L6, 23/3/2023 ①

Recall $\underset{\sim}{\Sigma} = (\Sigma, a, \Theta)$ rk n irreg. class Θ at a

$\Theta = \sum n_i I_i, \quad I_i \subset \mathcal{I} \to \partial$

$\Rightarrow \tilde{\Sigma} \subset \hat{\Sigma} \to \Sigma$

$\hat{\Sigma} \supset \Sigma^\circ = \Sigma \setminus a$

$\mathbb{A} \subset \partial$ singular directions

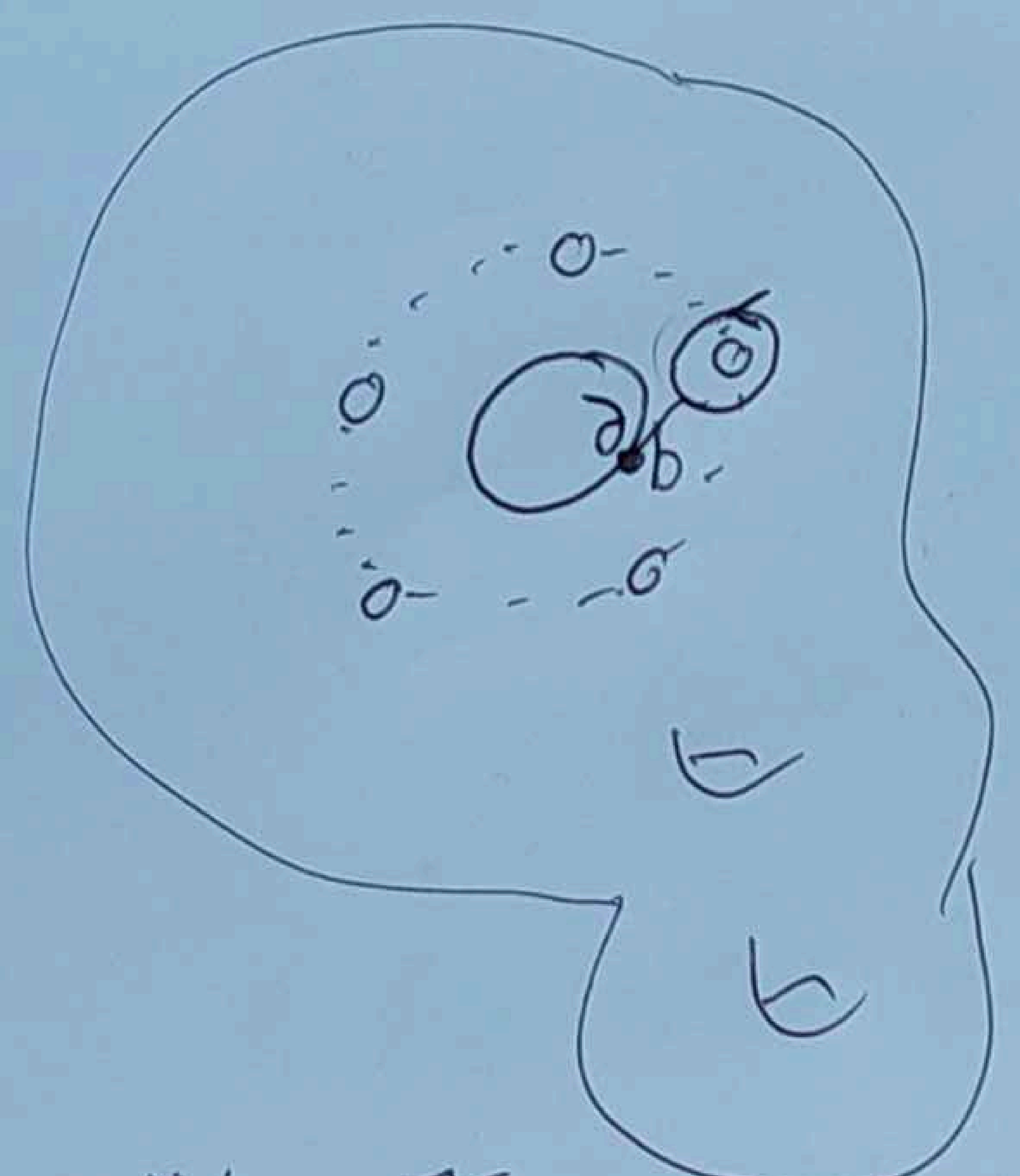
$<_d \subset I_d \times I_d \qquad I_d = \pi^{-1}(d)$

