Fission varieties

P. Boalch (ENS Paris)

Complex character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

Riemann surface   Poisson variety

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \sqcup \dots \sqcup \partial_m$ $\quad (\partial_i \cong S^1)$

Choose basepoints $b_i \in \partial_i$

Let $\Pi = \pi_1(\Sigma, \{b_1, \dots, b_m\})$

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& symplectic leaves are $\mu^{-1}(\mathcal{C})/G^m$ $\quad (\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_m) \subset G^m)$

Further if $\Sigma \longrightarrow \mathbb{B}$ is a family of Riemann surfaces $\Sigma_p \quad p \in \mathbb{B}$

get algebraic Poisson action

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"The Betti moduli spaces $M_B(\Sigma_p, G) = \mathrm{Hom}(\pi_1(\Sigma_p), G)/G$ form a local system of varieties" (Simpson)

If Σ a punctured smooth complex algebraic curve

($\&$ $G = GL_n(\mathbb{C})$)

Deligne's Riemann-Hilbert correspondence $\Rightarrow$

The G orbits in $\mathrm{Hom}(\pi_1(\Sigma), G)$ correspond bijectively to isomorphism classes of regular singular connections on rank n algebraic vector bundles $V \longrightarrow \Sigma$

- want to extend previous story to irregular case

Fix $T \subset G$, Lie algebras $\mathfrak{t} \subset \mathfrak{g}$

Def^n Δ complex disc, $a \in \Delta$

An "irregular type" at a is an element

$$Q \in \mathfrak{t}(\hat{\mathcal{K}}) / \mathfrak{t}(\hat{\mathcal{O}})$$

[if z local coord vanishing at a, $\hat{\mathcal{K}} = \mathbb{C}((z))$, $\hat{\mathcal{O}} = \mathbb{C}[\![z]\!]$

so $Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$ for some $A_i \in \mathfrak{t}$]

Def^n An "irregular curve" Σ is a smooth compact cx algebraic curve, with distinct marked points $a_1, \ldots, a_m \in \Sigma$ and an irregular type Q_i at each marked point

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Aim: define (Poisson) Betti moduli spaces $\mathcal{M}_B(\Sigma, G)$ of irreg. curves Σ & show they form local system of Poisson varieties if Σ undergoes an admissible deformation.

Irregular Betti spaces

Irreg RH on curves worked out decades ago for $G = GL_n(\mathbb{C})$

(Balser Jurkat Lutz Malgrange Sibuya Deligne Martinet Ramis ...)

- will give explicit as possible approach using groupoids (for any reductive G)

Irregular Betti spaces

Let Σ be an irreg. curve (marked points $a_1, \dots, a_m$, irreg. types $Q_1, \dots, Q_m$)

Let $\hat{\Sigma} \longrightarrow \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial \hat{\Sigma} = \partial_1 \sqcup \dots \sqcup \partial_m$)

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Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $\mathbb{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathbb{A}_i$

3) A unipotent group $\mathbb{S}\mathrm{to}_d(Q_i) \subset G$ normalised by H_i

1) H_i = stabilizer of Q_i under adjoint action

$$\left(H_i = \{ g \in G \mid \mathrm{Ad}_g(A_i) = A_i \ \ \forall i \} \right)$$

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so $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [Y, X] = \alpha(Y) X \ \forall Y \in \mathfrak{t} \}$

Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

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then $d \in \partial$ is a singular direction supported by $\alpha \in R$

if $\exp(q_\alpha)$ has maximal decay as $z \to a$ along d

(leading term of q_α is real and negative along d)

& $\mathbb{A} \subset \partial$ is set of all sing. directions $(\forall \alpha \in R)$

3) Let $R(d) = \{ \alpha \mid \alpha \text{ supports } d \} \subset R$

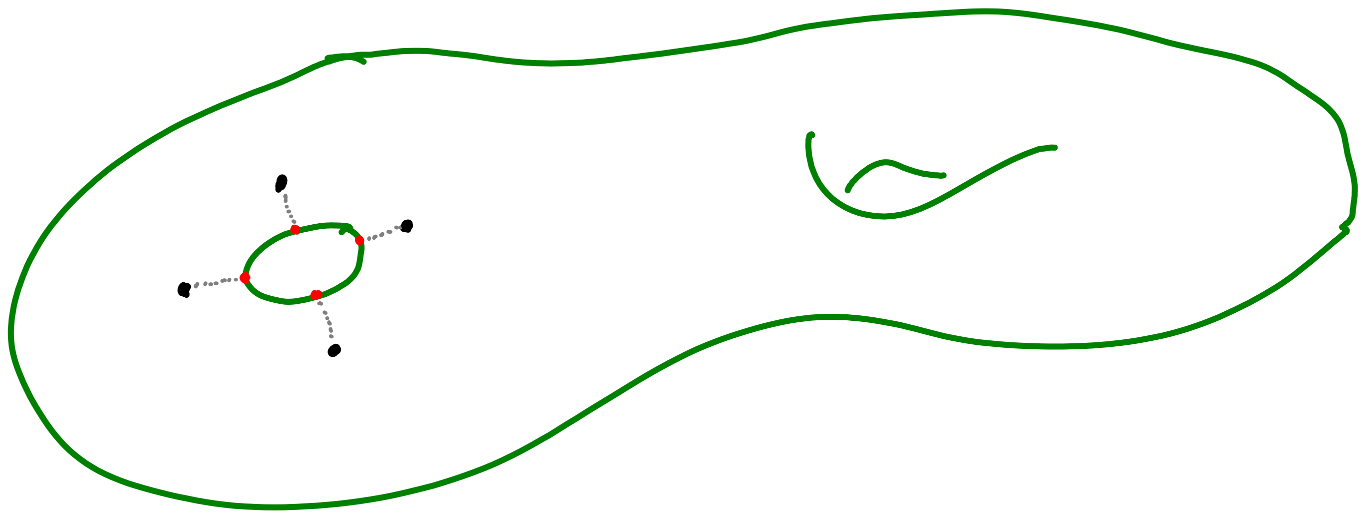
$$Sto_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma Sto_d is a well defined unipotent subgroup of G

