MEROMORPHIC CONNECTIONS IN 2D GAUGE THEORY PRELIMINARY LECTURE NOTES

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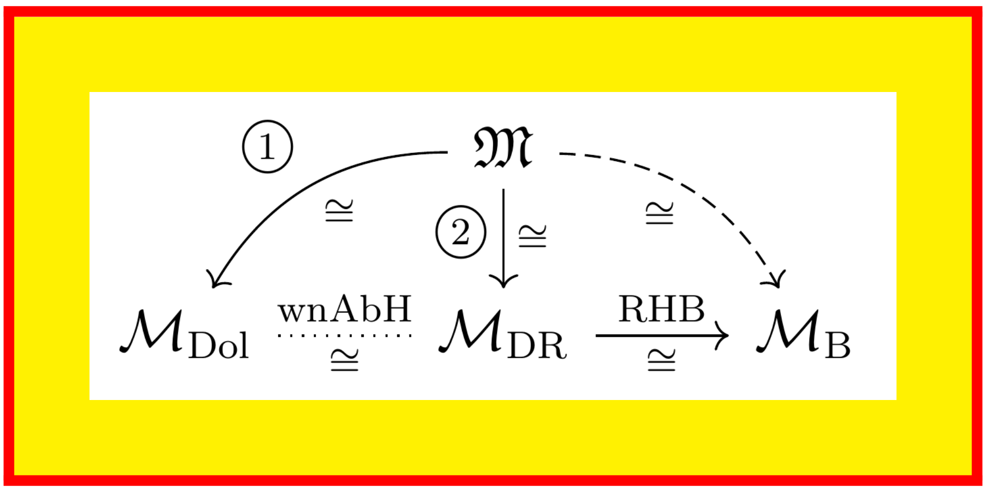

Non-Abelian Hodge space $\mathfrak{M}$,
three preferred algebraic structures:
Dolbeault, De Rham, Betti
mero. Higgs bundles, mero. connections, Stokes local systems

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Contents

Resumé

La but du cours est d'expliquer la lien entre les équations différentielles algébriques linéaires (les connexions méromorphes sur les fibrés vectoriels sur les courbes complexes lisses) et quelques exemples des équations différentielles non-linéaires.

L'idée de base, centrale dans la théorie de jauge, est que l'inconnu dans l'équation différentielle nonlineaire est mieux compris comme un connexion linéaire. Ici on utilise la même principe mais pour les connexions algébriques méromorphes.

Ceci donne l'opportunité d'étudier quelques exemples des jolies variétés algébriques (hyperkahlerienne) qui apparaissent comme espaces de modules dans cette histoire.

Contenu

• Espaces de modules de connexions additifs et multiplicatifs (géométrie symplectique holomorphe)

• Systèmes locaux de Stokes, variétés de caractères sauvages

• Application de Riemann-Hilbert-Birkhoff, théorie de Lie globale

• Équations de Yang-Mills autodual, équations de Hitchin, fibrés harmoniques sur les surfaces de Riemann non-compact (rotation hyperkahlerienne, correspondance de Hodge nonabelienne sauvage)

• Systèmes d'isomondromie (Painlevé, Schlesinger, Jimbo-Miwa-Mori-Sato, simplement lacé,...), connexions d'Ehresmann nonlineaires et lien avec les groupes de tresse

• Fibres de Higgs méromorphes et systèmes intégrables algèbro-géométriques (Garnier, Mumford, Hitchin, Bottacin-Markman, ...)

List of key points:

Lecture 1: Definition of meromorphic connection. View as global/intrinsic linear differential systems. Gauge transformations, gauge action. Algebraic versions. Definition of meromorphic Higgs bundles and ζ-connections.

Sketch of big picture (at symbolic level). Two key correspondences.

Start list of key examples: Painlevé's discovery of natural deformations of the theory of elliptic functions (simplification of $\mathrm{P_I}$, $\mathrm{P_{II}}$). Link to gauge theory (R. Fuchs): First steps in description of geometry of Painlevé VI.

Lecture 2: Definition of curvature. Flatness in terms of commuting operators. Holomorphic structures via $\overline{\partial}$-operators and Koszul–Malgrange statement. Definition of local system of sets and of vector spaces, relation to covering spaces. Transport and monodromy of a local system. Equivalence of five viewpoints on connections in the compact case (no poles).

Representation varieties $\mathcal{R}$ as framed moduli spaces, and as affine varieties. Classification of solutions of the flatness equation. Character variety/Betti moduli space.

Dimension counting in the Riemann problem (Hilbert 21). Relation to matrix exponential map. The question that Birkhoff's invariants answered $\rightsquigarrow$ global Lie theory. Example of Painlevé 2 ($\widehat{A}_1$) wild character variety (Flaschka–Newell surface).

Lecture 3: Surface groups. Geometric local systems (linear and nonlinear); Local systems of character varieties. Hurwitz action and Fricke–Klein–Vogt example.

Dubrovin's Markoff example (braiding of BPS states). Klein example (nonlinear representation theory).

Abelian example: Legendre family, explicit equation for flat sections.

Lecture 4: Basic definitions: Logarithmic, generic and very good connections. Exponential local system, Stokes circles. Modular parameters (irregular types, irregular classes) $\rightsquigarrow$ notion of rank n wild Riemann surface.

Dominance orderings, Stokes/oscillating directions, points of maximal decay, singular directions, Stokes arrows.

Stokes diagrams of one-level irregular classes (simple examples such as Airy, Bessel, Kummer). Fabry's theorem.

Lecture 5:

Return to generic case: Rephrase Birkhoff and define simple fission spaces.

1. Lecture 1: Basic examples, questions and definitions

1.1. **What is a meromorphic connection?**

We will need several different flavours (categories) of connections, and some confusion in the subject comes from different authors having different default definitions. The relation between various definitions will be crucial to understand. Thus we'll start with the central notions we will use, and then later discuss variations and their relation.

The starting point is a first order linear differential operator of the form

$$\frac{d}{dz} - B(z)$$

where $B(z)$ is an $n \times n$ matrix of holomorphic functions on an open subset $U \subset \mathbb{C}$. As a first example one might consider a polynomial system:

$$\frac{d}{dz} - (A_0 + A_1 z + \cdots A_m z^m) \tag{1.1}$$

for $n \times n$ matrices A_i. As a second example one might consider:

$$\frac{d}{dz} - \left(\frac{A_1}{z - a_1} + \cdots + \frac{A_m}{z - a_m} \right) \tag{1.2}$$

for $n \times n$ matrices A_i, away from the poles (these are often called "Fuchsian systems").

This yields the linear system of differential equations

$$\frac{dv}{dz} = Bv$$

where v is a length n column vector of holomorphic functions. The coordinate-free version of this operator is got by "multiplying by dz", to get the *connection*

$$\nabla = d - A, \qquad A = B(z)dz$$

so that A is a matrix of holomorphic one-forms and d is the exterior derivative. This is a connection on the trivial rank n holomorphic vector bundle on U, i.e. on $E = \mathbb{C}^n \times U \to U$. Solutions v are now called *horizontal sections* and the equation $dv/dz = Bv$ is rewritten $\nabla(v) = 0$, i.e. $dv = Av$. We can remove the condition that E is trivial and consider connections on non-trivial vector bundles, leading to the following definition, first in the case with no poles.

Let Σ be a compact Riemann surface.

Definition 1.1. *A* <u>holomorphic connection</u> *is a pair* (E, ∇) *where* $E \to \Sigma$ *is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by holomorphic one-forms, such that the Leibniz rule is satisfied:

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.3}$$

for all local sections s of E and functions f.

This is a completely standard definition, going back to Koszul. A connection is a way to differentiate sections of E: If X is a vector field on Σ and s is a section of E then

$$\nabla_X(s) := \langle X, \nabla(s)\rangle$$

is again a section of E, the derivative by ∇ of s along X. Here the brackets $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the natural pairing between the tangent bundle and the cotangent bundle.

In a local trivialisation of E, over some open subset $U \subset \Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for an $n \times n$ matrix of holomorphic one-forms A, where n is the rank of E. If z is a local coordinate on U this means we can write $A = Bdz$ for a matrix B of holomorphic functions on U. Thus a connection $\nabla = d - Bdz$ is really just a global, coordinate-free version of the matrix differential operators $\frac{d}{dz} - B$ we first considered.

If we change the choice of local trivialisation of E then A changes by a *gauge transformation*:

$$A \;\mapsto\; g[A] := gAg^{-1} + (dg)g^{-1} \tag{1.4}$$

where $g : U \to \mathrm{GL}_n(\mathbb{C})$ is a holomorphic map. Our conventions are set-up such that this is a group action:

Exercise 1.2. Show that $(g \circ h)[A] = g[h[A]]$.

Exercise 1.3. Show that if $\mathbf{e} = (e_1, \ldots, e_n)$ is the initial basis of E and $\mathbf{e}' = (e'_1, \ldots, e'_n)$ is the new basis, and g is such that $\mathbf{e} = \mathbf{e}' \circ g$, then we do indeed get the formula (1.4) for $g[A]$.

Exercise 1.4. Choose an open covering $\Sigma = \bigcup_{i \in I} U_i$ of Σ and a trivialisation $\mathbf{e}_i$ of E over U_i for each i, and so the connection takes the form $d - A_i$ on U_i. Let $U_{ij} = U_i \cap U_j$ and define $g_{ij} : U_{ij} \to \mathrm{GL}_n(\mathbb{C})$ so that $\mathbf{e}_i = \mathbf{e}_j \circ g_{ji}$ on U_{ij}. Show that $g_{ij}[A_j] = A_i$ for all $i, j \in I$. Show that the connection (E, ∇) is completely determined by the cover, the clutching maps g_{ij} and the matrices A_i for all $i, j \in I$. How does this data change if we change trivialisation over each open set: $\mathbf{e}_i \mapsto \mathbf{e}_i \circ h_i$ for some $h_i : U_i \to \mathrm{GL}_n(\mathbb{C})$?

Write $G = \mathrm{GL}_n(\mathbb{C})$, let $\Delta \subset \mathbb{C}$ be an open disk, and let $\mathcal{G} = \mathrm{Map}_{\mathrm{hol}}(\Delta, G)$ be the group of all holomorphic maps from Δ to G. Also write $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$ and let

$$\mathcal{A} = \{A = B(z)dz \;\big|\; B : U \to \mathfrak{g}\}$$

be the space of all holomorphic connections on the trivial bundle on the open disk Δ, so that B is a holomorphic map. Thus by the exercise above the group $\mathcal{G}$ acts on the space $\mathcal{A}$ by gauge transformations:

$$\mathcal{G} \times \mathcal{A} \to \mathcal{A}; \qquad (g, A) \mapsto g[A] = gAg^{-1} + (dg)g^{-1}.$$

Holomorphic connections are not interesting locally since they are all isomorphic:

Lemma 1.5. *$\mathcal{G}$ acts transitively on $\mathcal{A}$. In particular for any $A \in \mathcal{A}$ there is a $g \in \mathcal{G}$ such that $g[A] = 0$ (every holomorphic connection is locally isomorphic to the trivial connection).*

Proof. Given A we wish to find g so that $gAg^{-1} + (dg)g^{-1} = 0$. In other words $gA + (dg) = 0$. If we write $h = g^{-1}$ and use the useful fact that $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ then we want $h : U \to G$ so that

$$\frac{dh}{dz} = B(z)h$$

where $B = A/dz$ as usual. In classical language this equation just says that h is a “fundamental solution” (or “fundamental matrix”) of the linear system $d/dz - B$. (By definition this means that the columns of h make up a basis of solutions of the system.) It is a classical fact (Cauchy?) that holomorphic systems have fundamental solutions[1]. In fact its easy to construct a series solution term by term, and then one proves the resulting series solution converges. □

Let $E = \mathbb{C}^n \times \Delta$ denote the trivial bundle. Note that a fundamental solution h is the same thing as an isomorphism $(E, d = d - 0) \to (E, d - A)$ from the trivial connection to the connection $d - A$. This just says $h[0] = A$, i.e. $A = (dh)h^{-1}$ or $dh = Ah$.

In general an isomorphism from $(E, d - A_1) \to (E, d - A_2)$ is a section h of $\mathrm{Hom}(E, E)$ that is invertible and satisfies $h[A_1] = A_2$ i.e.

$$hA_1h^{-1} + (dh)h^{-1} = A_2$$

or in other words:

$$dh = A_2h - hA_1.$$

Indeed it is natural to define a connection $\mathrm{Hom}(\nabla_1, \nabla_2)$ on $\mathrm{Hom}(E, E)$, whose horizontal sections are given by this equation. Similarly if there are two different vector bundles, and one can thus define dual connections etc.

Of course we can consider holomorphic connections on punctured Riemann surfaces but that won’t capture most of the properties of the first (polynomial) example, and

[1] See e.g. classical ODE books by Hartman, Coddington–Levinson, Ince, Hille, ...

not all the properties of the second (Fuchsian) example. Instead we proceed as follows to encompass them.

Now let Σ be a compact Riemann surface and $\mathbf{a} = (a_1, \dots, a_m) \subset \Sigma$ a finite subset. Let $D = \sum n_i(a_i)$ be an effective divisor on Σ supported on $\mathbf{a}$, so that $n_i \geq 1$ are integers.

Definition 1.6. *A* meromorphic connection *with poles bounded by D is a pair (E, ∇) where $E \to \Sigma$ is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$-linear operator, from the sheaf of sections $\mathcal{E}$ of E to the sections of E twisted by meromorphic one-forms with poles bounded by D, such that the Leibniz rule is satisfied:

$$\nabla(fs) = (df)s + f\nabla(s) \tag{1.5}$$

for all local sections s of E and functions f.

In a local trivialisation of E, over some open subset $U \subset \Sigma$ the operator ∇ takes the form

$$\nabla = d - A$$

for a matrix of meromorphic one-forms A (with poles bounded by D). E.g. if $a_1 \in U$ and z is a local coordinate vanishing at a_1 then

$$\nabla = d - \frac{B(z)dz}{z^{n_1}}$$

in a neighbourhood of a_1, where B is holomorphic across a_1.

Remark 1.7. Note that:

1) $E \to \Sigma$ is a holomorphic vector bundle on the *compact* surface, so this is a genuine generalisation of a holomorphic connection.

2) this notion is well defined, but it would not be if D was not effective. The point is that the Leibniz rule tacitly uses the inclusion $\Omega^1 \subset \Omega^1(D)$ of the holomorphic one forms into the meromorphic one forms. If D was not effective, say $n_1 < 0$, then the Leibniz rule would not make sense (as there is then no such inclusion: df would not necessarily be a section of $\Omega^1(D)$).

3) this is not a completely standard definition (although we have been happily using it since 1999 or so). One can also define the notion of "meromorphic connection on a meromorphic bundle", where a "meromorphic bundle" is a locally free $\mathcal{O}(*D)$ module. In practice this means that one allows meromorphic gauge transformations with any order pole at the points of D. This definition is also useful, but is less convenient for gauge theory or moduli theory.

Remark 1.8. The nonabelian cohomology set $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathcal{O}))$ is *slick notation* for the set of isomorphism classes of rank n holomorphic vector bundles on Σ. Here $\mathcal{O}$ is the sheaf of holomorphic functions and $\mathrm{GL}_n(\mathcal{O})$ is the sheaf of holomorphic maps in to the group $\mathrm{GL}_n(\mathbb{C})$. The (Cech) definition of $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathcal{O}))$ involves equivalence classes of 1-cocycles, and this really is the same thing as expressing a vector bundle in terms of clutching maps, by choosing local trivialisations on each open set of an open covering (and the equivalence relation comes from changing the choice of trivialisation). See for example J. Frenkel's 1957 paper *Cohomologie non abélienne et espaces fibrés*:

3. *Cohomologie de dimension* 1. — Soit F un faisceau de *groupes* sur X, $\mathfrak{U} = \{U_i\}_{i \in I}$ un recouvrement ouvert de X. Nous dirons qu'une 1-cochaîne $f : (i, j) \to f_{ij}$ de $\mathfrak{U}$ à valeurs dans F est un 1-cocycle si l'on a

$$(3.1) \qquad f_{ij}(x) f_{jk}(x) = f_{ik}(x) \qquad \text{pour tout } x \text{ de } U_{ijk}.$$

Deux cochaînes $\{f_{ij}\}$, $\{g_{ij}\}$ seront dites *cohomologues* s'il existe une 0-cochaîne $h = \{h_i\}$ de $\mathfrak{U}$ à valeurs dans F telle que

$$(3.2) \qquad f_{ij}(x) = h_i^{-1}(x) g_{ij}(x) h_j(x) \qquad \text{pour tout } x \text{ de } U_{ij}.$$

La cohomologie est une relation d'*équivalence* dans $C^1(\mathfrak{U}, F)$ respectant l'ensemble des cocycles. L'ensemble $H^1(\mathfrak{U}, F)$ des classes de cocycles de $\mathfrak{U}$ à valeurs dans F qui sont cohomologues s'appelle le *premier ensemble de cohomologie du recouvrement* $\mathfrak{U}$ *à valeurs dans* F. Cet ensemble n'a de structure de groupe naturelle que si F est un faisceau de groupes *abéliens*, auquel cas c'est le premier groupe de cohomologie classique de $\mathfrak{U}$ à valeurs dans le faisceau abélien F. Il a cependant un élément privilégié, que nous conviendrons d'appeler l'*élément neutre* de $H^1(\mathfrak{U}, F)$, savoir la classe du cocycle

$$f_{ij}(x) = e_x \qquad \text{pour tout } x \text{ de } U_{ij},$$

où e_x est l'élément neutre de F_x [l'application $x \to e_x$ d'un ouvert U de X dans F est bien continue en vertu de l'axiome (II)].

FIGURE 1. The definition of $\mathrm{H}^1(\Sigma, \mathcal{G})$ in Frenkel 1957.

Later on we will need $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathbb{C}))$, which is slick notation for the set of isomorphism classes of local systems of n-dimensional complex vector spaces on Σ; the clutching maps on double intersections are now *constant* maps to $\mathrm{GL}_n(\mathbb{C})$. Of course it is very suggestive notation, and leads to the idea that moduli spaces of local systems should have other motivic incarnations analogous to the De Rham and Dolbeault approaches in the abelian case.

1.2. **Some variations: algebraicity.** Suppose Σ is actually a smooth compact complex algebraic curve.

•$v1$) Algebraic connections (E, ∇) (if E is algebraic and ∇ is algebraic). Thus there is a Zariski open covering $\Sigma = \bigcup U_i$ so that the restriction of E to each open set U_i is trivialisable. (Recall Zariski open subset are just the complements of finite subsets of points.) Then by choosing such trivialisations the bundle E is determined by algebraic clutching maps $g_{ij} : U_{ij} \to G$, where $U_{ij} = U_i \cap U_j$. Then on U_i we have $\nabla = d - A_i$ where A_i is a matrix of regular differentials (algebraic one-forms) on U_i. On the double intersections the A_i are related by gauge transformations as usual

$$g_{ij}[A_j] = A_i.$$

These are just the algebraic version of holomorphic connections. In fact some form of GAGA implies the analytification functor gives an equivalence of categories (Algebraic connections) $\to$ (holomorphic connections), in this setting where Σ is compact.

