IMRN International Mathematics Research Notices
2002, No. 22

G-Bundles, Isomonodromy, and Quantum Weyl Groups

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1 Introduction

It is now twenty years since Jimbo, Miwa, and Ueno [23] generalized Schlesinger's equations (governing isomonodromic deformations of logarithmic connections on vector bundles over the Riemann sphere) to the case of connections with arbitrary order poles. An interesting feature was that new deformation parameters arose: one may vary the *irregular type* of the connections at each pole of order two or more (irregular pole), as well as the pole positions. Indeed, for each irregular pole the fundamental group of the space of deformation parameters was multiplied by a factor of

$$P_n = \pi_1\big(\mathbb{C}^n \setminus \text{diagonals}\big), \tag{1.1}$$

where n is the rank of the vector bundles. (This factor arose because the connections must be *generic*; the leading term at each irregular pole must have distinct eigenvalues.)

The motivation behind the first part of this paper is the question of how to generalize the work of Jimbo, Miwa, and Ueno (and also [9, 8]) to the case of meromorphic connections on principal G-bundles for complex reductive groups G. For simple poles (Schlesinger's equations) this generalization is immediate, but in general one needs to understand the "G-valued" Stokes phenomenon in order to proceed (i.e., one needs to understand the local moduli of meromorphic connections on G-bundles). This is done in Section 2. Naturally enough a good theory is obtained provided the leading term at each irregular pole is *regular semisimple* (i.e., lies on the complement of the

Received 3 December 2001. Revision received 25 January 2002.
Communicated by Corrado De Concini.

root hyperplanes in some Cartan subalgebra). The main result of Section 2 is an irregular Riemann-Hilbert correspondence describing the local moduli in terms of G-valued Stokes multipliers, and is the natural generalization of the result of Balser, Jurkat, and Lutz [7] in the $GL_n(\mathbb{C})$ case. The proof is necessarily quite different from that of [7] however.

In the rest of the paper, we consider isomonodromic deformations of such connections in the simplest case: that of connections with one order-two pole over the unit disc. The main things we will prove are: (1) that the classical actions of quantum Weyl groups found by De Concini, Kac, and Procesi [13] do arise from isomonodromy (and so have a purely geometrical origin) and (2) that a certain flat connection appearing in the work of De Concini and Toledano Laredo arises directly from the isomonodromy Hamiltonians, indicating that the previous result is the classical analogue of their conjectural Kohno-Drinfeld theorem for quantum Weyl groups.

In more detail, in this "simplest case" the fundamental group of the space of deformation parameters is the generalized pure braid group associated to $\mathfrak{g} = \mathrm{Lie}(G)$:

$$P_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{\mathrm{reg}}), \tag{1.2}$$

where $\mathfrak{t}_{\mathrm{reg}}$ is the regular subset of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. By considering isomonodromic deformations, one obtains a nonlinear (Poisson) action of $P_{\mathfrak{g}}$ as follows (this is purely geometrical—as explained in [9] the author likes to think of isomonodromy as a natural analogue of the Gauss-Manin connection in non-abelian cohomology): there is a moduli space $\mathcal{M}$ of generic (compatibly framed) meromorphic connections on G-bundles over the unit disc and having order-two poles over the origin (see Section 3 for full details). Taking the leading coefficients (irregular types) at the pole gives a map $\mathcal{M} \to \mathfrak{t}_{\mathrm{reg}}$ which in fact expresses $\mathcal{M}$ as a fibre bundle. Performing isomonodromic deformations of the connections then amounts precisely to integrating a natural flat connection on this fibre bundle (the isomonodromy connection). Thus, upon choosing a basepoint $A_0 \in \mathfrak{t}_{\mathrm{reg}}$, a natural $P_{\mathfrak{g}}$ action is obtained on the fibre $\mathcal{M}(A_0)$, by taking the holonomy of the isomonodromy connection.

Now, in [8], the author found that (for $G = GL_n(\mathbb{C})$) the fibres $\mathcal{M}(A_0)$ are isomorphic to the Poisson Lie group G^* dual to G (and that the natural Poisson structures then coincide). The results of Section 2 enable this to be extended easily to general G. Thus isomonodromy gives a natural (Poisson) $P_{\mathfrak{g}}$ action on G^*.

On the other hand, in their work on representations of quantum groups at roots of unity, De Concini, Kac, and Procesi [13] have written down explicitly a Poisson action of the full braid group $B_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{\mathrm{reg}}/W)$ on G^*. This was obtained by taking the classical limit of the explicit $B_{\mathfrak{g}}$ action—the *quantum Weyl group* action—on the corresponding quantum group, due to Lusztig [26] and independently Kirillov-Reshetikhin [24] and Soĭbel'man [35]. In this paper, it is explained how to convert the fibre bundle $\mathcal{M} \to \mathfrak{t}_{reg}$ into a bundle $\mathcal{M}' \to \mathfrak{t}_{reg}/W$ with flat connection (and standard fibre G^*), by twisting by a finite group (Tits' extension of the Weyl group by an abelian group). Then the main result of Section 3 is the following theorem.

Theorem. The holonomy action of the full braid group $B_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{reg}/W)$ on G^* (obtained by integrating the flat connection on $\mathcal{M}'$) is the same as the $B_{\mathfrak{g}}$ action on G^* of De Concini-Kac-Procesi [13]. □

Thus the geometrical origins of the quantum Weyl group actions are in the geometry of meromorphic connections having order-two poles.

In Section 4, a Hamiltonian description is given for the equations governing the isomonodromic deformations of Section 3. Then it is shown how this leads directly to a certain flat connection appearing in the recent paper [37] and featuring in the conjectural "Kohno-Drinfeld theorem for quantum Weyl groups"; see [37], where this conjecture is explained—and proved for $\mathfrak{sl}_n(\mathbb{C})$. The history of this, given in [37], is a little complicated: De Concini discovered the connection and conjecture in unpublished work around 1995. Next Millson and Toledano Laredo jointly rediscovered the connection. Then Toledano Laredo rediscovered the conjecture and found how to prove it for $\mathfrak{sl}_n(\mathbb{C})$ by translating it into the usual Kohno-Drinfeld theorem.

Our derivation of this connection of De Concini-Millson-Toledano Laredo (DMT) suggests that the theorem of Section 3 here should be interpreted as the classical analogue (for any $\mathfrak{g}$) of the aforementioned conjectural Kohno-Drinfeld theorem for quantum Weyl groups. The background for this interpretation comes from [29] (and also [4, 18]). In [29], Reshetikhin explained how Knizhnik- Zamolodchikov type equations arise as deformations of the isomonodromy problem. Although poles of order two or more are considered in [29], the extra deformation parameters are not considered and so the braiding due to the irregular types did not appear. The derivation that is given here of the DMT connection amounts to the following statement. If the idea of [29] is extended to deformations of the isomonodromy problem for connections on $\mathbb{P}^1$ with just two poles (of orders one and two, respectively), then the DMT connection arises, rather than the Knizhnik-Zamolodchikov equations.

The organisation of this paper is as follows. Section 2 swiftly states all the required results concerning the moduli of meromorphic connections on principal G-bundles, the main proofs being deferred to an appendix. Section 3 then addresses isomonodromic deformations and proves the main theorem (Theorem 3.6), relating quantum Weyl group actions to meromorphic connections. Section 4 gives the Hamiltonian approach to the isomonodromic deformations considered and shows how this leads directly to the DMT connection. Appendix A gives the proofs for Section 2. Finally, Appendix B explains how, using the results of Section 2, one may extend to the current setting some closely related theorems of [8] showing that certain monodromy maps are Poisson.

2 G-valued Stokes multipliers

Let G be a connected complex reductive Lie group. Fix a maximal torus $T \subset G$ and let $\mathfrak{t} \subset \mathfrak{g}$ be the corresponding Lie algebras. Let $\mathcal{R} \subset \mathfrak{t}^*$ be the roots of G relative to T, so that as a vector space $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the one-dimensional subalgebra of elements $X \in \mathfrak{g}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{t}$.

Let A be a meromorphic connection on a principal G-bundle $P \to \Delta$ over the closed unit disc $\Delta \subset \mathbb{C}$, having a pole of order $k \geq 2$ over the origin and no others. We view A as a $\mathfrak{g}$-valued meromorphic one-form on P satisfying the usual conditions [25, page 64]; in particular the vertical component of A is nonsingular. Upon choosing a global section $s : \Delta \to P$ of P (which we may since every G-bundle over Δ is trivial), A is determined by the $\mathfrak{g}$-valued meromorphic one-form $A^s := -s^*(A)$ on Δ. (The minus sign is introduced here simply to agree with notation in the differential equations literature.) In turn $A^s = A^h dz/z^k$ for a holomorphic map $A^h : \Delta \to \mathfrak{g}$, where z is a fixed coordinate on Δ vanishing at 0.

By a *framing* of P at 0 we mean a point $s_0 \in P_0$ of the fibre of P at 0. This determines the leading coefficient $A_0 := A^h(0) \in \mathfrak{g}$ of A independently of the choice of a section s through s_0. The framed connection (P, A, s_0) is said to be *compatibly framed* if $A_0 \in \mathfrak{t}$. A compatibly framed connection is *generic* if $A_0 \in \mathfrak{t}_{reg}$, that is, if $\alpha(A_0) \neq 0$ for all $\alpha \in \mathcal{R}$. Let $\mathcal{G}$ denote the group of holomorphic maps $g : \Delta \to G$ and let $G[[z]]$ be the completion at 0 of $\mathcal{G}$.

Lemma 2.1. Let (P, A, s_0) be a generic compatibly framed connection with leading coefficient A_0. Choose a trivialization s of P with $s(0) = s_0$ and let $A^s = -s^*(A)$ as above. Then there is a unique formal transformation $\widehat{F} \in G[[z]]$ and unique elements $A_0^0, \ldots, A_{k-2}^0, \Lambda \in \mathfrak{t}$ such that $\widehat{F}(0) = 1$, $A_0^0 = A_0$, and

$$\widehat{F}[A^0] = A^s \tag{2.1}$$

where $A^0 := (A_0^0/z^k + \cdots + A_{k-2}^0/z^2 + \Lambda/z)dz$, and $\widehat{F}[A^0]$ denotes the gauge action (which, in any representation, is $\widehat{F}A^0\widehat{F}^{-1} + d\widehat{F}\widehat{F}^{-1}$). Moreover, changing the trivialization does not change A^0, and changes $\widehat{F}$ to $\widehat{g} \cdot \widehat{F}$ where $\widehat{g} \in G[[z]]$ is the Taylor expansion of some $g \in \mathcal{G}$ with $g(0) = 1$. □

The proof is given in Appendix A. We refer to A^0 as the *formal type* of (P, A, s_0) and to Λ as the *exponent of formal monodromy*. The primary aim of this section is to describe (in terms of Stokes multipliers) the set $\mathcal{H}(A^0)$ of isomorphism classes of generic compatibly framed connections on principal G-bundles over Δ with a fixed formal type A^0:

$$\mathcal{H}(A^0) = \left\{ (P, A, s_0) \mid \text{ formal type } A^0 \right\} / (\text{isomorphism}). \tag{2.2}$$

We remark that there are groups G for which this description cannot be reduced (for any A^0) to the $GL_n(\mathbb{C})$ case by choosing a representation $G \subset GL_n(\mathbb{C})$ (see Lemma A.2).

Since each such principal bundle is trivial, our task is equivalent to describing the quotient $\{A^s \mid \widehat{F}[A^0] = A^s \text{ for some } \widehat{F} \in G[[z]] \text{ with } \widehat{F}(0) = 1\}/\{g \in \mathcal{G} \mid g(0) = 1\}$. This will involve *summing* the (generally divergent) series $\widehat{F}$ on various sectors at 0, bounded by *anti-Stokes directions* which are defined as follows.

Let the circle S^1 parameterise rays (directed lines) emanating from $0 \in \mathbb{C}$. (Intrinsically this can be thought of as the boundary circle of the real oriented blowup of $\mathbb{C}$ at 0.) Note that $A^0 = dQ + \Lambda dz/z$ where $Q := \sum_{j=1}^{k-1} (z^{j-k}/(j-k)) A^0_{j-1}$ and let $q := A_0 z^{1-k}/(1-k)$ be the leading term of Q. Since A_0 is regular, for each root $\alpha \in \mathcal{R}$, there is a nonzero complex number c_α such that $\alpha \circ q = c_\alpha z^{1-k}$.

Definition 2.2. The *anti-Stokes directions* $\mathbb{A} \subset S^1$ are the directions along which $\exp(\alpha \circ q)$ decays most rapidly as $z \to 0$, that is, the directions along which $\alpha \circ q(z)$ is real and negative.

For $k = 2$ (which will be prominent in Section 3), $\mathbb{A}$ simply consists of the directions from 0 to $\alpha(A_0)$ for all $\alpha \in \mathcal{R}$. (In general $\mathbb{A}$ is just the inverse image under the $k-1$ fold covering map $z \to z^{k-1}$ of the directions to the points of the set $\langle A_0, \mathcal{R} \rangle \subset \mathbb{C}^*$.) Clearly, $\mathbb{A}$ has $\pi/(k-1)$ rotational symmetry and so $l := \#\mathbb{A}/(2k-2)$ is an integer. We refer to an l-tuple $\mathbf{d} \subset \mathbb{A}$ of consecutive anti-Stokes directions as a *half-period*.

Definition 2.3. Let $d \in \mathbb{A}$ be an anti-Stokes direction.

- The *roots* $\mathcal{R}(d)$ of d are the roots $\alpha \in \mathcal{R}$ *supporting* d:

$$\mathcal{R}(d) := \left\{ \alpha \in \mathcal{R} \mid (\alpha \circ q)(z) \in \mathbb{R}_{<0} \text{ for } z \text{ along } d \right\}. \tag{2.3}$$

- The *multiplicity* of d is the number $\#\mathcal{R}(d)$ of roots supporting d.

• The *group of Stokes factors* associated to d is the group

$$\mathbb{S}\mathrm{to}_d\left(A^0\right) := \prod_{\alpha\in\mathcal{R}(d)} U_\alpha \subset G, \tag{2.4}$$

where $U_\alpha = \exp(\mathfrak{g}_\alpha) \subset G$ is the one-dimensional unipotent group associated to $\mathfrak{g}_\alpha$, and the product is taken in any order.

• If $\mathbf{d} \subset \mathbb{A}$ is a half-period then the *group of Stokes multipliers* associated to $\mathbf{d}$ is

$$\mathbb{S}\mathrm{to}_{\mathbf{d}}\left(A^0\right) := \prod_{d\in\mathbf{d}} \mathbb{S}\mathrm{to}_d\left(A^0\right) \subset G. \tag{2.5}$$

To understand this we note the following facts (which are proved in Appendix A).

