# NONCOMPACT COMPLEX SYMPLECTIC AND HYPERKÄHLER MANIFOLDS PRELIMINARY NOTES 

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PHILIP BOALCH

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## 1. Introduction

The aim of this course is to introduce some aspects of the geometry of hyperkähler manifolds (and more general complex symplectic manifolds) focusing on basic ideas and examples.

The principal motivation is to give some of the background material for further study of complex symplectic and hyperkähler manifolds and their applications. Some examples of such applications that we have in mind (but will not be covered here!) include:

[^0]- Nakajima's work on the representation theory of quantum algebras [Nak94, Nak98, Nak01],
- The approach of Witten and collaborators [KW07, GW06, Wit08] to the geometric Langlands program.
- The approach of Gaiotto-Moore-Neitzke [GMN08] to certain "wall-crossing formulae" of Kontsevich-Soibelman [KS08], as consistency conditions for the existence of a hyperkähler metric.

Indeed for the most part we will try to be as down to earth as possible and focus on basic examples. (A surprisingly large number of features of the more complicated spaces appear even in the simplest cases.)

## Berger's List.

From a mathematical perspective hyperkähler manifolds first appeared in M.Berger's work on Riemannian holonomy groups, which we will now briefly recall. Given a Riemannian manifold $(M, g)$ and a point $p \in M$ we can associate an orthogonal transformation

$$
\tau(\gamma) \in \mathrm{O}\left(T_{p} M\right)
$$

of the tangent space $T_{p} M$ to any loop $\gamma: S^{1} \rightarrow M$ in $M$ based at $p$ (i.e. so that $\gamma(1)=p)$. This is defined by restricting (pulling back) the tangent bundle of $M$ to the loop $S^{1}$. The Levi-Civita connection of $g$ then becomes a flat connection over $S^{1}$ and we may use it to parallel translate any vector $v \in T_{p} M$ around the loop and back to $T_{p} M$. This defines an orthogonal transformation $\tau(\gamma)$ of $T_{p} M$. The holonomy group $\mathrm{Hol}_{p}$ is then defined to be the group generated by all these transformations as the loop $\gamma$ varies:

$$
\operatorname{Hol}_{p}=\left\langle\tau(\gamma) \mid \gamma: S^{1} \rightarrow M, \gamma(1)=p\right\rangle \subset \mathrm{O}\left(T_{p} M\right)
$$

If we change base points then the holonomy groups are identified, once we make a choice of path between the two points, and we may speak of "the" holonomy group Hol.

The holonomy groups were classified as follows:

Theorem 1.1 (M. Berger 1955). Let $(M, g)$ be an oriented simply connected Riemannian manifold of dimension $n$ which is neither locally a product nor symmetric. Then Hol is one of:


Thus hyperkähler manifolds are those lying in the intersection of all the boxes: they have dimension divisible by 4 and holonomy group the compact symplectic group (the quaternionic unitary group). The first example was constructed in dimension 4 in 1978 by Eguchi-Hanson, and then in 1979 Calabi found examples in every dimension (and coined the name "hyperkähler" for them). These examples will be described in some detail later on.

The emphasis on the noncompact case is mainly because that is what arises in the applications we are interested in (namely quiver varieties in [Nak94, Nak98, Nak01], moduli spaces of Higgs bundles in [KW07, GW06, Wit08], and a certain hyperkähler four manifold in [GMN08]). Also there are many more known examples and constructions. Indeed in the compact case one has the $K 3$ surfaces, the abelian surfaces (with the flat metric), two infinite families constructed out of Hilbert schemes of points on these hyperkähler surfaces, and two other (deformation classes of) examples of complex dimension 6 and 10 due to O'Grady [O'G99, O'G03]. See Beauville [Bea84] for the construction of the families, which rests on Yau's solution of the Calabi conjecture for the existence of the metric. In the noncompact case there are other ways of obtaining hyperkähler manifolds (i.e. constructive methods), which often give a lot more information about the hyperkähler metric than just an existence theorem.

## 2. Basic examples: Calogero-Moser spaces and Hilbert schemes

One of the key features of hyperkähler geometry is that hyperkähler manifolds have families of complex structures. In other words one may have two non-isomorphic complex manifolds, which are naturally isomorphic as real manifolds, due to the fact that they are the same hyperkähler manifold simply viewed in two different complex structures. Thus we are able to see relations between certain complex manifolds which are simply "hidden" from a purely complex viewpoint. In this section we will describe two classes of complex manifolds of independent interest (one from integrable systems and the other from algebraic geometry). We will see they are not isomorphic as complex manifolds, but later in the course we will see they are the same hyperkähler manifold
viewed in two different complex structures. This gives some concrete motivation for a lot of the course.

## Calogero-Moser spaces.

Let $V=\mathbb{C}^{n}$ and consider the space

$$
C_{n}=\left\{(X, Z) \in \operatorname{End}(V) \times \operatorname{End}(V) \mid[X, Z]+\mathrm{Id}_{V} \text { has rank one }\right\} / \mathrm{GL}_{n}(\mathbb{C})
$$

consisting of two square matrices $X, Z$ whose commutator $[Z, X]$ differs from the identity by a rank one matrix. Note (by taking the trace) that it is impossible to find two square matrices whose commutator is the identity. Thus we are asking, in some sense, for the "best approximation". [Here $\mathrm{GL}_{n}(\mathbb{C})$ is acting by conjugation, and the quotient means simply taking the set of orbits.]

An alternative description will also be useful. Consider the space

$$
\left\{(X, Z, v, \alpha) \in \operatorname{End}(V)^{2} \times V \times V^{*} \mid[X, Z]+\operatorname{Id}_{V}=v \otimes \alpha\right\} / \mathrm{GL}_{n}(\mathbb{C})
$$

where $g \in \mathrm{GL}_{n}(\mathbb{C})$ acts in the natural way, as

$$
g(X, Z, v, \alpha)=\left(g X g^{-1}, g Z g^{-1}, g v, \alpha \circ g^{-1}\right)
$$

It is easy to show that this action is free and one may show (cf. [Wil98] p.5) the result is a smooth (connected) affine variety of dimension $2 n$-it is the affine variety associated to the ring of invariant functions.

To go between the two descriptions consider the map from the second description to the first got by forgetting $v, \alpha$. This is clearly surjective, and moreover (since the rank one matrix is always nonzero-having trace $n) v_{1} \otimes \alpha_{1}=v_{2} \otimes \alpha_{2}$ iff there is a nonzero scalar relating the pairs $(v, \alpha)$. Now the scalar subgroup of $\mathrm{GL}_{n}(\mathbb{C})$ acts trivially on the pair $X, Z$ so we see the map on orbits is bijective.

Let us describe a big open subset of $C_{n}$. (We will later relate it to the CalogeroMoser integrable system.) Consider the subset $C_{n}^{\prime} \subset C_{n}$ where $X$ is diagonalizable. Then, moving within the orbit we may assume $X$ is diagonal, and write $X=\operatorname{diag}\left(x_{1}, \ldots, x_{n}\right)$. Thus

$$
[X, Z]_{i j}=\left(x_{i}-x_{j}\right) z_{i j}
$$

which should equal the $i j$ entry of $-\operatorname{Id}_{V}+v \alpha$. Taking $i=j$ we see $v_{i} \alpha_{i}=1$, so no components of $v$ or $\alpha$ vanish. Thus if $i \neq j$ we see $\left(x_{i}-x_{j}\right) z_{i j}=v_{i} \alpha_{j} \neq 0$ so $x_{i} \neq x_{j}$, i.e. $X$ is regular semisimple (its stabilizer is the diagonal maximal torus). Within the orbits of this torus action there is a unique point with $v_{i}=1$ for all $i$ (and then since $v_{i} \alpha_{i}=1$ we will also have $\alpha_{i}=1$ ). [Then $v \alpha$ is the rank one matrix with a 1 in every entry.] Now note that the off-diagonal parts of $Z$ are determined uniquely: $z_{i j}=1 /\left(x_{i}-x_{j}\right)$. Thus if we let $p_{i}=Z_{i i}$ we have coordinates $\left(x_{i}, p_{j}\right)$ on $C_{n}^{\prime}$ defined upto permutation of the indices, i.e. the map

$$
\begin{gathered}
\left(\mathbb{C}^{n} \backslash \operatorname{diagonals}\right) \times \mathbb{C}^{n} \rightarrow C_{n}^{\prime} \\
\left(x_{i}, p_{j}\right) \mapsto\left(\operatorname{diag}\left(x_{i}\right), Z,(1, \ldots, 1)^{T},(1, \ldots, 1)\right)
\end{gathered}
$$

is a covering with group $\operatorname{Sym}_{n}$, where $Z$ is as above with diagonal part $\operatorname{diag}\left(p_{j}\right)$ and offdiagonal entries $1 /\left(x_{i}-x_{j}\right)$.

Thus $C_{n}^{\prime} \cong\left(\left(\mathbb{C}^{n} \backslash\right.\right.$ diagonals $\left.) \times \mathbb{C}^{n}\right) / \operatorname{Sym}_{n}=T^{*}\left(\left(\mathbb{C}^{n} \backslash\right.\right.$ diagonals $\left.) / \operatorname{Sym}_{n}\right)$ is the cotangent bundle of the configuration space of $n$-identical particles on the complex plane $\mathbb{C}$, i.e. the phase space for $n$ distinct identical particles, and so $C_{n}$ is a partial compactification of this. This is interesting since $C_{n}$ contains the trajectories as the particle collide: As will be explained later (Exercise 3.29), if we consider $n$ particles on $\mathbb{C}$ with an inverse square potential, then for real initial positions there are no collisions, but on the complex plane there can be collisions, where the flows enter $C_{n} \backslash C_{n}^{\prime}$ : the flows are incomplete on $C_{n}^{\prime}$, but complete on $C_{n}$.

## Hilbert Scheme of points on $\mathbb{C}^{2}$.

Now let us describe the algebraic geometer's way to partially compactify the set of $n$-tuples of distinct unordered point of $\mathbb{C}^{2}$ (this will contain the collisions, but not points tending to infinity in $\mathbb{C}^{2}$ ).

Set $X=\mathbb{C}^{2}$. Given $n$ distinct points $x_{i} \in X$ we can consider the corresponding subscheme $Z \subset X$. This has structure sheaf

$$
\mathcal{O}_{Z}=\bigoplus_{i=1}^{n} \mathbb{C}_{i}
$$

where $\mathbb{C}_{i}$ is the skyscraper sheaf supported at $x_{i}$, and corresponds to the ideal

$$
I=\left\{f \in \mathbb{C}\left[z_{1}, z_{2}\right] \mid f\left(x_{1}\right)=\cdots=f\left(x_{n}\right)=0\right\} \subset \mathbb{C}\left[z_{1}, z_{2}\right]
$$

of functions vanishing at these points. Such an ideal is an example of an ideal "of colength $n$ " i.e. such that the $\mathbb{C}$-vector space

$$
\mathbb{C}\left[z_{1}, z_{2}\right] / I
$$

has dimension $n$. Thus the natural algebro-geometric partial compactification of ( $X^{n} \backslash$ diagonals) $/ \operatorname{Sym}_{n}$ is to consider the set of all such ideals:

$$
X^{[n]}:=\left\{\text { ideals } I \subset \mathbb{C}\left[z_{1}, z_{2}\right] \mid \operatorname{dim}_{\mathbb{C}}\left(\mathbb{C}\left[z_{1}, z_{2}\right] / I\right)=n\right\}
$$

Such an ideal corresponds to a subscheme of $X$ with fixed (constant) Hilbert polynomial $P(t)=n$ (and so justifies the name "Hilbert scheme"). In general $P(m)=$ $\chi\left(\mathcal{O}_{Z} \otimes \mathcal{O}_{X}(m)\right)$.

Consider a point $x \in X$ and a tangent vector $v \in T_{x} X$ to $X$ at $x$. Then we may define an ideal

$$
I=\left\{f \mid f(x)=0, d f_{x}(v)=0\right\}
$$

of functions vanishing at $x$ and having zero derivative in the direction $v$. This is an ideal of colength 2, so represents a point of $X^{[2]}$. In particular we see that the Hilbert scheme retains more information than just the positions of the points as they collide-e.g. this ideal determines the direction of $v$ as well.

Some basic properties (due to Fogarty in general) of the Hilbert scheme of points on $X$ are as follows:

1) $X^{[n]}$ is smooth and has dimension $2 n$,
2) There is a map (the Hilbert-Chow morphism) $\pi: X^{[n]} \rightarrow S^{n} X$ to the symmetric product given by

$$
Z \mapsto \sum_{x \in X} \operatorname{dim}_{\mathbb{C}}\left(\mathcal{O}_{Z, x}\right)[x]
$$

where $Z$ is the subscheme corresponding to an ideal $I$. We will give a direct definition of the multiplicities for this situation below, here we just define it as the dimension over $\mathbb{C}$ of the stalk at $x$ of the structure sheaf of $Z$. The symmetric product ${ }^{1} S^{n} X=$ $X^{n} / \operatorname{Sym}_{n}$ is singular (if $n>1$ ) and we have that:
3) $\pi$ is a resolution of singularities (in particular it is an isomorphism away from the singularities).

A point to note is that $X^{[n]}$ is not affine. For example in the case $n=2$ we have $X^{[n]} \cong \mathbb{C}^{2} \times T^{*} \mathbb{P}^{1}\left(\right.$ which has a compact complex submanifold, $\left.\mathbb{P}^{1}\right)$.

Let us give a more explicit description of $X^{[n]}$. Given a point $I$ of $X^{[n]}$ we can associate an $n$-dimensional complex vector space $V=\mathbb{C}\left[z_{1}, z_{2}\right] / I$. This has the following properties:

1) The action of $z_{i}$ (by multiplication) on $\mathbb{C}\left[z_{1}, z_{2}\right]$ yields elements

$$
B_{i} \in \operatorname{End}(V)
$$

for $i=1,2$.
2) The elements $B_{i}$ commute: $\left[B_{1}, B_{2}\right]=0$.
3) The element $1 \in \mathbb{C}\left[z_{1}, z_{2}\right]$ maps to a vector $v \in V$
4) $v$ is a cyclic vector: any element of $V$ is a linear combination of elements of the form $w v$ where $w$ is a word in the $B_{i}$.

In other words $V$ is a "cyclic $\mathbb{C}\left[z_{1}, z_{2}\right]$-module". Now the fact is that this data determines $I$ : Given data ( $V, B_{1}, B_{2}, v$ ) satisfying these conditions we may define a $\operatorname{map} \varphi: \mathbb{C}\left[z_{1}, z_{2}\right] \rightarrow V$ by setting

$$
\varphi(f)=f\left(B_{1}, B_{2}\right) \cdot v \in V .
$$

[^1]Since $v$ is cyclic this is surjective and we take $I$ to be the kernel of $\varphi$. It is an ideal of colength $n$. This establishes the following [cf. [Nak99], chapter 1]:

Proposition 2.1. $X^{[n]}$ is isomorphic to the set of $\mathrm{GL}_{n}(\mathbb{C})$ orbits in the space of matrices

$$
\left\{\begin{array}{cc}
{\left[B_{1}, B_{2}\right]=0, \text { and if }} \\
\left(B_{1}, B_{2}, v\right) \in \operatorname{End}(V)^{2} \times V & \cup \subset V \text { with } v \in U \text { and } \\
B_{i}(U) \subset U, i=1,2, \text { then } U=V
\end{array}\right\}
$$

where $V=\mathbb{C}^{n}$.
Thus $C_{n}$ and $X^{[n]}$ are both smooth (algebraic) complex manifolds of dimension $2 n$ and both involve "adding extra material to account for collisions" in one sense or another. But they are not isomorphic as complex manifolds, since one is affine and the other is not. Later in this course we will see however that they are in fact the same hyperkähler manifold viewed in two different complex structures. One may say there is a "hidden" symmetry group changing the complex structure moving from one space to the other.

To end this section we will make some aspects more explicit in terms of this "matricial" description of the Hilbert scheme. In terms of this description the Hilbert-Chow map (to the symmetric product) is as follows. Given commuting matrices $B_{1}, B_{2}$ we may simultaneously put them in upper triangular form, with diagonal entries $\lambda_{1}, \ldots, \lambda_{n}$ and $\mu_{1}, \ldots, \mu_{n}$ respectively say. The corresponding point of $S^{n} X$ is then

$$
\sum_{i=1}^{n}\left[\left(\lambda_{i}, \mu_{i}\right)\right]
$$

Note that if all these $n$ points of $X$ are distinct then (in the basis in which they are upper triangular) both $B_{1}, B_{2}$ are necessarily diagonal matrices (stabilized by the maximal diagonal torus of $\mathrm{GL}_{n}(\mathbb{C})$ ). In this basis, each component of the cyclic vector $v$ must be nonzero, and so there is a unique element of the torus mapping $v$ to the vector $(1,1, \ldots, 1)^{T}$. This shows the open part of $X^{[n]}$ where both $B_{1}, B_{2}$ are diagonal maps isomorphically onto the smooth locus of the symmetric product (where the points are distinct).

Let us look at the simplest case where two points coincide, in $X^{[2]}$. Let us consider the locus lying over the point $2[(0,0)] \in S^{2} X$. These will be represented by matrices of the form

$$
B_{1}=\left(\begin{array}{cc}
0 & \alpha \\
0 & 0
\end{array}\right), \quad B_{2}=\left(\begin{array}{cc}
0 & \beta \\
0 & 0
\end{array}\right)
$$

for some $\alpha, \beta \in \mathbb{C}$. Clearly if $\alpha=\beta=0$ then $v$ is not cyclic (whatever $v$ is). Otherwise one easily sees the set of cyclic vectors is $\left\{\binom{p}{q}\right\}$ with $q \neq 0$. The subgroup stabilizing $B_{1}, B_{2}$ is $\left\{g=\left(\begin{array}{ll}x & y \\ 0 & x\end{array}\right)\right\}$ with $x$ nonzero, and this acts simply transitively on the set of
cyclic vectors. The larger group $\left\{h=\left(\begin{array}{cc}x & y \\ 0 & z\end{array}\right)\right\}$ with both $x, z$ nonzero preserves the form of the matrices $B_{1}, B_{2}$ (i.e. maps them to a pair of matrices of the same form). This larger group acts on the pair $\alpha, \beta$ as follows:

$$
h(\alpha, \beta)=(t \alpha, t \beta)
$$

where $t=x / z \in \mathbb{C}^{*}$ for $h=\left(\begin{array}{ll}x & y \\ 0 & z\end{array}\right)$. This shows that the set of orbits of such matrices (the fibre of $\pi: X^{[2]} \rightarrow S^{2} X$ ) is a projective line $\mathbb{P}^{1}(\mathbb{C})$, with homogeneous coordinates $[\alpha: \beta]$. By definition (after choosing $v=\binom{0}{1}$ ) the corresponding ideal is the kernel of the map

$$
f \in \mathbb{C}\left[z_{1}, z_{2}\right] \mapsto f\left(B_{1}, B_{2}\right)\binom{0}{1}
$$

Expanding such $f$ in a Taylor expansion at 0 we see (easily) that the kernel $I$ consists of functions $f$ with $f(0)=0$ and first derivatives at zero constrained so that:

$$
\alpha \frac{\partial f}{\partial z_{1}}(0)+\beta \frac{\partial f}{\partial z_{2}}(0)=0 .
$$

The left hand side of this is $\left\langle\alpha \frac{\partial}{\partial z_{1}}+\beta \frac{\partial}{\partial z_{2}}, d f\right\rangle$ and so $[\alpha: \beta]$ should be viewed as a (complex) tangent direction; i.e. a point of the projectivised tangent space $\mathbb{P} T_{0} X$ to $X=\mathbb{C}^{2}$ at the origin; it corresponds to the ray in $T_{0} X \cong \mathbb{C}^{2}$ through the vector

$$
\alpha \frac{\partial}{\partial z_{1}}+\beta \frac{\partial}{\partial z_{2}} .
$$

A little bit more work will enable us to identify $X^{[2]} \cong \mathbb{C}^{2} \times T^{*} \mathbb{P}^{1}$. First note there is a free action of the additive group $\mathbb{C}^{2}$ on $X^{[2]}$ by translating $z_{1}, z_{2}$. Quotienting by this action amounts to restricting to the subset $X_{0}^{[2]} \subset X^{[2]}$ (the "punctual" Hilbert scheme) where the points have centre of mass zero. One has $X^{[2]} \cong \mathbb{C}^{2} \times X_{0}^{[2]}$ (and similarly on the symmetric products). Then one observes that $S_{0}^{2} X$ is just $\mathbb{C}^{2} /\{ \pm 1\}$. This is the $A_{1}$ singularity and the resolution of this (got by simply blowing up the singular point once) is the total space of the bundle $\mathcal{O}(-2) \rightarrow \mathbb{P}^{1}$, which is isomorphic to the cotangent bundle of $\mathbb{P}^{1}$.

Exercise 2.2 (Nakajima [Nak99]). Suppose we have $B_{1}, B_{2} \in \operatorname{End}(V)$, a cyclic vector $v \in V$ and an element $\phi \in V^{*}$ such that

$$
\left[B_{1}, B_{2}\right]+v \otimes \phi=0 .
$$

Show, as follows, that $\phi=0$, so in fact $\left[B_{1}, B_{2}\right]=0$.

1) Show that $\phi(v)=0$,

Suppose now that $\phi(w v)=0$ for all products (words) $w$ of the $B_{i}$ of length $<k$.
2) Deduce that $\phi \circ w_{1} B_{2} B_{1} w_{2}=\phi \circ w_{1} B_{1} B_{2} w_{2}$ for any words $w_{1}, w_{2}$ such that $w_{1}$ has length $<k$.
3) Deduce that $\phi \circ w=\phi \circ B_{1}^{k_{1}} B_{2}^{k_{2}}$ for any word $w$ of length $k$, where $k_{i}$ is the number of $B_{i}$ 's occuring in $w$ (for $i=1,2$ ).
4) Verify that $\left[X^{k}, Y\right]=\sum_{l=0}^{k-1} X^{l}[X, Y] X^{k-l-1}$ for any square matrices $X, Y$.
5) Use 4) to show that

$$
\begin{gathered}
\phi w v=\operatorname{Tr}(w v \otimes \phi)=-\operatorname{Tr}\left(w\left[B_{1}, B_{2}\right]\right) \\
=-\operatorname{Tr}\left(B_{1}^{k_{1}}\left[B_{2}^{k_{2}}, B_{1}\right] B_{2}\right)=\cdots=-\sum_{l=0}^{k_{2}-1} \phi B_{2}^{k_{2}-l} B_{1}^{k_{1}} B_{2}^{l} v
\end{gathered}
$$

where $w=B_{1}^{k_{1}} B_{2}^{k_{2}}$ as in 3$)$,
6) Deduce from 3) and 5) that $\phi w v=-k_{2} \phi w v$, so $\phi w v=0$
7) Deduce that $\phi=0$.

This immediately yields the following alternative description of the Hilbert scheme, looking a little more like the definition of the Calogero-Moser spaces.

Corollary 2.3. $X^{[n]}$ is isomorphic to the set of $\mathrm{GL}_{n}(\mathbb{C})$ orbits in the space

$$
\left\{\left(B_{1}, B_{2}, v, \phi\right) \in \operatorname{End}(V)^{2} \times V \times V^{*} \left\lvert\, \begin{array}{c}
{\left[B_{1}, B_{2}\right]+v \otimes \phi=0, \text { and if }} \\
U \subset V \text { with } v \in U \text { and } \\
B_{i}(U) \subset U, i=1,2, \text { then } U=V
\end{array}\right.\right\}
$$

where $V=\mathbb{C}^{n}$.

## 3. REAL AND COMPLEX SYMPLECTIC GEOMETRY

We will quickly cover the basics of symplectic (and holomorphic symplectic) geometry. The aim is to get to the definition of the moment map, consider some examples and define the symplectic quotient construction.

## Symplectic vector spaces.

First it is useful to recall (from basic linear algebra) the canonical form of a skewsymmetric bilinear form on a vector space.
Lemma 3.1. Let $V$ be a finite dimensional vector space (over a field of characteristic not equal to 2) and let $\omega$ be a skew-symmetric bilinear form on $V$. Then there is a basis

$$
u_{1}, \ldots, u_{k}, v_{1}, \ldots, v_{k}, e_{1}, \ldots, e_{l}
$$

of $V$ such that

$$
\omega\left(u_{i}, v_{i}\right)=1=-\omega\left(v_{i}, u_{i}\right)
$$

for $i=1, \ldots, k$ and $\omega$ is zero on all other pairs of basis vectors.
Proof. If $\omega \neq 0$ then there are $u, v \in V$ such that $\omega(u, v) \neq 0$ and we may scale $u$ such that $\omega(u, v)=1$. Clearly $u, v$ are linearly independent (since $\omega(u, u)=0$ if char $\neq 2$ ), so we may set $u_{1}=u, v_{1}=v$. Let $V_{1} \subset V$ be the span of $u_{1}, v_{1}$, and set

$$
U=V_{1}^{\perp}=\{x \in V \mid \omega(x, u)=\omega(x, v)=0\} .
$$

If $x \in V$ then $x^{\prime}:=x-u_{1} \omega\left(x, v_{1}\right)+v_{1} \omega\left(x, u_{1}\right)$ is in $U$, and so $V=V_{1} \oplus U$. Now if $\left.\omega\right|_{U} \neq 0$ we may iterate until we find $V=V_{1} \oplus \cdots V_{k} \oplus U$ and $\left.\omega\right|_{U}=0$.

Here we are only interested in the real and complex cases.
A (real) symplectic vector space $(V, \omega)$ is a real vector space $V$ together with a skew-symmetric bilinear form

$$
\omega: V \otimes V \rightarrow \mathbb{R}
$$

which is nondegenerate in the sense that the associated linear map

$$
\begin{equation*}
\omega^{b}: V \rightarrow V^{*} ; \quad v \mapsto \iota_{v} \omega=\omega(v, \cdot) \tag{3.1}
\end{equation*}
$$

from $V$ to its dual space, is an isomorphism. By Lemma 3.1 we may then find a basis $p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{n}$ of $V^{*}$ such that

$$
\omega=\sum_{1}^{n} d p_{i} \wedge d q_{i}
$$

for some integer $n$. In particular $V$ has even real dimension $2 n$.

Similarly a complex symplectic vector space $\left(V, \omega_{\mathbb{C}}\right)$ is a complex vector space $V$ together with a skew-symmetric $\mathbb{C}$-bilinear form

$$
\omega_{\mathbb{C}}: V \otimes V \rightarrow \mathbb{C}
$$

which is nondegenerate in the sense that the associated linear map

$$
\begin{equation*}
\omega_{\mathbb{C}}^{b}: V \rightarrow V^{*} ; \quad v \mapsto \iota_{v} \omega_{\mathbb{C}}=\omega_{\mathbb{C}}(v, \cdot) \tag{3.2}
\end{equation*}
$$

from $V$ to its dual space, is an isomorphism. (Beware now $V^{*}$ denotes the complex dual space, of $\mathbb{C}$-linear maps $V \rightarrow \mathbb{C}$.) Again by Lemma 3.1 we can find a basis $p_{1}, \ldots, p_{n}, q_{1}, \ldots, q_{n}$ of $V^{*}$ such that

$$
\omega_{\mathbb{C}}=\sum_{1}^{n} d p_{i} \wedge d q_{i}
$$

for some integer $n$. In particular $V$ has even complex dimension $2 n$.
A simple example should clarify the distinction between real and complex symplectic vector spaces (the simplest examples of real and complex symplectic manifolds). Take $V=\mathbb{C}^{2} \cong \mathbb{R}^{4}$ with complex (linear) coordinates $z, w \in \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Then $\omega_{\mathbb{C}}=d z \wedge d w$ is a complex symplectic form. If we write $z=x+i y, w=u+i v$ for real coordinates $x, y, u, v \in \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ then

$$
\omega_{\mathbb{C}}=d x \wedge d u+d v \wedge d y+i(d y \wedge d u+d x \wedge d v)
$$

and we see that both the real and imaginary parts of $\omega_{\mathbb{C}}$ are real symplectic forms on $\mathbb{R}^{4}$.

## Basic definitions for symplectic manifolds.

Definition 3.2. A (real) symplectic manifold is a pair $(M, \omega)$ consisting of a differentiable manifold $M$ and a real two-form $\omega$, such that:

- $\omega$ is closed: i.e. $d \omega=0$, and
- $\omega$ is nondegenerate: the associated linear map $T_{m} M \rightarrow T_{m}^{*} M ; v \mapsto \omega_{m}(v, \cdot)$ from the tangent space to the cotangent space, is an isomorphism at each point $m \in M$.

Thus firstly the tangent space $T_{m} M$ to a symplectic manifold is a symplectic vector space at each point $m \in M$, but there is also the nonalgebraic condition that $\omega$ is closed.

In the holomorphic (or complex algebraic) category one uses a different notion (and one should not confuse the two), as follows:

Definition 3.3. A complex symplectic manifold is a pair $\left(M, \omega_{\mathbb{C}}\right)$ consisting of $a$ complex manifold $M$ and $a$ holomorphic two-form $\omega_{\mathbb{C}}$ (of type $(2,0)$ ) such that:

- $\omega_{\mathbb{C}}$ is closed: i.e. $d \omega_{\mathbb{C}}=0$, and
- $\omega_{\mathbb{C}}$ is nondegenerate: the associated linear map $T_{m} M \rightarrow T_{m}^{*} M ; v \mapsto\left(\omega_{\mathbb{C}}\right)_{m}(v, \cdot)$ from the holomorphic tangent space to the holomorphic cotangent space, is an isomorphism at each point $m \in M$.

These are sometimes also referred to as holomorphic symplectic manifolds. (Note that if one just stipulates that $\omega_{\mathbb{C}}$ is a $C^{\infty}$ global $(2,0)$ form, then requiring it to be closed implies it is in fact holomorphic.)

