MEROMORPHIC CONNECTIONS IN 2D GAUGE THEORY PRELIMINARY LECTURE NOTES

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PHILIP BOALCH

$$
\begin{tikzcd}
 & \mathfrak{M} \arrow[dl, bend right, "\cong", "1"'] \arrow[d, "\cong", "2"'] \arrow[dr, dashed, bend left, "\cong"] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\text{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\text{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}
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Non-Abelian Hodge space $\mathfrak{M}$,
three preferred algebraic structures:
Dolbeault, De Rham, Betti
mero. Higgs bundles, mero. connections, Stokes local systems

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Contents

Resumé

La but du cours est d'expliquer la lien entre les équations différentielles algébriques linéaires (les connexions méromorphes sur les fibrés vectoriels sur les courbes complexes lisses) et quelques exemples des équations différentielles non-linéaires.

L'idée de base, centrale dans la théorie de jauge, est que l'inconnu dans l'équation différentielle nonlineaire est mieux compris comme un connexion linéaire. Ici on utilise la même principe mais pour les connexions algébriques méromorphes.

Ceci donne l'opportunité d'étudier quelques exemples des jolies variétés algébriques (hyperkahlerienne) qui apparaissent comme espaces de modules dans cette histoire.

Contenu

• Espaces de modules de connexions additifs et multiplicatifs (géométrie symplectique holomorphe)

• Systèmes locaux de Stokes, variétés de caractères sauvages

• Application de Riemann-Hilbert-Birkhoff, théorie de Lie globale

• Équations de Yang-Mills autodual, équations de Hitchin, fibrés harmoniques sur les surfaces de Riemann non-compact (rotation hyperkahlerienne, correspondance de Hodge nonabelienne sauvage)

• Systèmes d'isomondromie (Painlevé, Schlesinger, Jimbo-Miwa-Mori-Sato, simplement lacé,...), connexions d'Ehresmann nonlineaires et lien avec les groupes de tresse

• Fibres de Higgs méromorphes et systèmes intégrables algèbro-géométriques (Garnier, Mumford, Hitchin, Bottacin-Markman, ...)

List of key points:

Lecture 1: Definition of meromorphic connection. View as global/intrinsic linear differential systems. Gauge transformations, gauge action. Algebraic versions. Definition of meromorphic Higgs bundles and ζ-connections.

Sketch of big picture (at symbolic level). Two key correspondences.

Start list of key examples: Painlevé's discovery of natural deformations of the theory of elliptic functions (simplification of $\mathrm{P_I}$,$\mathrm{P_{II}}$). Link to gauge theory (R. Fuchs): First steps in description of geometry of Painlevé VI.

1. Lecture 1: Basic examples, questions and definitions

1.1. What is a meromorphic connection?

We will need several different flavours (categories) of connections, and some confusion in the subject comes from different authors having different default definitions. The relation between various definitions will be crucial to understand. Thus we'll start with the central notions we will use, and then later discuss variations and their relation.

The starting point is a first order linear differential operator of the form

$$\frac{d}{dz} - B(z)$$

where $B(z)$ is an $n \times n$ matrix of holomorphic functions on an open subset $U \subset \mathbb{C}$. As a first example one might consider a polynomial system:

$$\frac{d}{dz} - (A_0 + A_1 z + \cdots A_m z^m) \tag{1.1}$$

for $n \times n$ matrices A_i. As a second example one might consider:

$$\frac{d}{dz} - \left(\frac{A_1}{z - a_1} + \cdots + \frac{A_m}{z - a_m} \right) \tag{1.2}$$

for $n \times n$ matrices A_i, away from the poles (these are often called "Fuchsian systems").

This yields the linear system of differential equations

$$\frac{dv}{dz} = Bv$$

where v is a length n column vector of holomorphic functions. The coordinate-free version of this operator is got by "multiplying by dz", to get the *connection*

$$\nabla = d - A, \qquad A = B(z)dz$$

so that A is a matrix of holomorphic one-forms and d is the exterior derivative. This is a connection on the trivial rank n holomorphic vector bundle on U, i.e. on $E = \mathbb{C}^n \times U \to U$. Solutions v are now called *horizontal sections* and the equation $dv/dz = Bv$ is rewritten $\nabla(v) = 0$, i.e. $dv = Av$. We can remove the condition that E is trivial and consider connection on non-trivial vector bundles, leading to the following definition, first in the case with no poles.

Let Σ be a compact Riemann surface.

Definition 1.1. *A* <u>holomorphic connection</u> *is a pair* (E, ∇) *where* $E \to \Sigma$ *is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$*-linear operator, from the sheaf of sections* $\mathcal{E}$ *of* E *to the sections of* E *twisted by holomorphic one-forms, such that the Leibniz rule is satisfied:*

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.3}$$

for all local sections s *of* E *and functions* f*.*

This is a completely standard definition, going back to Koszul. A connection is a way to differentiate sections of E: If X is a vector field on Σ and s is a section of E then

$$\nabla_X(s) := \langle X, \nabla(s)\rangle$$

is again a section of E, the derivative by ∇ of s along X. Here the brackets $\langle\,\cdot\,,\,\cdot\rangle$ denote the natural pairing between the tangent bundle and the cotangent bundle.