$d \in \partial$

$\tilde{\Sigma} = \hat{\Sigma} \setminus e(\mathbb{A})$ tangential punctures

$\pi : I \to \partial$

$(I = \cup I_i \subset \mathcal{I})$



$b \in \partial$ tangential basepoint

$R = \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{Hom}(\Pi, G)$

$\Pi = \pi_1(\tilde{\Sigma}, b)$

$H(\partial) \subset GL(\mathbb{C}^{\Theta})$ $\mathbb{T}$

$H(\partial) \cong \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ e.g. Airy

$\mathbb{V} \to \tilde{\Sigma}$

$\mathrm{Stod} \subset GL(\mathbb{V}_d) \qquad \mathbb{V}_d = \bigoplus_{i \in I_d} \mathbb{V}_d(i)$

<u>Prop</u> • R is an affine complex variety naturally equipped with an action of H

$$H = \mathrm{GrAut}(\mathcal{F}) \qquad \mathcal{F} = \mathbb{C}^{H} = \bigoplus_{i \in I_b} \mathbb{C}^{H(i)} = \mathbb{C}^{n}$$

(framing group)

$$\cong \prod_{i \in I_b} GL(\mathbb{C}^{H(i)}) \subset GL(\mathcal{F}) = GL_n(\mathbb{C})$$

- Isomorphism classes of framed Stokes local system, i.e pair $(\mathbb{V}, \phi)$ where ϕ is framing of $\mathbb{V}$ at b

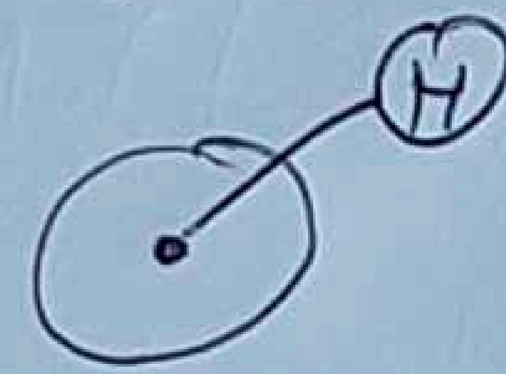
  i.e $\phi : \mathcal{F} \xrightarrow{\simeq} \mathbb{V}_b$ (graded isom)

  is in bijection with the points of R

- thus set of isom. classes of St. Loc. Systems on $\underset{\sim}{\Sigma}$ is in bijection with the set of H-orbits in R (H action ~ forgetting ϕ)