## Basic examples of symplectic manifolds.

Example 3.4 (Cotangent bundles). Let $N$ be a manifold and let $M=T^{*} N$ be the total space of its cotangent bundle. This has a natural symplectic structure, which may be defined locally as follows. Choose local coordinates $x_{1}, \ldots, x_{n}$ on $N$. Then the one-forms $d x_{1}, \ldots, d x_{n}$ provide a local trivialisation of $T^{*} N$, so we obtain local coordinate functions $p_{1}, \ldots p_{n}$ on the fibres of $T^{*} N$ (fixing the values of these determine the point $\sum p_{i} d x_{i}$ of the fibre). Thus $M$ has local coordinates $x_{i}, p_{i}$. We may define a one form

$$
\theta=\sum_{1}^{n} p_{i} d x_{i}
$$

locally on $M$ and it turns out that this local definition in fact defines a global one-form (the "Liouville form") on $M$. The exterior derivative

$$
\omega=d \theta
$$

is a natural symplectic form on $M$. Clearly it is closed (since it is exact), and it is nondegenerate because in local coordinates it is just

$$
\sum_{1}^{n} d p_{i} \wedge d x_{i}
$$

This is the basic class of symplectic manifolds crucial to classical mechanics and much else, and indeed any symplectic manifold is locally of this form (the Darboux theorem). However there are many other symplectic manifolds which are not cotangent bundles globally.

The intrinsic definition of the Liouville form $\theta$ is as follows. Given a point $m=$ $(p, x) \in M=T^{*} N$ (with $x \in N, p \in T_{x}^{*} N$ ) and a tangent vector $v \in T_{m} M$ the one form $\theta$ should produce a number $\langle\theta, v\rangle_{m}$. This number is obtained from $v$ as follows: the derivative at $m$ of the projection $\pi: M \rightarrow N$ is a map $d \pi_{m}: T_{m} M \rightarrow T_{x} N$, and we simply pair the image of $v$ with $p$ :

$$
\langle\theta, v\rangle_{m}:=\left\langle p, d \pi_{m}(v)\right\rangle .
$$

This example may be read in both a real fashion (to obtain a real symplectic form) or in a complex fashion, if $M$ is a complex manifold, using the holomorphic
(co)tangent bundles and coordinates, to get a complex symplectic form on the total space of the holomorphic cotangent bundle.

Example 3.5 (Coadjoint orbits). Let $G$ be a Lie group and let $\mathfrak{g}=T_{e} G$ be its Lie algebra. Let $\mathfrak{g}^{*}=T_{e}^{*} G$ be the dual vector space to $\mathfrak{g}$. If $G$ acts on itself by conjugation, this fixes the identity $e \in G$ and so we have an induced action on $\mathfrak{g}$ and $\mathfrak{g}^{*}$, the adjoint action Ad and the coadjoint action Ad*. These are related as follows, where $X \in \mathfrak{g}, \alpha \in \mathfrak{g}^{*}, g \in G$ :

$$
\left\langle\operatorname{Ad}_{g}^{*}(\alpha), X\right\rangle=\left\langle\alpha, \operatorname{Ad}_{g^{-1}}(X)\right\rangle,
$$

where the brackets denote the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^{*}$. Infinitesimally (writing $g=\exp (X t)$ for $X \in \mathfrak{g}$ and taking the derivaive at $t=0$ ) yields the corresponding Lie algebra actions:

$$
\begin{aligned}
\operatorname{ad}_{X}: \mathfrak{g} & \rightarrow \mathfrak{g} ; \quad Y \mapsto \operatorname{ad}_{X}(Y)=[X, Y] \\
\operatorname{ad}_{X}^{*}: \mathfrak{g}^{*} \rightarrow \mathfrak{g}^{*} ; & \alpha \mapsto \operatorname{ad}_{X}^{*}(\alpha)
\end{aligned}
$$

which in turn are related by:

$$
\left\langle\operatorname{ad}_{X}^{*}(\alpha), Y\right\rangle=-\left\langle\alpha, \operatorname{ad}_{X}(Y)\right\rangle .
$$

It follows immediately from these definitions that if $\mathcal{O} \subset \mathfrak{g}$ is an arbitrary orbit for the adjoint action, and $Y \in \mathcal{O}$ then the tangent space to $\mathcal{O}$ at $Y$ is $\left\{\operatorname{ad}_{X}(Y) \mid X \in \mathfrak{g}\right\}$. E.g. for $G=\mathrm{GL}_{n}(\mathbb{C})$ the orbit through $Y$ is just the set of conjugate $n \times n$ matrices:

$$
\mathcal{O}=\left\{g Y g^{-1} \mid g \in G\right\} \subset \mathfrak{g l}_{n}(\mathbb{C})
$$

Then writing $g=\exp (X t)$ for $X \in \mathfrak{g}$ and taking the derivaive at $t=0$ yields

$$
T_{Y} \mathcal{O}=\{X Y-Y X \mid X \in \mathfrak{g}\} \subset \mathfrak{g l}_{n}(\mathbb{C})
$$

as stated, since here $\operatorname{ad}_{X}(Y)=[X, Y]$ agrees with the commutator $X Y-Y X$ of matrices. Indeed in general one has

$$
\left.\frac{d}{d t}\left(\operatorname{Ad}_{\exp (X t)}(Y)\right)\right|_{t=0}=[X, Y] \in \mathfrak{g}
$$

Similarly (basically by definition) if $\mathcal{O} \subset \mathfrak{g}^{*}$ is an arbitrary orbit for the coadjoint action, and $\alpha \in \mathcal{O}$ then the tangent space to $\mathcal{O}$ at $\alpha$ is

$$
T_{\alpha} \mathcal{O}=\left\{\operatorname{ad}_{X}^{*}(\alpha) \mid X \in \mathfrak{g}\right\}
$$

An important fact about coadjoint orbits is the following:
Theorem 3.6 (Kostant-Kirillov-Souriau). Coadjoint orbits are symplectic manifolds: Any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^{*}$ has a natural $G$-invariant symplectic structure, given by the formula:

$$
\omega_{\alpha}\left(\operatorname{ad}_{X}^{*}(\alpha), \operatorname{ad}_{Y}^{*}(\alpha)\right)=\langle\alpha,[X, Y]\rangle=\alpha([X, Y])
$$

for all $\alpha \in \mathcal{O} \subset \mathfrak{g}^{*}, X, Y \in \mathfrak{g}$.

Proof. To simplify notation, for $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^{*}$, write $[X, \alpha]:=\operatorname{ad}_{X}^{*}(\alpha) \in \mathfrak{g}^{*}$. First we check $\omega$ is well defined and nondegenerate. Note that

$$
\begin{aligned}
{\left[X_{1}, \alpha\right]=\left[X_{2}, \alpha\right] } & \Leftrightarrow\left\langle\left[X_{1}, \alpha\right], Y\right\rangle=\left\langle\left[X_{2}, \alpha\right], Y\right\rangle \quad \text { for all } Y \in \mathfrak{g} \\
& \Leftrightarrow \alpha\left(\left[Y, X_{1}\right]\right)=\alpha\left(\left[Y, X_{2}\right]\right) \quad \text { for all } Y \in \mathfrak{g} .
\end{aligned}
$$

Thus $\omega$ is independent of the choice of $X, Y$ representing the tanget vectors. Moreover (putting $X_{2}=0$ ) we see $\omega$ is nondegenerate. It is straightforward to see $\omega$ is $G$ invariant, so we need just check it is closed. Given $X \in \mathfrak{g}$ write $v_{X}$ for the vector field on $\mathcal{O}$ taking the value $v_{X, \alpha}:=-[X, \alpha] \in T_{\alpha} \mathcal{O}$ at $\alpha$ for each $\alpha \in \mathcal{O}$. Also let $H_{X}$ be the function on $\mathcal{O}$ defined by $H_{X}(\alpha)=\alpha(X)$. We then claim that

$$
d H_{X}=\omega\left(\cdot, v_{X}\right)
$$

as one-forms on $\mathcal{O}$, for all $X \in \mathfrak{g}$. Indeed any tangent vector is of the form $v_{Y}$ for some $Y \in \mathfrak{g}$ and

$$
\omega\left(v_{Y}, v_{X}\right)=\alpha([Y, X])=-\langle[Y, \alpha], X\rangle=H_{X}(-[Y, \alpha])
$$

which equals the derivaive of $H_{X}$ at $\alpha$ along $v_{Y}$ (since it is linear). Now since $\omega$ is $G$-invariant and the infinitesimal $G$-action maps $X \in \mathfrak{g}$ to (minus) the vector field $v_{X}$ (it is the "fundamental vector field" of the action, to be defined below) we have

$$
0=\mathcal{L}_{v_{X}} \omega=\left(d \iota_{v_{X}}+\iota_{v_{X}} d\right) \omega=-d\left(d H_{X}\right)+\iota_{v_{X}} d \omega=\iota_{v_{X}} d \omega .
$$

But the vector fields of the form $v_{X}$ span the tangent space to $\mathcal{O}$ at each point so we deduce $d \omega=0$.

This becomes more explicit in the case where $\mathfrak{g}$ admits a nondegenerate invariant symmetric bilinear form $\mathcal{B}$. Then the adjoint orbits and coadjoint orbits may be identified, with $\alpha \in \mathfrak{g}^{*}$ corresponding to $A \in \mathfrak{g}$ such that $\alpha=\mathcal{B}(A, \cdot)$. The adjoint and coadjoint actions then correspond to each other. It follows then that the adjoint orbits obtain a symplectic structure, and this may be written as

$$
\omega_{A}([X, A],[Y, A])=\mathcal{B}(A,[X, Y])
$$

for $A \in \mathcal{O} \subset \mathfrak{g}$.
For example taking $G=\mathrm{SO}_{3}\left(\mathbb{R}\right.$ ) we have $\mathfrak{g} \cong \mathbb{R}^{3}$ (with Lie bracket given by the cross-product) and the adjoint action corresponds to the standard action of $G$ on $\mathbb{R}^{3}$. The orbits are two-spheres of fixed radius (and the origin). These have a symplectic structure since $\mathcal{B}(A, B)=\operatorname{Tr}(A B)$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ the $3 \times 3$ real skew-symmetric matrices.

These are compact so clearly not isomorphic to cotangent bundles.
Exercise 3.7. Check that $\mathfrak{s o}_{3}(\mathbb{R}) \cong \mathbb{R}^{3}$ (with Lie bracket given by the cross-product).

Another example is to choose $n$ real numbers $\lambda_{1}, \ldots, \lambda_{n}$ and consider the set $\mathcal{O}$ of $n \times n$ Hermitian matrices with these eigenvalues. Multiplying by $i$ identifies the Hermitian matrices with the skew-Hermitian matrices, the Lie algebra of the unitary group $\mathrm{U}(n)$. This has nondegenerate invariant symmetric bilinear form $\mathcal{B}(A, B)=$ $\operatorname{Tr}(A B)$. Moreover the adjoint action is simply given by matrix conjugation, so $\mathcal{O}$ is indeed an adjoint orbit and thus a symplectic manifold.

Exercise $3.8\left(^{*}\right)$. Show that one obtains the projective spaces $\mathbb{P}^{n-1}$ (the space of lines in $\mathbb{C}^{n}$ ) and the Grassmannians $\operatorname{Gr}_{k}\left(\mathbb{C}^{n}\right)$ (the spaces of $k$-dimensional subspaces of $\mathbb{C}^{n}$ ) as examples of adjoint orbits of $\mathrm{U}(n)$. Identify the other adjoint orbits of $\mathrm{U}(n)$ with flag manifolds.

If $G$ is a complex Lie group then we obtain a complex symplectic structure on its coadjoint orbits. For example take $G=\mathrm{GL}_{n}(\mathbb{C})$, so $\mathfrak{g}$ is just the set of $n \times n$ complex matrices. The pairing $\mathcal{B}(A, B)=\operatorname{Tr}(A B)$ is a nondegenerate invariant symmetric complex bilinear form, and the adjoint action is given by matrix conjugation: $\operatorname{Ad}_{g}(A)=g A g^{-1}$ (and $\left.[X, A]=X A-A X\right)$. (Beware that this is not the Killing form, which is degenerate for $\mathrm{GL}_{n}(\mathbb{C})$.) Thus the orbit $\mathcal{O} \subset \mathfrak{g}$ of a matrix $A \in \mathfrak{g}$ is simply the set of matrices with the same Jordan normal form as $A$. This has a complex symplectic structure, given by the same formula as above:

$$
\omega_{A}([X, A],[Y, A])=\operatorname{Tr}(A[X, Y])
$$

These are basic examples of complex symplectic manifolds.
Exercise 3.9. Take $G=\mathrm{SL}_{2}(\mathbb{C})$, so $\mathfrak{g} \cong \mathbb{C}^{3}$ is the space of of $2 \times 2$ tracefree complex matrices. Set $\mathcal{B}(A, B)=\operatorname{Tr}(A B)$. (Co)adjoint orbits of $G$ are the simplest nontrivial examples of complex symplectic manifolds.

1) Choose a diagonal matrix $A \in \mathfrak{g}$ and consider its orbit $\mathcal{O}$. Write down an algebraic equation for $\mathcal{O} \subset \mathbb{C}^{3}$, showing it is a (smooth) affine surface (i.e. a real four-manifold). Show, by considering the holomorphic map $\mathcal{O} \rightarrow \mathbb{P}^{1}$ taking the first eigenspace (or otherwise), that $\mathcal{O}$ is not isomorphic to $\mathbb{C}^{2}$.
2) Choose a nonzero nilpotent matrix $A \in \mathfrak{g}$ and consider its orbit $\mathcal{O}$. Write down an equation satisfied by the points of $\mathcal{O} \subset \mathbb{C}^{3}$, and observe that zero is also a point of the zero locus of this equation. Deduce that $\mathcal{O}$ is the smooth locus of a singular affine surface. (This is the $A_{1}$ surface singularity $\cong\left\{x y=z^{2}\right\} \subset \mathbb{C}^{3}$.)
Exercise 3.10. Consider the adjoint orbits of the group of matrices of the form $\left(\begin{array}{ll}a & b \\ 0 & 1\end{array}\right)$ with $a, b \in \mathbb{R}, a>0$. Deduce that not all adjoint orbits are symplectic.

One motivation for coadjoint orbits is the (heuristic) "orbit method" of representation theory: one would like to quantise coadjoint orbits of $G$ to obtain representations (see e.g. [Kir99, Vog00]). For example for a compact group Borel-Weil theory may be viewed as saying that any irreducible representation arises by geometric quantisation
of the coadjoint orbit through the highest weight. Our direct motivation is however more geometric: complex coadjoint orbits will provide basic examples of hyperkähler manifolds.

## Hamiltonian vector fields and Poisson brackets.

Given a function $f$ on a symplectic manifold $(M, \omega)$, we obtain a vector field $v_{f}$, by taking the derivative of $f$ and then using the isomorphism between the tangent and cotangent bundle furnished by the symplectic form. This is the Hamiltonian vector field of the function $f$. Explicitly $v_{f}$ is defined by the formula:

$$
d f=\omega\left(\cdot, v_{f}\right)
$$

In other words $d f=-\iota_{v_{f}} \omega=-\omega^{b}\left(v_{f}\right)$. (The minus sign is put so that Lemma 3.12 below holds-i.e. we get a Lie algebra morphism.) Thus we obtain a map from the functions on $M$ to the Lie algebra of vector fields on $M$. Not all vector fields arise in this way, since for example we have:

Lemma 3.11. The flow of a Hamiltonian vector field preserves the symplectic form.
Proof. This follows from Cartan's formula:

$$
\begin{aligned}
\mathcal{L}_{v_{f}} \omega & =\left(d \iota_{v_{f}}+\iota_{v_{f}} d\right) \omega \\
& =d \iota_{v_{f}} \omega \\
& =-d(d f) \\
& =0
\end{aligned}
$$

However in many instances the vector fields we are interested in are Hamiltonian, and so their study is essentially reduced to studying functions. Indeed the notion of Poisson bracket lifts the Lie algebra structure from the vector fields to the functions on $M$. More precisely, the symplectic form determines a bilinear bracket operation on the functions

$$
\{\cdot, \cdot\}: F u n(M) \otimes F u n(M) \rightarrow F u n(M)
$$

defined by

$$
\{f, g\}=\omega\left(v_{f}, v_{g}\right)
$$

This is the Poisson bracket associated to $\omega$. In the real symplectic case we may take Fun $(M)=C^{\infty}(M)$. In the complex symplectic case we should take the holomorphic functions, or better (since there may not be many global holomorphic functions), the
sheaf of holomorphic functions. In this setting the Poisson bracket provides, for any open subset $U \subset M$, an operation

$$
\{\cdot, \cdot\}: \operatorname{Fun}(U) \otimes \operatorname{Fun}(U) \rightarrow \operatorname{Fun}(U)
$$

where $F u n(U)$ denotes the holomorphic functions on $U$. (Since the proofs are the same we will generally omit this extra level of complexity from the notation/statements.) The main properties of the Poisson bracket are as follows.
Lemma 3.12. 0) $\{f, g\}=-\{g, f\}$,

1) $v_{f}=\{f, \cdot\}$ as derivations acting on functions,
2) The map $f \rightarrow v_{f}$ from functions to vector fields satisfies $v_{\{f, g\}}=\left[v_{f}, v_{g}\right]$,
3) The Poisson bracket makes the functions Fun $(M)$ on $M$ into a Lie algebra (and Fun(•) in to a sheaf of Lie algebras).

Proof. 0) is clear, and 1) is immediate since:

$$
v_{f}(g)=\left\langle v_{f}, d g\right\rangle=\left\langle v_{f}, \omega\left(\cdot, v_{g}\right)\right\rangle=\omega\left(v_{f}, v_{g}\right)=\{f, g\}
$$

For 2) we need to show $\omega\left(\cdot,\left[v_{f}, v_{g}\right]\right)=d\{f, g\}$, i.e. that $\iota_{\left[v_{f}, v_{g}\right]} \omega=-d\{f, g\}$. However the standard formula $\left[\mathcal{L}_{X}, \iota_{Y}\right]=\iota_{[X, Y]}$ combined with the Cartan formula $\mathcal{L}_{X}=$ $d \iota_{X}+\iota_{X} d$ for the Lie derivative acting on forms implies:

$$
\iota_{[X, Y]}=d \iota_{X} \iota_{Y}-\iota_{X} d \iota_{Y}-\iota_{Y} d \iota_{X}+\iota_{X} \iota_{Y} d
$$

as operators acting on forms for any vector fields $X, Y$. Thus in our situation we obtain

$$
\iota_{\left[v_{f}, v_{g}\right]} \omega=d\left(\omega\left(v_{g}, v_{f}\right)\right)=-d\{f, g\}
$$

as required (since the other terms vanish).
For 3) we need just to check the Jacobi identity. This now follows directly from 1) and 2):
$\{\{f, g\}, h\}=v_{\{f, g\}}(h)=\left[v_{f}, v_{g}\right](h)=v_{f}\left(v_{g}(h)\right)-v_{g}\left(v_{f}(h)\right)=\{f,\{g, h\}\}-\{g,\{f, h\}\}$.

Thus the map from function to vector fields $f \mapsto v_{f}$ is a Lie algebra homomorphism. Note that the Poisson bracket is only bilinear over the base field $(\mathbb{R}$ or $\mathbb{C})$; indeed 1 ) here implies that $\{f, g h\}=\{f, h\} g+\{f, g\} h$.

## Lie group actions on symplectic manifolds.

Suppose a Lie group $G$ acts on a manifold $M$. Let $\mathfrak{g}$ denote the Lie algebra of $G$. For any $X \in \mathfrak{g}$ we denote by $v_{X}$ the fundamental vector field of $X$. By definition this is obtained by taking (minus) the tangent vector to the flow on $M$ generated by $X$.

Explicitly $X$ determines a one-parameter subgroup $e^{X t}$ of $G$, and we act with this on M:

$$
m \mapsto e^{X t} \cdot m
$$

The derivative of this at $t=0$ is a vector field on $M$, and (by convention) we define $v_{X}$ to be minus this vector field:

$$
v_{X}=-\left.\frac{d}{d t}\left(e^{X t} \cdot m\right)\right|_{t=0} .
$$

The reason for the sign added here is as follows:
Lemma 3.13. The map $\mathfrak{g} \rightarrow \mathcal{X}(M) ; X \mapsto v_{X}$ from the Lie algebra of $\mathfrak{g}$ to the set of global vector fields on $M$ is a Lie algebra homomorphism. In other words $\left[v_{X}, v_{Y}\right]=v_{[X, Y]}$ for all $X, Y \in \mathfrak{g}$.

Sketch. Suppose $X \in \mathfrak{g}$ and $g \in G$. Then the action of $G$ on $M$ enables us to view $g$ as an automorphism of $M$, and the induced action on vector fields will be denoted $g \cdot v$ (for $v$ a vector field on $M$ ). First one may show directly that

$$
\begin{equation*}
v_{\operatorname{Ad}_{g}(X)}=g \cdot v_{X} \tag{3.3}
\end{equation*}
$$

(this is basically the chain rule). Then, from the definition of the Lie bracket of vector fields, we have:

$$
\begin{array}{rlr}
{\left[v_{X}, v_{Y}\right]} & =\left.\frac{d}{d t}\right|_{t=0} g \cdot v_{Y} \quad \text { where } g=e^{X t} \\
& =\left.\frac{d}{d t}\right|_{t=0} v_{\operatorname{Ad}_{g}(Y)} & \text { by }(3.3) \\
& =v_{[X, Y]}
\end{array}
$$

as required. The point is that there is a minus sign in the definition of the Lie bracket of two vector fields $v, w$ :

$$
[v, w]_{m}=\mathcal{L}_{v}(w)_{m}=\left.\frac{d}{d t}\right|_{t=0} d \Phi_{-t}\left(w_{\Phi_{t}(m)}\right)
$$

(where $\Phi_{t}$ is the (local) flow of $v$ and $m \in M$ ). We put a minus sign in the definition of $v_{X}$ so that $\Phi_{-t}$ is the action of $e^{X t}$ when $v=v_{X}$.

On the other hand recall that given a function $f$ on a symplectic manifold $(M, \omega)$, we obtain a Hamiltonian vector field $v_{f}$ which preserves the symplectic form.

Now suppose a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ preserving the symplectic form. The role of a moment map is to combine the above two situations, i.e. we would like a collection of (Hamiltonian) functions, one for each element of the Lie algebra of $G$, whose Hamiltonian vector fields are the corresponding fundamental vector fields. This is encompassed by the following definition.

Definition 3.14. A moment map for the $G$ action on $(M, \omega)$ is a $G$-equivariant map

$$
\mu: M \rightarrow \mathfrak{g}^{*}
$$

from $M$ to the dual of the Lie algebra of $G$, such that:

$$
d\langle\mu, X\rangle=\omega\left(\cdot, v_{X}\right) \quad \text { for all } X \in \mathfrak{g}
$$

Here $G$ acts on $\mathfrak{g}^{*}$ by the coadjoint action.
Thus for any $X \in \mathfrak{g}$ we obtain a function $\mu^{X}:=\langle\mu, X\rangle$ on $M$, the $X$-component of $\mu$, and we demand that this has Hamiltonian vector field equal to the fundamental vector field $v_{X}$ of $X$.

Said differently the Poisson bracket makes the functions $F u n(M)$ on $M$ into a Lie algebra. Taking Hamiltonian vector fields yields a Lie algebra map $F u n(M) \rightarrow \mathcal{X}(M)$ to the vector fields on $M$. The moment map provides a lift of the map $\mathfrak{g} \rightarrow \mathcal{X}(M)$ to a Lie algebra morphism $\mathfrak{g} \rightarrow \operatorname{Fun}(M)$.

We will see the notion of moment map is crucial for many reasons (in particular for constructing hyperkähler manifolds).

Remark 3.15 (Shifting moment maps). Note that if $\mu: M \rightarrow \mathfrak{g}^{*}$ is a moment map and we choose any element $\lambda \in \mathfrak{g}^{*}$ which is preserved by the coadjoint action (i.e. $\operatorname{Ad}_{g}^{*}(\lambda)=\lambda$ for all $\left.g \in G\right)$, then the map $\mu-\lambda$ :

$$
m \mapsto \mu(m)-\lambda
$$

is also a moment map, since it is still equivariant and has the same derivative.
Exercise 3.16. Suppose $G$ acts on $(M, \omega)$ with moment map $\mu$, and that $H \subset G$ is a Lie subgroup. Thus the derivative at the identity of the inclusion is a map $\mathfrak{h} \rightarrow \mathfrak{g}$, and the dual linear map is a map $\pi: \mathfrak{g}^{*} \rightarrow \mathfrak{h}^{*}$. Show that $\pi \circ \mu$ is a moment map for the action of the subgroup $H$ on $M$.

Exercise 3.17. Suppose $G$ acts on two symplectic manifold $\left(M_{i}, \omega_{i}\right)$ with moment maps $\mu_{i}$, for $i=1,2$. Show that

$$
\begin{equation*}
\mu_{1}+\mu_{2}: M_{1} \times M_{2} \rightarrow \mathfrak{g}^{*} ;\left(m_{1}, m_{2}\right) \mapsto \mu_{1}\left(m_{1}\right)+\mu_{2}\left(m_{2}\right) \tag{3.4}
\end{equation*}
$$

is a moment map for the diagonal action of $G$ on the product $M_{1} \times M_{2}$, defined by $g\left(m_{1}, m_{2}\right)=\left(g \cdot m_{1}, g \cdot m_{2}\right)$.

## Examples of moment maps.

Lemma 3.18. Let $\mathcal{O} \subset \mathfrak{g}^{*}$ be a coadjoint orbit of a Lie group $G$. Let $\mu$ be inclusion map

$$
\mu: \mathcal{O} \rightarrow \mathfrak{g}^{*}
$$

Then $\mu$ is a moment map for the coadjoint action of $G$ on $\mathcal{O}$.
Proof. This was established in the proof of Theorem 3.6. In any case it is a straightforward (if confusing) unwinding of the definitions: given $X \in \mathfrak{g}$ we must show $d\langle\mu, X\rangle=\omega_{\alpha}\left(\cdot, v_{X}\right)$ at each point $\alpha \in \mathcal{O}$. In other words

$$
\langle\beta, d\langle\mu, X\rangle\rangle=\omega_{\alpha}\left(\beta, v_{X}\right)
$$

for any $\beta \in T_{\alpha} \mathcal{O} \subset \mathfrak{g}^{*}$. Since $\mu$ is the inclusion, the left-hand side is just $\beta(X)$. On the other hand $v_{X}=-\operatorname{ad}_{X}^{*}(\alpha)$ and so by definition the right-hand side is

$$
\omega_{\alpha}\left(\operatorname{ad}_{X}^{*}(\alpha), \beta\right)=\langle\alpha,[X, Y]\rangle
$$

for any $Y \in \mathfrak{g}$ such that $\beta=\operatorname{ad}_{Y}^{*}(\alpha)$. Now $\langle\alpha,[X, Y]\rangle=-\left\langle\alpha, \operatorname{ad}_{Y}(X)\right\rangle=\left\langle\operatorname{ad}_{Y}^{*}(\alpha), X\right\rangle=$ $\beta(X)$ as required.

Lemma 3.19. Let $(V, \omega)$ be a symplectic vector space and let $G=\operatorname{Sp}(V)$ be the group of linear automorphisms of $V$ preserving the symplectic form. Then the map

$$
\mu: V \rightarrow \mathfrak{g}^{*} ; \quad v \mapsto " A \mapsto \frac{1}{2} \omega(A v, v) "
$$

is a moment map for the action of $G=\operatorname{Sp}(V)$ on $V$.
Proof. Given $A \in \mathfrak{g}$ we should verify that $d\langle\mu, A\rangle=\omega\left(\cdot, v_{A}\right)$. Since $\omega$ is closed the left hand side is

$$
\frac{1}{2}(\omega(A d v, v)+\omega(A v, d v))=\omega(A v, d v)
$$

since $A$ is in the Lie algebra of $\operatorname{Sp}(V)$. Thus we must show $\omega(A v, w)=\omega\left(w, v_{A}\right)$ for any $w \in V$. Now at $v \in V$ the vector field $v_{A}$ takes the value $-A v$ (which is minus tangent to the flow), so the result follows by skew-symmetry.

Example 3.20. The name "moment map" (or momentum map), comes from the following example. Suppose $G$ acts on a manifold $N$. Then there is an induced action on the symplectic manifold $M=T^{*} N$, the cotangent lift of the action on $N$. The action of $G$ on $M$ is then Hamiltonian and the moment map is given by pairing the momentum (the fibre coordinates in $M \rightarrow N$ ) with the fundamental vector field for the $G$ action on $N$. More precisely:
Lemma 3.21. The map

$$
\mu: M=T^{*} N \rightarrow \mathfrak{g}^{*} ; \quad(p, x) \mapsto " X \mapsto\left\langle p, v_{X}^{N}\right\rangle "
$$

is a moment map for the action of $G$ on $M$, where $x \in N, p \in T_{x}^{*} N$ and $v_{X}^{N} \in \Gamma(T N)$ is the fundamental vector field for the action of $G$ on $N$.

Proof. More generally suppose we are in the situation where the symplectic form $\omega$ is exact: $\omega=d \theta$, and the $G$ action preserves $\theta$ (so $\mathcal{L}_{v_{X}} \theta=0$ for all $X \in \mathfrak{g}$ ). Then consider the map $\mu: M \rightarrow \mathfrak{g}^{*}$ defined by

$$
m \mapsto " X \mapsto\left\langle\theta, v_{X}\right\rangle_{m} "
$$

where $v_{X}$ is the fundamental vector field on $M$. This is a moment map since, by Cartan's formula

$$
\iota_{v_{X}} \omega=\iota_{v_{X}} d \theta=\mathcal{L}_{v_{X}} \theta-d \iota_{v_{X}} \theta=-d\langle\mu, X\rangle .
$$

(See e.g. [LM87] p. 192 for proof of equivariance, omitted here.) Now in our situation, where $\theta=\sum p_{i} d x_{i}$ is the Liouville form, we must check this definition of $\mu$ coincides with that above, i.e. that

$$
\left\langle\theta, v_{X}\right\rangle_{(p, x)}=\left\langle p, v_{X}^{N}\right\rangle_{x}
$$

where $x \in N, p \in T_{x}^{*} N$. This however is immediate by definition of $\theta$, and the fact that $v_{X}$ is a lift of $v_{X}^{N}$. ( $\theta$ the projection of $v_{X}$ to $N$, and pairs it with $p$.)