Now puncture $\hat{\Sigma}$ once in its interior near each singular direction $d \in \mathbb{A}_i$, $i=1,\dots,m$

and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface

$\tilde{\Sigma}$



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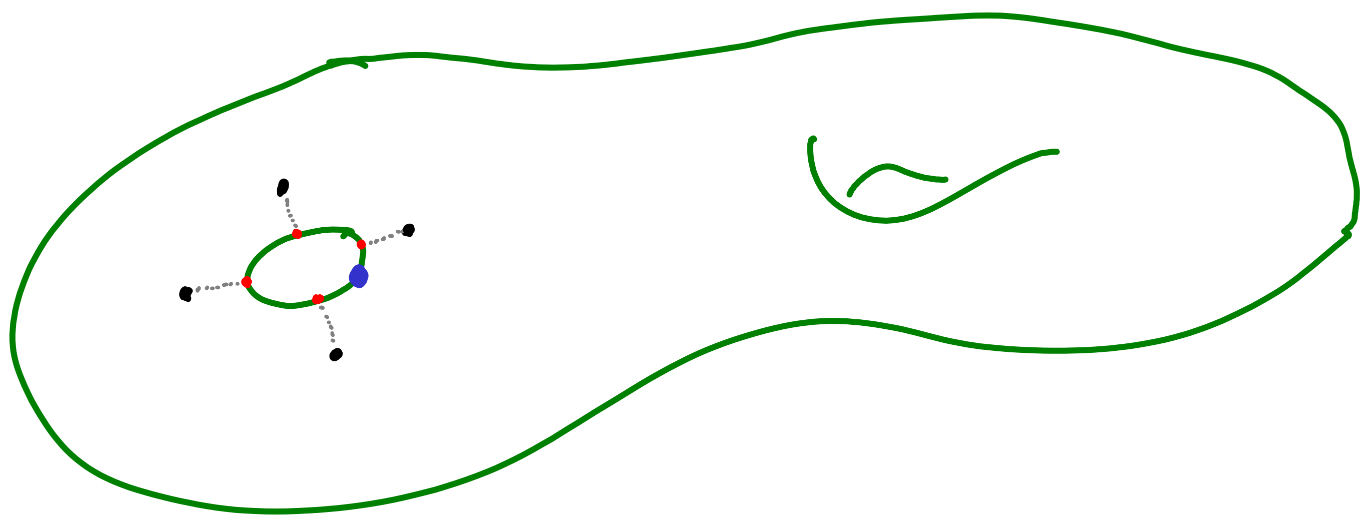

Choose a base point $b_i \in \partial_i$ in each boundary circle

Let $\Pi = \Pi_1(\tilde{\Sigma}, \{b_1, \dots, b_m\})$

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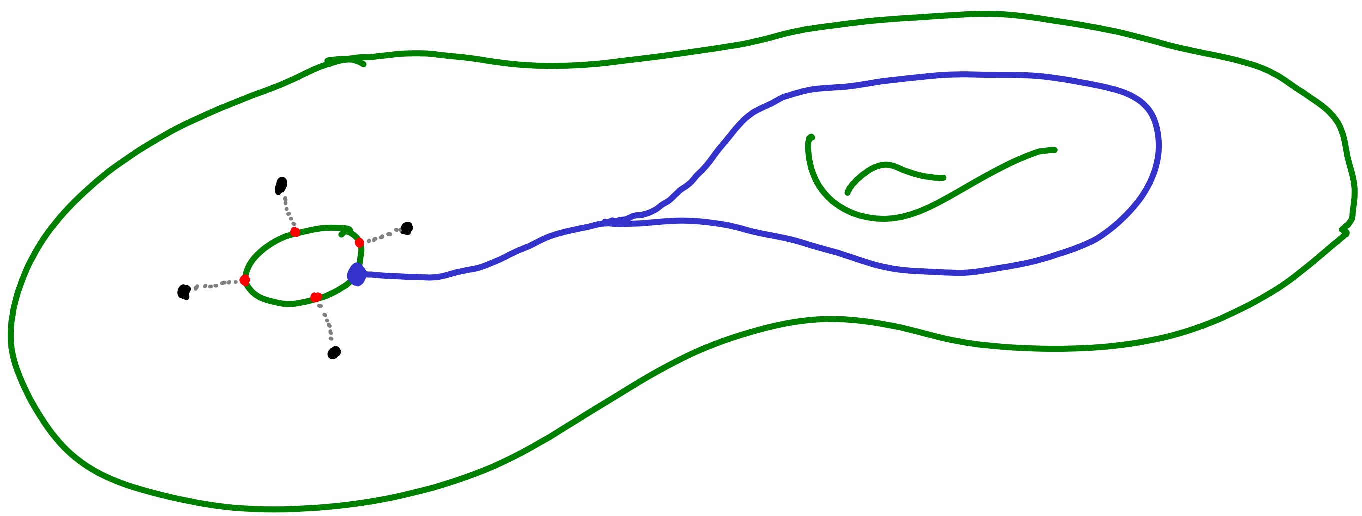

