MEROMORPHIC CONNECTIONS IN 2D GAUGE THEORY
PRELIMINARY LECTURE NOTES

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PHILIP BOALCH

$$\begin{tikzcd}
 & \mathfrak{M} \arrow[dl, bend right, "1"', "\cong"] \arrow[d, "2"', "\cong"] \arrow[dr, dashed, bend left, "\cong"] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\mathrm{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\mathrm{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}$$

Non-Abelian Hodge space $\mathfrak{M}$,
three preferred algebraic structures:
Dolbeault, De Rham, Betti
mero. Higgs bundles, mero. connections, Stokes local systems

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Contents

Resumé

La but du cours est d'expliquer la lien entre les équations différentielles algébriques linéaires (les connexions méromorphes sur les fibrés vectoriels sur les courbes complexes lisses) et quelques exemples des équations différentielles non-linéaires.

L'idée de base, centrale dans la théorie de jauge, est que l'inconnu dans l'équation différentielle nonlineaire est mieux compris comme un connexion linéaire. Ici on utilise la même principe mais pour les connexions algébriques méromorphes.

Ceci donne l'opportunité d'étudier quelques exemples des jolies variétés algébriques (hyperkahlerienne) qui apparaissent comme espaces de modules dans cette histoire.

Contenu

• Espaces de modules de connexions additifs et multiplicatifs (géométrie symplectique holomorphe)

• Systèmes locaux de Stokes, variétés de caractères sauvages

• Application de Riemann-Hilbert-Birkhoff, théorie de Lie globale

• Équations de Yang-Mills autodual, équations de Hitchin, fibrés harmoniques sur les surfaces de Riemann non-compact (rotation hyperkahlerienne, correspondance de Hodge nonabelienne sauvage)

• Systèmes d'isomondromie (Painlevé, Schlesinger, Jimbo-Miwa-Mori-Sato, simplement lacé,...), connexions d'Ehresmann nonlineaires et lien avec les groupes de tresse

• Fibres de Higgs méromorphes et systèmes intégrables algèbro-géométriques (Garnier, Mumford, Hitchin, Bottacin-Markman, ...)

List of key points:

Lecture 1: Definition of meromorphic connection. View as global/intrinsic linear differential systems. Gauge transformations, gauge action. Algebraic versions. Definition of meromorphic Higgs bundles and ζ-connections.

Sketch of big picture (at symbolic level). Two key correspondences.

Start list of key examples: Painlevé's discovery of natural deformations of the theory of elliptic functions (simplification of $\mathrm{P_I}$, $\mathrm{P_{II}}$). Link to gauge theory (R. Fuchs): First steps in description of geometry of Painlevé VI.

Lecture 2: Definition of curvature. Flatness in terms of commuting operators. Holomorphic structures via $\overline{\partial}$-operators and Koszul–Malgrange statement. Definition of local system of sets and of vector spaces, relation to covering spaces. Transport and monodromy of a local system. Equivalence of five viewpoints on connections in the compact case (no poles).

Representation varieties $\mathcal{R}$ as framed moduli spaces, and as affine varieties. Classification of solutions of the flatness equation. Character variety/Betti moduli space.

Dimension counting in the Riemann problem (Hilbert 21). Relation to matrix exponential map. The question that Birkhoff's invariants answered $\rightsquigarrow$ global Lie theory. Example of Painlevé 2 wild character variety (Flaschka–Newell surface).

1. Lecture 1: Basic examples, questions and definitions

1.1. **What is a meromorphic connection?**

We will need several different flavours (categories) of connections, and some confusion in the subject comes from different authors having different default definitions. The relation between various definitions will be crucial to understand. Thus we'll start with the central notions we will use, and then later discuss variations and their relation.

The starting point is a first order linear differential operator of the form

$$\frac{d}{dz} - B(z)$$

where $B(z)$ is an $n \times n$ matrix of holomorphic functions on an open subset $U \subset \mathbb{C}$. As a first example one might consider a polynomial system:

$$\frac{d}{dz} - (A_0 + A_1 z + \cdots A_m z^m) \tag{1.1}$$

for $n \times n$ matrices A_i. As a second example one might consider:

$$\frac{d}{dz} - \left(\frac{A_1}{z - a_1} + \cdots + \frac{A_m}{z - a_m}\right) \tag{1.2}$$

for $n \times n$ matrices A_i, away from the poles (these are often called "Fuchsian systems").

This yields the linear system of differential equations

$$\frac{dv}{dz} = Bv$$

where v is a length n column vector of holomorphic functions. The coordinate-free version of this operator is got by "multiplying by dz", to get the *connection*

$$\nabla = d - A, \qquad A = B(z)dz$$

so that A is a matrix of holomorphic one-forms and d is the exterior derivative. This is a connection on the trivial rank n holomorphic vector bundle on U, i.e. on $E = \mathbb{C}^n \times U \to U$. Solutions v are now called *horizontal sections* and the equation $dv/dz = Bv$ is rewritten $\nabla(v) = 0$, i.e. $dv = Av$. We can remove the condition that E is trivial and consider connections on non-trivial vector bundles, leading to the following definition, first in the case with no poles.

Let Σ be a compact Riemann surface.

Definition 1.1. *A* <u>holomorphic connection</u> *is a pair* (E, ∇) *where* $E \to \Sigma$ *is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by holomorphic one-forms, such that the Leibniz rule is satisfied:

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.3}$$

for all local sections s of E and functions f.

This is a completely standard definition, going back to Koszul. A connection is a way to differentiate sections of E: If X is a vector field on Σ and s is a section of E then

$$\nabla_X(s) := \langle X, \nabla(s)\rangle$$

is again a section of E, the derivative by ∇ of s along X. Here the brackets $\langle\,\cdot\,,\,\cdot\rangle$ denote the natural pairing between the tangent bundle and the cotangent bundle.

In a local trivialisation of E, over some open subset $U \subset \Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for an $n \times n$ matrix of holomorphic one-forms A, where n is the rank of E. If z is a local coordinate on U this means we can write $A = Bdz$ for a matrix B of holomorphic functions on U. Thus a connection $\nabla = d - Bdz$ is really just a global, coordinate-free version of the matrix differential operators $\frac{d}{dz} - B$ we first considered.

If we change the choice of local trivialisation of E then A changes by a *gauge transformation*:

$$A \;\mapsto\; g[A] := gAg^{-1} + (dg)g^{-1} \tag{1.4}$$

where $g : U \to \mathrm{GL}_n(\mathbb{C})$ is a holomorphic map. Our conventions are set-up such that this is a group action:

Exercise 1.2. Show that $(g \circ h)[A] = g[h[A]]$.

Exercise 1.3. Show that if $\mathbf{e} = (e_1, \ldots, e_n)$ is the initial basis of E and $\mathbf{e}' = (e'_1, \ldots, e'_n)$ is the new basis, and g is such that $\mathbf{e} = \mathbf{e}' \circ g$, then we do indeed get the formula (1.4) for $g[A]$.

Exercise 1.4. Choose an open covering $\Sigma = \bigcup_{i\in I} U_i$ of Σ and a trivialisation $\mathbf{e}_i$ of E over U_i for each i, and so the connection takes the form $d - A_i$ on U_i. Let $U_{ij} = U_i \cap U_j$ and define $g_{ij} : U_{ij} \to \mathrm{GL}_n(\mathbb{C})$ so that $\mathbf{e}_i = \mathbf{e}_j \circ g_{ji}$ on U_{ij}. Show that $g_{ij}[A_j] = A_i$ for all $i, j \in I$. Show that the connection (E, ∇) is completely determined by the cover, the clutching maps g_{ij} and the matrices A_i for all $i, j \in I$. How does this data change if we change trivialisation over each open set: $\mathbf{e}_i \mapsto \mathbf{e}_i \circ h_i$ for some $h_i : U_i \to \mathrm{GL}_n(\mathbb{C})$?

Write $G = \mathrm{GL}_n(\mathbb{C})$, let $\Delta \subset \mathbb{C}$ be an open disk, and let $\mathcal{G} = \mathrm{Map}_{\mathrm{hol}}(\Delta, G)$ be the group of all holomorphic maps from Δ to G. Also write $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ and let

$$\mathcal{A} = \{A = B(z)dz \mid B : U \to \mathfrak{g}\}$$

be the space of all holomorphic connections on the trivial bundle on the open disk Δ, so that B is a holomorphic map. Thus by the exercise above the group $\mathcal{G}$ acts on the space $\mathcal{A}$ by gauge transformations:

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A}; \qquad (g, A) \mapsto g[A] = gAg^{-1} + (dg)g^{-1}.$$

Holomorphic connections are not interesting locally since they are all isomorphic:

Lemma 1.5. *$\mathcal{G}$ acts transitively on $\mathcal{A}$. In particular for any $A \in \mathcal{A}$ there is a $g \in \mathcal{G}$ such that $g[A] = 0$ (every holomorphic connection is locally isomorphic to the trivial connection).*

Proof. Given A we wish to find g so that $gAg^{-1} + (dg)g^{-1} = 0$. In other words $gA + (dg) = 0$. If we write $h = g^{-1}$ and use the useful fact that $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ then we want $h : U \to G$ so that

$$\frac{dh}{dz} = B(z)h$$

where $B = A/dz$ as usual. In classical language this equation just says that h is a "fundamental solution" (or "fundamental matrix") of the linear system $d/dz - B$. (By definition this means that the columns of h make up a basis of solutions of the system.) It is a classical fact (Cauchy?) that holomorphic systems have fundamental solutions[1]. In fact its easy to construct a series solution term by term, and then one proves the resulting series solution converges. □

Let $E = \mathbb{C}^n \times \Delta$ denote the trivial bundle. Note that a fundamental solution h is the same thing as an isomorphism $(E, d = d - 0) \to (E, d - A)$ from the trivial connection to the connection $d - A$. This just says $h[0] = A$, i.e. $A = (dh)h^{-1}$ or $dh = Ah$.