For example let $V$ be a complex vector space, and take $G=\mathrm{GL}(V)$ acting on $N=V$ in the natural way. Then $M=T^{*} V \cong V^{*} \times V$ and the moment map is $\mu(\alpha, v)(X)=-\alpha(X v)$ where $\alpha \in V^{*}, X \in \mathfrak{g}=\operatorname{End}(V)$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^{*}$ using the pairing $\operatorname{Tr}(A B)$ then we have:

$$
\begin{equation*}
\mu(\alpha, v)=-v \otimes \alpha \in \operatorname{End}(V) \tag{3.5}
\end{equation*}
$$

Similarly if $G=\mathrm{GL}(V)$ acts on $N=\operatorname{End}(V)$ by conjugation. Then $M=$ $T^{*} \operatorname{End}(V) \cong \operatorname{End}(V) \times \operatorname{End}(V)$ and the moment map is

$$
\begin{equation*}
\mu\left(B_{1}, B_{2}\right)(X)=-\operatorname{Tr}\left(B_{1}\left[X, B_{2}\right]\right)=\operatorname{Tr}\left(X\left[B_{1}, B_{2}\right]\right) \tag{3.6}
\end{equation*}
$$

where $B_{i} \in \operatorname{End}(V)\left(B_{2} \in N, B_{1} \in T_{B_{2}}^{*} \operatorname{End}(V)\right), X \in \mathfrak{g}=\operatorname{End}(V)$. If we identify $\mathfrak{g}$ with $\mathfrak{g}^{*}$ using the pairing $\operatorname{Tr}(A B)$ then we have:

$$
\mu\left(B_{1}, B_{2}\right)=\left[B_{1}, B_{2}\right] \in \operatorname{End}(V),
$$

i.e. the moment map is given by the commutator.

## Symplectic Quotient Construction.

The aim of this subsection is to define a way construct new symplectic manifolds from old ones. Note that in general the quotient of a symplectic manifold by a Lie group may not be even dimensional (e.g. $S^{2} / S^{1}$ is an interval, where the circle acts by rotation), so cannot be symplectic in general. The following construction shows how the use of the moment map guides the way to obtain symplectic manifolds.

First some linear algebra. Let $(V, \omega)$ be a symplectic vector space and $W \subset V$ a subspace. Define the symplectic orthogonal $W^{\perp} \subset V$ as

$$
W^{\perp}=\{v \in V \mid \omega(w, v)=0 \text { for all } w \in W\} \subset V
$$

and the annihilator $W^{\circ} \subset V^{*}$ as

$$
W^{\circ}=\left\{\alpha \in V^{*} \mid \alpha(w)=0 \text { for all } w \in W\right\} \subset V^{*}
$$

Let $\omega^{b}: V \rightarrow V^{*}$ be the linear map determined by $\omega$ :

$$
\omega^{b}(v)=\iota_{v} \omega=" u \mapsto \omega(v, u) "
$$

where $u, v \in V$. The following easy observation will be useful:
Lemma 3.22. Let $V$ be a symplectic vector space and $W \subset V$ a subspace. Then

$$
\omega^{b}(W)=\left(W^{\perp}\right)^{\circ}
$$

as subspaces of $V^{*}$.
Proof. Clearly $\omega^{b}$ is bijective, and if $\alpha=\omega^{b}(w)$ for some $w \in W$, then $\alpha(u)=$ $\omega(w, u)=0$ for any $u \in W^{\perp}$. Thus $\omega^{b}(W) \subset\left(W^{\perp}\right)^{\circ}$, so they must be equal as they have the same dimension.

The basic idea of symplectic quotients is as follows. Suppose a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ and this action admits a moment map $\mu$. Let

$$
Z=\mu^{-1}(0)=\{m \in M \mid \mu(m)=0\}
$$

be the inverse image of zero under the moment map. Since the moment map is equivariant and zero is preserved by the coadjoint action, then $Z$ is $G$ invariant and we can consider the quotient

$$
M / / G:=Z / G=\mu^{-1}(0) / G
$$

The point is that under some mild conditions this is again a symplectic manifold, the symplectic quotient (or Marsden-Weinstein quotient) of $M$ by $G$.

First we will explain how the quotient inherits a symplectic form. Let us denote by $\iota$ the inclusion $\iota: Z \hookrightarrow M$ and by $\pi$ the projection $\pi: Z \rightarrow M / / G$.

Choose a point $m \in Z$. If $Z$ is a submanifold of $M$ then the tangent space to $Z$ is

$$
\operatorname{Ker}\left(d \mu_{m}\right)=T_{m} Z \subset T_{m} M
$$

Moreover the $G$-orbit through $m$ is within $Z$, so we can consider the tangent space to the orbit. Since the action is Hamiltonian this tangent space is simply the span of the Hamiltonian vector fields at $m$. We will denote it by $\mathfrak{g}_{m}$ :

$$
\mathfrak{g}_{m}=\left\{v_{X} \mid X \in G\right\} \subset T_{m} Z \subset T_{m} M
$$

Now the quotient $Z / G$ parameterises the set of $G$ orbits in $Z$, so if it is a manifold (and we have a local slice to the action) then its tangent space at $\pi(m)$ will be isomorphic to the quotient $T_{m} Z / \mathfrak{g}_{m}$. Thus the main step is to show that this vector space inherits a symplectic form from that on $T_{m} M$. Clearly we can restrict $\omega$ to $T_{m} Z$, where it will become degenerate. We need to show that the subspace we quotient by ( $\mathfrak{g}_{m}$ ) coincides with the degenerate directions, i.e. that

Proposition 3.23. $\mathfrak{g}_{m}=\left(\operatorname{Ker} d \mu_{m}\right)^{\perp}$.
Proof. This is equivalent to showing $\mathfrak{g}_{m}^{\perp}=\operatorname{Ker} d \mu_{m}$, or in turn that $\left(\mathfrak{g}_{m}^{\perp}\right)^{\circ}=$ $\left(\operatorname{Ker} d \mu_{m}\right)^{\circ}$. Now by definition the dual linear map to the map $d \mu_{m}: T_{m} M \rightarrow$ $T_{0} \mathfrak{g}^{*}=\mathfrak{g}^{*}$ is the map $\mathfrak{g} \rightarrow T_{m}^{*} M ; X \mapsto \omega\left(\cdot, v_{X}\right)$. In general the kernel of a linear map is the annihilator of the image of the dual map, so (taking annihilators) we have

$$
\begin{aligned}
\left(\operatorname{Ker} d \mu_{m}\right)^{\circ} & =\left\{\omega\left(\cdot, v_{X}\right) \mid X \in \mathfrak{g}\right\} \\
& =\left\{\omega(\cdot, v) \mid v \in \mathfrak{g}_{m}\right\} \\
& =\omega^{b}\left(\mathfrak{g}_{m}\right) \subset T_{m}^{*} M
\end{aligned}
$$

Now we use Lemma 3.22 (with $W=\mathfrak{g}_{m}$ ) to see $\omega^{b}\left(\mathfrak{g}_{m}\right)=\left(\mathfrak{g}_{m}^{\perp}\right)^{\circ}$ and so the result follows.

Thus $T_{m} Z / \mathfrak{g}_{m}$ is a symplectic vector space. Moreover since $\omega$ is $G$ invariant, for any $g \in G$ the action of $G$ yields a symplectic isomorphism $T_{m} Z / \mathfrak{g}_{m} \cong T_{g(m)} Z / \mathfrak{g}_{g(m)}$ and so we get a well defined nondegenerate two-form $\bar{\omega}$ on $Z / G$. To see it is closed we argue as follows. It is defined such that

$$
\pi^{*} \bar{\omega}=\iota^{*} \omega
$$

Moreover $\pi^{*} d \bar{\omega}=d \pi^{*} \bar{\omega}=d \iota^{*} \omega=\iota^{*} d \omega=0$. But this forces $d \bar{\omega}=0$ since $\pi$ is surjective on tangent vectors.

This establishes the following:
Theorem 3.24. Suppose a Lie group $G$ acts on a symplectic manifold $(M, \omega)$ with moment map $\mu$ such that $Z=\mu^{-1}(0)$ is a smooth submanifold of $M$, and the quotient $Z / G$ is a smooth manifold. Then $M / / G:=Z / G=\mu^{-1}(0) / G$ is a symplectic manifold.

There are various conditions that may be added to ensure the various criteria are met, for example if $G$ is a compact group acting freely and 0 is a regular value of the moment map. Note in this case one has:

$$
\operatorname{dim}(M / / G)=\operatorname{dim}(M)-2 \operatorname{dim}(G)
$$

The same argument works in the complex symplectic category (with the words holomorphic/complex added throughout):

Theorem 3.25. Suppose a complex Lie group $G$ acts holomorphically on a complex symplectic manifold $\left(M, \omega_{\mathbb{C}}\right)$ with moment map $\mu_{\mathbb{C}}$ such that $Z=\mu_{\mathbb{C}}^{-1}(0)$ is a smooth complex submanifold of $M$, and the quotient $Z / G$ is a smooth complex manifold. Then $M / / G:=Z / G=\mu_{\mathbb{C}}^{-1}(0) / G$ is a complex symplectic manifold.

Remark 3.26. Recall from Remark 3.15 that the moment map is not uniquely determined: if $\mu: M \rightarrow \mathfrak{g}^{*}$ is a moment map then so is $\nu:=\mu-\lambda$ for any $\lambda \in \mathfrak{g}^{*}$ which is fixed by the coadjoint action. Now

$$
\nu^{-1}(0) / G=\mu^{-1}(\lambda) / G
$$

so we see that we may perform symplectic reduction at any invariant value $\lambda$ of the moment map and not necessarily at zero (provided the various conditions hold to ensure the quotient is smooth).

Remark 3.27. More generally we may perform "symplectic reduction at any coadjoint orbit", as follows. Let $\mathcal{O} \subset \mathfrak{g}^{*}$ be a coadjoint orbit. Let $\mathcal{O}^{-}$denote the orbit $\mathcal{O}$ with the symplectic form negated. Then

$$
\begin{aligned}
\mu^{-1}(\mathcal{O}) / G & =\{(m, x) \in M \times \mathcal{O} \mid \mu(m)=x\} / G \\
& =\{(m, x) \in M \times \mathcal{O} \mid \mu(m)+(-x)=0\} / G \\
& =\left(M \times \mathcal{O}^{-}\right) / / G
\end{aligned}
$$

is the symplectic quotient by $G$ of the product of $M$ and $\mathcal{O}^{-}$, so is again symplectic in general. (An invariant element $\lambda \in \mathfrak{g}^{*}$ is the simplest example of a coadjoint orbit-just a single point.)

Example 3.28. We can now obtain the Calogero-Moser spaces as complex symplectic quotients. Let $V=\mathbb{C}^{n}$ and consider $N=\operatorname{End}(V) \times V$, with the natural action of $G=\operatorname{GL}(V)$. Let $M=T^{*} N \cong \operatorname{End}(V) \times \operatorname{End}(V) \times V \times V^{*}$ be the total space of the holomorphic cotangent bundle of $N$ with its standard complex symplectic form. By (3.4), (3.5) and (3.6) this has moment map

$$
(X, Z, v, \alpha) \mapsto[X, Z]-v \otimes \alpha \in \operatorname{End}(V)
$$

Clearly the complex symplectic quotient $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id}) / G$ is the Calogero-Moser space $C_{n}$, which we thus see has a natural complex symplectic structure.

Exercise 3.29 (Calogero-Moser flows).

1) Compute the restriction of this symplectic form to the open part $C_{n}^{\prime}$ of $C_{n}$ where $X$ is diagonalisable (explicitly in terms of the eigenvalues of $X$ ). Show it agrees with the standard symplectic structure on $T^{*}\left(\left(\mathbb{C}^{n} \backslash\right.\right.$ diagonals $\left.) / \mathrm{Sym}_{n}\right)$.
2) Consider the function $H=\operatorname{Tr}\left(Z^{2}\right) / 2$ on $M$. Show (upto signs) that the flow of the corresponding Hamiltonian vector field on $M$ is given by

$$
\begin{equation*}
(X, Z, v, \alpha) \mapsto(X+t Z, Z, v, \alpha) \tag{3.7}
\end{equation*}
$$

for $t \in \mathbb{C}$. Observe that this flow commutes with the action of $G$ and preserves the level set $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id})$ of the moment map.
3) Show that the restriction of $H$ to $\mu_{\mathbb{C}}^{-1}(-\mathrm{Id})$ descends to a function on $C_{n}$, and on the open part $C_{n}^{\prime}$ equals

$$
H=\operatorname{Tr}\left(Z^{2}\right) / 2=\sum_{i} \frac{1}{2} p_{i}^{2}-\sum_{i<j}\left(x_{i}-x_{j}\right)^{2}
$$

(where $p_{i}, x_{i}$ are the coordinates on $C_{n}^{\prime}$ ). This is the Calogero-Moser Hamiltonian, modelling the flows of $n$ identical particles on the complex plane with an inverse square potential. Whilst on $C_{n}^{\prime}$ the flows are complicated and incomplete, we see by "unwinding" the symplectic quotient the flows become as in (3.7) very simple (and thus complete on the partial compactification $\left.C_{n}\right) .{ }^{2}$

## 4. Quick review of KÄhler geometry

We will quickly review (mostly without proof) some of the basics of Kähler geometry. This will be useful to highlight the analogies with the hyperkähler case later.

## Linear algebra.

A complex structure on a real vector space $V$ is an real linear endomorphism whose square is minus the identity: $\mathbf{I}: V \rightarrow V, \mathbf{I}^{2}=-1$. This enables to view $V$ as a complex vector space, with $i \in \mathbb{C}$ acting on $V$ as $\mathbf{I}$ (this entails that $V$ has even real dimension).

A Kähler vector space is a real vector space $V$ together with a real symplectic form $\omega \in \bigwedge^{2} V^{*}$ and a complex structure I such that:
a) the complex structure preserves the symplectic form

$$
\omega(\mathbf{I} v, \mathbf{I} w)=\omega(v, w)
$$

and
b) the associated real bilinear form $g$ defined by

$$
g(v, w):=\omega(v, \mathbf{I} w)
$$

is positive definite (i.e. a metric-it is necessarily symmetric due to the invariance and antisymmetry of $\omega$ ).

Now let $V$ be a finite dimensional complex vector space. Recall that a (positive definite) Hermitian form on $V$ associates a complex number $h(v, w)$ to a pair of

[^2]$v, w \in V$. It is $\mathbb{C}$-linear in the first slot and is such that
$$
h(v, w)=\overline{h(w, v)}
$$
for all $v, w \in V$, and such that $h(v, v)>0$ for all nonzero $v \in V$. Note that this implies $h(\mathbf{I} v, \mathbf{I} w)=h(v, w)$, i.e. the complex structure preserves the Hermitian form.

Note that the real part of a positive definite Hermitian form is a real metric (i.e. a positive definite real symmetric bilinear form):

$$
g(v, w):=\operatorname{Re} h(v, w)=\operatorname{Re} \overline{h(v, w)}=\operatorname{Re} h(w, v)=g(w, v)
$$

Also the imaginary part (and any nonzero real multiple of it) is a (real) symplectic form:

$$
\operatorname{Im} h(v, w)=-\operatorname{Im} \overline{h(v, w)}=-\operatorname{Im} h(w, v)
$$

To check this skew-form is nondegenerate note that

$$
\begin{equation*}
\operatorname{Im} h(v, w)=-\operatorname{Re}(i h(v, w))=-\operatorname{Re} h(\mathbf{I} v, w)=-g(\mathbf{I} v, w) \tag{4.1}
\end{equation*}
$$

so $\operatorname{Im} h(\mathbf{I} v, v)=g(v, v)$ is zero only if $v=0$.
Note that (4.1) also shows that specifying a positive definite Hermitian form is the same as specifying a real metric $g$ such that $g(\mathbf{I} v, \mathbf{I} w)=g(v, w)$. Such a real metric is called a Hermitian metric; this may cause some confusion since it is not a Hermitian form, however little confusion is possible since the corresponding Hermitian form $h$ is determined by $g$ (from (4.1): if $g=\operatorname{Re}(h)$ and $\omega(v, w)=g(\mathbf{I} v, w)$ then $\omega=-\operatorname{Im} h)$ :

$$
\begin{gathered}
h=g-\sqrt{-1} \omega \\
\omega(v, w)=g(\mathbf{I} v, w), \quad g(v, w)=\omega(v, \mathbf{I} w)
\end{gathered}
$$

Said differently a Kähler vector space is the same thing as a (positive definite) Hermitian vector space (i.e. a finite dimensional Hilbert space).

Standard/Example Formulae: $V=\mathbb{C}^{n}$, Real coordinates $x_{i}, y_{i}$ (in real dual), with derivatives $d x_{i}, d y_{i}$. Complex coordinates $z_{i}=x_{i}+\sqrt{-1} y_{i}$.

Standard symplectic form:

$$
\omega=\sum d x_{i} \wedge d y_{i}=\frac{\sqrt{-1}}{2} \sum d z_{i} \wedge d \bar{z}_{i}
$$

since $d z_{i} \wedge d \bar{z}_{i}=2 \sqrt{-1} d x_{i} \wedge d y_{i}$. Hermitian form

$$
h=\sum d z_{i} \otimes d \bar{z}_{i}
$$

so $h(v, w)=\sum v_{i} \bar{w}_{i}$ where $v_{i}=z_{i}(v), \bar{w}_{i}=\bar{z}_{i}(w)$. Now

$$
d z \otimes d \bar{z}=d x d x+d y d y+i d y d x-i d x d y
$$

so the real part of the Hermitian form is

$$
g=\sum d x_{i}^{2}+d y_{i}^{2}
$$

and the imaginary part is

$$
-\omega=\sum d y_{i} \wedge d x_{i}
$$

i.e. $\omega$ is minus the imaginary part of the Hermitian form, as expected.

## Some group theory I.

The unitary group $\mathrm{U}(n)$ is the subgroup of $\mathrm{GL}_{n}(\mathbb{C})$ preserving the standard positive definite Hermitian form on $\mathbb{C}^{n}$. This is equivalent to preserving both the real and imaginary parts, so we have

$$
\mathrm{U}(n)=\mathrm{O}_{2 n}(\mathbb{R}) \cap \mathrm{Sp}_{2 n}(\mathbb{R})
$$

(since the real part is the standard positive definite bilinear form on $\mathbb{R}^{2 n}$ and the imaginary part is the standard real symplectic form, and moreover these parts determine the complex structure). Note that the real symplectic group $\mathrm{Sp}_{2 n}(\mathbb{R})$ is not compact, e.g for $n=1$ it is just $\mathrm{SL}_{2}(\mathbb{R})$ (since a symplectic form on a real two dimensional vector space is just a volume form). Do not confuse it with the compact group $\operatorname{Sp}(n)$, the "quaternionic unitary group" to be defined below- they are two different real forms of the complex symplectic group $\mathrm{Sp}_{2 n}(\mathbb{C})$.

Since the real symplectic form is preserved, a real volume form is preserved (an orientation-here the top exterior power of the symplectic form), i.e. $\operatorname{Sp}_{2 n}(\mathbb{R}) \subset$ $\mathrm{SL}_{2 n}(\mathbb{R})$, so in fact

$$
\mathrm{U}(n) \subset \mathrm{SO}_{2 n}(\mathbb{R})
$$

The complexification of this fact will be important below.

## Kähler manifolds.

Recall that an almost-complex structure on a real manifold $M$ is an endomorphism $\mathbf{I} \in \operatorname{End}(T M)$ of the tangent bundle such that $\mathbf{I}^{2}=-1$. (Thus each tangent space is a complex vector space via the action of $\mathbf{I}$.) A complex structure on $M$ is an integrable almost-complex structure, i.e. $M$ is a complex manifold (we can find local I-holomorphic coordinates etc.)

A Kähler manifold is a complex manifold $M$ together with a (real) symplectic form $\omega$ on $M$ which is compatible with the complex structure and induces a metric: if $\mathbf{I} \in \operatorname{End}(T M)$ is the complex structure, then

$$
\omega(\mathbf{I} v, \mathbf{I} w)=\omega(v, w)
$$

and such that the associated real (symmetric) bilinear form

$$
g(v, w)=\omega(v, \mathbf{I} w)
$$

is positive definite, at each point of $M$. (One may easily check that the skew-symmetry of $\omega$ and the invariance of $\omega$ under I implies the symmetry of $g$.) Thus $g$ is a Riemannian metric and is invariant under $\mathbf{I}$.

Note that Theorem 4.3 on p. 148 of [KN69] (combined with the Newlander-Nirenberg theorem) says that:

Theorem 4.1. If $M$ is a Riemannian manifold with metric $g$ and a global section $\mathbf{I} \in$ $\Gamma(\operatorname{End}(T M))$ such that $\mathbf{I}^{2}=-1, g(\mathbf{I} u, \mathbf{I} v)=g(u, v)$ then the following are equivalent:
a) $\nabla \mathbf{I}=0$,
b) $\mathbf{I}$ is an integrable complex structure (making $M$ into a complex manifold) and the associated two form $\omega$ (defined by $\omega(\cdot, \cdot)=g(\mathbf{I} \cdot, \cdot)$ ) is closed.

If either of these hold then $M$ is a Kähler manifold.
Corollary 4.2. Let $M$ be a complex manifold with complex structure $\mathbf{I} \in \operatorname{End}(T M)$ and with a Hermitian metric $g$. Then $g$ is Kähler (i.e. the associated $(1,1)$-form $\omega$ is closed) iff the complex structure $\mathbf{I}$ is parallel for the Levi-Civita connection.

Kähler manifolds arise in abundance since complex submanifolds of Kähler manifolds are Kähler:

Proposition 4.3. Let $M$ be a Kähler manifold with Kähler two-form $\omega$ and let $N$ be a complex manifold. Suppose that

$$
f: N \rightarrow M
$$

is a complex immersion (holomorphic map injective on tangent vectors). Then $N$ is a Kähler manifold with two-form $f^{*} \omega$.

Proof. Clearly $f^{*} \omega$ is closed. The holomorphicity of $f$ means that $d f_{*}$ intertwines the complex structures. The corresponding symmetric bilinear form is $g_{N}(v, w)=$ $\left(f^{*} \omega\right)(v, \mathbf{I} w)=\omega\left(d f_{*} v, d f_{*}(\mathbf{I} w)\right)$ and this equals $\omega\left(d f_{*} v, \mathbf{I} d f_{*}(w)\right)=g_{M}\left(d f_{*} v, d f_{*} w\right)=$ $f^{*}\left(g_{M}\right)(v, w)$ since $f$ is holomorphic. Thus $g_{N}=f^{*} g_{M}$ is positive definite since $f$ is an immersion. This also shows $f^{*} \omega$ is nondegenerate (by setting $w=v$ for example). Again since $f$ is holomorphic $g_{N}$ is preserved by the complex structure on $N$.

## Kähler quotients.

The main result is the following

Theorem 4.4 (see [HKLR87]). If $M / / G$ is a symplectic manifold obtained as the (real) symplectic quotient of a Kähler manifold $M$, by the action of a group $G$ that preserves the Kähler structure on $M$, then $M / / G$ is also naturally Kähler.

The proof is similar to the hyperkähler case we will give in detail below. Let us discuss some aspects of this situation which will be helpful later. Suppose $G$ is a compact group acting freely and 0 is a regular value of $\mu$ so that $Z=\mu^{-1}(0)$ is a smooth submanifold of $M$. Choose a point $m \in Z$, and write $\pi: Z \rightarrow N:=Z / G=$ $M / / G$ for the quotient. Thus the tangents to the action (the span of the fundamental vector fields at $m$ ) $\mathfrak{g}_{m}$ is a subspace of $T_{m} Z$ isomorphic to $\mathfrak{g}$, via the map $X \mapsto\left(v_{X}\right)_{m}$. Since we now have a metric, we can define the orthogonal subspace $H_{m}=\mathfrak{g}_{m}^{\perp} \subset T_{m} Z$ to $\mathfrak{g}_{m}$ (using the metric, not the symplectic form). Thus

$$
T_{m} Z=H_{m} \oplus \mathfrak{g}_{m}
$$

and the projection $\pi$ identifies $H_{m}$ with the tangent space $T_{\pi(m)} N$ of the quotient. On the other hand using the metric we can consider the normal directions to $Z$ in $M$ : these arise as the gradient vector fields $\operatorname{grad} \mu^{X}$ of components $\mu^{X}$ of $\mu$ (for $X \in \mathfrak{g}$ ). If $Y$ is any vector field, since $\mu$ is a moment map, we have

$$
g\left(\operatorname{grad} \mu^{X}, Y\right)=d \mu^{X}(Y)=-\omega\left(v_{X}, Y\right)=-g\left(\mathbf{I} v_{X}, Y\right)
$$

so $\operatorname{grad} \mu^{X}=-\mathbf{I} v_{X}$; the gradients are obtained by rotating the fundamental vector fields by $-\mathbf{I}$. Thus $\mathbf{I g}_{m}$ is the set of normal directions to $T_{m} Z$ in $T_{m} M$, and we have a decomposition:

$$
T_{m} M=H_{m} \oplus \mathfrak{g}_{m} \oplus \mathbf{I g}_{m}
$$

Clearly the last two factors make up a complex vector space $\cong \mathfrak{g}_{\mathbb{C}}=\mathfrak{g} \otimes \mathbb{C}$ and in particular $H_{m}$ is identified with the quotient of two complex vector spaces $T_{m} M / \mathfrak{g}_{\mathbb{C}}$ so is complex itself. [It follows that the metric on $N$ (defined by restricting from $T_{m} M$ to $H_{m}$ ) is Kähler since the Levi-Civita connection is obtained by orthogonal projection to $H_{m}$, and this commutes with the complex structures.] The point we wish to emphasize here is that $H_{m}$ arises by quotienting $T_{m} M$ by $\mathfrak{g}_{m} \otimes \mathbb{C}$. Thus $N$ appears as a quotient by a local (Lie algebra) action of $G_{\mathbb{C}}$ (the complexification of $G)$. If this comes from an action of $G_{\mathbb{C}}$ then we expect that

$$
N=M^{s s} / G_{\mathbb{C}}
$$

where $M^{s s} \subset M$ is the union $G_{\mathbb{C}} \cdot Z$ of the $G_{\mathbb{C}}$ orbits in $M$ which meet $Z$. In other words we obtain an alternate viewpoint on the Kähler quotient as the quotient of an (open) subset $M^{s s} \subset M$ by the complexified group. This will be made precise when we discuss GIT quotients later. For now we will give a simple example.

The simplest examples come from Kähler vector spaces. Let $V$ be a complex vector space with a positive definite Hermitian form $(\cdot, \cdot)$. Let $\omega$ be minus the imaginary part of this Hermitian form. Let $G=\mathrm{U}(V)$ be the group of linear automorphisms
of $V$ preserving the Hermitian form. Clearly this is a subgroup of the group of symplectic automorphisms of $V$. The Lie algebra $\mathfrak{g}$ of $G$ consists of the skew-Hermitian endomorphisms, i.e. those satisfying

$$
(A v, w)+(v, A w)=0
$$

for all $v, w \in V$.
Lemma 4.5. The map

$$
\mu: V \rightarrow \mathfrak{g}^{*} ; \quad v \mapsto " A \mapsto \frac{i}{2}(A v, v) "
$$

is a moment map for the action of $G=\mathrm{U}(V)$ on $V$.
Proof. This is immediate from Lemma 3.19 and Exercise 3.16. Here is the direct verification anyway: First note that $i(A v, v)$ is indeed real, since $A$ satisfies $(A v, v)+$ $(v, A v)=0$ and $(v, A v)=\overline{(A v, v)}$ so $(A v, v)$ is pure imaginary. Thus $\mu$ indeed maps $V$ to the real dual of $\mathfrak{g}$. Given $A \in \mathfrak{g}$ we should verify that $d\langle\mu, A\rangle=\omega\left(\cdot, v_{A}\right)$. The left hand side is $\frac{i}{2}((A d v, v)+(A v, d v))$, so we must show

$$
\frac{i}{2}((A w, v)+(A v, w))=\omega\left(w, v_{A}\right)
$$

for any $w \in V$. Now at $v \in V$ the vector field $v_{A}$ takes the value $-A v$ (which is minus tangent to the flow), so the right hand side is: $\operatorname{Im}(w, A v)=\frac{1}{2 i}((w, A v)-(A v, w))$ which indeed equals the left-hand side, since $A$ is skew Hermitian.

Note that if we use the standard Hermitian structure $h(v, w)=v^{T} \bar{w}$ on $V=\mathbb{C}^{n}$ and use the pairing $(A, B) \mapsto \operatorname{Tr}(A B)$ to identify $\mathfrak{g}$ with its dual, then the moment map is given by

$$
\mu: V \rightarrow \mathfrak{g} ; \quad \mu(v)=\frac{i}{2} v \otimes v^{\dagger}
$$

(where $\dagger$ dentoes the conjugate transpose) since $\frac{i}{2} \operatorname{Tr}\left(A v \otimes v^{\dagger}\right)=\frac{i}{2} v^{\dagger} A v=\frac{i}{2} v^{T} A^{T} \bar{v}=$ $\frac{i}{2}(A v, v)$.

Example 4.6. Take $M=\mathbb{C}^{n}$ with its standard Kähler structure. Let $S^{1} \subset U(n)$ be the circle subgroup of diagonal scalar unitary matrices, acting on $M$. Since $U(n)$ has moment map $v \mapsto \frac{i}{2} v \otimes v^{\dagger}$, and the dual of the derivative of the inclusion $S^{1} \subset U(n)$ is given by the trace, this action has moment map $v \in M \mapsto \frac{i}{2}\|v\|^{2}$. Now since the coadjoint action of $S^{1}$ is trivial we may perform reduction at any value of the moment map (all coadjoint orbits are points). Taking the value $i / 2$ we find

$$
\mu^{-1}(i / 2)=\{v \in M \mid\|v\|=1\}
$$

is the unit sphere in $\mathbb{C}^{n}$, and the symplectic quotient is

$$
M / / S^{1}=\mu^{-1}(i / 2) / S^{1} \cong \mathbb{P}^{n-1}
$$

the complex projective space of dimension $n-1$, since each complex one dimensional subspace of $\mathbb{C}^{n}$ intersects the sphere in a circle, and this circle is an $S^{1}$ orbit. Thus the projective space inherits a Kähler structure. This is the (well-known) Fubini-Study Kähler structure; the key point here is that we have obtained a nontrivial Kähler structure from the standard Kähler structure on a vector space. In this example the complexified group $G_{\mathbb{C}}=\mathbb{C}^{*}$ does indeed act holomorphically (by scalar multiplication), and $M^{s s}$, the union of the $\mathbb{C}^{*}$ orbits which meet $\mu^{-1}(i / 2)$, is just $V \backslash\{0\}$. Thus in this case it is easy to see

$$
M / / S^{1}=M^{s s} / \mathbb{C}^{*}=(V \backslash\{0\}) / \mathbb{C}^{*}
$$

which is closer to the usual description of the projective space as the space of one dimensional subspaces of $V$.

