

Poisson geometry of moduli spaces of
meromorphic connections on curves
and their Stokes data

P. Boalch (ENS Paris)

Complex character varieties

($G =$ connected complex reductive gp)

Σ



$\text{Hom}(\pi_1(\Sigma), G) / G$

Riemann surface

Poisson variety

Airyah-Bott, Goldman, Karshon, Farkas, Weinstein,

Guruprasad-Huebschmann-Jeffrey-Weinstein, Andersen-Mattes-Reshetikhin ...

generic symplectic leaves are hyperkähler manifolds (Hitchin)

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \cup \dots \cup \partial_m$ ($\partial_i \cong S^1$)

Choose basepoints $b_i \in \partial_i$

Let $\Pi = \Pi_1(\Sigma, \{b_1, \dots, b_m\})$

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& symplectic leaves are $\mu^{-1}(e)/\mathcal{G}^m$ ($e = (e_1, \dots, e_m) \in \mathcal{G}^m$)

Further if $\Sigma \rightarrow B$ is a family of Riemann surfaces
 $\Sigma_p \quad p \in B$

get algebraic Poisson action

$$\pi_1(B, p) \curvearrowright \text{Hom}(\pi_1(\Sigma_p), G)/G$$

"The Betti moduli spaces $M_B(\Sigma_p, G) = \text{Hom}(\pi_1(\Sigma_p), G)/G$
form a local system of varieties" (Simpson)

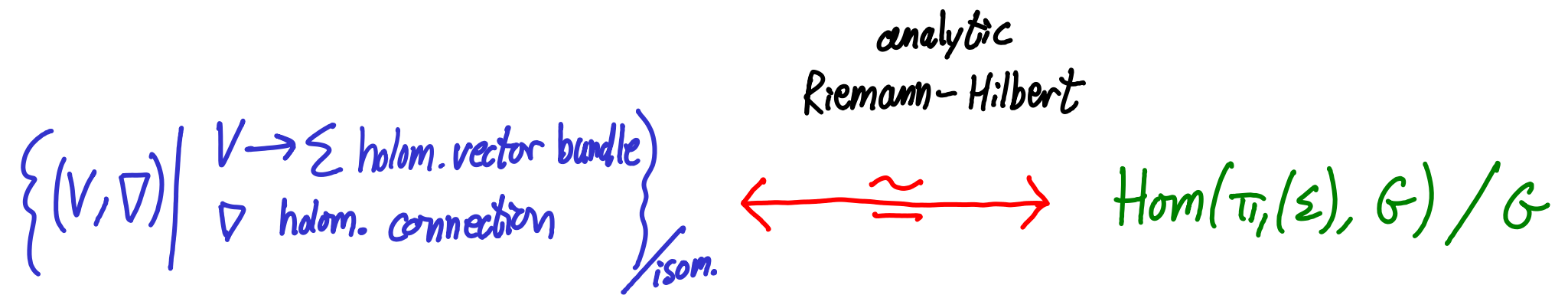
$\Sigma = \bar{\Sigma} \setminus \{a_1, \dots, a_m\}$ punctured smooth algebraic curve/ \mathbb{C}
 $G = \text{GL}_n(\mathbb{C})$

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$$\left\{ (V, \nabla) \mid \begin{array}{l} V \rightarrow \Sigma \text{ } \mathbb{C}^\infty \text{ vector bundle} \\ \nabla \text{ flat connection} \end{array} \right\} \xrightarrow[\text{isom.}]{} \overset{\mathbb{C}^\infty \text{ Riemann-Hilbert}}{\mathrm{Hom}(\pi_1(\Sigma), G) / G}$$

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$\left\{ (V, \nabla) \mid \begin{array}{l} V \rightarrow \Sigma \text{ alg. vector bundle} \\ \nabla \text{ alg. connection} \\ \text{with } \underline{\text{regular sing. s}} \end{array} \right\} / \text{isom.}$

Deligne - Plemelj;
 Riemann - Hilbert




$\mathrm{Hom}(\pi_1(\Sigma), G) / G$

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restrict
to Σ 

Deligne - Plemelj;
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$$\mathrm{Hom}(\pi_1(\Sigma), G) / G$$


- can now study transcendental aspects of RH map

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Riemann-Hilbert
-Birkhoff 

$\left\{ \text{Monodromy and Stokes data} \right\}$

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Riemann-Hilbert
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$\left\{ \text{Monodromy and Stokes data} \right\}$

- Irregular Atiyah-Bott (-'99, '01)
- Hyperkähler metrics (Biquard-B, '04)
- today: Poisson/sympl. structures algebraically

Definition

A holomorphic quasi-Hamiltonian G-space is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu: M \rightarrow \mathfrak{g}$ (G acts on \mathfrak{g} by conjugation)

such that

$$\textcircled{1} \quad d\omega = \mu^*(\eta)$$

$$\textcircled{2} \quad \forall X \in \mathfrak{g} \quad \omega(\nu_X, \cdot) = \frac{1}{2} \mu^*(\theta + \bar{\theta}, X)$$

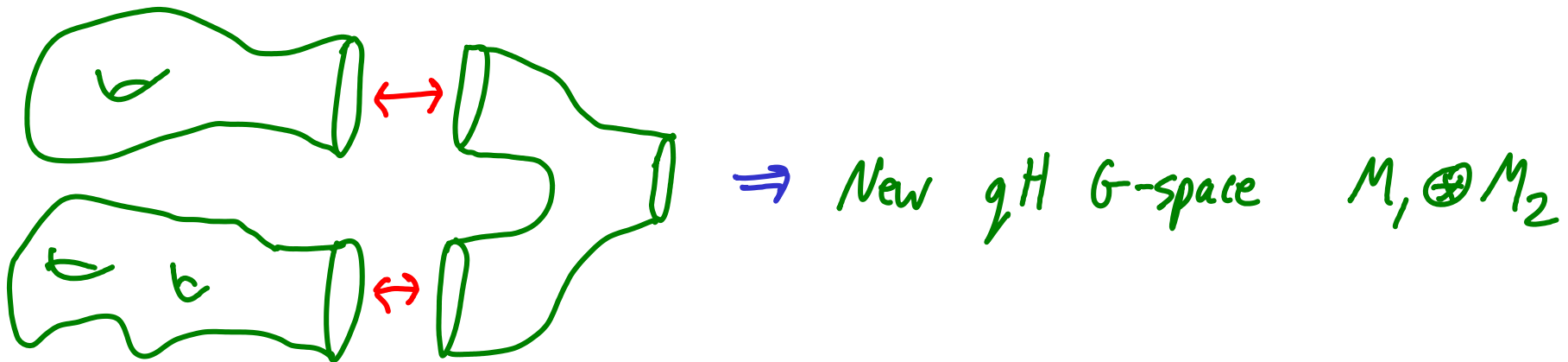
$$\textcircled{3} \quad \forall m \in M \quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$$

where $\eta =$ biinvariant 3-form on G , $\theta, \bar{\theta}$ Maurer-Cartan forms on G

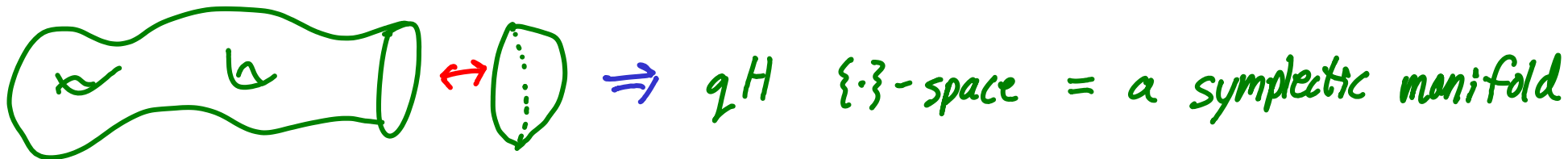
- These axioms are 'what we get from ω -d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with \mathfrak{g}^* -valued moment map)

Operations

① Can 'fuse' 2 qHamiltonian G-spaces:

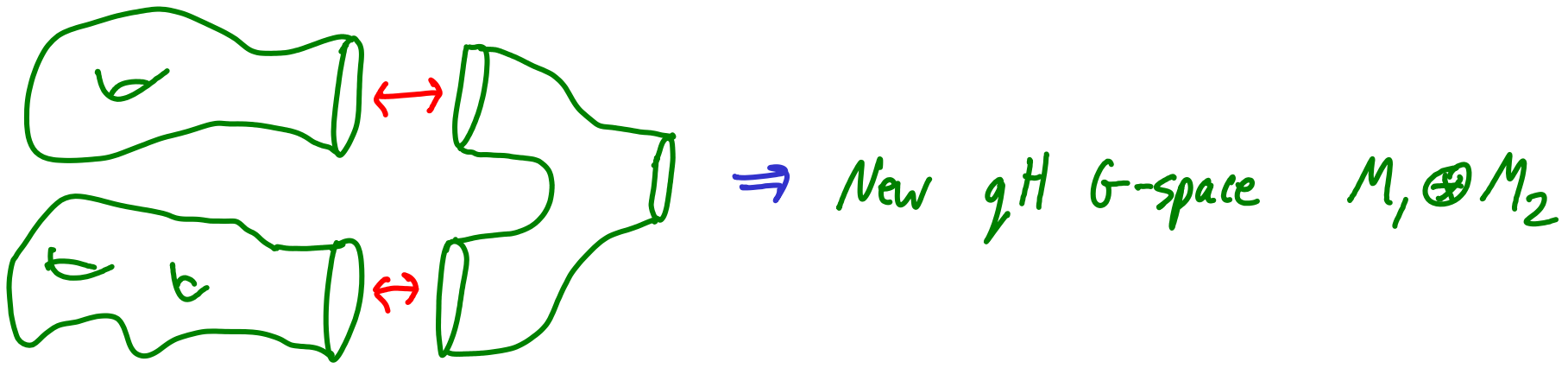


② & reduce:

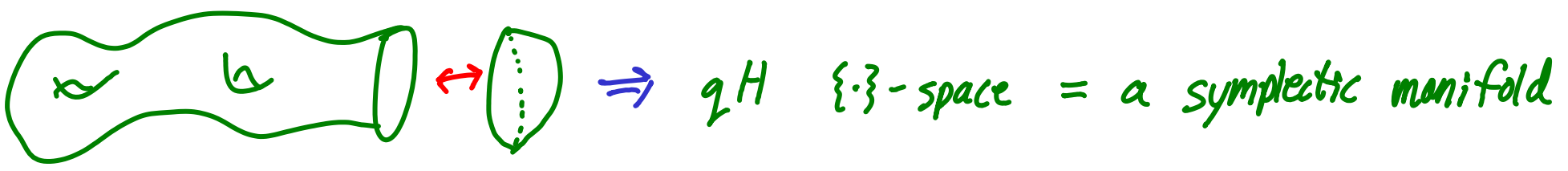


Operations

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Basic examples

① Conjugacy classes $\mathcal{C} \subset G$

② $D = G \times G$ qH $G \times G$ space (double)

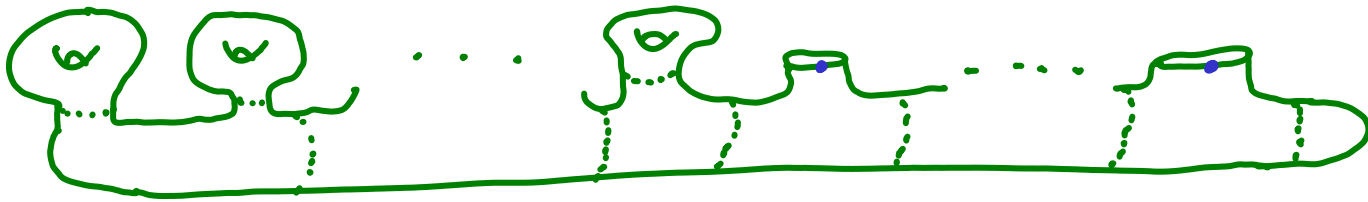


③ $1D = G \times G$ qH G-space (internally fused double)



Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

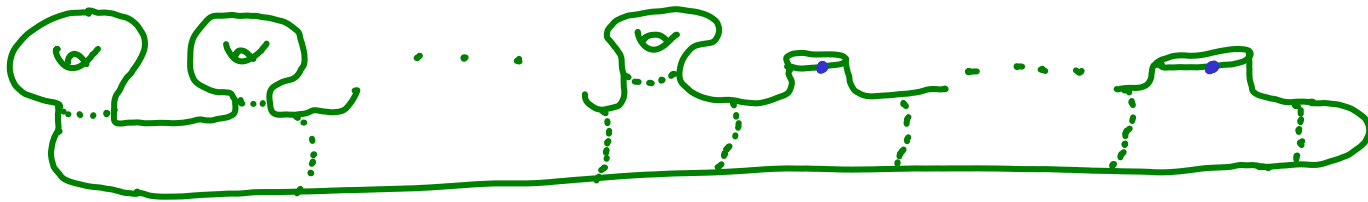
$$\underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_g \otimes \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_m // G \cong \text{Hom}(\pi, G)$$



$$\mu^{-1}(e) / G^m \cong \left\{ (A, B, M) \mid \prod_{i=1}^g [A_i, B_i] \prod_{i=1}^m M_i = 1, M_i \in e_i \right\} / G$$

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_g \otimes \underbrace{\mathbb{D} \otimes \dots \otimes \mathbb{D}}_m // G \cong \text{Hom}(\pi, G)$$



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Aim: New pieces to construct irregular Betti spaces?

New holomorphic q -Hamiltonian spaces

Choose $B_{\pm} \subset G$ opposite Borel subgroups

$T = B_{+} \cap B_{-}$ maximal torus

$U_{\pm} \subset B_{\pm}$ unipotent radicals

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$U_{\pm} \subset B_{\pm}$ unipotent radicals

e.g. $G = GL_n(\mathbb{C})$

$$U_+ \sim \begin{pmatrix} 1 & & & * \\ & \ddots & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$U_- \sim \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ * & & & 1 \end{pmatrix}$$

$$T \sim \begin{pmatrix} * & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & * \end{pmatrix}$$

New holomorphic \mathfrak{g} -Hamiltonian spaces

Choose $B_{\pm} \subset G$ opposite Borel subgroups

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$U_{\pm} \subset B_{\pm}$ unipotent radicals

Thm (- '02)

The space $G A_T^r := G \times (U_+ \times U_-)^r \times T$

is a quasi-Hamiltonian $G \times T$ space for any $r = 1, 2, \dots$

- moment map $\mu(C, s_1, \dots, s_{2r}, t) = (C^{-1} t s_{2r} \cdots s_1 C, t^{-1})$
- $G A_T^r / G \cong G^*$ as Poisson space (dual Poisson Lie group)