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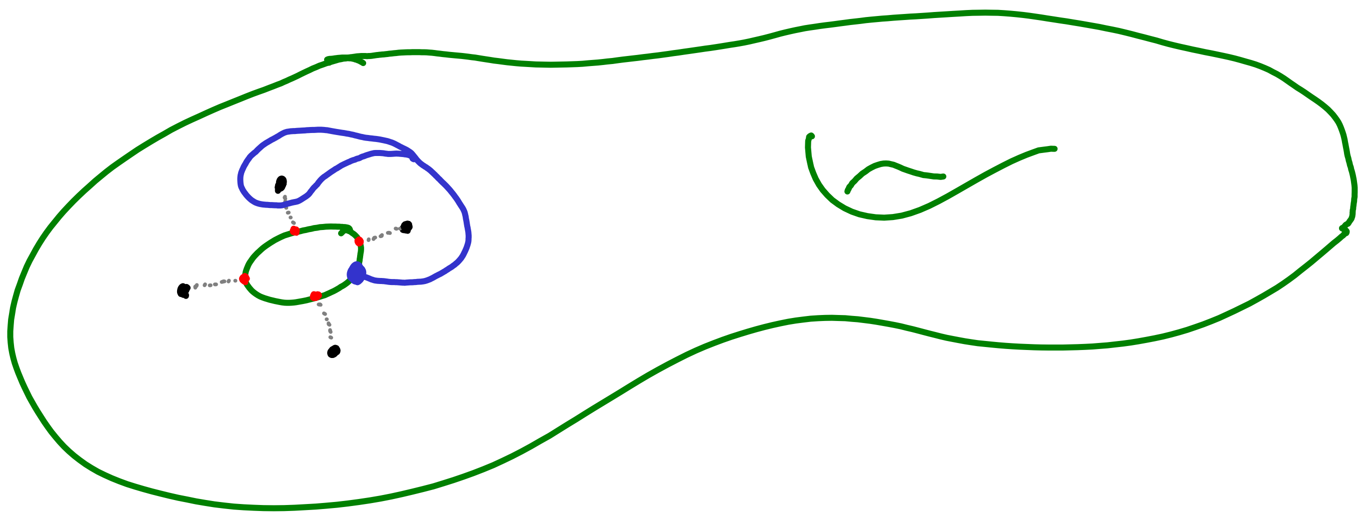

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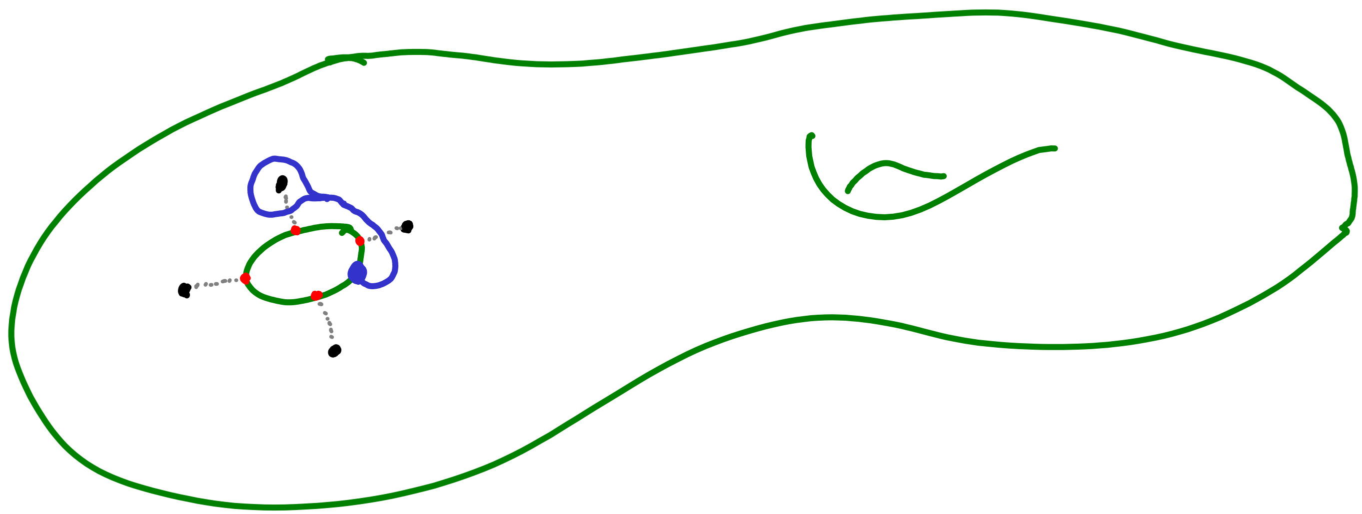

