those distributions which extend to give bounded linear functionals on $W^{2,p}_{\delta}$, endowed with the dual norm. If δ is nonexceptional and moreover $1 < p \leqq q$, $p' = p/(p-1)$, then Proposition 1.6 and the Sobolev inequality (1.8) show that $\ker(P^*) \subset W^{2,q}_{2-n-\delta}$ and hence

$$\dim\operatorname{coker} P = \dim\ker P^* = N(P^*, 2 - n - \delta).$$

This is finite by Theorem 1.10; hence P is Fredholm with *Fredholm index*

$$\iota(P,\delta) = N(P,\delta) - N(P^*, 2-n-\delta). \tag{1.34}$$

Since the index is locally constant in the space of Fredholm operators with the $\|\cdot\|_{\mathrm{op}}$ topology (see [26], [19]), we have

$$\iota(P,\delta) = \iota(\Delta,\delta) = \iota_0(\delta)$$

and the index $\iota_0(\delta)$ of the flat Laplacian can be explicitly computed. Let $\{\varphi_{k,a}\}$, $k \in \mathbf{Z}^+$, $1 \leqq a \leqq n_k$, be a basis for the eigenfunctions of $\Delta_{S^{n-1}}$ with eigenvalue $\lambda_k = -k(n+k-2)$, so that (see [7])

$$n_k = \dim \mathbf{H}_k = (n-2+2k)(n-3+k)!/k!(n-2)!, \tag{1.35}$$

where $\mathbf{H}_k = \{$homogeneous harmonic polynomials of degree k in $\mathbf{R}^n\}$. Then, defining $N_0(\delta) = N(\Delta,\delta)$ we have

(1.36)

$$N_0(\delta) = n_0 + n_1 + n_2 + \cdots + n_k = (n-1+2k)(n-2+k)!/k!(n-1)!$$

when $k = k^-(\delta) > 0$ and thus

$$\iota_0(\delta) = \begin{cases} N_0(\delta) & \text{if } \delta > 0, \\ -N_0(2-n-\delta) & \text{if } \delta < 0. \end{cases}$$

The above remarks are summarised as

PROPOSITION 1.14. *Suppose that P and P^* both satisfy conditions* (1.18) *and that δ is nonexceptional, $1 < p \leqq q$. Then P: $W_\delta^{2,p} \to W_{\delta-2}^{0,p}$ is a Fredholm operator and $N(P,\delta) = \dim\ker P$ is independent of p. If $k^-(\delta) < \delta' \leqq \delta$, then $N(P,\delta) = N(P,\delta')$ and if $u \in W_\delta^{2,p}$ and $Pu \in W_{\delta'-2}^{0,p}$, then $u \in W_{\delta'}^{2,p}$.*

Proof: Previous remarks show that P is Fredholm and $N(P,\delta)$ does not depend on p. From the invariance of the index and $k^-(\delta) < \delta' \leqq \delta$ we have

$$\iota(P,\delta) - \iota(P,\delta') = 0$$

and writing this in terms of $N(P,\delta)$ gives

$$N(P,\delta) - N(P,\delta') = N(P^*, 2-n-\delta) - N(P^*, 2-n-\delta').$$

The inclusion $L_{\delta_1}^p \subset L_{\delta_2}^p$ for $\delta_1 < \delta_2$ implies that the right-hand side of this equality is non-positive whilst the left-hand side is non-negative, so $N(P,\delta) = N(P,\delta')$. Now if $Pu \in W_{\delta'}^{0,p}$, then $\int vPu\,dx = 0$ for all $v \in \ker(P^*, 2-n-\delta')$ so that $Pu \in \mathrm{Ran}(P,\delta')$ and there is $w \in W_{\delta'}^{2,p}$ such that $Pu = Pw$. Then $(u-w) \in \ker(P,\delta)$, but $\ker(P,\delta) = \ker(P,\delta')$ and hence $u \in W_{\delta'}^{2,p}$.

When $P = \Delta_g$, then $P^* = P$ (where the adjoint is determined by the pairing $(u, v) = \int uv\sqrt{g}\, dx$) and some more information is available.

PROPOSITION 1.15. *Suppose that $g_{ij}(x)$ is uniformly elliptic in $\mathbf{R}^n$ and $(g_{ij} - \delta_{ij}) \in W_0^{1,q}$ for some $n < q < \infty$, and that δ is nonexceptional and $1 < p \leqq q$. Then Δ_g: $W_\delta^{2,p} \to W_\delta^{0,p}{}_2$ is Fredholm and $N(\Delta_g, \delta) = N_0(\delta)$.*

Proof: In view of the invariance of the index and the selfadjointness of Δ_g it will suffice to show that $N(\Delta_g, \delta) = 0$ for $\delta < 0$. But if $\Delta_g u = 0$ and $u \in W_\delta^{2,p}$, $\delta < 0$, then $u = o(1)$ at infinity and the strong maximum principle for weak solutions (see [17], Theorem 8.19) shows that $u \equiv 0$.

COROLLARY 1.16. *Let δ be nonexceptional, $n < q < \infty$, and $1 < p \leqq q$. Then there are constants C, $\varepsilon > 0$ such that, for any metric g_{ij} with $\|g_{ij} - \delta_{ij}\|_{1,q,0} \leqq \varepsilon$,*

$$\|u - \ker \Delta_g\|_{2,p,\delta} \leqq C\|\Delta_g u\|_{0,p,\delta\ 2} \quad \textit{for all} \quad u \in W_\delta^{2,p}. \tag{1.37}$$

Proof: This follows immediately from Propositions 1.12 and 1.15.

It would be interesting to find a direct PDE proof of the estimate (1.37). When $2 - n < \delta < 0$ and Δ is an isomorphism, this can be done using the weighted Poincaré inequality (1.15) but in general it seems quite difficult.

The classical expansion of harmonic functions in terms of the spherical harmonics $\varphi_{k,a}$ can be adapted to operators asymptotic to Δ at rate $\tau > 0$. A bootstrap argument based on the Schauder estimates was given in [24] and easily adapts to the present situation.

THEOREM 1.17. *Suppose that P is asymptotic to Δ at rate $\tau > 0$ and $u \in W_\delta^{2,q}$, δ nonexceptional, satisfies $Pu = 0$ in $E_R = \mathbf{R}^n \setminus B_R$ for some $R \geqq 1$. Then there is an exceptional value $k \leqq k\ (\delta)$ and $h_k \in C^\infty(\mathbf{R}^n)$ such that h_k is harmonic and homogeneous of degree k in E_R and*

$$u - h_k = o(r^{k-\tau}) \quad \textit{as} \quad r \to \infty. \tag{1.38}$$

Remarks. 1. The method also applies to $Pu = f$ and, by using the explicit kernel, lower-order terms of the expansion (1.38) can be estimated as in [24].

2. If $\tau = 0$, then Proposition 1.14 can be used to infer that $u = o(r^{k+\varepsilon})$ for any $\varepsilon > 0$, but the exact analogue of (1.38) is false since $u = \log \sigma$, $P = \Delta - (n-2)\sigma^{-2}(\log 2\sigma)^{-1}$ provides a counter-example.