Let us record another example of moment map for later use:
Example 4.7. Take $V=\operatorname{End}\left(\mathbb{C}^{n}\right)$ with the Hermitian form $(X, Y)=\operatorname{Tr}\left(X Y^{\dagger}\right)$, with $G=\mathrm{U}(n)$ acting by conjugation (we are thus embedding $\mathrm{U}(n)$ in $\mathrm{U}\left(n^{2}\right)$ ). Then a moment map $\mu: V \rightarrow \mathfrak{g}$ is given by

$$
\mu(X)=\frac{i}{2}\left[X, X^{\dagger}\right]
$$

(where again we use $(A, B) \mapsto \operatorname{Tr}(A B)$ to identify $\mathfrak{g}$ with its dual) since for $X \in$ $V, A \in \mathfrak{g}$, the adjoint action of $A$ on $X$ is $[A, X]$, so we just observe that

$$
\mu(X)(A)=\frac{i}{2} \operatorname{Tr}\left(A\left[X, X^{\dagger}\right]\right)=\frac{i}{2} \operatorname{Tr}\left([A, X] X^{\dagger}\right)=\frac{i}{2}([A, X], X)
$$

as in Lemma 4.5.

## 5. Quaternions and hyperkähler vector spaces

## Quaternions.

The quaternions are the real (noncommutative) algebra

$$
\mathbb{H}=\left\{q=x_{0}+x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k} \mid x_{i} \in \mathbb{R}\right\} \cong \mathbb{R}^{4}
$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the quaternion identities:

$$
\mathrm{i}^{2}=\mathrm{j}^{2}=\mathrm{k}^{2}=\mathbf{i j k}=-1
$$

Thus each generator $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is a square root of -1 , and we have $\mathbf{i j}=\mathbf{k}=-\mathbf{j} \mathbf{i}, \mathbf{j} \mathbf{k}=$ $\mathbf{i}, \mathbf{k i}=\mathbf{j}$ etc. Given a quaternion $q \in \mathbb{H}$ we define its real part to be $x_{0}$ and its imaginary part to be $x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k}$, so that

$$
\operatorname{Im} \mathbb{H}=\left\{x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k} \mid x_{i} \in \mathbb{R}\right\} \cong \mathbb{R}^{3} .
$$

An important property is the existence of a conjugation on $\mathbb{H}$ :

$$
\bar{q}=q-2 \operatorname{Im} q, \quad \overline{x_{0}+x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k}}=x_{0}-x_{1} \mathbf{i}-x_{2} \mathbf{j}-x_{3} \mathbf{k}
$$

negating the imaginary part. Thus $q$ is real iff $q=\bar{q}$. Moreover the conjugation satisfies $\overline{(p q)}=(\bar{q})(\bar{p})$ so that $q \bar{q}$ is real. This enables us to define a norm:

$$
|q|^{2}=q \bar{q}=x_{0}^{2}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2} .
$$

The nonzero quaternions are precisely those with nonzero norm which in turn are precisely those with a multiplicative inverse $q^{-1}=\bar{q} /|q|^{2}$. They constitute a four dimensional Lie group

$$
\mathrm{GL}_{1}(\mathbb{H})=\{q \in \mathbb{H} \mid q \neq 0\}=\mathbb{H}^{*} .
$$

The unit quaternions (those of norm one) clearly constitute a three sphere, and moreover they form a subgroup:

$$
\operatorname{Sp}(1)=\{q \in \mathbb{H}| | q \mid=1\} \cong \operatorname{SU}(2) \cong S^{3} .
$$

Note that the generators $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of $\mathbb{H}$ are not really distinguished: there are lots of ordered triples of mutually orthogonal elements of norm one with the same algebraic properties: Their choice amounts to choosing an orthonormal basis of $\operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}$; the set of such choices is a torsor for $\mathrm{SO}_{3}(\mathbb{R}) \cong \mathrm{Sp}(1) /\{ \pm 1\}$. Indeed any such triple is of the form

$$
q(\mathbf{i}, \mathbf{j}, \mathbf{k}) q^{-1}=\left(q \mathbf{i} q^{-1}, q \mathbf{j} q^{-1}, q \mathbf{k} q^{-1}\right)
$$

for some unit quaternion $q$ and the stabiliser of $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is $\{ \pm 1\} \subset \operatorname{Sp}(1)$. In particular any element of the form $q \mathbf{i} q^{-1}$ is a square root of -1 ; the set of such constitutes the orbit of $\mathrm{SO}_{3}(\mathbb{R})$ through (the nonzero vector) $\mathbf{i} \in \mathbb{R}^{3}$, so is a two-sphere:

$$
\left\{x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k} \mid x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=1\right\} \subset \operatorname{Im} \mathbb{H}
$$

In the sequel this will be the two-sphere of complex structures on a hyperkähler manifold (and in general they will not all be equivalent).

If we use the norm to define the constant flat metric $g(p, q)=\operatorname{Re}(p \bar{q})$ on $\mathbb{H}$ we then get a triple of symplectic form $\omega_{\mathbf{i}}, \omega_{\mathbf{j}}, \omega_{\mathbf{k}}$ defined as usual: $\omega_{\mathbf{i}}(v, w)=g(\mathbf{i} v, w)$ etc. The formulae are as follows:

$$
\begin{gathered}
g=\sum_{0}^{3} d x_{i}^{2} \\
\omega_{\mathbf{i}}=d x_{0} \wedge d x_{1}+d x_{2} \wedge d x_{3}
\end{gathered}
$$

$$
\begin{aligned}
& \omega_{\mathbf{j}}=d x_{0} \wedge d x_{2}+d x_{3} \wedge d x_{1} \\
& \omega_{\mathbf{k}}=d x_{0} \wedge d x_{3}+d x_{1} \wedge d x_{2}
\end{aligned}
$$

Given that $\mathbb{C}$ with coordinate $z=x+i y$ has symplectic structure $d x \wedge d y$, one can "see" these formulae directly by writing:

$$
\begin{aligned}
q=x_{0}+x_{1} \mathbf{i}+x_{2} \mathbf{j}+x_{3} \mathbf{k} & =\left(x_{0}+x_{1} \mathbf{i}\right)+\left(x_{2}+x_{3} \mathbf{i}\right) \mathbf{j} \\
& =\left(x_{0}+x_{2} \mathbf{j}\right)+\left(x_{3}+x_{1} \mathbf{j}\right) \mathbf{k} \\
& =\left(x_{0}+x_{3} \mathbf{k}\right)+\left(x_{1}+x_{2} \mathbf{k}\right) \mathbf{i}
\end{aligned}
$$

respectively, since then the action of the complex structures (acting by left multiplication on $\mathbb{H}$ ) is clear, decomposing $\mathbb{H} \cong \mathbb{C} \times \mathbb{C}$ in three different ways.

Note that this may be encoded succinctly as:

$$
g-\mathbf{i} \omega_{\mathbf{i}}-\mathbf{j} \omega_{\mathbf{j}}-\mathbf{k} \omega_{\mathbf{k}}=d q \otimes d \bar{q}
$$

and using the quaternion multiplication to expand the right-hand side. This should be compared to $h=g-\sqrt{-1} \omega$.

Exercise: Check this, and show $g+\mathbf{i} \omega_{\mathbf{i}}+\mathbf{j} \omega_{\mathbf{j}}+\mathbf{k} \omega_{\mathbf{k}} \neq d \bar{q} \otimes d q$.
Remark 5.1. Recall the Hodge star operator, acting on forms, defined by the equation $a \wedge * b=(a, b)$ vol. This squares to one when restricted to two-forms on a four dimensional space, so the two-forms break up in to the $\pm 1$ eigenspaces. The three real two forms appearing here are a basis of the self-dual two forms on $\mathbb{R}^{4}$, i.e. they are preserved by the Hodge star. This enables us to identify the imaginary quaternions with the self-dual two forms.

Suppose we work in the complex structure $\mathbf{i}$, and write $q=z+w \mathbf{j}$ with $z=$ $x_{0}+\mathbf{i} x_{1}, w=x_{2}+\mathbf{i} x_{3}($ so $z, w: \mathbb{H} \rightarrow \mathbb{C}$ are now holomorphic coordinates on $\mathbb{H})$. Then we claim that

$$
\omega_{\mathbb{C}}:=\omega_{\mathbf{j}}+\sqrt{-1} \omega_{\mathbf{k}} \quad \in \bigwedge^{2}\left(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}\right)^{*}
$$

is a complex symplectic form (i.e. of type $(2,0)$ ). Indeed it is just

$$
\begin{aligned}
d z \wedge d w & =\left(d x_{0}+\sqrt{-1} d x_{1}\right) \wedge\left(d x_{2}+\sqrt{-1} d x_{3}\right) \\
& =\left(d x_{0} \wedge d x_{2}-d x_{1} \wedge d x_{3}\right)+\sqrt{-1}\left(d x_{0} \wedge d x_{3}+d x_{1} \wedge d x_{2}\right)
\end{aligned}
$$

This will be seen much more generally below. Note that this corresponds to viewing $\mathbb{H} \cong \mathbb{C}^{2}$ as the cotangent bundle of the $w$-line: $\mathbb{H}=T^{*} \mathbb{C}_{w}$ (since this has symplectic form $d(z d w))$.

Moreover the quaternionic Hermitian form

$$
d q \otimes d \bar{q}=g-\mathbf{i} \omega_{\mathbf{i}}-\mathbf{j} \omega_{\mathbf{j}}-\mathbf{k} \omega_{\mathbf{k}}=\left(g-\mathbf{i} \omega_{\mathbf{i}}\right)-\left(\omega_{\mathbf{j}}+\mathbf{i} \omega_{\mathbf{k}}\right) \mathbf{j}=h-\omega_{\mathbb{C}} \mathbf{j}
$$

is the sum of a (usual) Hermitian form minus $\mathbf{j}$ times a complex symplectic form. (This is the quaternionic analogue/complexification of the relation $h=g-\mathbf{i} \omega$.) In particular preserving $d q \otimes d \bar{q}$ is equivalent to preserving both $h$ and $\omega_{\mathbb{C}}$. In particular we see

$$
\mathrm{Sp}(1)=\mathrm{U}(2) \cap \mathrm{Sp}_{2}(\mathbb{C})
$$

is the intersection of the unitary group (preserving $h$ ) and the complex symplectic group $\mathrm{Sp}_{2}(\mathbb{C})=\mathrm{SL}_{2}(\mathbb{C})$ preserving $\omega_{\mathbb{C}}$ (which here is a holomorphic volume form). This will be extended below.

Note that when working in complex structure $\mathbf{i}$ the extra structure of the quaternions on the complex vector space $\mathbb{C}^{2}$ is all encoded in the action of $\mathbf{j}$ by left multiplication (since $\mathbf{k}=\mathbf{i} \mathbf{j}$ ). Explicitly this extra complex structure on $\mathbb{C}^{2}$ is given as follows

$$
\mathbf{j}(z, w)=(-\bar{w}, \bar{z})
$$

(as is easily seen by computing $\mathbf{j} \cdot(z+w \mathbf{j})$, given that $\mathbf{j} z=\bar{z} \mathbf{j}$ etc.)
$2 \times 2$ Matrices. Under the isomorphism $\mathbb{C}^{2} \cong \mathbb{H} ;(z, w) \mapsto q=z+w \mathbf{j}$ the right action of $\mathbb{H}$ on itself $(x(q)=q x)$, translates into a (right) action of $\mathbb{H}$ on $\mathbb{C}^{2}$ by $\mathbb{C}$ linear maps. In this way we get an embedding $\mathbb{H} \hookrightarrow \operatorname{End}\left(\mathbb{C}^{2}\right)$ of algebras. Viewing $(z, w) \in \mathbb{C}^{2}$ as row vectors, one may readily verify that the elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ become the matrices

$$
\left(\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right), \quad\left(\begin{array}{cc}
i & 0 \\
0 & -i
\end{array}\right), \quad\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right), \quad\left(\begin{array}{cc}
0 & i \\
i & 0
\end{array}\right) \in \operatorname{End}\left(\mathbb{C}^{2}\right)
$$

respectively (e.g. the computation $(z+w \mathbf{j}) \mathbf{k}=i w+(i z) \mathbf{j}$, yields the formula for the image of $\mathbf{k}$ ). In particular $\operatorname{Im} \mathbb{H}$ corresponds to the set of skew-Hermitian matrices with trace zero-the Lie algebra of $\mathfrak{s u}(2) \cong \mathfrak{s p}(1)$.

## Quaternionic vector spaces, some group theory 2.

More generally let $V=\mathbb{H}^{n} \cong \mathbb{R}^{4 n}$ be a "quaternionic vector space". We view it as a left $\mathbb{H}$-module by left multiplication. Thus $V$ has three complex structures $\mathbf{I}, \mathbf{J}, \mathbf{K}$ given by left (component-wise) multiplication by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively. Then we obtain a group

$$
\mathrm{GL}_{n}(\mathbb{H}) \subset \mathrm{GL}_{2 n}(\mathbb{C}) \subset \mathrm{GL}_{4 n}(\mathbb{R})
$$

of invertible quaternionic matrices (having real dimension $4 n^{2}<8 n^{2}<16 n^{2}$ ). If we let $g \in \mathrm{GL}_{4 n}(\mathbb{R})$ act on $\mathbf{q} \in V$ as $\mathbf{q} g^{-1}$ (where we view $\mathbf{q}$ as a row vector), then the action of $\mathrm{GL}_{n}(\mathbb{H})$ commutes with the action of the complex structures, i.e. it is "quaternionic-linear".

We may define a standard "quaternionic Hermitian product"

$$
((\mathbf{p}, \mathbf{q}))=\mathbf{p q}^{\dagger}=\sum p_{i} \bar{q}_{i} \in \mathbb{H}
$$

where $\mathbf{p} \in V$ has components $p_{1}, \ldots p_{n}, \in \mathbb{H}$ etc. Note that $\overline{((\mathbf{p}, \mathbf{q}))}=((\mathbf{q}, \mathbf{p}))$. Equivalently $((\cdot, \cdot))=d \mathbf{q} \otimes d \mathbf{q}^{\dagger}=\sum d q_{i} \otimes d \bar{q}_{i}$. This decomposes as above in the $n=1$ case

$$
d \mathbf{q} \otimes d \mathbf{q}^{\dagger}=g-\mathbf{i} \omega_{\mathbf{i}}-\mathbf{j} \omega_{\mathbf{j}}-\mathbf{k} \omega_{\mathbf{k}}=\left(g-\mathbf{i} \omega_{\mathbf{i}}\right)-\left(\omega_{\mathbf{j}}+\mathbf{i} \omega_{\mathbf{k}}\right) \mathbf{j}=h-\omega_{\mathbb{C}} \mathbf{j}
$$

yielding the flat metric $g$, Kähler forms, Hermitian metric and complex symplectic form. (Compare with the relation $h=g-\mathbf{i} \omega$.) In local holomorphic coordinates (for I) we have $\omega_{\mathbb{C}}=\sum d z_{i} \wedge d w_{i}$.

More abstractly one may define a hyperkähler vector space $V$ to be a free left $\mathbb{H}$-module together with a (positive definite) quaternionic Hermitian form (i.e. an $\mathbb{R}$-bilinear map $((\cdot, \cdot)): V \otimes_{\mathbb{R}} V \rightarrow \mathbb{H}$, which is $\mathbb{H}$ linear in the first slot, satisfies $\overline{((\mathbf{p}, \mathbf{q}))}=((\mathbf{q}, \mathbf{p}))$, and is such that the real number $((\mathbf{q}, \mathbf{q}))$ is positive unless $\mathbf{q}=0)$.

We define the quaternionic unitary group $\operatorname{Sp}(n)$ to be

$$
\operatorname{Sp}(n)=\left\{g \in M_{n}(\mathbb{H}) \mid g g^{\dagger}=1\right\}
$$

where $g^{\dagger}$ is the transpose of the quaternionic conjugate matrix. This is the group preserving the quaternionic inner product, if we make $g \in \mathrm{GL}_{n}(\mathbb{H})$ act on $\mathbf{q} \in V$ as $\mathbf{q} g^{-1}$, where we view $\mathbf{q}$, as a row vector of quaternions. Thus $g \in \operatorname{Sp}(n)$ iff

$$
\left(\left(\mathbf{p} g^{-1}, \mathbf{q} g^{-1}\right)\right)=((\mathbf{p}, \mathbf{q}))
$$

for all $\mathbf{p}, \mathbf{q} \in V$. The Lie algebra of $\operatorname{Sp}(n)$ is thus the set of quaternionic skew-adjoint matrices:

$$
\mathfrak{s p}(n)=\left\{X \in M_{n}(\mathbb{H}) \mid X+X^{\dagger}=0\right\}
$$

In particular we deduce that $\operatorname{dim}_{\mathbb{R}}(\operatorname{Sp}(n))=3 n+4 n(n-1) / 2=2 n^{2}+n$.
Since $d \mathbf{q} \otimes d \mathbf{q}^{\dagger}=h-\omega_{\mathbb{C}} \mathbf{j}$ is the sum of a Hermitian form minus the complex symplectic form (times $\mathbf{j}$ ), preserving $d \mathbf{q} \otimes d \mathbf{q}^{\dagger}$ is equivalent to preserving both $h$ and $\omega_{\mathbb{C}}$. It follows that the quaternionic unitary group of $\mathbb{H}^{n} \cong \mathbb{C}^{2 n}=T^{*} \mathbb{C}^{n}$ is the intersection

$$
\operatorname{Sp}(n)=\mathrm{U}(2 n) \cap \operatorname{Sp}_{2 n}(\mathbb{C})
$$

of the unitary group (stabilising $h$ ) and the complex symplectic group (stabilising $\omega_{\mathbb{C}}$ ). Moreover clearly $\mathrm{Sp}_{2 n}(\mathbb{C}) \subset \mathrm{SL}_{2 n}(\mathbb{C})$ (since the top exterior power of the complex symplectic form is a holomorphic volume form) so that in fact

$$
\mathrm{Sp}(n) \subset \mathrm{SU}(2 n)
$$

This means that hyperkähler manifolds are Ricci flat-they are Calabi-Yau manifolds of even complex dimension

Note also that $\operatorname{Sp}(1) \cong \mathrm{SU}(2)$, but (e.g. by comparing dimensions $2 n^{2}+n$ and $4 n^{2}-1$ resp.) this inclusion is strict if $n>1$.

## 6. HyperkÄHLER MANIFOLDS

Let $M$ be a manifold of dimension $4 n$.
Definition 6.1. $M$ is a hyperkähler manifold if it is equipped with

1) A triple of global sections $\mathbf{I}, \mathbf{J}, \mathbf{K} \in \Gamma(\operatorname{End}(T M))$ of the tangent bundle, satisfying the quaternion identities:

$$
\mathbf{I}^{2}=\mathbf{J}^{2}=\mathbf{K}^{2}=\mathbf{I} \mathbf{J K}=-1
$$

2) A Riemannian metric $g$ such that

$$
g(\mathbf{I} u, \mathbf{I} v)=g(\mathbf{J} u, \mathbf{J} v)=g(\mathbf{K} u, \mathbf{K} v)=g(u, v)
$$

for all tangent vectors $u, v \in T_{m} M$ for all $m \in M$, and moreover $\mathbf{I}, \mathbf{J}, \mathbf{K}$ are covariant constant, i.e.

$$
\nabla \mathbf{I}=\nabla \mathbf{J}=\nabla \mathbf{K}=0
$$

where $\nabla$ is the covariant derivative of the Levi-Civita connection of $g$.
Thus on a hyperkähler manifold, given three real numbers $a_{i}$ whose squares sum one, we may define

$$
\mathbf{I}_{a}=a_{1} \mathbf{I}+a_{2} \mathbf{J}+a_{3} \mathbf{K} \in \Gamma(\operatorname{End}(T M))
$$

and verify that $\nabla \mathbf{I}_{a}=0, \mathbf{I}_{a}^{2}=-1, g\left(\mathbf{I}_{a} u, \mathbf{I}_{a} v\right)=g(u, v)$ so that, by Theorem 4.1

$$
\left(M, g, \mathbf{I}_{a}\right) \text { is a Kähler manifold for any } a=\left(a_{1}, a_{2}, a_{3}\right) \in S^{2}
$$

i.e. it has a whole $S^{2}$ family of Kähler structure, hence the name.

Let us denote the triple of Kähler forms as follows:

$$
\omega_{\mathbf{I}}(u, v)=g(\mathbf{I} u, v), \quad \omega_{\mathbf{J}}(u, v)=g(\mathbf{J} u, v), \quad \omega_{\mathbf{K}}(u, v)=g(\mathbf{K} u, v)
$$

and we will also sometime write $\omega_{1}=\omega_{\mathbf{I}}, \omega_{2}=\omega_{\mathbf{J}}, \omega_{3}=\omega_{\mathbf{K}}$.
A trivial example of hyperkähler manifold is a hyperkähler vector space, i.e. the flat metric and triple of complex structures on the $\mathbb{H}^{n}$ described earlier.

If we view $M$ as a complex manifold using the complex structure $I$ then we claim that the the complex two-form

$$
\omega_{\mathbb{C}}=\omega_{\mathbb{C}}^{(\mathbf{I J K})}:=\omega_{\mathbf{J}}+\sqrt{-1} \omega_{\mathbf{K}}
$$

is a complex symplectic form. (Note that this depends the choice of the triple $\mathbf{I}, \mathbf{J}, \mathbf{K}$, an orthonormal basis of $\operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}$, and not just $\mathbf{I}$.) We will first check it is of type
$(2,0)$. Indeed

$$
\begin{aligned}
\omega_{\mathbb{C}}(\mathbf{I} p, q) & =g(\mathbf{J} \mathbf{} p, q)+\sqrt{-1} g(\mathbf{K I} p, q) \\
& =-g(\mathbf{K} p, q)+\sqrt{-1} g(\mathbf{J} p, q) \\
& =\sqrt{-1}(g(\mathbf{J} p, q)+\sqrt{-1} g(\mathbf{K} p, q)) \\
& =\sqrt{-1} \omega_{\mathbb{C}}(p, q)
\end{aligned}
$$

which means that $\omega_{\mathbb{C}}$ is of type $(2,0)$. [Immediately this shows that the complex one-form $p \mapsto \omega_{\mathbb{C}}(p, q)$ is $(1,0)$ for any fixed $q$. Then by skew-symmetry $\omega_{\mathbb{C}}$ is $(2,0)$.] Clearly $\omega_{\mathbb{C}}$ is closed and it is nondegenerate since e.g. the above shows $\omega_{\mathbb{C}}(\mathbf{I} p, \mathbf{K} p)$ has nonzero real part whenever $p$ is nonzero. Thus $(M, \mathbf{I})$ is a complex symplectic manifold. (Similarly, for example, $(M, \mathbf{J})$ and $(M, \mathbf{K})$ are complex symplectic with complex symplectic forms $\omega_{\mathbb{C}}^{\mathrm{JKI}}, \omega_{\mathbb{C}}^{\mathrm{KIJ}}$.) Thus the notion of hyperkähler manifold is an enrichment of the notion of complex symplectic manifold. This is a useful viepoint since most examples arise in the first instance as complex symplectic manifolds.

It turns out that one obtains the same structure (of hyperkähler manifold on $M$ ) by asking for apparently weaker conditions: First note that the metric and the complex structures are determined by the three Kähler two-forms alone: For example since we have

$$
\begin{equation*}
\omega_{\mathbf{J}}(u, v)=g(\mathbf{J} u, v)=g(\mathbf{K I} u, v)=\omega_{\mathbf{K}}(\mathbf{I} u, v) \tag{6.1}
\end{equation*}
$$

it follows that $\mathbf{I}=\left(\omega_{\mathbf{K}}^{b}\right)^{-1} \circ \omega_{\mathbf{J}}^{b} \in \operatorname{End}(T M)$ and similarly we may obtain $\mathbf{J}, \mathbf{K}$. These now yield the metric also since e.g. $g(u, v)=\omega_{\mathbf{I}}(u, \mathbf{I} v)$. Thus we can ask for conditions in terms of the triple of forms to determine the hyperkähler structure:

Lemma 6.2 ([Hit87]). Suppose ( $M, g, \mathbf{I}, \mathbf{J}, \mathbf{K}$ ) is a Riemannian manifold with a triple of skew-adjoint endomorphism of the tangent bundle satisfying the quaternion identities. Then $M$ is hyperkähler iff the corresponding triple of two-forms are closed.

Proof. Clearly if it is hyperkähler then the forms are closed. Conversely by Theorem 4.1 it is sufficient to show the almost complex structures are integrable. For this we will use the Newlander-Nirenberger theorem in the following form: an almost-complex structure is integrable iff the Lie bracket of any two vector fields of type $(1,0)$ is again of type ( 1,0 ) (see [KN69], Theorems 2.8 and $2.7 \mathrm{pp} .124-126$ ). Now if $u, v$ are sections of the complexified tangent bundle of $M$ then from (6.1) we see $u$ is of type $(1,0)$ in the complex structure $\mathbf{I}$ (i.e. $\mathbf{I} u=+\sqrt{-1} u$ ) iff $\iota_{u} \omega_{\mathbf{J}}=\sqrt{-1} \iota_{u} \omega_{\mathbf{K}}$ i.e. iff

$$
\iota_{u} \bar{\omega}_{\mathbb{C}}=0
$$

where $\bar{\omega}_{\mathbb{C}}=\omega_{\mathbf{J}}-\sqrt{-1} \omega_{\mathbf{K}}$. Thus if $v, w$ are both of type $(1,0)$ for $\mathbf{I}$ we must show $\iota_{[v, w]} \bar{\omega}_{\mathbb{C}}=0$. This is now straightforward:

$$
\begin{aligned}
\iota_{[v, w]} \bar{\omega}_{\mathbb{C}} & =\left[\mathcal{L}_{v}, \iota_{w}\right] \bar{\omega}_{\mathbb{C}} & & \\
& =-\iota_{w} \mathcal{L}_{v} \bar{\omega}_{\mathbb{C}} & & \text { since } w \text { is }(1,0) \\
& =-\iota_{w}\left(d \circ \iota_{v}\right) \bar{\omega}_{\mathbb{C}} & & \text { by the Cartan formula and } d \bar{\omega}_{\mathbb{C}}=0 \\
& =0 & & \text { since } v \text { is }(1,0)
\end{aligned}
$$

Thus $\mathbf{I}$ is integrable and similarly for $\mathbf{J}, \mathbf{K}$.

This will be useful since often it is easier to check if forms are closed than complex structures being covariant constant.

## Hyperkähler quotients.

In this section we will define the hyperkähler quotient - a generalization of the symplectic (and Kähler) quotient, enabling lots of new examples of noncompact hyperkähler manifolds to be constructed.

Let $(M, g, \mathbf{I}, \mathbf{J}, \mathbf{K})$ be a hyperkähler manifold. Suppose $G$ is a compact Lie group acting on $M$ preserving the metric and the triple of complex structures. Thus $G$ preserves the triple of Kähler forms and we suppose further that there is a triple of equivariant moment maps $\mu_{\mathbf{I}}, \mu_{\mathbf{J}}, \mu_{\mathbf{K}}: M \rightarrow \mathfrak{g}^{*}$. More invariantly these constitute the components of a single map, the hyperkähler moment map:

$$
\mu: M \rightarrow \mathfrak{g}^{*} \otimes \mathbb{R}^{3} .
$$

Theorem 6.3 ([HKLR87]). Let $\zeta \in \mathfrak{g}^{*} \otimes \mathbb{R}^{3}$ be a $G$-invariant point (where $G$ acts by the coadjoint action component-wise). Suppose that $G$ acts freely on the subset $\mu^{-1}(\zeta) \subset M$. Then the quotient

$$
M / / /{ }_{\zeta} G:=\mu^{-1}(\zeta) / G
$$

is a (smooth) manifold with a natural hyperkähler structure induced from that of $M$. Its real dimension is $\operatorname{dim}_{\mathbb{R}}(M)-4 \operatorname{dim}_{\mathbb{R}}(G)$. If $M$ is complete then so is $M / / / G$.

Proof. First we check it is a smooth manifold. For this it is sufficient to check the derivative of $\mu$ is surjective at each point of $Z:=\mu^{-1}(\zeta)$ (so that $Z$ is a smooth submanifold) and then the slice theorem implies the quotient, by a free action of a compact group, exists and is smooth.

Choose $m \in Z$. Given $X \in \mathfrak{g}$ let $v_{X}$ denote the corresponding fundamanetal vector field on $M$ and let $\mathfrak{g}_{m} \subset T_{m} M$ denote the tangent space to the orbit through $m$. Since the action is free $\mathfrak{g} \cong \mathfrak{g}_{m} ; X \mapsto\left(v_{X}\right)_{m}$. We wish to show that $d \mu_{m}: T_{m} M \rightarrow \mathfrak{g}^{*} \otimes \mathbb{R}^{3}$ is surjective. This follows from the following.

Lemma 6.4. $d \mu_{m}$ maps $\mathbf{I g}_{m} \subset T_{m} M$ onto $(*, 0,0) \subset \mathfrak{g}^{*} \otimes \mathbb{R}^{3}$, and similarly $d \mu_{m}\left(\mathbf{J} \mathfrak{g}_{m}\right)=$ $(0, *, 0)$ etc. Moreover the four subspaces $\mathfrak{g}_{m}, \mathbf{I} \mathfrak{g}_{m}, \mathbf{J} \mathfrak{g}_{m}, \mathbf{K g}_{m}$ are orthogonal.

Proof. The second statement follows from the moment map property: Clearly $d \mu_{m}\left(v_{X}\right)=0$ for all $X \in \mathfrak{g}$ (since $\mu$ maps the orbit though $m$ onto the point $\zeta$ ). Given $Y \in \mathfrak{g}$ let $\mu^{Y}=\langle\mu, Y\rangle: M \rightarrow \mathbb{R}^{3}$ be the $Y$-component of $\mu$. The moment map condition implies that, for all $X, Y \in \mathfrak{g}$ :

$$
\begin{gathered}
0=d \mu_{m}^{Y}\left(v_{X}\right)=\left(\omega_{\mathbf{I}}\left(v_{X}, v_{Y}\right), \omega_{\mathbf{J}}\left(v_{X}, v_{Y}\right), \omega_{\mathbf{K}}\left(v_{X}, v_{Y}\right)\right) \\
=\left(g\left(\mathbf{I} v_{X}, v_{Y}\right), g\left(\mathbf{J} v_{X}, v_{Y}\right), g\left(\mathbf{K} v_{X}, v_{Y}\right)\right)
\end{gathered}
$$

Thus $\mathfrak{g}_{m}$ is orthogonal to each of $\mathbf{I} \mathfrak{g}_{m}, \mathbf{J} \mathfrak{g}_{m}, \mathbf{K} \mathfrak{g}_{m}$ and the full statement follows by invariance of $g$.

