# Transformation groups for isomonodromy equations

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According to Hartshorne the guiding problem of algebraic geometry is to classify algebraic varieties up to isomorphism, i.e. spaces defined locally by polynomial equations. To make it more manageable one often restricts to special classes of varieties. We are interested in the more general problem of classifying algebraic differential equations, i.e. we allow derivatives in the equations. This could be viewed as a half-way step to "noncommutative algebraic geometry". To make it more manageable we will restrict to the class of isomonodromy systems, i.e. the nonlinear differential equations which control monodromy preserving deformations of linear differential systems. This class of systems is known to have many nice properties and applications, and they may be viewed as nonlinear analogues of the Gauss-Manin connections.

The best-known isomonodromy system, the system of Schlesinger equations, appeared in Schlesinger's ICM talk in 1908. The next significant generalisation appeared in the work of Jimbo-Miwa-Mori-Sato (JMMS) in 1980, in relation to the quantum nonlinear Schrodinger equation. In 1994 Harnad showed that the JMMS system admits a symmetry, not shared by the Schlesinger equations, nor by the isomonodromy systems of Jimbo-Miwa-Ueno (1981).

In this talk I will describe some of the theory of simply-laced isomonodromy systems and their automorphisms/isomorphisms. These systems generalise the JMMS system (and are not included in the JMU system), and the isomorphisms generalise Harnad's duality. As a special case we recover the Okamoto symmetries of the fourth, fifth and sixth Painlevé equations, and put these symmetries into the larger context of Weyl groups for not-necessarily-affine Kac-Moody root systems. On one hand this explains why there are such symmetries, via the Fourier-Laplace transform, and on the other hand it shows where such exotic root systems occur in nature. The appearance of such root systems and Weyl groups seems to distinguish this theory from earlier work on soliton equations.

As a corollary we may attach an isomonodromy system to any complete $k$-partite graph (for any $k$), and more generally to any "supernova" graph (and some data on the graph). In particular the Painlevé equations IV, V, VI are attached to the triangle, square and four legged star respectively, as suggested by Okamoto's work (that the affine Weyl groups of these graphs give the symmetry groups of these Painlevé equations). The bipartite case $k = 2$ gives the JMMS system. Further, by considering hyperbolic (doubly extended) Dynkin graphs, this viewpoint yields a higher Painlevé X system of order $2n$, for each $n = 1, 2, 3$,..., where X=I,II,...,VI is any of the Painlevé equations (which appear when $n = 1$). These higher Painlevé systems are distinct from the so-called "Painlevé hierarchies". As part of this project we also (define and) solve many irregular additive Deligne-Simpson problems, in terms of the Kac-Moody root system determined by the graph, extending Crawley-Boevey's results in the case of star-shaped graphs.

**Main references**:
P. Boalch, Simply-laced isomonodromy systems, Pub. Math. IHES (to appear), `arXiv:1107.0874`.
P. Boalch, Irregular connections and Kac-Moody root systems, `arXiv:0806.1050`.