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Now consider $\mathrm{Hom}(\Pi, G)$

and the subset $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of "Stokes representations"

$$\mathrm{Hom}(\Pi, G) \supset \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$$

satisfying:

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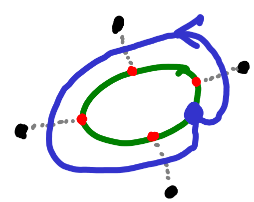
$\cup$

and the subset $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ of "Stokes representations" satisfying:

1) If $\gamma = \partial_i$ then $\rho(\gamma) \in H_i$



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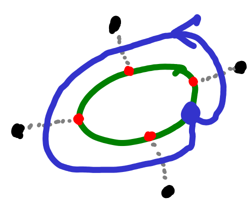
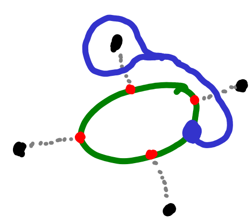


2) If γ goes around ∂_i from b_i until $d \in \mathbb{A}_i$ then loops around the corresponding puncture before returning to b_i, then $\rho(\gamma) \in \mathbb{S}\mathrm{to}_d$



Thm

The space of Stokes representations $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\underset{\sim}{H}$-space $(\underset{\sim}{H} = H_1 \times \cdots \times H_m)$

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Corollary $\mathcal{M}_B(\Sigma, G) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$

inherits an intrinsic Poisson structure (algebraically) with symplectic leaves $\mu^{-1}(e)/\underset{\sim}{H}$ for $e = (e_1, \dots, e_m) \subset \underset{\sim}{H}$

[$\mathcal{M}_B$ classifies irreg. connections with the given irreg. types

$A \cong dQ$ + less singular terms, locally]

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$$

Irregular curve

Poisson variety

Def^n

A family $\Sigma \to \mathbb{B}$ of irregular curves $(\Sigma_p, a_i, Q_i)_{p \in \mathbb{B}}$ is "admissible" if

1) The fibres Σ_p remain smooth

2) None of the marked points a_i coalesce

3) For each root $\alpha \in R$

$$\mathrm{PoleOrder}(\alpha \circ Q_i) \in \mathbb{Z}_{\geq 0}$$

is a constant function on $\mathbb{B}$

Thm

If $\Sigma \longrightarrow \mathbb{B}$ is an admissible family of irregular curves

$\Sigma_p = \pi^{-1}(p)$, $p \in \mathbb{B}$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowright \mathrm{Hom}_{\mathbb{S}}(\Pi(p), G)/\underset{\sim}{H}$$

"The Betti moduli spaces $M_B(\Sigma_p, G)$ form a local system of (Poisson) varieties"

Definition A *holomorphic quasi-Hamiltonian G-space* is a complex G-manifold M with a G-invariant two form ω and a G-equivariant map $\mu: M \longrightarrow G$ (G acts on G by conjugation) such that

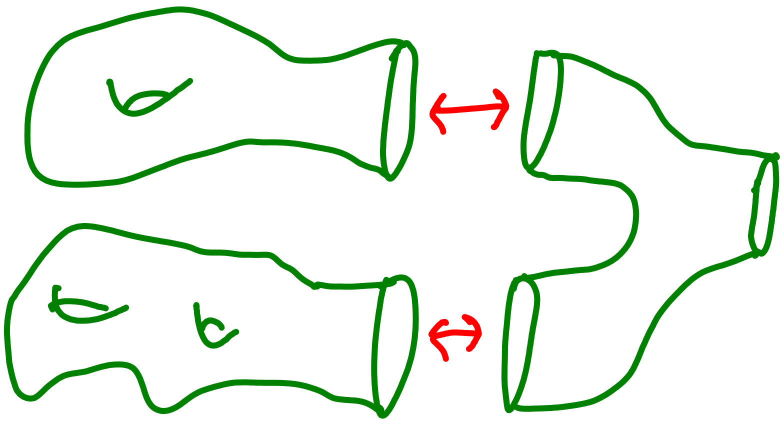
1. $d\omega = \mu^*(\eta)$
2. $\forall X \in \mathfrak{g} \quad \omega(v_X, \cdot) = \frac{1}{2}\mu^*(\theta + \bar{\theta}, X)$
3. $\forall m \in M \quad \ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$

where η = biinvariant 3-form on G, $\theta, \bar{\theta}$ Maurer-Cartan forms on G

- These axioms are 'what we get from ∞-d viewpoint'
- Multiplicative analogue of Hamiltonian G-space (with $\mathfrak{g}^*$-valued moment map)

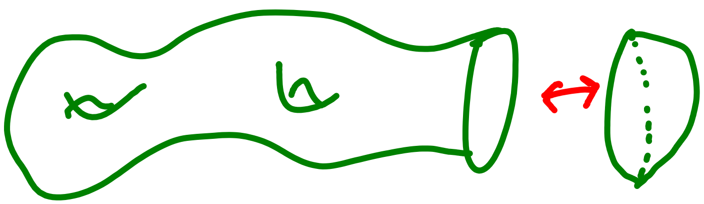
Operations

1. Can 'fuse' 2 q Hamiltonian G-spaces:



$\Rightarrow$ New qH G-space $M_1 \circledast M_2$

2. & reduce:



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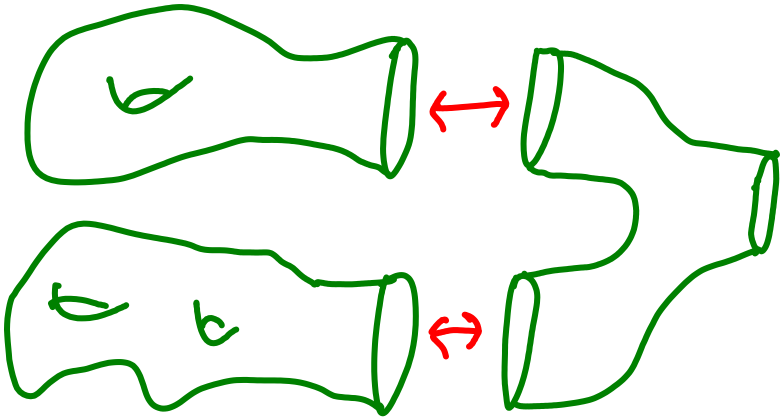

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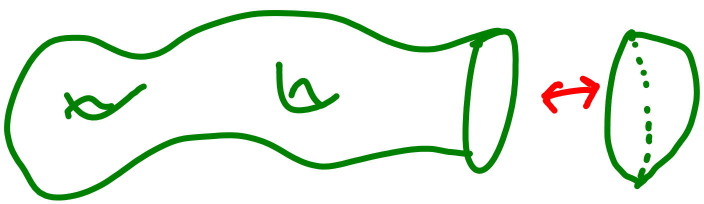

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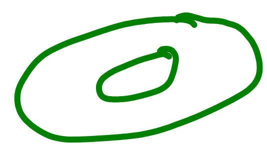
Basic examples ① Conjugacy classes $\mathcal{C} \subset G$

② $D = G \times G$ qH $G \times G$ space (double)

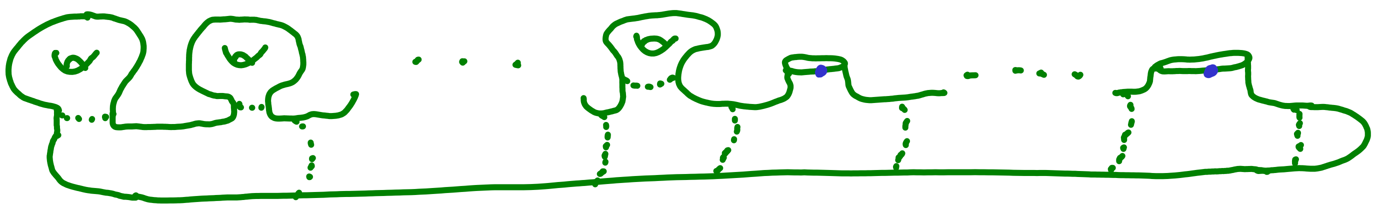


Annulus

③ $\mathbb{D} = G \times G$ qH G-space (internally fused double)

Can construct all moduli spaces of holomorphic connections on Riemann surfaces from these pieces:

$$\underbrace{\mathbb{D} \circledast \cdots \circledast \mathbb{D}}_{g} \circledast \underbrace{D \circledast \cdots \circledast D}_{m} /\!\!/ G \cong \mathrm{Hom}(\pi, G)$$



$$\mu^{-1}(\mathcal{C})/G^m \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1 \,, \quad M_i \in \mathcal{C}_i \right\} \Big/ G$$

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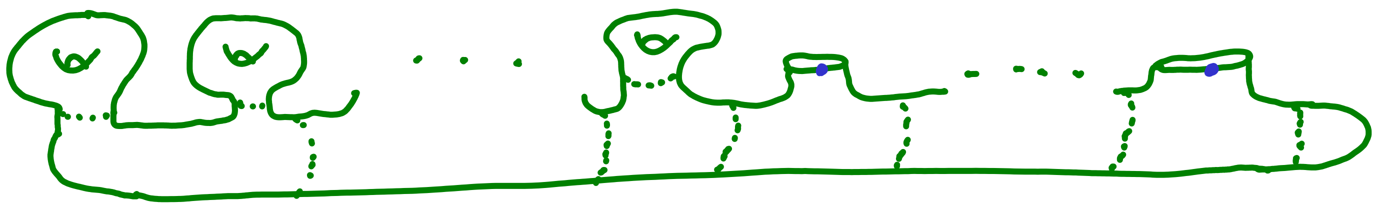

$$\mu^{-1}(e)/G^m \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{M}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} M_i = 1, \; M_i \in \mathcal{C}_i \right\} / G$$

Aim: New pieces to construct irregular Betti spaces?