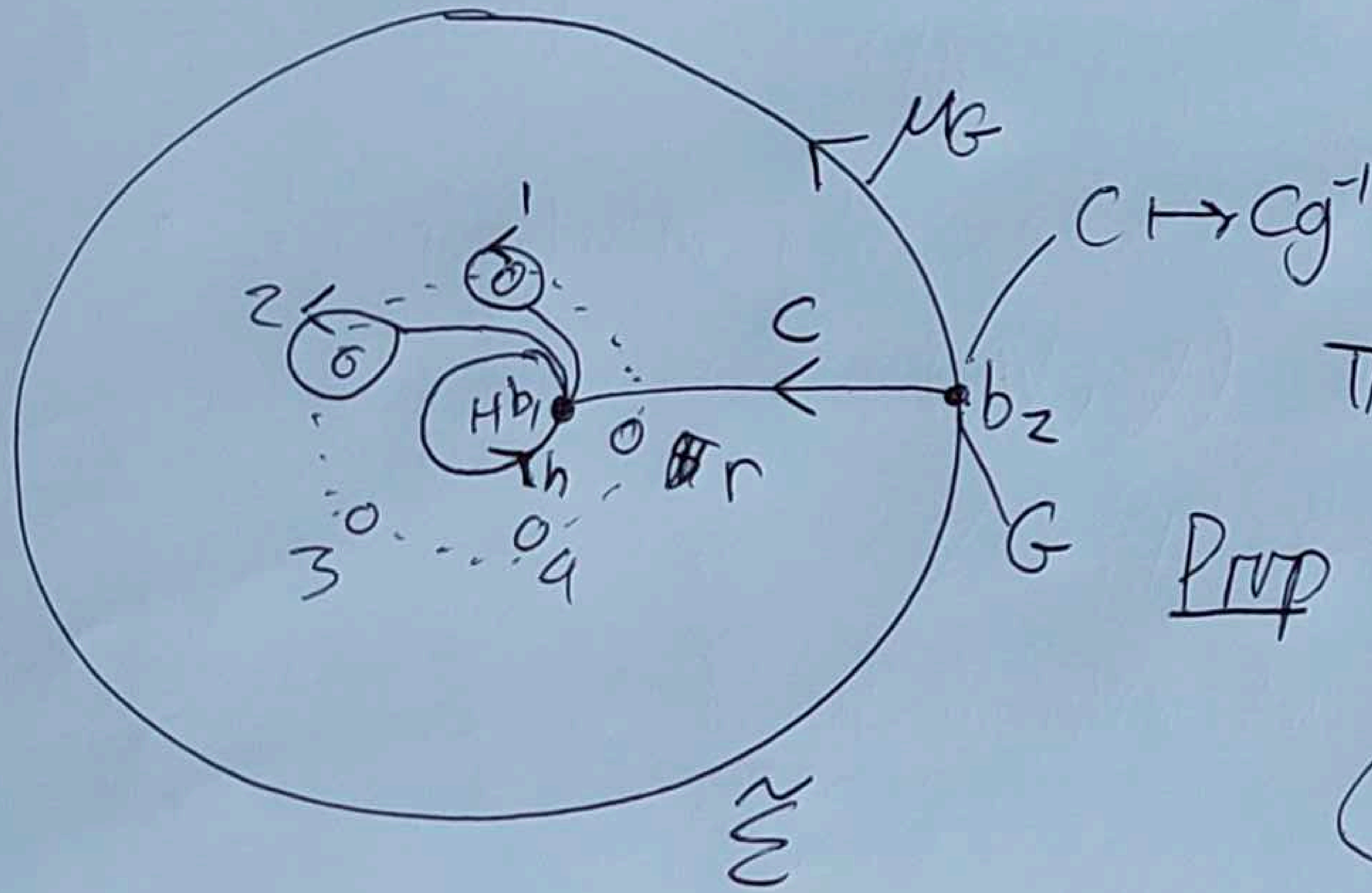
$\mathcal{M}_B = \mathcal{R} / \underset{\sim}{H}$ in general (wild char. var.) ③

<u>Simple pieces</u> (fission spaces)

$\mathcal{A}(\textcircled{H}) = \mathcal{R}\Big($  $\Big) = \mathcal{R}(\underset{\sim}{\Sigma}) \qquad (\textcircled{H}_0 \,\&\, \textcircled{H}_\infty)$

$\underset{\sim}{\Sigma} = \mathbb{P}^1, \{0, \infty\}, \underset{\sim}{\textcircled{H}}$

$\textcircled{H}_0 = \textcircled{H}, \quad \textcircled{H}_\infty = n\langle 0 \rangle$ (tame)



$\Pi = \Pi_1(\tilde{\Sigma}, \beta), \quad \beta = \{b_1, b_2\}$

<u>Prop</u> $\mathcal{A} \cong \left\{ C, h, S_1, \ldots, S_r \;\middle|\; \begin{matrix} h \in H(\partial) \\ S_i \in \mathbb{S}\mathrm{to}_{d_i} \end{matrix} \right\}$

$$\mathcal{A} \cong G \times H(\partial) \times \mathbb{S}\mathrm{to}$$

$\mathbb{S}\mathrm{to} = \prod_1^r \mathbb{S}\mathrm{to}_{d_i} \subset G^r$

$G = GL_n(\mathbb{C})$

<u>note</u> $\overbrace{G \times H}^{\underset{\sim}{H} \,=}$ acts on $\mathcal{A}$ $\quad (\mathcal{M}_B = \mathcal{A} / G \times H)$