•$v2$) Similarly there is a notion of "Algebraic meromorphic connections", as above but allowing the A_i to be matrices of rational differentials (algebraic one-forms with poles), with poles bounded by the fixed effective divisor D. Again a version of GAGA implies the analytification functors gives an equivalence (to the meromorphic connections on holomorphic vector bundles). These will actually be the realm for most of our examples, with nonlinear differential equations flowing in their spaces of coefficients.

•$v3$) If we now take $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ to be an open curve (in fact any smooth complex algebraic curve takes this form for some finite set $\mathbf{a}$). Then we can consider algebraic connections $(E, \nabla) \to \Sigma^\circ$ on the open curve. This category is actually very close to being a subcategory of the category of meromorphic connections on holomorphic vector bundles on Σ with poles on $\mathbf{a}$ (if we allow any pole orders). There is in fact a version of GAGA that shows this category is equivalent to the "meromorphic connections on meromorphic bundles" on Σ with any order poles on $\mathbf{a}$.

In the next lecture we will consider holomorphic connections on vector bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$; this is relatively trivial and all the extra structure "hidden" in the poles at the punctures is lost. However we won't get to hermitian metrics for some time so its worth noting here:

> The relevance in "hardcore analytic" gauge theory (on the punctured surface), of having (meromorphic connections on) holomorphic vector bundles on the compact surface, comes from the fact that the addition of a hermitian metric controls the growth of sections at the punctures, leading to preferred extensions across the punctures, and thus holomorphic vector bundles (or parabolic vector bundles) on the compact surface, and in turn this leads to algebraicity.

1.3. **Some more variations: Higgs bundles and ζ-connections.**

Definition 1.9. *A* <u>meromorphic Higgs bundle</u> *with poles bounded by D is a pair (E, Φ) where $E \to \Sigma$ is a holomorphic vector bundle and* $\Phi \in \mathrm{H}^0(\Sigma, \mathrm{End}(E) \otimes \Omega^1(D))$

is the Higgs fields, a meromorphic one-form with values in $\mathrm{End}(E)$*, with poles bounded by* D*.*

Thus locally we can write $\Phi = Bdz$ for a matrix B of meromorphic functions on U.

In a sense Higgs bundles have two origins: just as an operator $d/dz - B$ led to a connection $d - Bdz$, any matrix $L(z)$ of rational functions (aka a "rational Lax matrix") leads to a Higgs field $L(z)dz$ (on the trivial vector bundle on $\mathbb{P}^1$). On the other hand holomorphic Higgs fields on higher genus Riemann surfaces were introduced by Hitchin and Simpson. These two viewpoints were "put together" in the definition of meromorphic Higgs bundle (Nitsure, Bottacin, Markman, ...).

Exercise 1.10. Suppose ∇_1, ∇_2 are meromorphic connections on $E \to \Sigma$ with poles on D. Show that $\Phi := \nabla_1 - \nabla_2$ is a meromorphic Higgs field.

Now choose a complex number $\zeta \in \mathbb{C}$.

Definition 1.11. *A* meromorphic ζ-connection *with poles bounded by* D *is a pair* (E, ∇) *where* $E \to \Sigma$ *is a holomorphic vector bundle, and*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$$

is a $\mathbb{C}$*-linear operator, from the sheaf of sections* $\mathcal{E}$ *of* E *to the sections of* E *twisted by meromorphic one-forms with poles bounded by* D*, such that the* ζ*-Leibniz rule is satisfied:*

$$\nabla(fs) = \zeta(df)s + f\nabla(s) \tag{1.6}$$

for all local sections s *of* E *and functions* f*.*

Exercise 1.12. Study ζ-connections in a local trivialisation, and show that the gauge action is modified to: $g[A]_\zeta = gAg^{-1} + \zeta(dg)g^{-1}$.

Exercise 1.13. Show that for $\zeta = 0$ a meromorphic ζ-connection is the same thing as a meromorphic Higgs bundle.

Exercise 1.14. Show that for $\zeta = 1$ a meromorphic ζ-connection is the same thing as a meromorphic connection.

Thus there is a "continuous deformation" from connections to Higgs bundles.

Exercise 1.15. Write down the algebraic versions of the definitions of Higgs bundles and ζ-connections.

This is often referred to as the "autonomous limit" in the integrable systems literature. We will eventually see it is the same as the "Painlevé simplification" of the Painlevé equations, and see it as a hyperkähler rotation.

Of course, physicists have been putting numbers (like $\hbar$) in front of their differential operators for a long time.

And it is essentially the same as the deformation from loop algebras into affine Kac–Moody algebras (although in the full story there is also a central extension, dual to this deformation).

Remark 1.16. Note that for any line bundle $\mathcal{L} \to \Sigma$ one can define a $\mathcal{L}$-valued Higgs bundle as pair (E, Φ) with Φ a section of $\mathrm{End}(E) \otimes \mathcal{L}$. However unless this is secretly a meromorphic Higgs bundle (i.e. there is an isomorphism $\mathcal{L} \cong \Omega^1(D)$ for some effective D) then there is no analogous notion of "$\mathcal{L}$-valued connections" (Rmk. 1.7 2).

2. Sketch of big picture

2.1. **Three algebraic worlds:** Before delving into the details lets try to signpost where we want go (at least symbolically for the moment). Much of the story we want to describe can be summarised in the (slightly oversimplified) diagram:

Dolbeault		De Rham		Betti
Rational Lax matrices L $\dot{L} = [P, L]$		Rational diff. op.s $\frac{d}{dz} - B$		Stokes and monodromy data
$\cap$		$\cap$		$\cap$
Mero. Higgs bundles (E, Φ)	$\overset{\text{wnAbH}}{\longleftrightarrow}$	Mero. connections (E, ∇)	$\overset{\text{RHB}}{\longleftrightarrow}$	Stokes local systems
$\cup$		$\cup$		$\cup$
Holom. Higgs bundles (E, Φ)	$\overset{\text{nAbH}}{\longleftrightarrow}$	Holom. connections (E, ∇)	$\overset{\text{RH}}{\longleftrightarrow}$	Local systems/ π_1-rep.s

The main aim is to describe the central row, and, as the diagram indicates, it is set-up to include both the rich class of examples of rational Lax matrices and the sophisticated nonabelian Hodge setting of holomorphic Higgs bundles (no poles), related to the Hitchin integrable systems.

Just as an operator $\frac{d}{dz} - B$ becomes a connection by multiplying by dz (and one can study its *isomonodromic deformations*), a rational Lax matrix $L(z)$ becomes a Higgs field Ldz by multiplying by a rational one-form, such as dz. The Lax matrices appear in Lax equations, which are equation of the form $\dot{L} = [P, L]$ controlling *isospectral deformations* of L, and are the bread and butter of the theory of integrable systems; the solution of the system comes from a straight line flow on the Jacobian of the spectral curve defined by $\det(L - \lambda) = 0$. We will discuss some examples in detail but for now note there are lots, as listed for example in the book of Babelon et al, or basic sources such as:

Adler and van Moerbeke (1980) Completely integrable systems, Euclidean Lie algebras, and curves, Adv. in Math. 38, no. 3, 267-317.

Reyman and Semenov-Tian-Shansky (1994) Integrable systems II group theoretical methods in the theory of finite dimensional integrable systems

Phillip Griffiths (1985) Linearizing Flows and a Cohomological Interpretation of Lax Equations

Mumford (1984) "Tata Lectures on Theta II, Jacobi theta functions and differential equations" (this book is devoted to a class of examples involving 2×2 Lax matrices).

Dubrovin–Krichever–Novikov (1985) Integrable systems I

Adams–Harnad–Previato (1988) Isospectral flows in finite and inifinite dimensions

The first large class of examples seems to be due to Garnier 1919 (and we will discuss the "Painlevé simplification" method he used to discover them, taking the autonomous limit of the Schlesinger equations):

Garnier (1919) Sur une classe de systèmes différentiels abéliens déduits de la théorie des équations linéaires, (Rendiconti del Circolo Matematico di Palermo 43, pp.155-191).

Several of the key ideas of Garnier's paper were rediscovered as an offshoot of soliton theory, before Garnier's work was rediscovered and widely disseminated[2], around 1980.

[2]E.g. there is a section on it in the well-known paper of Flaschka-Newell on isomonodromy (Comm. Math. Phys. 76 (1980), 65-116), and it is mentioned in Dubrovin's 1981 paper on theta functions, the 1980 Krichever-Novikov review (Russian Math. Surveys 35:6 (1980), 53-79) and in the footnote p.156 of the 1980 paper of Jimbo-Miwa-Mori-Sato. D.V. Chudnovsky wrote a paper on it (Let. Nuovo Cimento 26 (14) 1979), and M. Gaudin cited that in his 1983 book (La fonction d'onde de Bethe), having discovered the quantum version in 1976.

2.2. **Two organisational diagrams.** Mathematically this story leads to an interesting class of *moduli spaces*, i.e. spaces whose points correspond to isomorphism classes of certain meromorphic connections (or Higgs bundles, or Stokes local systems). This goes slightly beyond the objects usually studied by algebraic geometers, and one of the main inputs is to write down the moduli problem that encompasses this picture.

In particular we will fix a Riemann surface, some marked points and some precisely defined boundary data. This will determine a hyperkähler manifold $\mathfrak{M}$ with three preferred algebraic structures, corresponding to the three columns of the above table. We label the columns "Dolbeault, De Rham, Betti" as they are precise analogues of the Dolbeault, De Rham and Betti approaches to linear cohomology (it was first abstracted to the context of nonabelian cohomology by Simpson, and then later extended to the meromorphic case relevant to Lax matrices). The result, to be explained, is a diagram as follows:

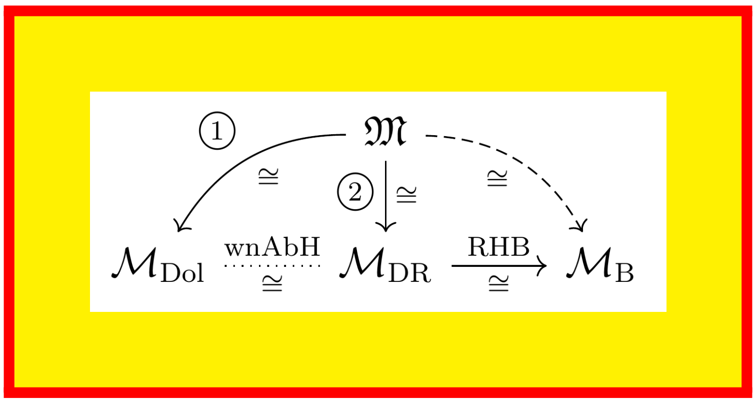

FIGURE 2. Nonabelian Hodge space $\mathfrak{M}$, with three preferred algebraic structures.

However this does not capture the full story and in practice people work with simpler open parts $\mathcal{M}^*$ of the moduli spaces in genus zero, where things can be made explicit, and actual nonlinear differential equations can be obtained. We will explain that the classical Riemann–Hilbert map is a holomorphic map

$$\boxed{\mathcal{M}^* \hookrightarrow \mathcal{M}_{\mathrm{B}}} \tag{2.1}$$

where $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}} \subset \mathcal{M}_{\mathrm{DR}}$ is the open part of the full De Rham moduli space where the bundles E are trivial. As we will see the spaces $\mathcal{M}^*$ have the flavour of the "Lie algebra" of the full nonabelian Hodge space $\mathfrak{M} \cong \mathcal{M}_{\mathrm{B}}$, and the Riemann–Hilbert map is a natural generalisation of the exponential map.

However this still does not capture the full story as we also wish to *vary the modular parameters*, changing the complex structure on the Riemann surface, the pole positions, and the "irregular class" of the connections at each pole. These parameters will lead to the independent variables ("times") in the isomonodromy equations.

3. Glimpses of the elephant

The next few sections will describe a few simple pieces of the full picture, that provided motivation.

3.1. **Painlevé's deformation of the theory of elliptic functions.**

Painlevé discovered most of the Painlevé equations as deformations of differential equations for elliptic functions, i.e. as equations that limit to equations for elliptic functions. He used the term "simplification" (*simplifié*) for the limiting differential equation, solvable in terms of elliptic functions.

In more detail Painlevé was looking for new special functions, defined as solutions to non-linear algebraic differential equations. He looked for equations whose solutions had good meromorphic continuation properties: outside a fixed critical set, any local solution should have arbitrary meromorphic continuation. If $D \subset \mathbb{C}$ is the fixed critical set (a finite set in all examples here), then any local solution $y(t)$ should extend to a meromorphic function on the universal cover $\widetilde{\mathbb{C} \setminus D}$. This is known as the Kowalevski–Painlevé (KP) property (and can also be expressed as saying there are no "movable singularities" apart from poles).

The KP property is preserved under any deformation of the differential equation[3]. Thus to rule out many possible forms of differential equations, Painlevé would add parameters by hand and then take limits to get simpler equations (Painlevé's α method). If he could recognise or prove the limiting equation did not have the KP property then he could ignore the putative equation, and thus get a short list of possibilities, that then could be proved to have the KP property directly.

For example $\mathrm{P_I}$, the first Painlevé equation, $y'' = 6y^2 + t$ is a deformation of the equation for the Weierstrass $\wp$ function:

First recall the standard differential equation satisfied by $\wp$ is

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

for constants $g_2, g_3 \in \mathbb{C}$. Thus $2\wp'\wp'' = 12\wp^2\wp' - g_2\wp'$ so that

$$\wp'' = 6\wp^2 - g_2/2$$

Lemma 3.1 (cf. Painlevé 1900 p.226, Ince [] pp.321 and 329)**.** *Suppose $y(t)$ satisfies $\mathrm{P_I}$ so that $y'' = 6y^2 + t$. If $t = \alpha x, y = w(x)/\alpha^2$ for a constant α then*

$$w'' = 6w^2 + \alpha^5 w. \tag{3.1}$$

In particular if $\alpha^5 = 1$ then this is a symmetry of $\mathrm{P_I}$. If we take the limit $\alpha \to 0$ then we get $w'' = 6w^2$. This integrates once to $(w')^2 = 4w^3 + c$, which can be solved

[3]see e.g. paragraph 1 p.319 in Ince's book "ordinary differential equations" [?].

in terms of the Weierstrass $\wp$ function: $w = \wp(x+k)$ (where $\wp$ has $g_2 = 0, g_3 = -c$, and $k \in \mathbb{C}$ is arbitrary).

Proof. Write $v = y'$ so that $dy = vdt, dv = (6y^2+t)dt$. Now put $t = \alpha x, y = w/\alpha^2$ (as on Painlevé 1900 p.226, Ince p.329 [], or Valiron p.410 []). Thus $dt = \alpha dx, dy = dw/\alpha^2$, so $w' = dw/dx = \alpha^3 dw/dz = \alpha^3 v$. Thus

$$dv = (6y^2+t)dt = \alpha(6w^2/\alpha^4 + \alpha w)dx$$

and so $w'' = \alpha^3 v' = \alpha^5 w + 6w^2$ which is (3.1). The last statement is straightforward, recalling that in general $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ for constants g_2, g_3. □

Thus Painlevé discovered a natural deformation of the theory of elliptic functions!

—see Alves `https://arxiv.org/abs/2103.02697v1` §3 for a recent discussion of this story.

The next example is the case of Painlevé II:

Lemma 3.2. *Suppose $y(t)$ satisfies the $\mathrm{P_{II}}$ equation $y'' = 2y^3 + ty + \alpha$. If $t = \gamma x, y = w(x)/\gamma$ for a constant γ then*

$$w'' = 2w^3 + \gamma^3 xw + \gamma^2\alpha. \tag{3.2}$$

Thus if we now take the limit $\gamma = 0$ then $w'' = 2w^3$, which integrates once to $(w')^2 = w^4 + c$, and can be solved in terms of the Jacobi sn function: $w = c_1\mathbf{sn}(c_1(ix+c_2), i)$.

Proof. Write $v = y'$ so that $dy = vdt, dv = (2y^3 + ty + \alpha)dt$. Now put $t = \gamma x, y = w/\gamma$. Thus $dt = \gamma dx, dy = dw/\gamma$, so $w' = dw/dx = \gamma^2 dw/dz = \gamma^2 v$. Thus

$$dv = (2y^3 + ty + \alpha)dt = \gamma(2w^3/\gamma^3 + xw + \alpha)dx$$

and so $w'' = \gamma^2 v' = 2w^3 + \gamma^3 xw + \gamma^2\alpha$ which is (3.2). The last statement is straightforward. □

In this way Painlevé discovered some very interesting nonlinear differential equations, the Painlevé equations 1,2,3,4.

Later on (late 1970s) the Painlevé equations, and their solutions, the Painleve transcendents, started appearing in physics problems such as the Ising model[4] (in some sense physics got sufficiently nonlinear to catch up with the mathematics...).

Note that so-far these equations have no link to gauge theory: there are no linear differential equations in the story. That link came about via a 1905 paper of R. Fuchs where he discovered a new Painlevé equation, called Painlevé six, $\mathrm{P_{VI}}$, controlling the

[4] E.g. Wu-McCoy-Tracy-Barouch (1976) "Spin-spin correlation functions for the two-dimensional Ising model, Exact theory in the scaling region"

isomonodromic deformations of a linear differential equation. This is a completely different way to get nonlinear differential equations[5].

One can find the standard list of Painlevé equations in many places (e.g. wikipedia), but we really want to think of them as geometric objects, and this is obscure in their explicit expression. They will each lead to a deformation class of nonabelian Hodge spaces of complex dimension two, the minimal possible nonzero dimension, so they give the simplest examples.

The basic features are summarised in the table below:

Painlevé equation:	1	2	3	4	5	6
Domain of t:	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{C} \setminus \{0,1\}$
No. of constant parameters:	0	1	2	2	3	4
(Affine Dynkin) Diagram :	$\widehat{A}_0$	$\widehat{A}_1$	$\widehat{D}_2$	$\widehat{A}_2$	$\widehat{A}_3 = \widehat{D}_3$	$\widehat{D}_4$
Okamoto Diagram :	$\widehat{E}_8$	$\widehat{E}_7$	$\widehat{D}_6$	$\widehat{E}_6$	$\widehat{D}_5$	$\widehat{D}_4$

TABLE 1. Basic data for Painlevé equations

(Here we have omitted two degenerate versions of Painlevé 3.)

The diagrams can be drawn as follows (the number of nodes is one plus the number of constants):

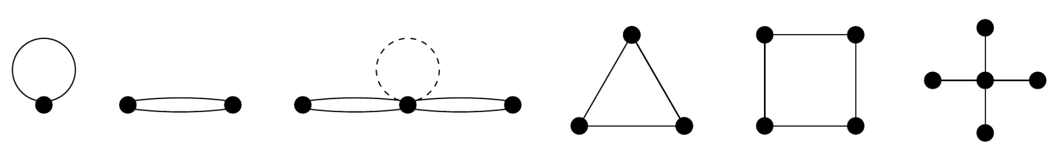

FIGURE 3. The diagrams of the six Painlevé equations.