non commutative
non cocommutative

$U_q \mathfrak{g}$ Drinfeld-Jimbo

De Concini-Kac-Processi

$\text{Fun}(G^*)$

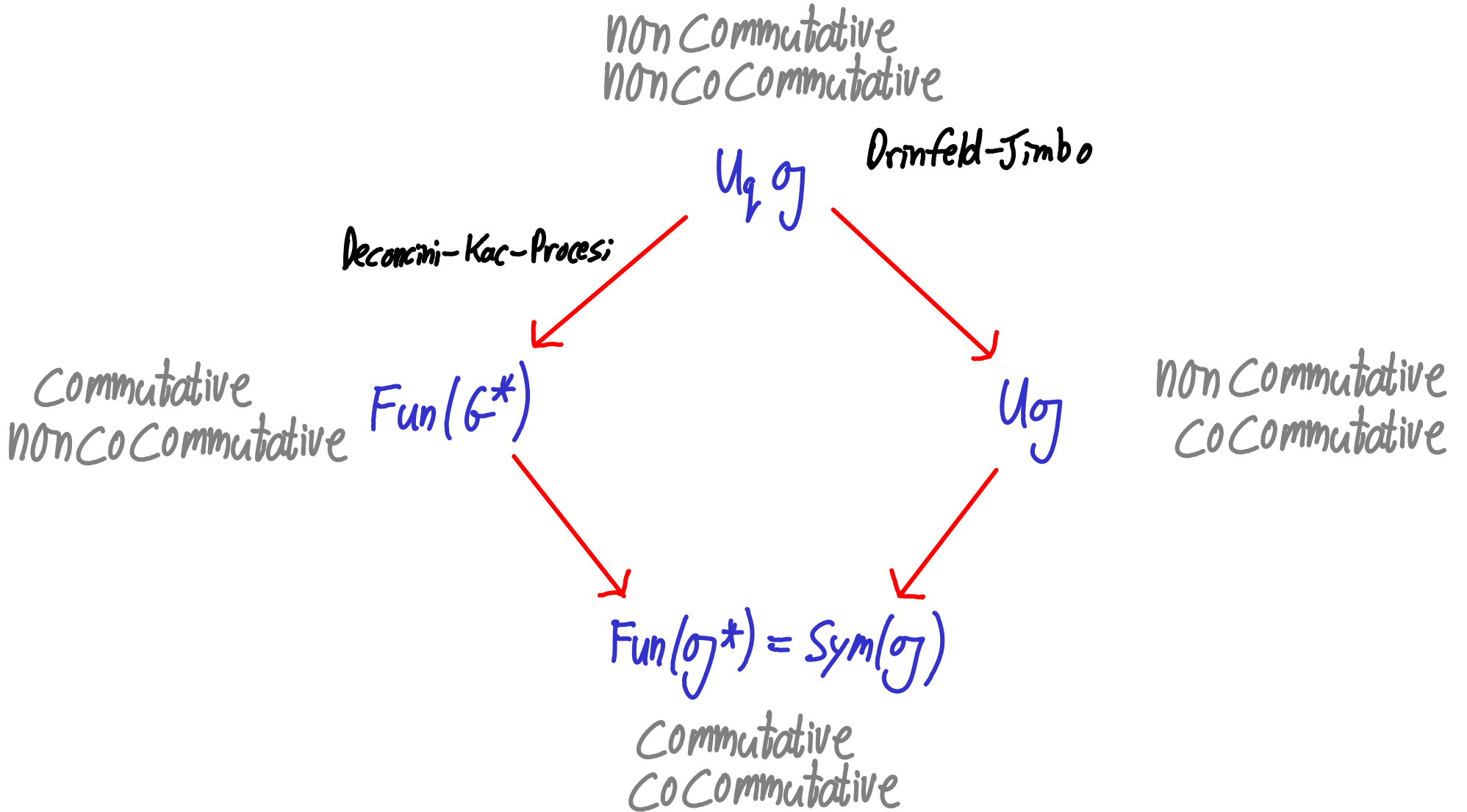
$U\mathfrak{g}$

non commutative
cocommutative

$\text{Fun}(\mathfrak{g}^*) = \text{Sym}(\mathfrak{g})$

commutative
cocommutative

commutative
non cocommutative



New holomorphic q -Hamiltonian spaces

Van den Bergh spaces:

Choose V, W complex vector spaces

$$\mathcal{B}(V, W) = \{(a, b) \in \text{Hom}(W, V) \oplus \text{Hom}(V, W) \mid \det(1 + ab) \neq 0\}$$

New holomorphic q -Hamiltonian spaces

Van den Bergh spaces:

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Thm (M. Van den Bergh '04, '07)

$\mathcal{B}(V, W)$ is a q -Hamiltonian $GL(V) \times GL(W)$ space

- moment map $\mu(a, b) = ((1 + ab)^{-1}, 1 + ba)$
- defines holom. symplectic structures on multiplicative quiver varieties
- such spaces classify regular holonomic \mathcal{D} -modules on a disc

Fission spaces

Choose $P_{\pm} \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_{\pm} \subset P_{\pm}$ unipotent radicals

Thm (- '02, '09, '11)

The "fission space" $G A_H^r := G \times (U_+ \times U_-)^r \times H$

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- moment map $\mu(C, s_1, \dots, s_{2r}, h) = (C^{-1} h s_{2r} \cdots s_1 C, h^{-1})$
- $(U_+ \times U_-)^r \cong$ Stokes data of meromorphic connections with one level

Example

Suppose $G = GL(V \oplus W)$, $H = GL(V) \times GL(W)$, $U_+ = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cong \text{Hom}(W, V)$
 $U_- = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \cong \text{Hom}(V, W)$

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Then $G \backslash A_H^2 // G$

$$\cong \left\{ (S_1, S_2, S_3, S_4, h) \in U_+ \times U_- \times U_+ \times U_- \times H \mid h S_4 S_3 S_2 S_1 = 1 \right\}$$

is a q-Hamiltonian $H = GL(V) \times GL(W)$ space ($\mu = h^{-1}$)

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If we write $S_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $S_2 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$, $h = \begin{pmatrix} x & \\ & y \end{pmatrix}$

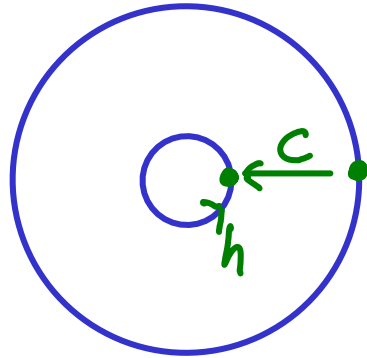
then

$$x = 1 + ab, \quad y = (1 + ba)^{-1} \quad \& \quad G \mathcal{A}_H^2 // G \cong \mathcal{B}(V, W)$$

Picture

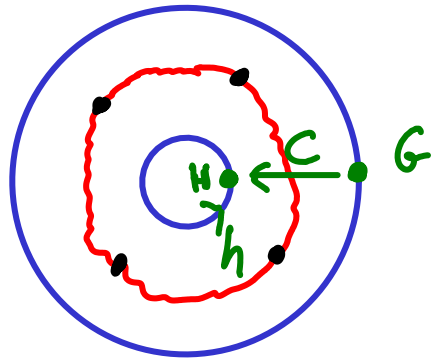
If $P_{\pm} = G = H$

$G \mathcal{A}_H = G \times G$ is the double
 \downarrow
 (C, h)



$$\mu = (C^{-1}hC, h^{-1})$$

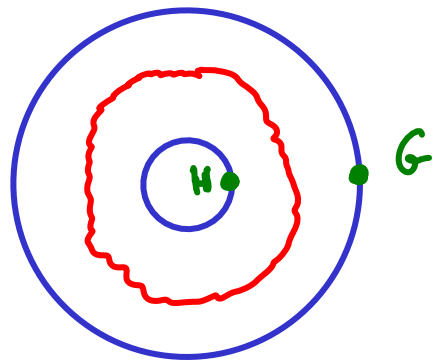
General case can be pictured similarly (breaking group from G to H)



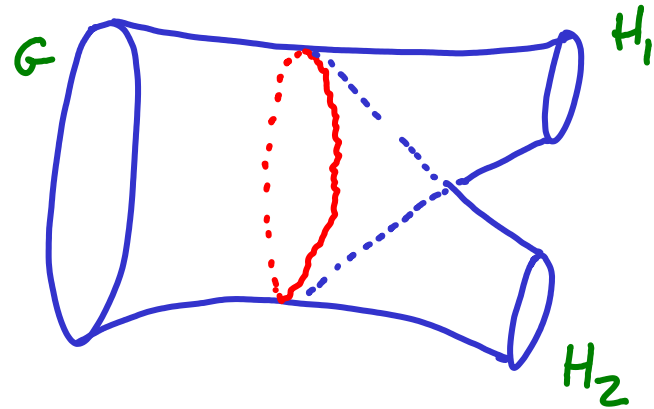
$$\mu = (C^{-1}hS_{2r} \dots S_1, C, h^{-1})$$