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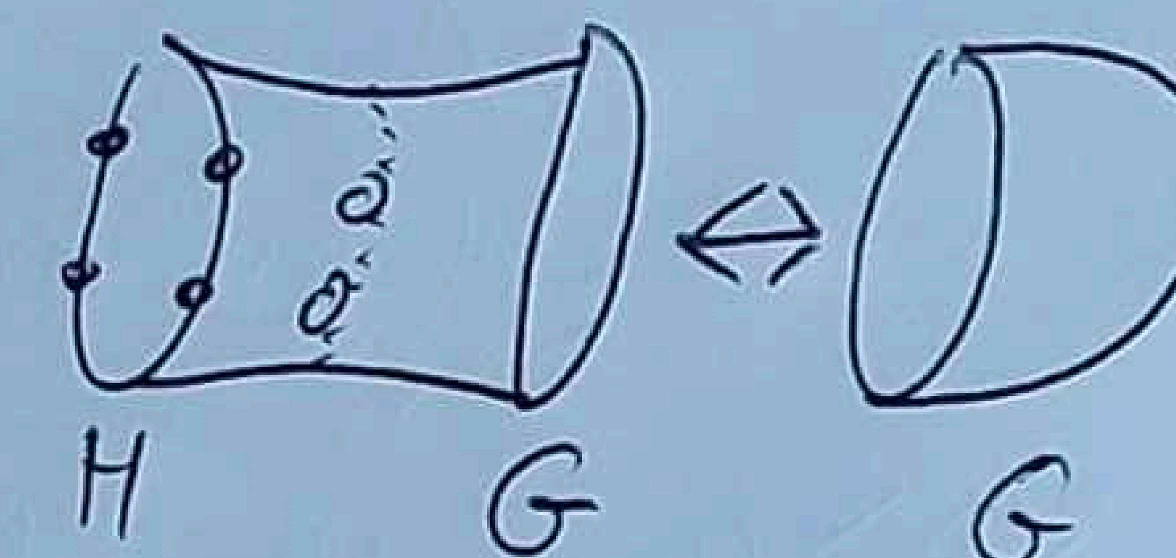
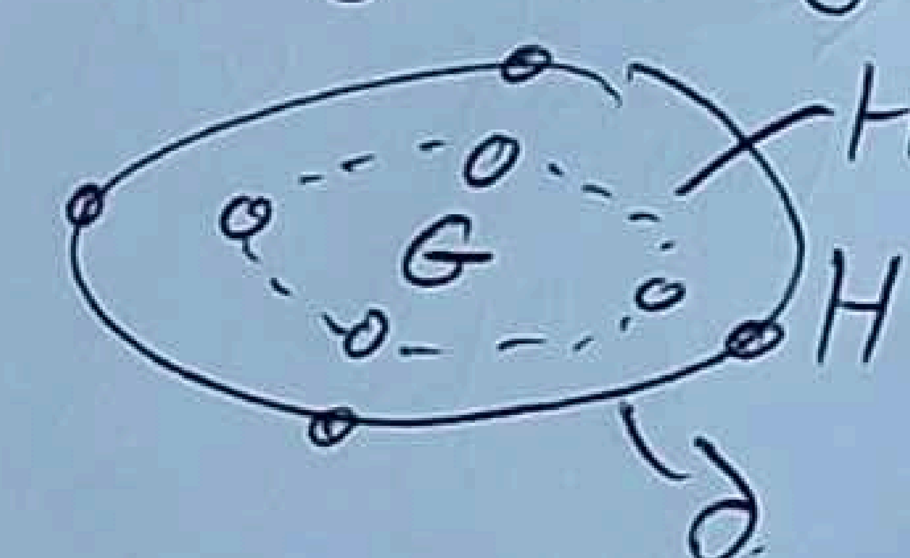
$G \times H \circlearrowright \mathcal{A} = G \times H(\partial) \times \mathbb{S}to$

$$\left.\begin{aligned} \mu_G &= C^{-1} h\, S_r \cdots S_2 S_1 C \\ \mu_H &= h^{-1} \end{aligned}\right\} \quad \mu : \mathcal{A} \to G \times H(\partial)$$