Lemma 2.4. If $\mathbf{d} \subset \mathbb{A}$ is a half-period then $\bigcup_{d\in\mathbf{d}} \mathcal{R}(d)$ is a system of positive roots in some (uniquely determined) root ordering.

• For any anti-Stokes direction d, the corresponding group of Stokes factors is a unipotent subgroup of G of dimension equal to the multiplicity of d.

• For any half-period $\mathbf{d}$, the corresponding group of Stokes multipliers is the unipotent part of the Borel subgroup of G determined by the positive roots above.

• The groups of Stokes multipliers corresponding to consecutive half-periods are the unipotent parts of opposite Borel subgroups. □

Now choose a sector $\mathrm{Sect}_0 \subset \Delta$ with vertex 0 bounded by two consecutive anti-Stokes directions. Label the anti-Stokes directions $d_1, \ldots, d_{\#\mathbb{A}}$ in a positive sense starting on the positive edge of Sect_0. Let $\mathrm{Sect}_i := \mathrm{Sect}(d_i, d_{i+1})$ denote the i*th sector* (where the indices are taken modulo $\#\mathbb{A}$) and define the i*th supersector* to be $\widehat{\mathrm{Sect}}_i := \mathrm{Sect}(d_i - \pi/(2k-2), d_{i+1} + \pi/(2k-2))$. All of the sectors Sect_i and $\widehat{\mathrm{Sect}}_i$ are taken to be *open* as subsets of Δ.

Theorem 2.5. Suppose that $\widehat{F} \in G[[z]]$ is a formal transformation as produced by Lemma 2.1. Then there is a unique holomorphic map $\Sigma_i(\widehat{F}) : \mathrm{Sect}_i \to G$ for each i such that (1) $\Sigma_i(\widehat{F})[A^0] = A$, and (2) $\Sigma_i(\widehat{F})$ can be analytically continued to the supersector $\widehat{\mathrm{Sect}}_i$ and then $\Sigma_i(\widehat{F})$ is asymptotic to $\widehat{F}$ at 0 within $\widehat{\mathrm{Sect}}_i$.

Moreover, if $t \in T$ and $g \in \mathcal{G}$ with $g(0) = t$ then $\Sigma_i(\widehat{g} \circ \widehat{F} \circ t^{-1}) = g \circ \Sigma_i(\widehat{F}) \circ t^{-1}$, where $\widehat{g}$ is the Taylor expansion of g at 0. □

This is proved in Appendix A. The point is that on a narrow sector there are generally many holomorphic isomorphisms between A^0 and A which are asymptotic to $\widehat{F}$ and one is being chosen in a canonical way.

It is now easy to construct canonical A-horizontal sections of P over the sectors, using these holomorphic isomorphisms and the fact that $z^\Lambda e^Q$ is horizontal for A^0 (which is viewed as a meromorphic connection on the trivial G-bundle).

For this, we need to choose a branch of $\log(z)$ along d_1 which we then extend in a positive sense across $\mathrm{Sect}_1, d_2, \mathrm{Sect}_2, d_3, \ldots, \mathrm{Sect}_0$ in turn. It will be convenient later (when A^0 varies) to encode the (discrete) choice of initial sector Sect_0 and branch of $\log(z)$ in terms of the choice of a single point $\widetilde{p} \in \widetilde{\Delta^*}$ of the universal cover of the punctured disc, lying over Sect_0.

Definition 2.6. Fix data $(A^0, z, \widetilde{p})$ as above and suppose that (P, A, s_0) is a compatibly framed connection with formal type A^0.

• The *canonical fundamental solution* of A on the ith sector is the holomorphic map

$$\Phi_i := \Sigma_i(\widehat{F}) z^\Lambda e^Q : \mathrm{Sect}_i \longrightarrow G, \tag{2.6}$$

where z^Λ uses the choice (determined by $\widetilde{p}$) of $\log(z)$ on Sect_i.

• The *Stokes factors* K_i $(i = 1, \ldots, \#\mathbb{A})$ of A are defined as follows. If Φ_i is continued across the anti-Stokes ray d_{i+1}, then on Sect_{i+1} we have $K_{i+1} := \Phi_{i+1}^{-1} \circ \Phi_i$ for $1 \le i < \#\mathbb{A}$ and $K_1 := \Phi_1^{-1} \circ \Phi_{\#\mathbb{A}} \circ M_0^{-1}$, where $M_0 := e^{2\pi\sqrt{-1}\cdot\Lambda} \in T$ is the *formal monodromy*.

• The *Stokes multipliers* S_i $(i = 1, \ldots, 2k-2)$ of A are

$$S_i := K_{il} \cdots K_{(i-1)l+1}\,, \tag{2.7}$$

where $l = \#\mathbb{A}/(2k-2)$. Equivalently, if Φ_{il} is continued across $d_{il+1}, \ldots, d_{(i+1)l}$ and onto $\mathrm{Sect}_{(i+1)l}$, then $\Phi_{il} = \Phi_{(i+1)l} S_{i+1}$ for $i = 1, \ldots, 2k-3$, and $\Phi_{il} = \Phi_l S_1 M_0$ for $i = 2k-2$.

Note that the canonical solutions Φ_i are appropriately equivariant under change of trivialization, so are naturally identified with A-horizontal sections of P. It follows that the Stokes factors and Stokes multipliers are constant (z-independent) elements of G. Also, from the proof of Lemma 2.4, note that S_i uniquely determines each Stokes factor appearing in (2.7). In Appendix A we establish the following basic lemma.

Lemma 2.7. $K_i \in \mathbb{S}\mathrm{to}_{d_i}(A^0)$ and $S_j \in \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$ where $\mathbf{d} = (d_{(j-1)l+1}, \ldots, d_{jl})$. □

It is immediate from Definition 2.6 and the last part of Theorem 2.5 that the Stokes multipliers are independent of the trivialization choice in Lemma 2.1, and so are well-defined (group-valued) functions on $\mathcal{H}(A^0)$. The main result of this section is then the following theorem.

Theorem 2.8. Fix the data $(A^0, z, \widetilde{p})$ as above. Let $U_+ = \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$ where $\mathbf{d} = (d_1, \dots, d_l)$ is the first half-period and let U_- denote the opposite full unipotent subgroup of G. Then the *irregular Riemann-Hilbert map* taking the Stokes multipliers induces a *bijection*

$$\mathcal{H}(A^0) \xrightarrow{\cong} (U_+ \times U_-)^{k-1}; \qquad [(P, A, s_0)] \longmapsto (S_1, \dots, S_{2k-2}). \tag{2.8}$$

In particular $\mathcal{H}(A^0)$ is isomorphic to a complex vector space of dimension $(k-1)\cdot(\#\mathcal{R})$. □

Proof. For injectivity, suppose that we have two compatibly framed meromorphic connections with $\widehat{F}_1[A^0] = A^s_1$ and $\widehat{F}_2[A^0] = A^s_2$ and having the same Stokes multipliers. Therefore, the Stokes factors are also equal and it follows immediately that $\Sigma_i(\widehat{F}_2) \circ \Sigma_i(\widehat{F}_1)^{-1}$ has no monodromy around 0 and does not depend on i, and thereby defines a holomorphic map $g : \Delta^* \to G$. Thus on any sector g has asymptotic expansion $\widehat{F}_2 \circ \widehat{F}_1^{-1}$ and so (by Riemann's removable singularity theorem), we deduce that the formal series $\widehat{F}_2 \circ \widehat{F}_1^{-1}$ is actually convergent with the function g as sum. This gives an isomorphism between the connections we began with: they represent the same point in $\mathcal{H}(A^0)$. Surjectivity follows from the G-valued analogue of a theorem of Sibuya, which we prove in Appendix A. ■

To end this section we show that the Stokes multipliers of a holomorphic family of connections vary holomorphically with the parameters of the family. Suppose that we have a family of compatibly framed meromorphic connections on principal G-bundles over the disc Δ, parameterised by some polydisc X. Upon choosing compatible trivializations, this family may be written as

$$A^s = A^h \frac{dz}{z^k} \tag{2.9}$$

for a holomorphic map $A^h : \Delta \times X \to \mathfrak{g}$ with leading coefficient a holomorphic map $A_0 = A^h|_{z=0} : X \to \mathfrak{t}_{\mathrm{reg}}$. The proof of Lemma 2.1 is completely algebraic and remains unchanged upon replacing the coefficient ring $\mathbb{C}$ by the ring $\mathcal{O}(X)$ of holomorphic functions on X; there is a unique formal transformation $\widehat{F} \in G(\mathcal{O}(X)[[z]])$ and unique holomorphic maps $A^0_0, \dots, A^0_{k-2}, \Lambda : X \to \mathfrak{t}$ such that $\widehat{F}|_{z=0} = 1$, $A^0_0 = A_0$ and $\widehat{F}[A^0] = A^s$, where $A^0 := (A^0_0/z^k + \cdots + A^0_{k-2}/z^2 + \Lambda/z)dz$. Given a point $x \in X$, let $\widehat{F}_x \in G[[z]]$ denote the corresponding formal bundle automorphism.

Now choose any basepoint $x_0 \in X$ and let $\mathbb{A}_0 \subset S^1$ denote the anti-Stokes directions associated to $A_0(x_0)$. Let $\check{S} \subset \Delta$ be any sector (with vertex 0) and whose closure contains none of the directions in $\mathbb{A}_0$. By continuity, there is a neighbourhood $U \subset X$ of

x_0 such that none of the anti-Stokes directions associated to $A_0(x)$ lie in $\check{S}$, for any $x \in U$. We always label the sectors such that $\check{S} \subset \mathrm{Sect}_0$. The following lemma will be proved in Appendix A.

Lemma 2.9. In the situation above, the holomorphic maps $\Sigma_0(\widehat{F}_x) : \check{S} \to G$ (defined for each x in Theorem 2.5 and restricted to $\check{S}$) vary holomorphically with $x \in U$ and so constitute a holomorphic map

$$\Sigma_0\left(\widehat{F}\right) : \check{S} \times U \longrightarrow G. \tag{2.10}$$

□

Corollary 2.10. In the situation above, taking Stokes multipliers define a holomorphic map $U \to (U_+ \times U_-)^{k-1}$ from the parameter space U to the space of Stokes multipliers. In particular, if A^0 is any formal type then $\mathcal{H}(A^0)$ is a coarse moduli space in the analytic category. □

Proof. Lemma 2.9 implies each of the sums $\Sigma_{il}(\widehat{F}_x)$ varies holomorphically with $x \in U$ (even though the integer l may jump; $\Sigma_{il}(\widehat{F}_x)$ is defined invariantly as the "sum" of $\widehat{F}_x$ on the sector $\check{S} \cdot \exp((i\pi\sqrt{-1})/(k-1))$). Thus, once a branch of $\log(z)$ is chosen on $\check{S}$, the canonical solutions Φ_{il} also vary holomorphically with x. The Stokes multipliers are defined directly in terms of these canonical solutions and so also vary holomorphically. That $\mathcal{H}(A^0)$ is a coarse moduli space is immediate from this and Theorem 2.8. ■

3 Isomonodromic deformations

In this section, we define and study isomonodromic deformations of generic compatibly framed meromorphic connections on principal G-bundles over the unit disc, having an order-two pole at the origin. Due to the results of Section 2 the definition is now a straightforward matter. (The $GL_n(\mathbb{C})$ case over $\mathbb{P}^1$, with arbitrary many poles of arbitrary order was defined in [23] and studied further in [9].)

The main aim here is to describe a relationship between isomonodromic deformations and certain braid group actions arising in the theory of quantum groups. In brief this relationship is as follows. In [8], the author identified the Poisson Lie group G^* dual to $G = GL_n(\mathbb{C})$ with a certain moduli space $\mathcal{M}(A_0)$ of meromorphic connections on vector bundles (principal $GL_n(\mathbb{C})$ bundles) over the unit disc and having an order-two pole at the origin and *irregular type* $A_0 \in \mathfrak{t}_{reg}$. Section 2 enables us to extend this identification easily to arbitrary G.

By considering *isomonodromic deformations* of such connections (where A_0 plays the role of deformation parameter), one obtains an action of the pure braid group

$P_\mathfrak{g} = \pi_1(\mathfrak{t}_{reg})$ on $\mathcal{M}(A_0) \cong G^*$. This is purely geometrical: there is a moduli space $\mathcal{M}$ of meromorphic connections fibring over $\mathfrak{t}_{reg}$ (with fibre $\mathcal{M}(A_0) \cong G^*$ over A_0) and having a natural flat (Ehresmann) connection—the isomonodromy connection. The $P_\mathfrak{g}$ action is just the holonomy of this connection.

On the other hand, De Concini-Kac-Procesi [13] have described explicitly an action of the full braid group $B_\mathfrak{g} = \pi_1(\mathfrak{t}_{reg}/W)$ on G^* in their work on representations of quantum groups at roots of unity. (This is for simple $\mathfrak{g}$, which is certainly the most interesting case.) This action is the classical version of the *quantum Weyl group* actions of $B_\mathfrak{g}$ on a quantum group (the quantization of G^*) which were defined by Lusztig, Kirillov-Reshetikhin, and Soĭbel'man (see [14] for more details; in particular Section 12 gives the definition of the quantum group having classical limit G^*).

In this section, we explain how to convert $\mathcal{M}$ into a fibre bundle $\mathcal{M}' \to \mathfrak{t}_{reg}/W$ with flat connection, using Tits' extended Weyl group [36], and then prove that the holonomy action of $B_\mathfrak{g}$ on the fibres of $\mathcal{M}'$ (which are still isomorphic to G^*) is precisely the action of De Concini-Kac-Procesi. Thus we have a geometrical description of their action; roughly speaking the infinite part (related to $P_\mathfrak{g}$) of the $B_\mathfrak{g}$ action comes from geometry whereas the rest (related to the Weyl group) is put in by hand.

We have restricted to the order-two pole case over a disc here since that is what is required for the application we have in mind here. However, the results of Section 2 do immediately facilitate the definition of isomonodromic deformations in much more generality.

The fibration $\mathcal{M} \to \mathfrak{t}_{reg}$. Fix a connected complex simple Lie group G and a maximal torus $T \subset G$. In terms of the definitions of Section 2 we have the following.

Definition 3.1. The moduli space $\mathcal{M}$ is the set of isomorphism classes of triples (P, A, s_0) of generic compatibly framed meromorphic connections A on principal G-bundles $P \to \Delta$ having an order-two pole at the origin.