In a local trivialisation of E, over some open subset $U \subset \Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for an $n\times n$ matrix of holomorphic one-forms A, where n is the rank of E. If z is a local coordinate on U this means we can write $A = Bdz$ for a matrix B of holomorphic functions on U. Thus a connection $\nabla = d - Bdz$ is really just a global, coordinate-free version of the matrix differential operators $\frac{d}{dz} - B$ we first considered

If we change the choice of local trivialisation of E then A changes by a *gauge transformation*:

$$A \;\mapsto\; g[A] := gAg^{-1} + (dg)g^{-1} \tag{1.4}$$

where $g : U \to \mathrm{GL}_n(\mathbb{C})$ is a holomorphic map. Our conventions are set-up such that this is a group action:

Exercise 1.2. Show that $(g\circ h)[A] = g[h[A]]$.

Exercise 1.3. Show that if $\mathbf{e} = (e_1, \dots, e_n)$ is the initial basis of E and $\mathbf{e}' = (e'_1, \dots, e'_n)$ is the new basis, and g is such that $\mathbf{e} = \mathbf{e}'\circ g$, then we do indeed get the formula (1.4) for $g[A]$.

Write $G = \mathrm{GL}_n(\mathbb{C})$, let $\Delta \subset \mathbb{C}$ be an open disk, and let $\mathcal{G} = \mathrm{Map}_{\mathrm{hol}}(\Delta, G)$ be the group of all holomorphic maps from Δ to G. Also write $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ and let

$$\mathcal{A} = \{A = B(z)dz \;\big|\; B : U \to \mathfrak{g}\}$$

be the space of all holomorphic connections on the trivial bundle on the open disk Δ, so that B is a holomorphic map. Thus by the exercise above the group $\mathcal{G}$ acts on the space $\mathcal{A}$ by gauge transformations:

$$\mathcal{G}\times\mathcal{A} \to \mathcal{A}; \qquad (g, A) \mapsto g[A] = gAg^{-1} + (dg)g^{-1}.$$

Holomorphic connections are not interesting locally since they are all isomorphic:

Lemma 1.4. *$\mathcal{G}$ acts transitively on $\mathcal{A}$. In particular for any $A \in \mathcal{A}$ there is a $g \in \mathcal{G}$ such that $g[A] = 0$ (every holomorphic connection is locally isomorphic to the trivial connection).*

Proof. Given A we wish to find g so that $gAg^{-1} + (dg)g^{-1} = 0$. In other words $gA + (dg) = 0$. If we write $h = g^{-1}$ and use the useful fact that $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ then we want $h : U \to G$ so that

$$\frac{dh}{dz} = B(z)h$$

where $B = A/dz$ as usual. In classical language this equation just says that h is a "fundamental solution" (or "fundamental matrix") of the linear system $d/dz - B$. (By definition this means that the columns of h make up a basis of solutions of the system.) It is a classical fact (Cauchy?) that holomorphic systems have fundamental solutions[1]. In fact its easy to construct a series solution term by term, and then one proves the resulting series solution converges. □

Let $E = \mathbb{C}^n \times \Delta$ denote the trivial bundle. Note that a fundamental solution h is the same thing as an isomorphism $(E, d = d - 0) \to (E, d - A)$ from the trivial connection to the connection $d - A$. This just says $h[0] = A$, i.e. $A = (dh)h^{-1}$ or $dh = Ah$.

In general an isomorphism from $(E, d - A_1) \to (E, d - A_2)$ is a section h of $\mathrm{Hom}(E, E)$ that is invertible and satisfies $h[A_1] = A_2$ i.e.

$$hA_1h^{-1} + (dh)h^{-1} = A_2$$

or in other words:

$$dh = A_2h - hA_1.$$

Indeed it is natural to define a connection $\mathrm{Hom}(\nabla_1, \nabla_2)$ on $\mathrm{Hom}(E, E)$, whose horizontal sections are given by this equation. Similarly if there are two different vector bundles, and one can thus define dual connections etc.

Of course we can consider holomorphic connections on punctured Riemann surfaces but that won't capture most of the properties of the first (polynomial) example, and not all the properties of the second (Fuchsian) example. Instead we proceed as follows to encompass them.

Now let Σ be a compact Riemann surface and $\mathbf{a} = (a_1, \ldots, a_m) \subset \Sigma$ a finite subset. Let $D = \sum n_i(a_i)$ be an effective divisor on Σ supported on $\mathbf{a}$, so that $n_i \geq 1$ are integers.

[1]See e.g. classical ODE books by Hartman, Coddington–Levinson, Ince, Hille, ...

Definition 1.5. *A* <u>meromorphic connection</u> *with poles bounded by D is a pair (E,∇) where $E\to\Sigma$ is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E}\to\mathcal{E}\otimes\Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by meromorphic one-forms with poles bounded by D, such that the Leibniz rule is satisfied:

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.5}$$

for all local sections s of E and functions f.

In a local trivialisation of E, over some open subset $U\subset\Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for a matrix of meromorphic one-forms A (with poles bounded by D). E.g. if $a_1\in U$ and z is a local coordinate vanishing at a_1 then

$$\nabla = d - \frac{B(z)dz}{z^{n_1}}$$

in a neighbourhood of a_1, where B is holomorphic across a_1.