In general an isomorphism from $(E, d - A_1) \to (E, d - A_2)$ is a section h of $\mathrm{Hom}(E, E)$ that is invertible and satisfies $h[A_1] = A_2$ i.e.

$$hA_1h^{-1} + (dh)h^{-1} = A_2$$

or in other words:

$$dh = A_2h - hA_1.$$

Indeed it is natural to define a connection $\mathrm{Hom}(\nabla_1, \nabla_2)$ on $\mathrm{Hom}(E, E)$, whose horizontal sections are given by this equation. Similarly if there are two different vector bundles, and one can thus define dual connections etc.

Of course we can consider holomorphic connections on punctured Riemann surfaces but that won't capture most of the properties of the first (polynomial) example, and

[1]See e.g. classical ODE books by Hartman, Coddington–Levinson, Ince, Hille, ...

not all the properties of the second (Fuchsian) example. Instead we proceed as follows to encompass them.

Now let Σ be a compact Riemann surface and $\mathbf{a} = (a_1, \ldots, a_m) \subset \Sigma$ a finite subset. Let $D = \sum n_i(a_i)$ be an effective divisor on Σ supported on $\mathbf{a}$, so that $n_i \geq 1$ are integers.

Definition 1.6. *A* meromorphic connection *with poles bounded by D is a pair (E, ∇) where $E \to \Sigma$ is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by meromorphic one-forms with poles bounded by D, such that the Leibniz rule is satisfied:

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.5}$$

for all local sections s of E and functions f.

In a local trivialisation of E, over some open subset $U \subset \Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for a matrix of meromorphic one-forms A (with poles bounded by D). E.g. if $a_1 \in U$ and z is a local coordinate vanishing at a_1 then

$$\nabla = d - \frac{B(z)dz}{z^{n_1}}$$

in a neighbourhood of a_1, where B is holomorphic across a_1.

Remark 1.7. Note that:

1) $E \to \Sigma$ is a holomorphic vector bundle on the *compact* surface, so this is a genuine generalisation of a holomorphic connection.

2) this notion is well defined, but it would not be if D was not effective. The point is that the Leibniz rule tacitly uses the inclusion $\Omega^1 \subset \Omega^1(D)$ of the holomorphic one forms into the meromorphic one forms. If D was not effective, say $n_1 < 0$, then the Leibniz rule would not make sense (as there is then no such inclusion: df would not necessarily be a section of $\Omega^1(D)$).

3) this is not a completely standard definition (although we have been happily using it since 1999 or so). One can also define the notion of "meromorphic connection on a meromorphic bundle", where a "meromorphic bundle" is a locally free $\mathcal{O}(*D)$ module. In practice this means that one allows meromorphic gauge transformations with any order pole at the points of D. This definition is also useful, but is less convenient for gauge theory or moduli theory.

Remark 1.8. The nonabelian cohomology set $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathcal{O}))$ is *slick notation* for the set of isomorphism classes of rank n holomorphic vector bundles on Σ. Here $\mathcal{O}$ is the sheaf of holomorphic functions and $\mathrm{GL}_n(\mathcal{O})$ is the sheaf of holomorphic maps in to the the group $\mathrm{GL}_n(\mathbb{C})$. The (Cech) definition of $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathcal{O}))$ involves equivalence classes of 1-cocycles, and this really is the same thing as expressing a vector bundle in terms of clutching maps, by choosing local trivialisations on each open set of an open covering (and the equivalence relation comes from changing the choice of trivialisation). See for example J. Frenkel's 1957 paper *Cohomologie non abélienne et espaces fibrés*:

3. ***Cohomologie de dimension*** **1. — Soit F un faisceau de *groupes* sur X, $\mathfrak{U} = \{U_i\}_{i\in I}$ un recouvrement ouvert de X. Nous dirons qu'une 1-cochaîne $f : (i, j) \to f_{ij}$ de $\mathfrak{U}$ à valeurs dans F est un 1-cocycle si l'on a**

$$(3.1) \qquad f_{ij}(x) f_{jk}(x) = f_{ik}(x) \qquad \text{pour tout } x \text{ de } U_{ijk}.$$

Deux cochaînes $\{f_{ij}\}$, $\{g_{ij}\}$ seront dites *cohomologues* s'il existe une 0-cochaîne $h = \{h_i\}$ de $\mathfrak{U}$ à valeurs dans F telle que

$$(3.2) \qquad f_{ij}(x) = h_i^{-1}(x) g_{ij}(x) h_j(x) \qquad \text{pour tout } x \text{ de } U_{ij}.$$

La cohomologie est une relation d'*équivalence* dans $C^1(\mathfrak{U}, F)$ respectant l'ensemble des cocycles. L'ensemble $H^1(\mathfrak{U}, F)$ des classes de cocycles de $\mathfrak{U}$ à valeurs dans F qui sont cohomologues s'appelle le *premier ensemble de cohomologie du recouvrement* $\mathfrak{U}$ *à valeurs dans* F. Cet ensemble n'a de structure de groupe naturelle que si F est un faisceau de groupes *abéliens*, auquel cas c'est le premier groupe de cohomologie classique de $\mathfrak{U}$ à valeurs dans le faisceau abélien F. Il a cependant un élément privilégié, que nous conviendrons d'appeler l'*élément neutre* de $H^1(\mathfrak{U}, F)$, savoir la classe du cocycle

$$f_{ij}(x) = e_x \qquad \text{pour tout } x \text{ de } U_{ij},$$

où e_x est l'élément neutre de F_x [l'application $x \to e_x$ d'un ouvert U de X dans F est bien continue en vertu de l'axiome (II)].

FIGURE 1. The definition of $\mathrm{H}^1(\Sigma, \mathcal{G})$ in Frenkel 1957.

Later on we will need $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathbb{C}))$, which is slick notation for the set of isomorphism classes of local systems of n-dimensional complex vector spaces on Σ; the clutching maps on double intersections are now *constant* maps to $\mathrm{GL}_n(\mathbb{C})$. Of course it is very suggestive notation, and leads to the idea that moduli spaces of local systems should have other motivic incarnations analogous to the De Rham and Dolbeault approaches in the abelian case.

1.2. **Some variations: algebraicity.** Suppose Σ is actually a smooth compact complex algebraic curve.

$\bullet v1$) Algebraic connections (E, ∇) (if E is algebraic and ∇ is algebraic). Thus there is a Zariski open covering $\Sigma = \bigcup U_i$ so that the restriction of E to each open set U_i is trivialisable. (Recall Zariski open subset are just the complements of finite subsets of points.) Then by choosing such trivialisations the bundle E is determined by algebraic clutching maps $g_{ij} : U_{ij} \to G$, where $U_{ij} = U_i \cap U_j$. Then on U_i we have $\nabla = d - A_i$ where A_i is a matrix of regular differentials (algebraic one-forms) on U_i. On the double intersections the A_i are related by gauge transformations as usual

$$g_{ij}[A_j] = A_i.$$

These are just the algebraic version of holomorphic connections. In fact some form of GAGA implies the analytification functors gives an equivalence of categories (Algebraic connections) $\to$ (holomorphic connections), in this setting where Σ is compact.

$\bullet v2$) Similarly there is a notion of "Algebraic meromorphic connections", as above but allowing the A_i to be matrices of rational differentials (algebraic one-forms with poles), with poles bounded by the fixed effective divisor D. Again a version of GAGA implies the analytification functors gives an equivalence (to the meromorphic connections on holomorphic vector bundles). These will actually be the realm for most of our examples, with nonlinear differential equations flowing in their spaces of coefficients.

$\bullet v3$) If we now take $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ to be an open curve (in fact any smooth complex algebraic curve takes this form for some finite set $\mathbf{a}$). Then we can consider algebraic connections $(E, \nabla) \to \Sigma^\circ$ on the open curve. This category is actually very close to being a subcategory of the category of meromorphic connections on holomorphic vector bundles on Σ with poles on $\mathbf{a}$ (if we allow any pole orders). There is in fact a version of GAGA that shows this category is equivalent to the "meromorphic connections on meromorphic bundles" on Σ with any order poles on $\mathbf{a}$.

In the next lecture we will consider holomorphic connections on vector bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$; this is relatively trivial and all the extra structure "hidden" in the poles at the punctures is lost. However we won't get to hermitian metrics for some time so its worth noting here:

> The relevance in "hardcore analytic" gauge theory (on the punctured surface), of having (meromorphic connections on) holomorphic vector bundles on the compact surface, comes from the fact that the addition of a hermitian metric controls the growth of sections at the punctures, leading to preferred extensions across the punctures, and thus holomorphic vector bundles (or parabolic vector bundles) on the compact surface, and in turn this leads to algebraicity.

1.3. **Some more variations: Higgs bundles and ζ-connections.**

Definition 1.9. *A* meromorphic Higgs bundle *with poles bounded by D is a pair (E, Φ) where $E \to \Sigma$ is a holomorphic vector bundle and* $\Phi \in \mathrm{H}^0(\Sigma, \mathrm{End}(E) \otimes \Omega^1(D))$

is the Higgs fields, a meromorphic one-form with values in $\mathrm{End}(E)$*, with poles bounded by* D*.*

Thus locally we can write $\Phi = Bdz$ for a matrix B of meromorphic functions on U.

In a sense Higgs bundles have two origins: just as an operator $d/dz - B$ led to a connection $d - Bdz$, any matrix $L(z)$ of rational functions (aka a "rational Lax matrix") leads to a Higgs field $L(z)dz$ (on the trivial vector bundle on $\mathbb{P}^1$). On the other hand holomorphic Higgs fields on higher genus Riemann surfaces were introduced by Hitchin and Simpson. These two viewpoints were "put together" in the definition of meromorphic Higgs bundle (Nitsure, Bottacin, Markman, ...).

Exercise 1.10. Suppose ∇_1, ∇_2 are meromorphic connections on $E \to \Sigma$ with poles on D. Show that $\Phi := \nabla_1 - \nabla_2$ is a meromorphic Higgs field.

Now choose a complex number $\zeta \in \mathbb{C}$.