Proof: From $Pu = f$ in E_R we have $\Delta u = F$, where $F = (\delta_{ij} - a^{ij})\partial_{ij}^2 u - b^i \partial_i u - cu$ in $|x| > R$; thus $F \in W_{\delta\ 2\ -\tau}^{2,q}$ by the decay assumptions on P. Since

Δ: $W^{2,q}_{\delta-\tau} \to W^{0,q}_{\delta-\tau-2}$ is Fredholm, there is a $v \in W^{2,q}_{\delta-\tau}$ such that

$$\Delta(u - v) = 0 \quad \text{for} \quad |x| > R.$$

(It may not be possible to have $\Delta(u - v) = 0$ in all $\mathbf{R}^n$ since Δ may have non-trivial cokernel.) The classical expansion for harmonic functions now shows that

$$u - v = h_k + O(r^{k-1})$$

for some $k \leqq k^-(\delta)$ and h_k as above. The decay estimate for v is improved by iteration: $u - h_k \in W^{2,q}_{\delta-\tau}$ implies $F \in W^{0,q}_{\delta-2-2\tau}$ and hence, by Proposition 1.14, $v \in W^{2,q}_{\delta-2\tau}$. This argument can be repeated until we obtain $(u - h_k) \in W^{2,q}_{k-\tau}$.

Observe that this result and the following corollary depend only on the structure of P at infinity and thus generalise to asymptotically flat manifolds.

COROLLARY 1.18. *Suppose that P is asymptotic to Δ at rate $\tau > 0$.*

(i) *If $k > 0$ is exceptional and $0 < \varepsilon < 1$, then*

$$\begin{aligned} N(P, k + \varepsilon) - N(P, k - \varepsilon) &\leqq n_k, \\ N(P, 2 - n - k + \varepsilon) - N(P, 2 - n - k - \varepsilon) &\leqq n_k, \end{aligned} \tag{1.39}$$

where $n_k = N_0(k + \varepsilon) - N_0(k - \varepsilon)$ is given explicitly by (1.35).

(ii) *There is an exceptional value $K < 0$ such that*

$$N(P, \delta) = 0 \quad \textit{for all} \quad \delta < K \textit{ nonexceptional}. \tag{1.40}$$

Proof: (i) follows immediately from the expansion (1.38) and the definition of n_k. Theorem 1.17 shows that $u \in \ker(P, \delta)$ grows like an integral power of r and (ii) then follows from the inclusion $\ker(P, \delta) \subset \ker(P, \frac{1}{2})$ for $\delta < \frac{1}{2}$ and the finite-dimensionality of $\ker(P, \frac{1}{2})$.

2. Asymptotically Flat Manifolds

DEFINITION 2.1. A smooth n-dimensional manifold (M, g) with complete Riemannian metric $g \in W^{1,q}_{\text{loc}}(M)$ for some $n < q < \infty$ is said to be *asymptotically flat* if there is a compact $K \subset\subset M$ such that $M \setminus K$ has a *structure of infinity*:—there is $R \geqq 1$ and a C^∞ diffeomorphism Φ: $M \setminus K \to E_R$ which satisfies

(i) $(\Phi_* g)_{ij}$ is uniformly equivalent to the flat metric δ_{ij} on E_R, so that there is a $\lambda \geqq 1$ such that

$$\lambda^{-1}|\xi|^2 \leqq (\Phi_* g)_{ij}(x)\xi^i\xi^j \leqq \lambda|\xi|^2 \quad \text{for all} \quad x \in E_R,\ \xi^i \in \mathbf{R}^n, \tag{2.1}$$

(ii)

$$(\Phi_* g)_{ij} - \delta_{ij} \in W^{1,q}_{-\tau}(E_R) \quad \text{for some } \textit{decay rate} \quad \tau > 0. \tag{2.2}$$

Alternatively we may consider Φ as defined by the *coordinates at infinity* $x^i = \Phi^i(m)$, $m \in M$, and write the structure at infinity as (Φ, x).

For simplicity of presentation we have chosen not to consider manifolds with more than one infinity ("end") and incomplete manifolds. The extensions generally require only minor modifications, which we leave to the reader. In the next section we shall show that the conditions (2.1), (2.2) essentially determine the structure at infinity, but for the present we fix a structure Φ and consider the properties of the operator Δ_g.

To define suitable function spaces let $\sigma \in C^\infty(M)$ be a strictly positive function satisfying

$$\sigma(m) = |\Phi(m)| = \left(\sum_1^n (x^i(m))^2\right)^{1/2} \quad \text{for} \quad m \in M \setminus K, \tag{2.3}$$

and define the weighted spaces $L^p_\delta(M)$ using the weight function $\sigma(m)$ and the natural volume form of (M, g) in the same manner as before. Condition (2.1) guarantees that $L^p_\delta(M)$ is independent of the structure (Φ, x) used to define it, but the spaces $W^{k,p}_\delta(\Phi)$, $k \geqq 1$, defined using partial derivatives with respect to the coordinates (x^i) will depend on Φ. This is unavoidable since the metric is not smooth enough to define higher covariant derivatives and also because of the rather weak assumption (2.2) on the derivatives of Φ. Note however that Proposition 1.6 shows that $\ker(\Delta_g, \delta)$ is independent of Φ.

With straightforward modifications most of the results of Section 1 are also valid for elliptic operators on M and we have in particular:

PROPOSITION 2.2. *Suppose that (M, g, Φ) are as above, $1 < p \leqq q$, and δ is nonexceptional. Then*

$$\Delta_g\colon W^{2,p}_\delta(\Phi) \to W^{0,p}_{\delta-2}(\Phi) \tag{2.4}$$

is Fredholm and $u \in \ker(\Delta_g, \delta)$ admits an expansion (1.38) *at infinity. Furthermore,*

$$N(\Delta_g, \delta) = \dim\ker\left(\Delta_g\colon W^{2,p}_\delta(\Phi) \to W^{0,p}_{\delta-2}(\Phi)\right) = N_0(\delta), \tag{2.5}$$

where $N_0(\delta)$ is defined by (1.36), *and thus $\Delta_g\colon W^{2,p}_\delta(\Phi) \to W^{0,p}_{\delta-2}(\Phi)$ is an isomorphism if $2 - n < \delta < 0$.*

Proof: The Fredholm property and the expansion at infinity are proved as in Section 1, while if $\delta < 0$, then (2.5) follows from the strong maximum principle as in Proposition 1.5. Now suppose $k = k^-(\delta) \geqq 0$ and let $h_k \in C^\infty(M)$ be a harmonic polynomial in E_R. Then $\Delta_g h_k \in W^{0,q}_{k-2-\tau}$ (the dependence on Φ here is implicit) and since $\Delta^*_g\colon W^{0,q}_{2-n-k+\tau} \to W^{-2,q}_{-n-k+\tau}$ has trivial kernel, we can

find $v_k \in W^{2,q}_{k-\tau}$ such that $\Delta_g(h_k - v_k) = 0$ and hence $\dim\ker(\Delta_g, \delta) \geqq N_0(\delta)$. The expansion (1.38) at infinity for arbitrary $u \in \ker(\Delta_g, \delta)$ shows however that $N(\Delta_g, \delta) \leqq N_0(\delta)$ which gives (2.5). The isomorphism follows from the triviality of the kernel of the adjoint Δ^*_g: $W^{2,p}_{\delta'} \to W^{0,p}_{\delta'-2}$ since $\delta' = 2 - n - \delta < 0$.