Remark 1.6. Note that:

1) $E\to\Sigma$ is a holomorphic vector bundle on the *compact* surface, so this is a genuine generalisation of a holomorphic connection.

2) this notion is well defined, but it would not be if D was not effective. The point is that the Leibniz rule tacitly uses the inclusion $\Omega^1\subset\Omega^1(D)$ of the holomorphic one forms into the meromorphic one forms. If D was not effective, say $n_1<0$, then the Leibniz rule would not make sense (as there is then no such inclusion: df would not necessarily be a section of $\Omega^1(D)$).

3) this is not a completely standard definition (although we have been happily using it since 1999 or so). One can also define the notion of "meromorphic connection on a meromorphic bundle", where a "meromorphic bundle" is a locally free $\mathcal{O}(*D)$ module. In practice this means that one allows meromorphic gauge transformations with any order pole at the points of D. This definition is also useful, but is less convenient for gauge theory or moduli theory.

1.2. **Some variations: algebraicity.**

Suppose Σ is actually a smooth compact complex algebraic curve.

•$v1$) Algebraic connections (E,∇) (if E is algebraic and ∇ is algebraic). Thus there is a Zariski open covering $\Sigma=\bigcup U_i$ so that the restriction of E to each open set U_i is trivialisable. (Recall Zariski open subset are just the complements of finite subsets of points.) Then by choosing such trivialisations the bundle E is determined by algebraic clutching maps $g_{ij}: U_{ij} \to G$, where $U_{ij} = U_i \cap U_j$. Then on U_i we have $\nabla = d - A_i$ where A_i is a matrix of regular differentials (algebraic one-forms) on U_i. On the double intersections the A_i are related by gauge transformations as usual

$$g_{ij}[A_j] = A_i.$$

These are just the algebraic version of holomorphic connections. In fact some form of GAGA implies the analytification functors gives an equivalence of categories (Algebraic connections) $\to$ (holomorphic connections), in this setting where Σ is compact.

$\bullet v2$) Similarly there is a notion of "Algebraic meromorphic connections", as above but allowing the A_i to be matrices of rational differentials (algebraic one-forms with poles), with poles bounded by the fixed effective divisor D. Again a version of GAGA implies the analytification functors gives an equivalence (to the meromorphic connections on holomorphic vector bundles). These will actually be the realm for most of our examples, with nonlinear differential equations flowing in their spaces of coefficients.

$\bullet v3$) If we now take $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ to be an open curve (in fact any smooth complex algebraic curve takes this form for some finite set $\mathbf{a}$). Then we can consider algebraic connections $(E, \nabla) \to \Sigma^\circ$ on the open curve. This category is actually very close to being a subcategory of the category of meromorphic connections on holomorphic vector bundles on Σ with poles on $\mathbf{a}$ (if we allow any pole orders). There is in fact a version of GAGA that shows this category is equivalent to the "meromorphic connections on meromorphic bundles" on Σ with any order poles on $\mathbf{a}$.

In the next lecture we will consider holomorphic connections on vector bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$; this is relatively trivial and all the extra structure "hidden" in the poles at the punctures is lost. However we won't get to hermitian metrics for some time so its worth noting:

> The relevance in "hardcore analytic" gauge theory (on the punctured surface), of having (meromorphic connections on) holomorphic vector bundles on the compact surface, comes from the fact that the addition of a hermitian metric controls the growth of sections at the punctures, leading to preferred extensions across the punctures, and thus holomorphic vector bundles (or parabolic vector bundles) on the compact surface.

1.3. **Some more variations: Higgs bundles and ζ-connections.**

Definition 1.7. *A* <u>meromorphic Higgs bundle</u> *with poles bounded by D is a pair (E, Φ) where $E \to \Sigma$ is a holomorphic vector bundle and $\Phi \in \mathrm{H}^0(\Sigma, \mathrm{End}(E) \otimes \Omega^1(D))$ is the Higgs fields, a meromorphic one-form with values in $\mathrm{End}(E)$, with poles bounded by D.*

Thus locally we can write $\Phi = Bdz$ for a matrix B of meromorphic functions on U.

In a sense Higgs bundles have two origins: just as an operator $d/dz - B$ led to a connection $d - Bdz$, any matrix $L(z)$ of rational functions (aka a "rational Lax matrix") leads to a Higgs field $L(z)dz$ (on the trivial vector bundle on $\mathbb{P}^1$). On the other hand holomorphic Higgs fields on higher genus Riemann surfaces were introduced by Hitchin and Simpson. These two viewpoints were "put together" in the definition of meromorphic Higgs bundle (Nitsure, Bottacin, Markman, ...).

Exercise 1.8. Suppose ∇_1, ∇_2 are meromorphic connections on $E \to \Sigma$ with poles on D. Show that $\Phi := \nabla_1 - \nabla_2$ is a meromorphic Higgs field.

Now choose a complex number $\zeta \in \mathbb{C}$.

Definition 1.9. *A* meromorphic ζ-connection *with poles bounded by D is a pair (E, ∇) where $E \to \Sigma$ is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by meromorphic one-forms with poles bounded by D, such that the ζ-Leibniz rule is satisfied:

$$\nabla(fs) = \zeta(df)s + f\nabla(s) \tag{1.6}$$

for all local sections s of E and functions f.

