Wild character varieties
and
multiplicative quiver varieties

P. Boalch

Complex character varieties

(G = connected complex reductive gp)
e.g $G = GL_n(\mathbb{C})$

$$\Sigma \longmapsto \mathrm{Hom}(\pi_1(\Sigma), G) / G$$

Riemann surface $\qquad\qquad$ Poisson variety

Atiyah-Bott, Goldman, Karshon, Fock-Rosly, Weinstein,
Guruprasad-Huebschmann-Jeffrey-Weinstein, Andersen-Mattes-Reshetikhin ...

generic symplectic leaves are hyperkähler manifolds (Hitchin)

Further if $\Sigma \longrightarrow \mathbb{B}$ is a family of Riemann surfaces Σ_p $p \in \mathbb{B}$

get algebraic Poisson action

$$\pi_1(\mathbb{B}, p) \circlearrowleft \mathrm{Hom}(\pi_1(\Sigma_p), G)/G$$

$\sim$ Mapping class group of Σ_p acts on character variety

"The Betti moduli spaces $M_B(\Sigma_p, G) = \mathrm{Hom}(\pi_1(\Sigma_p), G)/G$ form a local system of varieties" (Simpson)

(Character varieties as finite dimensional multiplicative symplectic quotients)

Quasi-Hamiltonian approach

Say $\partial\Sigma = \partial_1 \sqcup \dots \sqcup \partial_m$ $\quad (\partial_i \cong S^1)$

Choose basepoints $b_i \in \partial_i$

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& symplectic leaves are $\mu^{-1}(\mathcal{C})/G^m$ $\quad (\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_m) \subset G^m)$

If Σ a smooth complex algebraic curve and $G = GL_n(\mathbb{C})$

Deligne's (1970) Riemann-Hilbert correspondence $\Rightarrow$

$$\left\{ (V, \nabla) \;\middle|\; \begin{array}{l} V \to \Sigma \text{ rank } n, \text{ algebraic vector bundle} \\ \nabla \text{ a connection on } V \text{ with regular singularities} \end{array} \right\} \Big/ \text{isomorphism}$$

$\cong$ G^m orbits in $\mathrm{Hom}(\Pi, G)$

$\cong$ G orbits in $\mathrm{Hom}(\pi_1(\Sigma), G)$

– will extend previous story to irregular case

Summary of Main steps / Key ideas

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$\mathcal{M}_{DR}(\Sigma)$ $\mathcal{M}_B(\Sigma)$

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③ Define space $\mathbb{M}(\Sigma)$ of "admissible deformations" of Σ
(generalising moduli of curve with marked points)

Theorem (1999, ..., 2011)

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Let $\Sigma \to \mathbb{B} \longrightarrow \mathbb{M}(\Sigma)$ be an admissible family of irregular curves (Σ_b for $b \in \mathbb{B}$)

2. The spaces $\mathcal{M}_B(\Sigma_b)$ form a local system of Poisson varieties over $\mathbb{B}$, and so have an algebraic Poisson action

$$\pi_1(\mathbb{B}) \circlearrowright \mathcal{M}_B(\Sigma_b)$$

(wild mapping class group action)

Remarks

① Simplest example (Σ type 2+1) gives Poisson variety G^* underlying Drinfeld-Jimbo quantum group $U_q\mathfrak{g}$

– then wild mapping class gp is G-braid gp $\Rightarrow$ quantum Weyl gp action on G^*

$$\begin{array}{ccccc} & & U_q\mathfrak{g} & & \\ & \swarrow & & \searrow & \\ \mathrm{Fun}(G^*) & & & & U\mathfrak{g} \\ & \searrow & & \swarrow & \\ & & \mathrm{Fun}(\mathfrak{g}^*) = S\mathfrak{g} & & \end{array}$$

② Work with Biquard ('04) shows such spaces of irregular connections $\mathcal{M}_{DR}$ (with compatible parabolic structures and stability conditions) are (complete) hyperkähler manifolds and $\cong$ Higgs bundle moduli spaces (generalising Hitchin's approach in the case of holomorphic connections)

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H} = \mathcal{M}_B(\Sigma)$$

Irregular curve

Poisson variety

Fix $T \subset G$, Lie algebras $\mathfrak{t} \subset \mathfrak{g}$

Def^n Δ complex disc, $a \in \Delta$

An "irregular type" at a is an element

$$Q \in \mathfrak{t}(\hat{\mathcal{K}}) / \mathfrak{t}(\hat{\mathcal{O}})$$

[if z local coord vanishing at a, $\hat{\mathcal{K}} = \mathbb{C}((z))$, $\hat{\mathcal{O}} = \mathbb{C}[\![z]\!]$

so $$Q = \frac{A_r}{z^r} + \cdots + \frac{A_1}{z} \quad \text{for some } A_i \in \mathfrak{t}$$]

(no restrictions on A_r)

Def^n An "irregular curve" Σ is a smooth compact cx algebraic curve, with distinct marked points $a_1, \dots, a_m \in \Sigma$ and an irregular type Q_i at each marked point

$$\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$$

$$\underset{\sim}{a} = (a_1, \dots, a_m), \qquad \underset{\sim}{Q} = (Q_1, \dots, Q_m)$$

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Given $\boldsymbol{\Sigma} = (\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$, let $\Sigma^\circ = \Sigma \setminus \{a_1, \dots, a_m\}$

Def^n "connection on $\boldsymbol{\Sigma}^\circ$": (P, A) where

$$\begin{cases} P \to \Sigma^\circ & \text{algebraic } G\text{-bundle} \\ A & \text{connection on } P \text{ such that} \end{cases}$$

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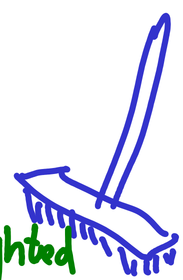

weighted version

$$A \cong dQ_i + (\text{logarithmic terms}) \quad \text{near } a_i \quad \forall i$$

Irregular Riemann Hilbert correspondence

Building on: Birkhoff, Jurkat, Sibuya, Deligne, Malgrange, Balser, Lutz, Babbitt, Varadarajan, Martinet, Ramis, Loday-Richaud...

Σ irreg. curve

$$\boxed{\text{Category of connections on } \Sigma^\circ \quad \cong \quad \text{Stokes } G\text{-local systems for } \Sigma}$$

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$$\Longrightarrow \left\{\text{Connections on } \Sigma^{\circ}\right\}/\text{isom.} \cong \left\{ \text{---} \;\;'' \;\; \text{---} \right\}/\text{isom.}$$

Stokes local systems (see arXiv 1111.6228 for details)

Let Σ be an irreg. curve (marked points $a_1, \dots, a_m$, irreg. types $Q_1, \dots, Q_m$)

Let $\hat{\Sigma} \longrightarrow \Sigma$ be real oriented blow up of Σ at a_i

(each a_i replaced by a circle ∂_i, so $\partial\hat{\Sigma} = \partial_1 \sqcup \dots \sqcup \partial_m$)

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Then each Q_i determines:

1) A connected complex reductive group $H_i \subset G$

2) A finite set $\mathbb{A}_i \subset \partial_i$ of singular directions at a_i

and for each $d \in \mathbb{A}_i$

3) A unipotent group $\mathbb{S}to_d(Q_i) \subset G$ normalised by H_i

1) H_i = stabilizer of Q_i under adjoint action

$\left(H_i = \{ g \in G \mid \mathrm{Ad}_g(A_i) = A_i \ \ \forall i \} \right)$

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2) Let $R \subset \mathfrak{t}^*$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$

so $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid [Y, X] = \alpha(Y) X \ \forall Y \in \mathfrak{t} \}$

Let $q_\alpha = \alpha \circ Q$ (mero. function near $a \in \Sigma$)

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then $d \in \partial$ is a singular direction supported by $\alpha \in R$

if $\exp(q_\alpha)$ has maximal decay as $z \to a$ along d

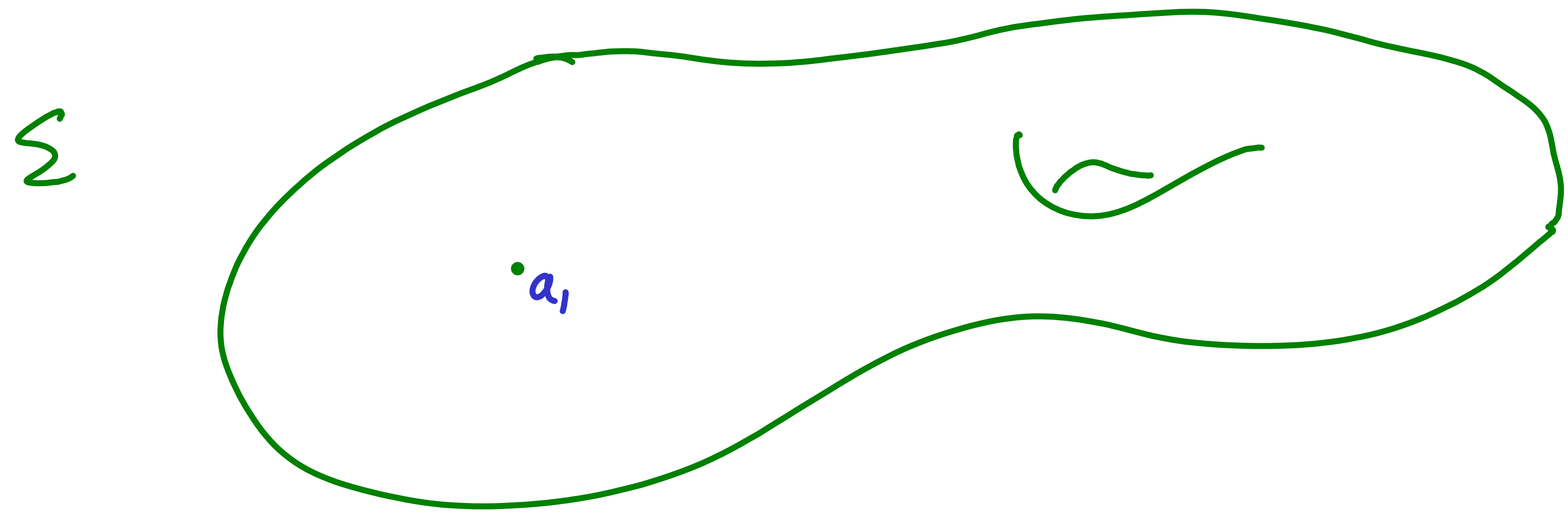
(leading term of q_α is real and negative along d)

& $\mathbb{A} \subset \partial$ is set of all sing. directions ($\forall \alpha \in R$)

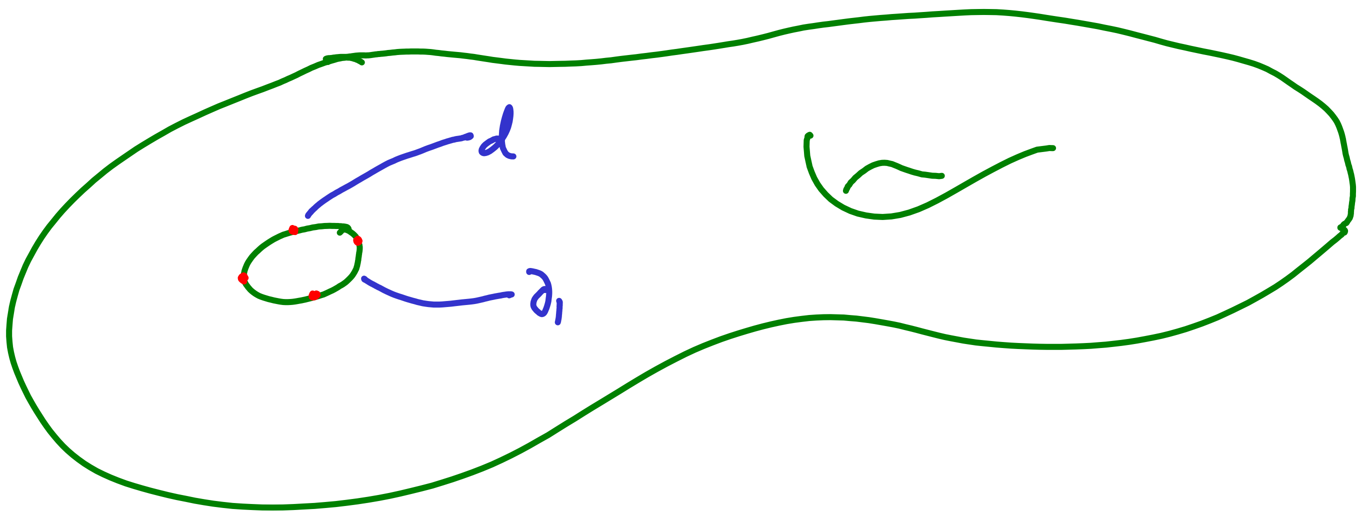
3) Let $R(d) = \{\alpha \mid \alpha \text{ supports } d\} \subset R$

$$\mathrm{Sto}_d = \prod_{\alpha \in R(d)} \exp(\mathfrak{g}_\alpha) \hookrightarrow G$$