This gives a nice description of $N(P, \delta)$:

COROLLARY 2.3. *Suppose that the elliptic operator P given by* (1.17) *is formally selfadjoint and is asymptotic to Δ at rate $\tau > 0$ (this entails a straightforward generalisation of Definition* 1.5). *Then $\iota(P, \delta) = \iota_0(\delta)$ and*

$$N(P, \delta) = N_0(\delta) + E(\delta), \tag{2.6}$$

where E: $\mathbf{R} \setminus \{\text{exceptional}\} \to \mathbf{Z}^+ \cup \{0\}$ is continuous and satisfies

(i) $E(\delta) = E(2 - n - \delta)$,

(ii) $E(\delta)$ *is increasing for* $\delta \leqq 1 - \frac{1}{2}n$ *and decreasing for* $\delta \geqq 1 - \frac{1}{2}n$,

(iii) $E(\delta) = 0$ *for* $\delta > K$ *and* $\delta < 2 - n - K$, *for some exceptional value* $K \geqq 0$.

Remark. When P is C^∞, Proposition 2.2 and the equality of the Fredholm indices follow from results in [22] and [21].

Proof: Let g_0 be a metric on M which is flat for $r > R$. Then the invariance of the index shows that $\iota(P, \delta) = \iota(\Delta_{g_0}, \delta)$ and the previous result implies that this equals $\iota_0(\delta)$. Statement (i) follows from the definition

$$\iota(P, \delta) = N(P, \delta) - N(P, 2 - n - \delta),$$

and $N_0(\delta) = 0$ for $\delta < 0$ gives the non-negativity of $E(\delta)$. The expansion (1.38) gives (iii) as in Corollary 1.18 and also implies that

$$N(P, k + \varepsilon) - N(P, k - \varepsilon) \leqq n_k$$

for any $0 < \varepsilon < 1$ and exceptional $k \geqq 0$, which gives (ii).

3. The Uniqueness of Infinity

In this section we show that the structure of infinity of (M, g) is essentially unique, in the sense that any two structures of infinity Φ, Ψ differ by a rigid motion and terms which are $o(r^{1-\tau})$. Since Definition 2.1 almost implies that $(\Phi^*\delta - \Psi^*\delta) \in W^{1,q}_{-\tau}(M)$, this conclusion is not unexpected.

As observed previously, the space $L^p_\delta(M)$ is invariantly defined while the higher derivative spaces $W^{k,p}_\delta(\Phi)$ will depend on the structure of infinity Φ

chosen, so the following result may be interpreted as saying that harmonic coordinates at infinity give a preferred C^∞ structure for M (with respect to the metric g_{ij}) which is $C^{1,\alpha}$-compatible with the original structure.

THEOREM 3.1. *Let (Φ, x) be a structure of infinity, $\Phi\colon M \setminus K \to E_R$ where $K \subset\subset M$, $R \geqq 1$, and fix $1 < \eta < 2$. There are functions $y^i \in L^q_\eta(M)$, $i = 1, \cdots, n$, such that $\Delta_g y^i = 0$ and $(x^i - y^i) \in W^{2,q}_{1-\tau}(\Phi; E_R)$ and hence there is $R_1 \geqq R$ such that (y^i) give coordinates in $\Phi^{-1}(E_{R_1})$ with*

$$\left.\begin{aligned} |x^i - y^i|(m) &= o\big(\sigma(m)^{1-\tau}\big) \\ \big|g(\partial_{x^i}, \partial_{x^j}) - g(\partial_{y^i}, \partial_{y^j})\big|(m) &= o\big(\sigma(m)^{-\tau}\big) \end{aligned}\right\} \quad \text{as} \quad \sigma(m) \to \infty. \tag{3.1}$$

Furthermore, the set of functions $\{1, y^1, \cdots, y^n\}$ is a basis for

$$\mathbf{H}_1 = \big\{u \in L^q_\eta(M)\colon \Delta_g u = 0\big\}. \tag{3.2}$$

Remark. By Proposition 1.6, $\mathbf{H}_1$ is intrinsic to (M, g) and does not depend on the structure of infinity Φ.

Proof: Extend the functions x^i in a C^∞ manner to all of M. Then $\Delta_g x^i \in W^{0,p}_{1-\tau}(M)$ since

$$\Delta_g x^i = \Gamma^i = g^{jk}\Gamma^i_{jk} = \tfrac{1}{2} g^{jk} g^{il}\big(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}\big) \tag{3.3}$$

in $M \setminus K$, where Γ^i_{jk} is the usual Christoffel symbol. Corollary 2.2 shows that $\Delta_g | W^{2,q}_{1-\tau}(\Phi)$ has trivial cokernel so there is $v^i \in W^{2,q}_{1-\tau}(\Phi)$ such that $\Delta_g(x^i - v^i) = 0$ in M. Setting $y^i = x^i - v^i$ gives (3.1), and (3.2) follows from the estimates of Theorem 1.2. Theorem 1.6 shows that

$$\mathbf{H}_1 = \ker\big(\Delta_g\colon W^{2,q}_\eta \to W^{0,q}_{\eta-2}\big)$$

and this has dimension $(n + 1)$ by Corollary 2.2. The functions y^i are linearly independent since the differentials dy^i form a basis for T^*M near infinity and hence $\{1, y^1, \cdots, y^n\}$ is a basis for $\mathbf{H}_1$.

COROLLARY 3.2 (Uniqueness of infinity). *Let (M, g) be an asymptotically flat manifold (possibly with boundary) and suppose that (Φ, x) and (Ψ, z) are two distinct structures at infinity with decay rates τ_1, τ_2, respectively. Then there is a rigid motion $(O^i_j, a^i) \in O(n, \mathbf{R}) \times \mathbf{R}^n$ of $\mathbf{R}^n$, and a compact $K \subset\subset M$ such that*

$$\big|x^i - \big(O^i_j z^j + a^i\big)\big|(m) = o\big(\sigma(m)^{1-\tau}\big) \quad \text{for} \quad m \in M \setminus K, \tag{3.4}$$

where $\tau = \min\{\tau_1, \tau_2\}$. *More precisely, the transition function* $F = \Psi \circ \Phi^{-1}$: $E_R \to \mathbf{R}^n$, *some* $R > 1$, *satisfies (with* $z^i = F^i(x)$*)*

$$(3.5) \qquad \left|x^i - \left(O^i_j F^j(x) + a^i\right)\right| \in W^{2,q}_{1-\tau}(\Phi; E_R) \quad \textit{for} \quad i = 1, \cdots, n.$$

Proof: Since this is a statement about the structures at infinity only, we can reduce the case where M has boundary to that where $M \approx \mathbf{R}^n$ by first excising the compact set $\{m \in M,\ \sigma(m) \leqq R\}$, for R sufficiently large. Compactness guarantees that glueing in $\Phi(B_R)$ and extending g_{ij} to $M \cup \Phi(B_R)$ does not change the asymptotic flatness with respect to either structure, so the result will follow from the case $M \approx \mathbf{R}^n$.

Let $(y^i), (w^i)$ be the harmonic coordinates constructed in the previous theorem, and corresponding to the structures Φ, Ψ, respectively. Then $\{1, y^1, \cdots, y^n\}, \{1, w^1, \cdots, w^n\}$ are both bases for $\mathbf{H}_1$ so there is an affine transformation $(A^i_j, a^i) \in \mathrm{GL}(n, \mathbf{R}) \times \mathbf{R}^n$ such that

$$y^i = A^i_j w^j + a^i.$$

Since y^i, w^i are asymptotically rectangular it follows that $A^i_j \in O(n, \mathbf{R})$ and the conclusions all follow now from the estimates of Theorem 3.1.