(have "irreg. Atiyah-Bott" from 1999)

Fission spaces

Choose $P_\pm \subset G$ opposite parabolics

$H = P_+ \cap P_-$ Levi subgroup

$U_\pm \subset P_\pm$ unipotent radicals

Thm (– '02, '09, '11)

The "fission space" ${}_G\mathcal{A}^r_H := G \times (U_+ \times U_-)^r \times H$

is a quasi-Hamiltonian $G \times H$ space

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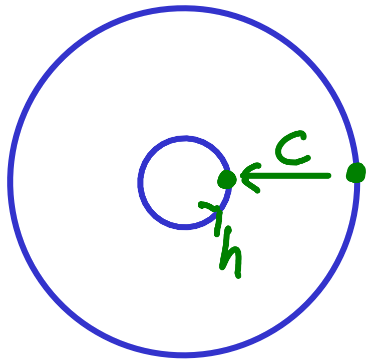
- moment map $\mu(C, S_1, \ldots, S_{2r}, h) = (C^{-1} h S_{2r} \cdots S_1 C, h^{-1})$
- $(U_+ \times U_-)^r \cong$ Stokes data of connections with $Q = \dfrac{A}{z^r}$, $C_G(A) = H$

Picture

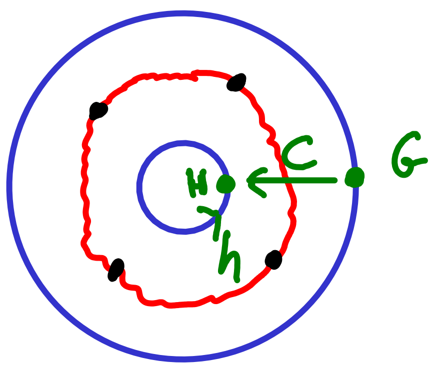
If $P_\pm = G = H$ $\qquad {}_G\mathcal{A}_H = G \times G$ is the double

$$\begin{array}{c} G \times G \\ \rotatebox{90}{$\in$} \\ (c, h) \end{array}$$



$$\mu = (c^{-1} h c, h^{-1})$$

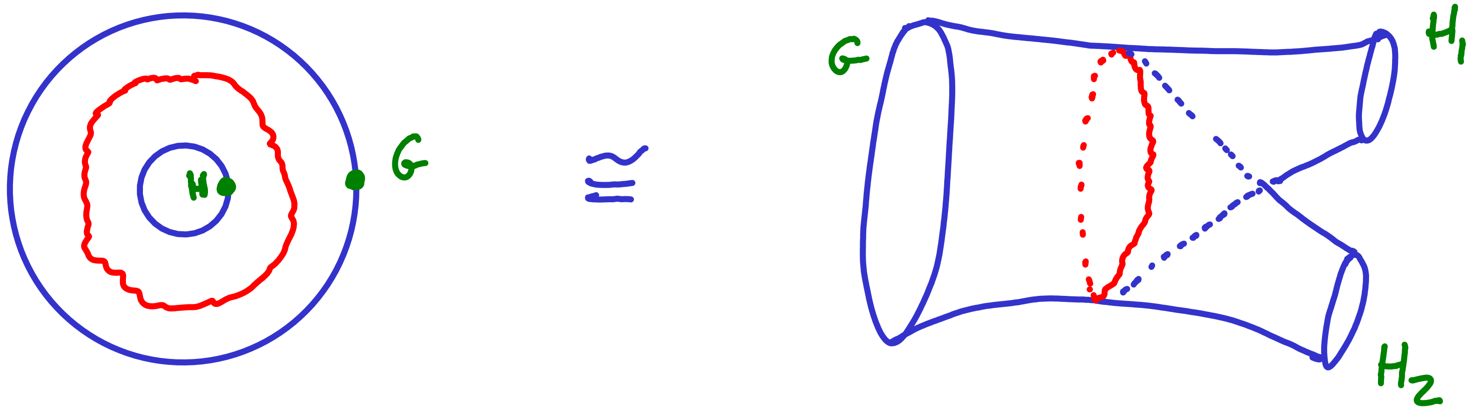
General case can be pictured similarly (breaking group from G to H)



$$\mu = (c^{-1} h s_{2r} \cdots s_1 c, h^{-1})$$

Typically H is a product e.g. $H = H_1 \times H_2$

- can glue on both a qH H_1-space & a qH H_2-space



"fission" operation ($\neq$ fusion)

Definition

A "fission variety" is a symplectic or quasi-Hamiltonian variety obtained via the fusion & reduction operations on spaces of the form

1) Conjugacy classes $\mathcal{C} \subset G$ in arbitrary complex reductive groups
2) Fission spaces ${}_G\mathcal{A}^r_H$

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Thm

If Σ is an irregular curve then $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ is a fission variety

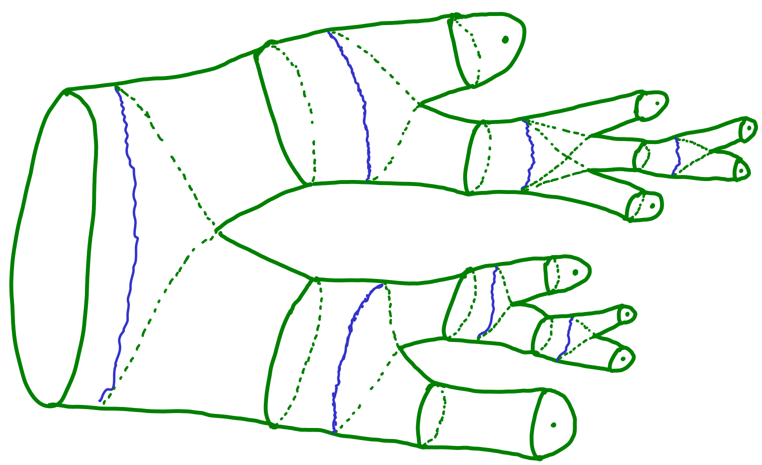
If $Q = \dfrac{A_r}{z^r} + \cdots + \dfrac{A_1}{z}$