Lemma Sto_d is a well defined unipotent subgroup of G

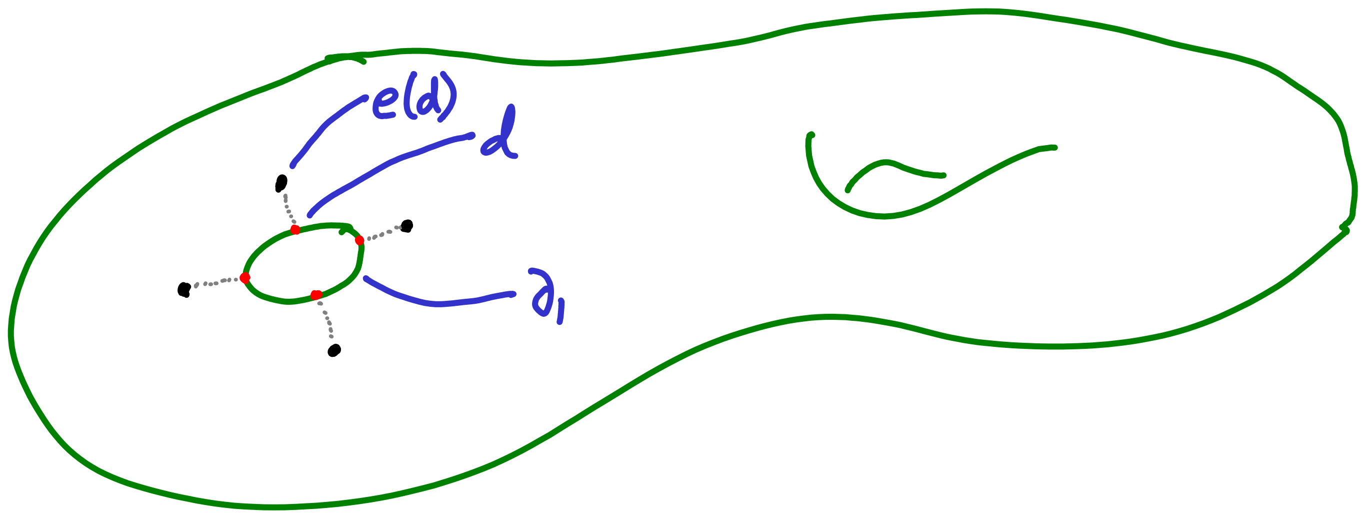

Σ
a_1

$\hat{\Sigma}$



Now puncture $\hat{\Sigma}$ at a point $e(d)$ near each singular direction $d \in \mathbb{A}_i$, $i=1,\dots,m$
and let $\tilde{\Sigma} \subset \hat{\Sigma}$ be resulting punctured surface:

$\tilde{\Sigma}$



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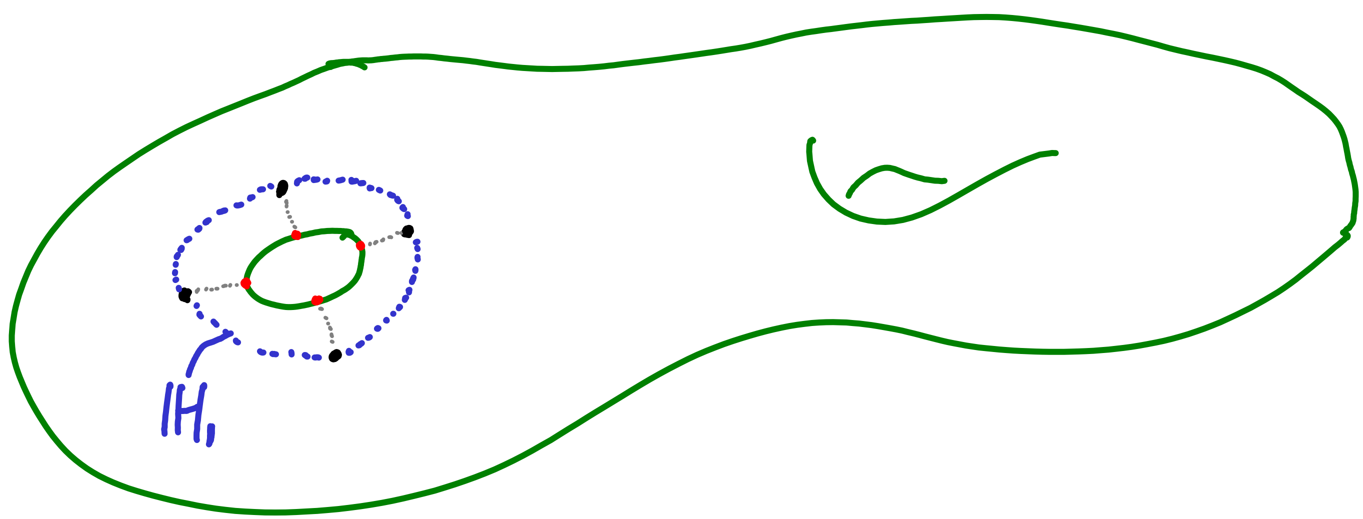

Draw "halo" $\mathbb{H}_i$ $(i=1,\dots,m)$

Def^{n} A Stokes G-local system for Σ is a G local system on $\tilde{\Sigma}$ with a flat reduction to H_i in $\mathbb{H}_i$ $(\forall i)$ such that:

monodromy around $e(d)$ (based in $\mathbb{H}_i$) is in $\mathbb{S}\mathrm{to}_d$ $\forall\, d \in \mathbb{A}_i$

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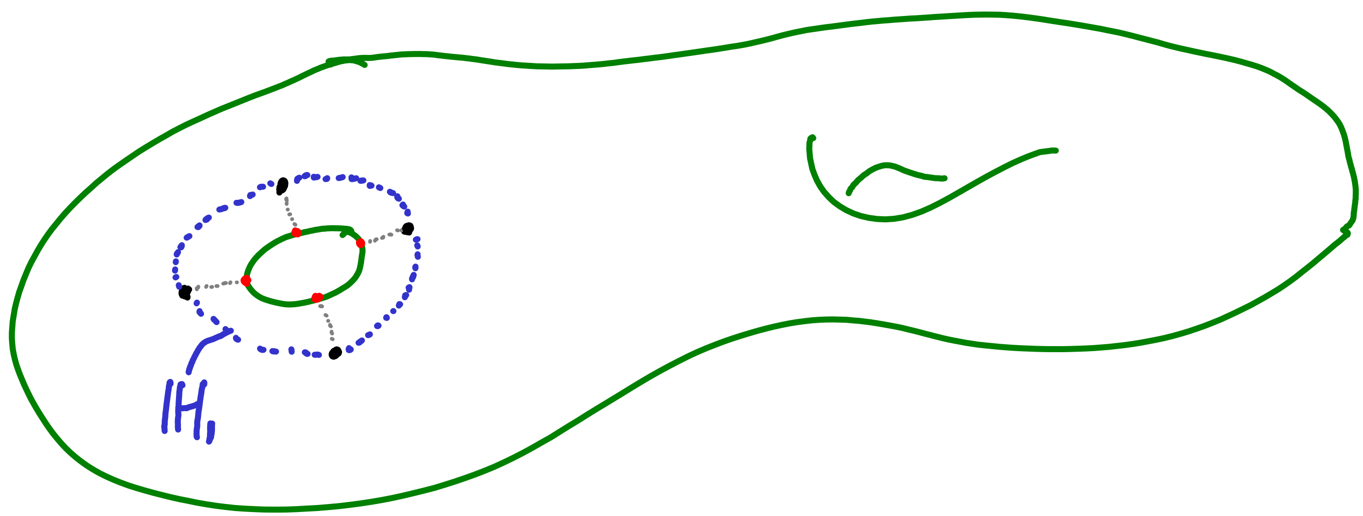

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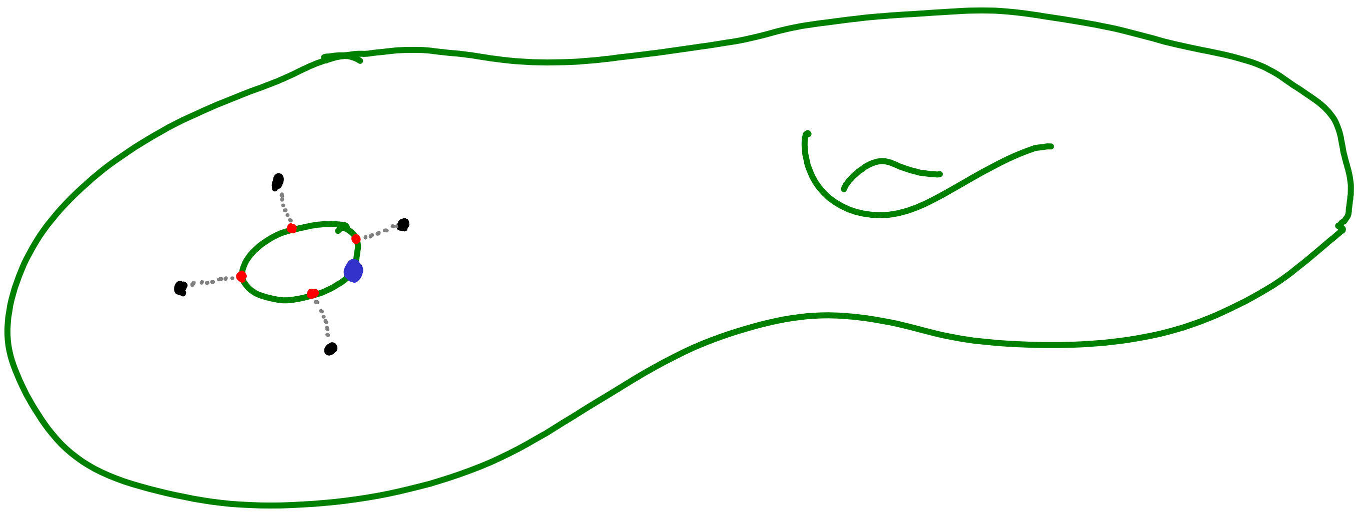
Choose a base point $b_i \in \partial_i$ in each boundary circle

Let $\Pi = \Pi_1(\widetilde{\Sigma}, \{b_1, \dots, b_m\})$

$\widetilde{\Sigma}$



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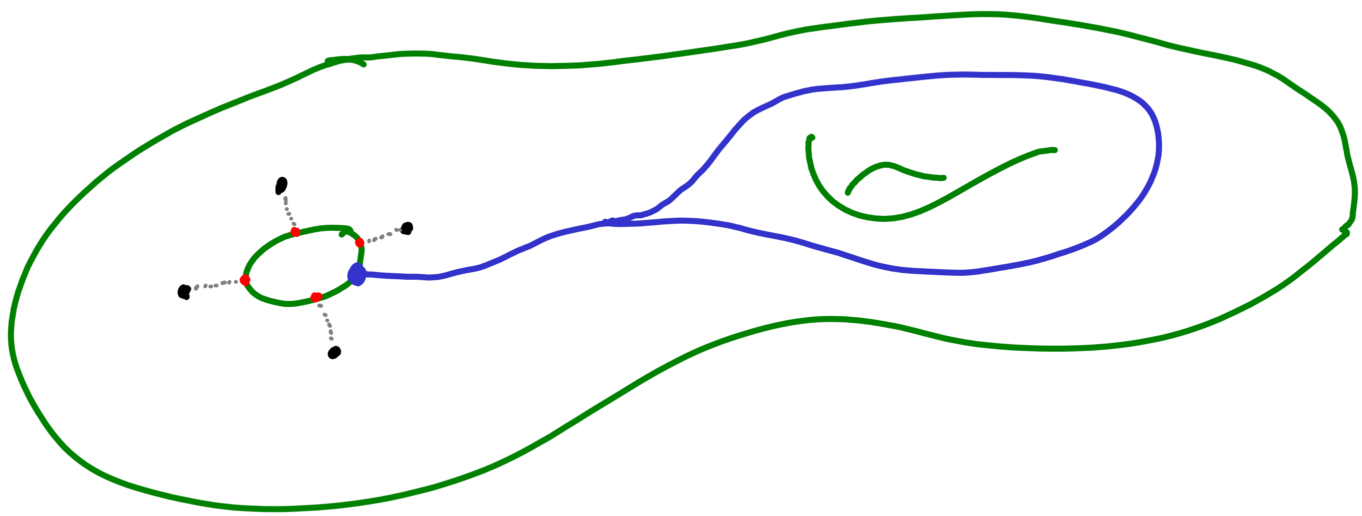

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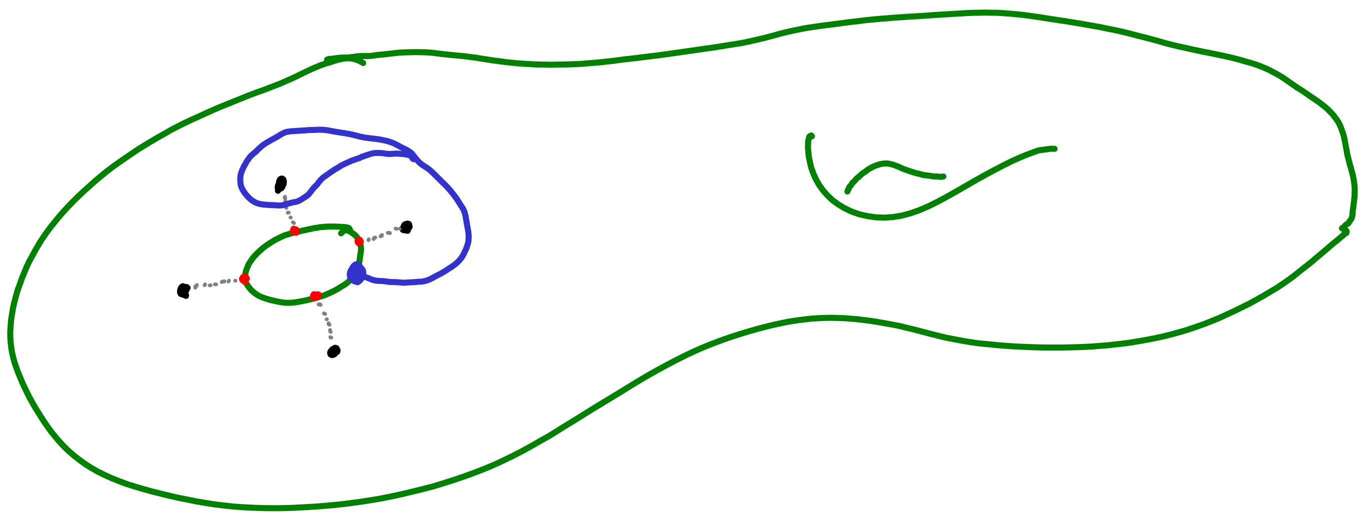