Definition 1.11. *A* meromorphic ζ-connection *with poles bounded by* D *is a pair* (E, ∇) *where* $E \to \Sigma$ *is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$*-linear operator, from the sheaf of sections* $\mathcal{E}$ *of* E *to the sections of* E *twisted by meromorphic one-forms with poles bounded by* D*, such that the* ζ*-Leibniz rule is satisfied:*

$$\nabla(fs) = \zeta(df)s + f\nabla(s) \tag{1.6}$$

for all local sections s *of* E *and functions* f*.*

Exercise 1.12. Study ζ-connections in a local trivialisation, and show that the gauge action is modified to: $g[A]_\zeta = gAg^{-1} + \zeta(dg)g^{-1}$.

Exercise 1.13. Show that for $\zeta = 0$ a meromorphic ζ-connection is the same thing as a meromorphic Higgs bundle.

Exercise 1.14. Show that for $\zeta = 1$ a meromorphic ζ-connection is the same thing as a meromorphic connection.

Thus there is a "continuous deformation" from connections to Higgs bundles.

Exercise 1.15. Write down the algebraic versions of the definitions of Higgs bundles and ζ-connections.

This is often referred to as the "autonomous limit" in the integrable systems literature. We will eventually see it is the same as the "Painlevé simplification" of the Painlevé equations, and see it as a hyperkähler rotation.

Of course, physicists have been putting numbers (like $\hbar$) in front of their differential operators for a long time.

And it is essentially the same as the deformation from loop algebras into affine Kac–Moody algebras (although in the full story there is also a central extension, dual to this deformation).

Remark 1.16. Note that for any line bundle $\mathcal{L} \to \Sigma$ one can define a $\mathcal{L}$-valued Higgs bundle as pair (E, Φ) with Φ a section of $\mathrm{End}(E) \otimes \mathcal{L}$. However unless this is secretly a meromorphic Higgs bundle (i.e. there is an isomorphism $\mathcal{L} \cong \Omega^1(D)$ for some effective D) then there is no analogous notion of "$\mathcal{L}$-valued connections" (Rmk. 1.7 2).

2. Sketch of big picture

2.1. **Three algebraic worlds:** Before delving into the details lets try to signpost where we want go (at least symbolically for the moment). Much of the story we want to describe can be summarised in the (slightly oversimplified) diagram:

Dolbeault		De Rham		Betti
Rational Lax matrices L $\dot{L} = [P, L]$		Rational diff. op.s $\frac{d}{dz} - B$		Stokes and monodromy data
$\cap$		$\cap$		$\cap$
Mero. Higgs bundles (E, Φ)	$\overset{\text{wnAbH}}{\longleftrightarrow}$	Mero. connections (E, ∇)	$\overset{\text{RHB}}{\longleftrightarrow}$	Stokes local systems
$\cup$		$\cup$		$\cup$
Holom. Higgs bundles (E, Φ)	$\overset{\text{nAbH}}{\longleftrightarrow}$	Holom. connections (E, ∇)	$\overset{\text{RH}}{\longleftrightarrow}$	Local systems/ π_1-rep.s

The main aim is to describe the central row, and, as the diagram indicates, it is set-up to include both the rich class of examples of rational Lax matrices and the sophisticated nonabelian Hodge setting of holomorphic Higgs bundles (no poles), related to the Hitchin integrable systems.

Just as an operator $\frac{d}{dz} - B$ becomes a connection by multiplying by dz (and one can study its *isomonodromic deformations*), a rational Lax matrix $L(z)$ becomes a Higgs field Ldz by multiplying by a rational one-form, such as dz. The Lax matrices appear in Lax equations, which are equation of the form $\dot{L} = [P, L]$ controlling *isospectral deformations* of L, and are the bread and butter of the theory of integrable systems; the solution of the system comes from a straight line flow on the Jacobian of the spectral curve defined by $\det(L - \lambda) = 0$. We will discuss some examples in detail but for now note there are lots, as listed for example in the book of Babelon et al, or basic sources such as:

Adler and van Moerbeke (1980) Completely integrable systems, Euclidean Lie algebras, and curves, Adv. in Math. 38, no. 3, 267-317.

Reyman and Semenov-Tian-Shansky (1994) Integrable systems II group theoretical methods in the theory of finite dimensional integrable systems

Phillip Griffiths (1985) Linearizing Flows and a Cohomological Interpretation of Lax Equations

Mumford (1984) "Tata Lectures on Theta II, Jacobi theta functions and differential equations" (this book is devoted to a class of examples involving 2×2 Lax matrices).

Dubrovin–Krichever–Novikov (1985) Integrable systems I

Adams–Harnad–Previato (1988) Isospectral flows in finite and inifinite dimensions

The first large class of examples seems to be due to Garnier 1919 (and we will discuss the "Painlevé simplification" method he used to discover them, taking the autonomous limit of the Schlesinger equations):

Garnier (1919) Sur une classe de systèmes différentiels abéliens déduits de la théorie des équations linéaires, (Rendiconti del Circolo Matematico di Palermo 43, pp.155-191).

Several of the key ideas of Garnier's paper were rediscovered as an offshoot of soliton theory, before Garnier's work was rediscovered and widely disseminated[2], around 1980.

[2]E.g. there is a section on it in the well-known paper of Flaschka-Newell on isomonodromy (Comm. Math. Phys. 76 (1980), 65-116), and it is mentioned in Dubrovin's 1981 paper on theta functions, the 1980 Krichever-Novikov review (Russian Math. Surveys 35:6 (1980), 53-79) and in the footnote p.156 of the 1980 paper of Jimbo-Miwa-Mori-Sato. D.V. Chudnovsky wrote a paper on it (Let. Nuovo Cimento 26 (14) 1979), and M. Gaudin cited that in his 1983 book (La fonction d'onde de Bethe), having discovered the quantum version in 1976.

2.2. **Two organisational diagrams.** Mathematically this story leads to an interesting class of *moduli spaces*, i.e. spaces whose points correspond to isomorphism classes of certain meromorphic connections (or Higgs bundles, or Stokes local systems). This goes slightly beyond the objects usually studied by algebraic geometers, and one of the main inputs is to write down the moduli problem that encompasses this picture.

In particular we will fix a Riemann surface, some marked points and some precisely defined boundary data. This will determine a hyperkähler manifold $\mathfrak{M}$ with three preferred algebraic structures, corresponding to the three columns of the above table. We label the columns "Dolbeault, De Rham, Betti" as they are precise analogues of the Dolbeault, De Rham and Betti approaches to linear cohomology (it was first abstracted to the context of nonabelian cohomology by Simpson, and then later extended to the meromorphic case relevant to Lax matrices). The result, to be explained, is a diagram as follows:

$$\begin{tikzcd}
 & \mathfrak{M} \arrow[dl, bend right, "1"', "\cong"] \arrow[d, "2"', "\cong"] \arrow[dr, dashed, bend left, "\cong"] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\mathrm{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\mathrm{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}$$

FIGURE 2. Nonabelian Hodge space $\mathfrak{M}$, with three preferred algebraic structures.

However this does not capture the full story and in practice people work with simpler open parts $\mathcal{M}^*$ of the moduli spaces in genus zero, where things can be made explicit, and actual nonlinear differential equations can be obtained. We will explain that the classical Riemann–Hilbert map is a holomorphic map

$$\boxed{\mathcal{M}^* \hookrightarrow \mathcal{M}_{\mathrm{B}}} \tag{2.1}$$

where $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}} \subset \mathcal{M}_{\mathrm{DR}}$ is the open part of the full De Rham moduli space where the bundles E are trivial. As we will see the spaces $\mathcal{M}^*$ have the flavour of the "Lie algebra" of the full nonabelian Hodge space $\mathfrak{M} \cong \mathcal{M}_{\mathrm{B}}$, and the Riemann–Hilbert map is a natural generalisation of the exponential map.

However this still does not capture the full story as we also wish to *vary the modular parameters*, changing the complex structure on the Riemann surface, the pole positions, and the "irregular class" of the connections at each pole. These parameters will lead to the independent variables ("times") in the isomonodromy equations.

3. Glimpses of the elephant

The next few sections will describe a few simple pieces of the full picture, that provided motivation.

3.1. Painlevé's deformation of the theory of elliptic functions.

Painlevé discovered most of the Painlevé equations as deformations of differential equations for elliptic functions, i.e. as equations that limit to equations for elliptic functions. He used the term "simplification" (*simplifié*) for the limiting differential equation, solvable in terms of elliptic functions.

In more detail Painlevé was looking for new special functions, defined as solutions to non-linear algebraic differential equations. He looked for equations whose solutions had good meromorphic continuation properties: outside a fixed critical set, any local solution should have arbitrary meromorphic continuation. If $D \subset \mathbb{C}$ is the fixed critical set (a finite set in all examples here), then any local solution $y(t)$ should extend to a meromorphic function on the universal cover $\widetilde{\mathbb{C} \setminus D}$. This is known as the Kowalevski–Painlevé (KP) property (and can also be expressed as saying there are no "movable singularities" apart from poles).

The KP property is preserved under any deformation of the differential equation[3]. Thus to rule out many possible forms of differential equations, Painlevé would add parameters by hand and then take limits to get simpler equations (Painlevé's α method). If he could recognise or prove the limiting equation did not have the KP property then he could ignore the putative equation, and thus get a short list of possibilities, that then could be proved to have the KP property directly.

For example $\mathrm{P_I}$, the first Painlevé equation, $y'' = 6y^2 + t$ is a deformation of the equation for the Weierstrass $\wp$ function:

First recall the standard differential equation satisfied by $\wp$ is

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for constants $g_2, g_3 \in \mathbb{C}$. Thus $2\wp'\wp'' = 12\wp^2\wp' - g_2\wp'$ so that

$$\wp'' = 6\wp^2 - g_2/2$$

Lemma 3.1 (cf. Painlevé 1900 p.226, Ince [] pp.321 and 329)**.** *Suppose $y(t)$ satisfies $\mathrm{P_I}$ so that $y'' = 6y^2 + t$. If $t = \alpha x, y = w(x)/\alpha^2$ for a constant α then*

$$w'' = 6w^2 + \alpha^5 w. \tag{3.1}$$

In particular if $\alpha^5 = 1$ then this is a symmetry of $\mathrm{P_I}$. If we take the limit $\alpha \to 0$ then we get $w'' = 6w^2$. This integrates once to $(w')^2 = 4w^3 + c$, which can be solved

[3] see e.g. paragraph 1 p.319 in Ince's book "ordinary differential equations" [**?**].

in terms of the Weierstrass $\wp$ function: $w = \wp(x+k)$ (where $\wp$ has $g_2 = 0, g_3 = -c$, and $k \in \mathbb{C}$ is arbitrary).