The example of [12] shows that different structures may have differing decay rates and one consequence of Theorem 3.1 is that harmonic coordinates have the best possible decay. By assuming more regularity we show that this decay rate is determined by the decay of the Ricci tensor.

PROPOSITION 3.3. *Suppose that* (M, g) *has a structure of infinity* Φ *with decay rate* $\eta > 0$, *so* $(\Phi_* g - \delta) \in W^{2,q}_{-\eta}(E_R)$ *for some* $q > n$, $R \geqq 1$, *and that the Ricci tensor of* (M, g) *satisfies*

$$\mathrm{Ric}(g) \in L^q_{-2-\tau}(M) \quad \textit{for some nonexceptional} \quad \tau > \eta.$$

Then there is a structure of infinity Θ *defined by coordinates harmonic near infinity which satisfies* $(\Theta_* g - \delta) \in W^{2,q}_{-\tau}(E_{R_1})$, *for some* $R_1 \geqq R$.

Proof: Defining Θ by harmonic coordinates near infinity as in Corollary 3.2, we have the identity

$$(3.6) \qquad \mathrm{Ric}_{ij} = -\tfrac{1}{2} g^{kl} \partial^2_{kl}(g_{ij}) + Q_{ij}(g, \partial g),$$

where $Q_{ij}(g, \partial g)$ is quadratic in ∂g. Since $\mathrm{Ric}(g)$ is a tensor, the left-hand side is in $L^q_{-2-\tau}(M)$ and the Sobolev inequality (1.9) shows that if $(\Theta_* g - \delta) \in W^{2,q}_{-\eta}(E_R)$, then $Q(\partial g) \in W^{1,q}_{-2-2\eta}(E_R)$. Proposition 1.14 now implies that $(\Theta_* g - \delta) \in W^{2,q}_{-\eta^*}(E_R)$, where $\eta^* = \min\{2\eta, \tau\} > \eta$ and the result follows by repeating this argument until $\eta^* = \tau$.

4. Uniqueness of the Mass

In this section we give sharp conditions on (M, g) under which the expression (0.1) for the mass of an asymptotically flat 3-manifold is well defined and does not depend on the (implicit) structure of infinity. The arguments work more generally for n-dimensional manifolds, $n \geqq 3$, where the mass is defined analogously by

$$c(n)\text{mass} = \oint_{S_\infty} (g_{ij,j} - g_{jj,i})\, dS^i, \tag{4.1}$$

and $c(n)$ is some normalising constant. This will show that the mass is a *geometric invariant* of (M, g).

The key is the identity for the scalar curvature $R(g)$ of (M, g) in local coordinates,

$$\begin{aligned} R(g) = |g|^{-1/2}\, \partial_i\big(|g|^{1/2} g^{ij}\big(\Gamma_j - \tfrac{1}{2}\, \partial_j(\log|g|)\big)\big) \\ - \tfrac{1}{2} g^{ij}\Gamma_i\, \partial_j(\log|g|) + g^{ij}g^{kl}g^{pq}\Gamma_{ikp}\Gamma_{jql}, \end{aligned} \tag{4.2}$$

where $\Gamma^i = g^{kl}\Gamma^i_{kl}$. This formula is a special case of the important expansion of the Einstein-Hilbert action (valid in all dimensions)

$$R * 1 = d\big(g^{ab}\omega^c_a \wedge \eta_{cb}\big) + g^{ab}\omega^c_d \wedge \omega^d_a \wedge \eta_{bc}, \tag{4.3}$$

where ω^b_a is the Levi-Civita connection 1-form of g_{ij} with respect to a frame $\{X_a\}$, $*1$ is the metric volume form and $\eta_{ab} = (X_a \wedge X_b) \lrcorner * 1$. This shows that the mass can be interpreted as a generalisation of the geodesic curvature term in the Gauss-Bonnet theorem.

We say that (M, g) satisfies the *mass decay conditions* if there is an asymptotic structure Φ such that

$$(\Phi_* g - \delta) \in W^{2,q}_{-\tau}(E_{R_0}) \text{ for some } R_0 > 1,\ q > n, \text{ and } \tau \geqq \tfrac{1}{2}(n-2), \tag{4.4a}$$

$$R(g) \in L^1(M). \tag{4.4b}$$

PROPOSITION 4.1. *Suppose that (M, g, Φ) satisfies the mass decay conditions (4.4). Let $\{D_k\}_1^\infty$ be an exhaustion of M by closed sets such that the sets $S_k = \Phi(\partial D_k)$ are connected $(n-1)$-dimensional C^1 submanifolds without boundary in $\mathbf{R}^n$ such that*

$$\begin{aligned} &R_k = \inf\{|x| : x \in S_k\} \to \infty \quad \textit{as} \quad k \to \infty, \\ &R_k{}^{(n-1)}\text{area}(S_k) \quad \textit{is bounded as} \quad k \to \infty, \end{aligned} \tag{4.5}$$

and $R_1 \geqq R_0$. *Then the mass* $\operatorname{mass}(g, \Phi)$ *of* (M, g, Φ), *defined by*

$$\operatorname{mass}(g, \Phi) = \lim_{k \to \infty} \oint_{S_k} \left((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i} \right) dS^i \tag{4.6}$$

is independent of the sequence $\{S_k\}$.

Proof: Working in the coordinates (x^i) at infinity, the asymptotic decay assumptions and the Sobolev estimate (1.9) imply that the boundary term in (4.2) can be written as

$$|g|^{1/2} g^{ij} \left(\Gamma_j - \tfrac{1}{2} \partial_j (\log|g|) \right) = g_{ij,j} - g_{jj,i} + o(r^{-1-2\tau}). \tag{4.7}$$

The condition $\tau \geqq \frac{1}{2}(n-2)$ implies that the error term in (4.7) is $o(r^{-(n-1)})$ and thus does not contribute in the limit of (4.6). Integration by parts of (4.2) over $D_k \setminus D_1$ now shows that

$$\begin{aligned} &\lim_{k \to \infty} \oint_{S_k} \left((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i} \right) dS^i \\ &\quad = \oint_{S_1} \left((\Phi_* g)_{ij,j} - (\Phi_* g)_{jj,i} \right) dS^i \\ &\qquad + \lim_{k \to \infty} \int_{D_k \setminus D_1} \left(R(g) + \tfrac{1}{2} g^{ij} \Gamma_1 \partial_j (\log|g|) + g^{ij} g^{kl} g^{pq} \Gamma_{ikp} \Gamma_{jql} \right) * 1, \end{aligned}$$

and since $g_{ij,k} \in L^2(M)$ and $R(g) \in L^1(M)$, the right-hand side has a limit which is independent of the sequence $\{S_k\}$.

We now show that $\operatorname{mass}(g, \Phi)$ as defined by (4.5) is in fact independent of the structure of infinity Φ. This relies on the uniqueness result of the previous section and a curious cancellation (4.9) which seems to be a generic phenomenon when dealing with the scalar curvature (see eg. [33], [28]). An infinitesimal form of this cancellation is folklore in the relativity community (see [34]).