[5]R.Fuchs' isomonodromy approach was extended to the original Painlevé equations by Garnier 1912 (so they too, in fact, are gauge theoretic equations).

3.2. Towards the Painlevé VI connections.

The 1905 paper of R. Fuchs[6], should probably be viewed as the true "start of 2d gauge theory" where a nonlinear differential equation arose naturally, controlling a linear differential equation (i.e. where the "unknown" is really a linear differential equation $\sim$ a meromorphic connection on a rank two vector bundle on $\mathbb{P}^1$). The underlying idea can be traced back to a suggestion of Riemann 1857.[7]

What is Painlevé VI, the Fuchsian Painlevé equation?

Definition 3.3. *Given constants $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the corresponding Painlevé VI equation $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$ is the algebraic differential equation:*

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

for a meromorphic function $y(t)$ where $t \in \mathbb{C} \setminus \{0,1\}$.

This frankly horrific expression does not express very well the true beauty of the underlying geometric object. The simplest encoding of it seems to be the following time-dependent Hamiltonian formulation, due to Malmquist 1922.

Proposition 3.4 (cf. [?] p.86). *If $a_1, a_2, a_3, b \in \mathbb{C}$ then the function $H(q,p,t)$ defined by*

$$t(t-1)H(q,p,t) = q(q-t)(q-1)\left(p^2 + p\left(\frac{a_1}{q} + \frac{a_2}{q-t} + \frac{a_3}{q-1}\right)\right) + b \cdot q$$

is a time-dependent Hamiltonian function for $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$, in the sense that if

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

then $y = q(t)$ is a solution to $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$ where

$$\alpha = (a_1 + a_2 + a_3)^2/2 - 2b, \ \beta = -a_1^2/2, \ \gamma = a_3^2/2, \ \delta = -a_2(a_2-2)/2.$$

Proof. These are a pair of coupled first order nonlinear differential equations. The first equation gives a direct relation between p and $q' = dq/dt$, and using this the second equation then yields a second order non-linear differential equation for q''. A direct computation (best done with a computer algebra package) shows this is $\mathrm{P}_{\mathrm{VI}}(\alpha, \beta, \gamma, \delta)$, with y replaced by q. $\square$

[6] https://webusers.imj-prg.fr/~philip.boalch/files/fuchs.r_1905_surquelquesequationsdifferentielleslineairesdusecondeordre_CRAS

[7] see the historical discussion in Jimbo–Miwa–Ueno 1981.

The modern geometric viewpoint on this (Schlesinger, Jimbo–Miwa–Ueno, Malgrange, Okamoto) goes as follows[8]:

Let $G = \mathrm{SL}_2(\mathbb{C}), \mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ (using $\mathrm{GL}_2(\mathbb{C})$ gives nothing extra)

$t \in \mathbb{B} := \mathbb{C} \setminus \{0, 1\}$

Thus the choice of t determines a four-tuple of points: $\mathbf{a} = \mathbf{a}(t) = (0, t, 1, \infty) \in (\mathbb{P}^1)^4 \setminus$ diagonals, where $\mathbb{P}^1$ is the Riemann sphere.

We want to consider simple moduli spaces $\mathcal{M}^* = \mathcal{M}^*_{\mathrm{DR}}$ of meromorphic connections on trivial vector bundles on $\mathbb{P}^1$ with poles at $D := \mathbf{a}$. They are Fuchsian systems, of the form

$$\nabla = d - A, \qquad A = \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1}\right) dz$$

where $A_i \in \mathfrak{sl}_2(\mathbb{C})$ are trace-less 2×2 matrices. This has a further pole at ∞ with residue $A_4 := -(A_1 + A_2 + A_3)$, so that

$$\sum_1^4 A_i = 0. \tag{3.3}$$

Two such Fuchsian systems are isomorphic if they are related by a global gauge transformation $g : \mathbb{P}^1 \to \mathrm{GL}_2(\mathbb{C})$. Any such holomorphic map is constant so the set of isomorphism classes is just the quotient by the conjugation action: $g[A] = gAg^{-1}$. Generically the projective group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$ acts freely so a rough dimension count shows the space of isomorphism classes of such Fuchsian systems should have dimension $3.3 - 3 = 6$ (there are 3 independent residues, and $\dim(\mathrm{PSL}_2(\mathbb{C})) = 3$).

To reduce the dimension we notice the action is really just conjugating the residues A_i, so we can fix their adjoint orbits.

Choose $\lambda_i \in \mathbb{C}$ for $i = 1, 2, 3, 4.$ and let

$$\mathcal{O}_i = \left\{ g \begin{pmatrix} \lambda_i & 0 \\ 0 & -\lambda_i \end{pmatrix} g^{-1} \;\middle|\; g \in \mathrm{SL}_2(\mathbb{C}) \right\} \subset \mathfrak{g}$$

be the adjoint orbit of matrices with eigenvalues $\pm\lambda_i$. We will assume $2\lambda_i$ is not an integer, so in particular $\mathcal{O}_i$ has complex dimension 2.

Then we can look at the set of isomorphism classes of such Fuchsian systems with $A_i \in \mathcal{O}_i$ for $i = 1, 2, 3, 4$.

$$\mathcal{M}^*(t) := \{A \mid A_i \in \mathcal{O}_i\}/\mathrm{SL}_2(\mathbb{C})$$

[8]rewritten in terms of moduli spaces, and Ehresmann connections, as in P.B. Adv. Math. 2001: https://webusers.imj-prg.fr/~philip.boalch/files/smid.pdf.

It turns out that if the constants $\boldsymbol{\lambda} = \{\lambda_i\} \in \mathbb{C}^4$ are off of some hyperplanes then the projective group $\mathrm{PSL}_2(\mathbb{C})$ acts freely and the quotient is an algebraic variety of dimension

$$4 \times 3 - 2 \times 3 = 2$$

so it is a complex surface[9] . Of course really this space does not depend on t and is described directly in terms of the residues.

Define a map

$$\mu : \mathcal{O}_1 \times \cdots \times \mathcal{O}_4 \to \mathfrak{g}; (A_1, \ldots, A_4) \mapsto \sum A_i.$$

Then we can write:

$$\mathcal{M}^* \cong \mu^{-1}(0)/G =: (\mathcal{O}_1 \times \cdots \times \mathcal{O}_4) /\!\!/ G$$

where the double slash $/\!\!/$ is just notation for the subquotient $\mu^{-1}(0)/G$, i.e. we consider the subvariety $\mu^{-1}(0)$ inside $\mathcal{O}_1 \times \cdots \times \mathcal{O}_4$ and then quotient that by G. (We will later see this is an example of a holomorphic symplectic quotient.)

Now we vary $t \in \mathbb{B} := \mathbb{C} \setminus \{0, 1\}$ and look at the relative situation. Thus we define a fibre bundle

$$\underline{\mathcal{M}}^* \to \mathbb{B}$$

such that the fibre over $t \in \mathbb{B}$ is the space $\mathcal{M}^*(t)$. This fibre bundle is trivial, it is just the product $\underline{\mathcal{M}}^* = \mathcal{M}^* \times \mathbb{B}$, since as we saw above the spaces $\mathcal{M}^*$ do not depend on t.

Now, geometrically, the Painevé VI equation that R. Fuchs discovered is a (nonlinear) Ehresmann connection on this bundle $\underline{\mathcal{M}}^*$, and the independent variable (the time) is the parameter t running over $\mathbb{B}$. It is a *second order* nonlinear differential equation, as the fibres have dimension 2.

—Quick aside on Ehresmann connections:

Suppose $\mathbb{B}$ is a complex manifold and $\pi : M \to \mathbb{B}$ is a fibre bundle, with fibres $M_t = \pi^{-1}(t)$ for $t \in \mathbb{B}$.

Definition 3.5. *A (holomorphic) Ehresmann connection on the bundle M is the choice, for any $p \in M$ of a linear subspace $H_p \subset T_pM$ that is transverse to the vertical subspace V_p, the tangent space of the fibres $V_p = \mathrm{Ker}(d\pi_p) \subset T_pM$, so that*

$$H_p \oplus V_p = T_pM$$

for all $p \in M$. These subspace should vary holomorphically (so the H_p form a holomorphic vector bundle on M, a subbundle of the tangent bundle TM).

[9] more on these hyperplanes (and surfaces) later, but the impatient could read section 2 of https://arxiv.org/pdf/0706.2634

If $U \subset \mathbb{B}$ then a local section $s : U \to M$ is *horizontal* if it is tangent to the Ehresmann connection, i.e. for any $t \in U$ and tangent vector $v \in T_t\mathbb{B}$ the corresponding vector $ds(v) \in T_pM$ is actually in the subspace $H_p \subset T_pM$, where $p = s(t) \in M$.

In brief whereas a Koszul connection on a vector bundle encodes linear differential systems in an intrinsic way, the notion of Ehresmann connection encodes non-linear differential equations. An Ehresmann connection is "complete" if any path in $\mathbb{B}$ between any two points $t_1, t_2 \in \mathbb{B}$ has a unique horizontal lift to a path in M starting at any point $p \in M_{t_1}$. Some authors put this condition in the definition of Ehresmann connection, but we will not.)

—

In our setting we can thus speak of the Painlevé VI connections, and then choose explicit coordinates to get the explicit differential equation. It is really the Ehresmann connection (or rather its extension from $\mathcal{M}^*$ to $\mathcal{M}_{\mathrm{DR}}$) that is the geometric object we want to understand.

There are two ways to get the Painlevé VI connection, and we'll just mention them here, and explain the details once we have set up the background:

1) De Rham approach, via Schlesinger's equations.

2) Betti approach passing to the other side of Riemann–Hilbert. In brief the corresponding character varieties $\mathcal{M}_{\mathrm{B}}$ also form a bundle $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$. However this bundle is not naturally trivial, but it is *canonically locally trivial*: if we choose any disk $\Delta \subset \mathbb{B}$ then there is a canonical identification of the fibres $\mathcal{M}_{\mathrm{B}}(t_1) \cong \mathcal{M}_{\mathrm{B}}(t_2)$ for $t_1, t_2 \in \Delta$ (this identification depends on the choice of the disk). This structure is encoded in the sentence:

"The spaces $\mathcal{M}_{\mathrm{B}}(t)$ form a *local system of varieties* over $\mathbb{B}$".

This will be spelt out in great detail, but for now we just note that implies that the bundle $\underline{\mathcal{M}}_{\mathrm{B}} \to \mathbb{B}$ has a natural complete flat Ehresmann connection. We can transfer this to the bundle $\underline{\mathcal{M}}^* \to \mathbb{B}$ and rewrite it in carefully chosen algebraic coordinates there to get a nonlinear differential equation, P_{VI}.

4. Lecture 2: Flat connections and the compact case

As we explained the general notion of meromorphic connection is essentially the simplest context that contains the three basic classes of connections, namely

- the polynomial connections,
- the Fuchsian systems, and
- the holomorphic connections on higher genus compact Riemann surfaces.

In this lecture we will discuss this last case in detail, and the corresponding monodromy data. This case is especially nice since it avoids discussing boundary conditions.

4.1. **The example of compact Riemann surfaces (no poles).**

Our first aim is to explain all the definitions and sketch some of the ideas of the proof the following statement:

Theorem 4.1. *Suppose Σ is a smooth compact complex algebraic curve. The following categories are equivalent (via specific functors that we will describe):*

1) Algebraic connections on algebraic vector bundles on Σ,

2) Holomorphic connections on holomorphic vector bundles on Σ,

3) Flat C^∞ connections on C^∞ complex vector bundles on Σ,

4) Local systems of finite dimensional complex vector spaces on Σ,

5) For any fixed basepoint $b \in \Sigma$, the category of finite dimensional complex $\pi_1(\Sigma, b)$ representations.

This has numerous consequences, for example: The equivalence 1) $\iff$ 5) gives a purely algebraic way to access the topological fundamental group (this is an example of the change in algebraic structure given by Riemann–Hilbert). The equivalence 3) $\iff$ 5) gives a completely explicit way to classify the set of solutions of a nonlinear differential equation. For example we will deduce the corollary:

Corollary 4.2. *For any integer $n \geq 1$ the set of isomorphism classes of rank n objects (in any of the five categories in the theorem) is naturally in bijection with the set of orbits of an action of the complex algebraic group $G = \mathrm{GL}_n(\mathbb{C})$ on an affine algebraic variety $\mathcal{R}$. Explicitly G acts by conjugation on the* <u>representation variety</u>*:*

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, b), G).$$

By performing this quotient in an algebraic fashion, this will lead to the first example of Betti moduli space $\mathcal{M}_{\mathrm{B}}$ (the character variety), and thus will give the simplest instance of the association of a variety $\mathcal{M}_{\mathrm{B}}$ with the choice of a surface Σ and a group G (no boundary conditions).

What is an *equivalence of categories*?
A functor $F : X \to Y$ between two categories is an equivalence of categories if 1) it is essentially surjective, and 2) it is fully faithful. This gives a convenient/precise/flexible language to see some things are "more or less the same".

1) means that for each object $y \in Y$ there exists is an object $x \in X$ and an isomorphism $y \cong F(x)$.

2) means that for any $x_1, x_2 \in X$ the functor F maps the space $\mathrm{Hom}_X(x_1, x_2)$ of morphisms (in X) bijectively onto the space of morphisms $\mathrm{Hom}_Y(F(x_1), F(x_2))$ between their images in Y.

In particular the choice of an equivalence F induces a bijection between the sets of isomorphism classes in X and Y. For more details and other formulations, see e.g. p.71 of Gelfand–Manin (Methods of homological algebra []).

The definitions of 1) and 2) have already been covered. The functor 1) $\to$ 2) is analytification $(E, \nabla) \mapsto (E^{an}, \nabla^{an})$ (An algebraic vector bundle is a special type of holomorphic vector bundle; algebraic clutching maps are in particular holomorphic. In terms of sheaves of sections we just take the holomorphic sections of the algebraic bundle E. Then the action of ∇^{an} on holomorphic sections is completely determined by the Leibniz rule $\nabla(fs) = (df)s + f\nabla(s)$ for holomorphic f and algebraic s). The equivalence between them is a version of GAGA, since Σ is compact (see e.g. Malgrange [**?**] p.152).

For 3), the definition of C^∞ connections is straightforward but it is worth noting that there is now an integrability condition: the connections should be *flat*, i.e. have vanishing curvature.

This is actually one of the central ideas (probably *the* central idea) in the subject of integrable systems, that the vanishing of curvature is a nonlinear differential equation, and this is the key mechanism how linear connections lead to nonlinear differential equations.

Given a connection $\nabla = d + A$ on a trivial vector bundle, then its curvature is the matrix of two-forms $\Omega = \nabla^2 = dA + A^2$. Here in the $\mathfrak{gl}_n(\mathbb{C})$ setting A^2 is well defined as a matrix of two-forms. In general (for other Lie algebras) we just define $A^2 = [A, A]/2$ and use the same notation. But we really want to see what this means:

Suppose the base is two-dimensional with coordinates x, y, and we write

$$A = X(x, y)dx + Y(x, y)dy$$

for matrix valued functions X, Y.

Then the key computation to do is to compute the commutator:

$$\left[\frac{\partial}{\partial x}+X,\frac{\partial}{\partial y}+Y\right] \;=\; \frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}+[X,Y] \tag{4.1}$$

and note that the connection $d+A$ is flat if and only if this commutator is zero:

Exercise 4.3. Show that $dA+A^2$ is proportional to the commutator above times $dx\wedge dy$.

Thus the flatness condition is a nonlinear (quadratic) differential equation:

$$\frac{\partial X}{\partial y}=\frac{\partial Y}{\partial x}+[X,Y].$$

For example, as we will see, all the isomonodromy equations arise from such curvature equations, and the isospectral (Lax) equation are autonomous limits of them (passing via ζ connections in one direction, to remove one of the derivatives).

Now to pass from 2) to 3) we just take the underlying C^∞ vector bundle, and use the Leibniz rule to define the action of ∇ on any C^∞ section $\nabla(fs)=(df)+f\nabla(s)$ for C^∞ function f and holomorphic sections s. Here d is the full exterior derivative on Σ and will have both $dz,d\overline{z}$ terms in general. Any local C^∞ section can be written as $\sum f_is_i$ for C^∞ functions and holomorphic sections. The resulting connection is clearly flat as any holomorphic connection on a Riemann surface is flat (and thus any gauge transformation of it too: $\Omega\mapsto g\Omega g^{-1}$ under a gauge transformation g).

To go backwards we need to see how a flat C^∞ connection determines the structure of holomorphic vector bundle. Locally such a connection has the form:

$$d-\alpha=(\partial-\alpha^{1,0})+(\overline{\partial}-\alpha^{0,1})$$

so the $0,1$ part is $\overline{D}:=\overline{\partial}-\alpha^{0,1}=\overline{\partial}-Bd\overline{z}$ for some matrix B of C^∞ functions.

A theorem of Koszul–Malgrange says that this determines the structure of holomorphic vector bundle, with the "holomorphic sections" defined to be the sections in the kernel of $\overline{D}$:

Theorem 4.4 ([?]). *If B is an $n\times n$ matrix of complex C^∞ functions on a disk Δ and $\overline{D}:=\overline{\partial}-Bd\overline{z}$, then there is a basis of sections $s_1,\dots,s_n$ in the kernel of $\overline{D}$. This implies the kernel of $\overline{D}$ is the sheaf of sections of a holomorphic vector bundle; a [locally] free $\mathcal{O}$-module, where $\mathcal{O}$ is the sheaf of holomorphic functions.*

If $g:\Delta\to G$ is the matrix with columns $s_1,\dots,s_n$ then $\overline{\partial}g=Bgd\overline{z}$ so that

$$g[0]^{0,1}:=g(0)g^{-1}+(\overline{\partial}g)g^{-1}=(\overline{\partial}g)g^{-1}=Bd\overline{z},$$

i.e. the inverse of g gives a gauge transformation converting $\overline{D}$ into $\overline{\partial}$, the "trivial" $\overline{\partial}$-operator.

If we pass to such a holomorphic basis then the flat connection will become a connection with zero $0,1$ part, so of the form

$$\nabla = d - Cdz$$

for a matrix C of C^∞ functions: since it is still flat the matrix C is actually holomorphic: $\nabla^2 = (dC)dz = (\overline{\partial}C)dz$ and the vanishing of this means that C is holomorphic. So we get a holomorphic connection on the holomorphic bundle determined by $\overline{D} = \nabla^{0,1}$. This is how to pass back and forth between holomorphic and flat C^∞ connections.