Exercise 1.10. Study ζ-connections in a local trivialisation, and show that the gauge action is modified to: $g[A]_\zeta = gAg^{-1} + \zeta(dg)g^{-1}$.

Exercise 1.11. Show that for $\zeta = 0$ a meromorphic ζ-connection is the same thing as a meromorphic Higgs bundle.

Exercise 1.12. Show that for $\zeta = 1$ a meromorphic ζ-connection is the same thing as a meromorphic connection.

Thus there is a "continuous deformation" from connections to Higgs bundles.

Exercise 1.13. Write down the algebraic versions of the definitions of Higgs bundles and ζ-connections.

This is often referred to as the "autonomous limit" in the integrable systems literature. We will eventually see it is the same as the "Painlevé simplification" of the Painlevé equations, and see it as a hyperkähler rotation.

Of course, physicists have been putting numbers (like $\hbar$) in front of their differential operators for a long time.

And it is essentially the same as the deformation from loop algebras into affine Kac–Moody algebras (although in the full story there is also a central extension, dual to this deformation).

Remark 1.14. Note that for any line bundle $\mathcal{L} \to \Sigma$ one can define a $\mathcal{L}$-valued Higgs bundle as pair (E, Φ) with Φ a section of $\mathrm{End}(E) \otimes \mathcal{L}$. However unless this is secretly a meromorphic Higgs bundle (i.e. there is an isomorphism $\mathcal{L} \cong \Omega^1(D)$ for some effective D) then there is no analogous notion of "$\mathcal{L}$-valued connections" (Rmk. 1.6 2).

2. Sketch of big picture

2.1. **Three algebraic worlds:** Before delving into the details lets try to signpost where we want go (at least symbolically for the moment). Much of the story we want to describe can be summarised in the (slightly oversimplified) diagram:

Dolbeault		De Rham		Betti
Rational Lax matrices L $\dot{L} = [P, L]$		Rational diff. op.s $\frac{d}{dz} - A$		Stokes and monodromy data
$\bigcap$		$\bigcap$		
Mero. Higgs bundles (E, Φ)	$\overset{\text{wnAbH}}{\longleftrightarrow}$	Mero. connections (E, ∇)	$\overset{\text{RHB}}{\longleftrightarrow}$	Stokes local systems
$\bigcup$		$\bigcup$		
Holom. Higgs bundles (E, Φ)	$\overset{\text{nAbH}}{\longleftrightarrow}$	Holom. connections (E, ∇)	$\overset{\text{RH}}{\longleftrightarrow}$	Local systems/ π_1-rep.s

The main aim is to describe the central row, and, as the diagram indicates, it is set-up to include both the rich class of examples of rational Lax matrices and the sophisticated nonabelian Hodge setting of holomorphic Higgs bundles (no poles), related to the Hitchin integrable systems.

For example a rational Lax matrix $L(z)$ becomes a Higgs field Ldz by multiplying by a rational one-form, such as dz. The Lax matrices appear in Lax equations $\dot{L} = [P, L]$ and are the bread and butter of the theory of integrable systems, the solution of the system comes from a straight line flow on the Jacobian of the spectral curve defined by $\det(L - \lambda) = 0$. We will discuss some examples in detail but for now note there are lots, as listed for example in the book of Babelon et al, or basic sources such as:

Adler and van Moerbeke (1980) Completely integrable systems, Euclidean Lie algebras, and curves, Adv. in Math. 38, no. 3, 267-317.

Griffiths (1985) Linearizing Flows and a Cohomological Interpretation of Lax Equations

Adams–Harnad–Previato (1988) Isospectral flows in finite and inifinite dimensions

Reyman and Semenov-Tian-Shansky (1994) Integrable systems II group theoretical methods in the theory of finite dimensional integrable systems

Mumford's 1984 book "Tata Lectures on Theta II, Jacobi theta functions and differential equations" is devoted to a class of examples involving 2×2 Lax matrices.

The first large class of examples seems to be due to Garnier 1919 (and we will discuss the "Painlevé simplification" method he used to discover them, taking the autonomous limit of the Schlesinger equations):

Garnier (1919) Sur une classe de systèmes différentiels abéliens déduits de la théorie des équations linéaires, (Rendiconti del Circolo Matematico di Palermo 43, pp.155-191).

Several of the key ideas of Garnier's paper were rediscovered as an offshoot of soliton theory, before Garnier's work was rediscovered and widely disseminated[2], around 1980.

[2]E.g. there is a section on it in the well-known paper of Flaschka-Newell on isomonodromy (Comm. Math. Phys. 76 (1980), 65-116), and it is mentioned in Dubrovin's 1981 paper on theta functions, the 1980 Krichever-Novikov review (Russian Math. Surveys 35:6 (1980), 53-79) and in the footnote p.156 of the 1980 paper of Jimbo-Miwa-Mori-Sato. D.V. Chudnovsky wrote a paper on it (Let. Nuovo Cimento 26 (14) 1979), and M. Gaudin cited that in his 1983 book (La fonction d'onde de Bethe), having discovered the quantum version in 1976.