THEOREM 4.2. *Let* $(\Phi, x), (\Psi, z)$ *be two structures of infinity for* (M, g) *satisfying the mass decay conditions* (4.4) *with decay rates* τ_1, τ_2, *respectively, so* $\tau = \min\{\tau_1, \tau_2\} \geqq \frac{1}{2}(n-2)$. *Then* $\operatorname{mass}(g, \Phi), \operatorname{mass}(g, \Psi)$ *are well defined and equal.*

Proof: Write the identity (4.3) as $R \bullet 1 = dA + B$, where the boundary term $A = g^{ij} \omega_i^k \wedge \eta_{kj}$ depends on the frame $\{X_i\}$. Under a frame change $X_i = Q_i^j X_j'$ defined by Q: $M \setminus K \to \mathrm{GL}(n, R)$, the change in boundary term is easily calculated,

$$A - A' = dQ_i^j \wedge \left(X_j' \wedge g^{ik} X_k \right) \lrcorner * 1. \tag{4.8}$$

Setting $X_i = \partial_{x^i}$, $X_i' = \partial_{z^i}$, the estimate (3.5) implies that $Q_i^j(x) = O_i^j + o(r^{-\tau})$ for some $O_i^j \in O(n, R)$ and $\partial Q_i^j = o(r^{-(\tau+1)})$. The coordinates (z^i) can be rotated so that $O_i^j \to \delta_i^j$ (note this does not change the mass (4.1)), and then

$$\left(X_j' \wedge g^{ik} X_k\right) \lrcorner * 1 = *_0(dx^j \wedge dx^i) + o(r^{-\tau}),$$

where $*_0$ is the Hodge star corresponding to the flat metric in the (x^i) coordinates. Letting A_X, A_Z denote the boundary terms of the frames $\{\partial_{x^i}\}$, $\{\partial_{z^i}\}$, respectively, we thus have

$$A_X - A_Z = d\left(*_0\left(Q_i^j\, dx^j \wedge dx^i\right)\right) + o(r^{-1-2\tau}); \tag{4.9}$$

thus the mass integrals over the boundaries S_k in (4.6) differ by $o(1)$ and a term which integrates to zero.

In [12], a family of coordinate systems on the standard Schwarzschild spacelike slice ($n = 3$) is constructed such that the metric has the form $\delta_{ij} + O(r^{-1/2})$ and the mass *varies* within the family, which shows that the conditions of Theorem 4.2 are exactly optimal for the mass to be uniquely defined. From Proposition 3.3 we have an intrinsic method of determining whether the mass can be properly defined.

THEOREM 4.3. *Suppose that* (M, g) *satisfies the decay conditions of Proposition* 3.3, *so that*

$$\mathrm{Ric}(g) \in L^q_{-2-\tau}(M).$$

(i) *If* $\tau \geqq \frac{1}{2}(n-2)$, *then the mass exists and is unique.*
(ii) *If* $\tau > n - 2$, *then the mass is zero.*

Proof: From Proposition 3.3, there are asymptotically flat coordinates with decay rate τ, so (i) follows from Theorems 4.1 and 4.2. If $\tau > n - 2$, then we use an observation of R. Schoen [30]: from Theorem 1.17 and Proposition 3.3 we have an expansion in harmonic coordinates (x^i):

$$g_{ij} = \delta_{ij} + A_{ij} r^{2-n} + O(r^{-\tau}),$$

where A_{ij} is a constant matrix. The harmonic condition implies that

$$0 = A_{ij} x^j - \tfrac{1}{2} A_{jj} x^i \quad \text{for all} \quad (x^i) \in \mathbf{R}^n,$$

so that $A_{ij} = 0$ and the mass vanishes.

We then have the following interesting consequence.

THEOREM 4.4. *Suppose that* (M, g) *satisfies the mass decay conditions with* $\tau > n - 2$ *and has* $\mathrm{Ric}(g) \in L^q_{-2-\tau}(M)$. *If either of the following holds*:

(i) $R(g) \geqq 0$ *and* (M, g) *is a manifold for which the positive mass theorem holds* (*eg. see Section* 6), *or*

(ii) $\mathrm{Ric}(g) \geqq 0$,

then $(M, g) = (\mathbf{R}^n, \delta)$.

Proof: (i) follows immediately from the above remark that $\mathrm{mass}(g) = 0$ by applying the uniqueness part of the positive mass theorem. To show (ii) we give an argument of [33]. Let (y^i) be globally defined harmonic functions forming rectangular coordinates at infinity and let

$$K^{(i)} = dy^i, \qquad i = 1, \cdots, n,$$

so that $|K^{(i)}|^2 = g^{ii}$. Now, in harmonic coordinates the mass integral simplifies to

$$-2c(n)\mathrm{mass} = \oint_{S_\infty} g_{jj,i}\, dS^1, \tag{4.10}$$

so that integrating the identity

$$\Delta_M |K^{(i)}|^2 = 2|\nabla K^{(i)}|^2 + 2\,\mathrm{Ric}(K^{(i)}, K^{(i)})$$

over M (the asymptotic conditions on $\mathrm{Ric}(g)$ and Theorem 3.1 ensure that all terms are integrable) and applying Stokes theorem one obtains

$$\begin{aligned} 2\sum_{i=1}^{n} \int_M \left(|\nabla K^{(i)}|^2 + \mathrm{Ric}(K^{(i)}, K^{(i)})\right) d\,\mathrm{vol} &= \oint_{S_\infty} \partial_j\left(|K^{(i)}|^2\right) dS^j \\ &= 2c(n)\mathrm{mass}(g). \end{aligned}$$

But the mass is zero by the previous theorem; thus the $K^{(i)}$, $i = 1, \cdots, n$, are globally parallel forms and hence (M, g) is flat.

5. Remarks on Harmonic Coordinates

The harmonic coordinates which entered peripherally into the proof of the uniqueness of the mass (Corollary 4.2) have some useful properties. As has been seen, they form an almost canonical coordinate system at infinity and in this section we shall describe their relation with the mass. This includes a new and elementary proof of the positive mass theorem for sufficiently flat initial data (with non-negative scalar curvature) and an estimate which should be useful when considering the Einstein conjecture (see [10]) (the instability of Minkowski space).

For this section we shall assume that (M, g) is an asymptotically flat 3-manifold with a structure of infinity (Φ, x) such that

$$(5.1)\qquad ((\Phi_* g) - \delta) \in W^{2,q}_{-\tau}(E_R) \quad \text{for some} \quad R > 1,\ q > 3 \text{ and } \tau > \tfrac{1}{2},$$

$$(5.2)\qquad R(g) \in W^{0,q}_{2-2\tau}(M).$$

The condition that the coordinates (x^i) be harmonic implies that

$$(5.3)\qquad \Gamma^i = g^{kl}\Gamma^i_{kl} = -\Delta_g x^i = g^{ij}g^{kl}\left(g_{jk,l} - \tfrac{1}{2}g_{kl,j}\right) = 0$$

and then the basic identity (4.2) simplifies to

$$(5.4)\qquad \tfrac{1}{2}\Delta_g(\log|g|) = -R(g) + g^{ij}\Gamma^l_{ik}\Gamma^k_{jl}.$$

Thus if (Φ, x) is a coordinate system harmonic at infinity, then the assumed decay conditions (5.1), (5.2) and Theorem 1.17 shows that

$$(5.5)\qquad \log|g| = c/r + o(r^{-1})$$

for some constant c. Using (4.2) and (5.3) we see that the definition of mass becomes

$$(5.6)\qquad \operatorname{mass}(g) = -\frac{1}{32\pi}\oint_{S_\infty} \partial_i(\log|g|)\, dS^i$$

and hence $c = 8m(g)$. If we write $g_{ij} = \delta_{ij} + h_{ij}$, where $h_{ij} = o(r^{-\tau})$, then (5.5) shows that

$$(5.7)\qquad \sum_1^3 h_{jj} = 8m(g)/r + o(r^{-1}),$$

which is a gauge condition that has been proposed by J. York [34].