Now for the first part. Choose $X \in \mathfrak{g}$. We will show $d \mu_{m}\left(\mathbf{I} v_{X}\right)=\left(-g\left(v_{(\cdot)}, v_{X}\right), 0,0\right) \in$ $\mathfrak{g}^{*} \otimes \mathbb{R}^{3}$. Indeed for any $Y \in \mathfrak{g}$ (replacing $v_{X}$ by $-\mathbf{I} v_{X}$ above):

$$
\begin{gathered}
-d \mu_{m}^{Y}\left(\mathbf{I} v_{X}\right)=\left(\omega_{\mathbf{I}}\left(v_{Y}, \mathbf{I} v_{X}\right), \omega_{\mathbf{J}}\left(v_{Y}, \mathbf{I} v_{X}\right), \omega_{\mathbf{K}}\left(v_{Y}, \mathbf{I} v_{X}\right)\right) \\
=\left(g\left(\mathbf{I} v_{Y}, \mathbf{I} v_{X}\right), g\left(\mathbf{J} v_{Y}, \mathbf{I} v_{X}\right), g\left(\mathbf{K} v_{Y}, \mathbf{I} v_{X}\right)\right) \\
=\left(g\left(v_{Y}, v_{X}\right), 0,0\right)
\end{gathered}
$$

Now the bilinear form $(X, Y) \mapsto g_{m}\left(v_{X}, v_{Y}\right)$ on $\mathfrak{g} \otimes \mathfrak{g}$ is nondegenerate (being the restriction of $g$ to $\mathfrak{g}_{m}$ composed with the action isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}_{m}$ ) so the first statement follows.

Thus $Z=\mu^{-1}(\zeta)$ is a smooth manifold. At $m \in Z$ its tangent space Ker $d \mu_{m}$ may be characterised as the orthogonal complement to $\mathbf{I g}_{m} \oplus \mathbf{J g}_{m} \oplus \mathbf{K g}_{m}$ since we have

$$
\begin{gathered}
-d \mu_{m}^{Y}(v)=\left(\omega_{\mathbf{I}}\left(v_{Y}, v\right), \omega_{\mathbf{J}}\left(v_{Y}, v\right), \omega_{\mathbf{K}}\left(v_{Y}, v\right)\right) \\
=\left(g\left(\mathbf{I} v_{Y}, v\right), g\left(\mathbf{J} v_{Y}, v\right), g\left(\mathbf{K} v_{Y}, v\right)\right)
\end{gathered}
$$

for all $v \in T_{m} M, Y \in \mathfrak{g}$.
Since the action is free $Z / G$ is a manifold and the tangent space to a point $[G \cdot m] \in$ $Z / G$ is given by the orthogonal complement $H_{m}$ of $\mathfrak{g}_{m}$ in $T_{m} Z$ (for any $m$ in the fibre). This is the same as the orthogonal complement to $\mathfrak{g}_{m} \oplus \mathbf{I g}_{m} \oplus \mathbf{J g}_{m} \oplus \mathbf{K} \mathfrak{g}_{m}$ in $T_{m} M$. As such it is preserved by $\mathbf{I}, \mathbf{J}, \mathbf{K}$ (so has a quaternionic triple of almost complex structures) and the metric $g$ may be restricted. By Lemma 6.2 it is sufficient now to check that the corresponding triple of two-forms $\omega_{\mathbf{I}}^{\prime}, \omega_{\mathbf{J}}^{\prime}, \omega_{\mathbf{K}}^{\prime}$ are closed.

Let $\iota: Z \rightarrow M$ be the inclusion and let $\pi: Z \rightarrow Z / G$ be the quotient map. Then by definition $\iota^{*}\left(\omega_{\mathbf{I}}\right)=\pi^{*}\left(\omega_{\mathbf{I}}^{\prime}\right)$ so we have

$$
\pi^{*}\left(d \omega_{\mathbf{I}}^{\prime}\right)=d \pi^{*}\left(\omega_{\mathbf{I}}^{\prime}\right)=d \iota^{*}\left(\omega_{\mathbf{I}}\right)=\iota^{*}\left(d \omega_{\mathbf{I}}\right)=0
$$

by the naturality of the exterior derivative. It follows that $d \omega_{\mathbf{I}}^{\prime}=0$ since $\pi$ is surjective on tangent vectors (by definition). Similarly for $\omega_{\mathbf{J}}, \omega_{\mathbf{K}}$, so we have ideed constructed a hyperkähler manifold.

Finally we briefly discuss completeness. Suppose $M$ is complete and we have a (maximal) geodesic $\gamma:[0, T) \rightarrow N:=Z / G$ with $\gamma(0)=\pi(m)$. Then we may lift $\gamma$ to a curve $\widetilde{\gamma}:[0, T) \rightarrow Z$ based at $m$, and tangent to the horizontal subspace $H_{z}$ (orthogonal to $\mathfrak{g}_{z}$ ) at each point. By definition of the metric on $N$ the length along $\widetilde{\gamma}$ is the same as the length along $\gamma$. This lifted curve may be extended in $M$ (as $M$ is complete) so has a limit at $t=T$. This limit is in $Z$ since $Z$ is closed in $M$, so projects to a point of $N$, so $\gamma$ can in fact be extended.

Remark 6.5. Suppose that we write a hyperkähler moment map $\mu$ as $\left(\mu_{\mathbb{R}}, \mu_{\mathbb{C}}\right)$ where $\mu_{\mathbb{R}}=\mu_{\mathbf{I}}: M \rightarrow \mathfrak{g}^{*}$ and $\mu_{\mathbb{C}}=\mu_{\mathbf{J}}+\sqrt{-1} \mu_{\mathbf{K}}: M \rightarrow \mathbb{C} \otimes \mathfrak{g}^{*}$ are built out of $\mu$. Then $\mu_{\mathbb{C}}$ is a holomorphic map on $M$ (in complex structure $\mathbf{I}$ ), since for all $X \in \mathfrak{g}$ and vector fields $Y$ :

$$
\begin{aligned}
d\left\langle\mu_{\mathbb{C}}, X\right\rangle(\mathbf{I} Y) & =g\left(\mathbf{J I} Y, v_{X}\right)+\sqrt{-1} g\left(\mathbf{K I} Y, v_{X}\right) \\
& =-g\left(\mathbf{K} Y, v_{X}\right)+\sqrt{-1} g\left(\mathbf{J} Y, v_{X}\right) \\
& =\sqrt{-1}\left(g\left(\mathbf{J} Y, v_{X}\right)+\sqrt{-1} g\left(\mathbf{K} Y, v_{X}\right)\right) \\
& =\sqrt{-1} d\left\langle\mu_{\mathbb{C}}, X\right\rangle(Y) .
\end{aligned}
$$

Thus, if we write $\zeta=\left(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}\right)$ similarly, then $\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) \subset M$ is a complex submanifold (in complex structure I) so inherits the induced Kähler structure, and we may view the hyperkähler quotient as the Kähler quotient of it by the action of $G$ with moment map $\mu_{\mathbb{R}}$ :

$$
M / / /{ }_{\zeta} G=\mu^{-1}(\zeta) / G=\left(\mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right) \cap \mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right)\right) / G=\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) \|_{\zeta_{\mathbb{R}}} G
$$

Note also that if the action of $G$ extends to a holomorphic action of the complexification $G_{\mathbb{C}}$ of $G$ then this action will have moment map $\mu_{\mathbb{C}}$ (with respect to the complex symplectic form $\omega_{\mathbb{C}}$ ). Later on, once we have studied quotients by such (noncompact) complex algebraic groups $G_{\mathbb{C}}$ we will relate this to the complex symplectic quotient $\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) / G_{\mathbb{C}}$. For now we will just look at an example where everything can be computed by hand.

The most basic example of a hyperkähler moment map is for $\operatorname{Sp}(n)$ acting on the hyperkähler vector space $\mathbb{H}^{n}$ (with quaternionic Hermitian form $\left.((\mathbf{p}, \mathbf{q}))=\mathbf{p q}^{\dagger} \in \mathbb{H}\right)$ :

$$
g(\mathbf{q}):=\mathbf{q} \cdot g^{-1}, \quad \mathbf{q} \in \mathbb{H}^{n}, g \in \operatorname{Sp}(n)
$$

where as usual we view $\mathbf{q}$ as a row vector (a $1 \times n$ quaternionic matrix), and $\dagger$ denotes the quaternionic conjugate of the transposed quaternionic matrix. This action
preserves the hyperkähler structure of $\mathbb{H}^{n}$ (since, for example the complex structures were defined in terms of left multiplication).

Lemma 6.6. A hyperkähler moment map for the above action is given by

$$
\mu(\mathbf{q})(X)=\frac{1}{2}((\mathbf{q} X, \mathbf{q}))=\frac{1}{2} \operatorname{Im}\left(\mathbf{q} X \mathbf{q}^{\dagger}\right) \in \operatorname{Im} \mathbb{H}
$$

where $\mathbf{q} \in \mathbb{H}^{n}, X \in \mathfrak{s p}(n)$ (so that $\mu(\mathbf{q}) \in \mathfrak{s p}(n)^{*} \otimes \operatorname{Im} \mathbb{H}$ ) and we identify $\mathbb{R}^{3} \cong \operatorname{Im} \mathbb{H}$ $\operatorname{via}\left(\mu_{1}, \mu_{2}, \mu_{3}\right)=\mathbf{i} \mu_{1}+\mathbf{j} \mu_{2}+\mathbf{k} \mu_{3}$.

Proof. This is clearly equivariant, i.e. $\mu\left(\mathbf{q} g^{-1}\right)\left(g X g^{-1}\right)=\mu(\mathbf{q})(X)$. Write $\widehat{h}=$ $((\cdot, \cdot))$ for the quaternionic form. Since $\widehat{h}=g-\mathbf{i} \omega_{1}-\mathbf{j} \omega_{2}-\mathbf{k} \omega_{3}$, the first component of $\mu$ is $\mu_{1}$ where

$$
\mu_{1}(\mathbf{q})(X)=-\frac{1}{2} \omega_{1}(\mathbf{q} X, \mathbf{q}) \in \mathbb{R}
$$

By Lemma 3.19 this is indeed a moment map for the action of $\operatorname{Sp}(n)$ on $\left(\mathbb{H}^{n}, \omega_{1}\right)$ since $-\mathbf{q} X$ is the derivative of the flow $\mathbf{q} \exp (-t X)$ generated by $X$. Similarly for the other components. Observe that for $X \in \mathfrak{s p}(n)$ (i.e. $X=-X^{\dagger}$ ) the expression $\mathbf{q} X \mathbf{q}^{\dagger} \in \mathbb{H}$ is automatically in $\operatorname{Im} \mathbb{H}$ (since it equals minus its quaternionic conjugate).

Note (more abstractly) that the same proof shows $\mu(\mathbf{q})(X)=\frac{1}{2}\left(\left(v_{X}(\mathbf{q}), \mathbf{q}\right)\right)$ is a moment map for the natural action of the compact symplectic group of any hyperkähler vector space $\mathbb{V}$ with form $((\cdot, \cdot))$.

Now we will give a first example of hyperkähler quotient.
Example 6.7. Let $V=\mathbb{C}^{n}$ be a complex vector space of dimension $n$, equipped with the standard Hermitian inner product. Let

$$
\mathbb{V}=V \times V^{*} \cong \mathbb{H}^{n}
$$

On $V \times V^{*}$ the structure of hyperkähler vector space is determined by the action of $\mathbf{j}$ given by $\mathbf{j}(v, \alpha)=\left(\alpha^{\dagger},-v^{\dagger}\right)$, and the metric, given by $\|(v, \alpha)\|^{2}=\|v\|^{2}+\|\alpha\|^{2}=$ $v^{\dagger} v+\alpha \alpha^{\dagger}$. (Equivalently by convention $(v, \alpha) \in V \times V^{*}$ corresponds to the point of $\mathbb{H}^{n}$ with components $v_{i}-\alpha_{i} \mathbf{j}$.) We consider the action of the circle $S^{1}$ on $\mathbb{V}$ as follows:

$$
g(v, \alpha)=(g v, \alpha / g)
$$

for $(v, \alpha) \in V \times V^{*}$, and $g \in S^{1}$. A moment map for this action is $\mu=\left(\mu_{\mathbb{R}}, \mu_{\mathbb{C}}\right)$ where

$$
\begin{gathered}
\mu_{\mathbb{R}}(v, \alpha)=\frac{i}{2}\left(\|v\|^{2}-\|\alpha\|^{2}\right) \in i \mathbb{R}=\operatorname{Lie}\left(S^{1}\right) \\
\mu_{\mathbb{C}}(v, \alpha)=-\alpha(v) \in \mathbb{C}=\operatorname{Lie}\left(\mathbb{C}^{*}\right)
\end{gathered}
$$

This may be checked directly, or deduced from the previous example (it will also follow from more general examples below).

The hyperkähler quotient at the value $\zeta=(i / 2,0,0)$ of the moment map is thus

$$
\left\{(v, \alpha) \in V \times V^{*} \mid\|v\|^{2}-\|\alpha\|^{2}=1, \alpha(v)=0\right\} / S^{1}
$$

It is easy to see this is equal to the complex quotient

$$
\left\{(v, \alpha) \in V \times V^{*} \mid v \neq 0, \alpha(v)=0\right\} / \mathbb{C}^{*}
$$

since $\mathbb{C}^{*} \cong S^{1} \times \mathbb{R}_{>0}^{\times}$, and for any pair $(v, \alpha)$ with $v$ nonzero there is a unique real $t>0$ such that $t^{2}\|v\|^{2}-t^{-2}\|\alpha\|^{2}=1$. In turn this is the standard description of the cotangent bundle $T^{*} \mathbb{P}(V)$ of the projective space of $V$. This follows from the following more general fact:

Lemma 6.8. Let $M=G r_{k}(V)$ be the Grassmannian of $k$ dimensional complex subspaces of $V$. Let $W \subset V$ be such subspace. Then the tangent space to $M$ at $W$ is naturally $T_{W} M=\operatorname{Hom}(W, V / W)=W^{*} \otimes(V / W)$ and in turn

$$
T_{W}^{*} M=W \otimes(V / W)^{*}=W \otimes W^{\circ}
$$

where $W^{\circ} \subset V^{*}$ is the annihilator of $W: W^{\circ}=\left\{\alpha \in V^{*} \mid \alpha(W)=0\right\}$.
Assuming this, and setting $k=1$ so $M=\mathbb{P}(V)$, we have

$$
T^{*} \mathbb{P}(V)=\left\{(L, v \otimes \alpha) \in \mathbb{P}(V) \times V \otimes V^{*} \mid v \in L, \alpha \in L^{\circ}\right\}
$$

which agrees with the above description of the hyperkähler quotient (the map $(v, \alpha) \mapsto$ ( $[v], v \otimes \alpha$ ) is surjective with the $\mathbb{C}^{*}$ orbits as fibres).

Thus the total space of the holomorphic cotangent bundle to any projective space is a complete hyperkähler manifold. In the case $V=\mathbb{C}^{2}$ we obtain the cotangent bundle $T^{*} \mathbb{P}^{1}$ to the Riemann sphere, of real dimension 4. This hyperkähler fourmanifold is the Eguchi-Hanson space, discovered in 1978 [EH78], and was the first nontrivial example of a hyperkähler manifold. The hyperkähler metrics on the higher dimensional cotangent bundles of projective spaces were constructed in 1979 by Calabi [Cal79] (by a different method to that above), who coined the term "hyperkähler".

## Other examples of hyperkähler moment maps.

We will derive some other specific examples of hyperkähler moment maps that will be useful both in the infinite dimensional situation later and in the realm of quiver varieties.

First let $G$ be a Lie group of unitary operators (e.g. any compact Lie group embedded in some $\mathrm{U}(n)$ ), with Lie algebra $\mathfrak{g}$ (consisting of skew-adjoint operators).

Consider the quaternionic vector space

$$
\mathbb{V}=\mathbb{H} \otimes_{\mathbb{R}} \mathfrak{g}
$$

with $\mathbb{H}$ acting by left multiplication as usual. This becomes a hyperkähler vector space upon defining the norm of $X=X_{0}+\mathbf{i} X_{1}+\mathbf{j} X_{2}+\mathbf{k} X_{3}$ (with $X_{i} \in \mathfrak{g}$ ) to be

$$
\|X\|^{2}=\sum_{0}^{3} \operatorname{Tr}\left(X_{i} X_{i}^{\dagger}\right)
$$

We may identify $\mathbb{V}$ in complex structure $\mathbf{I}$ with $\mathfrak{g}_{\mathbb{C}}^{2} \subset \operatorname{End}\left(\mathbb{C}^{n}\right)^{2}\left(\right.$ where $\left.\mathfrak{g}_{\mathbb{C}}=\mathfrak{g} \otimes \mathbb{C}\right)$ by mapping $X \in \mathbb{V}$ to

$$
Z=X_{0}+i X_{1}, \quad W=X_{2}+i X_{3} \in \mathfrak{g}_{\mathbb{C}} \subset \operatorname{End}\left(\mathbb{C}^{n}\right)
$$

(since this map $\mathbb{V} \rightarrow \mathfrak{g}_{\mathbb{C}}^{2}$ is $\mathbb{C}$-linear). The adjoint action of $G$ on $\mathfrak{g}$ induces an action of $G$ on $\mathbb{V}$ commuting with the $\mathbb{H}$-action: $g\left(X_{i}\right)=\operatorname{Ad}_{g}\left(X_{i}\right)$ for each $i$. Thus on $\mathfrak{g}_{\mathbb{C}}^{2}$ we just obtain two copies of Example 4.7, so this action has moment map

$$
\begin{aligned}
\mu_{\mathbb{R}}(X)=\mu_{1}(X) & =\frac{i}{2}\left[Z, Z^{\dagger}\right]+\frac{i}{2}\left[W, W^{\dagger}\right] \\
& =-\frac{i}{2}\left[X_{0}+i X_{1}, X_{0}-i X_{1}\right]-\frac{i}{2}\left[X_{2}+i X_{3}, X_{2}-i X_{3}\right] \\
& =-\left[X_{0}, X_{1}\right]-\left[X_{2}, X_{3}\right] \in \mathfrak{g} .
\end{aligned}
$$

Similarly by cyclic permutation of $1,2,3$ we obtain the other components of the hyperkähler moment map $\mu$ :

$$
\begin{align*}
& \mu_{1}(X)=-\left[X_{0}, X_{1}\right]-\left[X_{2}, X_{3}\right] \\
& \mu_{2}(X)=-\left[X_{0}, X_{2}\right]-\left[X_{3}, X_{1}\right]  \tag{6.2}\\
& \mu_{3}(X)=-\left[X_{0}, X_{3}\right]-\left[X_{1}, X_{2}\right]
\end{align*}
$$

The corresponding equations " $\mu=0$ " will be important when we consider infinite dimensional analogues later, yielding for example the self-dual Yang-Mills equations.

If we compute $[W, Z]$ we find that this is the complex moment map:

$$
\mu_{\mathbb{C}}(X):=\mu_{2}(X)+i \mu_{3}(X)=[W, Z] \in \mathfrak{g}_{\mathbb{C}}
$$

which is clearly holomorphic in complex structure $\mathbf{I}$, and is indeed the moment map for the action of the complexified group on $\mathfrak{g}_{\mathbb{C}}^{2}$.

Now we wish to restrict to the case where $\mathfrak{g}=\operatorname{LieU}(n)$ is the set of $n \times n$ skewhermitian matrices and take a more quaternionic view. The aim is to see what happens when we rotate the complex structures, and for this we need to compute $\mu(q X)$ for a unit quaternion $q \in \mathbb{H}$. Let us embed $\mathbb{C}$ in $\mathbb{H}$ by writing $\mathbb{C}=\mathbb{R} \oplus \mathbb{R}$, and thus embed $\mathfrak{g} \subset M_{n}(\mathbb{H})$. Then we may collapse the tensor product used above and identify $\mathbb{V}$ with the $n \times n$ quaternionic matrices:

$$
X \mapsto Q=X_{0}+\mathbf{i} X_{1}+\mathbf{j} X_{2}+\mathbf{k} X_{3}=Z-\mathbf{j} W^{\dagger} \in M_{n}(\mathbb{H})
$$

Note that the complex structure $\mathbf{j}$ thus acts as $\mathbf{j}(Z, W)=\left(W^{\dagger},-Z^{\dagger}\right)$. This is an isomorphism of hyperkähler vector spaces, where $M_{n}(\mathbb{H})$ has the usual left action of $\mathbb{H}$ and the quaternionic Hermitian form:

$$
((P, Q))=\operatorname{Tr}_{\mathbb{H}}\left(P Q^{\dagger}\right) \in \mathbb{H}
$$

for $P, Q \in M_{n}(\mathbb{H})$.
Lemma 6.9. For $A \in \mathfrak{g}$ we have

$$
\mu(Q)(A)=\frac{1}{2} \operatorname{Im} \operatorname{Tr}_{\mathbb{H}}\left(\left(Q A-\left(Q^{T} A^{T}\right)^{T}\right) Q^{\dagger}\right) \in \operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}
$$

where $Q^{T}$ denotes the transpose of the quaternionic matrix $Q$ and $Q^{\dagger}$ is the componentwise quaternionic conjugate of $Q^{T}$.

In particular it follows immediately from this that

$$
\begin{equation*}
\mu(q Q)(A)=q(\mu(Q)(A)) q^{-1} \in \operatorname{Im} \mathbb{H} \tag{6.3}
\end{equation*}
$$

for any unit quaternion $q \in \mathbb{H}$.
Lemma 6.9 is a special case of the following result, which will also be very useful. Consider more generally the set of $\mathbb{V}=M_{n \times m}(\mathbb{H})$ of $n \times m$ quaternionic matrices, with the same quaternionic Hermitian form $\operatorname{Tr}_{\mathbb{H}}\left(P Q^{\dagger}\right)$. In complex structure $\mathbf{I}$, via the expression $Q=Z-\mathbf{j} W^{\dagger}$, we may identify $\mathbb{V}$ with the set of

$$
(Z, W) \in \operatorname{Hom}\left(\mathbb{C}^{m}, \mathbb{C}^{n}\right) \times \operatorname{Hom}\left(\mathbb{C}^{n}, \mathbb{C}^{m}\right)
$$

Now $(g, h) \in \operatorname{Sp}(n) \times \operatorname{Sp}(m)$ acts on $\mathbb{V}$, commuting with the action of $\mathbb{H}$ (by left multiplication), and preserving $((\cdot, \cdot))$, as follows:

$$
g(Q)=\left(Q^{T} g^{T}\right)^{T}, \quad h(Q)=Q h^{-1} .
$$

This defines a map $\operatorname{Sp}(n) \times \operatorname{Sp}(m) \rightarrow \operatorname{Sp}(n m)$. From (the remark after) Lemma 6.6 the action of $\operatorname{Sp}(m)$ has moment map

$$
\mu_{m}(Q)(B)=\frac{1}{2} \operatorname{Im} \operatorname{Tr}_{\mathbb{H}}\left(Q B Q^{\dagger}\right) \in \operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}
$$

for $B \in \mathfrak{s p}(m)$ and the action of $\operatorname{Sp}(n)$ has moment map

$$
\mu_{n}(Q)(A)=-\frac{1}{2} \operatorname{Im} \operatorname{Tr}_{\mathbb{H}}\left(\left(Q^{T} A^{T}\right)^{T} Q^{\dagger}\right) \in \operatorname{Im} \mathbb{H} \cong \mathbb{R}^{3}
$$

for $A \in \mathfrak{s p}(n)$. Now we may restrict this action to the subgroup $\mathrm{U}(n) \times \mathrm{U}(m) \subset$ $\operatorname{Sp}(n) \times \operatorname{Sp}(m)$ where e.g. $\mathfrak{u}(n) \subset \mathfrak{s p}(n)$ is the subset of quaternionic skew-adjoint matrices which have no $\mathbf{j}$ or $\mathbf{k}$ components. Then the action may be rewritten as follows in terms of $Z, W$ using $Q=Z-\mathbf{j} W^{\dagger}$ :

$$
\begin{equation*}
g(Z, W)=\left(g Z, W g^{-1}\right), \quad h(Z, W)=\left(Z h^{-1}, h W\right) \tag{6.4}
\end{equation*}
$$

for $(g, h) \in \mathrm{U}(n) \times \mathrm{U}(m)$. If $n=m$, and we restrict to the diagonal subgroup, then Lemma 6.9 is now immediate. In general the real and imaginary part of the moment map may be computed directly, as follows:

Lemma 6.10. If $Q=Z-\mathbf{j} W^{\dagger} \in M_{n \times m}(\mathbb{H})$ and $A \in \mathfrak{u}(n), B \in \mathfrak{u}(m)$ then

$$
\begin{gathered}
\operatorname{Tr}_{\mathbb{H}}\left(\left(Q^{T} A^{T}\right)^{T} Q^{\dagger}\right)=\operatorname{Tr} A\left(Z Z^{\dagger}-W^{\dagger} W\right)+2 \operatorname{Tr}(A Z W) \mathbf{j} \in \operatorname{Im} \mathbb{H} \\
\operatorname{Tr}_{\mathbb{H}}\left(Q B Q^{\dagger}\right)=\operatorname{Tr} B\left(Z^{\dagger} Z-W W^{\dagger}\right)+2 \operatorname{Tr}(B W Z) \mathbf{j} \in \operatorname{Im} \mathbb{H}
\end{gathered}
$$

so that ${ }^{3}$

$$
\begin{array}{ll}
\left(\mu_{n}\right)_{\mathbb{R}}(Q)=\frac{i}{2}\left(Z Z^{\dagger}-W^{\dagger} W\right), & \left(\mu_{n}\right)_{\mathbb{C}}(Q)=-Z W \\
\left(\mu_{m}\right)_{\mathbb{R}}(Q)=\frac{i}{2}\left(W W^{\dagger}-Z^{\dagger} Z\right), & \left(\mu_{m}\right)_{\mathbb{C}}(Q)=W Z
\end{array}
$$

In particular if $m=n$ taking the sum of these (in each column) corresponds to restricting to the diagonal action, which then matches with the original formulae above. Also putting $m=1, Z=v, W=\alpha$ yields the formula used in the construction of the cotangent bundles of the projective spaces.

## Simplest hyperkähler rotation.[summary]

Via Equation (6.3) $T^{*} \mathbb{P}^{n-1}$ in complex structure $\mathbf{I}$, becomes an adjoint orbit of semisimple rank one matrices in $\mathfrak{g l}_{n}(\mathbb{C})$, in all other complex structures except $-\mathbf{I}$. When $n=2$ identify with affine quadric $x^{2}+y^{2}+z^{2}=1$ in $\mathbb{C}^{3}$ (view as alternative "complexification" of $S^{2}$ to $T^{*} \mathbb{P}^{1}$ ).

## Some facts about hyperkähler manifolds. [summary]

Compact+Kähler+complex symplectic implies hyperkähler, via Yau's theorem.

Guan's counterexample [Gua95]: there are compact, simply connected, complex symplectic manifolds which are not Kähler.

Alternative approach to hyperkähler metric on Kummer K3's by gluing 16 EguchiHanson spaces on to flat four-torus (made precise by Topiwala [Top87], Le BrunSinger [LS94]).

[^3]Cotangent bundles of Kähler manifolds: nonexistence on $T^{*} \Sigma$ if $\Sigma$ a compact Riemann surface of genus $g \geq 2$. Statement of the theorem of Feix-Kaledin. Donaldson's computation [Don03] for the cotangent bundle of the Poincaré disk.

## Twistor space.

Although we will not need it in this course a beautiful fact about hyperkähler manifolds is that all the complex structures fit together into a complex manifold of one (complex) dimension higher, and the metric itself may be encoded in this way, extending the original work of Penrose.

Let $M$ be a hyperkähler manifold. Let

$$
\mathcal{Z}=M \times S^{2}
$$

be the product of $M$ and the two-sphere. This defines $\mathcal{Z}$ as a smooth manifold, and we will give it a complex structure as follows. We use stereographic projection to identify ${ }^{4}$ the sphere $S^{2}$ (viewed as the sphere of complex structures on $M$ ) with the complex projective line $\mathbb{P}^{1}(\mathbb{C})$. Given a point $z=(m, a) \in \mathcal{Z}$ we define a complex structure $\mathbb{I}$ on the tangent space $T_{z} \mathcal{Z}=T_{m} M \oplus T_{a} \mathbb{P}^{1}$ as follows:

$$
\mathbb{I}(X, Y)=\left(\mathbf{I}_{a} X, I_{0} Y\right)
$$

where $\mathbf{I}_{a} \in \operatorname{End}\left(T_{m} M\right)$ is the complex structure corresponding to $a$ and $I_{0}$ is the standard complex structure on the Riemann sphere. These fit together to define an almost complex structure on $\mathcal{Z}$, and it is a theorem that it is integrable, making $\mathcal{Z}$ into a complex manifold (cf. [AHS78, Sal82, HKLR87]).

This has the following features:

1) The projection $\pi: \mathcal{Z} \rightarrow \mathbb{P}^{1}$ is a holomorphic map, and for each $m \in M$ the inclusion (section of $\pi) \mathbb{P}^{1} \rightarrow \mathcal{Z} ; \lambda \mapsto(m, \lambda)$ is holomorphic, and the normal bundle to each such embedded $\mathbb{P}^{1}$ is $\mathcal{O}(1) \otimes \mathbb{C}^{\operatorname{dim}_{\mathbb{C}}(M)}=\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$.
2) The antipodal map $\sigma(\lambda)=-1 / \bar{\lambda}$ on $\mathbb{P}^{1}$ (lifted to $\mathcal{Z}$ acting trivially on $M$ ) takes $\mathbf{I}_{a}$ to $-\mathbf{I}_{a}$ and $I_{0}$ to $-I_{0}$-i.e. it is an anti-holomorphic involution, giving $\mathcal{Z}$ a real structure, as one sees for example on a complex algebraic variety defined by real equations.
3) For any $\lambda \in \mathbb{C}$ the complex two-form

$$
\omega_{\mathbb{C}}^{\lambda}:=\omega_{\mathbf{J}}+\sqrt{-1} \omega_{\mathbf{K}}+2 \lambda \omega_{\mathbf{I}}-\lambda^{2}\left(\omega_{\mathbf{J}}-\sqrt{-1} \omega_{\mathbf{K}}\right)=\omega_{\mathbb{C}}+2 \lambda \omega_{\mathbf{I}}-\lambda^{2} \bar{\omega}_{\mathbb{C}}
$$

is a complex symplectic form on $M$ in complex structure $\mathbf{I}_{a}$ (where $a \in S^{2}$ is determined by $\lambda$ as above). It constitutes a holomorphic section of $\bigwedge^{2} T_{F}^{*}(2)$ over $\mathcal{Z}$. (Here $T_{F}$ is the vertical tangent bundle relative to the projection $\mathcal{Z} \rightarrow \mathbb{P}^{1}$, and the 2 means

[^4]we twist by the pull back to $\mathcal{Z}$ of the line bundle $\mathcal{O}(2)$ on $\mathbb{P}^{1}$ —this just expresses the fact $\omega_{\mathbb{C}}^{\lambda}$ is quadratic in $\lambda$.) [Beware that one needs the whole frame to define "the" complex two-form for a given complex structure.]