2.2. **Two organisational diagrams.** Mathematically this story leads to an interesting class of *moduli spaces*, i.e. spaces whose points correspond to isomorphism classes of certain meromorphic connections (or Higgs bundles, or Stokes local systems). This goes slightly beyond the objects usually studied by algebraic geometers, and one of the main inputs is to write down the moduli problem that encompasses this picture.

In particular we will fix a Riemann surface, some marked points and some precisely defined boundary data. This will determine a hyperkähler manifold $\mathfrak{M}$ with three preferred algebraic structures, corresponding to the three columns of the above table. We label the columns "Dolbeault, De Rham, Betti" as they are precise analogues of the Dolbeault, De Rham and Betti approaches to linear cohomology (it was first abstracted to the context of nonabelian cohomology by Simpson, and then later extended to the meromorphic case relevant to Lax matrices). The result, to be explained, is a diagram as follows:

$$\begin{tikzcd}
& \mathfrak{M} \arrow[dl, bend right, "1"', "\cong"] \arrow[d, "2"', "\cong"] \arrow[dr, dashed, bend left, "\cong"] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\mathrm{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\mathrm{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}$$

FIGURE 1. Nonabelian Hodge space $\mathfrak{M}$, with three preferred algebraic structures.

However this does not capture the full story and in practice people work with simpler open parts $\mathcal{M}^*$ of the moduli spaces in genus zero, where things can be made explicit, and actual nonlinear differential equations can be obtained. We will explain that the classical Riemann–Hilbert map is a holomorphic map

$$\boxed{\mathcal{M}^* \hookrightarrow \mathcal{M}_{\mathrm{B}}} \tag{2.1}$$

where $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}} \subset \mathcal{M}_{\mathrm{DR}}$ is the open part of the full De Rham moduli space where the bundles E are trivial. As we will see the spaces $\mathcal{M}^*$ have the flavour of the "Lie algebra" of the full nonabelian Hodge space $\mathfrak{M} \cong \mathcal{M}_{\mathrm{B}}$, and the Riemann–Hilbert map is a natural generalisation of the exponential map.

However this still does not capture the full story as we also wish to *vary the modular parameters*, changing the complex structure on the Riemann surface, the pole positions, and the "irregular class" of the connections at each pole. These parameters will lead to the independent variables ("times") in the isomonodromy equations.

3. Glimpses of the elephant

The next few sections will describe a few simple pieces of the full picture, that provided motivation.

3.1. **Painlevé's deformation of the theory of elliptic functions.**

Painlevé discovered most of the Painlevé equations as deformations of differential equations for elliptic functions, i.e. as equations that limit to equations for elliptic functions. He used the term "simplification" (*simplifié*) for the limiting differential equation, solvable in terms of elliptic functions.

In more detail Painlevé was looking for new special functions, defined as solutions to non-linear algebraic differential equations. He looked for equations whose solutions had good meromorphic continuations properties: outside a fixed critical set, any local solution should have arbitrary meromorphic continuation. If $D \subset \mathbb{C}$ is the fixed critical set (a finite set in all examples here), then any local solution $y(t)$ should extend to a meromorphic function on the universal cover $\widetilde{\mathbb{C} \setminus D}$. This is known as the Kowalevski–Painlevé (KP) property (and can also be expressed as saying there are no "movable singularities" apart from poles).

The KP property is preserved under any deformation of the differential equation[3]. Thus to rule out many possible forms of differential equations, Painlevé would add parameters by hand and then take limits to get simpler equations (Painlevé's α method). If he could recognise or prove the limiting equation did not have the KP property then he could ignore the putative equation, and thus get a short list of possibilities, that then could be proved to have the KP property directly.

For example $\mathrm{P_I}$, the first Painlevé equation, $y'' = 6y^2 + t$ is a deformation of the equation for the Weierstrass $\wp$ function:

First recall the standard differential equation satisfied by $\wp$ is

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for constants $g_2, g_3 \in \mathbb{C}$. Thus $2\wp'\wp'' = 12\wp^2\wp' - g_2\wp'$ so that

$$\wp'' = 6\wp^2 - g_2/2$$

Lemma 3.1 (cf. Painlevé 1900 p.226, Ince [] pp.321 and 329)**.** *Suppose $y(t)$ satisfies* $\mathrm{P_I}$ *so that $y'' = 6y^2 + t$. If $t = \alpha x, y = w(x)/\alpha^2$ for a constant α then*

$$w'' = 6w^2 + \alpha^5 w. \tag{3.1}$$

In particular if $\alpha^5 = 1$ then this is a symmetry of $\mathrm{P_I}$ *. If we take the limit $\alpha \to 0$ then we get $w'' = 6w^2$. This integrates once to $(w')^2 = 4w^3 + c$, which can be solved*

[3] see e.g. paragraph 1 p.319 in Ince's book "ordinary differential equations" [?].

in terms of the Weierstrass $\wp$ function: $w = \wp(x+k)$ (where $\wp$ has $g_2 = 0, g_3 = -c$, and $k \in \mathbb{C}$ is arbitrary).