Typically H is a product eg. $H = H_1 \times H_2$

- can glue on both a qH H_1 -space & a qH H_2 -space



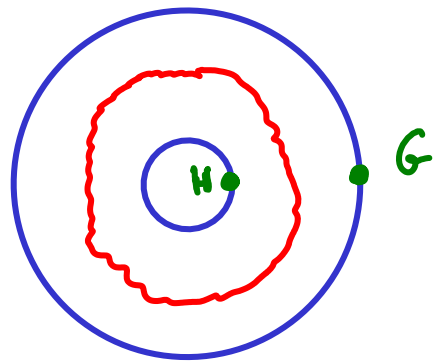
\cong



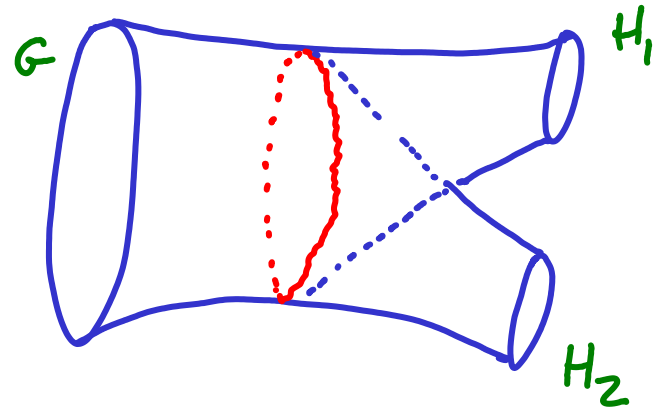
"fission" operation (\neq fusion)

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\cong



"fission" operation (\neq fusion)

Fusion \Rightarrow induction w.r.t genus & number of marked points

Fission \Rightarrow induction w.r.t order of poles (number of 'levels')

$$\theta \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\theta //_{\lambda} T$$

"complex weight variety"

$$\mathfrak{g} \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\mathfrak{g} //_1 T$$

$$\longleftrightarrow \approx \longrightarrow$$

$$\mathfrak{g}_1 \times \dots \times \mathfrak{g}_m //_{\check{\theta}} GL_m(\mathbb{C})$$

Gelfand-MacPherson^ℂ, Atiyah, Harnad

$$\theta \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\theta //_1 T$$

\Downarrow

$$\left(\frac{A_0}{z^2} + \frac{B}{z} \right) dz, \quad B \in \theta$$

$$A_0 = \text{diag}(a_i)$$

$$\longleftrightarrow \approx \longleftrightarrow$$

Gelfand-MacPherson^①, Atiyah, Harnad

Fourier-Laplace

$$\theta_1 \times \dots \times \theta_m //_{\check{\theta}} GL_m(\mathbb{C})$$

$\check{\theta}$

\cup

$$\sum_1^m \frac{A_i}{z-a_i} dz, \quad A_i \in \theta_i$$

$$\theta \subset \mathfrak{g}^*$$

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↓

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$$\sum_1^m \frac{A_i}{z-a_i} dz, \quad A_i \in \theta_i$$

Riemann
Hilbert

↓

$$\theta_1 \otimes \dots \otimes \theta_m //_{\check{\theta}} GL_n(\mathbb{C})$$

$$\theta \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\theta //_1 T$$

$$\xleftrightarrow[\text{Gelfand-MacPherson}^{\mathbb{C}}, \text{Atiyah, Harnad}]{\approx} \theta_1 \times \dots \times \theta_m //_{\check{\theta}} GL_n(\mathbb{C})$$

Fourier - Laplace

$$\left(\frac{A_0}{z^2} + \frac{B}{z} \right) dz, \quad B \in \theta$$

$$A_0 = \text{diag}(a_i)$$

$$\xleftrightarrow{\cup} \sum_1^m \frac{A_i}{z-a_i} dz, \quad A_i \in \theta_i$$

Riemann
Hilbert

$$e //_{\lambda} T$$

$$\xleftrightarrow{?} e_1 \otimes \dots \otimes e_m //_{\check{e}} GL_n(\mathbb{C})$$

$$\theta \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\theta //_1 T$$

$$\xleftrightarrow[\text{Gelfand-MacPherson}^{\mathbb{C}}, \text{Atiyah-Hitchin, Harnad}]{\approx} \theta_1 \times \dots \times \theta_m //_{\check{\theta}} GL_n(\mathbb{C})$$

Fourier-Laplace

$$\downarrow$$

$$\left(\frac{A_0}{z^2} + \frac{B}{z} \right) dz, \quad B \in \theta$$

$$A_0 = \text{diag}(a_i)$$

$$\downarrow$$

$$\sum_1^m \frac{A_i}{z-a_i} dz, \quad A_i \in \theta_i$$

Riemann
Hilbert

$$\left(e^{2\pi c} \mathfrak{A}'_T \right) //_{\lambda} T$$

$$\xleftrightarrow{?} e_1 \otimes \dots \otimes e_m //_{\check{\theta}} GL_n(\mathbb{C})$$

$$\theta \subset \mathfrak{g}^*$$

$$G = GL_m(\mathbb{C}) \supset T = (\mathbb{C}^*)^m$$

$$\theta //_{\lambda} T$$

$\longleftrightarrow \approx$
 Gelfand-MacPherson[Ⓢ], Atiyah-Hitchin, Harnad
 Fourier-Laplace

$$\theta_1 \times \dots \times \theta_m //_{\check{\theta}} GL_n(\mathbb{C})$$

$$\left(\frac{A_0}{z^2} + \frac{B}{z} \right) dz, \quad B \in \theta, \quad A_0 = \text{diag}(a_i)$$

$$\sum_{i=1}^m \frac{A_i}{z-a_i} dz, \quad A_i \in \theta_i$$

Stokes

Riemann
Hilbert

$$\left(\underbrace{e^{2\pi \sqrt{\lambda} A_T}}_{\cong \mathfrak{d} \subset \mathfrak{g}^*} \right) //_{\lambda} T$$

$$\longleftrightarrow \approx e_1 \otimes \dots \otimes e_m //_{\check{e}} GL_n(\mathbb{C})$$

ON THE SIMILARITY BETWEEN THE IWASAWA

PROJECTION AND THE DIAGONAL PART

by

J.J. Duistermaat

1. Statement of the result.

Let G be a real connected semisimple Lie group with finite center and $G = KAN$ its Iwasawa decomposition. Via the adjoint representation, and with respect to a suitable basis in \mathfrak{g} , K , resp. A , resp. N are the set of matrices in G which are orthogonal, resp. diagonal with positive entries, resp. upper triangular.

The Iwasawa projection H from G onto the Lie algebra \mathfrak{a} of A is defined by

$$(1.1) \quad x \in K \cdot \exp H(x) \cdot N, \quad x \in G.$$

Wild character varieties

($G =$ connected complex reductive gp)

Σ

\mapsto

$\text{Hom}_S(\Pi, G) / \underline{H}$

Irregular curve

Poisson variety

Fix $T \subset G$, Lie algebras $\mathfrak{t} \subset \mathfrak{g}$

Defⁿ Δ complex disc, $a \in \Delta$

An "irregular type" at a is an element

$$Q \in \mathfrak{t}(\hat{\kappa}) / \mathfrak{t}(\hat{\theta})$$

if z local coord vanishing at a , $\hat{\kappa} = \mathbb{C}[[z]]$, $\hat{\theta} = \mathbb{C}[[z]]$

so $Q = \frac{A_r}{z^r} + \dots + \frac{A_1}{z}$ for some $A_i \in \mathfrak{t}$

Defⁿ An "irregular curve" Σ is a smooth compact \mathbb{C} algebraic curve, with distinct marked points $a_1, \dots, a_m \in \Sigma$ and an irregular type Q_i at each marked point

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Aim: define (Poisson) Betti moduli spaces $M_B(\Sigma, G)$ of irreg. curves Σ & show they form local system of Poisson varieties if Σ undergoes an admissible deformation.