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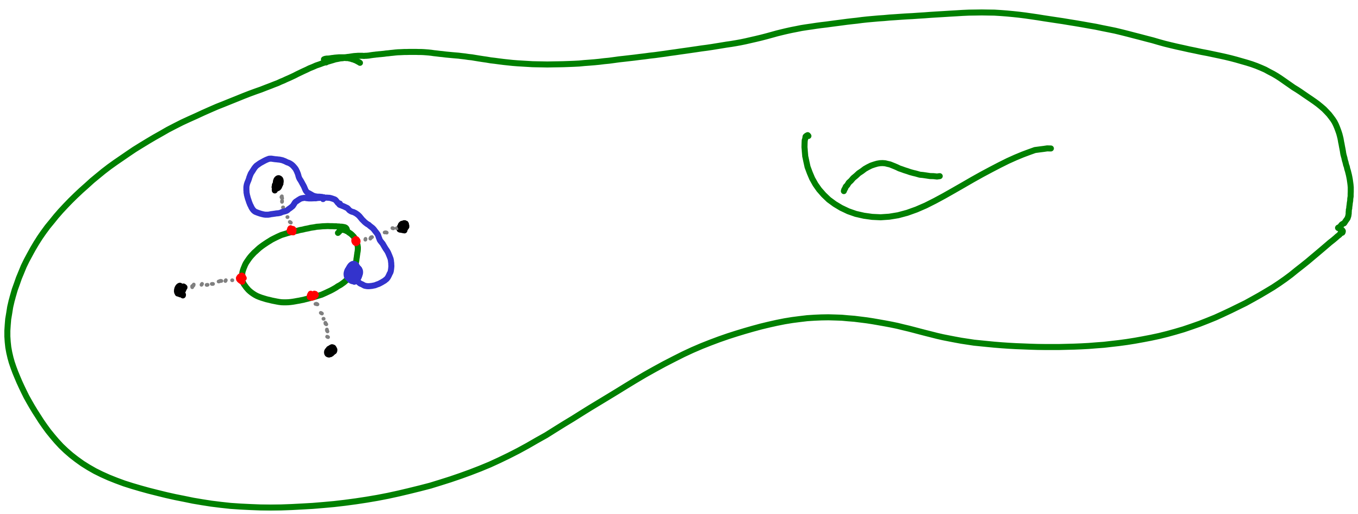

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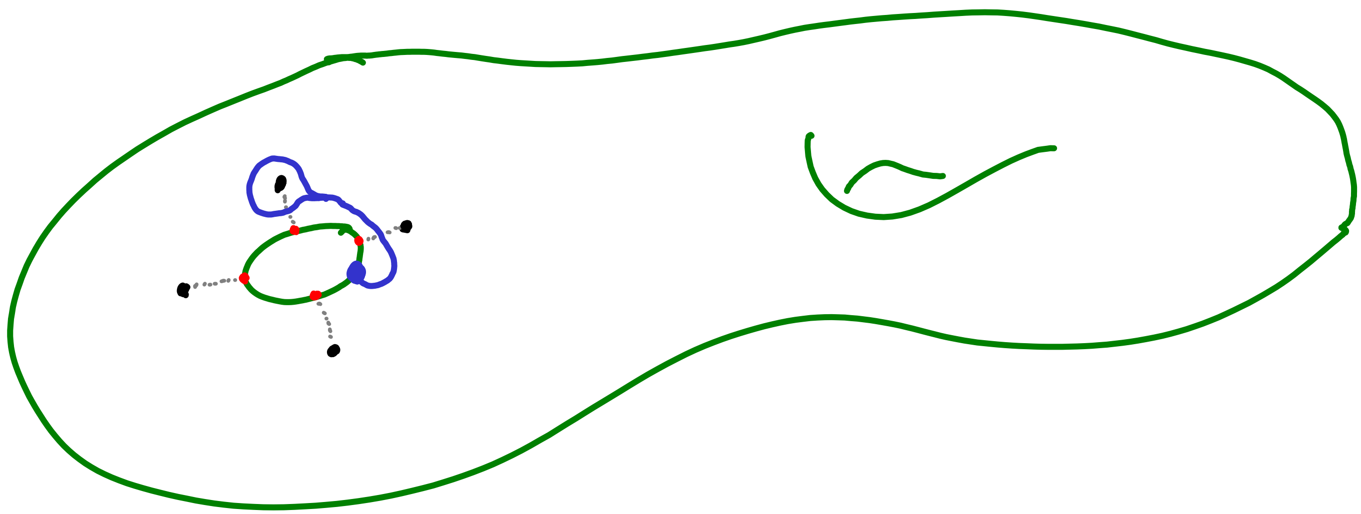

Now consider $\mathrm{Hom}(\Pi, G)$

and the subset $\mathrm{Hom}_{\mathbb{S}}(\Pi, G) \subset \mathrm{Hom}(\Pi, G)$ of "Stokes representations" satisfying:

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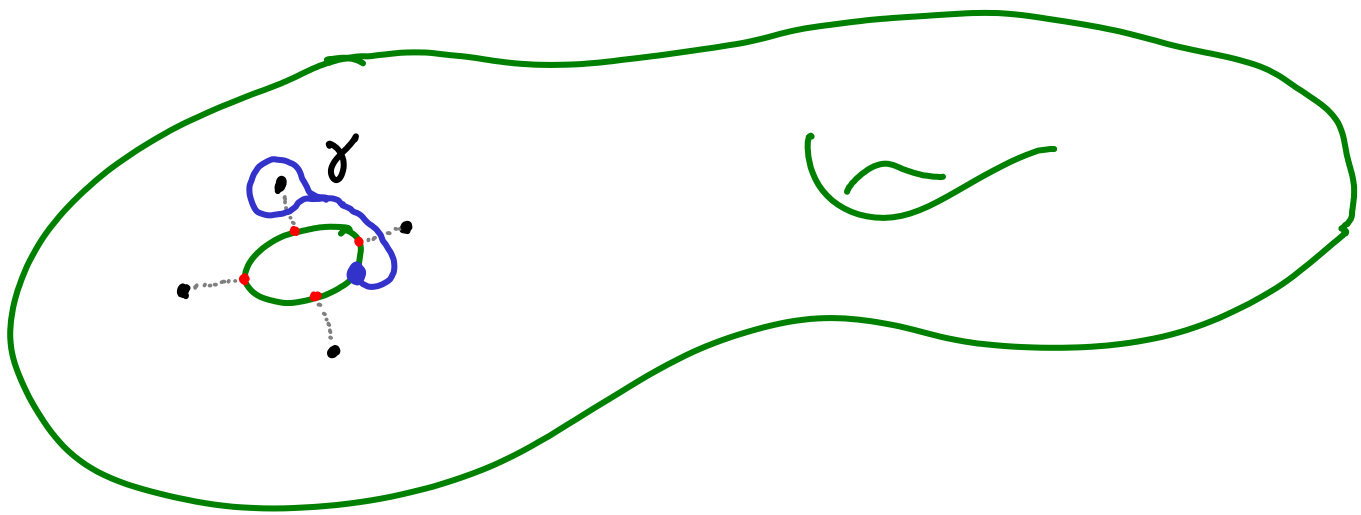

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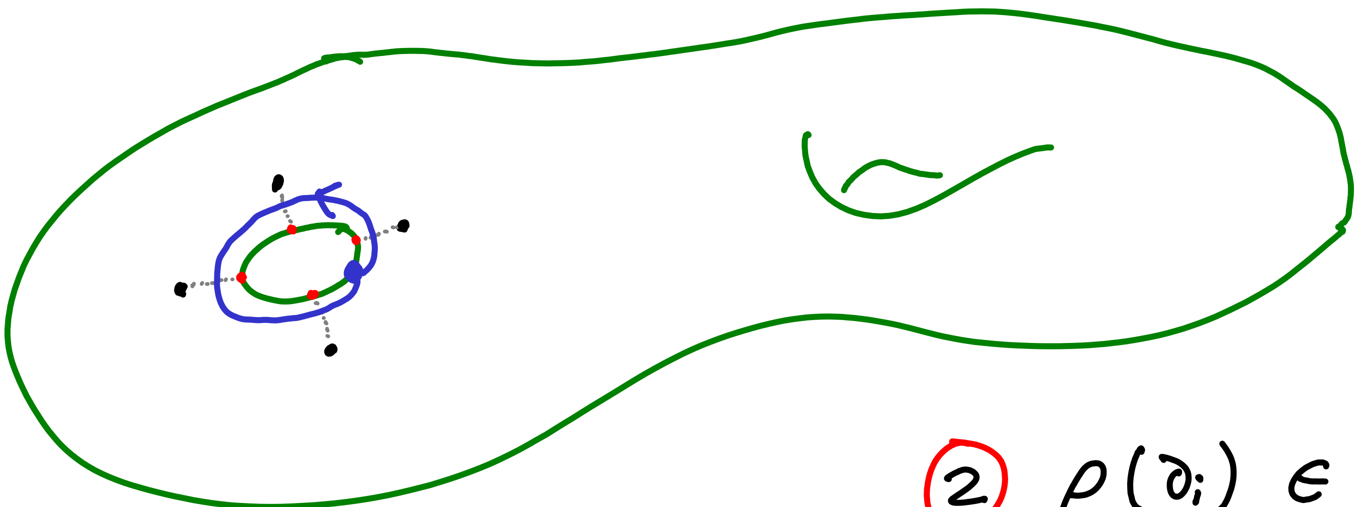

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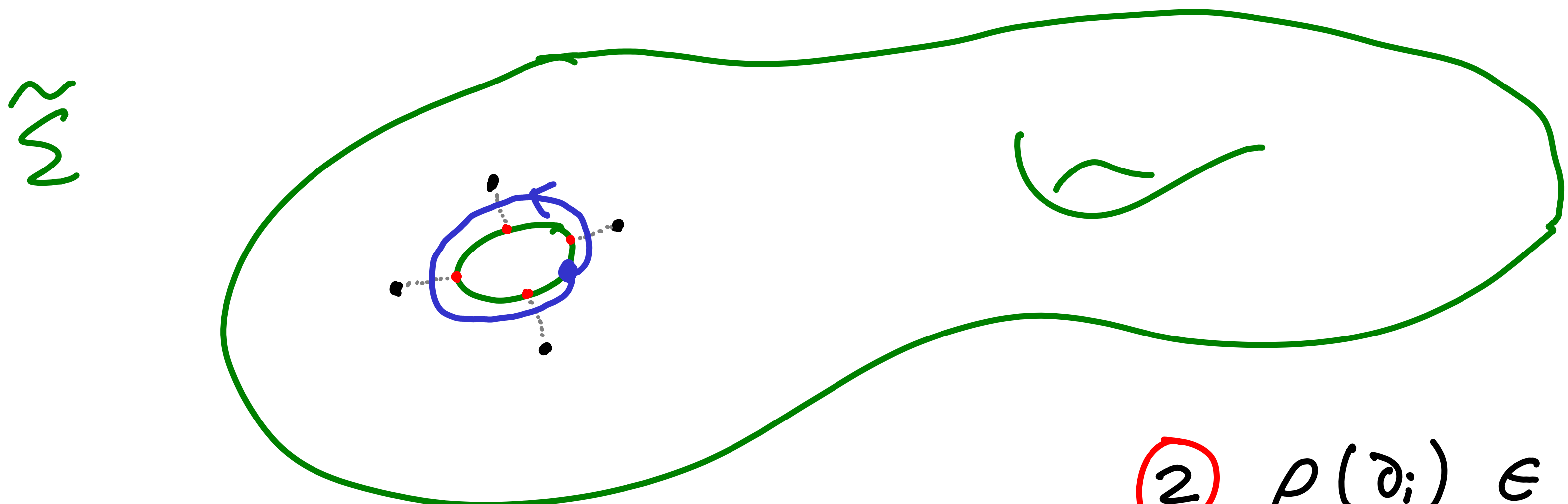
The group $\underset{\sim}{H} = H_1 \times \dots \times H_m$ acts on $\mathrm{Hom}_{\mathbb{S}}(\Pi, G)$ and

$$\cong \quad \{\underset{\sim}{H} \text{ orbits on } \mathrm{Hom}_{\mathbb{S}}(\Pi, G)\}$$



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Thm (– '11)

The space of Stokes representations $\mathrm{Hom}_{\mathbb{S}}(\Pi, \mathcal{G})$ is a smooth affine variety and is (naturally) a quasi-Hamiltonian $\underset{\sim}{H}$-space $\left(\underset{\sim}{H} = H_1 \times \cdots \times H_m\right)$

(proved in '02 in case of one level and $\underset{\sim}{H}$ abelian)

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Corollary

$$\mathcal{M}_B(\Sigma) := \mathrm{Hom}_{\mathbb{S}}(\Pi, G) / \underset{\sim}{H}$$

inherits an intrinsic Poisson structure (algebraically) with symplectic leaves $\mu^{-1}(e)/\underset{\sim}{H}$ for $e = (e_1, \ldots, e_m) \subset \underset{\sim}{H}$

$$\mu : \mathrm{Hom}_{\mathbb{S}}(\Pi, G) \longrightarrow \underset{\sim}{H} \qquad \text{moment map}$$

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \longmapsto \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H} = \mathcal{M}_B(\Sigma)$$

Irregular curve

Poisson variety

Wild character varieties

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Irregular curve — Poisson variety

and given conjugacy class $\mathcal{C} \subset \underset{\sim}{H}$ get

symplectic leaf $\mathcal{M}_B(\Sigma, \mathcal{C}) \subset \mathcal{M}_B(\Sigma)$