Of course in the current setting of flat connection we could bypass this and note that flatness implies that any flat connection $d - \alpha$ is locally trivial and has a basis of horizontal sections (the clutching map between such bases will be constant and so in particular holomorphic). This is the *nonabelian Poincaré lemma* (with one-forms replaced by connections and closedness by flatness):

Theorem 4.5. *Any flat connection has a fundamental solution (basis of horizontal sections) when restricted to any disk. In other words a (nonsingular) connection is flat if and only if it is locally isomorphic to the trivial connection. Explicitly in the current setting: If B, C are $n \times n$ matrices of complex C^∞ functions on a disk Δ and*

$$\nabla := d - \alpha = \partial - Cdz + \overline{\partial} - Bd\overline{z}$$

is a C^∞ connection that is flat*, then there is a basis of horizontal sections $s_1, \ldots, s_n$ on Δ. This implies the kernel of ∇ is a locally constant sheaf of n dimensional complex vector spaces.*

This can be proved directly (see e.g. [**?**]), or by passing to a holomorphic basis by Koszul–Malgrange, and then constructing a fundamental solution of the resulting holomorphic connection as we did before.

This leads to item 4) in the list, the local systems.

4.2. **Local systems.**

> Noter que *revêtement* et *faisceau localement constant* sont synonymes ([**?**] p.231)

Now we get to the intrinsic, purely topological, description of connections. A convenient framework to phrase this is *covering spaces* (often with uncountable fibres), or equivalently locally constant sheaves (with open sets in the usual topological sense).

Definition 4.6. *Suppose $\mathbb{B}$ is a topological manifold. A* <u>local system</u> *(of sets) on $\mathbb{B}$ is a locally constant sheaf of sets, and in turn it is the same thing as (the sheaf of sections of) a covering space of $\mathbb{B}$.*

This is really two definitions and an enlightening exercise shows they are the same.

In practice we can choose an open covering and a local system is then a bundle with local trivialisations (to a product with a fixed "standard fibre"), that can be defined by *constant* clutching maps on the double intersections of open sets in the covering.

Here, from flat connections, we have a local system of vector spaces, i.e. a locally constant sheaf of n dimensional complex vector spaces. This just means that the clutching maps are constant linear maps.

Exercise 4.7. Show that, by definition in the Cech approach, the set of isomorphism classes of local systems of n dimensional complex vector spaces on Σ is the nonabelian cohomology set $\mathrm{H}^1(\Sigma, \mathrm{GL}_n(\mathbb{C}))$.

Thus the passage from 3) to 4) is just to go from a flat connection (E, ∇) to its sheaf of horizontal sections V defined by

$$V(U) = \{\text{sections } s : U \to E \;\big|\; \nabla(s) = 0\}.$$

This is the desired local system.

To recover (E, ∇) from V we just tensor: the sheaf $\mathcal{E}$ of sections of E is

$$\mathcal{E}(U) = V(U) \otimes_{\mathbb{C}} C^\infty$$

and the connection can be defined on these sections via Leibniz, since $V(U)$ are the horizontal sections:

$$\nabla(vf) = (df)v + f\nabla(v) = (df)v$$

for C^∞ functions f. Similarly we could go directly back to a holomorphic vector bundle by tensoring with holomorphic functions $\mathcal{E}(U) := V(U) \otimes_{\mathbb{C}} \mathcal{O}$ is the sheaf of sections of a holomorphic vector bundle, and this gets a holomorphic connection in the same way: $\nabla(vf) = (df)v$ for holomorphic functions f.

4.3. **Monodromy of local systems.** Finally we can discuss *monodromy* and how to pass to representations of the fundamental group.

First of all there is a general statement.

Suppose $\mathbb{B}$ is a connected manifold and $\pi : C \to \mathbb{B}$ is any covering space (the fibres may be uncountable etc). Thus the sheaf of sections of C is a local system of sets.

For any two points $a, b \in \mathbb{B}$ the choice of a path $\gamma : [0, 1] \to \mathbb{B}$ in $\mathbb{B}$ from a to b, determines a bijection

$$T_\gamma(a, b) : C_a \cong C_b$$

the transport isomorphism, from the fibre $C_a = \pi^{-1}(a)$ of C at a, to the fibre C_b at b.

The transport map is defined as follows: For any point $c \in C_a$ the path γ has a unique lift to a path $\widetilde{\gamma} : [0, 1] \to C$ in C starting at c. This follows from the definition of covering space. Then $T_\gamma(a,b)(c)$ is defined to be the end point $\widetilde{\gamma}(1) \in C$ of this lifted path. From the definition of $\widetilde{\gamma}$ it is in C_b, i.e. it lies over b.

The map $T_\gamma(a,b)$ only depends on the homotopy class of γ (with fixed endpoints). Indeed any continuous deformation of γ cannot move $T_\gamma(a,b)(c)$ since it is constrained to be in the fibre C_b and the fibres are discrete.

Exercise 4.8. Rewrite this definition of transport in terms of locally constant sheaves of sections, and their clutching/restriction maps (passing from one open set to the next, via their intersection, covering the path γ), without first passing to the equivalent notion of covering spaces. If the local system is in fact the sheaf of horizontal sections of a holomorphic connection on a trivial vector bundle, show that this is the same thing as the analytic continuation of solutions.

In particular, considering loops based at b, this construction gives a homomorphism

$$\rho : \pi_1(\mathbb{B}, b) \to \mathrm{Aut}(C_b); \quad \rho(\gamma) = T_\gamma(b,b)$$

from the fundamental group of the base into the group of automorphisms of the fibre. This is just transport around loops. Said differently this is an *action* of $\pi_1(\mathbb{B}, b)$ on the fibre C_b, the *monodromy action*.

In the case that we started with a local system of vector spaces V (and not just sets) on $\mathbb{B} = \Sigma$ then this yields the monodromy representation $\rho : \pi_1(\mathbb{B}, b) \to \mathrm{Aut}(V_b) = \mathrm{GL}(V_b) \cong \mathrm{GL}_n(\mathbb{C})$. In other words the fibre V_b is a representation of the fundamental group. Thus the covering space and the basepoint determine a pair (V_b, ρ) consisting of a complex vector space equipped with a representation of $\pi_1(\mathbb{B}, b)$. This is an object of the category in 5), and this construction defines the desired functor 4) $\to$ 5).

Now we just need to check that this gives an equivalence. The key step is to define the inverse construction, from 5) to 4), which goes as follows.

Given $b \in \mathbb{B}$ let $\mathrm{pr} : \widetilde{\mathbb{B}} \to \mathbb{B}$ be the universal cover, based at b. By definition $\widetilde{\mathbb{B}}$ is the set of homotopy classes of paths in $\mathbb{B}$ starting at b, i.e. maps $\gamma : [0,1] \to \mathbb{B}$ such that $\gamma(0) = b$. Two such paths are identified if there is a homotopy between them, fixing both end points. The map pr takes the free end point of the path, $\mathrm{pr}(\gamma) = \gamma(1) \in \mathbb{B}$.

Write $\pi_1 = \pi_1(\mathbb{B}, b)$ for the fundamental group. This group acts on the fibres of $\widetilde{\mathbb{B}} \to \mathbb{B}$ freely and transitively, in other words:

Lemma 4.9. *The universal covering space $\widetilde{\mathbb{B}}$ is a principal π_1 bundle over $\mathbb{B}$.*

Proof. π_1 acts on $\widetilde{\mathbb{B}}$ in the natural way, composing a loop and a path: If $g \in \pi_1$ is a loop based at b and $\gamma \in \widetilde{\mathbb{B}}$ is a path starting at b then $\gamma \circ g \in \widetilde{\mathbb{B}}$ since it is clearly another path starting at b. The ordering of the composition $\gamma \circ g$ means "go around g and then go along γ".

Now it is easy to see that two paths $\gamma_1, \gamma_2 \in \widetilde{\mathbb{B}}$ have the same endpoint ($\gamma_1(1) = \gamma_2(1)$) if and only if they are related in this way by a loop based at b. Moreover two paths with the same end point are homotopic if and only if the loop relating them is contractible, so represents the identity in π_1. This says that π_1 acts freely and transitively on the fibre

$$\widetilde{\mathbb{B}}_c := \mathrm{pr}^{-1}(c) \subset \widetilde{\mathbb{B}}$$

of the universal covering map, for any $c \in \widetilde{\mathbb{B}}$.

□

Now for any representation of π_1 we can form the associated bundle (of the principal π_1 bundle $\widetilde{\mathbb{B}}$). If $\rho : \pi_1 \to V$ then the associated bundle is the quotient

$$\widetilde{\mathbb{B}} \times_\rho V := (\widetilde{\mathbb{B}} \times V)/\pi_1$$

where $g \in \pi_1$ acts on a pair $(c, v) \in \widetilde{\mathbb{B}} \times V$ as

$$g \cdot (c, v) = (cg^{-1}, \rho(g)v).$$

Since the action on $\widetilde{\mathbb{B}}$ is free, this quotient is well-defined, and it comes equipped with a map:

$$\widetilde{\mathbb{B}} \times_\rho V \to \mathbb{B} = \widetilde{\mathbb{B}}/\pi_1 \tag{4.2}$$

by projecting onto the first factor, with each fibre isomorphic to a copy of V.

Now we leave it as an exercise to check that the map (4.2) is a covering map (giving the fibres $\cong V$ the discrete topology), defining a local system of vector spaces, and moreover that its monodromy representation based at b is given by ρ.

4.4. **Representation varieties.** Let $G = \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^n)$ the group of linear automorphisms of a fixed (standard) copy of $\mathbb{C}^n$.

Let $\pi_1 = \pi_1(\Sigma, b)$ and suppose we are given a representation V of π_1, i.e. we are given a homomorphism

$$\rho : \pi_1 \to \mathrm{GL}(V).$$

Now if we choose a basis of V, i.e. an isomorphism $\phi : \mathbb{C}^n \xrightarrow{\cong} V$ with our standard copy of $\mathbb{C}^n$, then we get a "concrete" representation $\pi_1 \to G = \mathrm{GL}_n(\mathbb{C})$, into a fixed copy of the general linear group.

Lemma 4.10. *Let $\mathcal{R} = \mathrm{Hom}(\pi_1, G)$ be the set of group homomorphisms $\pi_1 \to G$. Consider the enriched category of triples (V, ρ, ϕ) where (V, ρ) is a π_1 representation as in 5), and $\phi : \mathbb{C}^n \xrightarrow{\cong} V$ is a framing of V. Then the set of isomorphism classes of such triples is naturally in bijection with the points of $\mathcal{R}$.*

Proof. This is a straightforward unwinding of the definitions. (The map ρ becomes a point of $\mathcal{R}$ once we use ϕ to identify V and $\mathbb{C}^n$.) □

Now $\mathcal{R}$ is naturally a complex affine algebraic variety. The easiest way to see this is to choose a presentation of π_1. The standard presentation is as follows (where g is the genus of Σ):

$$\pi_1(\Sigma, b) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

where $[a, b] = aba^{-1}b^{-1}$ is the multiplicative commutator. Given this presentation, it follows that $\mathcal{R}$ has the following presentation as an affine variety:

$$\mathcal{R} \cong \{(A_1, \dots, A_g, B_1, \dots, B_g) \in G^{2g} \mid [A_1, B_1] \cdots [A_g, B_g] = 1\} \subset G^{2g} \tag{4.3}$$

where $[A, B] = ABA^{-1}B^{-1}$ is the multiplicative commutator in the group G. Each matrix entry of the relation here is an algebraic equation, and so this defines $\mathcal{R}$ as a subvariety of G^{2g}. Of course $G = \mathrm{GL}_n(\mathbb{C})$ is itself an affine variety, for example defined by the equation:

$$\mathrm{GL}_n(\mathbb{C}) \cong \{(g, a) \in \mathrm{End}(\mathbb{C}^n) \times \mathbb{C} \mid a \det(g) = 1\} \subset \mathbb{C}^{n^2+1}.$$

Thus $\mathcal{R}$ is an affine variety, defined by the matrix equation in (4.3). A point of the right-hand side of (4.3) determines a unique representation ρ since it specifies where ρ sends generators of π_1, in $\mathrm{GL}_n(\mathbb{C})$.

Observe that G acts on $\mathcal{R}$ by diagonal conjugation of the matrices. This action corresponds to changing the choice of framing $\phi : \mathbb{C}^n \xrightarrow{\cong} V$; an element of $g \in G$ acts ion ϕ by pre-composition: $\phi \mapsto \phi \circ g^{-1}$.

Corollary 4.11. *The set of isomorphism classes of π_1 representation (V, ρ) of rank n, is naturally in bijection with the set of G orbits in $\mathcal{R}$.*

Proof. This now comes down to observing that $(V_1, \rho_1) \cong (V_2, \rho_2)$ if and only if we can choose framings $\phi_1 : \mathbb{C}^n \cong V_1$ and $\phi_2 : \mathbb{C}^n \cong V_2$ so that the two triples (V_1, ρ_1, ϕ_1) and (V_2, ρ_2, ϕ_2) determine the same point of $\mathcal{R}$. □

We are now in a very good position of a complex reductive group G acting on a complex affine variety $\mathcal{R}$ and there are standard tools (geometric invariant theory) to take the quotient of $\mathcal{R}$ by G in an algebraic way, thereby constructing the character variety $\mathcal{M}_{\mathrm{B}}$.

Let $\mathbb{C}[\mathcal{R}]$ denote the ring of regular functions on the affine variety $\mathcal{R}$ and let $\mathbb{C}[\mathcal{R}]^G \subset \mathbb{C}[\mathcal{R}]$ denote the subring of G invariant functions, where G acts by diagonal conjugation as above. Since G is reductive it is known that this ring is finitely generated and so determines an algebraic variety.

Definition 4.12. *The* character variety *(or* Betti moduli space*)* $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ *determined by the pair* (Σ, G) *is the variety associated to the ring* $\mathbb{C}[\mathcal{R}]^G$ *of* G *invariant functions on the representation variety* $\mathcal{R}$*. By construction the points of* $\mathcal{M}_{\mathrm{B}}(\Sigma, G)$ *correspond bijectively to the* closed *G-orbits in* $\mathcal{R}$*.*

We will write $\mathcal{M}_{\mathrm{B}} = \mathcal{R}^{\mathrm{ps}}/G$ where $\mathcal{R}^{\mathrm{ps}} \subset \mathcal{R}$ is the subset of points whose G-orbits are closed (since the closed orbits are often called the *polystable* point).

The book [] of Lubotzky–Magid "Varieties of representations of groups", studies the construction of character varieties of any finitely presented group in detail.

4.5. **Classification of solutions of the zero curvature equation.** As an application of the previous theorem, we can see that it gives a precise finite dimensional description of the space of equivalence classes of solutions of a nontrivial nonlinear differential equation in infinite dimensions.

Let $E = \mathbb{C}^n \times \Sigma \to \Sigma$ be the trivial complex vector bundle (that we view here as a C^∞ bundle).

Let

$$\mathcal{A} = \{d - \alpha \mid \alpha \in \Gamma(\Sigma, \mathrm{End}(E) \otimes (\Omega^{1,0} \oplus \Omega^{0,1}))\}$$

be the set of connections on E, so that α is an arbitrary $n \times n$ matrix of global C^∞ one-forms. Thus $\mathcal{A}$ is isomorphic to an infinite dimensional vector space.

Let $\mathcal{G} = C^\infty(\Sigma, \mathrm{GL}_n(\mathbb{C}))$ be the group of global gauge transformations of E, i.e. the C^∞ maps from Σ to $\mathrm{GL}_n(\mathbb{C})$. Thus $\mathcal{G}$ acts on $\mathcal{A}$ by gauge transformations as usual: $g[\alpha] = g\alpha g^{-1} + (dg)g^{-1}$.

Now consider the subset of connections which are flat, so α satisfies the nonlinear differential equation $d\alpha = \alpha^2$:

$$\mathcal{A}_{\mathrm{flat}} = \{d - \alpha \mid d\alpha = \alpha^2\} \subset \mathcal{A}.$$

This subset is preserved by the gauge action and the previous theorem implies the following classification of gauge orbits.

Corollary 4.13. *The set of* $\mathcal{G}$ *orbits in* $\mathcal{A}_{\mathit{flat}}$ *is naturally in bijection with the set of* G *orbits in the representation variety* $\mathcal{R}$*.*

Proof. Given what was proven in the theorem this amounts to observing that $\mathcal{A}_{\mathrm{flat}}/\mathcal{G}$ is the set of isomorphisms classes of flat connections on C^∞ vector bundles of rank n. This in turn follows from the fact that Chern-Weil theory implies any C^∞ complex vector bundle on a compact Riemann surface that admits a flat connection is trivial (in brief, it has degree zero). Thus if we choose a trivialisation, we see they all appear as points of $\mathcal{A}_{\mathrm{flat}}$. Moreover the notion of isomorphism of connections then comes down to the gauge action of $\mathcal{G}$ on $\mathcal{A}$. □

This may seem like it is just a tricky exercise in rephrasing the definitions but we will see below that this C^∞ viewpoint enables us to see, following Narasimhan and Atiyah–Bott, that the character variety has a holomorphic symplectic structure.

For now let us just quote a theorem that appears in Gunning's 1967 book "Lectures on vector bundles on Riemann surfaces":

Theorem 4.14 (Gunning [] p.196)**.** *Let* $\mathcal{M}_{\rm B}^s \subset \mathcal{M}_{\rm B}$ *be the subset of the character variety consisting of representations that are* irreducible*. Then* $\mathcal{M}_{\rm B}^s$ *is a (smooth) complex analytic manifold, of dimension* $2gn^2 - 2(n^2-1)$*.*

We won't prove this statement yet, as one of our aims will be to show how to prove that it has an algebraic symplectic structure at the same time, as well as many generalisations of it.

As a first step note that it is easy to explain the dimension formula since it is a subquotient of G^{2g}: it is the quotient of the subvariety $\mu^{-1}(1)^{\rm irr}$ by G where

$$\mu : G^{2g} \to G; \qquad (A_1, \dots, A_g, B_1, \dots, B_g) \mapsto [A_1, B_1] \cdots [A_g, B_g]$$

and $\mu^{-1}(1)^{\rm irr}$ is the subset of $\mu^{-1}(1)$ that are irreducible representations. The point is that $\mu^{-1}(1)^{\rm irr}$ has codimension (n^2-1) (since the determinant is already fixed to be 1), and $\mathrm{PGL}_n(\mathbb{C})$ acts freely on it, and that has dimension (n^2-1) as well, so we see the dimension is obtained by subtracting (n^2-1) twice from $2gn^2 = \dim(G^{2g})$.

Remark 4.15. Note that the equivalences between 2),3),4),5) work verbatim over any Riemann surface, not necessarily compact. This will be used to give *part* of the topological data of any meromorphic connection $(E, \nabla) \to \Sigma$ with poles on $\mathbf{a}$. Namely the restriction of (E, ∇) to $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ is a holomorphic connection and we can take the local system $V \to \Sigma^\circ$ of horizontal sections of that. We will see this is really only a very small part of the topological data attached to any meromorphic connection.