Define $G = H_r \supset H_{r-1} \supset \cdots \supset H_0 = H \supset T$

via $H_{i-1} = C_{H_i}(A_i)$

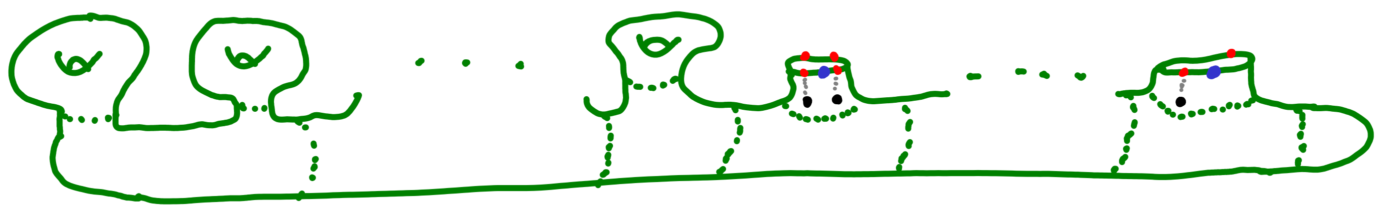
Then $\mathcal{A}(Q) := G \times \{\text{Stokes data for } Q\} \times H$ obtained by gluing

$$\mathcal{A}(Q) \cong {}_G\mathcal{A}^r_{H_{r-1}} \circledast {}_{H_{r-1}}\mathcal{A}_{H_{r-2}} \circledast \cdots \circledast {}_{H_1}\mathcal{A}_H$$



If Σ an irregular curve :

$$\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \cong \underbrace{\mathbb{D} \circledast \cdots\cdots \circledast \mathbb{D}}_{g} \circledast \mathcal{A}(Q_1) \circledast \cdots \circledast \mathcal{A}(Q_m) /\!\!/ G$$



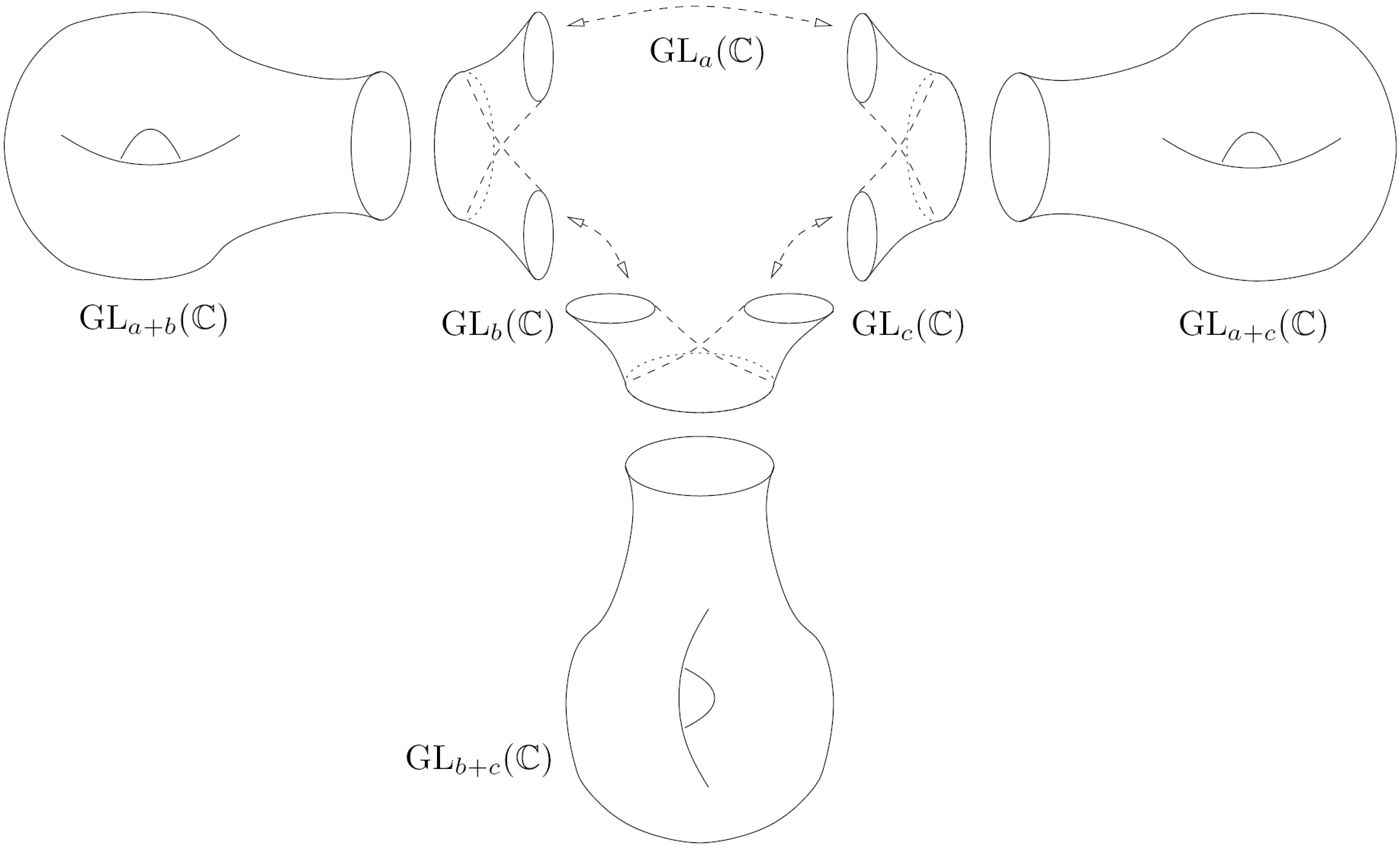
$$\mu^{-1}(e) \big/ \underset{\sim}{H} \cong \left\{ (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{h}, \underset{\sim}{S}) \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] \prod_{i=1}^{m} \mu_i = 1,\ h_i \in \mathcal{C}_i \right\} \big/ \underset{\sim}{H}$$