Denote by $\pi : \mathcal{M} \to \mathfrak{t}_{reg}$ the surjective map taking the leading coefficient of the connections in $\mathcal{M}$. Let $\mathcal{M}(A_0) \subset \mathcal{M}$ be the fibre of π over the point $A_0 \in \mathfrak{t}_{reg}$. ($A_0$ will be called the *irregular type*; it determines the irregular part of the formal type A^0 of a connection in $\mathcal{M}$.)

Proposition 3.2. The space $\mathcal{M}$ has the structure of complex analytic fibre bundle over $\mathfrak{t}_{reg}$ with standard fibre $U_+ \times U_- \times \mathfrak{t}$, where $U_\pm$ are the unipotent parts of a pair of opposite Borel subgroups $B_\pm \subset G$ containing T.

Moreover, there is a canonically defined flat (Ehresmann) connection on $\mathcal{M} \to \mathfrak{t}_{reg}$; the isomonodromy connection. □

Proof. Fix an irregular type $A_0 \in \mathfrak{t}_{\mathrm{reg}}$. This determines anti-Stokes directions at 0 as in Section 2. Choose $\widetilde{p} \in \widetilde{\Delta^*}$ as in Definition 2.6, determining an initial sector Sect_0 and branch of $\log(z)$. Then if we define $U_\pm$ in terms of the first half-period as in Theorem 2.8, this choice determines an isomorphism

$$\mathcal{M}(A_0) \cong U_+ \times U_- \times \mathfrak{t} \tag{3.1}$$

as follows. There is a surjective map $\mathcal{M}(A_0) \to \mathfrak{t}$ taking a connection to its exponent of formal monodromy Λ (the residue of its formal type), as defined in Lemma 2.1. By definition the fibre of this map over $\Lambda \in \mathfrak{t}$ is $\mathcal{H}(A^0)$ where $A^0 := (A_0/z^2 + \Lambda/z)dz$. Then by Theorem 2.8 each such fibre is canonically isomorphic to $U_+ \times U_-$ (using the choice of $\widetilde{p}$ made above) and so (3.1) follows.

Now if we vary A_0 slightly, since the anti-Stokes directions depend continuously on A_0 and Sect_0 is open, we may use the same $\widetilde{p}$ for all A_0 in some neighbourhood of the original one. The above procedure then gives a local trivialization of $\mathcal{M} \to \mathfrak{t}_{\mathrm{reg}}$ over this neighbourhood, implying it is indeed a fibre bundle.

If we repeat this for each $A_0 \in \mathfrak{t}_{\mathrm{reg}}$ and each choice of $\widetilde{p}$ we obtain an open cover of $\mathfrak{t}_{\mathrm{reg}}$ with a preferred trivialization of $\mathcal{M}$ over each open set. The clutching maps for this open cover are clearly constant (involving just rearranging the Stokes factors into Stokes multipliers in different ways and conjugating by various exponentials of Λ), and so we have specified a flat connection on the fibre bundle $\mathcal{M} \to \mathfrak{t}_{\mathrm{reg}}$, the local horizontal leaves of which contain meromorphic connections with the same Stokes multipliers and exponent of formal monodromy (for some—and thus any—choice of $\widetilde{p}$). ∎

Remark 3.3. The isomonodromy connection may be viewed profitably as an analogue of the Gauss-Manin connection in non-abelian cohomology (which has been studied by Simpson [33, 34]). Extending Simpson's terminology, we call the above definition the *Betti* approach to isomonodromy. There is also an equivalent *de Rham* approach involving flat meromorphic connections on G-bundles over products $\Delta \times U$ for open neighbourhoods $U \subset \mathfrak{t}_{\mathrm{reg}}$. (This is well known to isomonodromy experts in the $GL_n(\mathbb{C})$ case.) This point of view has been described by the author in [9, Section 7] for the $GL_n(\mathbb{C})$ case; this now extends immediately to arbitrary G (see [9, Theorem 7.2] in particular for the de Rham approach).

The next step is to convert $\mathcal{M}$ into a fibre bundle $\mathcal{M}' \to \mathfrak{t}_{\mathrm{reg}}/W$ with flat connection (where $W := N(T)/T$ is the Weyl group), so that one obtains a holonomy action of the full braid group $B_\mathfrak{g} := \pi_1(\mathfrak{t}_{\mathrm{reg}}/W)$ on the fibres, rather than just an action of the pure braid group $P_\mathfrak{g} := \pi_1(\mathfrak{t}_{\mathrm{reg}})$. (This step is closely related to a similar step taken by Toledano Laredo in [37].) One would like simply to quotient $\mathcal{M}$ by an action of W covering the standard free action on $\mathfrak{t}_{\mathrm{reg}}$. Indeed, if there was a homomorphic section $W \to N(T) \subset G$ of the canonical projection $\pi_N : N(T) \to W$, then we could simply act on $\mathcal{M}$ by constant gauge transformations. However there is no such section in general, even for $SL_2(\mathbb{C})$. (For $GL_n(\mathbb{C})$ one may use the section given by permutation matrices but here we require a general approach.)

The standard way around this problem was found by Tits [36]; there is a finite abelian extension

$$1 \longrightarrow \Gamma_1 \longrightarrow \Gamma \xrightarrow{\pi_\Gamma} W \longrightarrow 1 \tag{3.2}$$

of W (where Γ is finite and Γ_1 is abelian) and an inclusion $\iota : \Gamma \hookrightarrow N(T)$ covering the identity in W (i.e., so that $\pi_N \circ \iota = \pi_\Gamma$). The group Γ is Tits' extended Weyl group.

Remark 3.4. A construction of Γ is as follows. Choose a positive Weyl chamber, label the simple roots by $i = 1, \dots, n$ and choose Chevalley generators $\{e_i, f_i, h_i\}$ of $\mathfrak{g}$ as usual. Let

$$t_i := \exp\left(f_i\right) \exp\left(-e_i\right) \exp\left(f_i\right) \in G. \tag{3.3}$$

One then knows (from [36]) that: (1) these t_i satisfy the braid relations for $\mathfrak{g}$ and so determine a homomorphism $B_\mathfrak{g} \to G$, and (2) the image Γ of $B_\mathfrak{g}$ in G has the properties stated above. (We note for later use that replacing t_i by t_i^{-1} here determines another homomorphism $B_\mathfrak{g} \to G$ with the same image.)

We could now act with Γ on $\mathcal{M}$ by gauge transformations, but then the quotient would not be a fibre bundle over $\mathfrak{t}_{\mathrm{reg}}/W$, since this action is not free (e.g., Γ_1 acts trivially on formal types, but nontrivially on other connections). To get around this we first pull back $\mathcal{M} \to \mathfrak{t}_{\mathrm{reg}}$ to the Galois Γ_1 cover $\widehat{\mathfrak{t}}_{\mathrm{reg}}$ of $\mathfrak{t}_{\mathrm{reg}}$. (In other words $\widehat{\mathfrak{t}}_{\mathrm{reg}} := \widetilde{\mathfrak{t}}_{\mathrm{reg}}/K$, where $\widetilde{\mathfrak{t}}_{\mathrm{reg}}$ is the universal cover of $\mathfrak{t}_{\mathrm{reg}}$ and $K := \ker(B_\mathfrak{g} \to \Gamma) = \ker(P_\mathfrak{g} \to \Gamma_1)$.) Then define $\widehat{\mathcal{M}} := \mathrm{pr}^*(\mathcal{M})$ to be the pullback of the bundle $\mathcal{M}$ along the covering map $\mathrm{pr} : \widehat{\mathfrak{t}}_{\mathrm{reg}} \to \mathfrak{t}_{\mathrm{reg}}$. The connection on $\mathcal{M}$ pulls back to a flat connection on $\widehat{\mathcal{M}} \to \widehat{\mathfrak{t}}_{\mathrm{reg}}$.

Finally, we can now act with Γ on $\widehat{\mathcal{M}}$ by gauge transformations, covering the canonical free action of Γ on $\widehat{\mathfrak{t}}_{\mathrm{reg}}$, to obtain a fibre bundle $\mathcal{M}' := \widehat{\mathcal{M}}/\Gamma \to \mathfrak{t}_{\mathrm{reg}}/W$. In summary we have the commutative diagram

$$\begin{array}{ccccc} \widehat{\mathcal{M}} & \longrightarrow & \mathcal{M} & & \mathcal{M}' \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\mathfrak{t}}_{\mathrm{reg}} & \xrightarrow{\mathrm{pr}} & \mathfrak{t}_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}_{\mathrm{reg}}/W, \end{array} \tag{3.4}$$

where the horizontal maps are finite covering maps and the vertical maps are fibrations.

Lemma 3.5. The connection on $\widehat{\mathcal{M}} = \mathrm{pr}^*(\mathcal{M})$ is Γ invariant and so descends to a flat connection on $\mathcal{M}' \to \mathfrak{t}_{\mathrm{reg}}/W$. □

Proof. Choose $g \in \Gamma \subset N(T)$ and $\widehat{A}_0 \in \widehat{\mathfrak{t}}_{\mathrm{reg}}$. It is sufficient to show that, under the action of g, local horizontal sections of $\widehat{\mathcal{M}}$ over a neighbourhood of $\widehat{A}_0$ become horizontal sections over a neighbourhood of $g(\widehat{A}_0)$. To this end, choose open $U \subset \mathfrak{t}_{\mathrm{reg}}$ containing $A_0 := \mathrm{pr}(\widehat{A}_0)$ as in the proof of Proposition 3.2 and so small that $\mathrm{pr}^{-1}(U)$ consists of $\#\Gamma_1$ connected components. Let $\widehat{U}$ be the component containing $\widehat{A}_0$.

Now choose $\widetilde{p} \in \widetilde{\Delta^*}$ as in Proposition 3.2 and thereby obtain a (horizontal) trivialization of $\mathcal{M}$ over U and also of $\widehat{\mathcal{M}}$ over $\widehat{U}$

$$\widehat{\mathcal{M}}|_{\widehat{U}} \cong U_+ \times U_- \times \mathfrak{t} \times \widehat{U}, \tag{3.5}$$

where $U_\pm$ are determined by A_0 and $\widetilde{p}$ as in Theorem 2.8. Let $A_0' := gA_0g^{-1}$ so that $\mathrm{pr}(g(\widehat{A}_0)) = A_0'$. Thus $\mathrm{pr}(g(\widehat{U}))$ is a neighbourhood of A_0' over which we may trivialize $\mathcal{M}$ using the same choice of $\widetilde{p}$ as above (since A_0 and A_0' determine the same set of anti-Stokes directions). Thus in turn $\widehat{\mathcal{M}}|_{g(\widehat{U})} \cong U_+' \times U_-' \times \mathfrak{t} \times g(\widehat{U})$, where $U_\pm'$ are determined by A_0' and $\widetilde{p}$ as in Theorem 2.8.

Finally, we claim that $U_\pm' = gU_\pm g^{-1}$ and that, in terms of the above trivializations, the action of g on $\widehat{\mathcal{M}}$ is given by

$$g\big(S_+, S_-, \Lambda\big) = \big(gS_+g^{-1},\ gS_-g^{-1},\ g\Lambda g^{-1}\big) \tag{3.6}$$

(together with the standard action on the base $\widehat{\mathfrak{t}}_{\mathrm{reg}}$), where $(S_+, S_-, \Lambda) \in U_+ \times U_- \times \mathfrak{t}$. Since there is no dependence on the base, this clearly implies the proposition. The claim is established by a straightforward unwinding of the definitions. ■

In [8] (for $G = GL_n(\mathbb{C})$) it was found to be natural to identify the space $U_+ \times U_- \times \mathfrak{t}$ of monodromy data with the simply-connected Poisson Lie group G^* dual to G, which we will now do here in general (cf. also Appendix B for motivation). Given a choice $B_\pm$ of opposite Borel subgroups of G with $B_+ \cap B_- = T$, the group G^* is defined to be

$$G^* := \big\{\big(b_-, b_+, \Lambda\big) \in B_- \times B_+ \times \mathfrak{t} \mid \delta_-\big(b_-\big)\delta_+\big(b_+\big) = 1,\ \delta_+\big(b_+\big) = \exp\big(\pi i \Lambda\big)\big\}, \tag{3.7}$$

where $\delta_\pm : B_\pm \to T$ is the natural projection (with kernel the unipotent part $U_\pm$ of $B_\pm$) and $\exp : \mathfrak{t} \to T$ is the exponential map for T. This is a simply-connected (indeed contractible) subgroup of $B_- \times B_+ \times \mathfrak{t}$ (where $\mathfrak{t}$ is a group under $+$) of the same dimension as G.

The group G^* is then identified with $U_+ \times U_- \times \mathfrak{t}$ as follows (cf. [8, Definition 20])

$$U_+ \times U_- \times \mathfrak{t} \cong G^*; \qquad (S_+, S_-, \Lambda) \longmapsto (b_-, b_+, \Lambda), \tag{3.8}$$

where $b_- = e^{-\pi i \Lambda} S_-^{-1}$ and $b_+ = e^{-\pi i \Lambda} S_+ e^{2\pi i \Lambda}$, so that $b_-^{-1} b_+ = S_- S_+ \exp(2\pi i \Lambda)$.

Thus the fibrations (3.4) can now be viewed as having standard fibre G^* (although they are not principal G^*-bundles).

The final (trivial) complication is that we have G^* simply-connected, whereas [13, 14] use the quotient group defined by omitting the Λ component in (3.7) (or equivalently one only remembers $e^{\pi i \Lambda}$ rather than Λ). We will abuse notation and denote both groups G^*; in terms of the braid groups actions this is reasonable since (1) it is immediate that the connection on $\mathcal{M}'$ is invariant under the corresponding action of the lattice $\ker(\exp(\pi i \cdot) : \mathfrak{t} \to G)$ so descends to give a flat connection on the quotient bundle (still denoted $\mathcal{M}'$), and (2) the $B_{\mathfrak{g}}$ action of [13] lifts to an action on our G^* simply by acting on Λ via the standard Weyl group action.

The main result is then the following theorem.

Theorem 3.6. The holonomy action of the full braid group $B_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{\text{reg}}/W)$ on G^* (obtained by integrating the flat connection on $\mathcal{M}'$) is the same as the $B_{\mathfrak{g}}$ action on G^* of De Concini-Kac-Procesi [13]. □

Proof. Choose a real basepoint $A_0^* \in \mathfrak{t}_{\mathbb{R},\text{reg}} \subset \mathfrak{t}_{\text{reg}}$ in the Weyl chamber chosen in Remark 3.4 (cf. also (A.3)). Then the corresponding set $\mathbb{A}$ of anti-Stokes directions consists of just the two halves of the real axis. Let Sect_0 be the lower half disc, choose a point p on the negative imaginary axis and let $\widetilde{p} \in \widetilde{\Delta^*}$ be the point lying over p and on the branch of logarithm having $\log(-i) = 3\pi i/2$. Define the positive roots $\mathcal{R}_+$, the groups $B_\pm, U_\pm$, and in turn G^* to be those determined by A_0^* and $\widetilde{p}$. These choices determine an isomorphism $\mathcal{M}(A_0^*) \cong G^*$ via Theorem 2.8 and (3.8). (One may check $\mathcal{R}_+$ is the set of positive roots corresponding to the chosen positive Weyl chamber.)