Proof. Write $v = y'$ so that $dy = vdt, dv = (6y^2 + t)dt$. Now put $t = \alpha x, y = w/\alpha^2$ (as on Painlevé 1900 p.226, Ince p.329 [], or Valiron p.410 []). Thus $dt = \alpha dx, dy = dw/\alpha^2$, so $w' = dw/dx = \alpha^3 dw/dz = \alpha^3 v$. Thus

$$dv = (6y^2 + t)dt = \alpha(6w^2/\alpha^4 + \alpha w)dx$$

and so $w'' = \alpha^3 v' = \alpha^5 w + 6w^2$ which is (3.1). The last statement is straightforward, recalling that in general $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for constants g_2, g_3. □

Thus Painlevé discovered a natural deformation of the theory of elliptic functions!

—see Alves https://arxiv.org/abs/2103.02697v1 §3 for a recent discussion of this story.

The next example is the case of Painlevé II:

Lemma 3.2. *Suppose $y(t)$ satisfies the $\mathrm{P_{II}}$ equation $y'' = 2y^3 + ty + \alpha$. If $t = \gamma x, y = w(x)/\gamma$ for a constant γ then*

$$w'' = 2w^3 + \gamma^3 xw + \gamma^2\alpha. \tag{3.2}$$

Thus if we now take the limit $\gamma = 0$ then $w'' = 2w^3$, which integrates once to $(w')^2 = w^4 + c$, and can be solved in terms of the Jacobi sn function: $w = c_1\mathbf{sn}(c_1(ix + c_2), i)$.

Proof. Write $v = y'$ so that $dy = vdt, dv = (2y^3 + ty + \alpha)dt$. Now put $t = \gamma x, y = w/\gamma$. Thus $dt = \gamma dx, dy = dw/\gamma$, so $w' = dw/dx = \gamma^2 dw/dz = \gamma^2 v$. Thus

$$dv = (2y^3 + ty + \alpha)dt = \gamma(2w^3/\gamma^3 + xw + \alpha)dx$$

and so $w'' = \gamma^2 v' = 2w^3 + \gamma^3 xw + \gamma^2\alpha$ which is (3.2). The last statement is straightforward. □

In this way Painlevé discovered some very interesting nonlinear differential equations, the Painlevé equations 1,2,3,4.

Later on (late 1970s) the Painlevé equations, and their solutions, the Painleve transcendents, started appearing in physics problems such as the Ising model[4] (in some sense physics got sufficiently nonlinear to catch up with the mathematics...).

Note that so-far these equations have no link to gauge theory: there are no linear differential equations in the story. That link came about via a 1905 paper of R. Fuchs where he discovered a new Painlevé equation, called Painlevé six, $\mathrm{P_{VI}}$, controlling the

[4]E.g. Wu-McCoy-Tracy-Barouch (1976) "Spin-spin correlation functions for the two-dimensional Ising model, Exact theory in the scaling region"

isomonodromic deformations of a linear differential equation. This is a completely different way to get nonlinear differential equations[5].

One can find the standard list of Painlevé equations in many places (e.g. wikipedia), but we really want to think of them as geometric objects, and this is obscure in their explicit expression. They will each lead to a deformation class of nonabelian Hodge spaces of complex dimension two, the minimal possible nonzero dimension, so they give the simplest examples.

The basic features are summarised in the table below:

Painlevé equation:	1	2	3	4	5	6
Domain of t:	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C}\setminus\{0,1\}$
No. of constant parameters:	0	1	2	2	3	4
(Affine Dynkin) Diagram :	$\widehat{A}_0$	$\widehat{A}_1$	$\widehat{D}_2$	$\widehat{A}_2$	$\widehat{A}_3=\widehat{D}_3$	$\widehat{D}_4$
Okamoto Diagram :	$\widehat{E}_8$	$\widehat{E}_7$	$\widehat{D}_6$	$\widehat{E}_6$	$\widehat{D}_5$	$\widehat{D}_4$

TABLE 1. Basic data for Painlevé equations

(Here we have omitted two degenerate versions of Painlevé 3.)

The diagrams can be drawn as follows (the number of nodes is one plus the number of constants):

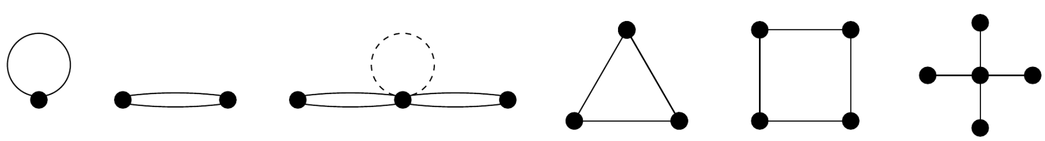

FIGURE 3. The diagrams of the six Painlevé equations.

[5]R.Fuchs' isomonodromy approach was extended to the original Painlevé equations by Garnier 1912 (so they too, in fact, are gauge theoretic equations).

3.2. Towards the Painlevé VI connections.

The 1905 paper of R. Fuchs[6], should probably be viewed as the true "start of 2d gauge theory" where a nonlinear differential equation arose naturally, controlling a linear differential equation (i.e. where the "unknown" is really a linear differential equation $\sim$ a meromorphic connection on a rank two vector bundle on $\mathbb{P}^1$). The underlying idea can be traced back to a suggestion of Riemann 1857.[7]

What is Painlevé VI, the Fuchsian Painlevé equation?

Definition 3.3. *Given constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the corresponding Painlevé VI equation $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$ is the algebraic differential equation:*

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

for a meromorphic function $y(t)$ where $t \in \mathbb{C} \setminus \{0, 1\}$.

This frankly horrific expression does not express very well the true beauty of the underlying geometric object. The simplest encoding of it seems to be the following time-dependent Hamiltonian formulation, due to Malmquist 1922.

Proposition 3.4 (cf. [?] p.86)**.** *If $a_1, a_2, a_3, b \in \mathbb{C}$ then the function $H(q,p,t)$ defined by*

$$t(t-1)H(q,p,t) = q(q-t)(q-1)\left(p^2 + p\left(\frac{a_1}{q} + \frac{a_2}{q-t} + \frac{a_3}{q-1}\right)\right) + b\cdot q$$

is a time-dependent Hamiltonian function for $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$, in the sense that if

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

then $y = q(t)$ is a solution to $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$ where

$$\alpha = (a_1 + a_2 + a_3)^2/2 - 2b,\ \ \beta = -a_1^2/2,\ \ \gamma = a_3^2/2,\ \ \delta = -a_2(a_2 - 2)/2.$$

Proof. These are a pair of coupled first order nonlinear differential equations. The first equation gives a direct relation between p and $q' = dq/dt$, and using this the second equation then yields a second order non-linear differential equation for q''. A direct computation (best done with a computer algebra package) shows this is $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$, with y replaced by q. $\square$

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[6] https://webusers.imj-prg.fr/~philip.boalch/files/fuchs.r_1905_surquelquesequationsdifferentielleslineairesdusecondeordre_CRAS

[7] see the historical discussion in Jimbo–Miwa–Ueno 1981.

The modern geometric viewpoint on this (Schlesinger, Jimbo–Miwa–Ueno, Malgrange, Okamoto) goes as follows[8]:

Let $G = \mathrm{SL}_2(\mathbb{C}), \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (using $\mathrm{GL}_2(\mathbb{C})$ gives nothing extra)

$t \in \mathbb{B} := \mathbb{C} \setminus \{0,1\}$

Thus the choice of t determines a four-tuple of points: $\mathbf{a} = \mathbf{a}(t) = (0, t, 1, \infty) \in (\mathbb{P}^1)^4 \setminus$ diagonals, where $\mathbb{P}^1$ is the Riemann sphere.

We want to consider simple moduli spaces $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}}$ of meromorphic connections on trivial vector bundles on $\mathbb{P}^1$ with poles at $D := \mathbf{a}$. They are Fuchsian systems, of the form

$$\nabla = d - A, \qquad A = \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1}\right) dz$$

where $A_i \in \mathfrak{sl}_2(\mathbb{C})$ are trace-less 2×2 matrices. This has a further pole at ∞ with residue $A_4 := -(A_1 + A_2 + A_3)$, so that

$$\sum_1^4 A_i = 0. \tag{3.3}$$

Two such Fuchsian systems are isomorphic if they are related by a global gauge transformation $g : \mathbb{P}^1 \to \mathrm{GL}_2(\mathbb{C})$. Any such holomorphic map is constant so the set of isomorphism classes is just the quotient by the conjugation action: $g[A] = gAg^{-1}$. Generically the projective group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$ acts freely so a rough dimension count shows the space of isomorphism classes of such Fuchsian systems should have dimension $3.3 - 3 = 6$ (there are 3 independent residues, and $\dim(\mathrm{PSL}_2(\mathbb{C})) = 3$).

To reduce the dimension we notice the action is really just conjugating the residues A_i, so we can fix their adjoint orbits.

Choose $\lambda_i \in \mathbb{C}$ for $i = 1, 2, 3, 4$. and let

$$\mathcal{O}_i = \left\{ g \left(\begin{smallmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{smallmatrix}\right) g^{-1} \;\middle|\; g \in \mathrm{SL}_2(\mathbb{C}) \right\} \subset \mathfrak{g}$$

be the adjoint orbit of matrices with eigenvalues $\pm\lambda_i$. We will assume $2\lambda_i$ is not an integer, so in particular $\mathcal{O}_i$ has complex dimension 2.