The definition of the mass via (5.6) has a purely geometric interpretation as a special case of the following result.

PROPOSITION 5.1. *Suppose that (N, h) is another asymptotically flat 3-manifold satisfying the conditions* (5.1), (5.2) *and that F: $M \to N$ is a harmonic map (see* [13]) *which is asymptotic to the identity; that is, there is a structure of infinity Φ on M such that*

$$(5.8)\qquad \left((F^*h)_{ij} - g_{ij}\right) \in W^{2,q}_{-\tau}(\Phi).$$

Denoting the Jacobian determinant of F by

$$J(F) = \det\left(\partial_{x^i}(F^p)g^{ij}h_{pq}\right),$$

we have

$$\text{mass}(g) - \text{mass}(h) = -\frac{1}{32\pi}\oint_{S_\infty} \partial_i(\log J(F))\, dS^i. \tag{5.9}$$

Proof: The condition (5.8) guarantees that the pullback $F^*y^p = y^p(F(x))$ of coordinates (y^p) at infinity for N are also coordinates at infinity for M. Then the harmonic mapping equation with respect to these coordinates $x^i = y^i(F(x))$ becomes

$$g^{ij}\big(\Gamma^p_{ij}(g) - \Gamma^p_{ij}(h)\big) = 0, \tag{5.10}$$

where

$$\Gamma^k_{ij}(h) = \tfrac{1}{2}(F^*h)^{kl}\big(\partial_{x^i}(F^*h)_{jl} + \partial_{x^j}(F^*h)_{il} - \partial_{x^l}(F^*h)_{ij}\big)$$

and

$$J(F) = \det\big((F^*h)_{ij}\big)/\det(g_{ij}).$$

Now, since $g^{ij} = (F^*h)^{ij} + o(r^{-\tau})$, the boundary integrand in (5.9) is

$$\begin{aligned}\partial_i(\log J(F)) &= g^{jk}\big(\partial_{x^i}(F^*h)_{jk} - \partial_i g_{jk}\big) + o(r^{-1-2\tau})\\ &= \partial_{x^i}(F^*h)_{jj} - \partial_i g_{jj} + 2\big(\Gamma_{jji}(g) - \Gamma_{jji}(h)\big) + o(r^{-1-2\tau}),\end{aligned}$$

using (5.10). Expanding the Christoffel symbols shows that the right-hand side is

$$2\big(\partial_j g_{ij} - \partial_i g_{jj}\big) - 2\big(\partial_{x^i}(F^*h)_{ij} - \partial_{x^j}(F^*h)_{jj}\big) + o(r^{-1-2\tau}),$$

and the result follows from the definition (4.6) of the mass.

It is worth noting that the identity (4.2) which underlies the definition of the mass has a counterpart here which explains to some extent the appearance of $J(F)$ and harmonic maps in the formula (5.9). Letting β^p_{ij} denote the second fundamental form of F: $M \to N$ (see [13]) we have the identity

$$\Delta_g \log(J(F)) = R(g) - g(dF^p, dF^q)\text{Ric}(h)_{pq} - g^{ij}\big((dF)^{-1}\beta\big)^l_{ik}\big((dF)^{-1}\beta\big)^k_{jl}$$

which simplifies when $(N, h) = (\mathbf{R}^3, \delta)$ and (x^i) are harmonic coordinates to (5.4).

The identity (5.4) can also be used to prove the positive mass theorem (see [31], [33]) if the initial data is sufficiently flat and $R(g)$ is non-negative. This applies particularly to spacetimes which are close to Minkowski space and satisfy the weak energy condition, since such spacetimes admit asymptotically flat maximal surfaces (see [6]). This proof should be compared with the rather more elaborate constructions of [11].

THEOREM 5.2. *There is an $\varepsilon > 0$ such that if (M, g) is any asymptotically flat 3-manifold with $M \approx \mathbf{R}^3$ and $R(g) \geqq 0$ and metric g_{ij} with respect to the natural global coordinates (x^i) derived from the diffeomorphism $M \approx \mathbf{R}^3$, which satisfies, for some $q > 3$, $\frac{1}{2} < \tau$,*

$$\|g_{ij} - \delta_{ij}\|_{2,q,-\tau} \leqq \varepsilon,$$

then

$$16\pi m(g) \geqq \int_{\mathbf{R}^3} \left(R(g) + \tfrac{1}{8}|\partial g|^2\right) dx. \tag{5.11}$$

Proof: Choose ε small enough so that Propositions 1.12 and 1.15 guarantee the existence of global asymptotically flat harmonic *coordinates* (y^i) which satisfy

$$|g_{ij} - \delta_{ij}|(m) \leqq 10^{-2} \quad \text{for all} \quad m \in M.$$

Now working in these harmonic coordinates we note that $\Gamma^i = 0$ implies that

$$\partial_i \log|g| = g^{jk}\partial_i g_{jk} = 2g^{jk}\,\partial_j g_{ik} = -2g_{ij}\,\partial_k g^{jk} \tag{5.12}$$

and hence

$$-\tfrac{1}{2}\Delta_g \log|g| = -\tfrac{1}{2}\partial_i\left(g^{ij}\log|g|\right) - \tfrac{1}{4}|\nabla \log|g||^2. \tag{5.13}$$

From the definition of the Christoffel symbols, the bad term in (5.4) may be written as

$$g^{ij}\Gamma^l_{ik}\Gamma^k_{jl} = -\tfrac{1}{4}g^{ij}g^{kl}g^{pq}\,\partial_i g_{kp}\,\partial_j g_{lq} + \tfrac{1}{2}g_{kl}\,\partial_i g^{jk}\,\partial_j g^{il} \tag{5.14}$$

and the first term on the right-hand side will be denoted by $|\partial g|^2$ for short. The boundary term in (5.13) gives the mass, so by (5.4) we have

$$16\pi m(g) = \int_{\mathbf{R}^3}\left(R(g) + \tfrac{1}{4}|\partial g|^2 + \tfrac{1}{4}|\nabla \log|g||^2 - \tfrac{1}{2}g_{kl}\,\partial_i g^{jk}\,\partial_j g^{il}\right) dx. \tag{5.15}$$

The remaining bad term is now handled by

$$\begin{aligned} g_{kl}\,\partial_i g^{jk}\,\partial_j g^{il} &= (g_{kl} - \delta_{kl})\,\partial_i g^{jk}\,\partial_j g^{il} + \partial_i g^{jk}\,\partial_j g^{ik} \\ &= (g_{kl} - \delta_{kl})\,\partial_i g^{jk}\,\partial_j g^{il} + \partial_i\left(g^{jk}\,\partial_j g^{ik} - g^{ik}\,\partial_j g^{jk}\right) \\ &\quad + \partial_i g^{ik}\,\partial_j g^{jk} \\ &\leqq 10^{-1}|\partial g|^2 + \tfrac{1}{2}|\nabla \log|g||^2 + \partial_i\left(g^{jk}\,\partial_j g^{ik} - g^{ik}\,\partial_j g^{jk}\right), \end{aligned}$$

using (5.12). The decay conditions ensure that the boundary integrand in this expression will be $o(r^{-2})$; thus, inserting this in (5.15), one obtains (5.11).