- such spaces will classifymero. connections locally isom. to $dQ_i +$ less singular terms
at each marked point
- local solutions involve $\exp(Q_i)$

Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = G_{2n}(\mathbb{C})$

(Birkhoff Bolser Jukatei Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

- will give explicit as possible approach using groupoids (for any reductive G)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points a_1, \dots, a_m , irreg. types Q_1, \dots, Q_m)

Let $\hat{\Sigma} \rightarrow \Sigma$ be real oriented blow up of Σ at a_i :

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(each a_i replaced by a circle ∂_i , so $\partial \hat{\Sigma} = \partial_1 \cup \dots \cup \partial_m$)

Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $A_i \subset \partial_i$ of singular directions at a_i

and for each $d \in A_i$

3) A unipotent group $\text{St}_d(Q_i) \subset G$ normalised by H_i

1) $H_i = \text{stabilizer of } Q_i \text{ under adjoint action}$
 $(H_i = \{g \in G \mid \text{Ad}_g(A_i) = A_i \ \forall i\})$

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Let $q_\alpha = d \circ Q$ (mero. function near $a \in \Sigma$)

then $d \in \partial$ is a singular direction supported by $\alpha \in \mathcal{R}$

if $\exp(q_\alpha)$ has maximal decay as $z \rightarrow a$ along d

(leading term of q_α is real and negative along d)

& $\mathcal{A} \subset \partial$ is set of all sing. directions ($\forall \alpha \in \mathcal{R}$)

3) Let $\mathcal{R}(d) = \{ \alpha \mid \alpha \text{ supports } d \} \subset \mathcal{R}$

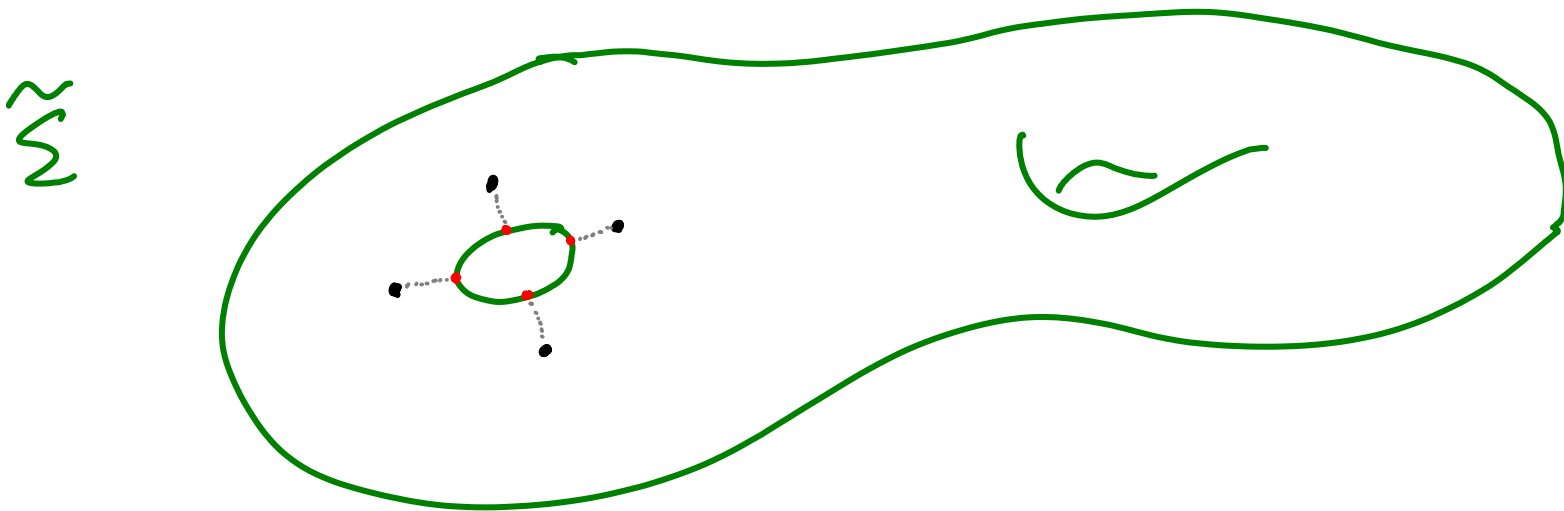
$$\mathcal{Sto}_d = \prod_{\alpha \in \mathcal{R}(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma \mathcal{Sto}_d is a well defined unipotent subgroup of G

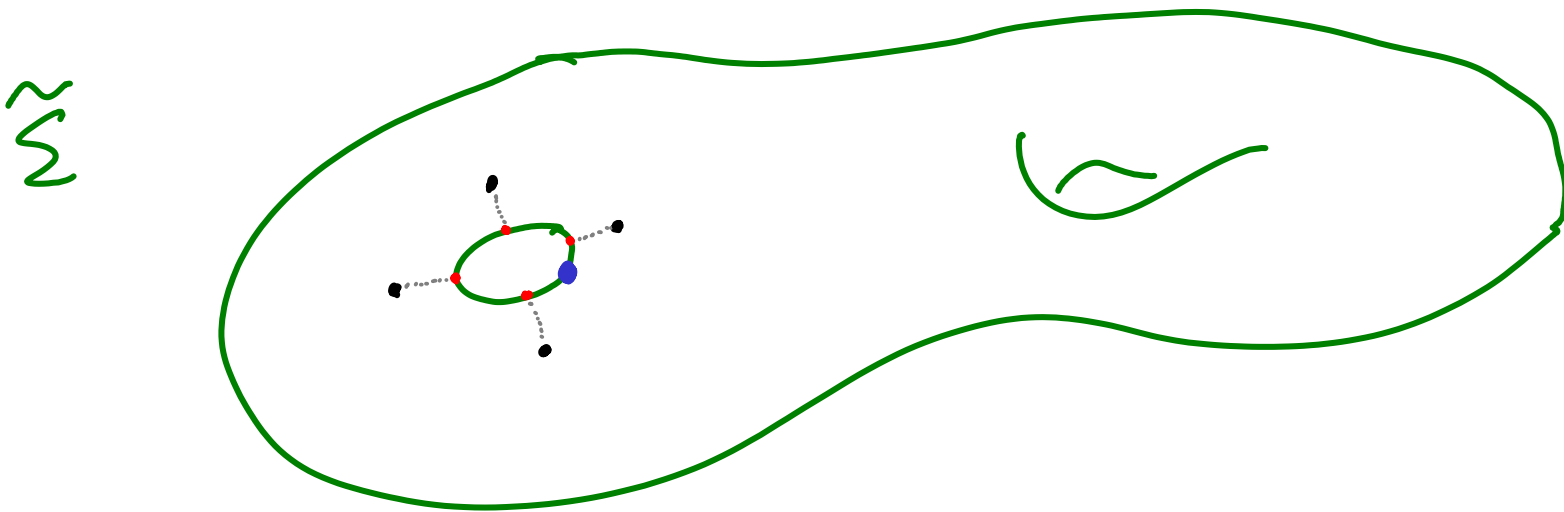
Now puncture $\hat{\Sigma}$ once in its interior near each singular

direction $d \in A_i$, $i=1, \dots, m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface

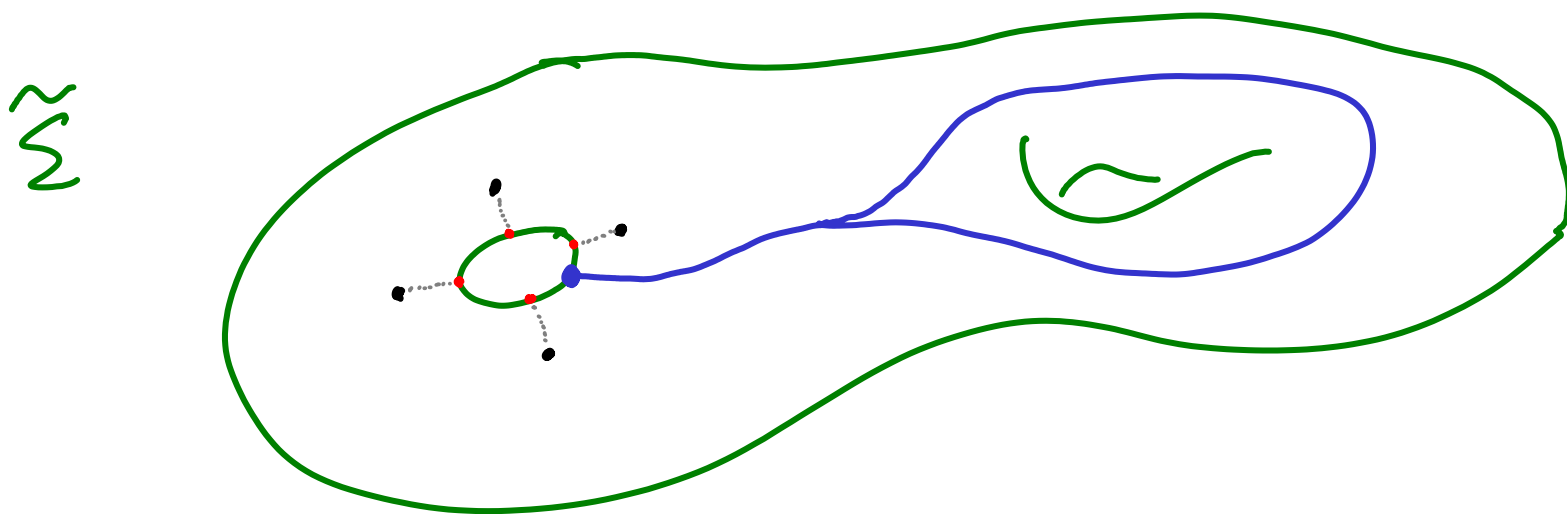


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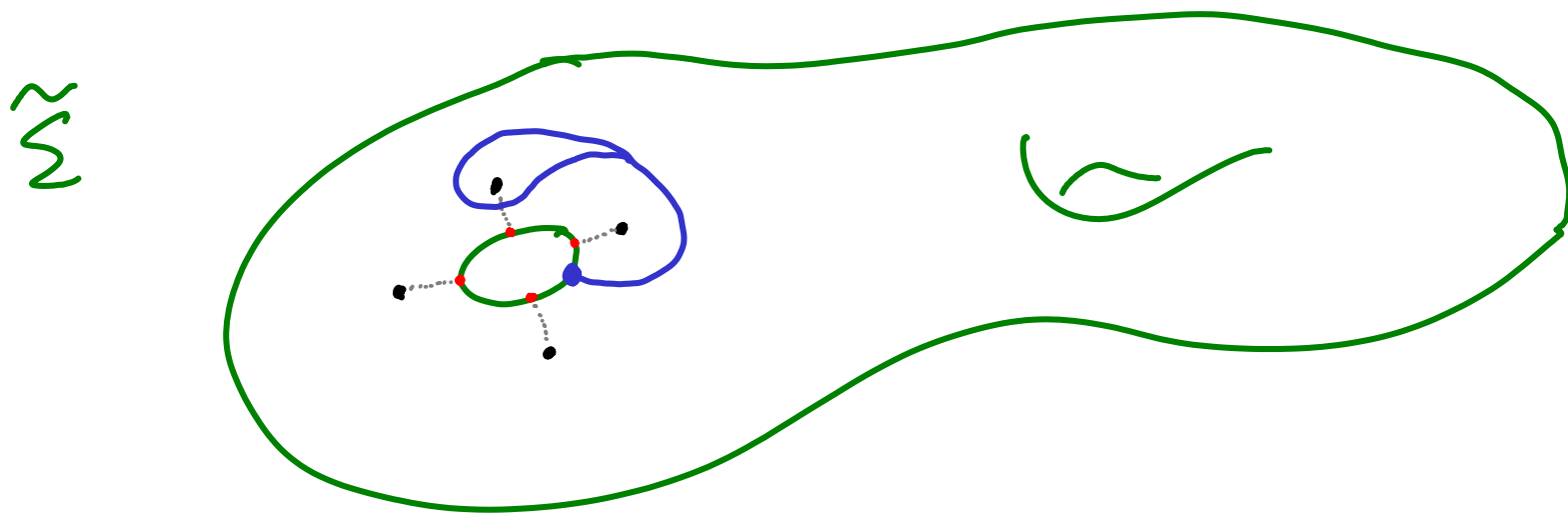


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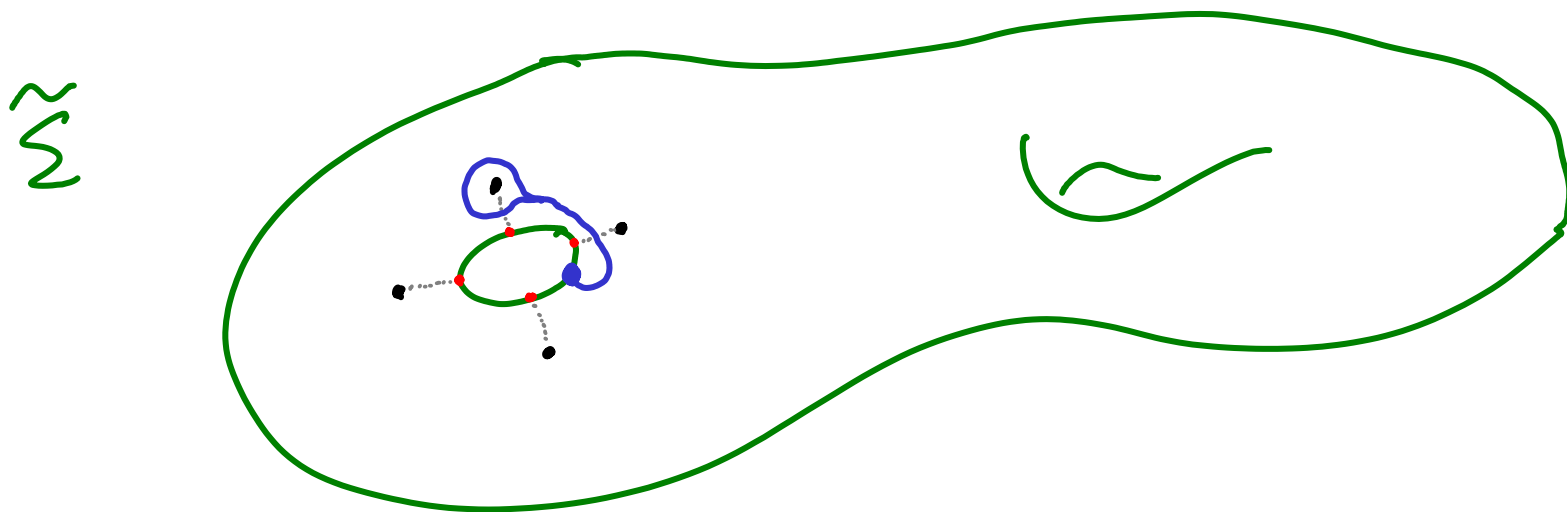
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Now consider $\text{Hom}(\Pi, G)$

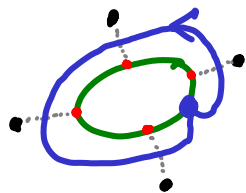
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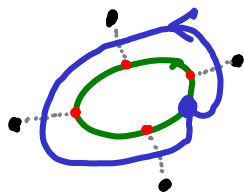


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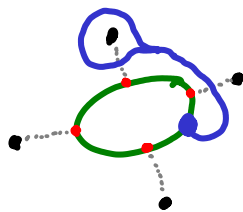
and the subset $\text{Hom}_S^U(\pi, G)$ of "Stokes representations"

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1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



2) If γ goes around ∂_i from b_i until $d \in A_i$ then loops around the corresponding puncture before returning to b_i , then $\rho(\gamma) \in \mathcal{S}to_d$



Thm (-'ii)

The space of Stokes representations $\text{Hom}_{\mathcal{S}}(\Pi, \mathcal{G})$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian \underline{H} -space ($\underline{H} = H_1 \times \dots \times H_m$)

Thm (-'ii)

The space of Stokes representations $\text{Hom}_{\mathcal{S}}(\Pi, \mathcal{G})$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian \underline{H} -space ($\underline{H} = H_1 \times \dots \times H_m$)

Corollary $M_B(\Sigma, \mathcal{G}) := \text{Hom}_{\mathcal{S}}(\Pi, \mathcal{G}) / \underline{H}$

inherits an intrinsic Poisson structure (algebraically) with

symplectic leaves $\mu^{-1}(e) / \underline{H}$ for $e = (e_1, \dots, e_m) \in \underline{H}$

M_B classifies irreg. connections with the given irreg. types
& Betti weights zero (else use \hat{e})

Also studied stability for $\underline{H} \curvearrowright \text{Hom}_G(\Pi, G)$:

Hilbert-Mumford + general quasi-Hamiltonian properties \implies

Thm If e sufficiently generic semisimple conjugacy class
then $(\mu^{-1}(e)/\underline{H})$ algebraic symplectic orbifold
(smooth symplectic algebraic variety if $G = \text{GL}_n(\mathbb{C})$)

Wild character varieties

($G =$ connected complex reductive gp)

 Σ \mapsto $\text{Hom}_S(\Pi, G) / \underline{H}$

Irregular curve

Poisson variety

Defⁿ

A family $\Sigma \rightarrow B$ of irregular curves (Σ_p, a_i, Q_i)
is "admissible" if $p \in B$