Now for each simple root $\alpha = \alpha_i \in \mathcal{R}_+$, Brieskorn [11] defines the following path γ_i in $\mathfrak{t}_{\text{reg}}$. Let s_i be the complex reflection acting on $\mathfrak{t}$ corresponding to α (the reflection fixing the hyperplane $\ker(\alpha)$ and respecting the Killing form). Let L_i be the complex line in $\mathfrak{t}$ containing A_0^* and $A_0' := s_i(A_0^*)$ and let I_i be the real line segment from A_0^* to A_0'. Then define the path $\gamma_i : [0,1] \to L_i$ from A_0^* to A_0' such that $[0, 1/3] \cup [2/3, 1]$ maps to I_i and $[1/3, 2/3]$ maps to a small semi-circle turning in a positive sense and centred on the midpoint of I_i. According to [11], if the semi-circles are sufficiently small, these paths γ_i are in $\mathfrak{t}_{\text{reg}}$ and descend to loops in $\mathfrak{t}_{\text{reg}}/W$ representing generators of $\pi_1(\mathfrak{t}_{\text{reg}}/W)$.

For our purposes here we choose the above semi-circles so small that, as A_0 moves along γ_i, precisely one anti-Stokes direction crosses over the point p—an anti-Stokes direction supported just by $-\alpha$ and moving in a positive sense. (To see this is possible observe that for any $\beta \in \mathcal{R}_+ \setminus \{\alpha\}$, $\beta(A_0^*)$ and $\beta(A_0')$ are real and positive, since α is the only positive root made negative by s_i. Thus by linearity $\beta(I_i) \subset \mathbb{R}_{>0}$. Hence if γ_i's semi-circle is sufficiently small $\beta(A_0)$ does not cross the imaginary axis for any A_0 on γ_i and therefore no anti-Stokes direction supported by $\pm\beta$ crosses p. Finally observe that, as A_0 moves along γ_i, $\alpha(A_0)$ starts in $\mathbb{R}_+$, moves towards 0, makes a positive semi-circle around 0 then moves away from 0 along $\mathbb{R}_-$. Since $q = -A_0/z$, here this implies the anti-Stokes direction supported by $-\alpha$ crosses p, and the one supported by α crosses the positive imaginary axis.)

We now wish to calculate the holonomy isomorphism $\mathcal{M}(A_0^*) \cong \mathcal{M}(A_0')$ obtained by integrating the isomonodromy connection along the path γ_i. As in Proposition 3.2 we have canonical descriptions of the fibre of $\mathcal{M}$ over both A_0^* and A_0':

$$\mathcal{M}(A_0^*) \cong U_+ \times U_- \times \mathfrak{t}, \qquad \mathcal{M}(A_0') \cong U_+' \times U_-' \times \mathfrak{t} \tag{3.9}$$

using the chosen $\widetilde{p}$ in both cases, where (as in Lemma 3.5) $U_\pm' = gU_\pm g^{-1}$ for any $g \in \Gamma$ with $\pi_\Gamma(g) = s_i \in W$. Thus we want to find the corresponding isomorphism $U_+ \times U_- \times \mathfrak{t} \cong U_+' \times U_-' \times \mathfrak{t}$. To describe it we need the following maps. Let $U_i = \exp(\mathfrak{g}_\alpha)$ be the root group corresponding to the simple root $\alpha = \alpha_i$. Then there is a homomorphism

$$\xi_i : U_+ \longrightarrow U_i \tag{3.10}$$

with the property that if any $S \in U_+$ is factorized (in any order) as a product of elements $u_\beta \in U_\beta$ for $\beta \in \mathcal{R}_+$ (with each β appearing just once) then $u_\alpha = \xi_i(S)$. (The existence of ξ_i may be seen as follows: the set $\Psi := \mathcal{R}_+ \setminus \{\alpha\}$ is a closed set of roots so $U_\Psi := \prod_{\beta\in\Psi} U_\beta$ is a subgroup of U_+. By [10, Proposition 14.5(3)] U_Ψ is a normal subgroup. (It is sufficient to prove that U_i normalizes U_Ψ.) Then ξ_i is taken to be the projection $U_+ \to U_+/U_\Psi$ where $U_i \cong U_+/U_\Psi$ via the inclusion $U_i \subset U_+$.) Similarly we have maps $\xi_{-i} : U_- \to U_{-\alpha}$.

Proposition 3.7. The holonomy isomorphism $\mathcal{M}(A_0^*) \cong \mathcal{M}(A_0')$ induced by the isomonodromy connection is given by $U_+ \times U_- \times \mathfrak{t} \to U_+' \times U_-' \times \mathfrak{t}$; $(S_+, S_-, \Lambda) \mapsto (S_+', S_-', \Lambda)$ where

$$S_+' := \xi_i(S_+)^{-1} S_+ M_0 \xi_{-i}(S_-) M_0^{-1}, \qquad S_-' := \xi_{-i}(S_-)^{-1} S_- \xi_i(S_+), \tag{3.11}$$

and $M_0 := \exp(2\pi i \Lambda)$. □

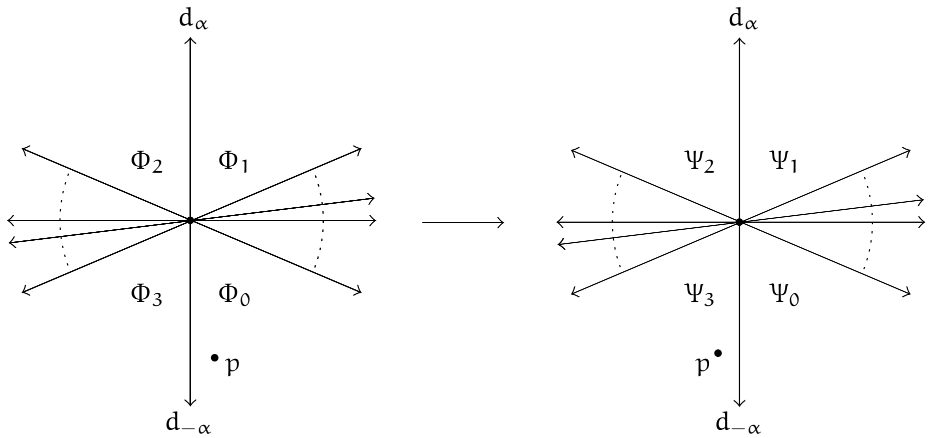

Figure 3.1

Proof. We must find the transition maps between the local trivializations just before and after the anti-Stokes direction $d_{-\alpha}$ supported by $-\alpha$ crosses p. By perturbing $\widetilde{p}$ (and therefore also p) slightly, this is equivalent to finding the transition map relating the two situations appearing in Figure 3.1, where p moves but A_0—and thus all the anti-Stokes directions—remain fixed. That is, we must find the composite map

$$U_+ \times U_- \times \mathfrak{t} \cong \mathcal{M}(A_0) \cong U'_+ \times U'_- \times \mathfrak{t}, \tag{3.12}$$

where the first (resp., second) isomorphism is determined by the $\widetilde{p}$ choice in the left (resp., right) diagram in Figure 3.1.

Choose arbitrary $(S_+, S_-, \Lambda) \in U_+ \times U_- \times \mathfrak{t}$ and let A be a connection on the trivial G-bundle over Δ with isomorphism class in $\mathcal{M}(A_0)$ corresponding to (S_+, S_-, Λ) under the left-hand isomorphism in (3.12). Let $\Phi_0, \dots, \Phi_3$ and $\Psi_0, \dots, \Psi_3$ be the canonical fundamental solutions of A on the sectors indicated in Figure 3.1. (Except for Φ_0 the indexing of these differs from Definition 2.6.) Since the $\log(z)$ choice on the sector containing p is extended to the other sectors in a negative sense, we immediately deduce $\Phi_3 = \Psi_3$, $\Phi_2 = \Psi_2$, $\Phi_1 = \Psi_1$, and $\Phi_0 = \Psi_0 M_0$.

Now let $K_\pm$ denote the Stokes factors of A across $d_{\pm\alpha}$ in the left diagram in Figure 3.1 (and similarly $K'_\pm$ for the right diagram). Clearly, $K_+ = K'_+$ since across d_α we have

$$K_+ := \Phi_2^{-1}\Phi_1 = \Psi_2^{-1}\Psi_1 =: K'_+. \tag{3.13}$$

Across $d_{-\alpha}$, $K_- := \Phi_0^{-1}\Phi_3$ and $K'_- := \Psi_0^{-1}\Psi_3$ so that $K'_- = M_0 K_- M_0^{-1}$.

In turn the Stokes multipliers are defined by the equations

$$\Phi_2 = \Phi_0 S_-, \qquad \Phi_0 = \Phi_2 S_+, \qquad \Psi_1 = \Psi_3 S'_-, \qquad \Psi_3 = \Phi_1 S'_+, \tag{3.14}$$

where in the left/right column the fundamental solutions are continued into the left/right half-plane before being compared, respectively. (Here we prefer to index the Stokes multipliers by $+$ and $-$ rather than 1 and 2 as in Section 2.) Combining this with the above expression for the Stokes factors we deduce

$$S'_+ = K_+^{-1} S_+ M_0 K_- M_0^{-1}, \qquad S'_- = K_-^{-1} S_- K_+. \tag{3.15}$$

Finally, from the alternative definition of the Stokes multipliers in terms of Stokes factors in Definition 2.6, we find $\xi_{\pm i}(S_\pm) = K_\pm$ thereby completing the proof of the proposition. ■

To rewrite this holonomy isomorphism in terms of the Poisson Lie groups it is convenient to introduce the following notation. If $b_\pm = v_\pm t^{\pm 1} = t^{\pm 1} u_\pm$ where $t \in T$ and $u_\pm, v_\pm \in U_\pm$, then

$${}^i b_\pm := \xi_{\pm i}(v_\pm)^{-1}, \qquad b^i_\pm := \xi_{\pm i}(u_\pm)^{-1}. \tag{3.16}$$

(The inverted left and right $\pm\alpha$ components of $b_\pm$, respectively.) Under the identification (3.8), the isomorphism (3.11) then simplifies to

$$(b_-, b_+, \Lambda) \longmapsto ({}^i b_+ b_- b^i_-, {}^i b_+ b_+ b^i_-, \Lambda). \tag{3.17}$$

Clearly we may quotient by the lattice $\ker(\exp(\pi i \cdot) : \mathfrak{t} \to G)$ (i.e., forget the Λ component above) since Λ only appears as $e^{2\pi i \Lambda}$ in the formulae and $t := e^{\pi i \Lambda} = \delta_+(b_+)$ is retained.

Now if we choose $\widehat{A}_0^* \in \mathrm{pr}^{-1}(A_0^*)$ and lift γ_i canonically to a path $\widehat{\gamma}_i$ in $\widehat{\mathfrak{t}}_{\mathrm{reg}}$ starting at $\widehat{A}_0^*$, then the holonomy of the connection on $\widehat{\mathcal{M}}$ along $\widehat{\gamma}_i$ is also given by Proposition 3.7 (since the connection is pulled back from $\mathcal{M}$). Then quotienting by Γ enables us to identify the fibres $\widehat{\mathcal{M}}(\widehat{A}_0^*)$ and $\widehat{\mathcal{M}}(\widehat{\gamma}_i(1))$ via the gauge action of t_i. (This uses the fact that the element of $B_\mathfrak{g}$ determined by γ_i maps to t_i under the surjection $B_\mathfrak{g} \to \Gamma$.) The Γ action on Stokes multipliers was given in (3.6), and so we deduce the following formula for the holonomy isomorphism $G^* \to G^*$ for the connection on $\mathcal{M}'$ around the loop γ_i/W:

$$(b_-, b_+) \longmapsto \left(t_i^{-1}{}^i b_+ b_- b_-^i t_i, t_i^{-1}{}^i b_+ b_+ b_-^i t_i\right). \tag{3.18}$$

Finally, we must compare (3.18) with the generators of the braid group action of De Concini-Kac-Procesi [13]. In [13] the braid group $B_{\mathfrak{g}}$ is defined abstractly by generators and relations, rather than as a fundamental group. Namely, one has generators T_i (one for each simple root α_i) and relations

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots \tag{3.19}$$

for $i \neq j$, where the number of factors on each side equals the order of the element $s_i s_j$ of the Weyl group. The action of $B_{\mathfrak{g}}$ on G^* is given in [13, Section 7.5] by the following formula:

$$T_i(t^{-1}u_-^{-1}, tu_+) = \left(t_i t^{-1}\left(u_-^{(i)}\right)^{-1}(\exp \tilde{x}_i e_i)t_i^{-1}, t_i t^{-1}\left(\exp \tilde{y}_i f_i\right)t^2 u_+^{(i)} t_i^{-1}\right), \tag{3.20}$$

where, in our notation, $\exp \tilde{x}_i e_i = \xi_i(u_+)^{-1}$, $\exp \tilde{y}_i f_i = \xi_{-i}(u_-)$, $u_+^{(i)} = u_+ \exp \tilde{x}_i e_i$, and $u_-^{(i)} = u_-(\exp \tilde{y}_i f_i)^{-1}$. One may readily check that this is the same as

$$(b_-, b_+) \longmapsto \left(t_i{}^i b_- b_- b_+^i t_i^{-1}, t_i{}^i b_- b_+ b_+^i t_i^{-1}\right), \tag{3.21}$$

where $(b_-, b_+) = (t^{-1}u_-^{-1}, tu_+) \in G^*$. In turn it is straightforward to check this is precisely the inverse map to (3.18). Thus, if we choose to identify the (abstractly presented) braid group with $\pi_1(\mathfrak{t}_{reg}/W)$ by mapping T_i to the *inverse* of the Brieskorn loop $[\gamma_i/W] \in \pi_1(\mathfrak{t}_{reg}/W)$, then we have established the theorem. ■

Remark 3.8. In the later paper [14] a slightly different formula appears and here we wish to clarify the (minor) discrepancy. The action on G^* descends along the map $\pi : G^* \to G^0; (b_-, b_+) \mapsto b_-^{-1}b_+$ to an action on the big cell $G^0 := U_- T U_+ \subset G$. [14, Corollary 14.4, page 97] gives the formula for this action on G^0 to be

$$a = u_- t^2 u_+ \longmapsto t_i^{-1}\xi_i(u_+) a \xi_i(u_+)^{-1} t_i. \tag{3.22}$$

Since π is a covering map (corresponding to replacing $t = e^{\pi i \Lambda}$ by t^2) and the action on t is the standard Weyl group action, we deduce the corresponding action on G^* is as in (3.21), except with each t_i replaced by t_i^{-1}. This action would be obtained from isomonodromy if we use the alternative construction of Tits' extended Weyl group noted at the end of Remark 3.4.