$\mathcal{A} \sim$ 

- can use to glue such pieces together
- get presentation of general $\mathcal{R}$



$$\underset{\left(\text{fission spaces}\right)}{\mathcal{A}} \rightsquigarrow \underset{\left(\underline{\text{reduced}}\ \text{fission spaces}\right)}{\mathcal{B}} = \mathcal{A} /\!\!/_{1} G = \{ a \in \mathcal{A} \mid \mu_G(a) = 1 \} / G$$

$$\cong \{ h, \underset{\sim}{S} \mid h S_r \cdots S_1 = 1 \}$$

$$(h \in H(\partial),\ S_i \in \mathbb{S}to_i)$$

$$= \mathcal{R} \text{ of } (\mathbb{P}^1, \underline{\underline{\infty}}, \Theta)$$

2 simplest classes of examples

1 ~ Birkhoff 1909, 1913 — GL_n, generic irr. class, $Q = A/z^k$, A diagonal with distinct evalues

Balser-Jurkat-Lutz ~1979

Jimbo-Miwa-Ueno 1981, (P.B '01)

$\Theta = \sum \langle a_i/z^k \rangle$ at $0 \in$ disk Δ $\quad A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

[2 Sibuya 1975, rk2 including twisted cases]

want to see the dimension count explicitly

Look at generic connections on $\mathbb{P}^1$

$d - \left(\sum_0^{k+1} A_i z^i\right) dz \qquad A_{k+1}$ generic $\Rightarrow$ very good

$\underset{G[[z^{-1}]]}{\sim} \quad dQ + \Lambda \frac{dz}{z} \qquad Q$ irr. type, Λ diagonal matrix

— want to fix Q & Λ

(6)

Fix $Q = \frac{A_k}{z^k} + \dots + \frac{A_1}{z}$ A_i diagonal, A_k has distinct eigenvalues

$\cancel{A}$ Λ diag.

$d - A$, $\underset{\text{hol-gauge}}{A \sim} dQ + \Lambda \frac{dz}{z} = \frac{B_{k+1}}{z^{k+1}}$ $(r = k+1)$

Q ir. type
Λ exp. of f. monod

$(g \in G[\![z]\!])$ $gAg^{-1} + (dg)g^{-1} = dQ + \Lambda \frac{dz}{z} + \text{hol.}$

lead term $g \frac{B_{k+1}}{z^{k+1}} g^{-1} + \dots$

Define $G_r = GL_n(\mathbb{C}[z]/z^r)$

$\mathfrak{g}_r = \mathrm{Lie}(G_r) = \Big(X_0 + X_1 z + \dots + \overset{X_k z^k}{\dots} + X_{r-1} z^{r-1} \Big)$ $X_i \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$

$\qquad = X$ $\qquad$ dim rn^2

Identify $\mathfrak{g}_r^*$ (dual of $\mathfrak{g}_r$ v.space) as

$\mathfrak{g}_r^* = \left\{ \underbrace{\left(\frac{B_r}{z^r} + \dots + \frac{B_1}{z} \right) dz}_{= B} \;\middle|\; B_i \in \mathfrak{g} \right\}$ (dim rn^2) $\ni dQ + \Lambda \frac{dz}{z}$

<u>Pairing</u> $\mathfrak{g}_r \times \mathfrak{g}_r^* \to \mathbb{C}$; $(X, B) \longmapsto \mathrm{Res}\,\mathrm{Tr}(XB) = \cancel{\mathrm{Tr}}\, B_1 X_0 + B_2 X_1 + \dots)$

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$dQ + \Lambda \frac{dz}{z} \in \mathfrak{g}_r^*$

Let $\mathcal{O} \subset \mathfrak{g}_r^*$ be the G_r orbit of $dQ + \Lambda \frac{dz}{z}$

where $g \in G_r$ acts as $(g, B) = g B g^{-1}$ (well-defined)

<u>Claim</u> $\dim \mathcal{O} = r n^2 - r(n) = r n(n-1)$

<u>Pf</u> check $\{ g \in G_r \mid g (dQ + \Lambda \frac{dz}{z}) g^{-1} = dQ + \Lambda \frac{dz}{z} \}$