$$\mu_i = C_i^{-1} h_i \cdots S_2^{(i)} S_1^{(i)} C_i$$

But there are many other examples of fission varieties

- e.g. can glue surfaces $\widetilde{\Sigma}$ along their boundaries (provided the groups H_i match up)

- can obtain all the so-called multiplicative quiver varieties



$\mathrm{GL}_a(\mathbb{C})$
$\mathrm{GL}_{a+b}(\mathbb{C})$
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$\mathrm{GL}_c(\mathbb{C})$
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Irregular Deligne-Simpson problem

Given irreg. curve Σ (& basepoints $\{b_i\}$) get

reductive group $\underset{\sim}{H}$ acting on smooth affine variety $M = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

Irregular Deligne-Simpson problem

Given irreg. curve Σ (& basepoints $\{b_i\}$) get

reductive group $\underset{\sim}{H}$ acting on smooth affine variety $M = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

In geometric invariant theory one then defines $\rho \in M$ to be "stable"

if its orbit $\underset{\sim}{H} \cdot \rho$ is closed and of maximal possible dimension.

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Question:

Given $e = (e_1, \ldots, e_m) \subset \underset{\sim}{H}$ is there a stable point $\rho \in \mu^{-1}(e)$?

Say $\rho \in M$ is "reducible" if there exists proper parabolics $P_1, \dots, P_m \subset G$ such that

1) $\rho(\gamma) P_i \rho(\gamma)^{-1} = P_j$ for all paths γ from b_i to b_j

2) $Z_i \subset P_i$ where $Z_i = Z(H_i)^0 \subset G$ (so $H_i = C_G(Z_i)$)

otherwise it is "irreducible".

Thm (if M nontrivial) ρ is stable $\iff$ ρ is irreducible

Example iDS

$$\left(G = GL(V),\ \Sigma = (\mathbb{P}^1, 0, Q),\ Q = \mathbb{A}/\mathbb{Z}^2 \right)$$

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$$V = V_1 \oplus \cdots \oplus V_k$$

$$H = \prod GL(V_i), \quad U_+ = \begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & * \\ & & & 1 \end{pmatrix}, \quad U_- = \begin{pmatrix} 1 & & & 0 \\ * & \ddots & & \\ \vdots & & \ddots & \\ * & \cdots & * & 1 \end{pmatrix}$$

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Let $\mathcal{B}(V) = A(Q)/\!/G = {}_G A_H /\!/G \cong \mathrm{Hom}_{\mathcal{S}}(\Pi, G)$

$$\cong \{ (h, \underset{+}{s_1}, \underset{-}{s_2}, \underset{+}{s_3}, \underset{-}{s_4}) \mid h s_4 s_3 s_2 s_1 = 1 \}$$

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- quasi Hamiltonian H-space $(\mu = h^{-1})$

Given $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_k) \subset H$ is there an irreducible point of $\mathcal{B}(V)$ s.t $h \in \mathcal{C}$?

$$\mathcal{B}(V) \cong \left\{ (S_1, S_2) \in U_+ \times U_- \;\middle|\; S_2 S_1 \in G^0 := U_+ H U_- \right\}$$

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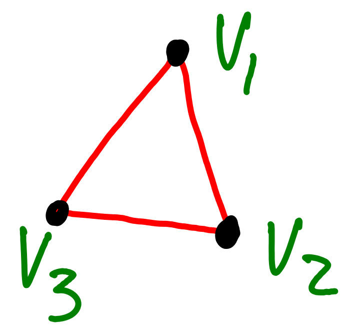
Γ = complete graph with k nodes

e.g. $k=3$ $\quad \Gamma =$



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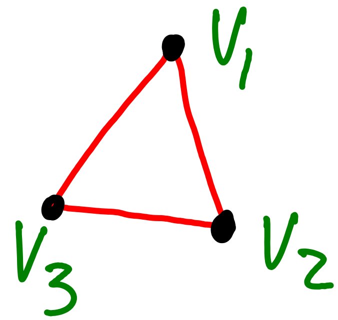

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Lemma Stokes rep. irred. iff corresponding graph rep. irred.

$iDS(\Gamma)$

Γ complete graph with k-nodes

$$\mu : \operatorname{Rep}^{\circ}(\Gamma, V) \longrightarrow H$$

For which conjugacy classes $\mathcal{C} \subset H$ is there an irreducible point in $\mu^{-1}(\mathcal{C})$?

iDS(Γ)

Γ complete graph with k-nodes

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If $Q = A/z^2 + B/z$ has second term get all complete k-partite graphs Γ

& then iDS(Γ) $\cong$ usual Deligne-Simpson problem if Γ star-shaped