4 Deformation of the isomonodromy Hamiltonians

In this section, the Hamiltonian description of the isomonodromic deformations of Section 3 is given. From this the connection of De Concini-Millson-Toledano Laredo will be derived directly.

Let $\mathcal{M}^* := \mathfrak{g}^* \times \mathfrak{t}_{\text{reg}}$ be the product of the dual of the Lie algebra of G with the regular subset of the chosen Cartan subalgebra. View $\mathcal{M}^*$ as a trivial fibre bundle over $\mathfrak{t}_{\text{reg}}$ with fibre $\mathfrak{g}^*$. Given $(B, A_0) \in \mathcal{M}^*$, consider the meromorphic connection A on the trivial G-bundle over $\mathbb{P}^1$ associated to the $\mathfrak{g}$-valued meromorphic one-form

$$A^s := \left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz \tag{4.1}$$

on $\mathbb{P}^1$. Restricting A to the unit disc Δ (and using the compatible framing coming from the given trivialization) specifies a point of the moduli space $\mathcal{M}(A_0)$. Thus there is a bundle map,

$$\nu : \mathcal{M}^* \longrightarrow \mathcal{M}; \qquad (B, A_0) \longmapsto A|_\Delta. \tag{4.2}$$

This map is holomorphic (by Corollary 2.10) and it is easy to prove it is generically a local analytic isomorphism. (It is studied fibrewise in Appendix B and in [8].) Thus the isomonodromy problem for the connections (4.1) is essentially equivalent to that considered in the previous section.

The proofs of the following two lemmas are not significantly different from the $GL_n(\mathbb{C})$ case and so are omitted here (cf. [15, 17, 19, 22]).

Lemma 4.1. The pullback along ν of the isomonodromy connection on $\mathcal{M}$ is given by the following nonlinear differential equation for sections $B : \mathfrak{t}_{\text{reg}} \to \mathfrak{g}^*$ of $\mathcal{M}^*$:

$$dB = \left[B, \operatorname{ad}_{A_0}^{-1}\left(\left[dA_0, B\right]\right)\right], \tag{4.3}$$

where d is the exterior derivative on $\mathfrak{t}_{\text{reg}}$ and $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ via the Killing form. (Note that $[dA_0, B]$ takes values in $\mathfrak{g}^{\text{od}} := \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$ and that ad_{A_0} is invertible on $\mathfrak{g}^{\text{od}}$.) □

Clearly B flows in a fixed coadjoint orbit in $\mathfrak{g}^*$. Thus, putting the standard Poisson structure on $\mathfrak{g}^*$, one would expect a symplectic interpretation. Indeed equation (4.3) has the following time-dependent Hamiltonian formulation. Consider the one-form

$$\varpi := \mathcal{K}\left(B, \operatorname{ad}_{A_0}^{-1}\left(\left[dA_0, B\right]\right)\right) \tag{4.4}$$

on $\mathcal{M}^*$, where $\mathcal{K}$ is the Killing form. Given a vector field ν on $\mathfrak{t}_{\mathrm{reg}}$ there is a corresponding vector field $\widetilde{\nu}$ on $\mathcal{M}^*$ (zero in $\mathfrak{g}^*$ directions) and thus a function

$$H_\nu := \langle \widetilde{\nu}, \varpi \rangle \tag{4.5}$$

on $\mathcal{M}^*$.

Lemma 4.2. The function H_ν is a time-dependent Hamiltonian for the flow of equation (4.3) along the vector field ν □

Remark 4.3. Usually one chooses a basis $\{\nu_i\}$ of $\mathfrak{t}$ and writes $\varpi = \sum H_i dt_i$ (where $A_0 = \sum t_i \nu_i \in \mathfrak{t}$). It is this one-form which is used to define the isomonodromy τ function.

Now observe that ϖ may equivalently be viewed as a one-form on $\mathfrak{t}_{\mathrm{reg}}$ whose coefficients are quadratic polynomials on $\mathfrak{g}^*$. Let us identify these quadratic polynomials with $S^2\mathfrak{g} = \mathrm{Sym}^2\,\mathfrak{g}$ and consider the natural symmetrisation map $\phi : S\mathfrak{g} \to U\mathfrak{g}$ from the symmetric algebra to the universal enveloping algebra. (This corresponds to deforming the isomonodromy Hamiltonians under the standard deformation *PBW quantisation* of $S\mathfrak{g}$ into $U\mathfrak{g}$.)

Proposition 4.4. The image of ϖ under ϕ is

$$\phi(\varpi) = \sum_{\alpha \in \mathcal{R}_+} \frac{\mathcal{K}(\alpha,\alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha)\frac{d\alpha}{\alpha} \in U\mathfrak{g} \otimes \Omega^1(\mathfrak{t}_{\mathrm{reg}}), \tag{4.6}$$

where $\{e_\alpha, f_\alpha, h_\alpha\}$ is the usual Chevalley basis for $\mathfrak{g}$, normalised so that $[e_\alpha, f_\alpha] = h_\alpha$. □

The proof is a straightforward calculation. This is precisely the $U\mathfrak{g}$ valued one-form appearing in [37]; Given a representation V of $\mathfrak{g}$ and thus an algebra homomorphism $\rho : U\mathfrak{g} \to \mathrm{End}(V)$, the flat connection of De Concini-Millson-Toledano Laredo (whose holonomy is conjectured to give the quantum Weyl group action) is

$$d - h\rho(\phi(\varpi)) \tag{4.7}$$

on the trivial vector bundle over $\mathfrak{t}_{\mathrm{reg}}$ with fibre V, where $h \in \mathbb{C}$ is constant.

Appendices

A Proofs for Section 2

In the $GL_n(\mathbb{C})$ case these results appear in [7] of Balser, Jurkat, and Lutz, which in turn uses a theorem of Sibuya (that the map taking the Stokes multipliers is surjective) and the main asymptotic existence theorem of Wasow [38] (in order to construct fundamental solutions). Here we follow the scheme of [7] wherever possible, but notable exceptions arise in the use of both the above theorems: (1) the reduction to the asymptotic existence theorem is completely different (see proof of Lemma 2.9). Also an independent construction of fundamental solutions is given using *multisummation* rather than the asymptotic existence theorem. (2) for surjectivity, we instead follow the approach of Malgrange [27] involving a $\overline{\partial}$-problem which extends easily to general groups.

Note that we must adapt the proofs from the $GL_n(\mathbb{C})$ case (rather than simply choosing a faithful representation $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{C})$ and using existing results) because there are no representations taking elements of $\mathfrak{t}_{reg}$ into regular diagonal elements of $\mathfrak{gl}_n(\mathbb{C})$ in general.

Example A.1. The standard representation of $\mathfrak{g} = \mathfrak{so}_4(\mathbb{C})$ is equivalent to writing

$$\mathfrak{g} \cong \left\{ X \in \mathfrak{gl}_4(\mathbb{C}) \mid X^T J + JX = 0 \right\} \subset \mathfrak{gl}_4(\mathbb{C}), \tag{A.1}$$

where $J = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ and I is the 2×2 identity matrix. This description is chosen so that we may take $\mathfrak{t} = \{\mathrm{diag}(a, b, -a, -b) \mid a, b \in \mathbb{C}\}$. Now the regular elements of $\mathfrak{t}$ (as a Cartan subalgebra of $\mathfrak{g}$) are precisely those with both $a + b$ and $a - b$ nonzero. However, they will not be regular for $\mathfrak{gl}_4(\mathbb{C})$ unless we also impose $a \neq 0$ and $b \neq 0$ as well.

One still may feel that for sufficiently generic values of the parameters one may always reduce to the $GL_n(\mathbb{C})$ case. Let us dispel this feeling.

Lemma A.2. There are reductive groups G with the following property: if $A_0 \in \mathfrak{t}_{reg}$ is any regular element of a Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$ and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is any nontrivial representation of $\mathfrak{g}$, then $\rho(A_0)$ does not have pairwise distinct eigenvalues. □

Proof. If $\rho(A_0)$ has pairwise distinct eigenvalues, clearly each weight space of V is one-dimensional; V is a multiplicity one representation. However, if for example $G = E_8$ then $\mathfrak{g}$ has no nontrivial multiplicity one representations. ■

Thus there are groups for which one may *never* reduce to the $GL_n(\mathbb{C})$ case via a representation.

For the purposes of this appendix we will use the notion of *Stokes directions* as well as the anti-Stokes directions already defined: $\sigma \in S^1$ is a *Stokes direction* if and only if $\sigma - \pi/(2k-2)$ is an anti-Stokes direction. (These are the directions along which the asymptotic behaviour of $\exp(\alpha \circ q)$ *changes* for some root α, and they arise as bounding directions of the supersectors.) A crucial step enabling us to generalize [7] is the following lemma.

Lemma A.3. Suppose that $\theta \in S^1$ is not a Stokes direction. Then the following hold:

• The element

$$\lambda := \mathrm{Re}\left(A_0 \exp\left(-i(k-1)\theta\right)\right) \in \mathfrak{t}_{\mathbb{R}} \tag{A.2}$$

is in the interior of a Weyl chamber, and so determines an ordering of the roots $\mathcal{R}$.

• The positive roots $\mathcal{R}_+(\lambda)$ (in this ordering) are precisely those roots supporting some anti-Stokes direction within $\pi/(2k-2)$ of θ.

• Let U_+ be the unipotent part of the Borel subgroup $B_+ \supset T$ determined by $\mathcal{R}_+(\lambda)$. The following conditions on an element $C \in G$ are equivalent:

(1) $z^\Lambda e^Q C e^{-Q} z^{-\Lambda}$ tends to $1 \in G$ as $z \to 0$ in the direction θ;
(2) $z^\Lambda e^Q C e^{-Q} z^{-\Lambda}$ is asymptotic to 1 as $z \to 0$ in the direction θ;
(3) $C \in U_+$. □

Proof. First we recall some group-theoretic facts (from, e.g., [10]). Let $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ be the character lattice of T, so that $\mathcal{R} \subset X(T)$ naturally (by thinking of the roots multiplicatively). In turn $\mathcal{R}$ is a subset of the real vector space $\mathfrak{t}^*_{\mathbb{R}} := X(T) \otimes_{\mathbb{Z}} \mathbb{R}$; This has (real) dual $\mathfrak{t}_{\mathbb{R}}$ and naturally $\mathfrak{t}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{t}$. By definition the Weyl chambers of G relative to T are the connected components of

$$\mathfrak{t}_{\mathbb{R},\mathrm{reg}} := \left\{\lambda \in \mathfrak{t}_{\mathbb{R}} \mid \alpha(\lambda) \neq 0 \ \forall \alpha \in \mathcal{R}\right\}. \tag{A.3}$$

Choosing a Weyl chamber is equivalent to choosing a system of positive roots; If $\lambda \in \mathfrak{t}_{\mathbb{R},\mathrm{reg}}$ then the system of positive roots corresponding to λ's connected component is

$$\mathcal{R}_+(\lambda) := \left\{\alpha \in \mathcal{R} \mid \alpha(\lambda) > 0\right\}. \tag{A.4}$$

Now suppose that $\lambda := \mathrm{Re}(A_0 \exp(-i(k-1)\theta))$ as above and $\alpha \in \mathcal{R}$. It is easy to check that $\alpha(\lambda) = 0$ if and only if $\theta - \pi/(2k-2)$ is an anti-Stokes direction, but by hypothesis this is not the case, so λ is indeed regular.

Now consider the *sine-wave* function $f_\alpha(\phi) := \mathrm{Re}(\alpha(A_0)\exp(-i(k-1)\phi))$ as ϕ varies, for any $\alpha \in \mathcal{R}$. It has period $2\pi/(k-1)$ and is maximal at each anti-Stokes direction supporting α. Thus $f_\alpha(\theta) > 0$ if and only if there is an anti-Stokes direction supported by α within $\pi/(2k-2)$ of θ. In turn this is equivalent to $\alpha \in \mathcal{R}_+(\lambda)$, yielding the second statement. (Note that, if $\arg(z) = \phi$, then $\mathrm{Re}(\alpha \circ q(z)) = -cf_\alpha(\phi)$ for some positive real c.)

For the third statement we use the Bruhat decomposition of G (cf., e.g., [10, Section 14.12]). Choose arbitrarily a lift $\widetilde{w} \in N(T)$ of each element $w \in W := N(T)/T$ of the Weyl group. The Bruhat decomposition says that G is the disjoint union of the double cosets $B_+\widetilde{w}B_+$ as w ranges over W. The dense open *big cell* is the largest such coset (corresponding to the *longest element* of W) and is equal to B_-B_+, where B_- is the Borel subgroup opposite to B_+. Moreover, the product map

$$U_- \times T \times U_+ \longrightarrow G; \qquad (u_-, t, u_+) \longmapsto u_- \cdot t \cdot u_+ \tag{A.5}$$

is a diffeomorphism onto the big cell, where $U_\pm$ is the unipotent part of $B_\pm$.

Now observe that each coset in the Bruhat decomposition is stable under conjugation by T, and that U_+ (and in particular the identity element of G) is in the big cell. Thus we can reduce to the case where C is in the big cell; otherwise (3) is clearly not true and also neither of (1) or (2) hold, since $z^\Lambda e^Q C e^{-Q} z^{-\Lambda}$ will remain outside of the big cell.

Therefore, if we label $\mathcal{R}_+(\lambda) = \{\alpha_1, \dots, \alpha_n\}$ and let $\alpha_{-i} = -\alpha_i$, then C has a unique decomposition

$$C = u_{-1} \cdots u_{-n} t u_1 \cdots u_n \tag{A.6}$$

with $u_i \in U_{\alpha_i}$ and $t \in T$ (see [10, Section 14.5]). Each of these components is independent and $z^\Lambda e^Q C e^{-Q} z^{-\Lambda} = u^z_{-1} \cdots u^z_{-n} t u^z_1 \cdots u^z_n$, where $u^z_i = z^\Lambda e^Q u_i e^{-Q} z^{-\Lambda} \in U_{\alpha_i}$. Now, given a root α and $X \in \mathfrak{g}_\alpha$, the key fact is that we know the behaviour of $\mathrm{Ad}_{z^\Lambda e^Q}(X)$ as $z \to 0$ in the direction θ; it decays exponentially if $\alpha \in \mathcal{R}_+(\lambda)$ and otherwise (if $X \neq 0$) it explodes exponentially. (The dominant term of $z^\Lambda e^Q$ is e^q and this acts on X by multiplication by $e^{\alpha \circ q}$, which has the said properties.) Finally, in any representation ρ, u_i is of the form $1 + \rho(X_i)$ with $X_i \in \mathfrak{g}_{\alpha_i}$, so that $C \in U_+$ if and only if $t = 1$ and $X_i = 0$ for all $i < 0$, and in turn (via the decomposition of $z^\Lambda e^Q C e^{-Q} z^{-\Lambda}$) this is equivalent to both (1) and (2). ■

Now we move onto the proofs of the results of Section 2.