5. The Riemann problem (Hilbert 21)

We will take the point of view that the Betti spaces (and eventually the whole nonabelian Hodge space) is like a "global version" of a Lie group, attached to a Lie group plus a surface (with suitable boundary condtions). This comes more into focus if we look at the genus zero case with poles, as, in effect, we then see the *Lie algebra* of the space as well. These are the additive moduli spaces $\mathcal{M}^*$ and the simplest (Fuchsian) examples motivated the famous Riemann problem appearing in Hilbert's 21st problem (the Riemann–Hilbert problem).

There is some controversy over the exact statement of the question, but the basic idea is very simple and clear, and comes down to the following matching of dimensions.

Choose an integer $n > 0$ and m distinct points $\mathbf{a} = (a_1, \dots, a_m) \subset \mathbb{C}$ in the complex plane.

On one hand consider the set of rank n Fuchsian systems with poles at these points:

$$\begin{aligned} \widetilde{\mathcal{M}}^* &= \left\{ \nabla = d - A \;\middle|\; A = \sum_1^m \frac{A_i}{z - a_i} dz, A_i \in \mathrm{End}(\mathbb{C}^n) \right\} \\ &\cong \left\{ (A_1, \dots, A_m) \;\middle|\; A_i \in \mathrm{End}(\mathbb{C}^n) \right\}. \end{aligned} \tag{5.1}$$

On the other hand, given any such connection we can restrict it to the complement of the poles to get a holomorphic connection on $\Sigma^\circ := \mathbb{C} \setminus \mathbf{a}$, noting that in general the connection will have a further pole at ∞. Then we can take the local system of horizontal sections of that and in turn get a representation of the fundamental group $\pi_1 = \pi_1(\Sigma^\circ, b)$ in $\mathrm{GL}_n(\mathbb{C})$, for any choice of basepoint $b \in \Sigma^\circ$. (It comes with a framing as the underlying bundle is the trivial bundle.) Thus we get a point of the representation variety

$$\begin{aligned} \mathcal{R} &= \mathrm{Hom}(\pi_1, \mathrm{GL}_n(\mathbb{C})) \\ &\cong \left\{ (M_1, \dots, M_m) \;\middle|\; M_i \in \mathrm{GL}_n(\mathbb{C}) \right\} \end{aligned} \tag{5.2}$$

where the last isomorphism arises by choosing a suitable presentation of π_1, with m loops around the points in $\mathbf{a}$, freely generating π_1.

The spaces (5.1) and (5.2) are clearly both of the same dimension mn^2, and (5.2) *looks like* the multiplicative version of (5.1), with the Lie algebra replaced by the corresponding Lie group, and the sum replaced by the product (in a certain fixed order).

Moreover the *Riemann–Hilbert map*

$$\widetilde{\mathcal{M}}^* \xrightarrow{\nu_{\mathbf{a}}} \mathcal{R}$$

taking a connection to its monodromy representation is a *holomorphic* map, which generalises the matrix exponential map, that appears in the case $m = 1$:

Exponential map as a simple Riemann–Hilbert map.
Given $X \in \mathfrak{g} = \mathrm{End}(\mathbb{C}^n)$ then the connection $d - A$ where

$$A = \frac{1}{2\pi i} X \frac{dz}{z}$$

has monodromy given by

$$\exp(X) \in \mathrm{GL}_n(\mathbb{C}).$$

Proof. For any $Y \in \mathfrak{g}$, the connection $d - Y dz/z$ has fundamental solution z^Y on any open sector at zero (using any choice of branch of $\log(z)$). This has monodromy $\exp(2\pi i Y)$ around zero. □

Thus it is tempting to study this map, for example can it be upgraded to a precise bijective correspondence? What happens if we move the points $\mathbf{a}$?

6. Birkhoff's generalised Riemann problem

There is of course a more basic question that one can ask:

Suppose we have an arbitrary effective divisor $D = \sum n_i(a_i)$ where $\mathbf{a} = (a_1, \ldots, a_m) \subset \mathbb{P}^1$ and $n_i > 0$.

Then we can consider the finite dimensional space $\widetilde{\mathcal{M}}^*(D)$ consisting of all the meromorphic connections

$$d - A$$

where A is a matrix of rational one-forms with poles bounded by the divisor D.

Counting coefficients and using the residue theorem shows that

$$\dim(\widetilde{\mathcal{M}}^*) = n^2 \left(\sum_1^m n_i \right) - n^2$$

Question: Can one define invariants of such connections with any order poles, thereby defining a space $\mathcal{R}$ of dimension equal to $\dim(\widetilde{\mathcal{M}}^*)$ and a holomorphic map

$$\widetilde{\mathcal{M}}^* \to \mathcal{R},$$

generalising the Riemann–Hilbert map taking the monodromy representation?

Birkhoff (1909, 1913) found that this can indeed be done for a dense open subset of $\widetilde{\mathcal{M}}^*$. He imposed a genericity condition on the connection ("Birkhoff-generic")[10] and then constructed some invariants making up a space of the desired dimension.

These data and their generalisation/modification leading up to the definition of the general notion of *Stokes data* and *wild character varieties* are what we want to study in detail. In a sense they are the general notions of *global Lie groups* that appear in this way.

As a simple example to illustrate how this goes, suppose $n = 2$ and $D = 4(\infty)$ so we look at rank two connections with one pole of order 4 at infinity

$$\nabla = d - A,\ A = (A_0 + A_1 z + A_2 z^2)dz.$$

We suppose that the leading term A_2 is diagonal with distinct eigenvalues and is fixed, so there are 8 remaining free parameters in A_0, A_1. The monodromy-type data this leads to have the following form: first we restrict ∇ to the formal disk at ∞ and find it can be put uniquely in the form:

$$\widehat{\nabla} = d - \widehat{A},\ \widehat{A} = dQ + \Lambda \frac{dz}{z},\ Q = B_3 z^3 + B_2 z^2 + B_1 z$$

[10] that the leading term at each pole has n distinct eigenvalues, and further that the eigenvalues are off of some real codimension one walls.

via a *formal* (not necessarily convergent) gauge transformation, for some diagonal matrices B_i, Λ, with $B_3 = A_2/3$ and $\mathrm{Tr}(\Lambda) = 0$. This gives 5 parameters, in Λ, B_1, B_2. The remaining parameters are more mysterious and can be understood in several ways. One way (essentially that of Birkhoff) is that there are "wild monodromy data" $S_1, \ldots S_6$ that obey a *wild monodromy relation*:

$$S_6 S_5 \cdots S_1 = h, \qquad h := \exp(2\pi i \Lambda). \tag{6.1}$$

Moreover the S_i are constrained to be in alternating unipotent groups:

$$S_1, S_3, S_5 \in U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad S_2, S_4, S_6 \in U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

These unipotent groups have a total dimension of 6 and the relation (6.1) imposes 3 constraints on them (as the determinant is 1), and so this yields the desired remaining three parameters, making up 8 in total. This example in fact leads to the wild character variety (of complex dimension two) underlying the Painlevé II equation, that takes the form of the affine surface (the *Flaschka–Newell surface*):

$$x\,y\,z + x + y + z = c \tag{6.2}$$

for a constant $c \in \mathbb{C}$ (directly related to the constant α in P_{II}).

This is of course, all incredibly strange and mysterious, and begs many questions (that we will endeavour to answer in the rest of the course): what are these matrices S_i? Why are they triangular? Why are there 6 of them? Are we really generalising the fundamental group? What is the generalisation of the intrinsic topological notion of local system? Why has no-one told me about this before? (etc)

Derivation of the Flaschka–Newell surface ($\widehat{A}_1$ wild character variety). Here is the derivation of the surface from the monodromy relation:
Let $T \subset \mathrm{GL}_2(\mathbb{C})$ be the diagonal torus, and define the space $\mathcal{B}$ of solutions of the wild monodromy relation:

$$\mathcal{B} = \left\{ (h, \mathbf{S}) \in T \times (U_+ \times U_-)^3 \;\middle|\; S_6 S_5 \cdots S_1 = h \right\}$$

where $\mathbf{S} = (S_1, \ldots, S_6)$ with $S_{\mathrm{odd}} \in U_+, S_{\mathrm{even}} \in U_-$. It is easy to see $\mathcal{B}$ is a smooth affine variety of complex dimension 4. Given $t \in T$ of determinant one, let $\mathcal{B}(t) \subset \mathcal{B}$ be the subset of $\mathcal{B}$ with $h = t$. The torus T acts on $\mathcal{B}$ by diagonal conjugation, preserving the three-fold $\mathcal{B}(t)$.

Lemma 6.1. *If $t \neq 1$ the quotient $\mathcal{M}_{\mathrm{B}}(\widehat{A}_1, t)$ of $\mathcal{B}(t)$ by T is isomorphic to the smooth complex affine surface*

$$x\,y\,z + x + y + z = b - b^{-1} \tag{6.3}$$

where $b \in \mathbb{C}^*$ is any complex number such that $t = -\mathrm{diag}(b^{-2}, b^2)$.

Proof. Let s_i be the nontrivial off-diagonal matrix entry of S_i. The T-action reduces to that of the subtorus $C = \mathrm{diag}(1, *) \subset T$. If $t \neq 1$ the C-action is free and every orbit is closed in $\mathcal{B}(t)$ since then some of the odd and even s_i must be nonzero. So we just need to compute the ring of invariant functions. The equation $S_6 \cdots S_1 = t$ is equivalent to the three equations: $s_1 = -q(s_3 s_4 s_5 + s_3 + s_5)$, $s_6 = -q(s_2 s_3 s_4 + s_2 + s_4)$ (allowing to eliminate s_1 and s_6), and $s_2 s_3 s_4 s_5 + s_2 s_3 + s_2 s_5 + s_4 s_5 + 1 = 1/q$ where $q = -b^2$. To quotient by T we pass to invariants $s_{23}, s_{25}, s_{34}, s_{45}$ where $s_{ij} = s_i s_j$, and thus find $s_{25} = 1/q - (1 + s_{45} + s_{23} + s_{23} s_{45})$. Substituting this in the relation $s_{23} s_{45} = s_{34} s_{25}$ yields (6.3) after relabelling $s_{45} = z/b - 1, s_{23} = x/b - 1$ and $s_{34} = -1 - b\,y$. In other words:

$$x = b(1 + s_2 s_3), \quad y = -(1 + s_3 s_4)/b, \quad z = b(1 + s_4 s_5).$$

Smoothness is easy to check (and we will see later how to deduce it in general from the freeness of the C action). □

These surfaces appear in Flaschka–Newell 1980 [**?**] (3.24) in relation to P_{II}. They underlie simple examples of the hyperkähler manifolds constructed in [**?**].

7. Lecture 3: Geometric local systems, nonlinear and linear

Having introduced local systems last time it is now easy to introduce a large class of nonlinear local systems, whose fibres are character varieties. This is the Betti approach to (tame) isomonodromy, and is a key way to get nonlinear local systems from linear local systems on curves.

Write $\mathbf{\Sigma} = (\Sigma, \mathbf{a})$ where Σ is a compact Riemann surface and $\mathbf{a} \subset \Sigma$ is a finite subset. Let $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ be the corresponding punctured surface. Let $G = \mathrm{GL}_n(\mathbb{C})$ be a general linear group. Given a basepoint $b \in \Sigma^\circ$ let $\pi_1 = \pi_1(\Sigma^\circ, b)$.

Definition 7.1. *A <u>surface group</u> is a discrete group that is isomorphic to $\pi_1(\Sigma^\circ, b)$ for some choice of compact Riemann surface Σ and finite subset $\mathbf{a} \subset \Sigma$.*

This is a nice class of groups, whose character varieties have wonderful properties. Note that there are of course Riemann surfaces with much more complicated fundamental groups (e.g. $\mathbb{C} \setminus \mathbb{Z}^2$). In essence the surface groups are the fundamental groups of "algebraic Riemann surfaces" (those that arise as the analytification of a smooth complex algebraic curve). They only depend on the genus g and the number $m = \#\mathbf{a}$ of marked points, and admit the presentation:

$$\pi_1(\Sigma^\circ, b) \;\cong\; \langle a_1, \ldots, b_g, \gamma_1, \ldots, \gamma_m \;\big|\; [a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_m = 1\rangle \tag{7.1}$$

where $[a, b] = aba^{-1}b^{-1}$ is the multiplicative commutator.

As in the last lecture we can define the G-character variety of $\mathbf{\Sigma}$ by taking the affine geometric invariant theory quotient

$$\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, G) = \mathcal{R}/G, \qquad \mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma^\circ, b), G)$$

of the representation variety $\mathcal{R}$ of the punctured Σ° be the conjugation action of G, for any choice of basepoint $b \in \Sigma^\circ$. In brief $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, G)$ is the variety (the maximal spectrum) of the ring $\mathbb{C}[\mathcal{R}]^G$ of G invariant regular functions on $\mathcal{R}$, and this means that the points of $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, G)$ are the closed G orbits in $\mathcal{R}$

Reductive groups and Hilbert's theorem. Hilbert proved that if a *linearly reductive group* acts on an affine variety $\mathcal{R}$ then the ring of invariant functions $\mathbb{C}[\mathcal{R}]^G$ is finitely generated (and so we can take its maximal spectrum to get an affine variety). Most of our constructions use this result, if only as a black box. (See e.g. Thm. 4.53 in Mukai's book "Introduction to invariants and moduli").

Since we are working over $\mathbb{C}$ the linearly reductive groups are exactly the *canonical complexifications* of the compact Lie groups. For example $\mathrm{GL}_n(\mathbb{C})$ is the complexification of the unitary group U_n.

Remark 7.2. Note that the character variety does not depend on the choice of basepoint $b \in \Sigma^\circ$. Moving the basepoint along a path just conjugates the representations, and so does not change the conjugacy class of the representation (= point of $\mathcal{M}_{\mathrm{B}}$).

Thus have a procedure

$$\mathbf{\Sigma} \ \mapsto \mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, G)$$

attaching a variety to a surface with some marked points. This shouldn't be viewed as a static procedure, but rather it works well in families, deforming $\mathbf{\Sigma}$.

Naively one might think that we just have a gadget that produces a variety for each choice of integers g, m, n (taking the $\mathrm{GL}_n(\mathbb{C})$ character variety of the surface group determined by g, m). However this misses the key (but rather subtle) way that $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}, G)$ depends on the choice of the surface. If we define $\Gamma_{g,m}$ to be the abstract group defined by the presentation above (the right-hand side of (7.1)) then we can indeed define a variety for each choice of g, m, n (taking the $\mathrm{GL}_n(\mathbb{C})$-character variety of $\Gamma_{g,m}$). However we defined the character variety in terms of the fundamental group of a surface, and the subtly is hidden in the choice of the isomorphism in (7.1), i.e. the choice of generators of π_1.

Exercise 7.3. 1) Choose a local system of rank two complex vector spaces on $\mathbb{C} \setminus \{1, 2, 3\}$. Ask a friend in a different room to do the same. Choose a presentation of $\pi_1 = \pi_1(\mathbb{C} \setminus \{1, 2, 3\}, 0)$ with a simple loop γ_i around the point i for $i = 1, 2, 3$. Thus your local system determines monodromy matrices M_i $(i = 1, 2, 3)$, well-defined up to overall conjugation, and these matrices determine the local system up to isomorphism. Get your friend to do the same, and to tell you their monodromy matrices N_1, N_2, N_3. Show that you do not have enough information to decide if your local system is isomorphic to theirs, unless they also tell you their choice of generating loops. Design a method for deciding if your local system is isomorphic to theirs or not.

2) Think about repeating the exercise in 1) but starting with each person choosing a genus two surface with four punctures. Show that the question of identifying the local systems does not even make sense unless some choice of isomorphism between the surfaces or their fundamental groups is given.

7.1. **Betti isomonodromy connections.**

Suppose $\mathbb{B}$ is a manifold and $\pi : \underline{\mathbf{\Sigma}} \to \mathbb{B}$ is a family of Riemann surfaces with marked points, over $\mathbb{B}$. If $b \in \mathbb{B}$ then the fibre $\pi^{-1}(b) = \mathbf{\Sigma}_b = (\Sigma_b, \mathbf{a}_b)$ is a Riemann surface Σ_b with marked points $\mathbf{a}_b \subset \Sigma$. We assume it is an "admissible family" in the sense that the surfaces remain smooth (and are the fibres of a smooth fibre bundle $\mathrm{pr} : \underline{\Sigma} \to \mathbb{B}$) and that the points do not coalesce (there are the same number of marked points in each fibre).

The main statement is then as follows:

Proposition 7.4. *Suppose we replace each fibre of the admissible family* $\pi : \underline{\mathbf{\Sigma}} \to \mathbb{B}$ *by the corresponding character variety* $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_b, G)$*, for all* $b \in \mathbb{B}$*. Then the resulting bundle of character varieties over* $\mathbb{B}$ *naturally has the structure of a* local system of varieties*. Thus it can be described in terms of an open cover of* $\mathbb{B}$ *with constant clutching maps (that are themselves algebraic automorphisms of the fibres). In particular for any base point* $b \in \mathbb{B}$ *the fundamental group of the base* $\pi_1(\mathbb{B}, b)$ *acts on the fibre* $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma}_b, G)$ *by algebraic automorphisms.*

Proof. The last statement is just taking the monodromy action of the local system, as defined in the last lecture.

The first statement comes down to a simple topological fact (essentially the "homotopy invariance of the fundamental group").

In detail suppose we have a contractible open subset $U \subset \mathbb{B}$ and a point $b \in U$. Let $\Sigma_b^\circ = \Sigma_b \setminus \mathbf{a}_b$ be the corresponding punctured surface and let

$$\Sigma_U^\circ = \bigcup_{u \in U} \Sigma_u^\circ \subset \mathrm{pr}^{-1}(U) \subset \underline{\Sigma}$$

be the union of all the punctured surfaces over U (which is itself a fibre bundle over U). Then we claim that the inclusion of the fibre:

$$\Sigma_b^\circ \hookrightarrow \Sigma_U^\circ$$

is a homotopy equivalence and so induces an isomorphism of fundamental groups and in turn induces an identification of the character varieties:

$$\mathrm{Hom}(\pi_1(\Sigma_b^\circ), G)/G \;\cong\; \mathrm{Hom}(\pi_1(\Sigma_U^\circ), G)/G$$

of the fibre with that of the total space of the fibration over U. Indeed this follows from the homotopy long exact sequence for the fibration $\Sigma_U^\circ \to U$, since the base U is contractible: the inclusion induces and isomorphism $\pi_1(\Sigma_b^\circ) \cong \pi_1(\Sigma_U^\circ)$.

In turn if we have two points $b_1, b_2 \in U$ then the choice of U induces a canonical identification

$$\mathrm{Hom}(\pi_1(\Sigma_{b_1}^\circ), G)/G \;\cong\; \mathrm{Hom}(\pi_1(\Sigma_U^\circ), G)/G \;\cong\; \mathrm{Hom}(\pi_1(\Sigma_{b_2}^\circ), G)/G$$

between the character varieties of the fibres over b_1 and b_2.