$= \{ g = t_0 + X_1 z + \dots + X_k z^k \mid t_0 \in T, X_i \in \mathfrak{t} \}$

$T \subset G$

$\mathfrak{t} = \mathrm{Lie}(T)$

or Lie alg. version $\{ X \in \mathfrak{g}_r \mid [X, dQ + \frac{\Lambda dz}{z}] = 0 \} = \mathrm{Lie} = \{ \sum_0^k X_i z^i \}$

$X_i \in \mathfrak{t}$

In general global picture $\left(\mathcal{O}_i \ni dQ_i + \Lambda_i \frac{dz}{z-a_i}\right)$

$\mathcal{M}^* = \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ \underline{\underline{G}}$

choose $Q_1, \ldots, Q_m$ at $a_1, \ldots, a_m \in \mathbb{C}$ (generic)

$\Lambda_1 \quad \Lambda_m$ pole order k_i (of Q_i)

$(r_i = k_i + 1)$

$d - A, \quad A = \sum_{i=1}^{m} \left(\frac{B^{(i)}_{r_i}}{(z-a_i)^{r_i}} + \cdots + \frac{B^{(i)}_1}{(z-a_i)} \right) dz$ (no pole at ∞)

$(x = 1/z)$

$\mathrm{Res}_\infty \frac{dz}{z^k} = 0$ if $k \geq 2$, -1 if $k=1$ $\quad \frac{dz}{z} = -\frac{dx}{x}$

So need to impose $\sum_{i=1}^{m} B^{(i)}_1 = 0$ in order for A to be nonsing. at ∞

e.g if one pole $\mathcal{M}^* = \mathcal{O} /\!/ \underline{\underline{G}} = \underline{\underline{\mu^{-1}(0)}} / \underline{\underline{G}}$ $\quad \mu(\underset{\sim}{B}) = \sum_1^m B^{(i)}_1$

$\mu^{-1}(0) \subset \mathcal{O}, \quad \mu : \mathcal{O} \to \mathfrak{gl}_n(\mathbb{C})$

<u>Note</u> Global gauge trfm^{n}s (automorphisms) of $\mathcal{O}^{\oplus n} \to \mathbb{P}^1$ are constant $g \in GL_n(\mathbb{C})$

$\dim \ \mathcal{O}/\!/G \qquad \dim \mathcal{O} \qquad rn(n-1)$

$\mu^{-1}(0)/G \qquad \dim \quad rn(n-1) - 2(n^2-1)$ (heuristic)

(if $\mathcal{O}$ generic (3))

e.g $P_2 \quad k=3, \ n=2 \qquad 4\cdot 2\cdot 1 - 2(4-1)$

$8 - 6 = 2$ = dim Flaschka Newell

more generally would like $\dim \mathcal{O} = \dim \mathcal{C}$ "deeper conjugacy class"

& then $\dim(\mathcal{O}/\!/G) = \dim(\mathcal{C}/\!/G)$ $\quad \mu_G^{-1}(1)/G$

$\mu_{\mathfrak{g}}^{-1}(0)/G$

<u>What is $\mathcal{C}$?</u> in irreg case? ? $\mathcal{C} \subset G_r$? no

- if simple poles (log.) $\mathcal{C} \subset G$ is the conjugacy class of $\exp(2\pi i A)$ (if non-resonant)

Fix $dQ, \Lambda \rightsquigarrow H$ from Q in generic case

$\mathcal{A}, \mathcal{B}, \mathcal{C} = \mathcal{A} /\!/_{\check{\mathcal{C}}} H$ $\left(\begin{array}{c} H = T = C_G(Q) \subset GL_n(\mathbb{C}) \\ \text{diag. torus} \end{array} \right)$

$\mathcal{A} = G \times H \times \underline{\underline{\mathcal{S}to}}$ $H(\partial) = H = T \cong (\mathbb{C}^*)^n$ as not twisted

$Q = \frac{A}{z^k} + \cdots, \quad H = \sum_1^n \langle a_i / z^k \rangle$ (a_i distinct)

$\mu : \mathcal{A} \to G \times H$ $I \to \partial$ is trivial deg. n cover

$\mu = (\mu_G, \mu_H) = (C^{-1} h S_r \cdots S_1 C, h^{-1})$

∂

$\mathcal{C} = \mathcal{A} /\!/_{\check{\mathcal{C}}} H = \mu_H^{-1}(\check{\mathcal{C}}) / H = \{ (C, \underset{\sim}{S}) \in G \times \mathcal{S}to \} / T$

fix $\check{\mathcal{C}} \subset H$, here $\exp(2\pi i \Lambda) \in H = T$

$h = \exp(-2\pi i \Lambda) \in T$ (formal monod.)