6. The Positive Mass Theorem

In 1982, E. Witten described a proof of the positive mass theorem using spinors (see [33], [28]). The techniques developed thus far enable us to generalise this proof to dimensions $n \geqq 3$ with the same asymptotic conditions needed to ensure that the mass is well defined (Theorem 4.2), under the assumption that the manifold M^n admits a spin structure. This is a topological condition ($\omega_2(M) = 0$) which is automatically satisfied for oriented 3-manifolds, but it is a nontrivial condition in dimensions $n \geqq 4$.

The approach basically generalises that of [28], with some differences. For example, we use only the pure Dirac operator; the original calculations and a number of recent papers (see eg. [27]) modified this by adding zero-order terms (spinor endomorphisms) which give rise to additional terms which can be physically interpreted (e.g., charge, momentum).

We start by recalling the construction of spinors: for more details see [35]. The Clifford algebra $\mathrm{Cl}(V)$ of a vector space V with inner product g is the algebra generated by V and the relations

$$v^2 = -g(v, v) \quad \text{for all} \quad v \in V. \tag{6.1}$$

If $\dim V = n$, then $\dim \mathrm{Cl}(V) = 2^n$ and $\mathrm{Cl}(V)$ is naturally isomorphic (as a graded vector space) with the exterior algebra $\Lambda(V)$. Now suppose that g is positive definite and $\{e_i\}$ is an orthonormal basis of V so that $g(e_i, e_j) = \delta_{ij}$, and denote $\mathrm{Cl}(V) = \mathrm{Cl}(n)$. There is an irreducible representation (not unique in general)

$$\tau\colon \mathrm{Cl}(n) \to \mathrm{End}(S)$$

of $\mathrm{Cl}(n)$ as linear transformations (matrices) on some complex vector space S such that V acts by skew-Hermitean matrices with respect to the usual Hermitean inner product on S. We shall often denote this action by $\tau(x) = x \cdot$ for $x \in \mathrm{Cl}(n)$.

The Lie group $\mathrm{Spin}(n)$ imbeds in $\mathrm{Cl}(n)$ as the subgroup

$$\mathrm{Spin}(n) = \exp\{\mathrm{span}_{\mathbf{R}}\{e_i e_j,\ i < j\}\} \tag{6.2}$$

and thus has a linear representation (not irreducible),

$$\tau\colon \mathrm{Spin}(n) \to \mathrm{Aut}(S),$$

so that S is called the space of *Dirac spinors*. That the group defined by (6.2) is

$\mathrm{Spin}(n)$ can be seen from the double covering

$$\pi\colon \mathrm{Spin}(n) \to \mathrm{so}(n),$$

defined by

$$\pi(\sigma)\colon v \to \sigma v \sigma^{-1},$$

for $v \in V \subset \mathrm{Cl}(n)$, $\sigma \in \mathrm{Spin}(n)$. Since V generates $\mathrm{Cl}(V)$, for $\sigma \in \mathrm{Spin}(n)$, the action

$$\sigma\colon V \times S \to V \times S, \qquad (v, \psi) \to (\pi(\sigma)v, \tau(\sigma)\psi)$$

extends to give a commuting diagram

$$\begin{tikzcd}
\mathrm{Cl}(n) \times S \arrow[rr] \arrow[d, "\tau(\sigma)"] \arrow[dr, phantom, "\text{Clifford multiplication} \rightarrow"] & & S \arrow[d, "\tau(\sigma)"] \\
\mathrm{Cl}(n) \times S \arrow[rr] & & S
\end{tikzcd}$$

which allows us to extend Clifford multiplication to bundles. Thus, suppose M^n is a Riemannian spin manifold, so that the frame bundle $\mathscr{F}$ of TM has a double cover (spin structure)

$$\mathscr{F}^{\sim} \to \mathscr{F} \to M.$$

Then there is a natural action $\mathrm{Cl}(M) \times \Delta \to \Delta$ of the Clifford bundle

$$\mathrm{Cl}(M) = \mathscr{F}^{\sim} \times_{\mathrm{Spin}(n)} \mathrm{Cl}(n) = \mathscr{F} \times_{\mathrm{SO}(n)} \mathrm{Cl}(n)$$

on the bundle of Dirac spinors

$$\Delta = \mathscr{F}^{\sim} \times_{\mathrm{Spin}(n)} S.$$

We say that $\psi \in \Delta$ is a *constant spinor* with respect to the frame $f\colon U \to \mathscr{F}$, U open in M, if $\psi = [f^{\sim}, \psi_0]$, the equivalence class in Δ determined by the lifting $f^{\sim}$ of f, with $\psi_0\colon U \to S$ constant.

The Lie algebra isomorphism $\mathrm{spin}(n) \approx \mathrm{so}(n)$ can be described by

$$\tfrac{1}{2} e_i e_j \leftrightarrow e_i \wedge e_j,$$

where $e_i \wedge e_j$ is the generator in $\mathrm{so}(n)$ of an anti-clockwise rotation in the

(e_i, e_j)-plane: $e_i \to e_j$, $e_j \to -e_j$. A connection on $\mathscr{F}$,

$$\omega = -\sum_{i<j} \omega_{ij} \otimes e_i \wedge e_j \in \Lambda^1(T*M) \otimes \mathrm{so}(n),$$
$$\omega_{ij} = \langle e_i, \nabla e_j \rangle, \tag{6.3}$$

lifts to a connection on $\mathscr{F}^{\sim}$ and its associated bundle Δ, given by

$$\omega^{\sim} = -\tfrac{1}{2} \sum_{i<j} \omega_{ij} \otimes e_i e_j,$$

so that the covariant derivative of a constant spinor is

$$\nabla \varphi = -\tfrac{1}{4} \sum_{i,j} \omega_{ij} \otimes e_i \cdot e_j \cdot \varphi. \tag{6.4}$$

The Dirac operator $\mathscr{D}$: $\Gamma(\Delta) \to \Gamma(\Delta)$ is defined by

$$\mathscr{D}\psi = \sum_i e_i \cdot \nabla_i \psi \tag{6.5}$$

and it is not hard to verify the Lichnerowicz identity

$$\mathscr{D}^2 \psi = -\nabla^2_{i,i} \psi + \tfrac{1}{4} R \psi, \tag{6.6}$$

where R is the scalar curvature. Using the Hermitean inner product on Δ one obtains

$$\left(|\nabla \psi|^2 + \tfrac{1}{4} R |\psi|^2 - |\mathscr{D}\psi|^2\right) * 1 = d\left(\langle \psi, \sigma_{ij} \cdot \nabla_j \psi \rangle * e_i\right), \tag{6.7}$$

where

$$\sigma_{ij} = \tfrac{1}{2}[e_i, e_j] = e_i e_j + \delta_{ij}.$$

The identity (6.7) is the key to Wittens method, which requires us first to find a spinor ψ satisfying $\mathscr{D}\psi = 0$ and $\psi \to \psi_0$, a constant spinor at infinity, and then to identify the boundary term in (6.7) with the mass. As emphasised in [27], this identification does not depend on the Dirac equation. Henceforth we suppose that (M, g, Φ) is a complete asymptotically flat n-dimensional spin manifold satisfying the mass decay conditions (4.4) and having non-negative scalar curvature (in the weak sense),

$$R(g) \geqq 0. \tag{6.8}$$

PROPOSITION 6.1. *For $0 < \eta < n - 1$, the Dirac operator*

$$\mathscr{D}: W^{2,q}_{\eta}(\Delta) \to W^{1,q}_{\eta-1}(\Delta) \tag{6.9}$$

is an isomorphism, where $W^{k,p}_{\varepsilon}(\Delta)$ denotes the weighted Sobolev space of sections of Δ.