Wild character varieties

(G = connected complex reductive gp)

$$\Sigma \quad \longmapsto \quad \mathrm{Hom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H} = \mathcal{M}_B(\Sigma)$$

Irregular curve   Poisson variety

and given conjugacy class $\mathcal{C} \subset \underset{\sim}{H}$ get

symplectic leaf $\mathcal{M}_B(\Sigma, \mathcal{C}) \subset \mathcal{M}_B(\Sigma)$

Could now look at admissible deformations of irreg. curve Σ:

$$\mathrm{PoleOrder}(\alpha \circ Q_i) \text{ constant} \in \mathbb{Z}_{\geq 0} \quad \begin{cases} \forall \text{ roots } \alpha \in R \subset \mathfrak{t}^* \\ \forall i \end{cases}$$

$\rightsquigarrow$ local system of Poisson varieties $\mathcal{M}_B(\Sigma)$

Wild character varieties

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Irregular curve — Poisson variety

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symplectic leaf $\mathcal{M}_B(\Sigma, \mathcal{C}) \subset \mathcal{M}_B(\Sigma)$

Question: Classify complex symplectic manifolds $\mathcal{M}_B(\Sigma, \mathcal{C})$ upto deformation / isomorphism

Bigger picture

$$\Sigma \quad \Rightarrow \quad \mathcal{M}(\Sigma)$$

irreg. curve

hyperkahler manifold
"wild Hitchin space"
(Biquard-B '04)

(Survey: 1203.6607)

Bigger picture

3 algebraic structures:

$\Sigma \Rightarrow \mathcal{M}(\Sigma)$

irreg. curve

hyperkahler manifold
"wild Hitchin space"
(Biquard-B '04)

$\mathcal{M}_{Dol}$ Higgs bundles/Hitchin system

$\mathcal{M}_{DR}$ Connections $\rightsquigarrow$ Isomonodromy systems

$\mathcal{M}_B$ Monodromy/Stokes data $\rightsquigarrow$ wild mapping class group actions

(Survey: 1203.6607)

E.g. $\dim_{\mathbb{C}} = 2$

Conjectural classification of deformation classes: (arXiv 1203.6607)

E_8 E_7 E_6 D_4 $D_3 = A_3$ D_2 D_1 D_0

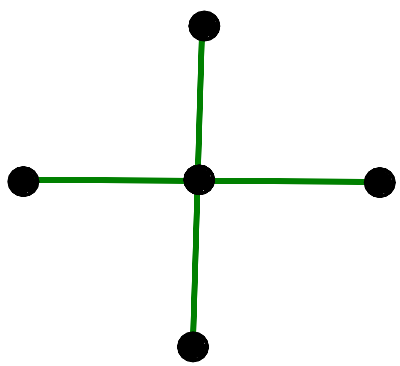
A_2 A_1 A_0

(and $\mathbb{C}^* \times \mathbb{C}^*$)

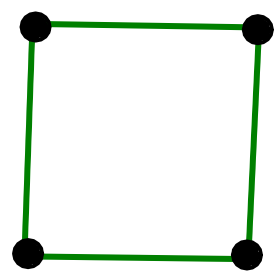
Graphical approach

Graphical approach

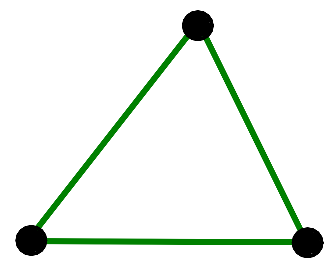
① Okamoto ('80s) related Painlevé equations to affine Weyl groups & thus to affine Dynkin diagrams



P_{VI}



P_{V}



P_{IV}

. . . .

Graphical approach

2 Nakajima and others (90's) developed a theory of (additive) "quiver varieties"

$$\Gamma \Rightarrow \mathcal{N}(\Gamma, \lambda, d)$$

graph

quiver variety

(hyperkähler / complex symplectic)

finite dimensional construction

Graphical approach

② Nakajima and others (90's) developed a theory of (additive) "quiver varieties"

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Γ graph with nodes I, $V = \bigoplus_{i \in I} V_i$ (I graded vector space)

$d = \{d_i\}$ $(d_i = \dim V_i) \in \mathbb{Z}^I$, $\lambda = \{\lambda_i\} \in \mathbb{C}^I$ parameters

$$\mathrm{Rep}(\Gamma, V) = \bigoplus_{e \in \Gamma} \mathrm{Hom}(V_{t(e)}, V_{h(e)}) \qquad t(e) \xrightarrow{e} h(e)$$

$G = \prod_I GL(V_i) \circlearrowright \mathrm{Rep}(\Gamma, V)$ & $\boxed{\mathcal{N}(\Gamma, \lambda, d) = \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} G}$

Graphical approach

3 Symplectic approach (– '99, '01) to Jimbo–Miwa–Ueno equations

$\Sigma = \mathbb{P}^1$

meromorphic connections on trivial bundle — $\mathcal{M}^* \xrightarrow{\text{JMU-iRH}} \mathcal{M}_B$ — monodromy / Stokes data

Graphical approach

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Factorises: $\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B$

$\mathcal{M}^*$: finite dim. symplectic quotient

$\mathcal{M}_{DR}$: ∞-dim. symplectic quotient

$\mathcal{M}_B$: finite dim. multiplicative symplectic quotient ('02)

[understand Painlevé property as "going off $\mathcal{M}^*$", but staying in $\mathcal{M}_{DR}$]

Graphical approach

Theorem (– '07, '08, '11 Simply-laced isomonodromy systems)

① Lots of spaces $\mathcal{M}^*$ are quiver varieties $\mathcal{N}(\Gamma)$ including those for P_2, P_4, P_5, P_6, and

② the Kac-Moody Weyl group of the graph lifts to give symplectic isom.s between $\mathcal{M}^*$'s, relating the isomonodromy equations

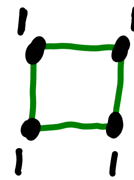
(so this generalises Okamoto's result)

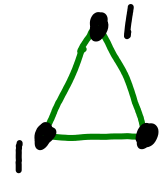
"Certain graph rep.s $\iff$ Certain $\mathbb{C}\langle z, \frac{d}{dz}\rangle$-module presentations"

- Fuchsian case of ① follows from Kraft-Procesi / Nakajima / Crawley-Boevey
- see Hiroe / Yamakawa for some extensions

Graphical approach

E.g.

$\mathcal{M}^*(P_5) \cong \mathcal{N}($  $)$

$\mathcal{M}^*(P_4) \cong \mathcal{N}($  $)$

Graphical approach

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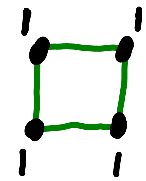
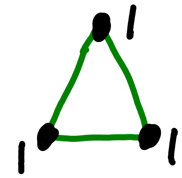
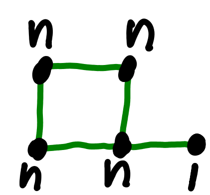
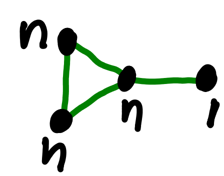
$\mathcal{M}^*(P_5) \cong \mathcal{N}\Big($  $\Big)$

$\mathcal{M}^*(P_4) \cong \mathcal{N}\Big($  $\Big)$

Corollary (of graphical approach, & extension of Jimbo-Miwa-Ueno)

Higher Painlevé systems (of order $2n$ for $n = 1, 2, \ldots$)

e.g. $\mathcal{M}^*(hP_5^n) \cong \mathcal{N}\Big($  $\Big)$

$\mathcal{M}^*(hP_4^n) \cong \mathcal{N}\Big($  $\Big)$

Graphical approach

works with any complete k-partite graph (more generally any "supernova graph") and they can be "read" in terms of connections in k+1 ways

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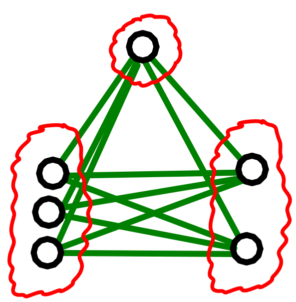
Complete k partite graphs $\Longleftrightarrow$ Integer partitions with k parts



$\Longleftrightarrow$ $1+2+3=6$

$\Gamma(3,2,1)$

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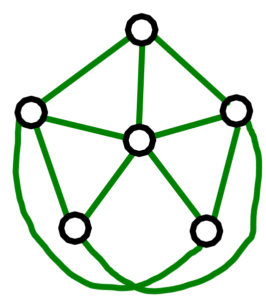

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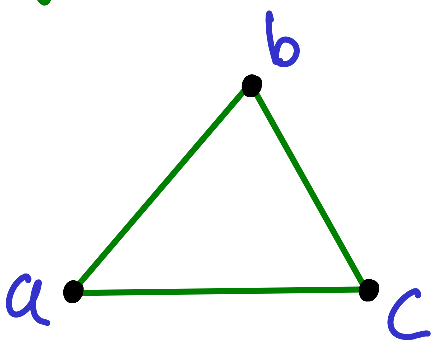
Example readings:
($k=3$)



$\Gamma(1,1,1)$

rank	pole orders
$a+b$	$3+1$
$b+c$	$3+1$
$c+a$	$3+1$
$a+b+c$	3

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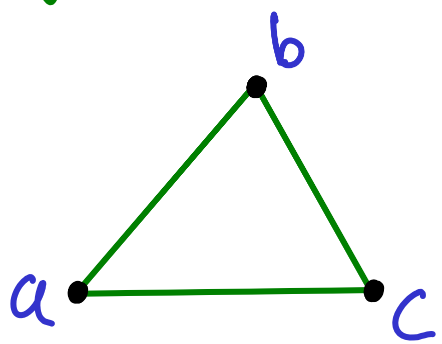

rank	pole orders
$a+b$	$3+1$
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$\rightsquigarrow$ Isomorphisms of corresponding spaces $\mathcal{M}^*$ & isomonodromy systems

Question:
$\exists$ corresponding algebraic symplectic isomorphisms of full moduli spaces $\mathcal{M}_B$?

$$\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B$$

Theorem (1307.1033)

YES!

Theorem (1307.1033)

YES! For example:

If Σ is a type $\underbrace{3+1^m}_{\text{(pole orders on } \mathbb{P}^1)}$ irregular curve for $G = GL(V)$

then $\exists$ a vector space $\hat{V}$ and an irregular curve $\hat{\Sigma}$ of type 3 for $\hat{G} = GL(\hat{V})$ such that

for any conjugacy class $\mathcal{C}$ for Σ

$$\boxed{\mathcal{M}_B^{st}(\Sigma, e) \cong \mathcal{M}_B^{st}(\hat{\Sigma}, \hat{e})}$$

as algebraic symplectic manifolds,

for some conjugacy class $\hat{e}$ for $\hat{\Sigma}$

Idea

In additive case one can simply "reorder the symplectic quotient"

(as in Harnad duality for $k=2$ / bipartite case)

and this can be expressed in terms of quiver varieties

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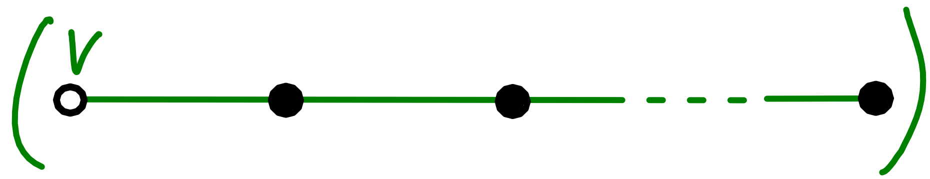
Recall

(Kraft-Procesi / Nakajima / Crawley-Boevey)

adjoint orbits $\mathcal{O} \subset \mathfrak{gl}(V)$ are quiver varieties:

$\mathcal{O} \cong \mathcal{N}$ 

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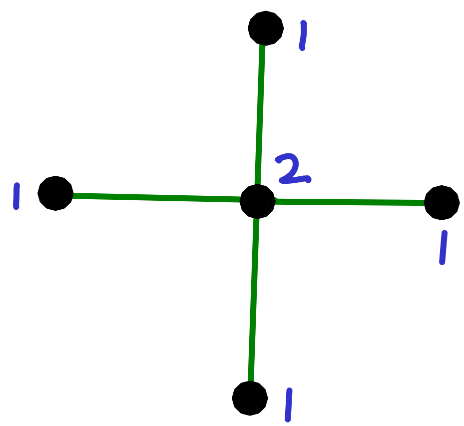

$G = Gl_2(\mathbb{C}) \times (\mathbb{C}^*)^4$

$V = \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

$\mathcal{M}^* \cong \mathrm{Rep}(\Gamma, V) /\!/_{\lambda} G$

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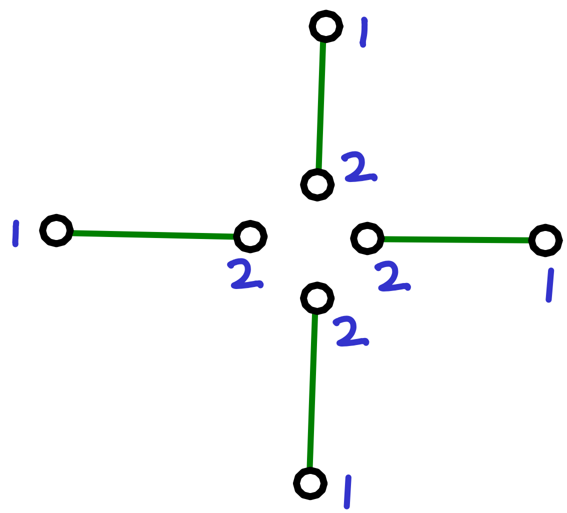
$\Gamma =$ 