$\mathcal{C} := \{ (C, \underset{\sim}{S}) \in G \times \mathcal{S}to \} / T$ "deeper conj class"

$t \in T$ acts by $t \cdot (C, \underset{\sim}{S}) = (tC, t \underset{\sim}{S} t^{-1})$ $\quad t \underset{\sim}{S} t^{-1} = (tS_1t^{-1} \ldots tS_rt^{-1})$

(free & so $\mathcal{C}$ is well defined) so $\mu_G : \mathcal{C} \to G$ well defined

$\mathcal{A} = G \times H \times \mathrm{Sto}$

$\mathcal{C} = \mathcal{A} /\!\!/_{\check{e}} T = \{ (C, \underset{\sim}{\xi}) \in G \times \mathrm{Sto} \} / T$

$$\dim = n^2 + \dim \mathrm{Sto} - n$$
$$= n^2 - n + \dim \mathrm{Sto} = n^2 - n + k(n^2 - n)$$

$\dim \theta = r(n^2 - n) = (n^2 - n) + (\cancel{k}\, r - 1)(n^2 - n)$ $\quad Q = A / z^k$

$\Big(Q$ had a pole of order k $\Big)$
$r = k + 1$

<u>Claim</u> $\mathrm{Sto} \cong (U_+ \times U_-)^k$

$\Rightarrow \dim \mathrm{Sto} = k(n^2 - n) \quad \Rightarrow \dim \theta = \dim \mathcal{C}$

$U_+ = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} = U_+^T$

$\dim U_+ \times U_- = \#$ roots $\quad \Big(\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \Big)$

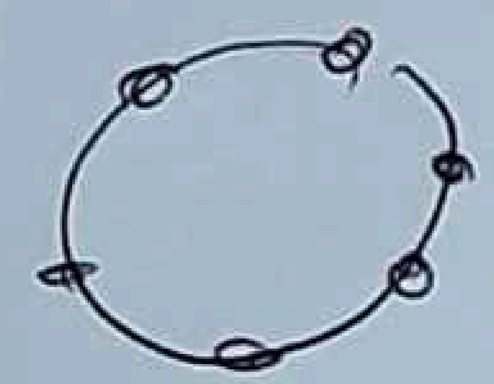
"Rootiness of Stokes data" if $Q = -A/z^k$ "take kth root of this picture"

e.g. $\nabla = \left(\frac{A}{z^2} + \frac{B}{z}\right) dz$, $Q = -A/z$, $A = \mathrm{diag}\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$

<u>Claim</u> $\mathbb{S}to \cong U_+ \times U_- \cong \prod_{\alpha \in R} G_\alpha$ $\left\{ \begin{array}{l} R \subset \mathfrak{t}^* \text{ roots of } \mathfrak{g} \\ \alpha_{ij}(A) = a_i - a_j \end{array} \right.$

(IMRN 2002)

- need to know Stokes arrows, $\mathbb{A} \subset \partial$ (at $z \to 0$)



<u>Claim</u> $\mathbb{A} \subset \partial$ are the directions $0 \to \langle R, A \rangle \subset \mathbb{C}^*$, $\langle R, A \rangle = \{\alpha(A)\}$

$= \{ a_i - a_j \mid i \neq j \in \{1, \ldots n\} \}$

<u>Key fact</u> $-\alpha(A)/z$ is real & negative when $\arg(z) = \arg(\alpha(A))$

Birkhoff generic case: these where all distinct ("wall crossing" ~definition of irreg. isomonodromy

(Cec.Vafa, Dubrovin ⟶ JMU '81 (generic GL_n))