The fact is then that the hyperkähler metric is completely encoded in this data:
Theorem 6.11 ([HKLR87]). Let $\mathcal{Z}$ be a complex manifold of complex dimension $2 n+1$ such that

1) $\mathcal{Z}$ is a holomorphic fibre bundle $\pi: \mathcal{Z} \rightarrow \mathbb{P}^{1}$ over the projective line,
2) $\pi$ admits a family of holomorphic sections each with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2 n}$,
3) there exists a holomorphic section $\omega$ of $\bigwedge^{2} T_{F}^{*}(2)$ defining a complex symplectic form on each fibre of $\pi$,
4) $\mathcal{Z}$ has a real structure $\sigma$ lifting the antipodal map on $\mathbb{P}^{1}$ and compatible with 1-3).

Then the parameter space of real sections of $\pi$ is a manifold (of real dimension $4 n$ ) with a natural hyperkähler metric for which $\mathcal{Z}$ is the twistor space.

We will just state two examples without giving full details.
Example 6.12. The twistor space of $\mathbb{H}$ is the total space of $\mathcal{O}(1) \oplus \mathcal{O}(1) \rightarrow \mathbb{P}^{1}$. This is isomorphic to $\mathbb{P}^{3}$ minus a line $\left(\cong \mathbb{P}^{1}\right)$ : First note that $S^{4}$ (the one point compactification of $\mathbb{H})$ is isomorphic to $\mathbb{P}^{1}(\mathbb{H})$ the "quaternionic projective line". Then observe that there is a natural map $\mathbb{P}^{3}(\mathbb{C}) \rightarrow \mathbb{P}^{1}(\mathbb{H})$ taking a complex ray in $\mathbb{C}^{4} \cong \mathbb{H}^{2}$ to its $\mathbb{H}$ span:

$$
[v]_{\mathbb{C}} \mapsto[v]_{\mathbb{H}} .
$$

This map has fibres isomorphic to $S^{2}$ and the twistor space of $\mathbb{H}$ arises by removing the fibre over $\infty \in \mathbb{P}^{1}(\mathbb{H})$.

Example 6.13. Let $H=\mathcal{O}(1)$ denote be the hyperplane bundle over $\mathbb{P}^{1}$, so global sections of $H^{k}$ correspond to degree $k$ complex polynomials. Choose a real quadratic polynomial $p \in \Gamma\left(H^{2}\right)$. The twistor space of the Eguchi-Hanson space $T^{*} \mathbb{P}^{1}$ arises by resolving the singularities of the hypersurface

$$
\left\{x y=z^{2}-p^{2}\right\} \subset H^{2} \oplus H^{2} \oplus H^{2} \rightarrow \mathbb{P}^{1}
$$

in the four-fold equal to the total space of $H^{2} \oplus H^{2} \oplus H^{2} \rightarrow \mathbb{P}^{1}$. Here $x, y, z$ each denote sections of $H^{2}$. In other words for each $\lambda \in \mathbb{P}^{1}$ we have a surface $x y=z^{2}-p^{2}(\lambda)$ in $\mathbb{C}^{3}$ and we are fitting them into a family as $\lambda$ varies. The surfaces will be singular when $\lambda$ equals one of the two roots of $p$ and they should be resolved to get the twistor space. (We recognise these surfaces as the smooth affine quadrics discussed earlier if $p(\lambda) \neq 0$ and the $A_{1}$ singularities if $p(\lambda)=0$, which resolve to give $T^{*} \mathbb{P}^{1} \cong \mathcal{O}(-2)$.)

This twistor space also has a natural compactification: one obtains the Nagata three-fold $W$, the first example of a compact algebraic manifold which is not projective (cf. [Nag58, Fuj98]). $W$ is defined starting from $\left(\mathbb{P}^{1}\right)^{3}$ by some explicit birational transformations. It admits a map to $\mathbb{P}^{1}$ such that the fibre $W_{\lambda}$ over $\lambda \in \mathbb{P}^{1}$ is isomorphic to $\mathbb{P}^{1} \times \mathbb{P}^{1}$ if $\lambda \neq 0, \infty$ and to the second Hirzebruch surface $\Sigma_{2}$ otherwise. Of course $\mathbb{P}^{1} \times \mathbb{P}^{1}$ is just a smooth quadric surface (a degree two surface in $\mathbb{P}^{3}$ e.g. the compactifications of our affine quadrics in $\mathbb{C}^{3}$ ) and $\Sigma_{2}$ is, by definition, the natural compactification of $T^{*} \mathbb{P}^{1}$, namely it is the projectivisation of the rank two bundle $\mathcal{O}(-2) \oplus \mathcal{O}$ on $\mathbb{P}^{1}$.

## 7. HyperkÄhler quiver varieties

## Hyperkähler varieties attached to quivers.

Our basic aim is to construct all the asymptotically locally Euclidean hyperkähler four-manifolds. [Definition of ALE omitted in notes.] These were constructed uniformly by Kronheimer [Kro89] and they involve the simply-laced affine Dynkin diagrams. However the construction admits a natural generalisation to arbitrary graphs so we will give the general construction. Such more general quiver varieties appear in work of Kronheimer-Nakajima [KN90] as moduli spaces of instantons on ALE hyperkähler four-manifolds and have been used extensively by Nakajima in work on the representation theory of quantum algebras cf. [Nak94, Nak98, Nak01]. In fact an apparently more general definition is used by Nakajima (involving weight spaces), but as we will see (cf. [CB01]) as hyperkähler manifolds they arise as special cases of the "weight-less" construction we will give here.
Choices. Let $\mathcal{Q}$ be a quiver, i.e. an oriented (finite) graph, with nodes $N$ and edges $\mathcal{Q}$. We will assume there are no edge loops (i.e. edges starting and ending at the same node), although many results generalise easily to this case. Multiple edges between the same pair of nodes are permitted. (We will say that $\mathcal{Q}$ is "simply-laced" if it has no multiple edges.) Choose an integer $d_{k} \geq 0$ and a triple of purely imaginary numbers $\zeta_{k} \in i \mathbb{R}^{3}$ for each node $k \in N$, such that

$$
\begin{equation*}
\sum \zeta_{k} d_{k}=0 \in i \mathbb{R}^{3} \tag{7.1}
\end{equation*}
$$

From this data we will construct a smooth hyperkähler manifold, a quiver variety, which will be complete under a genericity assumption on the data.

We will call the collection $\mathbf{d}=\left(d_{k}\right)_{k} \in \mathbb{Z}^{N}$ of integers $d_{k}$ the dimension vector, and the collection $\zeta=\left(\zeta_{k}\right)_{k} \in\left(i \mathbb{R}^{3}\right)^{N}$ the parameters. The condition (7.1) will be rewritten as $\zeta \cdot \mathbf{d}=0$.

We will say that $\mathbf{d}$ is indivisible if it is not a positive integer multiple of another element of $\mathbb{Z}^{N}$. From the quiver $\mathcal{Q}$ we may define a (generalised Cartan) matrix $\mathbf{C}$ : it is the $N \times N$ matrix

$$
\mathbf{C}=2 \operatorname{Id}-\mathbf{A}
$$

where $\mathbf{A}$ is the adjacency matrix of $\mathcal{Q}$ : the $j k$ matrix entry of $\mathbf{A}$ is the total number of edges between $j$ and $k$ in either direction (so it is a symmetric matrix). We will then say $\zeta$ is generic (with respect to $\mathbf{d}$ ) if

$$
\zeta \cdot v \neq 0 \in i \mathbb{R}^{3}
$$

for any $v$ in the finite set:

$$
R_{\oplus}(\mathbf{d}):=\left\{v=\left(v_{k}\right) \in \mathbb{Z}_{\geq 0}^{N} \mid v \cdot \mathbf{C} v \leq 2, \text { and } v_{k} \leq d_{k} \text { for all } k\right\} \backslash\{0, \mathbf{d}\}
$$

The main result of this section is then the following:
Theorem 7.1. There is a (possibly singular) space $\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)$ attached to the above data together with an open subset

$$
\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta) \subset \mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)
$$

If nonempty, $\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta)$ is a smooth hyperkähler manifold of complex dimension

$$
2-\mathbf{d} \cdot \mathbf{C d}
$$

and moreover if $\zeta$ is generic with respect to $\mathbf{d}$ then $\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)=\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta)$ is smooth and complete.

This will (basically) follow directly from the hyperkähler quotient theorem. Note that if $\mathbf{d}$ is divisible (i.e. not indivisible) then there are no generic parameters $\zeta$. Another trivial point that will be useful later is that if $d_{k}=1$ for some $k$ then $\mathbf{d}$ is indivisible.

The simplest examples (of minimal positive dimension) are of complex dimension 2 (i.e. real four-manifolds). These will be discussed later. They occur when $\mathbf{d} \cdot \mathbf{C d}=0$, i.e. $\mathbf{d}$ is "null", i.e. we are looking for a quiver whose Cartan matrix has a nontrivial kernel. Such matrices arise from the simply-laced affine Dynkin diagrams. They are of type $A_{n}, D_{n}, E_{6}, E_{7}, E_{8}$, and the corresponding hyperkähler four manifolds are the "ALE gravitational instantons". We will see that understanding this involves the McKay correspondence relating the affine Dynkin diagrams to the finite subgroups of $\mathrm{SL}_{2}(\mathbb{C})$. (In type $A$ these spaces are the "multi-Eguchi Hanson spaces" of GibbonsHawking [GH78], with the case $A_{1}$ being $T^{*} \mathbb{P}^{1}$ the Eguchi-Hanson space.)

## Construction.

Let $\widehat{\mathcal{Q}}$ denote the double of the quiver $\mathcal{Q}$, having the same nodes but with an extra edge added in the opposite direction for each edge of $\mathcal{Q}$. Thus if $\overline{\mathcal{Q}}$ denotes $\mathcal{Q}$ with each edge reversed then in terms of edges: $\widehat{\mathcal{Q}}=\mathcal{Q} \sqcup \overline{\mathcal{Q}}$. For each node $k \in N$ let
$V_{k}=\mathbb{C}^{d_{k}}$ be a complex vector space of dimension $d_{k}$, with the standard Hermitian form. Then as in Lemma 6.10 for each edge from $i \in N$ to $j \in N$ we can define a hyperkähler vector space $T^{*} \operatorname{Hom}\left(V_{i}, V_{j}\right)=\operatorname{Hom}\left(V_{i}, V_{j}\right) \times \operatorname{Hom}\left(V_{i}, V_{j}\right)$ isomorphic to the $d_{j} \times d_{i}$ quaternionic matrices. Taking the product of these for all edges, the quiver $\mathcal{Q}$ determines a large hyperkähler vector space:

$$
\mathbb{V}=\mathbb{V}_{\mathcal{Q}}(\mathbf{d}):=\prod_{e \in \mathcal{Q}} T^{*} \operatorname{Hom}\left(V_{t(e)}, V_{h(e)}\right)
$$

where $h, t: \mathcal{Q} \rightarrow N$ are the maps taking an edge to its head and tail respectively ${ }^{5}$. Thus a point of $\mathbb{V}$ is given by a collection of linear maps $\left(B_{e}\right)_{e}$ with $B_{e} \in$ $\operatorname{Hom}\left(V_{t(e)}, V_{h(e)}\right)$ for each edge $e \in \widehat{\mathcal{Q}}$ in the quiver $\widehat{\mathcal{Q}}$. (The case of Lemma 6.10 corresponds $\mathcal{Q}$ consisting of a single edge from $\mathbb{C}^{m}$ to $\mathbb{C}^{n}$.)

The complex dimension of $\mathbb{V}$ is

$$
2 \sum_{e \in \mathcal{Q}} d_{t(e)} d_{h(e)}=\sum_{i, j \in N} \mathbf{A}_{i j} d_{i} d_{j}
$$

where $\mathbf{A}$ is the adjacency matrix of $\mathcal{Q}$.
Now consider the group:

$$
\widetilde{G}=\mathrm{U}(\mathbf{d}):=\prod_{i \in N} \mathrm{U}\left(V_{i}\right) \subset \mathrm{GL}(\mathbf{d}):=\prod_{i \in N} \mathrm{GL}\left(V_{i}\right)
$$

of unitary automorphisms of the vector spaces at each node. (Here $\mathrm{U}\left(V_{i}\right) \subset \mathrm{GL}\left(V_{i}\right)$ is the group of automorphisms of $V_{i}$ preserving the inner product.) As in Equation (6.4) this acts on $\mathbb{V}$ in the natural way preserving the hyperkähler structure: if $g=\left(g_{i}\right)_{i} \in \widetilde{G}$ and $B_{j i} \in \operatorname{Hom}\left(V_{i}, V_{j}\right)$ then

$$
g(\phi)=g_{j} \circ B_{j i} \circ g_{i}^{-1}
$$

From Lemma 6.10 we see immediately that a hyperkähler moment map for this action is given explicitly by $\mu=\left(\mu_{\mathbb{R}}, \mu_{\mathbb{C}}\right)$ where

$$
\begin{gathered}
\mu_{\mathbb{R}}\left(\left(B_{e}\right)_{e}\right)=\frac{\sqrt{-1}}{2}\left(\sum_{e \in \widehat{\mathcal{Q}}, t(e)=i} B_{\bar{e}} B_{\bar{e}}^{\dagger}-B_{e}^{\dagger} B_{e}\right)_{i} \in \bigoplus_{i \in N} \mathfrak{u}\left(V_{i}\right)=\operatorname{Lie}(\widetilde{G})=\widetilde{\mathfrak{g}} \\
\mu_{\mathbb{C}}\left(\left(B_{e}\right)_{e}\right)=\left(\sum_{e \in \mathcal{\mathcal { Q }}, t(e)=i} \varepsilon(e) B_{\bar{e}} B_{e}\right)_{i} \in \bigoplus_{i \in N} \mathfrak{g l}\left(V_{i}\right)=\widetilde{\mathfrak{g}}_{\mathbb{C}}
\end{gathered}
$$

where $\varepsilon: \widehat{\mathcal{Q}} \rightarrow\{ \pm 1\}$ maps the edges of $\mathcal{Q}$ to +1 and $\overline{\mathcal{Q}}$ to -1 . For $e \in \widehat{\mathcal{Q}}, \bar{e}$ denotes the corresponding edge with the opposite orientation (so $\widehat{\mathcal{Q}}=\mathcal{Q} \sqcup \overline{\mathcal{Q}}$ ).

[^5]Slick notation. Let $V=\bigoplus V_{k}$ denote the direct sum of the vector spaces at the nodes. Thus $V$ is an " $N$-graded vector space". Given two such $N$-graded vector spaces $V, W$ define

$$
L(V, W)=\bigoplus_{k \in N} \operatorname{Hom}\left(V_{k}, W_{k}\right)
$$

and

$$
E(V, W)=\bigoplus_{e \in \widehat{\mathcal{Q}}} \operatorname{Hom}\left(V_{t(e)}, W_{h(e)}\right)
$$

(Thus $L(V, W)$ may be viewed as "block-diagonal matrices" in

$$
\operatorname{Hom}(V, W)=\bigoplus_{i, j \in N} \operatorname{Hom}\left(V_{i}, W_{j}\right)
$$

and, if $\mathcal{Q}$ is simply-laced, $E(V, W)$ may be viewed in terms of "off-diagonal block matrices", where the blocks are parameterised by $N$.)

Thus $\mathbb{V}=E(V, V)$ and $\widetilde{\mathfrak{g}}_{\mathbb{C}}=L(V, V)$. Given a third $N$-graded vector space $U$, there is a map

$$
E(V, W) \times E(U, V) \rightarrow L(U, W) ; \quad(A, B) \mapsto A B
$$

defined by

$$
A B=\left(\sum_{e \in \widehat{\mathcal{Q}}, t(e)=k} A_{\bar{e}} B_{e}\right)_{k} \in L(U, W)
$$

(In the simply-laced case, this amounts to matrix multiplication followed by projection to the block-diagonal part.) Then the moment maps may be rewritten as follows:

$$
\begin{gathered}
\mu_{\mathbb{R}}(B)=\frac{\sqrt{-1}}{2}\left(B B^{\dagger}-B^{\dagger} B\right) \in \widetilde{\mathfrak{g}} \\
\mu_{\mathbb{C}}(B)=B B \varepsilon \in \widetilde{\mathfrak{g}}_{\mathbb{C}}
\end{gathered}
$$

where $B \varepsilon \in E(V, V)$ is defined by $(B \varepsilon)_{e}=B_{e} \varepsilon(e)$.
Exercise 7.2. Suppose $N=\{1,2\}$ and $\mathcal{Q}$ has one edge from 1 to 2. Thus $V=V_{1} \oplus V_{2}$ and $B=\left(\begin{array}{cc}0 & W \\ Z & 0\end{array}\right) \in \operatorname{End}(V)$. Compute $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ and compare with Lemma 6.10.

Now to construct the quiver variety we perform the hyperkähler quotient at the value $\zeta$ of the moment map, after identifying the parameters $\zeta=\left(\zeta_{k}\right)_{k}$ with a triple of elements of the centre of the Lie algebra of $\widetilde{G}$ :

$$
\zeta_{k}=\left(\zeta_{k}^{1}, \zeta_{k}^{2}, \zeta_{k}^{3}\right) \in i \mathbb{R}^{3} \mapsto\left(\zeta_{k, \mathbb{R}}, \zeta_{k, \mathbb{C}}\right):=\left(\zeta_{k}^{1} \operatorname{Id}_{V_{k}},\left(\zeta_{k}^{2}+i \zeta_{k}^{3}\right) \operatorname{Id}_{V_{k}}\right) \in \mathfrak{u}\left(V_{k}\right) \oplus \mathfrak{g l}\left(V_{k}\right)
$$

so that the equation $\mu_{\mathbb{C}}(B)=\zeta_{\mathbb{C}}$ corresponds to setting $\mu_{2}(B)=\zeta^{2}, \mu_{3}(B)=\zeta^{3}$ if $\mu_{\mathbb{C}}=\mu_{2}+i \mu_{3}$.

Thus we may define:

$$
\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)=\mathbb{V}_{\mathcal{Q}}(\mathbf{d}) / / / \widetilde{\zeta} \widetilde{G}=\mu^{-1}(\zeta) / \widetilde{G}
$$

to be the hyperkähler quotient at the value $\zeta$ of the moment map. However $\widetilde{G}$ will never act freely since there is a central circle $\mathbb{T} \subset \widetilde{G}$ which always acts trivially on $\mathbb{V}: t \in \mathrm{U}(1)$ acts on $v \in V_{i}$ by scalar multiplication by $t$ for all $i$ (this defines a map $\mathrm{U}(1) \rightarrow \widetilde{G}$ and we define $\mathbb{T}$ to be its image - so each component $g_{i}$ of $g \in \mathbb{T}$ is "the same" scalar matrix, modulo the slight subtlety that some $V_{i}$ may be trivial: $g_{i}=t \cdot \operatorname{Id}_{V_{i}}$ if $\operatorname{dim}\left(V_{i}\right)>0$ else $\left.g_{i}=1\right)$. We define the smaller group

$$
G=\widetilde{G} / \mathbb{T}
$$

which also acts on $\mathbb{V}$ (and this action has a chance to be free at some points). The above map $\mu$ is also a hyperkähler moment map for $G$ : the dual of the Lie algebra of $G$ is identified with the hyperplane $\sum \operatorname{Tr}\left(X_{i}\right)=0$ in $\tilde{\mathfrak{g}}$ and from the explicit expression it is clear that $\mu$ takes values in this hyperplane. When restricted to the scalar matrices this hyperplane yields the condition $\zeta \cdot \mathbf{d}=0$ we already imposed on the parameters (said differently if we don't impose this condition then $\mathfrak{M}_{\mathcal{Q}}$ is empty). Thus we also have $\mathfrak{M}_{\mathcal{Q}}=\mathbb{V} / / / G$.

Also define

$$
\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta):=\mathbb{V}^{\mathrm{reg}} / / /{ }_{\zeta} G
$$

where $\mathbb{V}^{\text {reg }} \subset \mathbb{V}$ is the (open) subset of $\mathbb{V}$ where $G$ acts freely (i.e. the points having trivial stabiliser in $G$ ). [The openness follows from the slice theorem for smooth actions of compact groups.] From the hyperkähler reduction theorem we know that, if nonempty, $\mathfrak{M}_{\mathcal{Q}}^{\text {reg }}(\mathbf{d}, \zeta)$ will have complex dimension

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{C}}\left(\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta)\right) & =\operatorname{dim}_{\mathbb{C}}(\mathbb{V})-2 \operatorname{dim}_{\mathbb{R}}(G) \\
& =\operatorname{dim}_{\mathbb{C}}(\mathbb{V})-2 \operatorname{dim}_{\mathbb{R}}(\widetilde{G})+2 \\
& =\mathbf{d} \cdot \mathbf{A d}-2 \mathbf{d} \cdot \mathbf{d}+2 \\
& =2-\mathbf{d} \cdot \mathbf{C d}
\end{aligned}
$$

as expected.
We can now give the proof of Theorem 7.1:
Proof (of Theorem 7.1). It is enough to show that if $G$ does not act freely on $\mu^{-1}(\zeta)$ then the parameters $\zeta$ are not generic, i.e. satisfies an equation:

$$
\zeta \cdot v=0 \in i \mathbb{R}^{3}
$$

for some nonzero $v \in \mathbb{Z}^{N}$ distinct from $\mathbf{d}$ such that $v \cdot \mathbf{C} v \leq 2$, and $v_{i} \leq d_{i}$ for all $i$.

If this action is not free then there is some $g \in \widetilde{G}-\mathbb{T}$ which fixes some point $B=\left(B_{e}\right)_{e} \in \mathbb{V}$ with $\mu(B)=\zeta$. Now decompose each vector space

$$
V_{i}=\bigoplus_{\lambda} V_{i}(\lambda)
$$

into the eigenspaces of $g_{i}$ (where the $\lambda$ are the eigenvalues). It follows that

$$
\begin{equation*}
B_{e}\left(V_{t(e)}(\lambda)\right) \subset V_{h(e)}(\lambda), \quad \text { and } \quad B_{e}^{\dagger}\left(V_{h(e)}(\lambda)\right) \subset V_{t(e)}(\lambda) \tag{7.2}
\end{equation*}
$$

for all edges $e$. (E.g. if $i=h(e), j=t(e)$ then we know $g_{i} B_{e} g_{j}^{-1}=B_{e}$, so if $v \in V_{j}(\lambda)$ then $g_{i} B_{e}(v)=B_{e} g_{j}(v)=\lambda B_{e}(v)$, so $B_{e}(v) \in V_{i}(\lambda)$.)

Since $g \notin \mathbb{T}$ we may choose $\lambda$ such that $V_{i}(\lambda)$ is nontrivial and distinct from $V_{i}$ for some $i$. Define $V_{i}^{\prime}=V_{i}(\lambda)$ for all $i$, and define $\mathbf{d}^{\prime}$ so that $d_{i}^{\prime}=\operatorname{dim}\left(V_{i}^{\prime}\right)$. The equations in (7.2) show that the maps $B_{e}$ restrict to maps $B_{e}^{\prime}$ between the space $V_{i}^{\prime}$. They therefore define a point $B^{\prime}$ of $\mathbb{V}_{\mathcal{Q}}\left(\mathbf{d}^{\prime}\right)$. We may assume $G^{\prime}$ acts freely on the orbit of $B^{\prime} \in \mathbb{V}_{\mathcal{Q}}\left(\mathbf{d}^{\prime}\right)$, since otherwise we may repeat and decompose further (this includes the possibility that $G^{\prime}$ is the trivial group - which always acts freely). Looking at the moment map we see it follows from equation (7.2) that the value at $B^{\prime}$ of the moment map for $G^{\prime}$ is again $\zeta$ (the matrices $B_{e}$ are all "block-diagonal" with a rectangular block for each eigenvalue, and we are picking out the block corresponding to $\lambda$ ). It then follows that $\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}\left(\mathbf{d}^{\prime}, \zeta\right)$ is a nonempty hyperkähler manifold (since the subset of points at which the action is free is open and nonempty). The dimension formula (for this nonempty hyperkähler quotient) then implies

$$
2-\mathrm{d}^{\prime} \cdot \mathrm{Cd}^{\prime} \geq 0
$$

On the other hand since it is nonempty we must have $\zeta \cdot \mathbf{d}^{\prime}=0$ (as before, taking the trace of the moment map equation). Thus $v=\mathrm{d}^{\prime}$ contradicts the genericity of $\zeta$.

Note on signs: our moment map $\mu$ is minus that of [Nak03] but the parameters $\zeta$ should be identified: we do reduction at $\zeta$ whereas [Nak03] reduces at $-\zeta$, so the resulting spaces coincide. Beware that, as remarked in [Nak03], the sign of $\mu_{\mathbb{R}}$ there is opposite to that of [Nak94].

## 8. ALE HYPERKÄHLER FOUR MANIFOLDS

Now we will apply the above machinery to construct all the ALE hyperkähler four manifolds, following Kronheimer [Kro89]. The desired graphs and dimension vectors arise as follows.

## McKay's correspondence.

Let $\Gamma \subset \mathrm{Sp}(1) \cong \mathrm{SU}(2)$ be a finite group and let $N$ be the set of irreducible complex representations of $\Gamma$. Write $N=\left\{V_{0}, V_{1}, \ldots, V_{l}\right\}$ with $V_{0}$ being the trivial
one-dimensional representation. Now $\mathbb{H} \cong \mathbb{C}^{2}$ is the natural representation of $\Gamma$ and so we may decompose the tensor product

$$
\mathbb{H} \otimes_{\mathbb{C}} V_{i} \cong \bigoplus_{j \in N} A_{i j} V_{j}
$$

into irreducibles, where $A_{i j}$ is the multiplicity of $V_{j}$ in $\mathbb{H} \otimes V_{i}$. The corresponding $N \times N$ matrix $\mathbf{A}$ is symmetric, with each entry either 0 or 1 (unless $\Gamma=\mathbb{Z} / 2$ ) and with zeros on the diagonal. The McKay graph of $\Gamma$ is the graph with nodes $N$ and adjacency matrix A. McKay observed [McK80] that the graphs which arise in this way are precisely the extended (affine) Dynkin diagrams of the simply-laced simple Lie algebras: namely they are of type $A_{n}, D_{n}, E_{6}, E_{7}, E_{8}$ corresponding to the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral subgroups of $\mathrm{SU}(2)$ respectively. Moreover if we set $d_{i}=\operatorname{dim}\left(V_{i}\right)$ then the corresponding dimension vector $\mathbf{d}$ spans the kernel of the corresponding affine Cartan matrix $\mathbf{C}=2-\mathbf{A}$. On the Lie algebra side of the correspondence the nodes of the finite Dynkin diagram (obtained by removing the node 0 from the affine Dynkin diagram) label a choice of simple roots $\alpha_{1}, \ldots, \alpha_{l}$ and on this side of the correspondence the integers $d_{i}$ appear as the coefficients of the highest root in the basis of simple roots, namely the highest root is:

$$
\sum_{1}^{l} d_{i} \alpha_{i}
$$

and the numbers $d_{i}$ are readily listed in books on Lie algebras ([Bou81] Appendix, [Kac90] p.54). For example the cases of type $E_{6}, E_{7}, E_{8}$ are as in the following diagrams.


Extended $E_{6}$ Dynkin diagram
McKay graph of the binary tetrahedral group


Extended $E_{7}$ Dynkin diagram
McKay graph of the binary octahedral group


Extended $E_{8}$ Dynkin diagram
McKay graph of the binary icosahedral group

Thus we may take one of these graphs with the given dimension vector $\mathbf{d}$ and apply Theorem 7.1 to obtain a family of hyperkähler four manifolds (since $\mathbf{d}$ is in the kernel of $\mathbf{C}$, the complex dimension is 2 ).

Now we would like to understand the exceptional set of parameters $\zeta$ better, where the action may not be everywhere free. If we let

$$
X=\left\{x \in \mathbb{R}^{N} \mid x \cdot \mathbf{d}=0\right\} \cong \mathbb{R}^{l}
$$

then for each $v \in R_{\oplus}(\mathbf{d})$ we have a hyperplane

$$
D_{v}=\{x \in X \mid x \cdot v=0\}
$$

and by definition the parameters $\zeta$ are generic if they are in the set

$$
\begin{equation*}
\left(X \otimes i \mathbb{R}^{3}\right)^{o}:=X \otimes i \mathbb{R}^{3} \backslash \bigcup_{v \in R_{\oplus}(\mathbf{d})} D_{v} \otimes i \mathbb{R}^{3} \tag{8.1}
\end{equation*}
$$

-i.e. if they are off of a finite number of codimension three linear subspaces.
Lemma 8.1. We may identify $X$ with a real Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of the simple Lie algebra corresponding to $\Gamma$, and then the hyperplanes $D_{v}$ are the root hyperplanes of the corresponding (finite) root system.