Then we can look at the set of isomorphism classes of such Fuchsian systems with $A_i \in \mathcal{O}_i$ for $i = 1, 2, 3, 4$.

$$\mathcal{M}^*(t) := \{A \mid A_i \in \mathcal{O}_i\}/\mathrm{SL}_2(\mathbb{C})$$

[8]rewritten in terms of moduli spaces, and Ehresmann connections, as in P.B. Adv. Math. 2001: https://webusers.imj-prg.fr/~philip.boalch/files/smid.pdf.

It turns out that if the constants $\boldsymbol{\lambda} = \{\lambda_i\} \in \mathbb{C}^4$ are off of some hyperplanes then the projective group $\mathrm{PSL}_2(\mathbb{C})$ acts freely and the quotient is an algebraic variety of dimension

$$4 \times 3 - 2 \times 3 = 2$$

so it is a complex surface[9] . Of course really this space does not depend on t and is described directly in terms of the residues.

Define a map

$$\mu : \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 \to \mathfrak{g}; (A_1, \ldots, A_4) \mapsto \sum A_i.$$

Then we can write:

$$\mathcal{M}^* \cong \mu^{-1}(0)/G =: (\mathcal{O}_1 \times \cdots \times \mathcal{O}_4)/\!\!/G$$

where the double slash $/\!\!/$ is just notation for the subquotient $\mu^{-1}(0)/G$, i.e. we consider the subvariety $\mu^{-1}(0)$ inside $\mathcal{O}_1 \times \cdots \times \mathcal{O}_4$ and then quotient that by G. (We will later see this is an example of a holomorphic symplectic quotient.)

Now we vary $t \in \mathbb{B} := \mathbb{C} \setminus \{0, 1\}$ and look at the relative situation. Thus we define a fibre bundle

$$\underline{\mathcal{M}}^* \to \mathbb{B}$$

such that the fibre over $t \in \mathbb{B}$ is the space $\mathcal{M}^*(t)$. This fibre bundle is trivial, it is just the product $\underline{\mathcal{M}}^* = \mathcal{M}^* \times \mathbb{B}$, since as we saw above the spaces $\mathcal{M}^*$ do not depend on t.

Now, geometrically, the Painevé VI equation that R. Fuchs discovered is a (non-linear) Ehresmann connection on this bundle $\underline{\mathcal{M}}^*$, and the independent variable (the time) is the parameter t running over $\mathbb{B}$. It is a *second order* nonlinear differential equation, as the fibres have dimension 2.

—Quick aside on Ehresmann connections:

Suppose $\mathbb{B}$ is a complex manifold and $\pi : M \to \mathbb{B}$ is a fibre bundle, with fibres $M_t = \pi^{-1}(t)$ for $t \in \mathbb{B}$.

Definition 3.5. *A (holomorphic) Ehresmann connection on the bundle M is the choice, for any $p \in M$ of a linear subspace $H_p \subset T_pM$ that is transverse to the vertical subspace V_p, the tangent space of the fibres $V_p = \mathrm{Ker}(d\pi_p) \subset T_pM$, so that*

$$H_p \oplus V_p = T_pM$$

for all $p \in M$. These subspace should vary holomorphically (so the H_p form a holomorphic vector bundle on M, a subbundle of the tangent bundle TM).

[9] more on these hyperplanes (and surfaces) later, but the impatient could read section 2 of https://arxiv.org/pdf/0706.2634

If $U \subset \mathbb{B}$ then a local section $s : U \to M$ is *horizontal* if it is tangent to the Ehresmann connection, i.e. for any $t \in U$ and tangent vector $v \in T_t\mathbb{B}$ the corresponding vector $ds(v) \in T_pM$ is actually in the subspace $H_p \subset T_pM$, where $p = s(t) \in M$.

In brief whereas a Koszul connection on a vector bundle encodes linear differential systems in an intrinsic way, the notion of Ehresmann connection encodes non-linear differential equations. An Ehresmann connection is "complete" if any path in $\mathbb{B}$ between any two points $t_1, t_2 \in \mathbb{B}$ has a unique horizontal lift to a path in M starting at any point $p \in M_{t_1}$. Some authors put this condition in the definition of Ehresmann connection, but we will not.)

—

In our setting we can thus speak of the Painlevé VI connections, and then choose explicit coordinates to get the explicit differential equation. It is really the Ehresmann connection (or rather its extension from $\mathcal{M}^*$ to $\mathcal{M}_{\mathrm{DR}}$) that is the geometric object we want to understand.

There are two ways to get the Painlevé VI connection, and we'll just mention them here, and explain the details once we have set up the background:

1) De Rham approach, via Schlesinger's equations.

2) Betti approach passing to the other side of Riemann–Hilbert. In brief the corresponding character varieties $\mathcal{M}_{\mathrm{B}}$ also form a bundle $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$. However this bundle is not naturally trivial, but it is *canonically locally trivial*: if we choose any disk $\Delta \subset \mathbb{B}$ then there is a canonical identification of the fibres $\mathcal{M}_{\mathrm{B}}(t_1) \cong \mathcal{M}_{\mathrm{B}}(t_2)$ for $t_1, t_2 \in \Delta$ (this identification depends on the choice of the disk). This structure is encoded in the sentence:

> "The spaces $\mathcal{M}_{\mathrm{B}}(t)$ form a *local system of varieties* over $\mathbb{B}$".

This will be spelt out in great detail, but for now we just note that implies that the bundle $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$ has a natural complete flat Ehresmann connection. We can transfer this to the bundle $\underline{\mathcal{M}}^* \to \mathbb{B}$ and rewrite it in carefully chosen algebraic coordinates there to get a nonlinear differential equation, P_{VI}.

4. Lecture 2: Flat connections and the compact case

As we explained the general notion of meromorphic connection is essentially the simplest context that contains the three basic classes of connections, namely

- the polynomial connections,
- the Fuchsian systems, and
- the holomorphic connections on higher genus compact Riemann surfaces.

In this lecture we will discuss this last case in detail, and the corresponding monodromy data. This case is especially nice since it avoids discussing boundary conditions.

4.1. **The example of compact Riemann surfaces (no poles).** Our first aim is to explain all the definitions and sketch some of the ideas of the proof the following statement:

Theorem 4.1. *Suppose Σ is a smooth compact complex algebraic curve. The following categories are equivalent (via specific functors that we will describe):*

1) Algebraic connections on algebraic vector bundles on Σ,

2) Holomorphic connections on holomorphic vector bundles on Σ,

3) Flat C^∞ connections on C^∞ complex vector bundles on Σ,

4) Local systems of finite dimensional complex vector spaces on Σ,

5) For any fixed basepoint $b \in \Sigma$, the category of finite dimensional complex $\pi_1(\Sigma, b)$ representations.

This has numerous consequences, for example: The equivalence 1) $\Longleftrightarrow$ 5) gives a purely algebraic way to access the topological fundamental group (this is an example of the change in algebraic structure given by Riemann–Hilbert). The equivalence 3) $\Longleftrightarrow$ 5) gives a completely explicit way to classify the set of solutions of a nonlinear differential equation. For example we will deduce the corollary:

Corollary 4.2. *For any integer $n \geq 1$ the set of isomorphism classes of rank n objects (in any of the five categories in the theorem) is naturally in bijection with the set of orbits of an action of the complex algebraic group $G = \mathrm{GL}_n(\mathbb{C})$ on an affine algebraic variety $\mathcal{R}$. Explicitly G acts by conjugation on the* <u>representation variety</u>*:*

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, b), G).$$

By performing this quotient in an algebraic fashion, this will lead to the first example of Betti moduli space $\mathcal{M}_{\mathrm{B}}$ (the character variety), and thus will give the simplest instance of the association of a variety $\mathcal{M}_{\mathrm{B}}$ with the choice of a surface Σ and a group G (no boundary conditions).

What is an *equivalence of categories*?
A functor $F : X \to Y$ between two categories is an equivalence of categories if 1) it is essentially surjective, and 2) it is fully faithful. This gives a convenient/precise/flexible language to see some things are "more or less the same".

1) means that for each object $y \in Y$ there exists is an object $x \in X$ and an isomorphism $y \cong F(x)$.

2) means that for any $x_1, x_2 \in X$ the functor F maps the space $\mathrm{Hom}_X(x_1, x_2)$ of morphisms (in X) bijectively onto the space of morphisms $\mathrm{Hom}_Y(F(x_1), F(x_2))$ between their images in Y.

In particular the choice of an equivalence F induces a bijection between the sets of isomorphism classes in X and Y. For more details and other formulations, see e.g. p.71 of Gelfand–Manin (Methods of homological algebra []).

The definitions of 1) and 2) have already been covered. The functor 1) $\to$ 2) is analytification $(E, \nabla) \mapsto (E^{an}, \nabla^{an})$ (An algebraic vector bundle is a special type of holomorphic vector bundle; algebraic clutching maps are in particular holomorphic. In terms of sheaves of sections we just take the holomorphic sections of the algebraic bundle E. Then the action of ∇^{an} on holomorphic sections is completely determined by the Leibniz rule $\nabla(fs) = (df)s + f\nabla(s)$ for holomorphic f and algebraic s). The equivalence between them is a version of GAGA, since Σ is compact (see e.g. Malgrange [**?**] p.152).

For 3), the definition of C^∞ connections is straightforward but it is worth noting that there is now an integrability condition: the connections should be *flat*, i.e. have vanishing curvature.

This is actually one of the central ideas (probably *the* central idea) in the subject of integrable systems, that the vanishing of curvature is a nonlinear differential equation, and this is the key mechanism how linear connections lead to nonlinear differential equations.

Given a connection $\nabla = d + A$ on a trivial vector bundle, then its curvature is the matrix of two-forms $\Omega = \nabla^2 = dA + A^2$. Here in the $\mathfrak{gl}_n(\mathbb{C})$ setting A^2 is well defined as a matrix of two-forms. In general (for other Lie algebras) we just define $A^2 = [A, A]/2$ and use the same notation. But we really want to see what this means:

Suppose the base is two-dimensional with coordinates x, y, and we write

$$A = X(x, y)dx + Y(x, y)dy$$

for matrix valued functions X, Y.