This gives the canonical local trivialisations of the bundle, making it into a local system of varieties. □

Exercise 7.5. In the situation of the proposition, choose two intersecting contractible open sets $U_1, U_2 \subset \mathbb{B}$ with $U_{12} = U_1 \cap U_2$ contractible, and choose any points $b_1 \in U_1, b_2 \in U_2$. Choose presentations of the two surface groups $\pi_1(\Sigma_{b_1}^\circ), \pi_1(\Sigma_{b_2}^\circ)$ and thus describe the corresponding two character varieties explicitly in terms of matrices.

Now show that the clutching map on U_{12} is constant, since it comes from a (constant) identification $\pi_1(\Sigma^\circ_{b_1}) \cong \pi_1(\Sigma^\circ_{b_2})$.

Of course if we think in a more differential geometric way (assuming the character varieties are smooth) a local system of (smooth) varieties is the same as having a bundle equipped with a complete flat Ehresmann connection. Thus we are getting nonlinear flat connections in a completely geometric way. This becomes very interesting when we describe these connections in algebraic coordinates on the other side of the Riemann–Hilbert correspondence. The sixth Painlevé equation is the simplest example of a nonlinear differential equation that arises geometrically in this way (as we will see).

7.2. **Fricke–Klein–Vogt surfaces.** The simplest (tame) character varietes are the $\widehat{D}_4$ character varieties related to P_{VI}. Set $G = \mathrm{SL}_2(\mathbb{C})$, $\Sigma = \mathbb{P}^1$ and $\mathbf{a} \subset \Sigma$ a subset of four distinct points, and $\Sigma^\circ = \Sigma \setminus \mathbf{a}$.

The fundamental group $\pi_1(\Sigma^\circ)$ is the free group on 3 generators so $\mathcal{R} \cong G^3$ and we wish to compute the G invariant functions on this (where G acts by diagonal conjugation). This was done by Vogt in 1889 [] and rediscovered by Fricke–Klein.

Suppose $M_1, M_2, M_3 \in G = \mathrm{SL}_2(\mathbb{C})$ and define the seven G-invariant functions

$$\begin{gathered} m_1 := \mathrm{Tr}(M_1), \qquad m_2 := \mathrm{Tr}(M_2), \qquad m_3 := \mathrm{Tr}(M_3), \\ m_{12} := \mathrm{Tr}(M_1M_2), \qquad m_{23} := \mathrm{Tr}(M_2M_3), \qquad m_{13} := \mathrm{Tr}(M_1M_3) \\ m_4 = m_{321} := \mathrm{Tr}(M_3M_2M_1) \end{gathered} \tag{7.2}$$

so that $m_4 = \mathrm{Tr}(M_4)$ if $M_4 \in \mathrm{SL}_2(\mathbb{C})$ satisfies $M_4M_3M_2M_1 = 1$. The Fricke–Klein–Vogt relation is the relation

$$x\,y\,z + x^2 + y^2 + z^2 = b_1\,x + b_2\,y + b_3\,z + c \tag{7.3}$$

where $x = m_{23}, y = m_{13}, z = m_{12}$ and

$$\begin{aligned} b_1 &= (m_1m_4 + m_2m_3) \\ b_2 &= (m_2m_4 + m_1m_3) \\ b_3 &= (m_3m_4 + m_1m_2) \\ c &= 4 - (m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1m_2m_3m_4). \end{aligned}$$

Proposition 7.6. *These 7 functions subject to the relation (7.3) give a presentation of the ring of G invariant functions on $\mathcal{R} \cong G^3$. This identifies the character variety $\mathcal{M}_{\mathrm{B}} = \mathcal{R}/G$ as the hypersurface in $\mathbb{C}^7$ cut out by (7.3).*

A more recent exposition of this is in the 1980 paper of Magnus "Rings of Fricke characters and automorphism groups of free groups".

Often (in the context of P_{VI}) the traces m_i are written in terms of complex numbers θ_i as

$$m_i = 2\cos(\pi\theta_i) \tag{7.4}$$

for $i = 1, 2, 3, 4$, so that M_i has eigenvalues $\exp(\pm\pi\sqrt{-1}\theta_i)$.

We will see later that $\mathcal{M}_{\mathrm{B}}$ has a natural Poisson structure with *Casimir functions* given by m_1, m_2, m_3, m_4. This implies that all the interesting flows/automorphisms occur within the complex surfaces obtained by fixing these to be constants, i.e. with in the fibres of the map

$$\mathcal{M}_{\mathrm{B}} \to \mathbb{C}^4; \quad (x, y, z, \mathbf{m}) \mapsto \mathbf{m} := (m_1, m_2, m_3, m_4).$$

Notice that the b_i, c just depend on $\mathbf{m}$ and so the fibres of this map are the affine surfaces $\mathcal{M}_{\mathrm{B}}(\mathbf{m})$ defines by

$$x\,y\,z + x^2 + y^2 + z^2 = b_1\,x + b_2\,y + b_3\,z + c$$

the *Fricke–Klein–Vogt surfaces* (with b_1, b_2, b_3, c constant).

7.3. **Braid group action.** Now as we vary the four-tuple $\mathbf{a}$ the character varieties $\mathcal{M}_{\mathrm{B}}$ form a local system, so we can compute the monodromy automorphisms of $\mathcal{M}_{\mathrm{B}}$ that occur by varying the marked points. Consider the configuration space

$$\mathbb{C}^3 \setminus \text{diagonals} = \{(a_1, a_2, a_3) \in \mathbb{C}^3 \bigm| a_i \neq a_j\}$$

and let

$$\mathbb{B} = (\mathbb{C}^3 \setminus \text{diagonals})/\mathrm{Sym}_3$$

be the corresponding space of unordered 3 tuples of points. A point $b = \{a_1, a_2, a_3\} \in \mathbb{B}$ determines the (unordered) four-tuple

$$\mathbf{a} = \mathbf{a}_b = \{a_1, a_2, a_3, \infty\} \subset \Sigma = \mathbb{P}^1.$$

Thus $\mathbb{B}$ parameterises an admissible family of pointed surfaces

$$\pi : \underline{\boldsymbol{\Sigma}} \to \mathbb{B}$$

with $\pi^{-1}(b) = (\mathbb{P}^1, \mathbf{a}_b)$ for any $b \in \mathbb{B}$. Thus by Proposition 7.4 we get a local system

$$\underline{\mathcal{M}_{\mathrm{B}}} \to \mathbb{B}$$

of varieties, with fibres the corresponding Betti spaces $\mathcal{M}_{\mathrm{B}}$ (of complex dimension 6). The base space $\mathbb{B}$ can be described explicitly in terms of the space of coefficients of the polynomial

$$(x - a_1)(x - a_2)(x - a_3) = x^3 + s_1 x^2 + s_2 x + s_3$$

where $s_1 = -(a_1+a_2+a_3), s_2 = a_1a_2+a_2a_3+a_1a_3, s_3 = -a_1a_2a_3$. $\mathbb{B}$ is the complement of the discriminant which is

$$\Delta^2 = 18 s_1 s_2 s_3 - 4 s_1^3 s_3 + (s_1 s_2)^2 - 4 s_2^3 - 27 s_3^2 \;=\; ((a_1 - a_2)(a_2 - a_3)(a_1 - a_3))^2.$$

$$\mathbb{B} \cong \{(s_1, s_2, s_3) \in \mathbb{C}^3 \bigm| \Delta^2 \neq 0\} \subset \mathbb{C}^3.$$

In any case $\mathbb{B}$ is a smooth complex manifold of dimension 3, and by definition its fundamental group is the *three-string braid group*:

$$B_3 = \pi_1(\mathbb{B}).$$

This is a well-studied group, and has presentation

$$B_3 \cong \langle \beta_1, \beta_2 \bigm| \beta_1\beta_2\beta_1 = \beta_2\beta_1\beta_2 \rangle$$

with two generators and a single relation the *braid relation.* (Note that we can retract $\mathbb{B}$ onto the subset where $s_1 = 0$, and so identify $\pi_1(\mathbb{B})$ with the π_1 of the complement of the cuspidal curve $4s_2^3 + 27s_3^2 = 0$ in $\mathbb{C}^2$.)

The general theory above says we have a local system of Betti space over $\mathbb{B}$ and taking the monodromy of this gives an action of B_3 on the character variety $\mathcal{M}_{\rm B}$. This can be given explicitly as follows.

In terms of monodromy data this action can be given by the standard "Hurwitz" action:

$$\beta_1(M_3, M_2, M_1) = (M_2, M_2^{-1}M_3M_2, M_1)$$
$$\beta_2(M_3, M_2, M_1) = (M_3, M_1, M_1^{-1}M_2M_1)$$

which fixes the product $M_4M_3M_2M_1 = 1$. (Recall that the product $a \circ b$ in the fundamental group means "go around b and then a", and this explains why we put M_1 on the right.)

Exercise 7.7. Choose a basepoint $b \in \mathbb{B}$ and choose loops generating $\pi_1(\Sigma_b^\circ, e)$. Derive the formula for the Hurwitz action by "braiding" two of the points in $\mathbf{a}_b$, dragging the generating loops around, and then re-expressing the new loops in terms of the original loops.

Clearly this actions descends to the invariant functions, and acts on $\mathbf{m}$ as

$$\beta_1(m_1, m_2, m_3, m_4) = (m_1, m_3, m_2, m_4)$$
$$\beta_2(m_1, m_2, m_3, m_4) = (m_2, m_3, m_3, m_4)$$

and it turns out that the action on the remaining (quadratic) invariant functions is:

$$\beta_1(x, y, z) = (x, z, b_2 - y - xz)$$
$$\beta_2(x, y, z) = (y, b_1 - x - yz, z).$$

These formulae define an action of B_3 on $\mathcal{M}_{\rm B}$, and this action is the (nonlinear) monodromy of the local system

$$\underline{\mathcal{M}_{\rm B}} \to \mathbb{B}.$$

For example this entails reducing invariants functions such as

$$\mathrm{Tr}(M_2^{-1}M_3M_2M_1)$$

to polynomials in the chosen generating functions (in this example we get $b_2 - y - xz$). Methods to do this go back to Vogt (see Magnus op. cit.) and such formulae for the braid group action were found by Iwasaki []. A direct (more pedestrian) way to do such computations appeared in [**?**] by writing $M_i = \varepsilon_i(1 + e_i \otimes \alpha_i)$ for a scalar ε_i and a rank one matrix $e_i \otimes \alpha_i$ (so that e_i is a column vector and α_i is a row vector). This enables everything to be computed explicitly in terms of the numbers $\varepsilon_i, \alpha_i(e_j)$.

Remark 7.8. To relate this to $\mathrm{P_{VI}}$ we use a different base

$$\mathbb{B}_6 = \{(0, t, 1, \infty) \bigm| t \in \mathbb{C} \setminus \{0, 1\}\}.$$

The fundamental group of this is the free (nonabelian) group F_2 with two generators (and the variable t will be the t in $\mathrm{P_{VI}}$). One can show that the monodromy action of the of F_2 on $\mathcal{M}_{\rm B}$ is generated by the squares (β_1^2 and β_2^2) of the above automorphisms β_1, β_2 of $\mathcal{M}_{\rm B}$. This situation is especially nice since both β_1^2 and β_2^2 fix $\mathbf{m}$ and so this action now restricts to an action on the Fricke–Klein–Vogt surfaces. In other words for each $\mathbf{m} \in \mathbb{C}^4$ we have a local system

$$\underline{\mathcal{M}_{\rm B}}(\mathbf{m}) \to \mathbb{B}_6$$

whose fibres have complex dimension two, and the monodromy action of this local system is given by the automorphisms β_1^2 and β_2^2 of $\mathcal{M}_{\rm B}(\mathbf{m})$. It is this local system of surfaces that will give (the *second order* ODE) $\mathrm{P_{VI}}$, when we pass over to the De Rham side of the Riemann–Hilbert correspondence and rewrite this local system in algebraic coordinates there. In other words we will explain that $\mathrm{P_{VI}}$ is the explicit description of the corresponding (non-linear) De Rham local system $\underline{\mathcal{M}_{\rm DR}}(\theta) \to \mathbb{B}_6$ (where $\mathbf{m}$ and $\theta = (\theta_1, \ldots, \theta_4)$ are related as in (7.4).)

7.4. **The Klein cubic surface $\rightsquigarrow$ nonlinear representation theory.** The *Klein cubic surface* is the affine cubic surface $X \subset \mathbb{C}^3$ defined by

$$xyz + x^2 + y^2 + z^2 = x + y + z. \tag{7.5}$$

It is a smooth affine variety, so is a smooth noncompact complex manifold, and has complex dimension 2.

This surface is particularly interesting as it contains a braid group orbit S of size 7, given by the points whose coordinates are the binary numbers from 0 to 6:

$$S = \{(x, y, z) = (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\} \subset X.$$

It is easy to check this is indeed a finite braid group orbit by hand, recalling the braid group action is generated by the operations:

$$\beta_1 : (x, y, z) \ \mapsto\ (x, z, 1 - y - xz)$$
$$\beta_2 : (x, y, z) \ \mapsto\ (y, 1 - x - yz, z).$$

It is easy to check that the corresponding F_2 action generated by β_1^2, β_2^2 gives a degree 7, genus zero[11] Belyi cover of $\mathbb{B}_6 = \mathbb{C} \setminus \{0, 1\}$. In other words S is the fibre at some basepoint $b \in \mathbb{B}_6$ of a finite cover $\mathcal{S} \to \mathbb{B}_6$, and the action of β_1^2, β_2^2 on S is the monodromy of this finite cover (= local system of finite sets of size 7).

We are interested in the "meaning" of this pair $S \subset X$. Specifically we will see that X has several different "representations" as a moduli space.

First we can identify X with an $\mathrm{SL}_2(\mathbb{C})$ character variety:

Lemma 7.9. *The Klein surface (7.5) is the Fricke–Klein–Vogt surface with the parameters $\theta = (2, 2, 2, 4)/7$.*

Proof. From (7.3) we should check that if $m = 2\cos(2\pi/7), k = 2\cos(4\pi/7)$ then:

$$m^2 + mk = 1, \qquad 3m^2 + k^2 + km^3 = 4$$

which is an easy exercise, e.g. using $k = m^2-2, m^3+m^2-2m-1 = \sum_{j=-3}^{3} e^{\frac{2\pi i j}{7}} = 0.$ □

Of course, changing primitive seventh roots of unity, this lemma thus also holds for the parameters $\theta = (4, 4, 4, 6)/7$ and $\theta = (6, 6, 6, 2)/7$.

Remark 7.10. Given the data corresponding to any of these branches one can easily solve the seven equations (7.2) to find a corresponding $\mathrm{SL}_2(\mathbb{C})$ triple. For example for the branch with $(x, y, z) = (0, 0, 0)$ it is straightforward to find the triple:

$$M_1 = \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} w & x \\ -x & \overline{w} \end{pmatrix}, \qquad M_3 = \begin{pmatrix} w & \mu x \\ -x/\mu & \overline{w} \end{pmatrix}, \tag{7.6}$$

[11] Indeed the permutation of the branches around each of $0, 1, \infty$ has cycles of type $2+2+3$, so Riemann–Hurwitz says $2 - 2g = 7 \cdot (2 - 2g(\mathbb{P}^1)) - 3(1 + 1 + 2)$ so that $g = 0$.

where $\phi = \exp(\pi i\theta_1), \theta_1 = 2/7$, $w = \frac{1+\phi^2}{\phi-\phi^3}$, $x = \sqrt{1-|w|^2}$ and $\mu = (r + i\sqrt{4-r^2})/2$ where $r = \frac{1}{2} + 1/(4\cos(\pi\theta_1/2))$. The same formulae but with $\theta_1 = 6/7$ yields a triple for the parameters $\theta = (6,6,6,2)/7$. For $\theta_1 = 4/7$ so $\theta = (4,4,4,6)/7$ one should instead use: $r = \frac{1}{2} - 1/(4\cos(\pi\theta_1/2))$ and reverse the order of the triple (so that $\mathrm{Tr}(M_1M_2M_3) = 2\cos(6\pi/7)$). Note that we could also use the transcendental formulae in [**?**] Appendix B.

Lemma 7.11. *For $\theta_1 = 2/7$ or $4/7$ the three matrices M_i are in SU_2 whereas for $\theta_1 = 6/7$ they are in $\mathrm{SU}_{1,1} \cong \mathrm{SL}_2(\mathbb{R})$.*

Proof. We check directly that $M_iM_i^\dagger = 1$ in the first two cases and $M_i\Delta M_i^\dagger = \Delta$ in the last case, where $\Delta = \mathrm{diag}(1,-1)$. Recall also that $M \in \mathrm{SU}_{1,1}$ if and only if $CMC^{-1} \in \mathrm{SL}_2(\mathbb{R})$ where $C = \left(\begin{smallmatrix} i & 1 \\ 1 & i \end{smallmatrix}\right)$. □

Lemma 7.12. *The group generated by M_1, M_2, M_3 is an infinite group (in each of the three cases $\theta_1 = 2/7, 4/7, 6/7$), and they project to $\mathrm{PSL}_2(\mathbb{C})$ to generate a subgroup there isomorphic to the 237 triangle group.*

Proof. See [**?**] Lemma 7 and [**?**] Appx B, Prop. 6. □

The point here is that if (M_1, M_2, M_3) generated a finite group, then it would be clear that they live in a finite braid group, since there are only a finite number of triples in any finite group. Thus the fact that this orbit if finite is mysterious; it is one of only two such exotic braid group orbits of triples in $\mathrm{SL}_2(\mathbb{C})$.

However in this case we can explain this finite braid orbit S in terms of finite groups in two ways, by identifying X as a different moduli space:

•) X is also $\mathrm{GL}_3(\mathbb{C})$ character variety of a four-punctured sphere Σ°, and the points of S make up the braid orbit through the $\pi_1(\Sigma^\circ)$ representation determined by the triple of matrices:

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -\overline{a} \\ -1 & 1 & -\overline{a} \\ -a & -a & 0 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad r_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $a := (1+i\sqrt{7})/2$. These matrices (from (10.1) in the classic paper of Shephard–Todd [**?**]) are very special as they generate the Klein complex reflection group, a finite group of order 336 whose projectivisation in $\mathrm{PGL}_3(\mathbb{C})$ is Klein's simple group of order 168).