Proof. Write $\mathbb{Z}^{N}=\bigoplus_{i \in N} \mathbb{Z} \varepsilon_{i}$, so $\mathbf{d}=\sum d_{i} \varepsilon_{i}$, and $\varepsilon_{i} \cdot \varepsilon_{j}=\delta_{i j}$. Define $\alpha_{i}=\mathbf{C} \varepsilon_{i}$, so $\varepsilon_{i} \cdot \alpha_{j}=\mathbf{C}_{i j}$. Then observe $\left\{\alpha_{1}, \ldots, \alpha_{l}\right\}$ is a basis of $X$, since the span of $\varepsilon_{1}, \ldots, \varepsilon_{l}$ does not intersect the kernel of $\mathbf{C}$ (spanned by d). Now define a real bilinear form on $X$ via

$$
\left(\left(\alpha_{i}, \alpha_{j}\right)\right)=\mathbf{C}_{i j}
$$

for $i, j>0$. This is positive definite since $\mathbf{C}$ (with indices $>0$ ) is the Cartan matrix of the (finite) Dynkin diagram obtained by removing the extending vertex 0 from the original affine Dynkin diagram. Thus we may view the $\alpha_{i}$ as the simple roots of the corresponding finite root system. The full set of roots is then characterised as the $\mathbb{Z}$-linear combinations of the $\alpha_{i}$ having square norm equal to 2 :

$$
\Delta=\left\{\alpha=\sum_{i>0} n_{i} \alpha_{i} \mid n_{i} \in \mathbb{Z},((\alpha, \alpha))=2\right\}
$$

The subset of positive roots $\Delta_{+} \subset \Delta$ is the subset with $n_{i} \geq 0$ for all $i$. We wish to examine the set of hyperplanes cut out in $X$ by the exceptional set

$$
R_{\oplus}(\mathbf{d}):=\left\{v=\left(v_{k}\right) \in \mathbb{Z}_{\geq 0}^{N} \mid v \cdot \mathbf{C} v \leq 2, \text { and } v_{k} \leq d_{k} \text { for all } k\right\} \backslash\{0, \mathbf{d}\}
$$

where $v$ defines the linear function $x \mapsto v \cdot x$ on $X$. This gives a map $R_{\oplus}(\mathbf{d}) \rightarrow X^{*}$ and then we identify $X$ and its dual $X^{*}$ via $((\cdot, \cdot))$, so we have a map $R_{\oplus}(\mathbf{d}) \rightarrow X$. To complete the proof we just need to show:

Lemma 8.2. The image of this map $R_{\oplus}(\mathbf{d}) \rightarrow X$ is the set of roots $\Delta$.

Proof. If $v \cdot \mathbf{C} v<2$ then we know ([Kac90]) $v=n \mathbf{d}$, but none of these are in $R_{\oplus}(\mathbf{d})$. Thus suppose $v \cdot \mathbf{C} v=2$. Write $v=\sum v_{i} \varepsilon_{i}$. Clearly $v$ and $v+n \mathbf{d}$ map to the same point of $X$ for any $n \in \mathbb{Z}$. Thus we may replace $v$ by $v^{\prime}=v-v_{0} \mathbf{d}$, so that $v_{0}^{\prime}=0$ (note that $v_{0}=0$ or 1 ). Moreover clearly $v \cdot \mathbf{C} v=v^{\prime} \cdot \mathbf{C} v^{\prime}$, since $\mathbf{C d}=0$. On the other hand the image of $v$ in $X$ is $w:=\sum v_{i}^{\prime} \alpha_{i}$, since for $j>0$ :

$$
v \cdot \alpha_{j}=v^{\prime} \cdot \alpha_{j}=\sum_{i} v_{i}^{\prime} \varepsilon_{i} \cdot \alpha_{j}=\sum_{i} v_{i}^{\prime} \mathbf{C}_{i j}=\sum_{i} v_{i}^{\prime}\left(\left(\alpha_{i}, \alpha_{j}\right)\right)=\left(\left(w, \alpha_{j}\right)\right)
$$

and these $\alpha_{j}$ are a basis of $X$. Now $((w, w))=\sum v_{i}^{\prime} v_{j}^{\prime} \mathbf{C}_{i j}=v^{\prime} \cdot \mathbf{C} v^{\prime}=v \cdot \mathbf{C} v=2$ so $w$ is in $\Delta$. Clearly all positive roots arise in this way when $v_{0}=0$ and all negative roots when $v_{0}=1$.

Thus the hyperplanes we obtain are the standard root hyperplanes for the corresponding simply-laced simple Lie algebra.

Thus we have constructed a family of complete hyperkähler four-manifolds parameterised by $\left(X \otimes i \mathbb{R}^{3}\right)^{o}$. They fit into a smooth fibration and all the fibres are diffeomorphic, i.e. for each $\Gamma$ we get a family of hyperkähler metrics on a single differentiable manifold. To understand the underlying differentiable manifold we will now recall Kronheimer's original approach, which will then be identified (as in [Kro89]) with the above quiver approach using the McKay correspondence (this shows how the quiver framework - spaces of maps in both directions along edges of graphs- arose naturally). This then enables us to define a map, if $\zeta_{\mathbb{C}}=0$, from the corresponding quiver variety to $\mathbb{C}^{2} / \Gamma$, (which is a minimal resolution of singularities for generic $\zeta_{\mathbb{R}}$ ). Thus the underlying differentiable manifold is the minimal resolution of the rational double point (Kleinian singularity) $\mathbb{C}^{2} / \Gamma$. This extends the viewpoint descried earlier for the Eguchi-Hanson space $T^{*} \mathbb{P}^{1}$ as the resolution of the closure of the nontrivial nilpotent orbit in $\mathfrak{s l}_{2}(\mathbb{C})$ (indeed this orbit closure is $\mathbb{C}^{2} / \pm 1$, which is the $A_{1}$ case, with $\Gamma=\mathbb{Z} / 2 \mathbb{Z})$.

Kronheimer's original approach.[summary]

Let $V=\mathbb{C}[\Gamma]$ be the regular representation of $\Gamma$, with standard Hermitian form. Let $P=T^{*} \operatorname{End}(V) \cong \mathbb{H} \otimes\{$ skew adjoint elements of $\operatorname{End}(V)\}$, which has a natural structure of hyperkähler vector space as usual. Points are given by pairs of complex matrices $\alpha, \beta \in \operatorname{End}(V)$, and the moment maps are as usual with $\mu_{\mathbb{C}}=[\alpha, \beta]$.

Now $\Gamma$ acts on $P$ (via the standard representation on $\mathbb{H}$, and unitary automorphisms of $V$ ), and we consider the fixed point set

$$
\mathbb{V}=P^{\Gamma}
$$

which is again a hyperkähler vector space. Set

$$
\widetilde{G}=\{g \in \mathrm{U}(V) \mid g \text { commutes with the action of } \Gamma \text { on } V\}
$$

and define $G=\widetilde{G} / \mathbb{T}$ where $\mathbb{T}$ is the centre of $\mathrm{U}(V)$. Then consider the hyperkähler quotient $M=\mathbb{V} / / / G$ for $\zeta$ a coadjoint invariant element of the dual of the Lie algebra of $G$.

Lemma 8.3. $M$ is a quiver variety associated to the affine Dynkin diagram corresponding to $\Gamma$ by the McKay correspondence, with standard dimension vector $\mathbf{d}$.

Proof. We will just identify $\mathbb{V}$ and leave the rest to the reader. The regular representation $V$ decomposes into irreducible representations as $\bigoplus_{i \in N} V_{i} \otimes \mathbb{C}^{d_{i}}$. Here we view $\mathbb{H}=\mathbb{C}^{2}$ just as the defining representation of $\Gamma$. Then $P=\mathbb{H} \otimes_{\mathbb{C}} \operatorname{End}(V)=$ $\operatorname{Hom}(V, \mathbb{H} \otimes V)$ and so $P^{\Gamma}$ may be identified with the $\Gamma$-equivariant linear maps:

$$
\begin{aligned}
\mathbb{V}=P^{\Gamma} & =\operatorname{Hom}_{\Gamma}(V, \mathbb{H} \otimes V) \\
& =\bigoplus_{i, j \in N} \operatorname{Hom}_{\Gamma}\left(V_{i}, \mathbb{H} \otimes V_{j}\right) \otimes \operatorname{Hom}\left(\mathbb{C}^{d_{i}}, \mathbb{C}^{d_{j}}\right) \quad \text { expanding both copies of } V \\
& =\bigoplus_{i, j, k \in N} A_{j k} \operatorname{Hom}_{\Gamma}\left(V_{i}, V_{k}\right) \otimes \operatorname{Hom}\left(\mathbb{C}^{d_{i}}, \mathbb{C}^{d_{j}}\right) \quad \text { expanding } \mathbb{H} \otimes V_{j} \\
& =\bigoplus_{i, j \in N} A_{i j} \operatorname{Hom}\left(\mathbb{C}^{d_{i}}, \mathbb{C}^{d_{j}}\right) \quad \text { by Schur's lemma }
\end{aligned}
$$

which, by McKay, we identify as the space of linear maps in both direction along the edges of the affine Dynkin diagram.
—map to $\mathbb{C}^{2} / \Gamma$
-components of exceptional divisor as basis of $H_{2}(M, \mathbb{Z})$, and $X$ as $H^{2}(M, \mathbb{R})$,
-full statement of ALE classification (parameters as cohomology classes of triple of Kähler forms).

## Framed quiver varieties.

In brief given a quiver $\mathcal{Q}$ with nodes $N$, we defined

$$
\mathfrak{M}_{\mathcal{Q}}(\zeta, \mathbf{d})=E(V, V) \|_{\zeta} G(\mathbf{d})
$$

for an $N$-graded vector space $V$ of dimension $\mathbf{d}^{\zeta}$ and parameters $\zeta$ where $G(\mathbf{d})=$ $\mathrm{U}(\mathbf{d}) / \mathbb{T}$. If instead we have two $N$-graded vector spaces $V, W$ we can consider the framed quiver variety:

$$
\mathfrak{M}_{\mathcal{Q}}(\zeta, \mathbf{d}, \mathbf{w}):=\left(E(V, V) \times T^{*} L(V, W)\right) / / / / \mathrm{U}(\mathbf{d})
$$

where $\mathbf{w}$ is the dimension of $W$ and we give $T^{*} L(V, W)=L(V, W) \oplus L(W, V)$ the usual hyperkähler vector space structure, with the standard action of $\mathrm{U}(\mathbf{d}) \subset \operatorname{Aut}(V)$. (Note we do not quotient by automorphisms of $W$.)

Exercise 8.4 (Framing/Deframing). Suppose we are given data $\mathcal{Q}, N, V$ as above, such that $d_{0}=\operatorname{dim}_{\mathbb{C}}\left(V_{0}\right)=1$ for some node $0 \in N$. Let $N^{*}=N \backslash\{0\}$ and define a new quiver $\mathcal{Q}^{*}$ (with nodes $N^{*}$ ) by removing from $\mathcal{Q}$ all the edges involving the node 0 . Let $w_{i}$ be the number of edges between 0 and $i$ for any $i \in N^{*}$, and define $W_{i}=\mathbb{C}^{w_{i}}$. Thus we have two $N^{*}$-graded vector spaces:

$$
V^{*}=\bigoplus_{i \in N^{*}} V_{i}, \quad W=\bigoplus_{i \in N^{*}} W_{i}
$$

with dimension vectors $\mathbf{d}^{*}, \mathbf{w}$.

1) Observe that $G(\mathbf{d}) \cong \mathrm{U}\left(\mathbf{d}^{*}\right)$ and show, if $\mathcal{Q}$ is oriented suitably, that there is an isomorphism

$$
\mathfrak{M}_{\mathcal{Q}}(\zeta, \mathbf{d}) \cong \mathfrak{M}_{\mathcal{Q}^{*}}\left(\zeta^{*}, \mathbf{d}^{*}, \mathbf{w}\right)
$$

where $\zeta^{*}$ is obtained by forgetting the 0 -component of $\zeta$.
2) Show moreover that this process is invertible so that any framed quiver variety is isomorphic to an unframed quiver variety.
3) Observe in particular that the ALE spaces thus arise as framed quiver varieties for (finite/non-affine) Dynkin diagrams.

In applications it is often much more convenient to work with framed quiver varieties. For example they arise in the ADHM construction of instantons and its generalisations [ADHM78, KN90], and are important in representation theory [Nak94, Nak98, Nak01], where w often corresponds to the highest weight of a representation.

Open quiver varieties. Now consider a quiver $\mathcal{Q}$ with nodes $N$ which are partitioned into two sets, the "open" and "closed" nodes:

$$
N=N(o) \sqcup N(c) .
$$

Thus, given an $N$-graded vector space $V$ with dimension vector d then the group

$$
\mathrm{U}(\mathbf{d})=\mathrm{U}(\mathbf{d}(o)) \times \mathrm{U}(\mathbf{d}(c))
$$

factors as a product of the groups corresponding to the open and closed nodes, respectively. Then we may define an open quiver variety

$$
\mathfrak{M}_{\mathcal{Q}}^{o}(\zeta, V)=E(V, V) / / /{ }_{\zeta} \mathrm{U}(\mathbf{d}(c))
$$

by just quotienting by the group $\mathrm{U}(\mathbf{d}(c))$ corresponding to the closed nodes.
Exercise 8.5. 1) Show that any framed quiver variety $\mathfrak{M}_{\mathcal{Q}}(\zeta, \mathbf{d}, \mathbf{w})$ is an open quiver variety for a larger quiver obtained by adding a new open node for each node of $\mathcal{Q}$ and by adding a single extra edge from each node of $\mathcal{Q}$ to the corresponding new open node.
2) Show that any open quiver variety is isomorphic to the product of some (unframed) quiver variety with a hyperkähler vector space.

Such open quiver varieties are often convenient when describing the intermediate spaces which arise when one performs the hyperkähler reduction in stages.
-Example of cotangent bundles to type $A$ flag varieties, and hyperkähler rotation to semisimple orbits in $\mathfrak{g l}_{n}(\mathbb{C})$.
-Middle dimensional cohomology of framed quiver variety as a weight space of the highest weight representation of the Kac-Moody Lie algebra corresponding to the Cartan matrix $\mathbf{C}$ of $\mathcal{Q}$.

## 9. GIT quotients

The aim of this section is to identify some of the hyperkähler quotients we have been studying as (quasi-projective) algebraic varieties, and thereby give them a precise algebraic description.

The plan is as follows: 1) study some geometric invariant theory (GIT) quotients of complex affine varieties, 2) describe how the stability conditions appearing in GIT appear naturally as stability conditions on spaces of quiver representations, 3) explain how GIT quotients are related to Kähler quotients, and deduce the link with hyperkähler quotients. We will then for example be able to return to the example of the Hilbert scheme of points on $\mathbb{C}^{2}$, identify it as a hyperkähler manifold and see that upon rotating its complex structure the Calogero-Moser space appears.

GIT references: Mumford et al, Newstead, Mukai, Dolgachev, King,...

## Affine varieties.

First recall the basic correspondence ([Har77] Remark 1.4.6) between finitely generated rings over $\mathbb{C}$ which are integral domains and complex affine varieties: The points of the variety correspond to maximal ideals of the ring, and the ring is the affine coordinate ring of the variety. We will write $\operatorname{Spm}(R)$ for the variety corresponding to the $\operatorname{ring} R$, the "maximal spectrum" of $R$.

## Preliminaries on Proj.

Similarly to taking the maximal spectrum of a ring to obtain an affine variety, one may take the Proj of a graded ring to obtain a quasi-projective variety (cf. EGA II, §2, [Har77] pp.76-77).

Suppose $S$ is finitely generated integral domain, graded by $\mathbb{Z}_{\geq 0}$ and let $S_{+}$denote its positive part (which is an ideal):

$$
S=\bigoplus_{d \geq 0} S_{d} \quad \supset \quad S_{+}=\bigoplus_{d>0} S_{d}
$$

We will call $S_{+}$the irrelevant ideal. The grading of $S$ corresponds to an action of $\mathbb{C}^{*}$ on $\operatorname{Spm}(S)$ where $\mathbb{C}^{*}$ acts with weight $d$ on $S_{d}$ (in other words $t(f)(x)=t^{d} f(x)$ for $\left.f \in S_{d}, t \in \mathbb{C}^{*}, x \in \operatorname{Spm}(S)\right)$.

As a set $\operatorname{Proj}(S)$ is the set of homogeneous ${ }^{6}$ ideals of $S$ that do not contain $S_{+}$, and which are maximal amongst homogeneous ideals. This may be given the structure of a quasi-projective variety. (Note we write Proj rather than 'MaxProj' or 'Projm' etc.) These ideals are not actually maximal ideals in $S$, but rather correspond to the closures of the one-dimensional $\mathbb{C}^{*}$ orbits in the affine variety $\operatorname{Spm}(S)$.

The simplest example is to take $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ with the $x_{i}$ of degree 1 . Then $\operatorname{Proj}(S)=\mathbb{P}^{n}=(\operatorname{Spm}(S) \backslash\{0\}) / \mathbb{C}^{*}$. More generally if $S$ is the same ring but $\operatorname{deg}\left(x_{i}\right)=a_{i}$ with each $a_{i}$ an integer $\geq 0$, then $\operatorname{Proj}(S)$ is the weighted projective space $\mathbb{P}^{n}\left(a_{0}: \cdots: a_{n}\right)$.

The inclusion $S_{0} \rightarrow S$ yields a map

$$
\operatorname{Proj}(S) \rightarrow \operatorname{Spm}\left(S_{0}\right) ; \quad \mathfrak{m} \mapsto \mathfrak{m} \cap S_{0}
$$

and this is a projective map (i.e. it has projective fibres). Indeed if $S$ is generated by $n$ homogeneous generators $f_{i}$ in degrees $a_{1}, \ldots, a_{n}$ then the map

$$
\mathbb{C}\left[x_{1}, \ldots, x_{n}\right] \rightarrow S ; \quad x_{i} \mapsto f_{i}
$$

[^6]yields an embedding
$$
\operatorname{Proj}(S) \rightarrow \mathbb{P}\left(a_{1}: \cdots: a_{n}\right)
$$
into the weighted projective space. If $a_{1}=\cdots=a_{k}=0$ and the other degrees are larger, then $\mathbb{P}\left(a_{1}: \cdots: a_{n}\right)=\mathbb{C}^{k} \times \mathbb{P}\left(a_{k+1}: \cdots: a_{n}\right)$. Moreover $f_{1}, \ldots f_{k}$ generate $S_{0}$ and the map $\operatorname{Proj}(S) \rightarrow \operatorname{Spm}\left(S_{0}\right)$ is just the restriction of the projection $\mathbb{C}^{k} \times \mathbb{P}\left(a_{k+1}, \ldots, a_{n}\right) \rightarrow \mathbb{C}^{k}$. (In particular if $S_{0}=\mathbb{C}$ then $\operatorname{Proj}(S)$ is projective.)

Example 9.1. Let $R$ be a finitely generated integral domain and take $S=R[z]$ with $z$ degree one and $R$ in degree zero. Then $\operatorname{Proj}(S)=\operatorname{Spm}(R)$.

We also have a map $\operatorname{Spm}\left(S_{0}\right) \rightarrow \operatorname{Spm}(S)$ (corresponding to the fact that $S_{0}=$ $\left.S / S_{+}\right)$and its image is the fixed point set $F$ of the $\mathbb{C}^{*}$ action. Then as sets we have:

$$
\operatorname{Proj}(S)=(\operatorname{Spm}(S) \backslash F) / \mathbb{C}^{*}
$$

extending the picture for $\mathbb{P}^{n}$.
Remark 9.2.1) For any $m \geq 1$ define the following graded subring of $S$ :

$$
S^{[m]}=\bigoplus_{d \geq 0} S_{d m} \subset S
$$

Note that the map $\operatorname{Proj}(S) \rightarrow \operatorname{Proj}\left(S^{[m]}\right) ; \mathfrak{m} \mapsto \mathfrak{m} \cap S^{[j]}$ is an isomorphism.
2) We may always assume $S$ is generated in degrees 0 and 1 , by replacing $S$ by $S^{[m]}$ where $m$ is the least common multiple of the nonzero degrees of a set of generators of $S$. (Then we see $\operatorname{Proj}(S)$ is embedded in some product $\mathbb{C}^{k} \times \mathbb{P}^{l}$ of an affine space and a projective space.)

## Affine GIT quotients.

Now suppose $G$ is a reductive complex algebraic group (this just means $G$ is the complexification of a compact group $K$, i.e. $G$ is the complex (affine) algebraic group associated to the representative ring of $K$-the ring of functions which arise as matrix entries of finite dimensional representations of $K$ ). The main examples we will use are just products of general linear groups however. (Beware $G$ denoted a compact group in previous sections, but is complex here.)

Suppose $G$ acts linearly on an affine variety $X=\operatorname{Spm}(R)$. Thus $X \subset V$ for some vector space $V$ and $G$ acts on $X$ via a map $G \rightarrow \mathrm{GL}(V)$. We will not assume this representation of $G$ is faithful and let $\Delta=\operatorname{Ker}(G \rightarrow \mathrm{GL}(V))$. Then we may consider the ring of $G$ invariant functions on $X$ :

$$
R^{G}=\{f \in R \mid g(f)=f \text { for all } g \in G\} \subset R
$$

It is a theorem (of Hilbert) that this is again a finitely generated ring and we define the affine GIT quotient of $X$ by $G$ to be the associated affine variety: $\operatorname{Spm}\left(R^{G}\right)$.

Now we should consider to what extent this is a quotient of $X$ by $G$. First of all the inclusion $R^{G} \rightarrow R$ yields a map

$$
\phi: X \rightarrow \operatorname{Spm}\left(R^{G}\right)
$$

the affine quotient map. The basic facts about this map are as follows:

## Theorem 9.3.

1) The affine quotient map $\phi$ is surjective,
2) The points of $\operatorname{Spm}\left(R^{G}\right)$ correspond bijectively to the closed orbits in $X$,
3) Two points of $X$ map to the same point in $\operatorname{Spm}\left(R^{G}\right)$ if and only if the closures of their orbits intersect.

This leads to the definition of polystable points of $X$ to be those whose orbits are closed and the stable points of $X$ to be the polystable points whose orbits have dimension $\operatorname{dim}(G / \Delta)$. The subset $X^{s} \subset X$ of stable points is Zariski open in $X$ and the restriction of $\phi$ to $X^{s}$ is a geometric quotient (i.e. its fibres are $G$-orbits).

Remark 9.4. $X=X^{s}$ if and only if the stabilizer group in $G$ of every point $x \in X$ contains $\Delta$ with finite index.

Example 9.5. Recall the Calogero-Moser spaces were defined as the set of $\mathrm{GL}_{n}(\mathbb{C})$ orbits in the affine variety

$$
\left\{(X, Z, v, \alpha) \in \operatorname{End}(V)^{2} \times V \times V^{*} \mid[X, Z]+\operatorname{Id}_{V}=v \otimes \alpha\right\} \subset \operatorname{End}(V)^{2} \times V \times V^{*}
$$

where $V=\mathbb{C}^{n}$. Since the action is free, all points are stable and the set of orbits is isomorphic to the affine GIT quotient. Thus the set of orbits is naturally an affine variety (in other words the points of the variety associated to the ring of invariant functions correspond bijectively to the orbits).

## Proj quotients.

Now we will consider the slightly more involved case where, rather than just considering invariant functions on $X$ we consider invariant sections of a line bundle. This leads to a different choice of quotient for each choice of line bundle $L$ on $X$ with a lift of the $G$-action. In fact since $X$ is affine it is enough to consider the case where $L$ is trivial. Then the choice of lift of the $G$-action to $L$ corresponds to a choice of a character of $G$.

Choose a character $\chi: G \rightarrow \mathbb{C}^{*}$ and let

$$
L=X \times \mathbb{C}
$$

be the trivial line bundle over $X$ with an action of $G$ on $L$ defines as follows:

$$
g \cdot(x, z)=(g \cdot x, \chi(g) z) .
$$

(Such a choice of lift of the action of $G$ on $X$ is called a linearisation of the action.) We will always assume $\chi$ is chosen such that $\chi(\Delta)=1$, so that $\Delta$ acts trivially on $L$ (otherwise $L$ never has $G$-invariant sections). Similarly $G$ acts on the $n$th tensor power $L^{n}$ of $L$ for any $n \in \mathbb{Z}$; as a space we still have $L^{n}=X \times \mathbb{C}$ but the action is given by $g \cdot(x, z)=\left(g \cdot x, \chi^{n}(g) z\right)$.

If we let $R=\mathbb{C}[X]$ be the affine coordinate ring of $X$ then as in Example 9.1 we may view $X$ as $\operatorname{Proj}(R[z])$ with $z$ in degree one. $R[z]$ is naturally the ring of functions on the total space of $L^{-1}$. Now $G$ acts on $L^{-1}$ and thus on $R[z]=\mathbb{C}\left[L^{-1}\right]$ and so we may consider the ring of invariant functions

$$
S=R[z]^{G}
$$

which is again a graded ring (and finitely generated as before). Thus we may define the GIT quotient corresponding to $\chi$ as the proj of this graded ring:

$$
X / /(G, \chi):=\operatorname{Proj}(S)=\operatorname{Proj}\left(R[z]^{G}\right)
$$

Here are two other description of $S=R[z]^{G}$ :

1) Suppose we have a homogeneous element $f(x) z^{d} \in R[z]$ of degree $d$, with $f \in$ $\mathbb{C}[X]$. It is $G$ invariant if $f(g \cdot x) z^{d} / \chi(g)^{d}=f(x) z^{d}$ i.e. if

$$
f(g \cdot x)=\chi(g)^{d} f(x)
$$

for all $g \in G$. This says that $f$ is a "semi-invariant" of weight $\chi^{d}$ and we denote the set of such elements of $R$ by $R_{\chi^{d}}^{G}$. Equivalently this says $f$ is a $G$-invariant section of $L^{d}$ (more precisely the map $s: X \rightarrow L^{d} ; x \mapsto(x, f(x))$ is a $G$-invariant section-i.e. it is a $G$-equivariant map). Thus

$$
S=R[z]^{G}=\bigoplus_{d \geq 0} H^{0}\left(L^{d}\right)^{G}=\bigoplus_{d \geq 0} R_{\chi^{d}}^{G}
$$

is the sum of the $G$-invariant sections of powers of $L$.
2) Alternatively, if $\chi$ is nontrivial, consider the kernel $G_{\chi} \subset G$ of the character $\chi: G \rightarrow \mathbb{C}^{*}$. This subgroup $G_{\chi}$ acts on $R$ and we may consider the invariant ring $R^{G_{\chi}}$. Now this ring has an action of $\mathbb{C}^{*}=G / G_{\chi}$, and so is graded, and again the degree $d$ piece is $R_{\chi^{d}}^{G}$.

Thus the "quotient" $X / /(G, \chi)$ is now a quasi-projective variety (projective over the affine variety Spec $S_{0}$ corresponding to the degree zero subring).
e.g. if $\chi=1$ we just get the affine variety associated to the ring of invariant functions on $X .\left(S=\mathrm{H}^{0}\left(L^{0}\right)^{G} \otimes \mathbb{C}[z]\right.$, and Proj of this is Spec of $\left.\mathrm{H}^{0}\left(L^{0}\right)^{G}=R^{G}.\right)$

Now we wish to examine the extent to which $X / /(G, \chi)$ may be considered a quotient of $X$. Define the set of $\chi$-semistable points of $X$ as follows:

$$
X_{\chi}^{s s}=\left\{x \in X \mid f(x) \neq 0 \text { for some } f \in S_{d} \text { with } d>0\right\}
$$

In other words these are the points at which some homogeneous seminvariant of positive weight does not vanish. These are precisely the points at which the resulting rational map to the weighted projective space is well defined. Explicitly if we choose homogeneous generators $f_{1}, \ldots, f_{n}$ of $S$ then we have a rational map from $X$ to $X / /(G, \chi)=\operatorname{Proj}(S)$ given by

$$
x \mapsto\left(f_{1}(x): \cdots: f_{n}(x)\right) \in \mathbb{P}\left(a_{1}: \cdots: a_{n}\right)=\mathbb{C}^{k} \times \mathbb{P}\left(a_{k+1}: \cdots: a_{n}\right)
$$

where as before $\operatorname{deg}\left(f_{i}\right)=a_{i}$ and $a_{i}=0$ iff $i \leq k$. By definition this map is well defined precisely at the $\chi$-semistable points of $X$, i.e. where not all of the positive degree generators $f_{k+1}, \ldots, f_{n}$ vanish.

Thus $X_{\chi}^{s s}$ is a Zariski open subset of $X$ (whose complement is cut out by $f_{k+1}, \ldots, f_{n}$ ) and there is a morphism

$$
\phi: X_{\chi}^{s s} \rightarrow X / /(G, \chi)
$$

In particular we see $X / /(G, \chi)$ is the quotient of the open subset of semistable points by some equivalence relation (defined a priori by points being in the same fibre of $\phi$ ). As in the affine case this may be made more explicit:

Theorem 9.6. Two points of $X_{\chi}^{s s}$ map to the same point of $X / /(G, \chi)$ if and only if the closures of their orbits intersect in $X_{\chi}^{s s}$ :

$$
\begin{equation*}
\phi(x)=\phi(y) \quad \text { iff } \quad \overline{(G \cdot x)} \cap \overline{(G \cdot y)} \cap X_{\chi}^{s s} \neq \emptyset \tag{9.1}
\end{equation*}
$$

Indeed this may be deduced from the affine case by considering affine charts. Since each $G$-orbit in $X_{\chi}^{s s}$ contains a unique closed orbit in its closure, we see that the points of $X / /(G, \chi)$ correspond bijectively to the set of closed orbits in $X_{\chi}^{s s}$. We define $x \in X$ to be $\chi$-polystable if it is in $X_{\chi}^{s s}$ and if its orbit is closed in $X_{\chi}^{s s}$. Thus we have:

$$
X / /(G, \chi)=\left\{\text { closed orbits in } X_{\chi}^{s s}\right\}=\{\chi \text {-polystable points } x \in X\} / G
$$

In turn we may define $x \in X$ to be $\chi$-stable if it is $\chi$-polystable and its orbit has dimension $\operatorname{dim}(G / \Delta)$.

Example 9.7. Let us take $G=\mathbb{C}^{*}$ acting on $X=\mathbb{C}^{n}$ by $g(x)=g x$. If $\chi$ is trivial then $X_{\chi}^{s s}=X$ but there is just one closed orbit, the origin (all other orbits contain the origin in their closure). Thus $X / /(G, \chi)$ is just one point. On the other hand if $\chi(g)=g$ then $X_{\chi}^{s s}=X \backslash\{0\}$ and all the orbits of $\chi$-semistable points are closed in
$X_{\chi}^{s s}$. Thus $X / /(G, \chi)=\mathbb{P}(X)$ is the projective space of $X=\mathbb{C}^{n}$. (This illustrates how using a character can make a big difference to the resulting quotient.) Finally if $\chi(g)=g^{-1}$ then $X_{\chi}^{s s}$ (and thus the quotient) is empty.