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$G \circlearrowright \mathrm{Rep}(\Gamma, V)$

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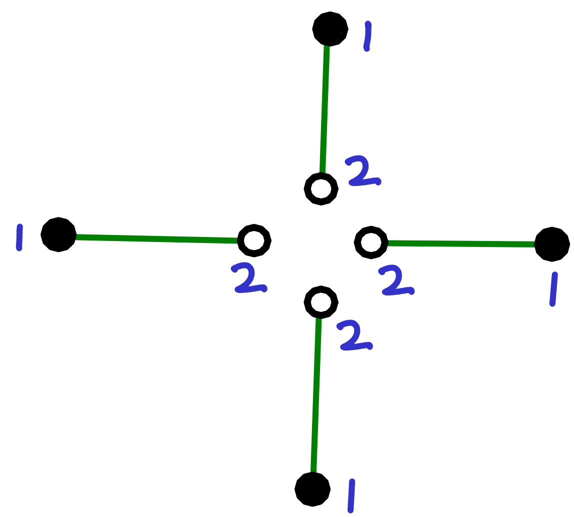

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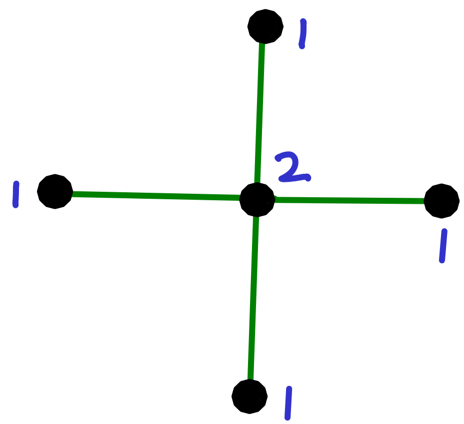
$\Gamma =$ 

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(usual Painlevé VI phase space)

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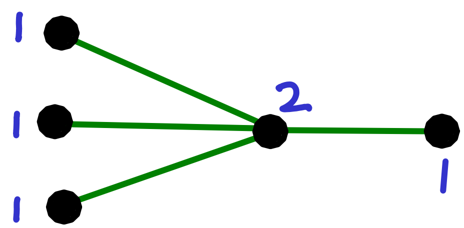

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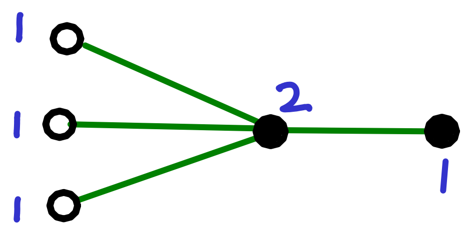
$\Gamma =$ 

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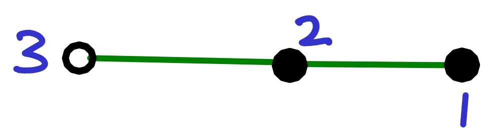

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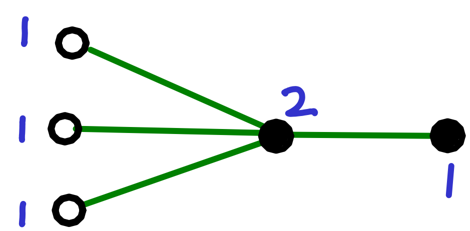
$\Gamma =$ 

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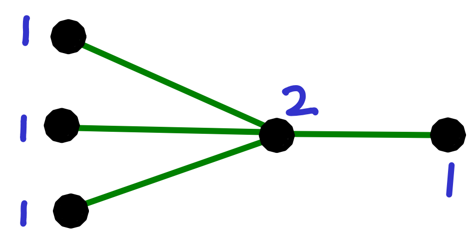
$\Gamma =$ 

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$T = (\mathbb{C}^*)^3 \circlearrowleft \mathcal{O} \subset \mathfrak{gl}_3(\mathbb{C})$

$\mathcal{M}^* \cong \mathcal{O} /\!\!/_{\lambda} T$

(phase space for Harnad's 2+1 dual Lax pair for P_{VI})

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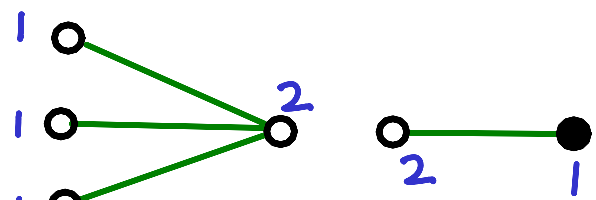

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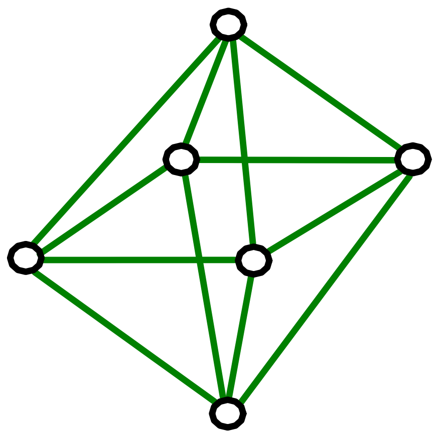
$T \times GL_2(\mathbb{C}) \circlearrowright T^*\mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^3) \times \mathcal{O}_4$ $T = (\mathbb{C}^*)^3, \quad \mathcal{O}_4 \subset \mathfrak{gl}_2(\mathbb{C})$

$\mathcal{M}^* \cong$ a space of type 3 connections on rank 5 bundles (new Lax pair for P_{VI} '08, '11)

Supernova example

$\Gamma = \Gamma(2,2,2) \cong$ 

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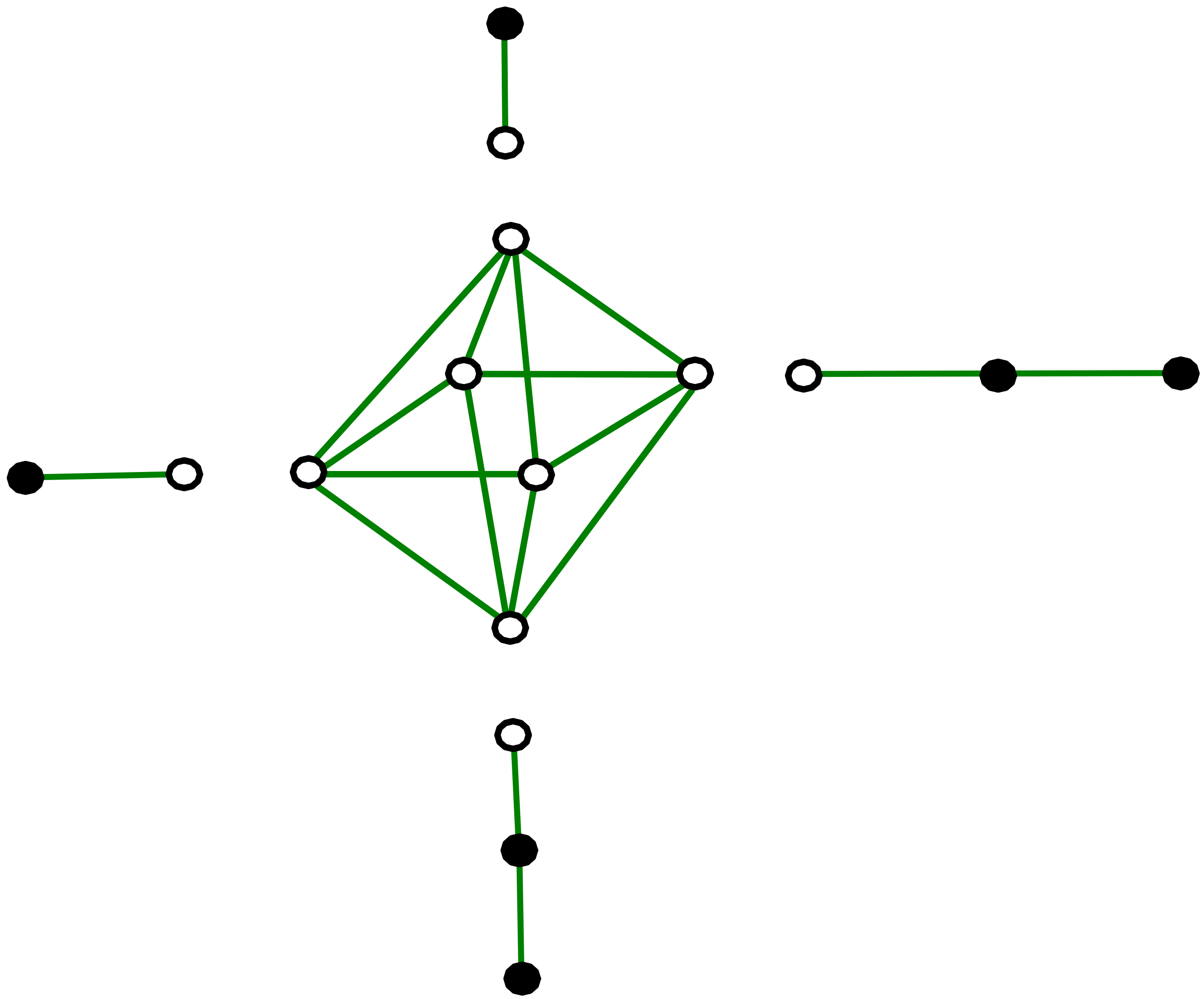
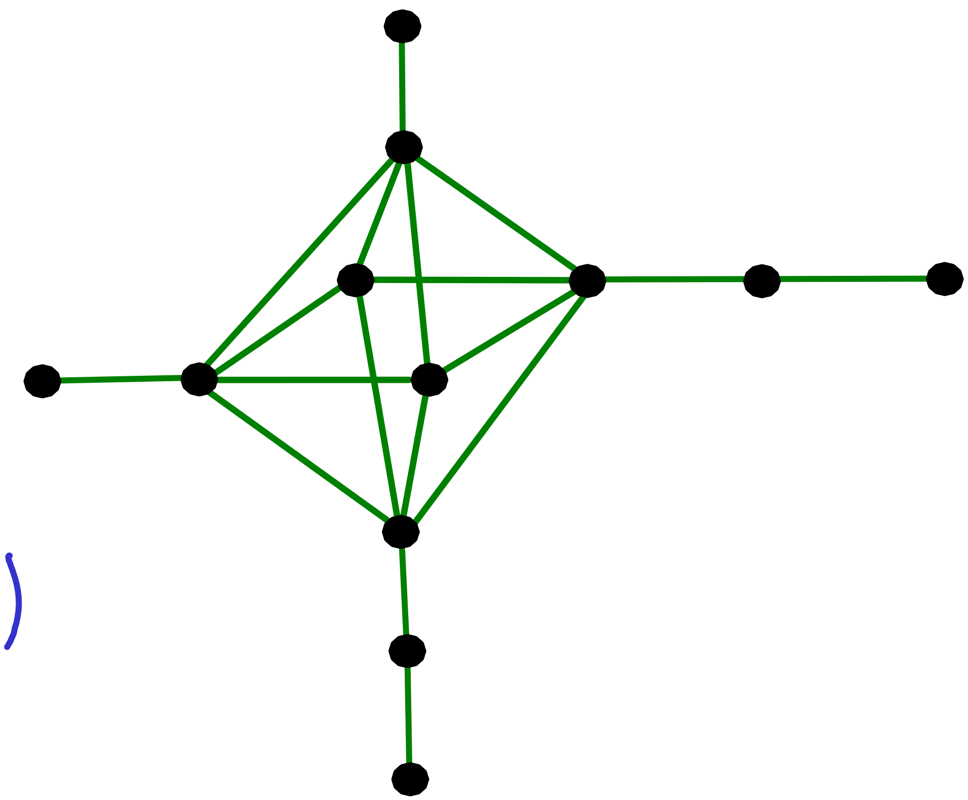
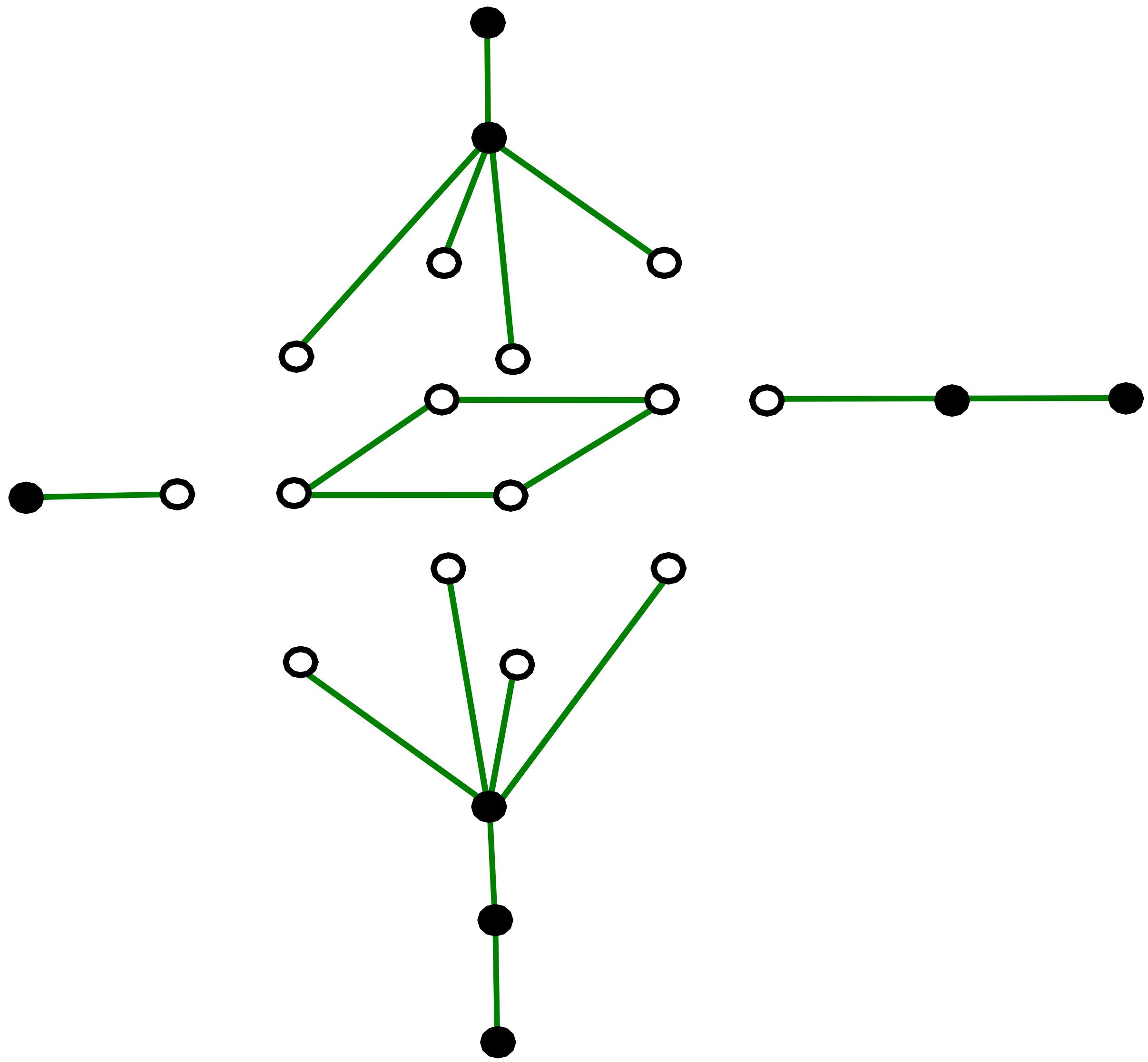