Then the key computation to do is to compute the commutator:

$$\left[\frac{\partial}{\partial x} + X, \frac{\partial}{\partial y} + Y\right] \;=\; \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} + [X, Y] \tag{4.1}$$

and note that the connection $d + A$ is flat if and only if this commutator is zero:

Exercise 4.3. Show that $dA + A^2$ is proportional to the commutator above times $dx \wedge dy$.

Thus the flatness condition is a nonlinear (quadratic) differential equation:

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} + [X, Y].$$

For example, as we will see, all the isomonodromy equations arise from such curvature equations, and the isospectral (Lax) equation are autonomous limits of them (passing via ζ connections in one direction, to remove one of the derivatives).

Now to pass from 2) to 3) we just take the underlying C^∞ vector bundle, and use the Leibniz rule to define the action of ∇ on any C^∞ section $\nabla(fs) = (df) + f\nabla(s)$ for C^∞ function f and holomorphic sections s. Here d is the full exterior derivative on Σ and will have both $dz, d\overline{z}$ terms in general. Any local C^∞ section can be written as $\sum f_i s_i$ for C^∞ functions and holomorphic sections. The resulting connection is clearly flat as any holomorphic connection on a Riemann surface is flat (and thus any gauge transformation of it too: $\Omega \mapsto g\Omega g^{-1}$ under a gauge transformation g).

To go backwards we need to see how a flat C^∞ connection determines the structure of holomorphic vector bundle. Locally such a connection has the form:

$$d - \alpha = (\partial - \alpha^{1,0}) + (\overline{\partial} - \alpha^{0,1})$$

so the $0,1$ part is $\overline{D} := \overline{\partial} - \alpha^{0,1} = \overline{\partial} - Bd\overline{z}$ for some matrix B of C^∞ functions.

A theorem of Koszul–Malgrange says that this determines the structure of holomorphic vector bundle, with the "holomorphic sections" defined to be the sections in the kernel of $\overline{D}$:

Theorem 4.4 ([?]). *If B is an $n \times n$ matrix of complex C^∞ functions on a disk Δ and $\overline{D} := \overline{\partial} - Bd\overline{z}$, then there is a basis of sections $s_1, \ldots, s_n$ in the kernel of $\overline{D}$. This implies the kernel of $\overline{D}$ is the sheaf of sections of a holomorphic vector bundle; a [locally] free $\mathcal{O}$-module, where $\mathcal{O}$ is the sheaf of holomorphic functions.*

If $g : \Delta \to G$ is the matrix with columns $s_1, \ldots, s_n$ then $\overline{\partial} g = Bgd\overline{z}$ so that

$$g[0]^{0,1} := g(0)g^{-1} + (\overline{\partial} g)g^{-1} = (\overline{\partial} g)g^{-1} = Bd\overline{z},$$

i.e. the inverse of g gives a gauge transformation converting $\overline{D}$ into $\overline{\partial}$, the "trivial" $\overline{\partial}$-operator.

If we pass to such a holomorphic basis then the flat connection will become a connection with zero $0,1$ part, so of the form

$$\nabla = d - Cdz$$

for a matrix C of C^∞ functions: since it is still flat the matrix C is actually holomorphic: $\nabla^2 = (dC)dz = (\overline{\partial}C)dz$ and the vanishing of this means that C is holomorphic. So we get a holomorphic connection on the holomorphic bundle determined by $\overline{D} = \nabla^{0,1}$. This is how to pass back and forth between holomorphic and flat C^∞ connections.

Of course in the current setting of flat connection we could bypass this and note that flatness implies that any flat connection $d - \alpha$ is locally trivial and has a basis of horizontal sections (the clutching map between such bases will be constant and so in particular holomorphic). This is the *nonabelian Poincaré lemma* (with one-forms replaced by connections and closedness by flatness):

Theorem 4.5. *Any flat connection has a fundamental solution (basis of horizontal sections) when restricted to any disk. In other words a (nonsingular) connection is flat if and only if it is locally isomorphic to the trivial connection. Explicitly in the current setting: If B, C are $n \times n$ matrices of complex C^∞ functions on a disk Δ and*

$$\nabla := d - \alpha = \partial - Cdz + \overline{\partial} - Bd\overline{z}$$

is a C^∞ connection that is flat*, then there is a basis of horizontal sections $s_1, \ldots, s_n$ on Δ. This implies the kernel of ∇ is a locally constant sheaf of n dimensional complex vector spaces.*

This can be proved directly (see e.g. [**?**]), or by passing to a holomorphic basis by Koszul–Malgrange, and then constructing a fundamental solution of the resulting holomorphic connection as we did before.

This leads to item 4) in the list, the local systems.

4.2. **Local systems.**

Noter que *revêtement* et *faisceau localement constant* sont synonymes ([**?**] p.231)

Now we get to the intrinsic, purely topological, description of connections. A convenient framework to phrase this is *covering spaces* (often with uncountable fibres), or equivalently locally constant sheaves (with open sets in the usual topological sense).

Definition 4.6. *Suppose $\mathbb{B}$ is a topological manifold. A* local system *(of sets) on $\mathbb{B}$ is a locally constant sheaf of sets, and in turn it is the same thing as (the sheaf of sections of) a covering space of $\mathbb{B}$.*

This is really two definitions and an enlightening exercise shows they are the same.

In practice we can choose an open covering and a local system is then a bundle with local trivialisations (to a product with a fixed "standard fibre"), that can be defined by *constant* clutching maps on the double intersections of open sets in the covering.

Here, from flat connections, we have a local system of vector spaces, i.e. a locally constant sheaf of n dimensional complex vector spaces. This just means that the clutching maps are constant linear maps.

Exercise 4.7. Show that, by definition in the Cech approach, the set of isomorphism classes of local systems of n dimensional complex vector spaces on Σ is the nonabelian cohomology set $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathbb{C}))$.

Thus the passage from 3) to 4) is just to go from a flat connection (E, ∇) to its sheaf of horizontal sections V defined by

$$V(U) = \{\text{sections } s : U \to E \bigm| \nabla(s) = 0\}.$$

This is the desired local system.

To recover (E, ∇) from V we just tensor: the sheaf $\mathcal{E}$ of sections of E is

$$\mathcal{E}(U) = V(U) \otimes_{\mathbb{C}} C^\infty$$

and the connection can be defined on these sections via Leibniz, since $V(U)$ are the horizontal sections:

$$\nabla(vf) = (df)v + f\nabla(v) = (df)v$$

for C^∞ functions f. Similarly we could go directly back to a holomorphic vector bundle by tensoring with holomorphic functions $\mathcal{E}(U) := V(U) \otimes_{\mathbb{C}} \mathcal{O}$ is the sheaf of sections of a holomorphic vector bundle, and this gets a holomorphic connection in the same way: $\nabla(vf) = (df)v$ for holomorphic functions f.

4.3. **Monodromy of local systems.** Finally we can discuss *monodromy* and how to pass to representations of the fundamental group.

First of all there is a general statement.

Suppose $\mathbb{B}$ is a connected manifold and $\pi : C \to \mathbb{B}$ is any covering space (the fibres may be uncountable etc). Thus the sheaf of sections of C is a local system of sets.

For any two points $a, b \in \mathbb{B}$ the choice of a path $\gamma : [0, 1] \to \mathbb{B}$ in $\mathbb{B}$ from a to b, determines a bijection

$$T_\gamma(a, b) : C_a \cong C_b$$

the transport isomorphism, from the fibre $C_a = \pi^{-1}(a)$ of C at a, to the fibre C_b at b.

The transport map is defined as follows: For any point $c \in C_a$ the path γ has a unique lift to a path $\widetilde{\gamma} : [0, 1] \to C$ in C starting at c. This follows from the definition of covering space. Then $T_\gamma(a,b)(c)$ is defined to be the end point $\widetilde{\gamma}(1) \in C$ of this lifted path. From the definition of $\widetilde{\gamma}$ it is in C_b, i.e. it lies over b.

The map $T_\gamma(a,b)$ only depends on the homotopy class of γ (with fixed endpoints). Indeed any continuous deformation of γ cannot move $T_\gamma(a,b)(c)$ since it is constrained to be in the fibre C_b and the fibres are discrete.

Exercise 4.8. Rewrite this definition of transport in terms of locally constant sheaves of sections, and their clutching/restriction maps (passing from one open set to the next, via their intersection, covering the path γ), without first passing to the equivalent notion of covering spaces. If the local system is in fact the sheaf of horizontal sections of a holomorphic connection on a trivial vector bundle, show that this is the same thing as the analytic continuation of solutions.

In particular, considering loops based at b, this construction gives a homomorphism

$$\rho : \pi_1(\mathbb{B}, b) \to \mathrm{Aut}(C_b); \quad \rho(\gamma) = T_\gamma(b, b)$$

from the fundamental group of the base into the group of automorphisms of the fibre. This is just transport around loops. Said differently this is an *action* of $\pi_1(\mathbb{B}, b)$ on the fibre C_b, the *monodromy action*.

In the case that we started with a local system of vector spaces V (and not just sets) on $\mathbb{B} = \Sigma$ then this yields the monodromy representation $\rho : \pi_1(\mathbb{B}, b) \to \mathrm{Aut}(V_b) = \mathrm{GL}(V_b) \cong \mathrm{GL}_n(\mathbb{C})$. In other words the fibre V_b is a representation of the fundamental group. Thus the covering space and the basepoint determine a pair (V_b, ρ) consisting of a complex vector space equipped with a representation of $\pi_1(\mathbb{B}, b)$. This is an object of the category in 5), and this construction defines the desired functor 4) $\to$ 5).

Now we just need to check that this gives an equivalence. The key step is to define the inverse construction, from 5) to 4), which goes as follows.