Proof of Lemma 2.1. (This is adapted from [23, Proposition 2.2], [30, Theorem B.1.3], and [2, Lemma 1, page 42].) For the existence of $\widehat{F}$ and A^0 we proceed as follows. Write

$$A^s = A_0 \frac{dz}{z^k} + \cdots + A_{k-1} \frac{dz}{z} + A_k dz + \cdots \tag{A.7}$$

with $A_i \in \mathfrak{g}$. First each A_i will be moved into $\mathfrak{t}$ and then the nonsingular part will be removed. Let $\mathfrak{g}^{od} = \bigoplus_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha$ (so that $\mathrm{ad}_{A_0} : \mathfrak{g}^{od} \to \mathfrak{g}^{od}$ is an isomorphism), and let $\mathrm{pr} : \mathfrak{g} \to \mathfrak{g}^{od}$ be the projection along $\mathfrak{t}$. Suppose inductively that the first p coefficients

$A_0, A_1, \ldots, A_{p-1}$ of A^s are in $\mathfrak{t}$ (so the $p = 1$ case holds by assumption). By applying the gauge transformation $\exp(z^p H_p)$ to A^s (where $H_p \in \mathfrak{g}$), we find

$$\exp\left(z^p H_p\right)\left[A^s\right] = A^s + \left[H_p, A_0\right] z^{p-k} dz + O\left(z^{p-k+1}\right) dz. \tag{A.8}$$

Thus, $A_p + [H_p, A_0]$ is the first coefficient which is not necessarily in $\mathfrak{t}$, and so by defining

$$H_p := \left(\mathrm{ad}_{A_0}\right)^{-1}\left(\mathrm{pr}\left(A_p\right)\right) \in \mathfrak{g}^{\mathrm{od}}, \tag{A.9}$$

we ensure that the first $p + 1$ coefficients of $\exp(z^p H_p)[A^s]$ are in $\mathfrak{t}$, completing the inductive step. Hence, if we define a formal transformation $\widehat{H} \in G[[z]]$ to be the infinite product

$$\widehat{H} := \cdots \exp\left(z^p H_p\right) \exp\left(z^{p-1} H_{p-1}\right) \cdots \exp\left(z H_1\right), \tag{A.10}$$

then each coefficient of $\widehat{H}[A^s]$ is in $\mathfrak{t}$. Now define A^0 to be the principal part of $\widehat{H}[A^s]$ so that $\widehat{H}[A^s] = A^0 + D$ with D nonsingular. Then define $\widetilde{F} := e^{(-\int_0^z D)} \in T[[z]]$ (where $\int_0^z D \in \mathfrak{t}[[z]]$ is the series obtained from D by replacing $z^p dz$ by $z^{p+1}/(p+1)$ for each $p \geq 0$), so that $d\widetilde{F}(\widetilde{F})^{-1} = d\log\widetilde{F} = -D$. Thus $(\widetilde{F}\widehat{H})[A^s] = A^0$ and so $\widehat{F} := (\widetilde{F}\widehat{H})^{-1} \in G[[z]]$ is the desired formal transformation.

For the uniqueness, it is clearly sufficient to show that if $\widehat{F}[A^0] = A^1$, where A^0 and A^1 are formal types and $\widehat{F}(0) = 1$, then $\widehat{F} = 1$. Now if $\widehat{F}[A^0] = A^1$, it follows that $\widehat{F}$ is actually convergent (since on using a faithful representation we see that $\widehat{F}$ solves the diagonal system $d\widehat{F} = A^0\widehat{F} - \widehat{F}A^1$). Let $F : \Delta \to G$ denote the sum of $\widehat{F}$. Also, if we write $A^i = dQ^i + \Lambda^i dz/z$ for $i = 0, 1$, then $d(e^{-Q^1} z^{-\Lambda^1} F z^{\Lambda^0} e^{Q^0}) = 0$, so that $F = z^{\Lambda^1} e^{Q^1} C e^{-Q^0} z^{-\Lambda^0}$ for some constant $C \in G$. Now $F \to 1$ on any sector and so (since Q^0 and Q^1 have the same leading term) as in Lemma A.3 we may deduce that $C \in U_- \cap U_+ = \{1\}$, and therefore $F = z^{\Lambda^1 - \Lambda^0} e^{Q^1 - Q^0}$. The only way this can have a Taylor expansion with constant term 1 at 0 is if $\Lambda^1 = \Lambda^0$, $Q^1 = Q^0$ and so $\widehat{F} = F = 1$. ■

Proof of Lemma 2.4. Given a half-period $\mathbf{d} \subset \mathbb{A}$, let $\theta(\mathbf{d})$ be the bisecting direction of the sector spanned by $\mathbf{d}$. By the symmetry of $\mathbb{A}$, $\theta(\mathbf{d}) - \pi/(2k-2)$ is half-way between two consecutive anti-Stokes directions, so $\theta(\mathbf{d})$ is not a Stokes direction. Therefore, we may feed $\theta(\mathbf{d})$ into Lemma A.3, the second part of which immediately yields the first statement of Lemma 2.4.

The third statement of Lemma 2.4 is now immediate from [10, Sections 14.5–14.8], (using the notion of *direct spanning* subgroups) and then the second statement follows provided we check that $\mathcal{R}(d)$ is a *closed* set of roots, in the sense that if $\alpha, \beta \in \mathcal{R}(d)$ and $\alpha + \beta \in \mathcal{R}$ then $\alpha + \beta \in \mathcal{R}(d)$. This however is immediate from the definition of $\mathcal{R}(d)$.

For the fourth statement simply observe λ is negated when $\theta(\mathbf{d})$ is rotated by $\pi/(k-1)$. ■

Proof of Theorem 2.5.

Uniqueness (cf. [7, Remark 1.4]). Suppose that $F_1, F_2 : \mathrm{Sect}_i \to G$ both have properties (1) and (2). Thus $(F_1^{-1}F_2)[A^0] = A^0$ and so

$$F_1^{-1}F_2 = z^{\Lambda}e^{Q}Ce^{-Q}z^{-\Lambda} \tag{A.11}$$

for some constant $C \in G$. By (2), $F_1^{-1}F_2$ extends to $\widehat{\mathrm{Sect}_i}$ and is asymptotic to 1 at zero there. But $\widehat{\mathrm{Sect}_i}$ has opening greater than $\pi/(k-1)$, so Lemma A.3 implies $C \in U_+ \cap U_- = \{1\}$ (by taking two non-Stokes directions in $\widehat{\mathrm{Sect}_i}$ differing by $\pi/(k-1)$). Hence $C = 1$ and $F_1 = F_2$.

Existence. Here we use multisummation (see proof of Lemma 2.9 for a more conventional approach). Choose a faithful representation $G \hookrightarrow GL_n(\mathbb{C})$ such that T maps to the diagonal subgroup. Let $\mathfrak{d} \subset \mathfrak{gl}_n(\mathbb{C})$ be the diagonal subalgebra and let $\alpha_{ij} : \mathfrak{d} \to \mathbb{C}; X \mapsto X_{ii} - X_{jj}$ be the roots of $GL_n(\mathbb{C})$. Everything now will be written in this representation. Thus $\widehat{F}$ is a formal solution to the system of linear differential equations:

$$d\widehat{F} = A\widehat{F} - \widehat{F}A^0. \tag{A.12}$$

This equation has *levels* $\mathbf{k} := \{-\deg(\alpha_{ij} \circ Q) \mid i, j = 1, \ldots, n\} \setminus \{0\}$. Note that the highest level is $k - 1$. (If $k = 2$ or if $G = GL_n(\mathbb{C})$ then this is the only level—however generally there may be lower levels as well.) The *singular directions* $\mathbb{A}_{\mathfrak{gl}}$ of (A.12) are the $GL_n(\mathbb{C})$ anti-Stokes directions, defined as follows. For each i, j, $\alpha_{ij} \circ Q$ is a polynomial in $1/z$ of degree at most $k-1$. If $\alpha_{ij} \circ Q$ is not zero, let $\mathbb{A}_{\mathfrak{gl}}^{ij}$ be the finite number of directions along which the leading term of $\alpha_{ij} \circ Q$ is real and negative and let $\mathbb{A}_{\mathfrak{gl}}^{ij}$ be empty otherwise. Then define $\mathbb{A}_{\mathfrak{gl}}$ to be the union of all these sets $\mathbb{A}_{\mathfrak{gl}}^{ij}$ as i and j vary. (One may check that $\mathbb{A} \subset \mathbb{A}_{\mathfrak{gl}}$.) Then the main theorem in multisummation theory implies the following.

Theorem A.4 (see [6, Theorem 4.1]). If $d \in S^1$ is not a singular direction then (each matrix entry of) $\widehat{F}$ is $\mathbf{k}$-summable in the direction d. The $\mathbf{k}$-sum of $\widehat{F}$ along d is holomorphic and asymptotic to $\widehat{F}$ at zero in the sector $\mathrm{Sect}(d - \pi/(2k-2) - \epsilon, d + \pi/(2k-2) + \epsilon)$ for some $\epsilon > 0$. □

Since they are unneeded here we omit the discussion of the finer Gevrey asymptotic properties that such sums possess, although we do need the fact that multisummation is a morphism of differential algebras (see [28, Theorem 1, page 348]). In more detail recall [6, Theorem 4.4] that the set $\mathbb{C}\{z\}_{k,d}$ of formal power series which are k-summable in the direction d is a differential subalgebra of $\mathbb{C}[[z]]$. Then k-summation maps this injectively onto some set $\mathcal{O}_{k,d}$ of germs at 0 of holomorphic functions on $\mathrm{Sect}(d-\pi/(2k-2), d+\pi/(2k-2))$. Quite generally the map taking asymptotic expansions is easily seen to be a differential algebra morphism, and so here it restrict to an *isomorphism* $\mathcal{O}_{k,d} \xrightarrow{\cong} \mathbb{C}\{z\}_{k,d}$ of differential algebras. By definition the multisummation operator is the inverse morphism.

Now to construct $\Sigma_i(\widehat{F})$, choose any direction d in Sect_i which is not in (the finite set) $\mathbb{A}_{\mathfrak{gl}}$. Let $\Sigma_i(\widehat{F})$ be the multisum of $\widehat{F}$ along the direction d from Theorem A.4 and let S be the sector appearing there. Since multisummation is a morphism of differential algebras, we deduce first that $\Sigma_i(\widehat{F})$ satisfies equation (A.12) (as is standard in the theory) and secondly we have the following lemma.

Lemma A.5. The map $\Sigma_i(\widehat{F})$ takes values in G. □

Proof. This is because G, being reductive, is an affine algebraic group and so the matrix entries of $\Sigma_i(\widehat{F})$ satisfy the same polynomial equations as the entries of $\widehat{F}$. In more detail, there are complex polynomials $\{p_j\}$ such that

$$G \cong \left\{(g, x) \in \mathbb{C}^{n\times n} \times \mathbb{C} \mid \det(g)\cdot x = 1,\ p_j(g) = 0\ \forall j\right\}, \tag{A.13}$$

as a subgroup of $GL_n(\mathbb{C})$, for some n. For any commutative algebra R over $\mathbb{C}$, the algebraic group $G(R)$ is defined simply by replacing the two occurrences of $\mathbb{C}$ in (A.13) by R. Thus we wish to show that $\Sigma_i(\widehat{F}) \in G(\mathcal{O}(S))$ (the group of holomorphic maps $S \to G$), given that $\widehat{F} \in G[[z]] := G(\mathbb{C}[[z]])$. But it is immediate that $p_j(\widehat{F}) = 0$ implies $p_j(\Sigma_i(\widehat{F})) = 0$ since multisummation is an algebra morphism. ■

Next, we must check that $\Sigma_i(\widehat{F})$ has property (2) of Theorem 2.5. The key point is that there are no Stokes directions in $\widehat{\mathrm{Sect}}_i \setminus S$; indeed the Stokes directions in the supersector $\widehat{\mathrm{Sect}}_i$ closest to the boundary rays are $d_{i+1} - \pi/(2k-2)$ and $d_i + \pi/(2k-2)$, both of which are in S. Thus the following G-valued analogue of the extension lemma of [7] will yield (2).