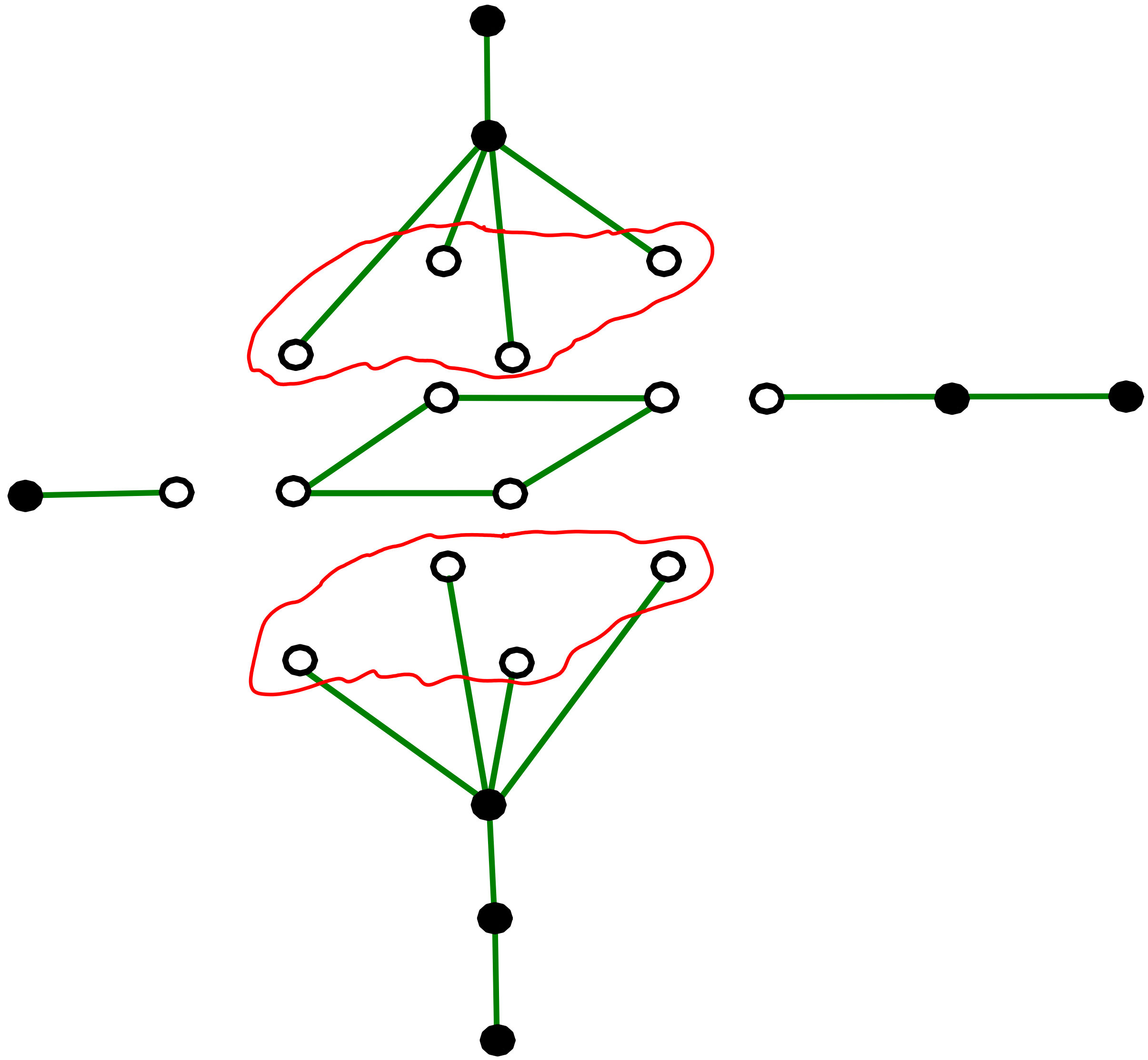
Supernova example

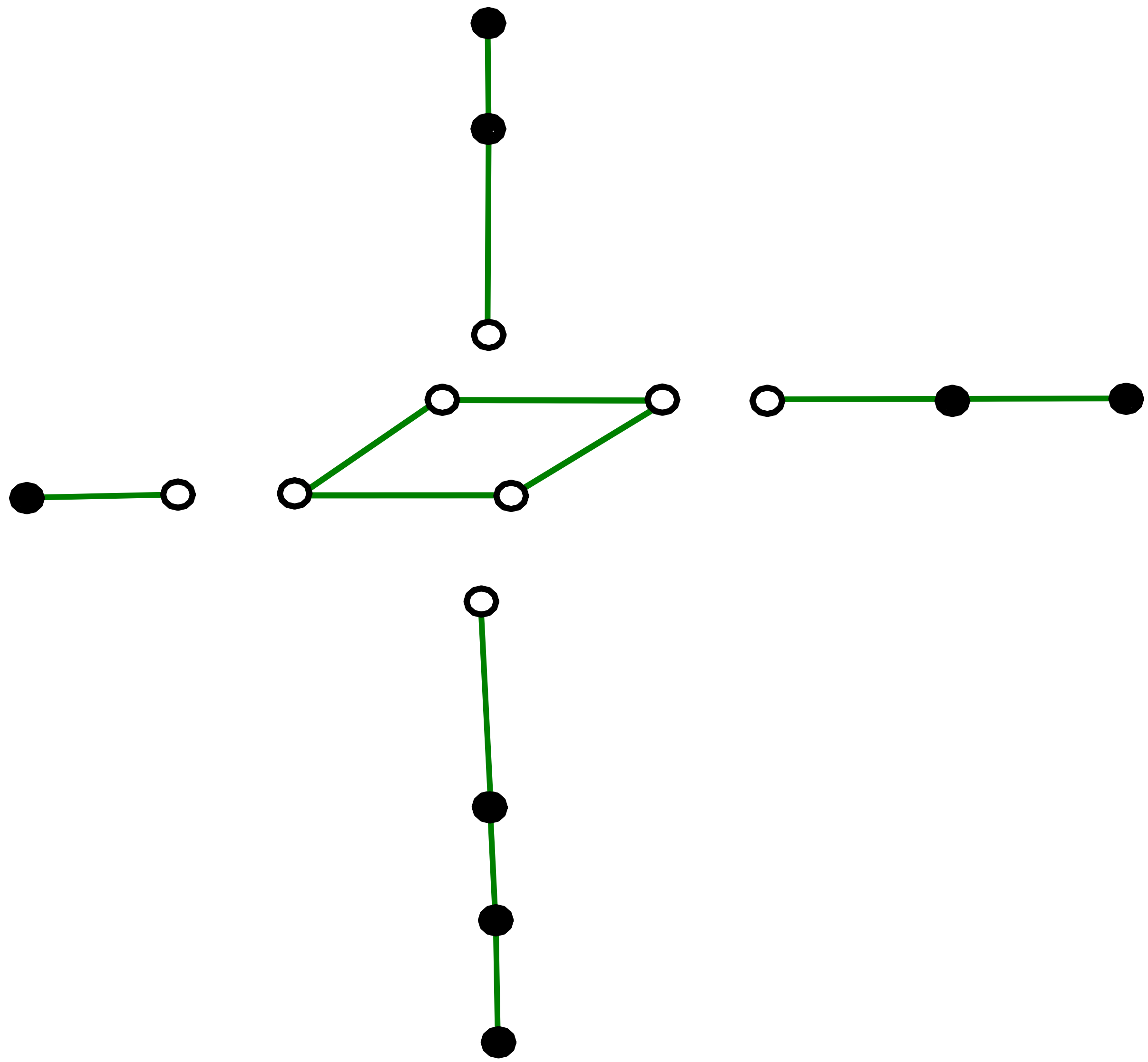
$\hat{\Gamma} =$

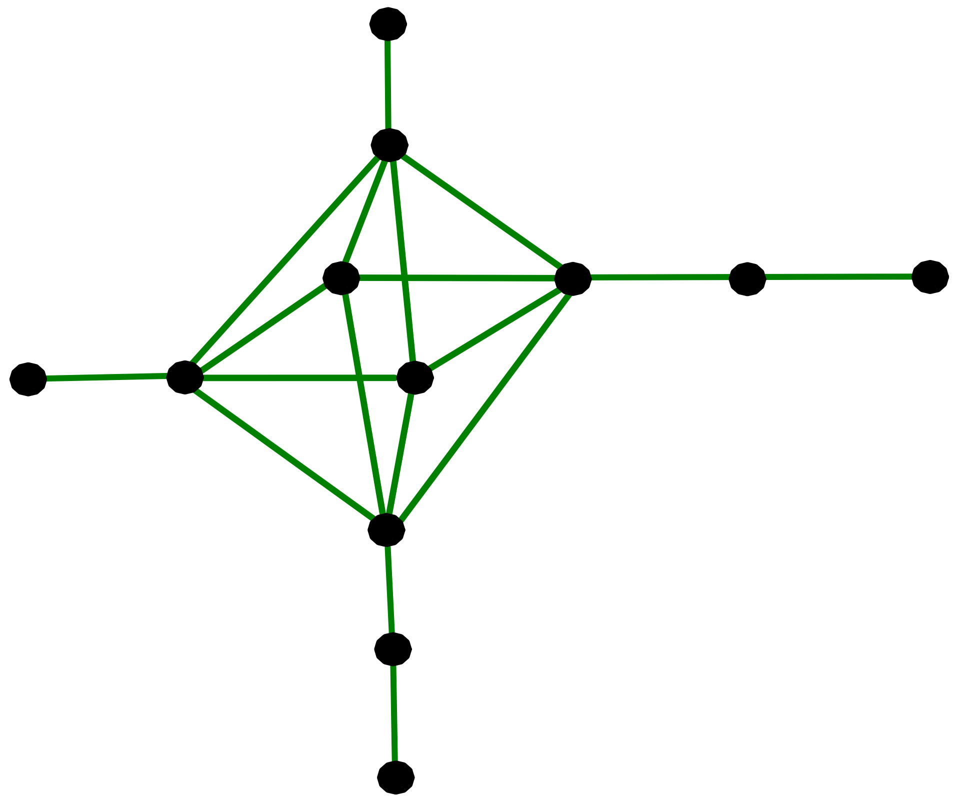


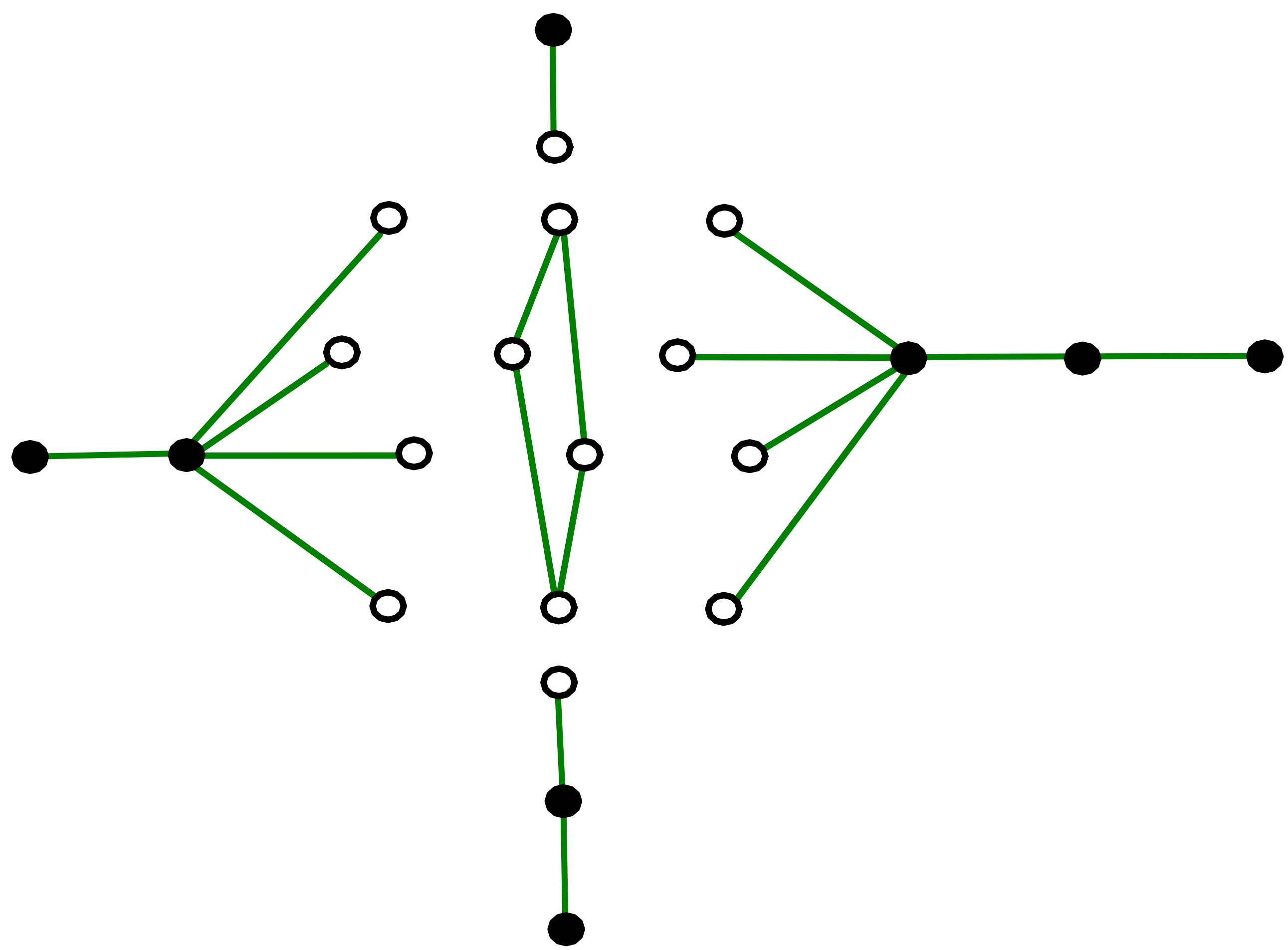
$\mathcal{M}^* \cong \mathcal{N}(\hat{\Gamma}, 1, d)$

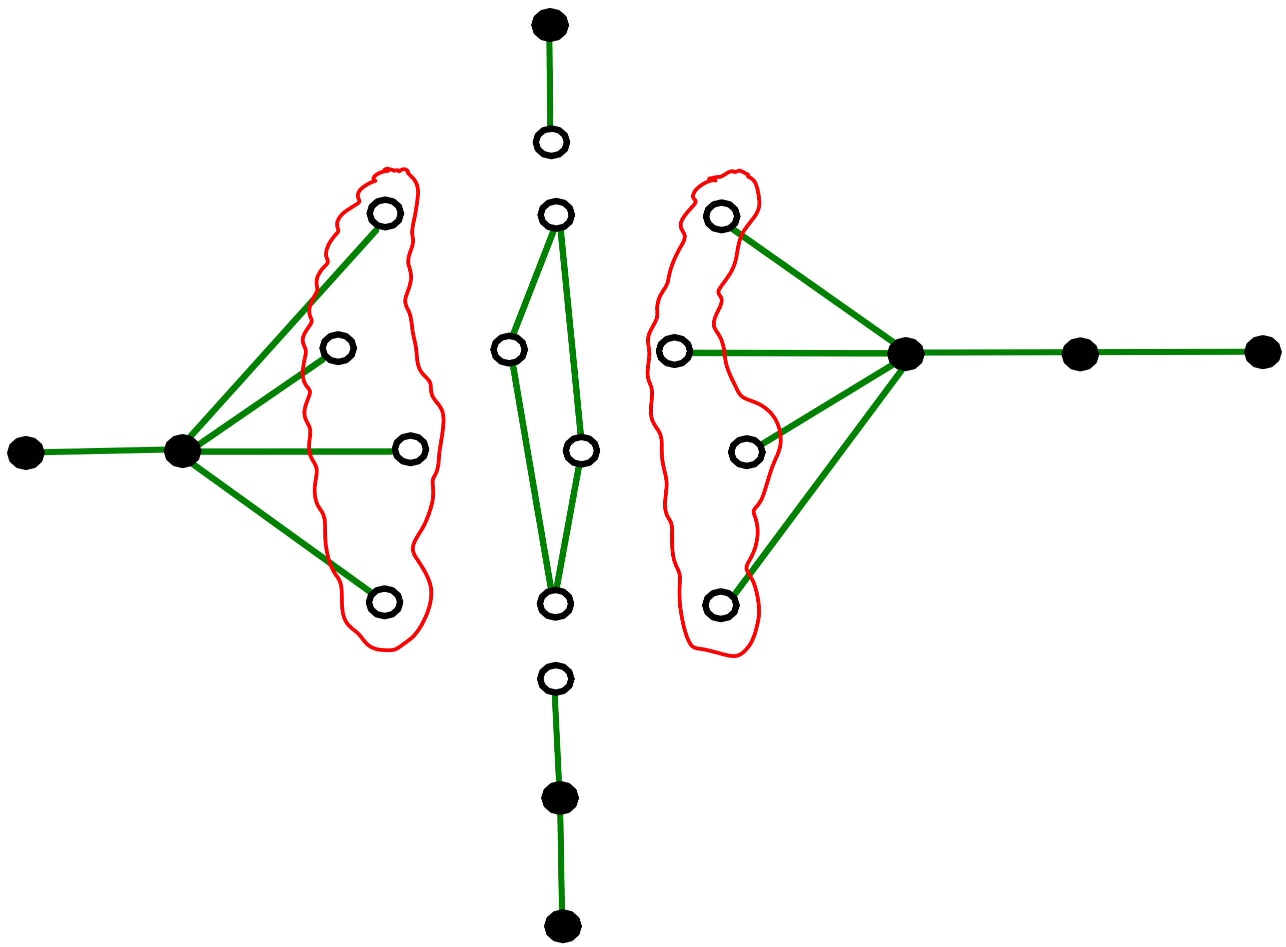


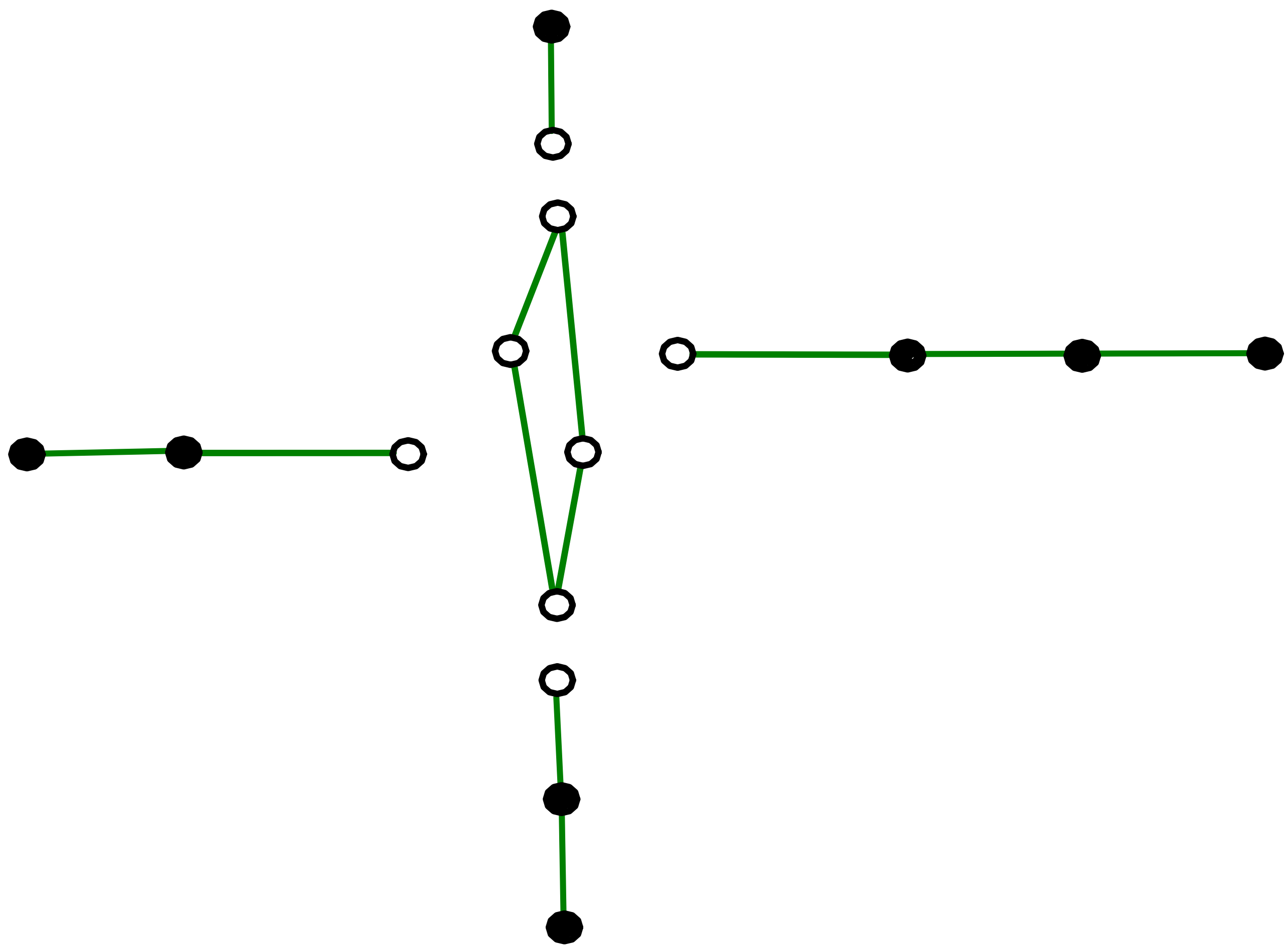


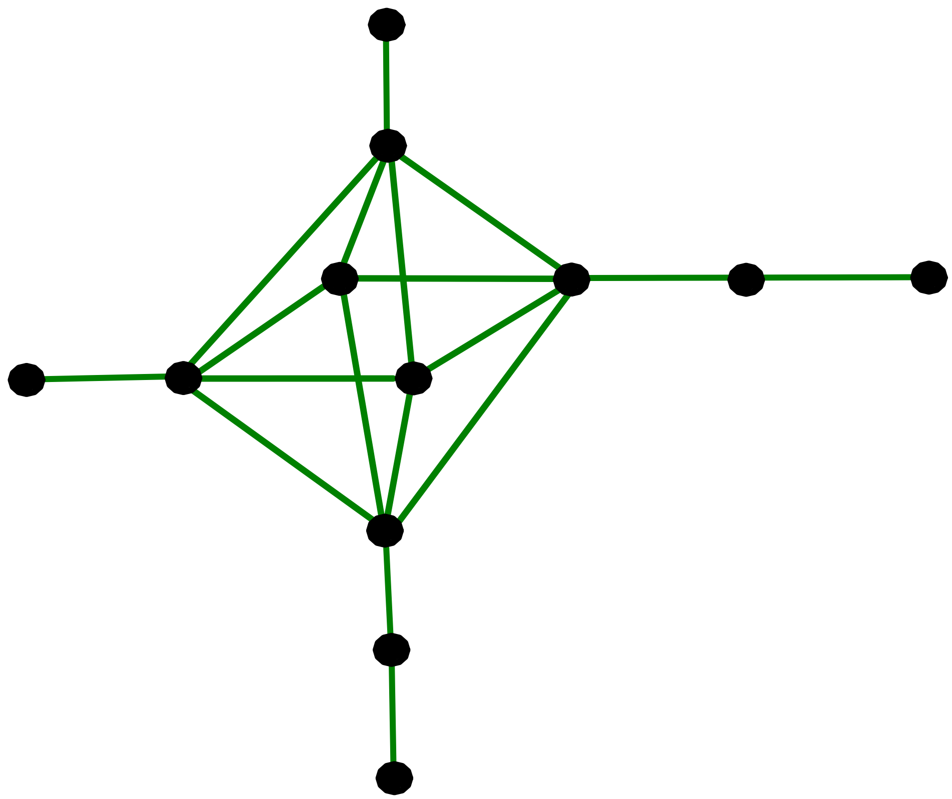












Idea

In additive case one can simply "reorder the symplectic quotient"

(as in Harnad duality for $k=2$ / bipartite case)

and this can be expressed in terms of quiver varieties

- Develop theory of "multiplicative quiver varieties" such that lots of wild character varieties $\mathcal{M}_B$ are multiplicative quiver varieties
- Obtain isomorphisms as before essentially by "reordering the multiplicative symplectic quotient"

Def^n A **coloured graph** is a graph Γ plus a map

$$\mathrm{Edges}(\Gamma) \xrightarrow{\gamma} C = \{\text{colours}\}$$

st $\quad \gamma^{-1}(c) = \coprod$ complete k-partite graphs $\quad \forall$ colours $c \in C$