- 1) The fibres Σ_p remain smooth
- 2) None of the marked points a_i coalesce
- 3) For each root $\alpha \in \mathcal{R}$

$$\text{PoleOrder}(\alpha \circ Q_i) \in \mathbb{Z}_{\geq 0}$$

is a constant function on B

Thm

If $\Sigma \rightarrow B$ is an admissible family of irregular curves

$$\Sigma_p = \pi^{-1}(p), \quad p \in B$$

get algebraic Poisson action

$$\pi, (B, p) \curvearrowright \text{Hom}_{\mathbb{S}}(\pi(p), \mathcal{G}) / \underline{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, \mathcal{G})$ form a local system of (Poisson) varieties"

E.g. 1) $G = \mathbb{C}^*$ all deformations admissible

Baker functions are horz. sections of such irreg.
connections (on spectral curve)

{ Krichever
↓

Solutions of KdV etc

"times of KdV hierarchy \leftrightarrow irreg. deformation parameters"

$$\exp(xw + t_2 w^2 + t_3 w^3 + \dots) \quad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

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Solutions of KdV etc

"times of KdV hierarchy \leftrightarrow irreg. deformation parameters"

$$\exp(xw + t_2 w^2 + t_3 w^3 + \dots) \quad \begin{cases} w = 1/z \\ x = t_1 \end{cases}$$

2) $Q = -\frac{A_0}{z}$, $A_0 \in \mathfrak{t}_{\text{reg}}$ $\left(dQ = \frac{A_0}{z^2} dz \right)$

$$\pi_1(\mathfrak{t}_{\text{reg}}) = \text{Pure } \sigma\text{-braid group}$$

Definition

A "fission variety" is a symplectic or quasi-Hamiltonian variety obtained via the fusion & reduction operations on spaces of the form

- 1) Conjugacy classes $\mathcal{C} \subset G$ in arbitrary complex reductive groups
- 2) Fission spaces $G \curvearrowright A_H^r$

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- 1) Conjugacy classes $\mathcal{C} \subset G$ in arbitrary complex reductive groups
- 2) Fission spaces $G \backslash A_H^r$

Thm

If Σ is an irregular curve then $\text{Hom}_S(\Pi, G)$ is a fission variety

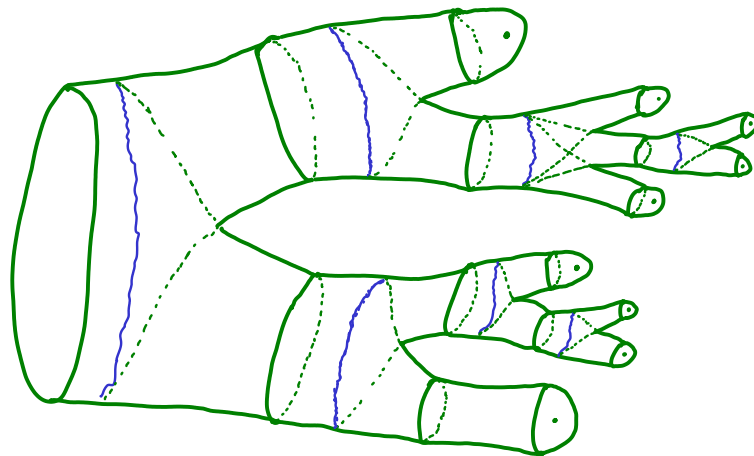
$$\text{If } Q = \frac{A_r}{z^r} + \dots + \frac{A_1}{z}$$

Define $G = H_r \supset H_{r-1} \supset \dots \supset H_0 = H \supset T$

$$\text{via } H_{i-1} = C_{H_i}(A_i)$$

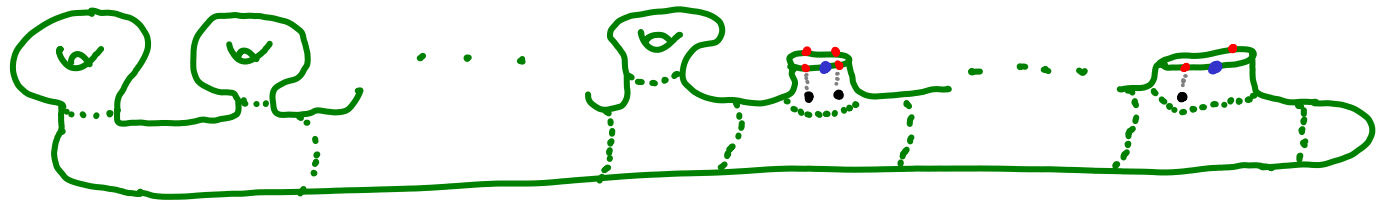
Then $A(Q) := G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$A(Q) \cong G \xrightarrow{A_{H_{r-1}}^r} \xrightarrow{C_{H_{r-1}}} A_{H_{r-2}}^{r-1} \xrightarrow{C_{H_{r-2}}} \dots \xrightarrow{C_{H_1}} A_H^1$$



If Σ an irregular curve :

$$\text{Hom}_{\mathcal{S}}(\pi, \mathcal{G}) \cong \underbrace{\mathbb{D} \otimes \cdots \otimes \mathbb{D}}_g \otimes A(Q_1) \otimes \cdots \otimes A(Q_m) // \mathcal{G}$$

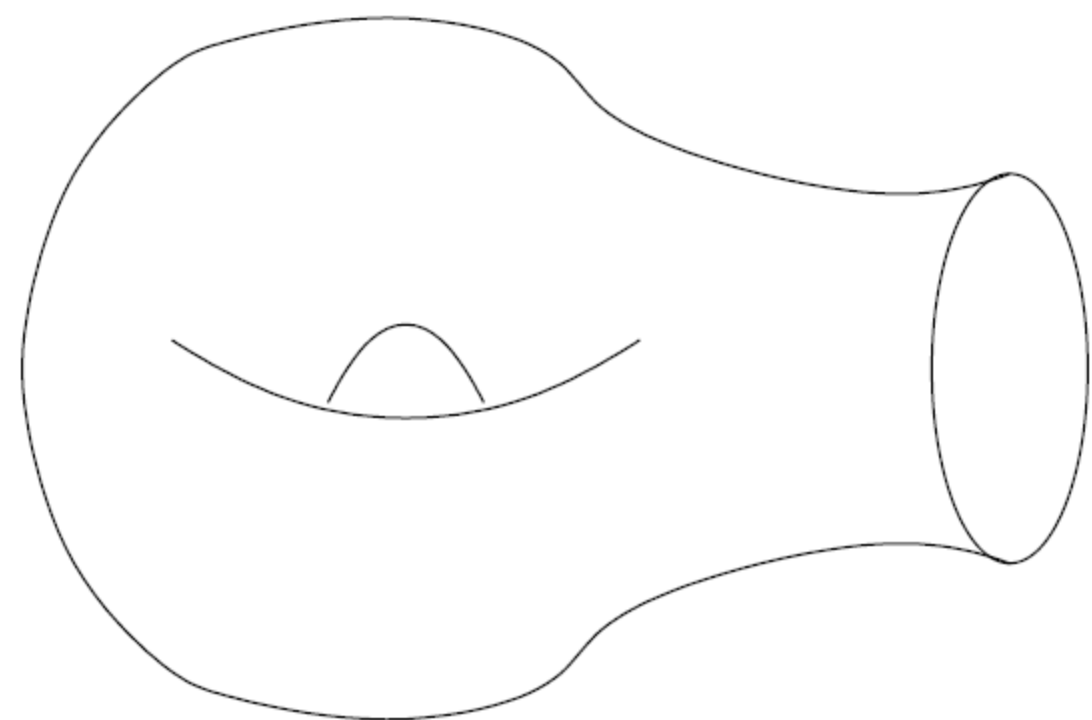


$$\mu^{-1}(e) / \underline{H} \cong \left\{ (A, B, C, h, S) \mid \prod_{i=1}^g [A_i, B_i] \prod_{i=1}^m \mu_i = 1, h_i \in \mathcal{C}_i \right\} / \underline{H}$$