## Hilbert-Mumford criterion.

Now we wish to characterise the $\chi$-semistable points in a simpler way. The HilbertMumford criterion enables us to just consider one parameter subgroups (1-PS) of $G$, i.e. homomorphisms $\lambda: \mathbb{C}^{*} \rightarrow G$.

For any 1-PS $\lambda$ define an integer $\langle\chi, \lambda\rangle$ to be the degree of the composition $\chi \circ \lambda$ : $\mathbb{C}^{*} \rightarrow \mathbb{C}^{*}$, i.e. it is the integer $m$ such that $\chi(\lambda(t))=t^{m}$ for all $t \in \mathbb{C}^{*}$.

The main statement is then:
Proposition 9.8. A point $x \in X$ is $\chi$-semistable iff every 1-PS $\lambda$ for which

$$
\lim _{t \rightarrow 0}(\lambda(t) \cdot x)
$$

exists satisfies $\langle\chi, \lambda\rangle \geq 0$.
In the context of representations of quivers this will allow us to translate the stability conditions into more concrete terms.

## Representations of quivers.

Now we will apply some of the above results to study spaces of quiver representations.

Let $\mathcal{Q}$ be a quiver with nodes $N$. (Here we allow $\mathcal{Q}$ to have edge loops.)
A representation $V$ of $\mathcal{Q}$ is a choice of a complex vector space $V_{i}$ for each node $i \in N$ and a linear map $\phi_{e}: V_{t(e)} \rightarrow V_{h(e)}$ for each oriented edge (arrow) $e \in \mathcal{Q}$. The dimension of $V$ is the vector $\mathbf{d} \in \mathbb{Z}_{\geq 0}^{N}$ with $d_{i}=\operatorname{dim}\left(V_{i}\right)$. One defines morphisms of representations and subrepresentations etc. in the obvious way.

One would like to define a moduli space of isomorphism classes of quiver representations with given dimension vector (i.e. a space whose points correspond to isomorphism classes of representations).

By choosing bases, i.e. isomorphisms $V_{i} \cong \mathbb{C}^{d_{i}}$, we see any representation of dimension $\mathbf{d}$ is given by specifying a point of the space

$$
\operatorname{Rep}(\mathcal{Q}, \mathbf{d})=\bigoplus_{e \in \mathcal{Q}} \operatorname{Hom}\left(W_{t(e)}, W_{h(e)}\right)
$$

where $W_{i}=\mathbb{C}^{d_{i}}$, i.e. by specifying the linear maps along each edge of $\mathcal{Q}$. In turn two repesentations are isomorphic if and only if the corresponding points of $\operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ are in the same orbit under the natural action of $\mathrm{GL}(\mathbf{d})=\prod_{i \in N} \mathrm{GL}_{d_{i}}(\mathbb{C})$. Thus we are left with the problem of forming the quotient of $\operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ by $\mathrm{GL}(\mathbf{d})$. However (as usual) there is not necessarily an algebraic variety whose points correspond to the orbits.

This situation was studied by King [Kin94], who considered the GIT quotients and described the points of the resulting spaces naturally in terms of quivers.

Any character $\chi$ of $\mathrm{GL}(\mathbf{d})$ is of the form

$$
\chi_{\theta}\left(\left\{g_{i}\right\}\right)=\prod_{i \in N} \operatorname{det}\left(g_{i}\right)^{-\theta_{i}}
$$

for some vector $\theta \in \mathbb{Z}^{N}$ of integers. Given a choice of such $\theta$ and a representation $V$ define

$$
\theta(V)=\theta \cdot \mathbf{d} \in \mathbb{Z}
$$

where $\mathbf{d}$ is the dimension vector of $V$. The subgroup $\Delta$ of $\mathrm{GL}(\mathbf{d})$ acting trivially on $\operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ is just the $\mathbb{C}^{*}$ subgroup of scalar matrices (the complexification of the circle $\mathbb{T}$ used earlier). Note that we have

$$
\chi_{\theta}(\Delta)=\{1\} \quad \text { if and only if } \quad \theta \cdot \mathbf{d}=0
$$

The key definition is then:

Definition 9.9 (King). A representation $V$ of a quiver $\mathcal{Q}$ is said to be

- $\theta$-semistable if $\theta(V)=0$ and for all subrepresentations $V^{\prime} \subseteq V$ we have $\theta\left(V^{\prime}\right) \leq 0$.
- $\theta$-stable if it is $\theta$-semistable and $\theta\left(V^{\prime}\right) \neq 0$ for all nontrivial proper subrepresentations $V^{\prime}$.

This is very similar to the case of degree zero vector bundles over compact Riemann surfaces (where one takes $\theta(V)$ to be the slope $\mu(V)=\operatorname{deg}(V) / \operatorname{rank}(V)$ ).

The first result is then
Theorem 9.10 ([Kin94]). A quiver representations $V \in \operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ is $\chi_{\theta}$-(semi)stable (for the action of $\mathrm{GL}(\mathbf{d})$ ) if and only if it is $\theta$-(semi)stable.

Secondly it is possible to reinterpret the GIT equivalence relation in Equation (9.1) purely in terms of quiver representations. For this first note that any $\theta$-semistable quiver representation $V$ has a Jordan-Holder series: There is an increasing sequence

$$
V_{1} \subset V_{2} \subset \cdots \subset V_{n}=V
$$

of $\theta$-semistable representations such that $V_{1}$ and each quotient $V_{i} / V_{i-1}$ is $\theta$-stable. Then we have an associated (graded) quiver representation:

$$
\operatorname{gr} V:=V_{1} \oplus V_{2} / V_{1} \oplus \cdots \oplus V_{n} / V_{n-1}
$$

whose summands are all $\theta$-stable, and $\operatorname{gr} V$ is uniquely determined by $V$ up to isomorphism ${ }^{7}$.

Definition 9.11. Two $\theta$-semistable quiver representations $V, V^{\prime}$ are said to be stonglyequivalent (or $S$-equivalent) if $\mathrm{gr} V$ and $\mathrm{gr}^{\prime}$ are isomorphic.

Then we also have:
Theorem 9.12 ([Kin94]). Two $\theta$-semistable quiver representations are GIT equivalent ((9.1) with $\chi=\chi_{\theta}$ ) if and only if they are $S$-equivalent.

Thus the points of the GIT quotient

$$
\operatorname{Rep}(\mathcal{Q}, \mathbf{d}) / /\left(\mathrm{GL}(\mathbf{d}), \chi_{\theta}\right)
$$

correspond bijectively to S -equivalence classes of $\theta$-semistable quiver representations with dimension d.

The main idea behind this is the relation between one parameter subgroups and filtrations of representations, as follows.

[^7]Given a quiver representation $V$, suppose we have a 1-PS $\lambda: \mathbb{C}^{*} \rightarrow \mathrm{GL}(\mathbf{d})$ such that the limit of $\lambda(t) V$ exists as $t \rightarrow 0$.

First we may break up the underlying vector spaces $V_{i}$ (at each node in $i \in N$ ) according to the weights of the $\mathbb{C}^{*}$ action. Let $V_{i}^{(n)} \subset V_{i}$ be the weight $n$ subspace (for $n \in \mathbb{Z}$ ) and for $e \in \mathcal{Q}$ let

$$
\phi_{e}^{(m n)}: V_{t(e)}^{(n)} \rightarrow V_{h(e)}^{(m)}
$$

be the induced linear map (from $V$ ) along the edge $e$ from the weight $n$ piece to the weight $m$ piece. This map varies with weight $m-n$ so the desired limit exists as $t \rightarrow 0$ iff $\phi_{e}^{(m n)}=0$ whenever $m-n<0$ (i.e. if $m<n$ ). Equivalently, if we define $V_{i}^{(\geq n)} \subset V_{i}$ to be the subspace with weights $\geq n$, this says that $\phi_{e}$ restricts to define a map

$$
\phi_{e}^{(n)}: V_{t(e)}^{(\geq n)} \rightarrow V_{h(e)}^{(\geq n)}
$$

for any $n$, i.e. that $V(n):=\left\{V_{i}^{\geq n}\right\}$ is a subrepresentation of $V$ for any $n$. Thus such 1-PS determines a filtration

$$
\cdots \subset V(n) \subset V(n-1) \subset V(n-2) \subset \cdots
$$

which will start at the zero representation and end at $V$. Conversely any such filtration will arise from at least one 1-PS for which the limit exists.

Note also that the limit representation $\lim _{t \rightarrow 0} \lambda(t) V$ is the associated graded

$$
\bigoplus_{n \in \mathbb{Z}} V(n) / V(n+1)
$$

of this filtration (since in the limit all maps between pieces with different weights will vanish; the vector space attached to the node $i$ by $V(n) / V(n+1)$ is just $\left.V_{i}^{(n)}\right)$.

Also $\left\langle\chi_{\theta}, \lambda\right\rangle=-\sum_{n} \theta(V(n))$ provided $\theta(V(n))$ is zero for all but finitely many values of $n$, since

$$
\left\langle\chi_{\theta}, \lambda\right\rangle=-\sum_{n \in \mathbb{Z}, i \in I} \theta_{i} \cdot n \operatorname{dim}\left(V_{i}^{(n)}\right) .
$$

This maybe computed explicitly by writing out $\lambda$ (which is diagonal in some basis) and $\chi_{\theta}$. In turn this is:

$$
-\sum_{n \in \mathbb{Z}} n \theta(V(n) / V(n+1))=-\sum_{n} \theta(V(n))
$$

since the function $\theta(\cdot)$ is additive on exact sequences.
Now to relate the notions of semistability we just compare the definition of $\theta$ stability and the statement of Proposition 9.8: firstly by definition $\theta(V)=0$ iff
$\chi_{\theta}(\Delta)=1$. Secondly given a proper subrepresentation $V^{\prime}$ we take any 1-PS $\lambda$ corresponding to the filtration $0 \subset V^{\prime} \subset V$. Then (by the previous remark) $\theta\left(V^{\prime}\right)=$ $-\left\langle\chi_{\theta}, \lambda\right\rangle$, so $\theta\left(V^{\prime}\right) \leq 0$ if and only if $\left\langle\chi_{\theta}, \lambda\right\rangle \geq 0$, as required.

Exercise 9.13. Consider the quiver $\mathcal{Q}$ with two nodes $N=\{1,2\}$ and $n$ edges from 1 to 2 . Set $\mathbf{d}=(k, 1)$, where $k<n$, so $\mathrm{GL}(\mathbf{d})=\mathrm{GL}_{k}(\mathbb{C}) \times \mathrm{GL}_{1}(\mathbb{C})$ and $\mathrm{GL}(\mathbf{d}) / \Delta \cong \mathrm{GL}_{k}(\mathbb{C})$.

1) Show that $\operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ is $\operatorname{GL}_{k}(\mathbb{C})$ equivariantly isomorphic to $\operatorname{Hom}\left(\mathbb{C}^{k}, \mathbb{C}^{n}\right)$,
2) Set $\theta=(-1, k)$ and show that $V \in \operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ is $\theta$-semistable iff it is $\theta$-stable iff the corresponding matrix in $\operatorname{Hom}\left(\mathbb{C}^{k}, \mathbb{C}^{n}\right)$ has rank $k$,
3) Identify the $\operatorname{GIT}$ quotient $\operatorname{Rep}(\mathcal{Q}, \mathbf{d}) / /\left(\mathrm{GL}(\mathbf{d}), \chi_{\theta}\right)$ with the Grassmannian of $k$ dimensional subspaces of $\mathbb{C}^{n}$.

Note: We have made two changes from [Kin94], to agree with conventions used for vector bundles: 1) our $\chi_{\theta}$ is his $\chi_{-\theta}$, and 2) we reversed the inequalities in the definition of $\theta$-semistability

## 10. Kempf-Ness relation between GIT and Real symplectic quotients

The general context we will consider is as follows. Let $V$ a complex vector space acted on linearly by a complex reductive $G$, i.e. $G$ acts via a representation $G \rightarrow$ $\mathrm{GL}(V)$. Let $X \subset V$ be an smooth affine variety preserved by $G$. Choose a Hermitian inner product (, ) on $V$ and a maximal compact subgroup $K<G$ preserving (, ), thereby making $V$ and $X$ into Kähler manifolds.

Recall that the action of $K$ on $V$ has moment map $\widehat{\mu}: V \rightarrow \mathfrak{k}^{*}$ such that

$$
\widehat{\mu}(x)(A)=\frac{i}{2}(A x, x) \in \mathbb{R}
$$

for all $x \in V, A \in \mathfrak{k}=\operatorname{Lie}(K)$. This restricts to a moment map $\mu_{\mathbb{R}}=\left.\widehat{\mu}\right|_{X}$ for the $K$-action on $X$.

Now let $\chi: G \rightarrow \mathbb{C}^{*}$ be a character of $G($ with $\chi(\Delta)=1)$. The derivative at the identity of $\chi$ is a $\mathbb{C}$-linear map $d \chi: \mathfrak{g} \rightarrow \mathbb{C}$. Moreover $\mathfrak{g}=\mathfrak{k} \oplus(i \mathfrak{k})$ and $\chi$ maps $\mathfrak{k}$ to the imaginary axis $i \mathbb{R} \subset \mathbb{C}$. Thus $\chi$ determines the point

$$
\zeta_{\mathbb{R}}:=\left.\frac{i}{2}(d \chi)\right|_{\mathfrak{k}} \in \mathfrak{k}^{*}
$$

in the (real) dual of the Lie algebra of $K$. The main result is then:
Theorem 10.1. Any point of $\mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right) \subset X$ is $\chi$-polystable and this inclusion induces a natural homeomorphism:

$$
X \int_{\zeta_{\mathbb{R}}} K=\mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right) / K \cong X / /(G, \chi)=\{\chi \text {-polystable points in } X\} / G
$$

between the symplectic quotient of $X$ by $K$ at the value $\zeta_{\mathbb{R}}$ of the moment map and the GIT quotient of $X$ by $G$ via $\chi$. In other words: Any closed $G$ orbit in $X_{\chi}^{s s}$ meets $\mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right)$ in exactly one $K$-orbit, and no other $G$ orbits intersect $\mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right)$.

Since $X$ is closed in $V$, this follows directly from the case $X=V$, which is due to Kempf-Ness [KN79] (in the case $\zeta_{\mathbb{R}}=0$ ) and the extension to $\zeta_{\mathbb{R}} \neq 0$ is in [Kin94].

Example 10.2. Suppose we take $G=\mathbb{C}^{*}$ acting on $V$ by $g(x)=g x$, with $\chi(g)=g$. Thus $K=S^{1} \subset G$, and $\zeta_{\mathbb{R}}=i / 2$ (using the usual identification of the Lie algebra and its dual). Then $\mu_{\mathbb{R}}(x)=\zeta_{\mathbb{R}}$ if and only if $(A x, x)=d \chi(A)=A$ for all $A \in \mathfrak{k}=i \mathbb{R}$, i.e. $\|x\|^{2}=1$. Thus the symplectic quotient is the projective space of $V$ as expected. On the other hand if $\chi(g)=g^{-1}$ we get the empty set (and one may check there are no semistable points).

Let us illustrate the basic idea behind this in the case when $\chi$ is trivial, so the GIT quotient is just the set of closed $G$ orbits in $X$. If an $G$ orbit is closed then we expect there to be a "best representative" in it: choose a Hermitian metric on $V$ and look for the vectors in the $G$ orbit with least norm. Thus we wish to minimize the function $x \mapsto\|x\|^{2}$ on the $G$ orbit. The crucial point is that the derivative of this function is basically the moment map: thus the vanishing of the moment map is the condition to be a critical point of this function:

Pick $x \in X$ and suppose $g \in G$ depends on $t \in \mathbb{C}$ and $g(0)=1$. Write $\delta$ for $\left.\frac{d}{d t}\right|_{t=0}$ :

$$
\begin{gathered}
\delta\|g \cdot x\|^{2}=\delta(g x, g x)=(\delta(g x), x)+(x, \delta(g x)) \\
=\left(\left(A+A^{*}\right) x, x\right)=0
\end{gathered}
$$

for all variations $A=\delta g \in \mathfrak{g}$. If $A \in \mathfrak{k}$ there is no condition, so $x$ is a critical point if $(B x, x)=0$ for all $B \in i \mathfrak{k}$. Clearly this is the same as saying $\mu_{\mathbb{R}}(x)=0$ with $\mu_{\mathbb{R}}$ the moment map for the $K$-action. One proves further that all critical points are minima and they are unique up to the action of $K$.

Example 10.3. For example consider the case of $\mathrm{U}(n)<\mathrm{GL}_{n}(\mathbb{C})$ acting by conjugation on $\operatorname{End}\left(\mathbb{C}^{n}\right)$. The action of $\mathrm{U}(n)$ has moment map $\frac{i}{2}\left[Z, Z^{*}\right]$ and so $\mu_{\mathbb{R}}^{-1}(0)$ is the set of normal matrices, i.e. those commuting with their Hermitian adjoint. These are the matrices which are diagonalizable via a unitary transformation, and so taking the quotient we get the set of diagonal matrices, modulo reordering their eigenvalues. On the other hand a matrix lives in a closed orbit under $\mathrm{GL}_{n}(\mathbb{C})$ iff it is diagonalisable, so we see the identification of the quotients clearly.

The general case $\zeta_{\mathbb{R}} \neq 0$ is similar, but instead using the function

$$
\frac{1}{2}\|x\|^{2}+\log |z|
$$

on $V \times \mathbb{C}=L^{-1}$. Differentiating as above one sees critical points are where $(B x, x)=$ $d \chi(B)$ for all $B \in i \mathfrak{k}$.

## GIT approach to quiver varieties.

We can now obtain some of the Nakajima quiver varieties as algebraic varieties. Recall the spaces $\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)$ were constructed as hyperkähler quotients. Write $\zeta=$ $\left(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}\right)$ so $\zeta_{\mathbb{R}} \in i \mathbb{R}^{N}$ and set

$$
\theta:=2 i \zeta_{\mathbb{R}} \in \mathbb{R}^{N}
$$

The hyperkähler quotient may be written as the Kähler quotient of the complex subvariety $\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) \subset \mathbb{V}$ :

$$
\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)=\mu^{-1}(\zeta) / \mathrm{U}(\mathbf{d})=\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) \cap \mu_{\mathbb{R}}^{-1}\left(\zeta_{\mathbb{R}}\right) / \mathrm{U}(\mathbf{d})=\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) /_{\zeta_{\mathbb{R}}} \mathrm{U}(\mathbf{d})
$$

In turn, if the components of $\theta$ are integral, by Theorem 10.1 this may be identified with the GIT quotient by the complex group GL(d) (the complexification of $\mathrm{U}(\mathbf{d})$ ) via the character $\chi_{\theta}$ :

$$
\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta) \cong \mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) / /\left(\mathrm{GL}(\mathbf{d}), \chi_{\theta}\right)
$$

which in turn is the set of $S$-equivalence classes of $\theta$-semistable representations in the subvariety $\mu_{\mathbb{C}}^{-1}\left(\zeta_{\mathbb{C}}\right) \subset \mathbb{V}$, where we identify $\mathbb{V}$ with the set $\operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$ of representations of the doubled quiver $\widehat{\mathcal{Q}}$ with dimension vector $\mathbf{d}$. In other words the quiver variety $\mathfrak{M}_{\mathcal{Q}}(\mathbf{d}, \zeta)$ may be viewed as the space of $S$-equivalence classes of $\theta$-semistable quiver representations $B \in \mathbb{V}=\operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$ satisfying just the complex moment map equations

$$
\begin{equation*}
\mu_{\mathbb{C}}(B)=B B \varepsilon=\zeta_{\mathbb{C}} . \tag{10.1}
\end{equation*}
$$

Further, one has that the regular locus $\mathfrak{M}_{\mathcal{Q}}^{\mathrm{reg}}(\mathbf{d}, \zeta)$ matches up with the (isomorphism classes of) $\theta$-stable representations satisfying (10.1).

Example 10.4. The above results also hold in the case where the quiver $\mathcal{Q}$ has edge loops. Consider the case with $N=\{0,1\}$ and $\mathcal{Q}$ consists of a single edge between 0 and 1 and a loop at 1 (i.e. an edge from 1 to 1 ). Define $\mathbf{d}$ by setting $d_{0}=1, d_{1}=n$. Thus a representation $V \in \operatorname{Rep}(\mathcal{Q}, \mathbf{d})$ consists of $B_{1}, B_{2} \in \operatorname{End}\left(V_{1}\right), v \in V_{1}, \phi \in V_{1}^{*}$ where $V_{1}=\mathbb{C}^{n}$. The $\left(\mathfrak{g l}_{n}(\mathbb{C})\right.$ part of the) complex moment map has the form

$$
\mu_{\mathbb{C}}(V)=\left[B_{1}, B_{2}\right]+v \otimes \phi
$$

Now recall the description of the Hilbert scheme of $n$-points on $\mathbb{C}^{2}$ from Corollary 2.3 , and note it may be rewritten as

$$
\left(\mathbb{C}^{2}\right)^{[n]} \cong\left\{\begin{array}{c|c}
\mu_{\mathbb{C}}(V)=0, \text { and if } \\
V \in \operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d}) & \begin{array}{c}
V^{\prime} \subset V \text { is a subrepresentation } \\
\text { with } V_{0}^{\prime}=V_{0}, \text { then } V^{\prime}=V
\end{array}
\end{array}\right\} / \mathrm{GL}(\mathbf{d}) .
$$

Now we just observe this condition is the same as both the $\theta$-semistability and $\theta$ stability condition on $V$, if we set $\theta=(n,-1)$ (i.e. $\theta_{0}=n, \theta_{1}=-1$ ). Indeed suppose $V^{\prime} \subset V$ is a subrepresentation of dimension $\mathbf{d}^{\prime}$. Then either i) $d_{0}^{\prime}=0$, so $\theta\left(V^{\prime}\right)=\theta \cdot \mathbf{d}^{\prime}=-d_{1}^{\prime}$ which is always less than 0 unless $V^{\prime}$ is trivial, or ii) $d_{0}^{\prime}=1$ so $\theta\left(V^{\prime}\right)=n-d_{1}^{\prime}$ which is $\leq 0$ iff $d_{1}^{\prime}=n$, i.e. $V^{\prime}=V$.

Thus if we define the affine variety $X=\mu_{\mathbb{C}}^{-1}(0) \subset \operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$ we deduce that $\left(\mathbb{C}^{2}\right)^{[n]}$ is the GIT quotient of $X$ by GL(d) via the character $\chi_{\theta}$ and in turn, by identifying $\operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$ as a hyperkähler vector space as usual, we have established:

Theorem 10.5 ([Nak99]). The Hilbert scheme of $n$ points on $\mathbb{C}^{2}$ is isomorphic to the hyperkähler manifold

$$
\operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d}) / / /{ }_{\zeta} \mathrm{U}(\mathbf{d})
$$

where $\zeta=\left(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}\right)$ with $\zeta_{\mathbb{C}}=0, \zeta_{\mathbb{R}}=\theta / 2 i$.
Exercise 10.6. Perform a hyperkähler rotation of the above description of the Hilbert scheme to identify it, in another complex structure, with the affine GIT quotient $Y / /(\mathrm{GL}(\mathbf{d}), 1)$ where $Y$ is the affine variety $Y=\mu_{\mathbb{C}}^{-1}\left(\frac{1}{2}\right) \subset \operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$. (Here $\mathcal{Q}, \mathbf{d}, \mu_{\mathbb{C}}$ are as in the above example.) Observe that the resulting affine variety is the CalogeroMoser space (after a minor rescaling of the variables).

The above description of the Hilbert scheme simplifies slightly if we pass to the corresponding framed quiver variety, and translate the stability conditions into that context. The general translation of the stability conditions is as follows:

Exercise 10.7 (Framed stability conditions). Let $\mathfrak{M}=\mathfrak{M}_{\mathcal{Q}}(\zeta, \mathbf{d})$ be a quiver variety with $d_{0}=1$ at some node $0 \in N$ and let $\mathfrak{M}_{\mathcal{Q}^{*}}\left(\zeta^{*}, \mathbf{d}^{*}, \mathbf{w}\right)$ be the corresponding (isomorphic) framed quiver variety, as in Exercise 8.4. Let $\theta=2 i \zeta_{\mathbb{R}}$ so that $\theta \in \mathbb{R}^{N}$. Thus $\theta \cdot \mathbf{d}=0$ and we define $\theta^{*} \in \mathbb{R}^{N^{*}}$ by forgetting the component $\theta_{0}$. Given $V \in \operatorname{Rep}(\widehat{\mathcal{Q}}, \mathbf{d})$ we obtain (as in Exercise 8.4) $V^{*} \in \operatorname{Rep}\left(\widehat{\mathcal{Q}}^{*}, \mathbf{d}^{*}\right)$, and an $N^{*}$-graded vector space $W$ (with summands $W_{i}=\mathbb{C}^{w_{i}}$ ). Also, from the maps to and from the node 0 , we obtain (grade preserving) linear maps $a: W \rightarrow V^{*}$ and $b: V^{*} \rightarrow W$ (i.e. maps $a_{i}: W_{i} \rightarrow V_{i}$ and $b_{i}: V_{i} \rightarrow W_{i}$ for each $\left.i \in N^{*}\right)$.

1) Show that $V$ is $\theta$-semistable if and only if both i) and ii) below hold:
i) If $S \subset V^{*}$ is a subrepresentation and $S \subset \operatorname{Ker}(b)$ then $\theta^{*} \cdot \operatorname{dim}(S) \leq 0$,
ii) If $T \subset V^{*}$ is a subrepresentation and $\operatorname{Im}(a) \subset T$ then $\theta^{*} \cdot \operatorname{dim}(T) \leq \theta^{*} \cdot \operatorname{dim}\left(V^{*}\right)$.
2) Show further that $V$ is $\theta$-stable if and only if the strict inequalities hold in i),ii) unless $S=0$ or $T=V$ respectively.

These are the stability conditions used often by Nakajima. Note that if each component of $\theta^{*}$ is $>0$ then condition ii) is superfluous and i) means that there are no nontrivial subrepresentations $S$ (of $V^{*}$ ) with $S \subset \operatorname{Ker}(b)$. On the other hand if each component of $\theta^{*}$ is $<0$ then condition i) is superfluous and ii) means that there are no proper subrepresentations $T$ of $V^{*}$ with $\operatorname{Im}(a) \subset T$. (The Hilbert scheme of points on $\mathbb{C}^{2}$ appears in this latter case taking the node at the foot to be open-i.e. the framing of the 1-loop quiver.)
-brief discussion of replacing the dimension 1 by $r$ :
—ADHM moduli spaces (regular part when $\zeta=0, \cong$ instantons on rank $r$ bundles on $\mathbb{R}^{4}$ with framing at infinity, empty if $r=1$ )
-Whole space (not just regular part) when $\zeta=0$ as singular partial compactification, "ideal instantons". (If $r=1$ get $\operatorname{Sym}^{n}\left(\mathbb{C}^{2}\right)$-have ideal rank one instantons, but no actual instantons.)
-Moduli spaces of torsion free sheaves on $\mathbb{P}^{2}$ with framing on line $\mathbb{P}^{1} \subset \mathbb{P}^{2}$ (when $i \zeta_{\mathbb{R}}<0$ on central node) and $\zeta_{\mathbb{C}}=0$ (Barth). (Generalization of Hilbert scheme of points via map $E \mapsto E /\left.E^{\vee \vee}\right|_{\mathbb{C}^{2}}$, for rank one torsion free sheaves $E$.) View as desingularization of space of ideal instantons, via map induced from inclusion $\mu^{-1}(\zeta) \subset \mu_{\mathbb{C}}^{-1}(0)$ since $\zeta_{\mathbb{C}}=0$.
—Now view $\mathbb{C}^{2}$ as $A_{0}$ ALE space (i.e. corresponding to the trivial subgroup of $\mathrm{SU}(2)$, whose affine Dynkin diagram has one node and one loop). Extension of above stories to general ALE spaces (see [KN90, Kuz07, Nak07]).

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École Normale Supérieure et CNRS, 45 rue d'Ulm, 75005 Paris, France
www.dma.ens.fr/~boalch
boalch@dma.ens.fr


[^0]:    Date: April 9, 2009.

[^1]:    $1_{\text {which can }}$ be viewed as the set of degree $n$ formal linear combination of points of $X$ i.e. finite formal sums of the form $\sum_{x \in X} n_{x}[x]$ with $n_{x} \in \mathbb{Z}_{\geq 0}$.

[^2]:    ${ }^{2}$ This "unwinding" is due to Kazhdan-Kostant-Sternberg [KKS78] and this complex case has been studied by G. Wilson [Wil98].

[^3]:    ${ }^{3}$ Beware of the signs here: by convention $\left(\mu_{m}\right)_{\mathbb{R}}(Q) \in \mathfrak{g}$ corresponds to $\mu_{1}(Q) \in \mathfrak{g}^{*}$ such that $\operatorname{Tr} B\left(\mu_{m}\right)_{\mathbb{R}}(Q)=\mu_{1}(Q)(B)$. Then the formula says $i \mu_{1}(Q)(B)=\operatorname{Tr} B\left(Z^{\dagger} Z-W W^{\dagger}\right) / 2$, and so we deduce $\left(\mu_{m}\right)_{\mathbb{R}}(Q)$.

[^4]:    ${ }^{4}$ more precisely we use the relation $\left(a_{1}, a_{2}, a_{3}\right)=\left(\frac{1-\lambda \bar{\lambda}}{1+\lambda \bar{\lambda}}, \frac{\lambda+\bar{\lambda}}{1+\lambda \bar{\lambda}}, \frac{\sqrt{-1}(\lambda-\bar{\lambda})}{1+\lambda \bar{\lambda}}\right)$ where $\lambda$ is a complex inhomogenous coordinate on the Riemann sphere. Inversely $\lambda=\left(a_{2}-\sqrt{-1} a_{3}\right) /\left(1+a_{1}\right)$.

[^5]:    ${ }^{5}$ Beware that instead of " $h, t$ " some authors use "in, out" or " $t, s$ " (for target and source) respectively.

[^6]:    ${ }^{6}$ i.e. generated by homogeneous elements

[^7]:    ${ }^{7}$ This is proved similarly to the usual Jordan-Holder theorem for modules - see [Ses67], since the category of $\theta$-semistable quiver representations is abelian, artinian and noetherian, and the simple objects (i.e. with no nontrivial proper subobjects) of this category are the $\theta$-stable representations. The construction goes by choosing any maximal $\theta$-semisimple subrepresentation and iterating.