Lemma A.6 (cf. [7, Lemma 1, page 73]). Suppose that $S, \widetilde{S}$ are two sectors with nonempty intersection and such that $\widetilde{S}$ contains no Stokes directions. If $F : S \to G$ is a holomorphic map asymptotic to $\widehat{F}$ at 0 in S and such that $F[A^0] = A$, then the analytic continuation of F to $S \cup \widetilde{S}$ is asymptotic to $\widehat{F}$ at 0 in $S \cup \widetilde{S}$. □

Proof. Choose any holomorphic map $\widetilde{F} : \widetilde{S} \to G$ asymptotic to $\widehat{F}$ at 0 and such that $F[A^0] = A$ (using multisummation for example—the hypotheses imply $\widetilde{S}$ has opening $< \pi/(k-1)$). Then (as in the uniqueness part above) there exists a constant $C \in G$ such that $F = \widetilde{F}z^{\Lambda}e^{Q}Ce^{-Q}z^{-\Lambda}$ in $S \cap \widetilde{S}$. Thus $\widetilde{F}z^{\Lambda}e^{Q}Ce^{-Q}z^{-\Lambda}$ is the analytic continuation of F to $\widetilde{S}$. Now since $\widetilde{F}^{-1}F$ is asymptotic to 1 at 0 in $S \cap \widetilde{S}$, Lemma A.3 implies that $C \in U_+$, where the root ordering is determined by any θ in $S \cap \widetilde{S}$. But since $\widetilde{S}$ contains no Stokes directions, Lemma A.3 implies that $z^{\Lambda}e^{Q}Ce^{-Q}z^{-\Lambda}$ is asymptotic to 1 at 0 in all of $\widetilde{S}$. In turn it follows that the analytic continuation of F is asymptotic to $\widehat{F}$ on all $S \cup \widetilde{S}$. ■

Finally, the last statement of Theorem 2.5 is immediate either from the morphism properties of multisummation, or from uniqueness. ■

Proof of Lemma 2.7. To see that $S_j \in \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$, recall from Lemma 2.4 that $\mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0) = U_+$ where the root order is determined by the bisecting direction $\theta(\mathbf{d})$ of $\mathbf{d}$. Now observe that $\widehat{\mathrm{Sect}}_{(j-1)l} \cap \widehat{\mathrm{Sect}}_{jl}$ contains $\theta(\mathbf{d})$ and so by Theorem 2.5 (if $j \neq 1$) $z^{\Lambda}e^{Q}S_je^{-Q}z^{-\Lambda} = \Sigma_{jl}(\widehat{F})^{-1}\Sigma_{(j-1)l}(\widehat{F})$ is asymptotic to 1 along $\theta(\mathbf{d})$. Thus Lemma A.3 implies $S_j \in \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0)$. (For $j = 1$ the argument is the same once the change in branch of $\log(z)$ is accounted for.) In turn to see that $K_i \in \mathbb{S}\mathrm{to}_{d_i}(A^0)$ simply observe that $\mathbb{S}\mathrm{to}_{d_i}(A^0) = \mathbb{S}\mathrm{to}_{\mathbf{d}}(A^0) \cap \mathbb{S}\mathrm{to}_{\mathbf{d}'}(A^0)$ (where $\mathbf{d} = (d_i, \dots, d_{i+l-1})$ and $\mathbf{d}' = (d_{i-l+1}, \dots, d_i)$ are the two half-periods ending on d_i), and that the above argument implies that K_i is in this intersection, since $\Sigma_i(\widehat{F})^{-1}\Sigma_{i-1}(\widehat{F})$ is asymptotic to 1 along both $\theta(\mathbf{d})$ and $\theta(\mathbf{d}')$. ■

Now we establish the surjectivity of the irregular Riemann-Hilbert map in Theorem 2.8. Fix a formal type A^0 and let $\mathbb{A}$ be the corresponding set of anti-Stokes directions. Also fix a choice of initial sector and branch of $\log(z)$ as in Section 2. Now choose arbitrarily a Stokes factor $K_d \in \mathbb{S}\mathrm{to}_d(A^0)$ for each $d \in \mathbb{A}$.

Theorem A.7. There exists a meromorphic connection A on the trivial principal G-bundle over Δ having formal type A^0 and Stokes factors $\{K_d\}$. □

Proof. (This is an adaptation of [5, Section 9.7], except we replace the key step with a $\overline{\partial}$-problem, as was suggested by Malgrange [27] and fleshed out in [3, Section 4.4].) First we remark that it is sufficient to construct A only in a neighbourhood of the origin because any such connection is gauge equivalent to a connection defined over the whole disc. (One may prove this as follows: given A over $\Delta_\epsilon := \{z \mid |z| \leq \epsilon\}$, choose any holomorphic connection A^1 on $G \times \Delta^*$ with the same monodromy as A around 0. The ratio $\Phi^1 \cdot \Phi^{-1}$ of corresponding fundamental solutions then defines a holomorphic map from Δ^*_ϵ to G which we use as a clutching function to define a principal G-bundle P over Δ. The connections A, A^1 define a single meromorphic connection on P. Moreover, P is trivial since all G-bundles over a disk are—cf. [16, page 370].)

Now choose j such that $K_{d_j} \neq 1$. By induction on the number of nontrivial Stokes factors we may assume that there is a connection B having Stokes factor K_d for each $d \neq d_j$ but having Stokes factor $1 \in \mathbb{S}\mathrm{to}_{d_j}(A^0)$ along d_j. Write $K = K_{d_j}$ for simplicity and let $B^s = -s^*(B)$ as usual. (Here s is the identity section of the trivial G-bundle $G \times \Delta$.) Let $\chi_j(z)$ be the canonical fundamental solution of B on Sect_j from Definition 2.6 and define $\chi(z) := \chi_j K \chi_j^{-1}$. Let $\widetilde{d}_j$ be the lift of the direction d_j to the universal cover $\widetilde{\Delta}^*$ of the punctured disk determined by the chosen branch of $\log(z)$ and let $\widetilde{\mathrm{Sect}}_j$ be the lift of Sect_j. Let $\mathcal{S}^*$ denote the sector in $\widetilde{\Delta}^*$ (of opening more than 2π) from $\widetilde{d}_j - \pi/(2k-2) + \delta$ to $\widetilde{d}_j + 2\pi + \pi/(2k-2) - \delta$, and let $\mathcal{S} = \mathrm{Sect}(\widetilde{d}_j - \pi/(2k-2) + \delta, \widetilde{d}_j + \pi/(2k-2) - \delta)$. Here $\delta > 0$ is fixed so that no Stokes directions lie in the interior of either component of $\mathrm{Sect}(\widetilde{d}_j - \pi/(2k-2), \widetilde{d}_j + \pi/(2k-2)) \setminus \mathcal{S}$ (i.e., $\delta < \min\{|d_j - d_{j\pm1}|\}$). We now claim that there exists a holomorphic map $\tau : \mathcal{S}^* \to G$ having an asymptotic expansion with constant term 1 in $\mathcal{S}^*$ and such that

$$\tau(ze^{2\pi i}) = \tau(z)\chi(z) \tag{A.14}$$

for any $z \in \widetilde{\mathrm{Sect}}_j \cap \mathcal{S}$, where χ is pulled up to $\widetilde{\mathrm{Sect}}_j$ in the obvious way.

Such τ may be constructed as follows. Let $\mathcal{S}' = \mathrm{Sect}(\widetilde{d}_j - \pi/(2k-2) + \delta, \widetilde{d}_j + \pi)$ and (as in [3, Lemma 4.3.2] and using the exponential map for G) extend χ to a C^∞ map $f : \mathcal{S}' \to G$ such that $f|_{\mathcal{S}} = \chi$, $f(z) = 1$ for $\arg(z)$ in some neighbourhood of $\widetilde{d}_j + \pi$ and $f \sim 1$ on all $\mathcal{S}'$. (By construction $\chi \sim 1$ on $\mathrm{Sect}(\widetilde{d}_j - \pi/(2k-2), \widetilde{d}_j + \pi/(2k-2))$.) Then define a C^∞ $\mathfrak{g}$-valued one-form α on Δ by letting $\alpha = f^{-1}\overline{\partial} f$ on $\mathcal{S}'$ and extending by zero. Now solve the $\overline{\partial}$-problem $g^{-1}\overline{\partial} g = \alpha$ for a smooth map g from some neighbourhood of $0 \in \Delta$ to G, with $g(0) = 1$. (This is possible for the same reasons as in the $GL_n(\mathbb{C})$ case, for which cf., e.g., [1, page 555].) Finally define $\tau : \mathcal{S}^* \to G$ by $\tau = gf^{-1}$ for $\arg(z) \le \widetilde{d}_j + \pi$ and $\tau = g$ for $\arg(z) \ge \widetilde{d}_j + \pi$; one easily checks this has the properties claimed.

To complete the proof define $\widetilde{\chi}(z) := \tau(z)\chi_j(z)$ for z in $\mathcal{S}^*$ (where χ_j is continued from $\widetilde{\mathrm{Sect}}_j$ as a fundamental solution of B). Then (A.14) implies $A^s := (d\widetilde{\chi})\widetilde{\chi}^{-1}$ is invariant under rotation by 2π and so defines a $\mathfrak{g}$-valued one-form on a neighbourhood of 0 in Δ^*. We will show that the corresponding connection A on the trivial principal G-bundle has the desired properties. First observe that $A^s = \tau[B^s]$ by holomorphicity, since this certainly holds near 0 in $\widetilde{\mathrm{Sect}}_j$. Since τ admits an asymptotic expansion $\widehat{\tau} \in G[[z]]$ in $\mathcal{S}^*$, it follows that A^s admits *Laurent* expansion $\widehat{\tau}[B^s]$. Thus if $B^s = \widehat{F}[A^0]$ (from Lemma 2.1) then $A^s = (\widehat{\tau} \circ \widehat{F})[A^0]$, and so A has formal type A^0 as required. Now from the range of validity of the asymptotic expansion of τ, and from the uniqueness of the sums in Theorem 2.5, we deduce that for $z \in \mathrm{Sect}_i$,

$$\Sigma_i\big(\widehat{\tau} \circ \widehat{F}\big)(z) = \tau\big(\widetilde{z}\big) \cdot \Sigma_i\big(\widehat{F}\big)(z), \tag{A.15}$$

where $\widetilde{z} \in \widetilde{\Delta}^*$ lies over z and between directions $\widetilde{d}_j$ and $\widetilde{d}_j + 2\pi$. (On Sect_{j-1} and Sect_j one needs to use the extension lemma, Lemma A.6, as well—which is applicable by the choice of δ.) In turn we immediately find that A has the same Stokes factors as B except in $\mathbb{S}\mathrm{to}_{d_j}(A^0)$. Here (across d_j) by definition A has Stokes factor

$$\Phi_j^{-1}\Phi_{j-1} = \chi_j^{-1}\tau^{-1}(z)\tau\big(ze^{2\pi i}\big)\chi_{j-1} \tag{A.16}$$

if $j \neq \#\mathbb{A}$, where Φ_i, χ_i denote canonical solutions of A, B, respectively. Since B has trivial Stokes factor here $\chi_{j-1} = \chi_j$ and so by (A.14) and the definition of χ, we find that $\Phi_j^{-1}\Phi_{j-1} = \chi_j^{-1}\chi\chi_j = K$ as required. (Similarly if $j = \#\mathbb{A}$.) ■

Proof of Lemma 2.9. Here an alternative construction of the sums of Theorem 2.5 will be given, closer to the usual approach in the $\mathrm{GL}_n(\mathbb{C})$ case. This is more direct than the multisummation approach above and enables us to prove that the sums vary holomorphically with parameters.

The usual construction of $\Sigma_0(\widehat{F})$ (cf., e.g., [38]) consists of two steps. Roughly speaking one converts the equation satisfied by $\widehat{F}$ into two independent nonlinear equations for "parts" of $\widehat{F}$. Then an asymptotic existence theorem is used to find analytic solutions to these two equations, that are asymptotic to the corresponding parts of $\widehat{F}$. Usually for $G = \mathrm{GL}_n(\mathbb{C})$ (see [38, Section 12.1] and [31]) the two equations involve the upper and lower triangular parts of $\widehat{F} - 1$. For general G this makes no sense: an alternative procedure will be used here to reduce the problem to the *same* asymptotic existence theorem.

If a and b are the boundary directions of the sector $\check{S}$ (so $\check{S} = \mathrm{Sect}(a, b)$), let $S := \mathrm{Sect}(b - \pi/(2k-2), a + \pi/(2k-2))$ (a sector of opening less than $\pi/(k-1)$ centred on $\check{S}$). For simplicity write $F = \Sigma_0(\widehat{F})$ for the G-valued map we are seeking. This should have asymptotic expansion $\widehat{F}$ as $z \to 0$ in S and should solve (for each $x \in U$) the equation

$$(dF)F^{-1} = A - FA^0F^{-1} \tag{A.17}$$

on the sector $S \subset \Delta$, where d is the exterior derivative on Δ. (Here $(dF)F^{-1}$ is defined in the usual way as the pullback of the right-invariant Maurer-Cartan form on G under the map F and $FA^0F^{-1} := \mathrm{Ad}_F\, A^0$.) Note that such F is unique since (for each x) the extension lemma (Lemma A.6) says F extends to $\widehat{\mathrm{Sect}}_0$ maintaining the asymptotic expansion $\widehat{F}$, and so the uniqueness part of Theorem 2.5 fixes F.

Now, because the big cell $G^0 = U_-TU_+ \subset G$ is open and contains the identity, we find the following lemma.

Lemma A.8. (1) $\widehat{F}$ admits a unique factorisation

$$\widehat{F} = \widehat{u}_- \cdot \widehat{t} \cdot \widehat{u}_+ \tag{A.18}$$

with $\widehat{u}_\pm \in U_\pm(\mathcal{O}(X)[[z]])$ and $\widehat{t} \in T(\mathcal{O}(X)[[z]])$.

(2) For $z \in S$ sufficiently close to 0, any solution F of (A.17) asymptotic to $\widehat{F}$ has a unique factorisation

$$F = u_- \cdot t \cdot u_+ \tag{A.19}$$

with $u_\pm, t$ taking values in $U_\pm, T$, respectively. □

Substituting $F = u_- \cdot t \cdot u_+$ into (A.17) and rearranging yields

$$u_-^{-1}du_- + (dt)t^{-1} + t\big(du_+\big)u_+^{-1}t^{-1} + tu_+A^0u_+^{-1}t^{-1} - u_-^{-1}Au_- = 0. \tag{A.20}$$

Taking the $\mathfrak{u}_-$ component of this gives the independent equation

$$u_-^{-1}du_- = \pi_-\big(u_-^{-1}Au_-\big) \tag{A.21}$$

for u_-, where $\pi_- : \mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}_+ \to \mathfrak{u}_-$ is the projection. We wish to solve (A.21) using the following asymptotic existence theorem with parameters, which is also used in the $GL_n(\mathbb{C})$ case.

Theorem A.9. Let S be an open sector in the complex z plane with vertex 0 and opening not exceeding $\pi/(k-1)$. Let $f(z, L, x) : \Delta \times \mathbb{C}^N \times U \to \mathbb{C}^N$ be a holomorphic map such that

(i) the Jacobian matrix $(\partial f_i/\partial(L)_j)|_{L=0,z=0}$ is invertible for all $x \in \overline{U}$, and

(ii) the differential equation

$$\frac{dL}{dz} = \frac{f(z, L, x)}{z^k} \tag{A.22}$$

admits a formal power series solution $\widehat{L} = \sum_1^\infty L_r(x)z^r \in \mathbb{C}^N[[z]] \otimes \mathcal{O}(U)$.

Then there exists, for sufficiently small $z \in S$, a holomorphic solution $L(z, x)$ of (A.22) having asymptotic expansion $\widehat{L}$ in S uniformly in some neighbourhood of $x_0 \in U$. □

Proof. Without parameters this is [38, Theorem 14.1]. The method of successive approximations used there extends immediately to the case with parameters since the uniform convergence of successive approximations does not destroy holomorphicity with respect to parameters (cf. [32, Remark 2, page 161]). Alternatively, Sibuya proves a very similar result (cf. [31, Lemma 2]) for the case where the parameters become singular on a sector. As remarked in [21, page 100] the case when the parameters are nonsingular and on a disc (as required here) is proved in exactly the same manner. ∎

The trick to convert (A.21) into the form (A.22) is as follows. (We will have $\mathbb{C}^N = \mathfrak{u}_-$.) Since U_- is unipotent the exponential map $\exp : \mathfrak{u}_- \to U_-$ is an algebraic isomorphism, and its derivative gives an isomorphism $\exp_* : T\mathfrak{u}_- \to TU_-$ of the tangent bundles. If we identify $T\mathfrak{u}_- \cong \mathfrak{u}_- \times \mathfrak{u}_-$ using the vector space structure of $\mathfrak{u}_-$ and $TU_- \cong U_- \times \mathfrak{u}_-$ using left multiplication in U_- then we deduce the following lemma.