Given $b \in \mathbb{B}$ let $\mathrm{pr} : \widetilde{\mathbb{B}} \to \mathbb{B}$ be the universal cover, based at b. By definition $\widetilde{\mathbb{B}}$ is the set of homotopy classes of paths in $\mathbb{B}$ starting at b, i.e. maps $\gamma : [0,1] \to \mathbb{B}$ such that $\gamma(0) = b$. Two such paths are identified if there is a homotopy between them, fixing both end points. The map pr takes the free end point of the path, $\mathrm{pr}(\gamma) = \gamma(1) \in \mathbb{B}$.

Write $\pi_1 = \pi_1(\mathbb{B}, b)$ for the fundamental group. This group acts on the fibres of $\widetilde{\mathbb{B}} \to \mathbb{B}$ freely and transitively, in other words:

Lemma 4.9. *The universal covering space $\widetilde{\mathbb{B}}$ is a principal π_1 bundle over $\mathbb{B}$.*

Proof. π_1 acts on $\widetilde{\mathbb{B}}$ in the natural way, composing a loop and a path: If $g \in \pi_1$ is a loop based at b and $\gamma \in \widetilde{\mathbb{B}}$ is a path starting at b then $\gamma \circ g \in \widetilde{\mathbb{B}}$ since it is clearly another path starting at b. The ordering of the composition $\gamma \circ g$ means "go around g and then go along γ".

Now it is easy to see that two paths $\gamma_1, \gamma_2 \in \widetilde{\mathbb{B}}$ have the same endpoint ($\gamma_1(1) = \gamma_2(1)$) if and only if they are related in this way by a loop based at b. Moreover two paths with the same end point are homotopic if and only if the loop relating them is contractible, so represents the identity in π_1. This says that π_1 acts freely and transitively on the fibre

$$\widetilde{\mathbb{B}}_c := \mathrm{pr}^{-1}(c) \subset \widetilde{\mathbb{B}}$$

of the universal covering map, for any $c \in \widetilde{\mathbb{B}}$.

□

Now for any representation of π_1 we can form the associated bundle (of the principal π_1 bundle $\widetilde{\mathbb{B}}$). If $\rho : \pi_1 \to V$ then the associated bundle is the quotient

$$\widetilde{\mathbb{B}} \times_\rho V := (\widetilde{\mathbb{B}} \times V)/\pi_1$$

where $g \in \pi_1$ acts on a pair $(c, v) \in \widetilde{\mathbb{B}} \times V$ as

$$g \cdot (c, v) = (cg^{-1}, \rho(g)v).$$

Since the action on $\widetilde{\mathbb{B}}$ is free, this quotient is well-defined, and it comes equipped with a map:

$$\widetilde{\mathbb{B}} \times_\rho V \to \mathbb{B} = \widetilde{\mathbb{B}}/\pi_1 \tag{4.2}$$

by projecting onto the first factor, with each fibre isomorphic to a copy of V.

Now we leave it as an exercise to check that the map (4.2) is a covering map (giving the fibres $\cong V$ the discrete topology), defining a local system of vector spaces, and moreover that its monodromy representation based at b is given by ρ.

4.4. **Representation varieties.** Let $G = \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^n)$ the group of linear automorphisms of a fixed (standard) copy of $\mathbb{C}^n$.

Let $\pi_1 = \pi_1(\Sigma, b)$ and suppose we are given a representation V of π_1, i.e. we are given a homomorphism

$$\rho : \pi_1 \to \mathrm{GL}(V).$$

Now if we choose a basis of V, i.e. an isomorphism $\phi : \mathbb{C}^n \xrightarrow{\cong} V$ with our standard copy of $\mathbb{C}^n$, then we get a "concrete" representation $\pi_1 \to G = \mathrm{GL}_n(\mathbb{C})$, into a fixed copy of the general linear group.

Lemma 4.10. *Let $\mathcal{R} = \mathrm{Hom}(\pi_1, G)$ be the set of group homomorphisms $\pi_1 \to G$. Consider the enriched category of triples (V, ρ, ϕ) where (V, ρ) is a π_1 representation as in 5), and $\phi : \mathbb{C}^n \xrightarrow{\cong} V$ is a framing of V. Then the set of isomorphism classes of such triples is naturally in bijection with the points of $\mathcal{R}$.*

Proof. This is a straightforward unwinding of the definitions. (The map ρ becomes a point of $\mathcal{R}$ once we use ϕ to identify V and $\mathbb{C}^n$.) $\square$

Now $\mathcal{R}$ is naturally a complex affine algebraic variety. The easiest way to see this is to choose a presentation of π_1. The standard presentation is as follows (where g is the genus of Σ):

$$\pi_1(\Sigma, b) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

where $[a, b] = aba^{-1}b^{-1}$ is the multiplicative commutator. Given this presentation, it follows that $\mathcal{R}$ has the following presentation as an affine variety:

$$\mathcal{R} \cong \{(A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g} \mid [A_1, B_1] \cdots [A_g, B_g] = 1\} \subset G^{2g} \tag{4.3}$$

where $[A, B] = ABA^{-1}B^{-1}$ is the multiplicative commutator in the group G. Each matrix entry of the relation here is an algebraic equation, and so this defines $\mathcal{R}$ as a subvariety of G^{2g}. Of course $G = \mathrm{GL}_n(\mathbb{C})$ is itself an affine variety, for example defined by the equation:

$$\mathrm{GL}_n(\mathbb{C}) \cong \{(g, a) \in \mathrm{End}(\mathbb{C}^n) \times \mathbb{C} \mid a \det(g) = 1\} \subset \mathbb{C}^{n^2+1}.$$

Thus $\mathcal{R}$ is an affine variety, defined by the matrix equation in (4.3). A point of the right-hand side of (4.3) determines a unique representation ρ since it specifies where ρ sends generators of π_1, in $\mathrm{GL}_n(\mathbb{C})$.

Observe that G acts on $\mathcal{R}$ by diagonal conjugation of the matrices. This action corresponds to changing the choice of framing $\phi : \mathbb{C}^n \xrightarrow{\cong} V$; an element of $g \in G$ acts ion ϕ by pre-composition: $\phi \mapsto \phi \circ g^{-1}$.

Corollary 4.11. *The set of isomorphism classes of π_1 representation (V, ρ) of rank n, is naturally in bijection with the set of G orbits in $\mathcal{R}$.*

Proof. This now comes down to observing that $(V_1, \rho_1) \cong (V_2, \rho_2)$ if and only if we can choose framings $\phi_1 : \mathbb{C}^n \cong V_1$ and $\phi_2 : \mathbb{C}^n \cong V_2$ so that the two triples (V_1, ρ_1, ϕ_1) and (V_2, ρ_2, ϕ_2) determine the same point of $\mathcal{R}$. $\square$

We are now in a very good position of a complex reductive group G acting on a complex affine variety $\mathcal{R}$ and there are standard tools (geometric invariant theory) to take the quotient of $\mathcal{R}$ by G in an algebraic way, thereby constructing the character variety $\mathcal{M}_{\mathrm{B}}$.

Let $\mathbb{C}[\mathcal{R}]$ denote the ring of regular functions on the affine variety $\mathcal{R}$ and let $\mathbb{C}[\mathcal{R}]^G \subset \mathbb{C}[\mathcal{R}]$ denote the subring of G invariant functions, where G acts by diagonal conjugation as above. Since G is reductive it is known that this ring is finitely generated and so determines an algebraic variety.

Definition 4.12. *The character variety (or Betti moduli space) $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ determined by the pair (Σ, G) is the variety associated to the ring $\mathbb{C}[\mathcal{R}]^G$ of G invariant functions on the representation variety $\mathcal{R}$. By construction the points of $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ correspond bijectively to the* closed *G-orbits in $\mathcal{R}$.*

We will write $\mathcal{M}_{\mathrm{B}} = \mathcal{R}^{\mathrm{ps}}/G$ where $\mathcal{R}^{\mathrm{ps}} \subset \mathcal{R}$ is the subset of points whose G-orbits are closed (since the closed orbits are often called the *polystable* point).

The book [] of Lubotzky–Magid "Varieties of representations of groups", studies the construction of character varieties of any finitely presented group in detail.

4.5. **Classification of solutions of the zero curvature equation.** As an application of the previous theorem, we can see that it gives a precise finite dimensional description of the space of equivalence classes of solutions of a nontrivial nonlinear differential equation in infinite dimensions.

Let $E = \mathbb{C}^n \times \Sigma \to \Sigma$ be the trivial complex vector bundle (that we view here as a C^∞ bundle).

Let
$$\mathcal{A} = \{d - \alpha \;\big|\; \alpha \in \Gamma(\Sigma, \mathrm{End}(E) \otimes (\Omega^{1,0} \oplus \Omega^{0,1}))\}$$
be the set of connections on E, so that α is an arbitrary $n \times n$ matrix of global C^∞ one-forms. Thus $\mathcal{A}$ is isomorphic to an infinite dimensional vector space.

Let $\mathcal{G} = C^\infty(\Sigma, \mathrm{GL}_n(\mathbb{C}))$ be the group of global gauge transformations of E, i.e. the C^∞ maps from Σ to $\mathrm{GL}_n(\mathbb{C})$. Thus $\mathcal{G}$ acts on $\mathcal{A}$ by gauge transformations as usual: $g[\alpha] = g\alpha g^{-1} + (dg)g^{-1}$.

Now consider the subset of connections which are flat, so α satisfies the nonlinear differential equation $d\alpha = \alpha^2$:
$$\mathcal{A}_{\text{flat}} = \{d - \alpha \;\big|\; d\alpha = \alpha^2\} \subset \mathcal{A}.$$
This subset is preserved by the gauge action and the previous theorem implies the following classification of gauge orbits.

Corollary 4.13. *The set of $\mathcal{G}$ orbits in $\mathcal{A}_{\mathit{flat}}$ is naturally in bijection with the set of G orbits in the representation variety $\mathcal{R}$.*

Proof. Given what was proven in the theorem this amounts to observing that $\mathcal{A}_{\text{flat}}/\mathcal{G}$ is the set of isomorphisms classes of flat connections on C^∞ vector bundles of rank n. This in turn follows from the fact that Chern-Weil theory implies any C^∞ complex vector bundle on a compact Riemann surface that admits a flat connection is trivial (in brief, it has degree zero). Thus if we choose a trivialisation, we see they all appear as points of $\mathcal{A}_{\text{flat}}$. Moreover the notion of isomorphism of connections then comes down to the gauge action of $\mathcal{G}$ on $\mathcal{A}$. □

This may seem like it is just a tricky exercise in rephrasing the definitions but we will see below that this C^∞ viewpoint enables us to see, following Narasimhan and Atiyah–Bott, that the character variety has a holomorphic symplectic structure.