Proof: The arguments of Section 1 extend readily to the Dirac operator (see e.g. [25], [9], [22]) and show that (6.9) is Fredholm with adjoint

$$\mathscr{D} = \mathscr{D}^*: W^{2,q'}_{\eta+1-n}(\Delta) \to W^{1,q'}_{\eta-n}(\Delta).$$

If $\psi \in \ker(\mathscr{D}, -\eta)$, then $|\psi|^2 \to 0$ at infinity and from (6.6) we have

$$\Delta|\psi|^2 = \tfrac{1}{4}R|\psi|^2 + |\nabla\psi|^2 \geqq 0,$$

and the strong maximum principle implies that $|\psi|^2 \equiv 0$, so that, for $0 < \eta < n - 1$, both $\ker \mathscr{D}$ and $\ker \mathscr{D}^*$ are trivial.

Given asymptotically flat coordinates (x^i) satisfying (4.4), we can easily construct an orthonormal frame $e_i = e_i^j \partial_{x^j}$ near infinity such that the "vielbein" e_i^j satisfies

$$e_i^j - \delta_{ij} \in W^{2,q}_{-\tau}(E_{R_0}). \tag{6.10}$$

This frame $\{e_i\}$ will be called *asymptotically constant* (with respect to the coordinates (x^i)) and the spinor ψ_0 is *constant near infinity* if it is constant with respect to such a frame. Note that from Corollary 3.2, up to terms in $W^{2,q}_{-\tau}(E_{R_0})$, ψ_0 is constant with respect to any other asymptotically constant frame.

COROLLARY 6.2. *Let ψ_0 be a spinor field on M which is constant near infinity. Then there is a spinor field ψ such that*

$$\begin{aligned} &\mathscr{D}\psi = 0, \\ &\psi - \psi_0 \in W^{2,q}_{-\tau}(\Delta). \end{aligned} \tag{6.11}$$

Proof: The asymptotic conditions ensure that $\mathscr{D}\psi_0 \in W^{1,q}_{1-\tau}(\Delta)$; thus Theorem 6.1 gives a unique $\psi_1 \in W^{2,q}_{-\tau}(\Delta)$ such that $\mathscr{D}\psi_1 = -\mathscr{D}\psi_0$, and then $\psi = \psi_0 + \psi_1$ is the required spinor.

We now have

THEOREM 6.3 (Positive Mass). *Suppose that (M, g) is a complete spin manifold satisfying the mass decay conditions* (4.4) *with non-negative scalar curvature* (6.8). *Let ψ_0 be a spinor, constant near infinity and normalised by $|\psi_0|^2 \to 1$ at*

infinity, and let ψ be the solution of Dirac's equation constructed in Corollary 6.2. *Then the mass of M is non-negative and is given by*

$$c(n)\mathrm{mass}(g) = \int_M (4|\nabla\psi|^2 + R|\psi|^2) * 1. \tag{6.12}$$

Furthermore, if $\mathrm{mass}(g) = 0$, *then M is flat.*

Proof: By virtue of Corollary 6.2, we must identify the boundary term in (6.7) with the mass. Following [28], the identity

$$d\big(\langle\varphi, \sigma_{ij}\cdot\chi\rangle * (e_i \wedge e_j)\big) = \big(\langle\varphi, \sigma_{ij}\cdot\nabla_j\chi\rangle - \langle\sigma_{ij}\cdot\nabla_j\varphi, \chi\rangle\big) * e_i$$

shows that the boundary term can be written as

$$\langle\psi_0, \sigma_{ij}\cdot\nabla_j\psi_0\rangle * e_i + d\big(\langle\psi_0, \sigma_{ij}\cdot\psi_1\rangle * (e_i \wedge e_j)\big)$$
$$+\big(\langle\psi, \sigma_{ij}\cdot\nabla_j\psi\rangle + \langle\sigma_{ij}\cdot\nabla_j\psi_0, \psi_1\rangle\big) * e_i.$$

The decay conditions and (6.11) ensure that the last two terms are $o(r^{-1-2\tau})$ and hence do not contribute to the boundary integral in the limit, while the second term drops out since $d^2 = 0$. We evaluate the remaining term using (6.4):

$$\langle\psi_0, \sigma_{ij}\cdot\nabla_j\psi_0\rangle * e_i = -\tfrac{1}{4}\sum_{i,j,k,l}\omega_{kl}(e_j)\langle\psi_0, \sigma_{ij}\cdot\sigma_{kl}\cdot\psi_0\rangle * e_i.$$

Since σ_{ij} is skew-Hermitean and we are only interested in real components, this simplifies to

$$\tfrac{1}{2}\omega_{ij}(e_j)|\psi_0|^2 * e_i - \tfrac{1}{4}\omega_{kl}(e_j)\langle\psi_0, \sigma_{ijkl}\cdot\psi_0\rangle * e_i, \tag{6.13}$$

where $\sigma_{ijkl} = e_i e_j e_k e_l$ if i, j, k, l are distinct and 0 otherwise. In terms of the coordinate Christoffel symbols Γ_{ijk} and vielbein e_i^j, the connection is

$$\omega_{ij}(e_k) = e_i^p e_j^q e_k^r \Gamma_{rqp} + e_i^p e_k^r g_{pq}\,\partial_r(e_j^q)$$
$$= \Gamma_{kji} + \partial_k(e_j^i) + o(r^{-1-2\tau}).$$

Decomposing $e = (e_j^i) = \delta + s + a$, where s, a are symmetric, antisymmetric, respectively, and $o(r^{-\tau})$, we see that

$$\partial g^{-1} = 2\,\partial s + o(r^{-1-2\tau})$$

and thus from (6.3), we have

$$\omega_{ij}(e_j) = \tfrac{1}{2}(\partial_j g_{ij} - \partial_i g_{jj}) + \partial_j a_{ij} + o(r^{-1-2\tau}).$$

Now

$$\partial_j a_{ij} * e_i = d\big(a_{ij} * (dx^i \wedge dx^j)\big) + o(r^{-1-2\tau}),$$

so we find that the mass is given by

$$2c(n)\mathrm{mass}(g) = 2\oint_{S_\infty} \omega_{ij}(e_j) * e_i. \tag{6.14}$$

A similar calculation using the antisymmetry of σ_{ijkl} shows that the second term of (6.13) is divergence $+ o(r^{-1-2\tau})$ and again does not contribute, so (6.12) follows. Now, "the square of a spinor is a vector", i.e.,

$$\langle v_\psi, X\rangle = \mathrm{Im}\langle \psi, X \cdot \psi\rangle \quad \text{for} \quad X \in \mathbf{R}^n$$

defines a vector $v_\psi \in \mathbf{R}^n$ from a spinor ψ and using the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ it is not hard to see that this map $S \to \mathbf{R}^n$ is onto. If the mass vanishes, then $\nabla\psi \equiv 0$ and hence $\nabla v_\psi \equiv 0$. Since ψ_0 is an arbitrary constant at infinity spinor, we can find a basis for TM consisting of covariantly constant vector fields. Thus M is flat.

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