Lemma A.10. There is an algebraic isomorphism

$$\psi : \mathfrak{u}_- \times \mathfrak{u}_- \xrightarrow{\cong} U_- \times \mathfrak{u}_- \tag{A.23}$$

which is linear in the second component and such that if $L(z) : \Delta \to \mathfrak{u}_-$ is any holomorphic map, then

$$\psi\left(L, \frac{dL}{dz}\right) = \left(e^L, e^{-L}\frac{d}{dz}\left(e^L\right)\right). \tag{A.24}$$

□

Thus, setting $u_- = e^L$, equation (A.21) is equivalent to (A.22) with the map f defined by

$$\big(L, f(z, L, x)\big) = \psi^{-1}\big(u_-, h(z, u_-, x)\big) \in \mathfrak{u}_- \times \mathfrak{u}_- \tag{A.25}$$

for any $z \in \Delta, L \in \mathfrak{u}_-, x \in U$, where $u_- = e^L$ and $h(z, u_-, x) := \langle z^k \pi_-(u_-^{-1} A(z, x) u_-), \partial/\partial z\rangle$ is from (A.21).

Clearly, the formal solution $\widehat{u}_-$ of (A.21) induces a formal solution of (A.22) of the desired form and so all that remains to solve (A.22) is to check the Jacobian condition in Theorem A.9. Geometrically this condition says precisely that the graph $\Gamma(f) \subset \mathfrak{u}_-^h \times \mathfrak{u}_-^v$ of the map $f(0, \cdot, x) : \mathfrak{u}_-^h \to \mathfrak{u}_-^v$ is transverse to the horizontal subspace $\mathfrak{u}_-^h$ at $L = 0$. (Here $\mathfrak{u}_-^h, \mathfrak{u}_-^v$ are just copies of $\mathfrak{u}_-$ labeled *horizontal* and *vertical*; f is viewed as a section of the tangent bundle to $\mathfrak{u}_-^h$.) Since ψ is a diffeomorphism it is sufficient to check this transversality condition on $U_- \times \mathfrak{u}_-$. By definition the graph of f corresponds to the graph of h under ψ. Clearly $L = 0$ if and only if $u_- = 1$ and $h(1) = \pi_-(A_0) = 0$ since $A_0 \in \mathfrak{t}$.

(Here we omit the arguments $z = 0$ and x of h for notational simplicity.) Thus $\psi(0, f(0)) = (1, 0)$ and we must check that the two vector spaces $\psi_*(\mathfrak{u}_- \times \{0\})$ and $T_{(1,0)}\Gamma(h)$ are transverse subspaces of the tangent space $T_{(1,0)}(U_- \times \mathfrak{u}_-) = \mathfrak{u}_- \times \mathfrak{u}_-$. The tangent space to the graph of h at $(1, 0)$ is

$$\{(X, \pi_-[X, A_0]) \mid X \in \mathfrak{u}_-\} \tag{A.26}$$

and one may calculate that the derivative of ψ maps the horizontal subspace $\mathfrak{u}_- \times \{0\}$ to

$$\{(X, 0) \mid X \in \mathfrak{u}_-\}. \tag{A.27}$$

Since $A_0(x) \in \mathfrak{t}_{\text{reg}}$, it follows immediately that these two subspaces are indeed transverse. Thus we may apply Theorem A.9 to obtain a holomorphic solution $L(z, x)$ of (A.22) and in turn obtain a solution $u_- = e^L$ of (A.21).

Given this solution u_-, now consider the $\mathfrak{t}$ component

$$(dt)t^{-1} = \delta(u_-^{-1} A u_-) - A^0 \tag{A.28}$$

of the full equation (A.20), where $\delta : \mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}_+ \to \mathfrak{t}$ is the projection. This equation has formal solution $\widehat{t}$ and so the right-hand side of (A.28) has *nonsingular* asymptotic expansion as $z \to 0$ in S. Immediately this implies (A.28) has a unique holomorphic solution tending to $1 \in T$ as $z \to 0$, given by

$$t(z, x) = \exp\left(\int_0^z \left(\delta(u_-^{-1} A u_-) - A^0\right)\right) \tag{A.29}$$

(cf. [38, Theorem 8.7, page 38]).

Thus we have obtained all except the u_+ component of the desired solution $F = u_- t u_+$. To obtain u_+ we repeat all the above procedure with the opposite factorisation $F = w_+ s w_-$ of F (with $w_\pm \in U_\pm$ and $s \in T$). This yields w_+ and s. Then, for sufficiently small z, the components u_+ and w_- are determined (holomorphically) from u_-, w_+, s, t by the equation

$$w_+^{-1} u_- t = s w_- u_+^{-1} \tag{A.30}$$

since both Bruhat decompositions are unique and the left-hand side is known. The resulting solution $F = u_- t u_+ = w_+ s w_-$ then has the desired properties. ∎

B Poisson properties of monodromy maps

In this appendix we explain how the results of Section 2 enable us to extend [8, Theorems 1 and 2] from $GL_n(\mathbb{C})$ to arbitrary connected complex reductive groups G. The main modifications of the proofs in [8] are purely notational and so here we will concentrate on giving a clear statement of the results. The setup is as follows.

Let K be any compact connected Lie group. Choose a maximal torus $T_K \subset K$ and a nondegenerate symmetric invariant bilinear form $\mathcal{K}$ on $\mathfrak{k} = \mathrm{Lie}(K)$. (Thus if K is semisimple we may take $\mathcal{K}$ to be the Killing form, or if $K = U(n)$, then $\mathcal{K}(A, B) = \mathrm{Tr}(AB)$ will do.)

Let G be the complex algebraic group associated to K (as in [12]; G is the variety associated to the complex representative ring of K). Any complex connected reductive Lie group G arises in this way (see [20]). We have $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$ with Cartan subalgebra $\mathfrak{t} = \mathfrak{t}_K \otimes \mathbb{C}$ and G has maximal torus $T = \exp(\mathfrak{t})$. Extend $\mathcal{K}$ $\mathbb{C}$-bilinearly to $\mathcal{K} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. The group G comes equipped with an involution (with fixed point set canonically isomorphic to K), which we will denote by $g \mapsto g^{-\dagger}$. (It is denoted ι in [12].) This induces an anti-holomorphic involution of $\mathfrak{g}$ (to be denoted $X \mapsto -X^\dagger$) fixing $\mathfrak{k}$ pointwise.

Note that $\mathfrak{t}$ comes with two real structures: one ($A \mapsto -A^\dagger$) from the identification $\mathfrak{t} = \mathfrak{t}_K \otimes \mathbb{C}$ and another (to be denoted $A \mapsto \overline{A}$) defined via the identification $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \otimes \mathbb{C}$ where $\mathfrak{t}_\mathbb{R} := X_*(T) \otimes_\mathbb{Z} \mathbb{R}$. (Here $X_*(T) := \mathrm{Hom}(\mathbb{C}^*, T)$ is embedded in $\mathfrak{t}$ by differentiation.) One may check that $\mathfrak{t}_K = i\mathfrak{t}_\mathbb{R}$.

Apart from the choices $K, \mathfrak{t}_K, \mathcal{K}$ made so far we need to make three further choices in order to define the monodromy map

$$\nu : \mathfrak{g}^* \longrightarrow G^*. \tag{B.1}$$

These are

(1) a regular element $A_0 \in \mathfrak{t}_{\mathrm{reg}}$. This determines anti-Stokes directions etc. as in Section 2 (taking the pole order $k = 2$),
(2) an initial sector Sect_0 bounded by two consecutive anti-Stokes directions, and
(3) a choice of branch of $\log(z)$ on Sect_0.

The choice of initial sector determines a system of positive roots $\mathcal{R}(d_1) \cup \cdots \cup \mathcal{R}(d_l)$ as in Lemma 2.4 (with $l = \#\mathbb{A}/2$) and thus a Borel subgroup $B_+ \subset G$ containing T. Let B_- be the opposite Borel subgroup and define the dual Poisson Lie group G^* as in (3.7) in terms of $B_\pm$. G^* is a contractible Lie group of the same dimension as G and has a natural Poisson Lie group structure which may be defined directly and geometrically as for $GL_n(\mathbb{C})$ in [8].

The monodromy map (B.1) is then defined as follows. Given $B \in \mathfrak{g}^*$, consider the meromorphic connection on the trivial principal G-bundle over the unit disc Δ determined by the $\mathfrak{g}$-valued meromorphic one-form

$$A^s := \left(\frac{A_0}{z^2} + \frac{B}{z}\right)dz, \tag{B.2}$$

where B is viewed as an element of $\mathfrak{g}$ via $\mathcal{K}$. This connection has Stokes multipliers $(S_+, S_-) = (S_1, S_2) \in U_+ \times U_-$ defined in Definition 2.6, using choices (2) and (3), and so determines an element

$$(b_-, b_+, \Lambda) \in G^* \tag{B.3}$$

via the formulae

$$b_- = e^{-\pi i \Lambda} S_-^{-1}, \quad b_+ = e^{-\pi i \Lambda} S_+ e^{2\pi i \Lambda}, \quad \Lambda = \delta(B) \tag{B.4}$$

so that $b_-^{-1} b_+ = S_- S_+ \exp(2\pi i \Lambda)$. (Here $\delta : \mathfrak{g} \to \mathfrak{t}$ is the projection with kernel $\mathfrak{u}_+ \oplus \mathfrak{u}_-$.) A slightly more direct/elegant definition of ν may be given, without first going through Stokes multipliers, exactly as before in [8, Section 4].

The monodromy map ν is a holomorphic map by Corollary 2.10 and it is easy to prove (as in [8]) it is generically a local analytic isomorphism and any generic symplectic leaf of $\mathfrak{g}^*$ maps into a symplectic leaf of G^*. The approach of [8] extends to yield the following theorem.

Theorem B.1. The monodromy map ν is a Poisson map for each choice of A_0, Sect_0, $\log(z)$, where $\mathfrak{g}^*$ has its standard complex Poisson structure and G^* has its canonical complex Poisson Lie group structure, but scaled by a factor of $2\pi i$. □

Proof. As in [8]; Just replace any expression of the form $\text{Tr}(AB)$ by $\mathcal{K}(A, B)$ and any reference to the *difference of eigenvalues* of any element $A \in \mathfrak{g}$ (which now makes no sense), by the *eigenvalues* of $\text{ad}_A \in \text{End}(\mathfrak{g})$. (The left- and right-invariant Maurer-Cartan forms on G make sense of expressions of the form $H^{-1}H'$ and $H'H^{-1}$ for maps H into G.) The only minor subtlety is in the proof of [8, Lemma 27] where one needs the fact that $\text{Ad}_{(e^Q)}(n_-)$ tends to zero as $z \to 0$ along a certain direction $(-\theta)$ for any fixed $n_- \in \mathfrak{u}_-$. However this follows directly from the third part of Lemma A.3 of the present paper. ■

Remark B.2. Thus locally the monodromy maps give appropriate "canonical" coordinate changes to integrate the explicit non-linear isomonodromy equations (4.3). This indicates just how complicated the monodromy maps are: for $G = SL_3(\mathbb{C})$ equation (4.3) is equivalent to the full family of Painlevé VI equations—generic solutions of which are known to involve "new" transcendental functions.

Now suppose A_0 is purely imaginary ($\overline{A}_0 = -A_0$). Then there are only two anti-Stokes directions; the two halves of the imaginary axis. Take Sect_0 to be the sector containing the positive real axis $\mathbb{R}_+$ and use the branch of $\log(z)$ which is real on $\mathbb{R}_+$. One may then check (as in [8, Lemma 29]) that if $(b_-, b_+, \Lambda) = \nu(B)$ then $\nu(-B^\dagger) = (b_+^{-\dagger}, b_-^{-\dagger}, -\Lambda^\dagger)$ so that ν restricts to a (real analytic) map

$$\nu|_{\mathfrak{k}^*} : \mathfrak{k}^* \longrightarrow K^*, \tag{B.5}$$

where $\mathfrak{k}^* \cong \mathfrak{k}$ via $\mathcal{K}$ and $K^* \subset G^*$ is defined to be the fixed point subgroup of the involution

$$\left(b_-, b_+, \Lambda\right) \longmapsto \left(b_+^{-\dagger}, b_-^{-\dagger}, -\Lambda^\dagger\right). \tag{B.6}$$

The group K^* has a natural (real) Poisson Lie group structure which may be defined as in [8] for $K = U(n)$. All of these restricted monodromy maps are Ginzburg-Weinstein isomorphisms.

Theorem B.3. If A_0 is purely imaginary, then the corresponding monodromy map restricts to a (real) Poisson diffeomorphism $\mathfrak{k}^* \cong K^*$ from the dual of the Lie algebra of K to the dual Poisson Lie group (with its standard Poisson structure, scaled by a factor of π). □

Proof. The proof in [8] goes through once the notational changes given in the previous proof are made. The fact that the unique Hermitian logarithms appearing in the proof of [8, Lemma 31] still exist (Hermitian now meaning $-X^\dagger = -X \in \mathfrak{g}$), follows easily from the fact that G has a faithful representation $\rho : G \hookrightarrow GL_N(\mathbb{C})$ with $\rho(K) = \rho(G) \cap U(N)$ (cf. [12, Lemma 2, page 201]). ■

Remark B.4. (1) The permutation matrices used in [8] have now been banished; Consequently the group G^* now depends on the choice of initial sector (a priori we make no choice of positive roots). The pleasant effect is that ν is now always T-equivariant, with T acting on $\mathfrak{g}^*$ via the coadjoint action and on G^* via the left or right dressing action. (The left and right dressing actions agree when restricted to T.) In turn the Ginzburg-Weinstein isomorphisms constructed above are all T_K-equivariant.

(2) The new proof of the theorem of Duistermaat given in [8, Section 6] for $GL_n(\mathbb{C})$ also extends immediately to connected complex reductive G.

Acknowledgments

I would like to thank B. Dubrovin and M. S. Narasimhan for many useful conversations and B. Kostant for kindly supplying a proof of the fact that E_8 has no nontrivial multiplicity one representations. The work for this paper was supported by grants from Scuola Internazionale Superiore di Studi Avanzati (SISSA), Trieste and the EDGE Research Training Network HPRN-CT-2000-00101 of the European Human Potential Programme.

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