Proof. Write $v = y'$ so that $dy = vdt, dv = (6y^2 + t)dt$. Now put $t = \alpha x, y = w/\alpha^2$ (as on Painlevé 1900 p.226, Ince p.329 [], or Valiron p.410 []). Thus $dt = \alpha dx, dy = dw/\alpha^2$, so $w' = dw/dx = \alpha^3 dw/dz = \alpha^3 v$. Thus

$$dv = (6y^2 + t)dt = \alpha(6w^2/\alpha^4 + \alpha w)dx$$

and so $w'' = \alpha^3 v' = \alpha^5 w + 6w^2$ which is (3.1). The last statement is straightforward, recalling that in general $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for constants g_2, g_3. $\square$

Thus Painlevé discovered a natural deformation of the theory of elliptic functions!

—see Alves `https://arxiv.org/abs/2103.02697v1` §3 for a recent discussion of this story.

The next example is the case of Painlevé II:

Lemma 3.2. *Suppose $y(t)$ satisfies the $\mathrm{P_{II}}$ equation $y'' = 2y^3 + ty + \alpha$. If $t = \gamma x, y = w(x)/\gamma$ for a constant γ then*

$$w'' = 2w^3 + \gamma^3 xw + \gamma^2\alpha. \tag{3.2}$$

Thus if we now take the limit $\gamma = 0$ then $w'' = 2w^3$, which integrates once to $(w')^2 = w^4 + c$, and can be solved in terms of the Jacobi sn function: $w = c_1\mathbf{sn}(c_1(ix + c_2), i)$.

Proof. Write $v = y'$ so that $dy = vdt, dv = (2y^3 + ty + \alpha)dt$. Now put $t = \gamma x, y = w/\gamma$. Thus $dt = \gamma dx, dy = dw/\gamma$, so $w' = dw/dx = \gamma^2 dw/dz = \gamma^2 v$. Thus

$$dv = (2y^3 + ty + \alpha)dt = \gamma(2w^3/\gamma^3 + xw + \alpha)dx$$

and so $w'' = \gamma^2 v' = 2w^3 + \gamma^3 xw + \gamma^2\alpha$ which is (3.2). The last statement is straightforward. $\square$

In this way Painlevé discovered some very interesting nonlinear differential equations, the Painlevé equations 1,2,3,4.

Later on (late 1970s) the Painlevé equations, and their solutions, the Painleve transcendents, started appearing in physics problems such as the Ising model[4] (in some sense physics got sufficiently nonlinear to catch up with the mathematics...).

Note that so-far these equations have no link to gauge theory: there are no linear differential equations in the story. That link came about via a 1905 paper of R. Fuchs where he discovered a new Painlevé equation, called Painlevé six, $\mathrm{P_{VI}}$, controlling the

[4]E.g. Wu-McCoy-Tracy-Barouch (1976) "Spin-spin correlation functions for the two-dimensional Ising model, Exact theory in the scaling region"

isomonodromic deformations of a linear differential equation. This is a completely different way to get nonlinear differential equations[5].

One can find the standard list of Painlevé equations in many places (e.g. wikipedia), but we really want to think of them as geometric objects, and this is obscure in their explicit expression. They will each lead to a deformation class of nonabelian Hodge spaces of complex dimension two, the minimal possible nonzero dimension, so they give the simplest examples.

The basic features are summarised in the table below:

Painlevé equation:	1	2	3	4	5	6
Domain of t:	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C}\setminus\{0,1\}$
No. of constant parameters:	0	1	2	2	3	4
(Affine Dynkin) Diagram :	$\widehat{A}_0$	$\widehat{A}_1$	$\widehat{D}_2$	$\widehat{A}_2$	$\widehat{A}_3=\widehat{D}_3$	$\widehat{D}_4$
Okamoto Diagram :	$\widehat{E}_8$	$\widehat{E}_7$	$\widehat{D}_6$	$\widehat{E}_6$	$\widehat{D}_5$	$\widehat{D}_4$

TABLE 1. Basic data for Painlevé equations

(Here we have omitted two degenerate versions of Painlevé 3.)

The diagrams can be drawn as follows (the number of nodes is one plus the number of constants):

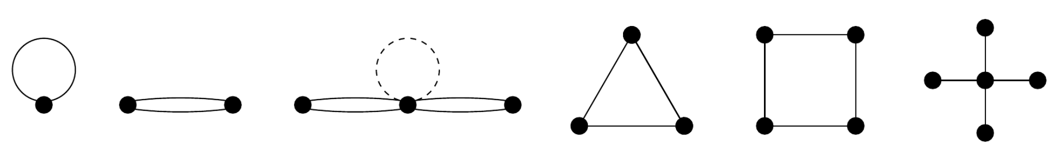

FIGURE 2. The diagrams of the six Painlevé equations.

[5]R.Fuchs' isomonodromy approach was extended to the original Painlevé equations by Garnier 1912 (so they too, in fact, are gauge theoretic equations).

3.2. **Towards the Painlevé VI connections.**

The 1905 paper of R. Fuchs[6], should probably be viewed as the true "start of 2d gauge theory" where a nonlinear differential equation arose naturally, controlling a linear differential equation (i.e. where the "unknown" is really a linear differential equation $\sim$ a meromorphic connection on a rank two vector bundle on $\mathbb{P}^1$). The underlying idea can be traced back to a suggestion of Riemann 1857.[7]

What is Painlevé VI, the Fuchsian Painlevé equation?