The fact that X is the corresponding $\mathrm{GL}_3(\mathbb{C})$ character variety of the four punctured sphere containing the π_1 representation determined by the r_1, r_2, r_3 is proved in [**?**]. As a sanity check we compute the dimension:

Let $\mathcal{C} \subset \mathrm{GL}_3(\mathbb{C})$ be the conjugacy class of order two complex reflections, i.e. matrices conjugate to $\mathrm{diag}(-1,1,1)$. Thus $\dim(\mathcal{C}) = 4$. Let $\mathcal{C}_\infty \subset \mathrm{GL}_3(\mathbb{C})$ be a generic conjugacy class, which has dimension 6. The corresponding character variety with local monodromy class $\mathcal{C}$ at 3 points and $\mathcal{C}_\infty$ at the fourth point has dimension:

$$\dim\{(g_1, g_2, g_3, g_4) \mid g_1, g_2, g_3 \in \mathcal{C}, g_4 \in \mathcal{C}_\infty, g_1g_2g_3g_4 = 1\}/G = 3 \cdot 4 + 6 - 2 \cdot 8 = 2.$$

•) X is also character variety of a four-punctured sphere Σ°, with group the complex simple algebraic group $G = G_2(\mathbb{C})$, through a very special triple of elements in the six-dimensional semisimple conjugacy class of G, that generate the finite simple group of order 6048. This was proved in B.–Paluba [**?**]. The dimension count in this case looks like:

$$3 \cdot 6 + 12 - 2 \cdot 14 = 2.$$

In any case these examples give the idea that the variety X should not be thought of as having a single fixed interpretation as an $\mathrm{SL}_2(\mathbb{C})$ character variety, but has other realisations/representations as well. In fact most examples of (symplectic) character varieties (in genus zero) have an infinite number of representations (most of which have not been classified).

Research problem: Finite representations for the elliptic 237 solutions
Consider the symmetric Fricke–Klein–Vogt surface $\mathcal{E}$

$$x\,y\,z + x^2 + y^2 + z^2 + 2\alpha = (1+\alpha)(x+y+z)$$

where $\alpha = 2\cos(\pi/7)$.
It is special as it contains a braid group orbit of size 18 that has no known relation to finite groups. The 18 points (x,y,z) of this orbit are:

$$\begin{array}{ccc} (0,1,1), & (1,0,1), & (1,1,0), \\ (0,1,\alpha), & (0,\alpha,1), & (\alpha,0,1), \\ (1,0,\alpha), & (1,\alpha,0), & (\alpha,1,0), \\ (1,1,\alpha), & (1,\alpha,1), & (\alpha,1,1), \\ (0,\alpha,\alpha), & (\alpha,0,\alpha), & (\alpha,\alpha,0), \\ (\alpha,\alpha,\beta), & (\alpha,\beta,\alpha), & (\beta,\alpha,\alpha), \end{array}$$

where $\beta = 2\cos(4\pi/7) = 1+\alpha-\alpha^2$. In other words these 18 points are the Sym_3 orbits of the 5 points: $(0,1,\alpha), (1,1,0), (1,1,\alpha), (\alpha,\alpha,0), (\alpha,\alpha,\beta)$. These Sym_3 orbits have sizes $6,3,3,3,3$ yielding $6+4\cdot 3 = 18$ points in total.
The braid group action is generated by the operations:

$$\beta_1 : (x,y,z) \;\mapsto\; (x, 1+\alpha-z-xy, y)$$

$$\beta_2 : (x,y,z) \;\mapsto\; (z, y, 1+\alpha-x-yz)$$

One can almost compute this orbit by hand using the fact that $\alpha^3 = \alpha^2+2\alpha-1$.

Problem: Find a representation of the surface $\mathcal{E}$ as a character variety of a four-punctured sphere for some (reductive) complex algebraic group G, such that the above 18 points of $\mathcal{E}$ correspond to triples of generators of a finite subgroup of G. Alternatively prove there is no such representation.

This orbit (and the corresponding explicit algebraic solution of $\mathrm{P_{VI}}$, living on the elliptic curve $u^2 = s\,(s^2+s+7)$) was found in [], and around the same time independently by A. V. Kitaev []. This orbit has two “siblings”, obtained by replacing α by $2\cos(3\pi/7)$ or $2\cos(5\pi/7)$, i.e. the other two roots of the polynomial $\alpha^3 - (\alpha^2+2\alpha-1)$ (and always defining $\beta = 1+\alpha-\alpha^2$).

7.5. **Dubrovin's example: braiding of BPS states.**

B. Dubrovin's 1995 paper "Geometry of topological field theories" contained an inspiring example. In brief he looked at $n \times n$ operators of the form

$$\Lambda = \frac{d}{dz} - U - \frac{1}{z}V(u)$$

where U is a diagonal matrix with entries $u = (u_1, \ldots, u_n), u_i \neq u_j$ and V was a skew-symmetric complex matrix. Viewed as a connection this has an irregular singularity at ∞, and Dubrovin defined its *Stokes matrix* S (a complex upper triangular unipotent matrix). He used them to classify *massive Frobenius manifolds* (a certain axiomatisation of certain 2d topological field theories).

On one hand, in earlier work of Cecotti–Vafa the matrix entries of S were integers counting BPS states (solitons between vacua), and a natural braid group action was defined on the space of S. On the other hand Dubrovin found a braid group invariant Poisson structure in the case $n = 3$, that "looked to be new". Here are some key excerpts of his paper:

Theorem 3.2. *There exists a local one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Massive Frobenius manifolds} \\ \text{modulo transformations (B.2)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Stokes matrices of differential} \\ \text{operators } \Lambda \text{ modulo transformations (3.164)} \end{array} \right\}.$$

Definition 3.5. The Stokes matrix S of the operator (3.120) considered modulo the transformations (3.164) will be called *Stokes matrix of the Frobenius manifold.*

Remark 3.10. In the paper [29] Cecotti and Vafa found a physical interpretation of the matrix entries S_{ij} for a Landau - Ginsburg TFT as the algebraic numbers of solitons propagating between classical vacua. In this interpretation S always is an integer-valued matrix. Due to (3.134) they arrive thus at the problem of classification of integral matrices S such that all the eigenvalues of $S^T S^{-1}$ are unimodular. This is the main starting point in the programme of classification of $N = 2$ superconformal theories proposed in [29].

It is interesting that *the same* Stokes matrix appears, according to [29], in the Riemann - Hilbert problem of [51] specifying the Zamolodchikov (or $t\,t^*$) hermitean metric on these Frobenius manifolds.

Example F.2. For $n = 3$ we put $s_{12} = x$, $s_{13} = y$, $s_{23} = z$. The transformations of the braid group act as follows:

$$\sigma_1 : \ (x, y, z) \mapsto (-x, z - xy, y), \tag{F.18a}$$

$$\sigma_2 : \ (x, y, z) \mapsto (y - xz, x, -z). \tag{F.18b}$$

These preserve the polynomial

$$x^2 + y^2 + z^2 - xyz. \tag{F.19}$$

Indeed, the characteristic equation of the matrix $S^T S^{-1}$ has the form

$$(\lambda - 1)[\lambda^2 + (x^2 + y^2 + z^2 - xyz - 2)\lambda + 1] = 0. \tag{F.20}$$

The action of the group B_3 (in fact, this can be reduced to the action of $PSL(2, \mathbf{Z})$) admits also an invariant Poisson bracket

$$\begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x\,. \\ \{z, x\} &= zx - 2y \end{aligned} \tag{F.21}$$

The polynomial (F.19) is the Casimir of the Poisson bracket. Thus an invariant symplectic structure is induced on the level surfaces

$$x^2 + y^2 + z^2 - xyz = \text{const}.$$

A B_n-invariant Poisson bracket exists also on the space of Stokes matrices of the order n. But it has more complicated structure.

For integer x, y, z this action on the invariant surface $x^2 + y^2 + z^2 = xyz$ was discussed first by Markoff in 1876 in the theory of Diophantine approximations [27]. The general

action (F.13b), (F.14) (still on integer valued matrices) appeared also in the theory of exceptional vector bundles over projective spaces [128]. Essentially it was also found from physical considerations in [29] (again for integer matrices S) describing "braiding of Landau - Ginsburg superpotential". The invariant Poisson structure (F.21) looks to be new.

These surfaces $x^2 + y^2 + z^2 - xyz = \text{const}.$ are of course easily seen to be isomorphic to Fricke–Klein–Vogt surfaces with the same braid group action, but Dubrovin's example was inspiring since the braid group action is not arising from the motion of the poles of the operator Λ (which has only two poles, at $0, \infty$). Rather the braid group action came from the motion of the matrix U, the leading coefficient of the leading term Udz of the connection, with a pole of order 2 at $z = \infty$, the irregular part of the connection.

This led to the realization that there is a whole new paradigm for (Poisson) braid group actions on spaces of monodromy data: if you include irregular connections there is a new type of braiding that is possible, where the space of deformation parameters is related to the structure group, not just the pole positions (and the moduli of Riemann surface). This idea led to the natural appearance of G-braid groups in 2d gauge theory [], the new topological symplectic structures on Stokes data in general [], and the wild nonabelian Hodge correspondence on curves [], that we want to describe in detail. Some aspects of such braiding had been studied earlier (by Garnier, Malgrange, Jimbo–Miwa–Ueno) for generic connections on vector bundles.

7.6. **Abelian Picard–Fuchs/Gauss-Manin example.** [[To Add]]

8. Lecture 4: Basic definitions

Before proceeding we will set-up some basic definitions, giving vocabulary for some types of meromorphic connections. (The simple definition of meromorphic connection we gave looks to be too naive for all the desired correspondences to work smoothly.)

$$\text{holomorphic} \;\subset\; \text{logarithmic} \;\subset\; \cdots \;\subset\; \text{very good} \;\subset\; \text{good} \subset\; \cdots$$

Suppose $(E, \nabla) \to \Sigma$ is a meromorphic connection with poles at the points $\mathbf{a} \subset \Sigma$.

Definition 8.1. *The meromorphic connection (E, ∇) is logarithmic if at each point $a \in \mathbf{a}$ it has a pole of order ≤ 1, so takes the form*

$$\nabla = d - A, \quad A = \Lambda \frac{dz}{z} + \textit{holomorphic}$$

is any local trivialisation, for some matrix $\Lambda \in \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$.

This is the simplest generalisation of holomorphic connections. Classically such connections were sometimes said to be of the "first kind". Note that a Fuchsian system is thus just a logarithmic connection on a trivial vector bundle on the Riemann sphere. (Beware that the definition logarithmic connections on higher dimensional complex manifolds is more subtle, and it is not enough to just have a simple pole.)

Definition 8.2. *The meromorphic connection (E, ∇) is tame or regular singular at a if any horizontal section on Σ° has at most polynomial growth at a along any ray (when working in any local trivialisation of E across the pole). A connection that is not tame is said to be wild or irregular singular.*

These are standard definitions. The basic facts one should know are that any logarithmic connection is tame, and a connection is tame if and only if there is a meromorphic gauge transformation relating it to a logarithmic connection. Thus we think of non-logarithmic tame connections as being "logarithmic connections written in a bad meromorphic trivialisation".

Exercise 8.3. Write down a tame meromorphic connection with a pole of order 2.

Definition 8.4. *If the meromorphic connection (E, ∇) has a pole of order ≥ 2 at a then it is said to be generic at a if the leading coefficient has n distinct eigenvalues. Thus in some trivialisation it takes the form:*

$$\nabla = d - A, \quad A = \left(\frac{A_k}{z^k} + \cdots \frac{A_1}{z} + \Lambda \right) \frac{dz}{z} + \textit{holomorphic}$$

for some matrix $A_k = \mathrm{diag}(c_1, \ldots, c_n)$ with $c_i \neq c_j$ for all $i \neq j$.

These are the simplest connections beyond the logarithmic case. We will see below that they are not tame. More generally:

Definition 8.5. *A meromorphic connection* (E, ∇) *is* *very good* *if at each point* $a \in \mathbf{a}$ *there is a local trivialisation of* E *such that* ∇ *takes the form*

$$\nabla = d - A, \quad A = dQ + \Lambda\frac{dz}{z} + \textit{holomorphic terms}$$

where:

- $Q = \sum_1^k A_i/z^i$ *(the irregular type) is a diagonal matrix of meromorphic functions,*
- Λ *(the formal residue) is a constant matrix that commutes with each coefficient of* Q *and* z *is a local coordinate vanishing at* a.

Thus the irregular type Q is a polynomial in $1/z$ with coefficients in the Lie algebra $\mathfrak{t} \subset \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ of diagonal matrices, and zero constant term.

For later use we will need a coordinate independent definition of irregular type:

Suppose $a \in \Sigma$ is a point. Then we can consider the formal completion

$$\widehat{\mathcal{O}}_a \cong \mathbb{C}[\![z]\!]$$

of the ring of functions at a and its field of fractions

$$\widehat{\mathcal{K}}_a \cong \mathbb{C}(\!(z)\!).$$

If we choose a coordinate then $\widehat{\mathcal{O}}_a$ is isomorphic to the ring of formal power series $\mathbb{C}[\![z]\!]$, i.e. the ring of expressions of the form $\sum_0^\infty a_i z^i$ where the $a_i \in \mathbb{C}$ are arbitrary. Similarly $\widehat{\mathcal{K}}_a \cong \mathbb{C}(\!(z)\!)$ and elements of this are formal Laurent series, so of the form f/z^k for some integer k and some $f \in \mathbb{C}[\![z]\!]$.

Definition 8.6. *Given* $\mathfrak{t} \subset \mathfrak{g}$ *as above and a point* $a \in \Sigma$ *then an* *irregular type* *at* a *is an element*

$$Q \in \mathfrak{t} \otimes \widehat{\mathcal{K}}_a/\widehat{\mathcal{O}}_a.$$

Of course if we choose a coordinate then $\widehat{\mathcal{K}}_a/\widehat{\mathcal{O}}_a \cong z^{-1}\mathbb{C}[z^{-1}]$ and we reduce to the original definition. The point is that we can now discuss deformations of an irregular type at a varying point a on the surface in an intrinsic fashion.

Later we will recall the notion of parabolic vector bundles, and extend the notion of "very good" to connections on parabolic vector bundles, and then define the slightly more general notion of "good" connections (those that reduce to very good connections after pullback along a finite cyclic cover).

9. Geometry at the boundary

Given a Riemann surface Σ and a point $a \in \Sigma$ then we will define a circle ∂ (the circle of directions at a) and a covering space $\pi : \mathcal{I} \to \partial$ of the circle of directions, i.e. a local systems of sets. $\mathcal{I}$ is called the exponential local system each component $I \subset \mathcal{I}$ is a called a *Stokes circle* and π restricts to a finite cover

$$\pi : I \to \partial.$$

As we will see, this is just a geometric way to think about all the possible exponential factors $\exp(q)$ appearing in solutions of meromorphic connections. It is very convenient and will be essential in order to understand the Stokes data intrinsically. (Examples of Stokes circles appeared in Stokes' 1857 paper and Fabry 1885 found formal solutions of arbitrary meromorphic linear differential equations.)

9.1. **Circle of directions.** If $T_a\Sigma \cong \mathbb{C}$ is the tangent space at a then

$$\partial = \partial_a = ((T_a\Sigma) \setminus \{0\})/\mathbb{R}_{>0} \cong S^1$$

is the set of real oriented directions at a. We picture a point $d \in \partial$ of the circle of directions as a little arrow at a pointing in the direction d.

Given a direction $d \in \partial$ let Sect_d denote a small open sector with vertex at a spanning the direction d. We will shrink this sector whenever convenient (both its radius and its opening)—it is a germ of an open sector (and is a subset of the open curve $\Sigma^\circ = \Sigma \setminus a$).

9.2. **Exponential local system.** The exponential local system is a natural covering space (local system of sets) $\pi : \mathcal{I} \to \partial$. It can be defined intrinsically but is much easier to define using a coordinate: If z is a local coordinate on Σ vanishing at a and we choose a direction $d \in \partial$ and a branch of $\log(z)$ on Sect_d then the germs of local sections of $\mathcal{I}$ over d are the functions on Sect_d that may be written as finite sums of the form

$$q = \sum a_i z^{-k_i} \tag{9.1}$$

where $a_i \in \mathbb{C}$, and $k_i \in \mathbb{Q}_{>0}$. As usual $z^{-k} := \exp(-k\log(z))$ in this expression. Thus any such q is a polynomial in t with zero constant term where $t = z^{-1/r} = \exp(-\log(z)/r)$ for some integer $r \geq 1$.

Given such a function q on Sect_d then the analytic continuation of q around a will have finite monodromy, so it will return to the same function on Sect_d after a finite number of turns. Let

$$I = \langle q \rangle \to \partial$$

denote the covering circle of ∂ that parameterises all these branches of the function q (essentially the germ of the Riemann surface of the function q as it is continued around a). This is the *Stokes circle* of q.

Thus a point $i \in I$ determines a function q_i on Sect_d where $d = \pi(i) \in \partial$ is the direction below the point $i \in I$ (and q_i is a certain branch of the continuation of the original function q).

Thus in brief the covering space $\mathcal{I} \to \partial$ is the disjoint union of all the Stokes circles that arise from all such functions q (and two such function determine the same circle if they are analytic continuations of each other, i.e. if they are in the same Galois orbit).

Remark 9.1. An intrinsic (coordinate independent) construction of $\mathcal{I}$ is given in [**?**] Rmk 3 (in which case sections of $\mathcal{I}$ are certain equivalence classes of functions on sectors, but that will make no difference in the use of these functions below).

An isomorphic local system "$d\mathcal{I}$" (whose sections are one-forms) was used by Deligne and Malgrange in [**?**, **?**].

9.3. **Numerical invariants of a Stokes circle.**

Any Stokes circle $I = \langle q \rangle \subset \mathcal{I}$ has three numbers attached to it:

- The ramification $\text{Ram}(q) \in \mathbb{Z}_{\geq 1}$ is the degree of the cover $\pi : \langle q \rangle \to \partial$, i.e the number of points in any fibre of this map. This is the number of branches that the function q has, and it is the lowest common multiple of the denominators of the k_i present in the expression for q. If $\text{Ram}(q) = 1$ then $I = \langle q \rangle$ is *unramified* or *untwisted.*
- $\text{slope}(q) \in \mathbb{Q}_{\geq 0}$ is the largest k_i occurring in (9.1), or $\text{slope}(q) = 0$ if $q = 0$.
- The irregularity $\text{Irr}(q) \in \mathbb{N}$ is the product:

$$\text{Irr}(q) = \text{slope}(q)\text{Ram}(q).$$

Some pictures of Stokes circles will be drawn below, and then it will become clear that the irregularity is the "number of wiggles" (in a precise sense) in the Stokes circle (or equivalently the number of points of maximal decay).

The circle $\langle 0 \rangle \subset \mathcal{I}$ is called the *tame circle* and it is the only Stokes circle with irregularity zero.

9.4. **Irregular classes.** An *irregular class* Θ (at a) is a finite multiset of Stokes circles at a (i.e. a set with positive integer multipicities). Thus it can be written as a finite formal sum

$$\Theta = n_1 I_1 + \cdots + n_s I_s$$

where $n_i \in \mathbb{N}$ and the $I_i \subset \mathcal{I}$ are distinct Stokes circles (but the ordering of the Stokes circles is not part of the data).

Said differently an irregular class is (the same thing as) a continuous map

$$\Theta : \mathcal{I} \to \mathbb{N}$$

(constant on each circle) assigning an integer to each point of $\mathcal{I}$, equal to zero for all but a finite number of circles. (Thus Θ amounts to a map $\pi_0(\mathcal{I}) \to \mathbb{N}$ on the set π_0 of connected components.)