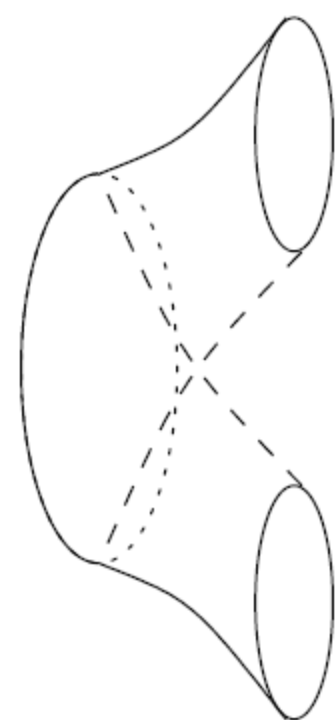
$$\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i$$

But there are many other examples of fission varieties

- e.g. can glue surfaces Σ along their boundaries
(provided the groups H_i match up)
- can obtain all the so-called multiplicative quiver varieties

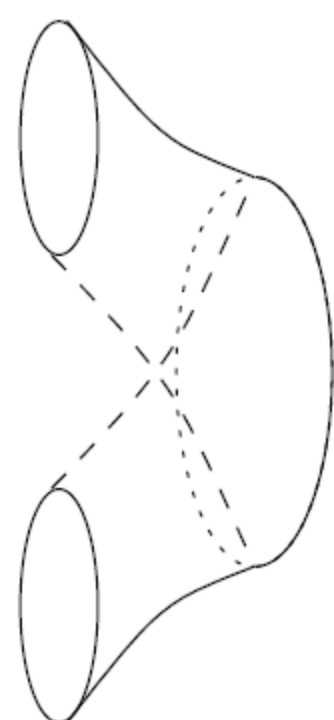


$GL_{a+b}(\mathbb{C})$

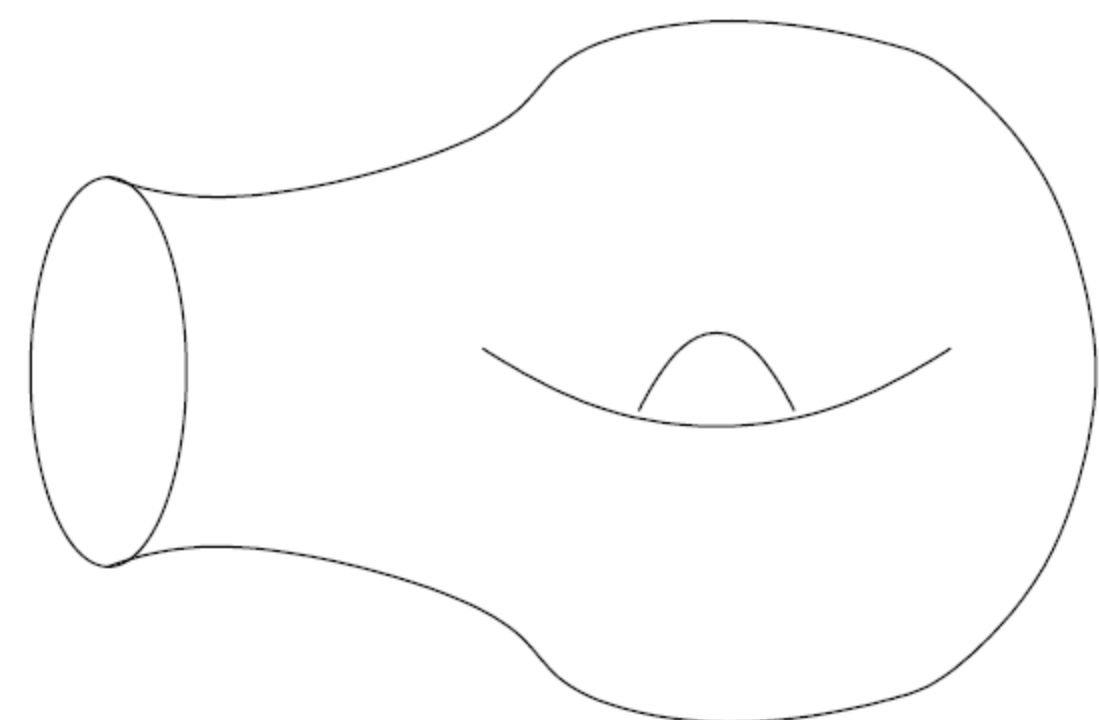


$GL_b(\mathbb{C})$

$GL_a(\mathbb{C})$



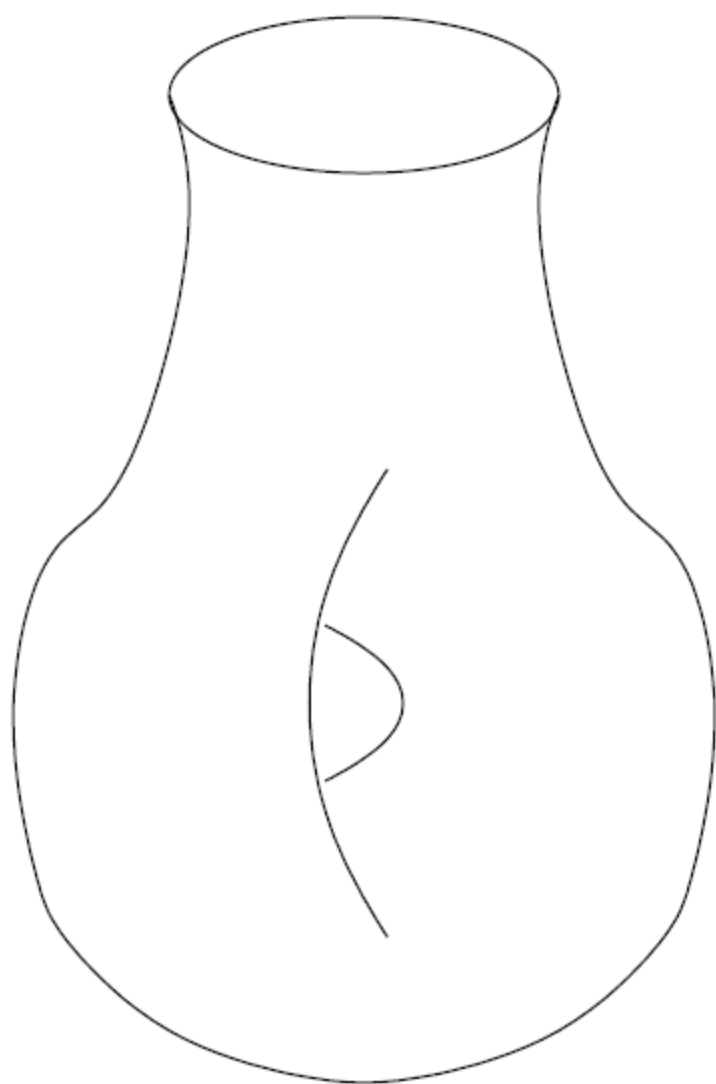
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$GL_{a+c}(\mathbb{C})$



$GL_{b+c}(\mathbb{C})$



(- An. Inst. Fourier 2009)

The amazingly rich structure of the solutions of the first Painlevé equation, the very simple differential equation

$$d^2y/dx^2 = 6y^2 + x.$$

Hans Duistermaat

Abstract: The differential equation $d^2y/dx^2 = 6y^2 + x$ is the first in Painlevé's classification of those algebraic second order ordinary differential equations such that every isolated singularity of moderate growth of every solution is a pole. Despite the simple form of the first Painlevé equation, the analysis of its solutions is very subtle. Painlevé thought in 1900 to have proved that every solution can be extended to a meromorphic function on the whole complex x -plane, but the first complete proofs were only given almost a century later. Boutroux discovered in 1913 a transformation to an approximately autonomous differential equation, which allowed him to draw spectacular conclusions about the asymptotic behavior of the locations of the poles near infinity in the complex plane. His proofs contained serious gaps and errors, but his conclusions were basically correct, and can be strengthened considerably. In this talk I would like to report on my joint work in progress with Nalini Joshi on this.

The first Painlevé equation can also be obtained as a compatibility condition between two linear systems of ordinary differential equations, one in the variable x with a parameter λ , and the other with the roles of x and λ interchanged. This has been used to study the asymptotic properties of the first Painlevé transcendents, but our analysis does not use this, in the literature called "isomonodromy method".