Definition 3.3. *Given constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the corresponding Painlevé VI equation $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$ is the algebraic differential equation:*

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y'$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

for a meromorphic function $y(t)$ where $t \in \mathbb{C} \setminus \{0, 1\}$.

This frankly horrific expression does not express very well the true beauty of the underlying geometric object. The simplest encoding of it seems to be the following time-dependent Hamiltonian formulation, due to Malmquist 1922.

Proposition 3.4 (cf. [?] p.86)**.** *If $a_1, a_2, a_3, b \in \mathbb{C}$ then the function $H(q, p, t)$ defined by*

$$t(t-1)H(q,p,t) = q(q-t)(q-1)\left(p^2 + p\left(\frac{a_1}{q} + \frac{a_2}{q-t} + \frac{a_3}{q-1}\right)\right) + b \cdot q$$

is a time-dependent Hamiltonian function for $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$, in the sense that if

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

then $y = q(t)$ is a solution to $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$ where

$$\alpha = (a_1 + a_2 + a_3)^2/2 - 2b,\ \ \beta = -a_1^2/2,\ \ \gamma = a_3^2/2,\ \ \delta = -a_2(a_2 - 2)/2.$$

Proof. These are a pair of coupled first order nonlinear differential equations. The first equation gives a direct relation between p and $q' = dq/dt$, and using this the second equation then yields a second order non-linear differential equation for q''. A direct computation (best done with a computer algebra package) shows this is $\mathrm{P_{VI}}(\alpha, \beta, \gamma, \delta)$, with y replaced by q. $\square$

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[6] https://webusers.imj-prg.fr/~philip.boalch/files/fuchs.r_1905_surquelquesequationsdifferentielleslineairesdusecondeordre_CRAS

[7] see the historical discussion in Jimbo–Miwa–Ueno 1981.

The modern geometric viewpoint on this (Schlesinger, Jimbo–Miwa–Ueno, Malgrange, Okamoto) goes as follows[8]:

Let $G = \mathrm{SL}_2(\mathbb{C}), \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (using $\mathrm{GL}_2(\mathbb{C})$ gives nothing extra)

$t \in \mathbb{B} := \mathbb{C} \setminus \{0, 1\}$

Thus the choice of t determines a four-tuple of points: $\mathbf{a} = \mathbf{a}(t) = (0, t, 1, \infty) \in (\mathbb{P}^1)^4 \setminus$ diagonals, where $\mathbb{P}^1$ is the Riemann sphere.

We want to consider a simple moduli spaces $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}}$ of meromorphic connections on trivial vector bundles on $\mathbb{P}^1$ with poles at $D := \mathbf{a}$. They are Fuchsian systems, of the form

$$\nabla = d - A, \qquad A = \left(\frac{A_1}{z} + \frac{A_2}{z - t} + \frac{A_3}{z - 1} \right) dz$$

where $A_i \in \mathfrak{sl}_2(\mathbb{C})$ are trace-less 2×2 matrices. This has a further pole at ∞ with residue $A_4 := -(A_1 + A_2 + A_3)$, so that

$$\sum_1^4 A_i = 0. \tag{3.3}$$

Two such Fuchsian systems are isomorphic if they are related by a global gauge transformation $g : \mathbb{P}^1 \to \mathrm{GL}_2(\mathbb{C})$. Any such holomorphic map is constant so the set of isomorphism classes is just the quotient by the conjugation action: $g[A] = gAg^{-1}$. Generically the projective group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$ acts freely so a rough dimension count shows the space of isomorphism classes of such Fuchsian systems should have dimension $3.3 - 3 = 6$ (there are 3 independent residues, and $\dim(\mathrm{PSL}_2(\mathbb{C})) = 3$).

To reduce the dimension we notice the action is really just conjugating the residues A_i, so we can fix their adjoint orbits.

Choose $\lambda_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$. and let

$$\mathcal{O}_i = \left\{ g \left(\begin{smallmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{smallmatrix} \right) g^{-1} \;\middle|\; g \in \mathrm{SL}_2(\mathbb{C}) \right\} \subset \mathfrak{g}$$

be the adjoint orbit of matrices with eigenvalues $\pm\lambda_i$. We will assume $2\lambda_i$ is not an integer, so in particular $\mathcal{O}_i$ has complex dimension 2.

Then we can look at the set of isomorphism classes of such Fuchsian systems with $A_i \in \mathcal{O}_i$ for $i = 1, 2, 3, 4$.

$$\mathcal{M}^*(t) := \{A \mid A_i \in \mathcal{O}_i\} / \mathrm{SL}_2(\mathbb{C})$$

[8]rewritten in terms of moduli spaces, and Ehresmann connections, as in P.B. Adv. Math. 2001: https://webusers.imj-prg.fr/~philip.boalch/files/smid.pdf.

It turns out that if the constants $\boldsymbol{\lambda} = \{\lambda_i\} \in \mathbb{C}^4$ are off of some hyperplanes then the projective group $\mathrm{PSL}_2(\mathbb{C})$ acts freely and the quotient is an algebraic variety of dimension

$$4 \times 3 - 2 \times 3 = 2$$

so it is a complex surface[9] . Of course really this space does not depend on t and is described directly in terms of the residues.