The *rank* of an irregular class Θ is the integer

$$\mathrm{rk}(\Theta) = \sum n_i \mathrm{Ram}(I_i) = \sum_{i \in \mathcal{I}_d} \Theta(i) \in \mathbb{N} \tag{9.2}$$

for any direction $d \in \partial$, where $\mathcal{I}_d$ denotes the fibre of the cover $\mathcal{I}$ over the direction d.

Given an irregular type

$$Q = \mathrm{diag}(q_1, \dots, q_n)$$

with each $q_i \in z^{-1}\mathbb{C}[z^{-1}]$ then the corresponding irregular class is

$$\Theta = \langle q_1 \rangle + \cdots + \langle q_n \rangle$$

which has rank n (this means we just remember the unordered set of q_i present and the multiplicities that they are repeated with in Q).

Proposition 9.2. *Any meromorphic connection $(E, \nabla) \to \Sigma$ has a rank n irregular class Θ_a at each singular point $a \in \Sigma$, where $n = \mathrm{rank}(E)$.*

Of course there is really an irregular class at any point of Σ, its just that it is "trivial" in the sense that it equals $n\langle 0 \rangle$ (the tame circle with multiplicity n) at all nonsingular points (as happens also at any logarithmic or tame singularity).

The irregular class is really just the Galois closed list of exponents of the exponential factors appearing in any (formal) basis of solutions at a, so this will follow when we review the formal classification later on. For example it is known that one can do a finite cyclic pullback and then a meromorphic gauge transformation to reduce any connection a very good connection, then the irregular type of this very good connection determines the irregular class of the original connection. The irregular class only depends on the restriction of the connection to the formal punctured disk at a (so any two connections related by $\mathrm{GL}_n(\mathbb{C}((z)))$ will have the same irregular class).

Exercise 9.3. Consider the rank two irregular class Θ at $x = \infty$ of $y'' = fy$ for a polynomial $f \in \mathbb{C}[x]$, i.e. the irregular class of the connection

$$d - A, \qquad A = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} dx.$$

Show, by pulling back under $x = t^2$ and diagonalising the irregular part, that $\Theta = \langle \pm \int \sqrt{f} dx \rangle$, or more precisely:

1) If f has even degree then we can choose $p \in \mathbb{C}((x^{-1}))$ such that $p^2 = f$, and in turn choose $q \in x\mathbb{C}[x]$ such that $dq - pdx \in \mathbb{C}[\![x^{-1}]\!]dx/x$. Then $\Theta = \langle q \rangle + \langle -q \rangle$.

2) If f has odd degree then we can choose $p \in \mathbb{C}((t^{-1}))$ such that $p^2 = f$ where $t^2 = x$, and in turn choose $q \in t\mathbb{C}[t]$ such that $dq - 2ptdt \in \mathbb{C}[\![t^{-1}]\!]dt/t$. Then $\Theta = \langle q \rangle$.

For example $y'' = 9xy$ has class $\Theta = \langle 2x^{3/2} \rangle$ at ∞ as in Stokes' 1857 paper [**?**].

Of course to get the irregular class in practice, for any given explicit connection, one can just use a computer algebra package to compute a basis of formal solutions and then look at the exponential factors $\exp(q)$ that appear.

There is a similar fact for meromorphic Higgs bundles:

Proposition 9.4. *Any meromorphic Higgs bundle* $(E, \Phi) \to \Sigma$ *has a rank* n *irregular class* Θ_a *at each singular point* $a \in \Sigma$*, where* $n = \mathrm{rank}(E)$.

Proof. In brief we take the integrals of the irregular part of the eigenforms of Φ. In more detail: Locally we can write $\Phi = B(z)dz/z$ for some $n \times n$ matrix of meromorphic functions B. Restricting to the formal punctured disk at a we can view the matrix entries as living in $\widehat{\mathcal{K}} = \mathbb{C}((z))$. Thus B has n eigenvalues in the algebraic closure of $\widehat{\mathcal{K}}$. This means there is an integer $s \geq 1$ and n eigenvalues $b_i \in \mathbb{C}((t))$ where $t^s = z$. Thus $\Phi = B(z)dz$ has n "eigenforms" $sb_i(t)dt/t$ (since $dz/z = sdt/t$). Then we can throw away the logarithmic parts and integrate to get $q_i(t) \in t^{-1}\mathbb{C}[t^{-1}]$ such that

$$dq_i = sb_i(t)dt/t + f_i(t)dt/t$$

for some nonsingular $f_i(t)$. Then the q_i (viewed as functions of z on any small sector) determine the function germs making up the irregular class of the Higgs bundle. Beware that s might be larger than the total ramification of the Stokes circles in the irregular class since we are truncating. □

9.5. **Wild Riemann surfaces.** It turns out that the irregular class makes up the basic "new modular parameters" that occur for irregular connections, behaving just like the modulus of the underlying Riemann surface and the location of the marked points $\mathbf{a}$. In particular it behaves completely differently to the formal residue Λ. This motivates the following definition.

Definition 9.5. *A rank* n *wild Riemann surface is a triple* $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ *where* Σ *is a Riemann surface,* $\mathbf{a} \subset \Sigma$ *is a finite subset and* $\Theta = \{\Theta_a \mid a \in \mathbf{a}\}$ *is the data of a rank* n *irregular class at each point* $a \in \mathbf{a}$.

Here we are mainly interested in the case where Σ is compact. We will define the character variety $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma})$ of any such wild Riemann surface, show that it is Poisson and forms a local system of varieties under any admissible deformation of $\boldsymbol{\Sigma}$.

Of course if all the irregular classes are trivial then $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ just amounts to choosing a Riemann surface with some marked points, and then $\mathcal{M}_{\mathrm{B}}(\boldsymbol{\Sigma})$ will be the usual (tame) character variety defined previously $\cong \mathrm{Hom}(\pi_1(\Sigma^\circ, b), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$.

Notes: This definition is from [**?**] Defn 8.1, Rmk 10.6, [**?**] §4. There are several minor variations that we won't worry about here, but are sometimes useful: One can work with irregular types instead of irregular classes (which were called "bare irregular types" in [**?**] Rmk 10.6); this is analogous to whether or not we order the points $\mathbf{a}$. Also one can work with smooth complex algebraic curves instead of Riemann surfaces (which doesn't make much difference in the compact case); the terms "irregular curve" or "wild curve" are sometimes used to replace the term "wild Riemann surface" in the algebraic case. Op. cit. give the definition for any complex reductive group, not just $\mathrm{GL}_n(\mathbb{C})$.

10. More geometry at the boundary

10.1. Points of maximal decay.

Given a Stokes circle $I = \langle q\rangle \subset \mathcal{I}$ at $a \in \Sigma$ then there is a distinguished finite subset the *points of maximal decay*:

$$\text{🍏}(q) \subset I$$

They will sometimes be called the "p.o.m.s" (or apples). The number of them is equal to the irregularity:

$$\boxed{\#\text{🍏}(q) = \mathrm{slope}(q)\cdot \mathrm{Ram}(q) = \mathrm{Irr}(q)}$$

so that $\text{🍏}(q)$ is empty if and only if I is the tame circle. A key point to note is that they are defined to be in $I \subset \mathcal{I}$ and not directly as a set of directions (in ∂).[12]

In brief if $I \neq \langle 0\rangle$ then $i \in I$ is a point of maximal decay if and only if the function $\exp(q_i)$ on Sect_d has maximal decay along the direction $d \in \partial$ (compared to the rate of decay along other nearby directions), where $d = \pi(i) \in \partial$ is the direction below $i \in I \subset \mathcal{I}$. Here q_i is the function germ on Sect_d determined by $i \in I$.

Putting all these together defines an infinite subset $\text{🍏} \subset \mathcal{I}$, so that $\text{🍏} \cap \langle q\rangle = \text{🍏}(q)$.

From this definition the set 🍏 is clearly well-defined (independent of any coordinate choice). If we have a coordinate z vanishing at a then the locations of the points of maximal decay only depend on the leading term of q (although lower terms may well affect the ramification degree of the cover $\langle q\rangle \to \partial$). Here are some examples.

• If $q = 1/z$ then $\langle q\rangle \to \partial$ is a trivial (degree one) cover, identifying $\langle q\rangle$ and ∂. $\text{🍏}(q)$ consists of the single direction where $1/z$ is real and negative, i.e. $\arg(z) = \pi$.

• If $q = z^{-k}$ with $k \in \mathbb{N}$ then $\langle q\rangle \to \partial$ is still a trivial degree 1 cover and $\text{🍏}(q)$ consists of the k directions where $1/z^k \in \mathbb{R}_-$, i.e. $z \in e^{(1+2j)\pi i/k}\cdot \mathbb{R}_+$ for $j = 1,\dots,k$.

• Similarly if $q = \lambda/z^k + \sum_1^{k-1} a_i/z^i$ with $\lambda \in \mathbb{C}^*, k \in \mathbb{N}$ then $\text{🍏}(q)$ consists of the k directions where λ/z^k is real and negative.

• If $q = z^{-1/r}$ then $\langle q\rangle \to \partial$ is a degree r cover. Lets choose a coordinate t upstairs, so $z = t^r$ and $q = 1/t$. Then $\text{🍏}(q) \subset \langle q\rangle$ consists of the single point where $t \in \mathbb{R}_-$.

• Similarly if $q = z^{-k/r}$ with k and r coprime, then $\langle q\rangle \to \partial$ is a degree r cover. Lets choose a coordinate t upstairs, so $z = t^r$ and $q = 1/t^k$. Then $\text{🍏}(q)$ consists of the k points where $1/t^k$ is real and negative, i.e. $t \in e^{(1+2j)\pi i/k}\cdot \mathbb{R}_+$ for $j = 1,\dots,k$.

[12] we have a mental picture of the apples being in the tree and not yet fallen to the ground...

• Similarly if $q = \lambda z^{-k/r} + \sum_1^{k-1} a_i z^{-i/r}$ with $\lambda \in \mathbb{C}^*$ and k, r coprime, then 🍎(q) consists of the k points where λ/t^k is real and negative. If k, r are not coprime but $\mathrm{Ram}(q) = r$ nonetheless, then 🍎(q) still consists of the k points where $\lambda/t^k \in \mathbb{R}_-$. For example $\langle x^3 + x^{1/2} \rangle$ has 6 apples and $\langle x^{5/3} + x^{1/18} \rangle$ has 30 (where $x = 1/z$).

Similarly one can define the "points of decay", "points of growth" and "points of indeterminacy" in any non-tame Stokes circle $I \subset \mathcal{I}$. These three sets of points partition I and it is easy to see there are $2\mathrm{Irr}(q)$ points of indeterminacy, interlaced with alternating open intervals of points of growth and decay. Each interval of decay contains a unique point of maximal decay (similarly each interval of growth contains a unique point of maximal growth, but we won't need to use those points).

In simple examples this growth/decay can be easily visualised in the Stokes diagram, as in the example of $q = x^{17}$ in Figure 4, where the singularity is at $a = \infty$ (so $z = x^{-1}$ is a local coordinate vanishing at a). For example we see on the positive real axis that the function $\exp(x^{17})$ has maximal growth there, and there are 16 other evenly spaced directions of maximal growth, interlaced with 17 directions of maximal decay, the first at $\arg(x) = \pi/17$.

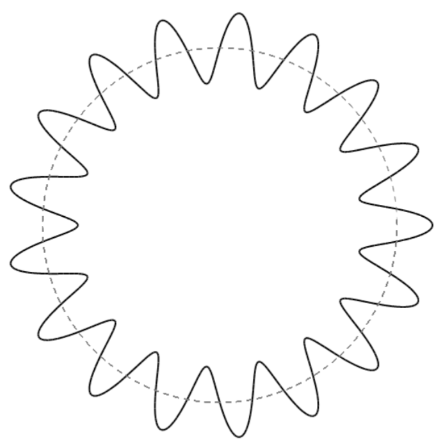

FIGURE 4. Stokes diagram for $\langle x^{17} \rangle$: the Stokes circle $\langle x^{17} \rangle$ is projected to the plane so as to indicate the growth/decay of $\exp(x^{17})$ near ∞.

10.2. **Dominance orderings and Stokes/oscillating directions.** By looking at the growth rates of the functions $\exp(q)$ near a, there is a partial ordering $<_d$ (exponential dominance) on each fibre of the cover $\pi : \mathcal{I} \to \partial$, defined as follows.

Suppose $d \in \partial$ and $i, j \in \mathcal{I}_d$ are distinct points of the fibre. Thus they correspond to function germs q_i, q_j on Sect_d. Then (by definition)

$$i <_d j, \qquad \text{or} \qquad q_i <_d q_j$$

if $\exp(q_i - q_j)$ is flat (has zero asymptotic expansion) on some open sectorial neighbourhood of d. In other words the point $q_i - q_j \in \mathcal{I}_d$ is a point of decay.

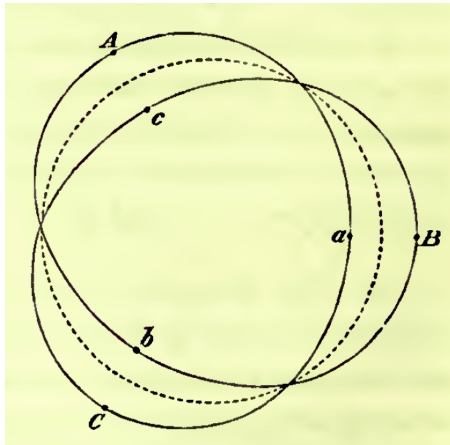

FIGURE 5. The Stokes diagram of $\langle 2x^{3/2} \rangle$, from Stokes' paper [?] on the Airy equation. The points a, b, c are the points of maximal decay.

The notation $i \leq_d j$ is defined to mean "either $i <_d j$ or $i = j$".

Given an irregular class $\Theta = \sum n_i I_i$ let

$$I = \bigcup I_i \subset \mathcal{I}$$

be the set of Stokes circles present in Θ, the "active exponents" or "finite subcover" of $\mathcal{I}$ determined by Θ. Thus $\pi : \mathcal{I} \to \partial$ restricts to I expressing it as a finite cover $\pi : i \to \partial$.

The dominance relation $<_d$ restricts to a partial order on each fibre $I_d = \pi^{-1}(d) \subset I$ of I. Since I is a finite cover of ∂, this is actually a total order on the finite set I_d for all but a finite number of directions

$$\mathbb{S} \subset \partial$$

the Stokes directions (or oscillating directions) of the irregular class Θ.

Thus if $d \in \partial \setminus \mathbb{S}$ is not a Stokes direction then I_d is totally ordered by the exponential dominance relation $<_d$.

In simple examples the changes of dominance ordering can be easily visualised by drawing the Stokes diagram of the irregular class, as in Fig. 5 for the example of $\Theta = \langle 2x^{3/2} \rangle$ for the version $y'' = 9xy$ of the Airy equation studied by Stokes in 1857. The Stokes directions are the three directions where the two strands cross, at $\arg(x) = \pm\pi/3, \pi$. This figure was also reproduced on the cover of [?].

There is a javascript program here:

https://webusers.imj-prg.fr/~philip.boalch/stokesdiagrams.html to draw lots of other examples of Stokes diagrams, the "symmetric Stokes diagrams" (see the explanation in the box at the bottom there).[13]

10.3. **Singular directions and Stokes arrows.**

The *Stokes arrows* lying over a direction $d \in \partial$ are the ordered pairs $(i, j) \in \mathcal{I}_d \times \mathcal{I}_d$ such that the difference

$$q_i - q_j \in \circlearrowleft \cap \mathcal{I}_d$$

is a point of maximal decay, where q_i, q_j are the functions on Sect_d determined by i, j respectively. In this case we will write

$$i \prec_d j$$

It is viewed as an arrow from the point j to the point i. This relation defines a partial order on each fibre $\mathcal{I}_d$ and exponential dominance refines it (if $q_i \prec_d q_j$ then $q_i <_d q_j$), since a point of maximal decay is, in particular, a point of decay. These arrows are important as they give the geometric way to define the non-trivial matrix entries in the Stokes matrices in the wild monodromy relations (in the presentations of the wild character varieties).

Given an irregular class Θ with active exponents $I \subset \mathcal{I}$ then there are only a finite number of Stokes arrows in $I \times I$.

These Stokes arrows lie over a finite set

$$\mathbb{A} \subset \partial$$

called the singular directions (or anti-Stokes directions [14]).

The Stokes quiver at a singular direction $d \in \mathbb{A}$ is the quiver given by the set of Stokes arrows $\prec_d$ at d (and nodes given by the finite set I_d).

10.4. **Simple example.**

Consider Weber's equation $y'' = (x^2/4 + \lambda)y$ where $\lambda \in \mathbb{C}$ (the equation for the parabolic cylinder functions) and the corresponding connection

$$\nabla = d - A, \quad A = \begin{pmatrix} 0 & 1 \\ x^2/4 + \lambda & 0 \end{pmatrix} dx.$$

This has just one singularity, at $x = \infty$. A short computation, or a glance at [?] §19.8, shows the formal solutions at ∞ involve the multivalued functions $f_\pm = \exp(q_\pm)x^{\pm\lambda - 1/2}$ where $q_\pm = \pm x^2/4$. Thus $\Theta = \langle q_+ \rangle + \langle q_- \rangle \to \partial$, with each circle $\langle q_\pm \rangle$ a trivial degree one cover.

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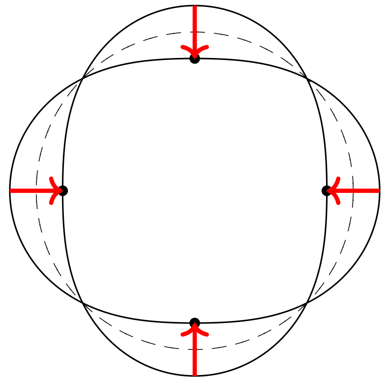
[13] Note that such diagrams are only useful for sufficiently simple examples (those with just "one level" in the sense that $\mathrm{slope}(q_i - q_j)$ is constant for all $i, j \in I_d$).

[14] in fact the terms "Stokes directions" and "anti-Stokes directions" are swapped in some papers, so we will try to prefer the unambiguous terms "oscillating directions" and "singular directions".



Stokes diagram of the Weber equation, with Stokes arrows drawn.

The exponential factors $\exp(q_\pm)$ here are the main contributors to the behaviour of solutions near $x = \infty$, and their dominance is encoded in the Stokes diagram in the figure drawn.

From this we see immediately the oscillating directions $\mathbb{S} \subset \partial$ are the four directions with argument $\pi/4 + k\pi/2$ (where the dominance changes), and the singular directions $\mathbb{A} \subset \partial$ are the real and imaginary axes (where the ratio of dominances is largest).

The apples (points of maximal decay) are the four points of I that project to the four marked points on the diagram, at the heads of the Stokes arrows.

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