For now let us just quote a theorem that appears in Gunning's 1967 book "Lectures on vector bundles on Riemann surfaces":

Theorem 4.14 (Gunning [] p.196)**.** *Let $\mathcal{M}_{\mathrm{B}}^s \subset \mathcal{M}_{\mathrm{B}}$ be the subset of the character variety consisting of representations that are* irreducible*. Then $\mathcal{M}_{\mathrm{B}}^s$ is a (smooth) complex analytic manifold, of dimension $2gn^2 - 2(n^2-1)$.*

We won't prove this statement yet, as one of our aims will be to show how to prove that it has an algebraic symplectic structure at the same time, as well as many generalisations of it.

As a first step note that it is easy to explain the dimension formula since it is a subquotient of G^{2g}: it is the quotient of the subvariety $\mu^{-1}(1)^{\mathrm{irr}}$ by G where

$$\mu : G^{2g} \to G; \qquad (A_1, \dots, A_g, B_1, \dots, B_g) \mapsto [A_1, B_1] \cdots [A_g, B_g]$$

and $\mu^{-1}(1)^{\mathrm{irr}}$ is the subset of $\mu^{-1}(1)$ that are irreducible representations. The point is that $\mu^{-1}(1)^{\mathrm{irr}}$ has codimension (n^2-1) (since the determinant is already fixed to be 1), and $\mathrm{PGL}_n(\mathbb{C})$ acts freely on it, and that has dimension (n^2-1) as well, so we see the dimension is obtained by subtracting (n^2-1) twice from $2gn^2 = \dim(G^{2g})$.

Remark 4.15. Note that the equivalences between 2),3),4),5) work verbatim over any Riemann surface, not necessarily compact. This will be used to give *part* of the topological data of any meromorphic connection $(E, \nabla) \to \Sigma$ with poles on $\mathbf{a}$. Namely the restriction of (E, ∇) to $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ is a holomorphic connection and we can take the local system $V \to \Sigma^\circ$ of horizontal sections of that. We will see this is really only a very small part of the topological data attached to any meromorphic connection.

5. The Riemann problem (Hilbert 21)

We will take the point of view that the Betti spaces (and eventually the whole nonabelian Hodge space) is like a "global version" of a Lie group, attached to a Lie group plus a surface (with suitable boundary condtions). This comes more into focus if we look at the genus zero case with poles, as, in effect, we then see the *Lie algebra* of the space as well. These are the additive moduli spaces $\mathcal{M}^*$ and the simplest (Fuchsian) examples motivated the famous Riemann problem appearing in Hilbert's 21st problem (the Riemann–Hilbert problem).

There is some controversy over the exact statement of the question, but the basic idea is very simple and clear, and comes down to the following matching of dimensions.

Choose an integer $n > 0$ and m distinct points $\mathbf{a} = (a_1, \dots, a_m) \subset \mathbb{C}$ in the complex plane.

On one hand consider the set of rank n Fuchsian systems with poles at these points:

$$\begin{aligned}\widetilde{\mathcal{M}}^* &= \left\{ \nabla = d - A \;\middle|\; A = \sum_1^m \frac{A_i}{z - a_i} dz, A_i \in \mathrm{End}(\mathbb{C}^n) \right\} \\ &\cong \left\{ (A_1, \dots, A_m) \;\middle|\; A_i \in \mathrm{End}(\mathbb{C}^n) \right\}. \end{aligned} \tag{5.1}$$

On the other hand, given any such connection we can restrict it to the complement of the poles to get a holomorphic connection on $\Sigma^\circ := \mathbb{C} \setminus \mathbf{a}$, noting that in general the connection will have a further pole at ∞. Then we can take the local system of horizontal sections of that and in turn get a representation of the fundamental group $\pi_1 = \pi_1(\Sigma^\circ, b)$ in $\mathrm{GL}_n(\mathbb{C})$, for any choice of basepoint $b \in \Sigma^\circ$. (It comes with a framing as the underlying bundle is the trivial bundle.) Thus we get a point of the representation variety

$$\begin{aligned}\mathcal{R} &= \mathrm{Hom}(\pi_1, \mathrm{GL}_n(\mathbb{C})) \\ &\cong \left\{ (M_1, \dots, M_m) \;\middle|\; M_i \in \mathrm{GL}_n(\mathbb{C}) \right\} \end{aligned} \tag{5.2}$$

where the last isomorphism arises by choosing a suitable presentation of π_1, with m loops around the points in $\mathbf{a}$, freely generating π_1.

The spaces (5.1) and (5.2) are clearly both of the same dimension mn^2, and (5.2) *looks like* the multiplicative version of (5.1), with the Lie algebra replaced by the corresponding Lie group, and the sum replaced by the product (in a certain fixed order).

Moreover the *Riemann–Hilbert map*

$$\widetilde{\mathcal{M}}^* \xrightarrow{\nu_{\mathbf{a}}} \mathcal{R}$$

taking a connection to its monodromy representation is a *holomorphic* map, which generalises the matrix exponential map, that appears in the case $m = 1$:

Exponential map as a simple Riemann–Hilbert map.
Given $X \in \mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ then the connection $d - A$ where

$$A = \frac{1}{2\pi i} X \frac{dz}{z}$$

has monodromy given by

$$\exp(X) \in \mathrm{GL}_n(\mathbb{C}).$$

Proof. For any $Y \in \mathfrak{g}$, the connection $d - Y dz/z$ has fundamental solution z^Y on any open sector at zero (using any choice of branch of $\log(z)$). This has monodromy $\exp(2\pi i Y)$ around zero. □

Thus it is tempting to study this map, for example can it be upgraded to a precise bijective correspondence? What happens if we move the points $\mathbf{a}$?

6. Birkhoff's generalised Riemann problem

There is of course a more basic question that one can ask:

Suppose we have an arbitrary effective divisor $D = \sum n_i(a_i)$ where $\mathbf{a} = (a_1, \ldots, a_m) \subset \mathbb{P}^1$ and $n_i > 0$.

Then we can consider the finite dimensional space $\widetilde{\mathcal{M}}^*(D)$ consisting of all the meromorphic connections

$$d - A$$

where A is a matrix of rational one-forms with poles bounded by the divisor D.

Counting coefficients and using the residue theorem shows that

$$\dim(\widetilde{\mathcal{M}}^*) = n^2 \left(\sum_1^m n_i \right) - n^2$$

Question: Can one define invariants of such connections with any order poles, thereby defining a space $\mathcal{R}$ of dimension equal to $\dim(\widetilde{\mathcal{M}}^*)$ and a holomorphic map

$$\widetilde{\mathcal{M}}^* \to \mathcal{R},$$

generalising the Riemann–Hilbert map taking the monodromy representation?

Birkhoff (1909, 1913) found that this can indeed be done for a dense open subset of $\widetilde{\mathcal{M}}^*$. He imposed a genericity condition on the connection ("Birkhoff-generic")[10] and then constructed some invariants making up a space of the desired dimension.

These data and their generalisation/modification leading up to the definition of the general notion of *Stokes data* and *wild character varieties* are what we want to study in detail. In a sense they are the general notions of *global Lie groups* that appear in this way.

As a simple example to illustrate how this goes, suppose $n = 2$ and $D = 4(\infty)$ so we look at rank two connections with one pole of order 4 at infinity

$$\nabla = d - A, \; A = (A_0 + A_1 z + A_2 z^2)dz.$$

We suppose that the leading term A_2 is diagonal with distinct eigenvalues and is fixed, so there are 8 remaining free parameters in A_0, A_1. The monodromy-type data this leads to have the following form: first we restrict ∇ to the formal disk at ∞ and find it can be put uniquely in the form:

$$\widehat{\nabla} = d - \widehat{A}, \; \widehat{A} = dQ + \Lambda \frac{dz}{z}, \; Q = B_3 z^3 + B_2 z^2 + B_1 z$$

[10] that the leading term at each pole has n distinct eigenvalues, and further that the eigenvalues are off of some real codimension one walls.

via a *formal* (not necessarily convergent) gauge transformation, for some diagonal matrices B_i, Λ, with $B_3 = A_2/3$ and $\mathrm{Tr}(\Lambda) = 0$. This gives 5 parameters, in Λ, B_1, B_2. The remaining parameters are more mysterious and can be understood in several ways. One way (essentially that of Birkhoff) is that there are "wild monodromy data" $S_1, \ldots S_6$ that obey a *wild monodromy relation*:

$$S_6 S_5 \cdots S_1 = h, \qquad h := \exp(2\pi i \Lambda). \tag{6.1}$$

Moreover the S_i are constrained to be in alternating unipotent groups:

$$S_1, S_3, S_5 \in U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad S_2, S_4, S_6 \in U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

These unipotent groups have a total dimension of 6 and the relation (6.1) imposes 3 constraints on them (as the determinant is 1), and so this yields the desired remaining three parameters, making up 8 in total. This example in fact leads to the wild character variety (of complex dimension two) underlying the Painlevé II equation, that takes the form of the affine surface (the *Flaschka–Newell surface*):

$$x\, y\, z + x + y + z = c$$

for a constant $c \in \mathbb{C}$ (directly related to the constant α in P_{II}).

This is of course, all incredibly strange and mysterious, and begs many questions (that we will endeavour to answer in the rest of the course): what are these matrices S_i? Why are they triangular? Why are there 6 of them? Are we really generalising the fundamental group? What is the generalisation of the intrinsic topological notion of local system? Why has no-one told me about this before? (etc)

Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, 75013 Paris, France.
https://webusers.imj-prg.fr/~philip.boalch/ boalch@imj-prg.fr