Define a map

$$\mu : \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 \to \mathfrak{g}; (A_1, \ldots, A_4) \mapsto \sum A_i.$$

Then we can write:

$$\mathcal{M}^* \cong \mu^{-1}(0)/G =: (\mathcal{O}_1 \times \cdots \times \mathcal{O}_4)/\!\!/G$$

where the double slash $/\!\!/$ is just notation for the subquotient $\mu^{-1}(0)/G$, i.e. we consider the subvariety $\mu^{-1}(0)$ inside $\mathcal{O}_1 \times \cdots \times \mathcal{O}_4$ and then quotient that by G. (We will later see this as an examples of a holomorphic symplectic quotient.)

Now we vary $t \in \mathbb{B} := \mathbb{C} \setminus \{0, 1\}$ and look at the relative situation. Thus we define a fibre bundle

$$\underline{\mathcal{M}}^* \to \mathbb{B}$$

such that the fibre over $t \in \mathbb{B}$ is the space $\mathcal{M}^*(t)$. This fibre bundle is trivial, it is just the product $\underline{\mathcal{M}}^* = \mathcal{M}^* \times \mathbb{B}$, since as we saw above the spaces $\mathcal{M}^*$ do not depend on t.

Now, geometrically, the Painevé VI equation that R. Fuchs discovered is a (nonlinear) Ehresmann connection on this bundle $\underline{\mathcal{M}}^*$, and the independent variable (the time) is the parameter t running over $\mathbb{B}$. It is a *second order* nonlinear differential equation, as the fibres have dimension 2.

—Quick aside on Ehresmann connections:

Suppose $\mathbb{B}$ is a complex manifold and $\pi : M \to \mathbb{B}$ is a fibre bundle, with fibres $M_t = \pi^{-1}(t)$ for $t \in \mathbb{B}$.

Definition 3.5. *A (holomorphic) Ehresmann connection on the bundle M is the choice, for any $p \in M$ of a linear subspace $H_p \subset T_pM$ that is transverse to the vertical subspace V_p, the tangent space of the fibres $V_p = \mathrm{Ker}(d\pi_p) \subset T_pM$, so that*

$$H_p \oplus V_p = T_pM$$

for all $p \in M$. These subspace should vary holomorphically (so the H_p form a holomorphic vector bundle on M, a subbundle of the tangent bundle TM).

[9]more on these hyperplanes (and surfaces) later, but the impatient could read section 2 of https://arxiv.org/pdf/0706.2634

If $U \subset \mathbb{B}$ then a local section $s : U \to M$ is *horizontal* if it is tangent to the Ehresmann connection, i.e. for any $t \in U$ and tangent vector $v \in T_t\mathbb{B}$ the corresponding vector $ds(v) \in T_pM$ is actually in the subspace $H_p \subset T_pM$, where $p = s(t) \in M$.

In brief whereas a Koszul connection on a vector bundle encodes linear differential systems in an intrinsic way, the notion of Ehresmann connection encodes non-linear differential equations. An Ehresmann connection is "complete" if any path in $\mathbb{B}$ between any two points $t_1, t_2 \in \mathbb{B}$ has a unique horizontal lift to a path in M starting at any point $p \in M_{t_1}$. Some authors put this condition in the definition of Ehresmann connection, but we will not.)

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In our setting we can thus speak of the Painlevé VI connections, and then choose explicit coordinates to get the explicit differential equation. It is really the Ehresmann connection (or rather its extension from $\mathcal{M}^*$ to $\mathcal{M}_{\mathrm{DR}}$) that is the geometric object we want to understand.

There are two ways to get the Painlevé VI connection, and we'll just mention them here, and explain the details once we have set up the background:

1) De Rham approach, via Schlesinger's equations.

2) Betti approach passing to the other side of Riemann–Hilbert. In brief the corresponding character varieties $\mathcal{M}_{\mathrm{B}}$ also form a bundle $\underline{\mathcal{M}_{\mathrm{B}}} \to \mathbb{B}$. However this bundle is not naturally trivial, but it is *canonically locally trivial*: if we choose any disk $\Delta \subset \mathbb{B}$ then there is a canonical identification of the fibres $\mathcal{M}_{\mathrm{B}}(t_1) \cong \mathcal{M}_{\mathrm{B}}(t_2)$ for $t_1, t_2 \in \Delta$ (this identification depends on the choice of the disk). This structure is encoded in the sentence:

"The spaces $\mathcal{M}_{\mathrm{B}}(t)$ form a *local system of varieties* over $\mathbb{B}$".

This will be spelt out in great detail, but for now we just note that implies that the bundle $\underline{\mathcal{M}_{\mathrm{B}}} \to \mathbb{B}$ has a natural complete flat Ehresmann connection. We can transfer this to the bundle $\underline{\mathcal{M}}^* \to \mathbb{B}$ and rewrite it in carefully chosen algebraic coordinates there to get a nonlinear differential equation, P_{VI} .

Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, 75013 Paris, France.
https://webusers.imj-prg.fr/~philip.boalch/ boalch@imj-prg.fr