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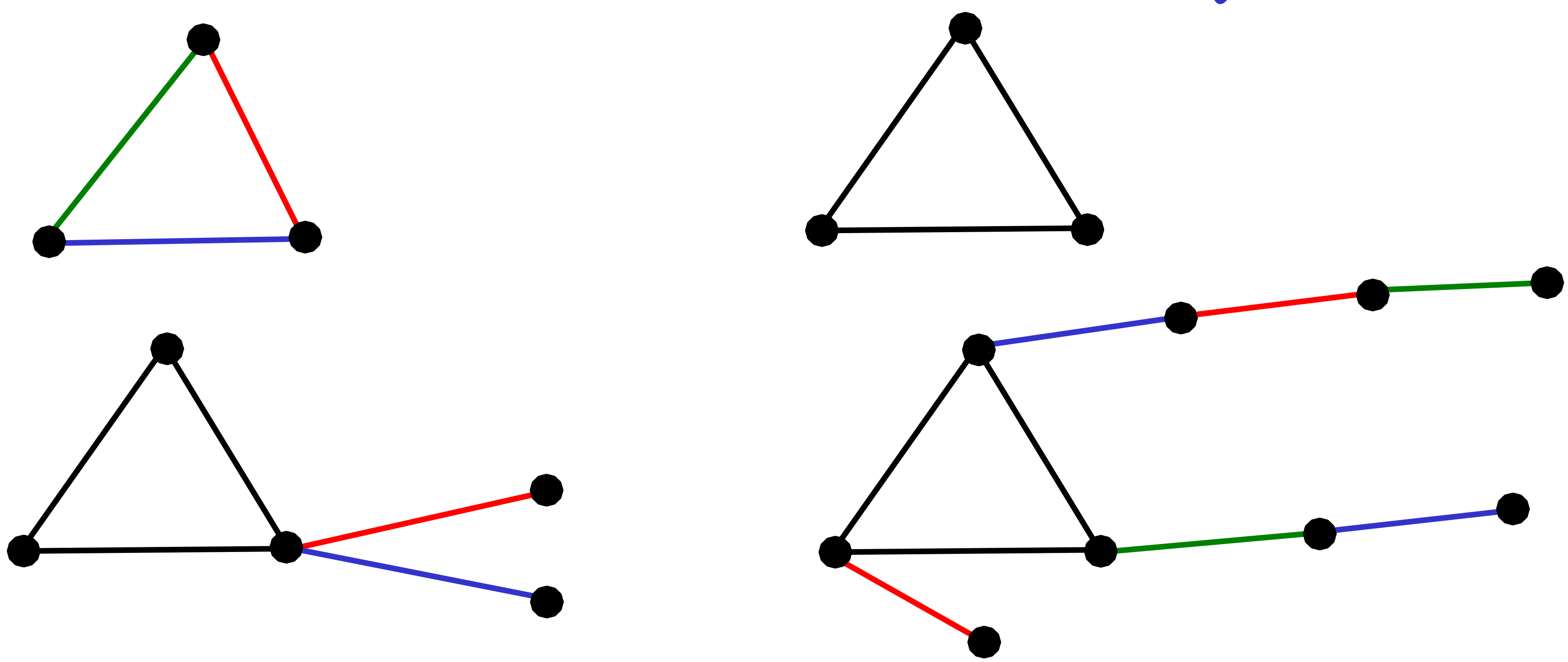
- It is "classical" if all edges different colours
- It is "simply-laced supernova" if monochromatic core + legs



Main results (1307.1033)

Γ coloured graph with nodes I , $V = \bigoplus_I V_i$ $d = (\dim V_i) \in \mathbb{Z}^I$

(+ ordering choice) $q \in (\mathbb{C}^*)^I$

Determines open subset $\mathrm{Rep}^*(\Gamma, V) \subset \mathrm{Rep}(\Gamma, V)$ of invertible graph rep.s

$$\downarrow \mu$$

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If Γ simply laced supernova graph (with k-partite core) have $k+1$ "readings"

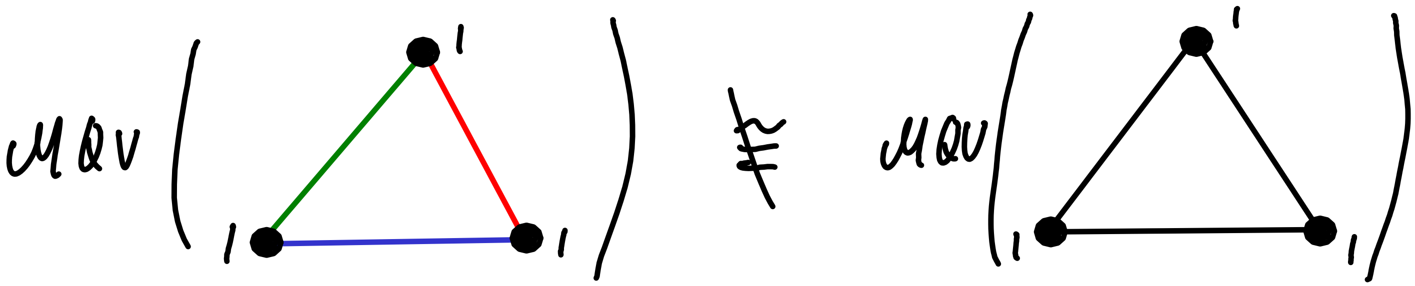
$$\mathcal{M}QV(\Gamma, q, d) \cong \mathcal{M}_B(\Sigma_i) \qquad \text{(algebraic symplectic isomorphisms)}$$

as wild char. varieties for $k+1$ irregular curves $\Sigma_1, \dots, \Sigma_{k+1}$

Remarks

- Proof uses fission operation to give simple inductive approach
- These isomorphisms $\rightsquigarrow$ isomorphisms covering Weyl gp $W(\Gamma)$ action on $\begin{cases} q \in (\mathbb{C}^*)^I \\ d \in \mathbb{Z}^I \end{cases}$
- Link to graphs $\rightsquigarrow$ Kac-Moody root system $\rightsquigarrow$ irregular Deligne-Simpson conjecture
- Classical case studied before (CB-Shaw, VanDenBergh, Yamakawa)

  does not give right spaces beyond star-shaped case:



- Get new noncommutative algebras "fission algebras" generalising the multiplicative preprojective algebras (thus presumably generalising the generalised DAHAs of Etingof-Oblomkov-Rains)

Key step

Classical case

$$\Gamma = \begin{tikzcd} V_1 \arrow[r, dash] \arrow[r, bend right, "b"'] & V_2 \arrow[l, bend right, "a"'] \end{tikzcd}$$

$$(a,b) \in \mathrm{Rep}(\Gamma, V) = T^*\mathrm{Hom}(V_1, V_2) \qquad V = V_1 \oplus V_2$$

$$\begin{tikzcd} \mathrm{Rep}(\Gamma, V) \\ \mathcal{B}(V_1, V_2) := \mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \} \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "\mu"'] \arrow[d, shift left=30ex, "{(1+ab), (1+ba)^{-1}}"] \\ G = GL(V_1) \times GL(V_2) \end{tikzcd}$$

Build general graph out of such pieces, for each edge

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$$G = GL(V)$$

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Thm (– '02, '11) (G complex reductive group)

If $P_\pm \subset G$ opposite parabolics with Levi decomposition $P_\pm = \mathcal{G} \cdot U_\pm$

then $\mathcal{A}^n = \mathcal{G} \times (U_+ \times U_-)^n \times G$ is a q-Hamiltonian $\mathcal{G} \times G$-space

($\forall n = 1, 2, \dots$)

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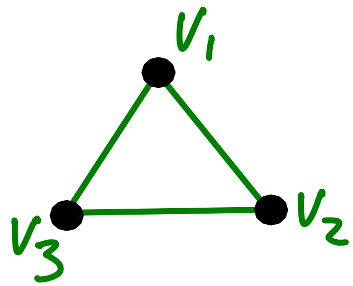
$$\mathcal{B} = \mathcal{A}^n /\!/ H = \{(h, S_1, \dots, S_{2n}) \in H \times (U_+ \times U_-)^n \mid h S_{2n} \cdots S_1 = 1\}$$ is q-Hamiltonian H-space

Key step

$\rightsquigarrow$ Natural generalizations, e.g. $\Gamma =$



$V = V_1 \oplus V_2 \oplus V_3$

$\mathbb{G} = \prod_1^3 GL(V_i)$

$\cap$

$G = GL(V)$

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[Different to classical case]

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$$B(V_1, V_2, V_3) := \mathrm{Rep}^*(\Gamma, V) \xrightarrow{\mu} \mathbb{G} \qquad (\mu = h^{-1})$$

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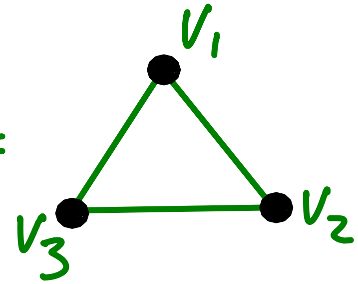

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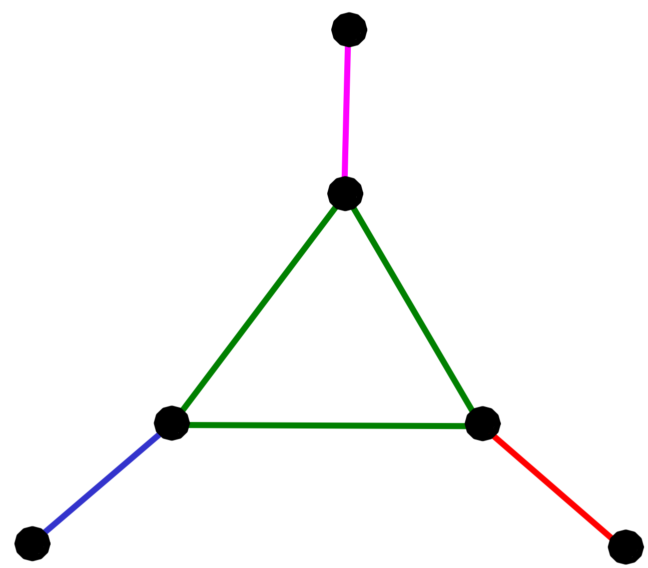
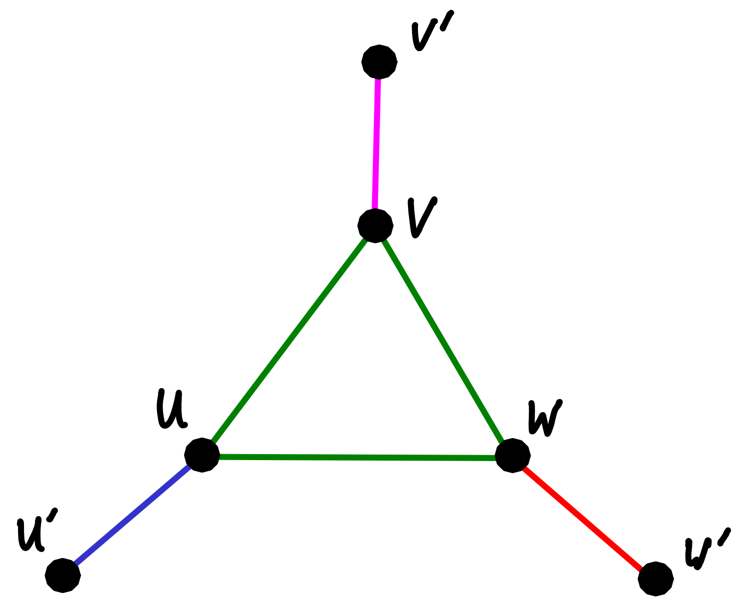
Build general coloured graph out of such pieces

Recall (Kraft–Procesi / Nakajima / Crawley-Boevey + Shaw)

conjugacy classes $\mathcal{C} \subset GL(V)$ are classical mult. quiver varieties:

$$\mathcal{C} \cong MQV\left(\overset{V}{\circ} \text{---} \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet\right)$$

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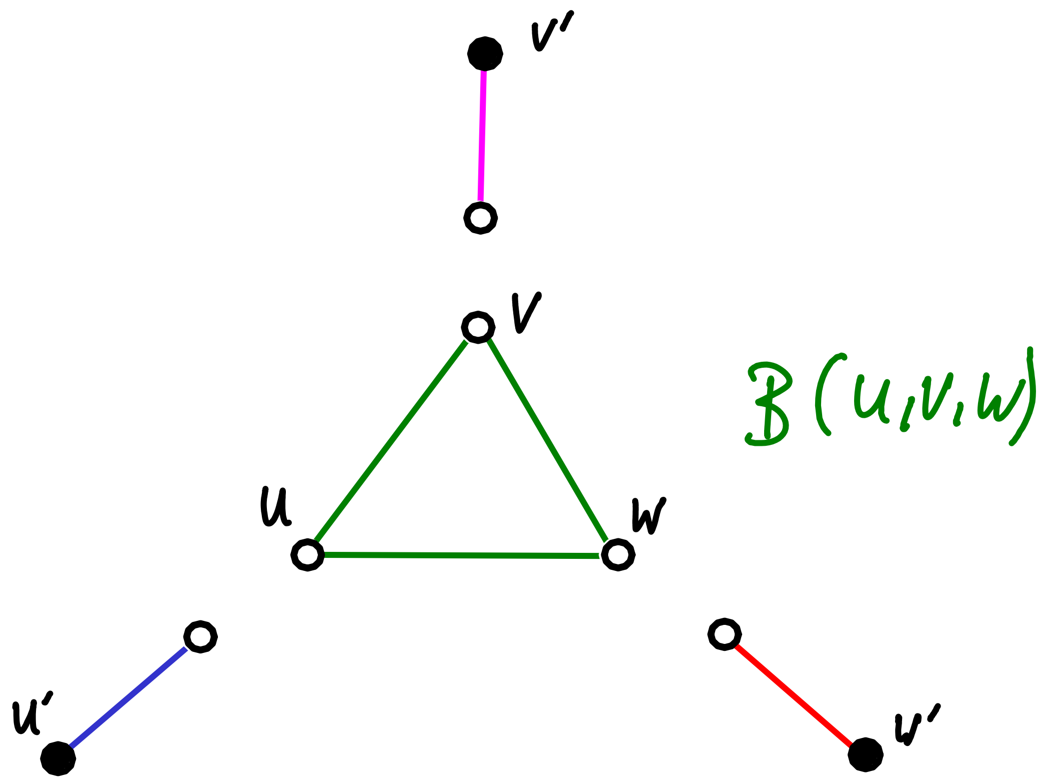
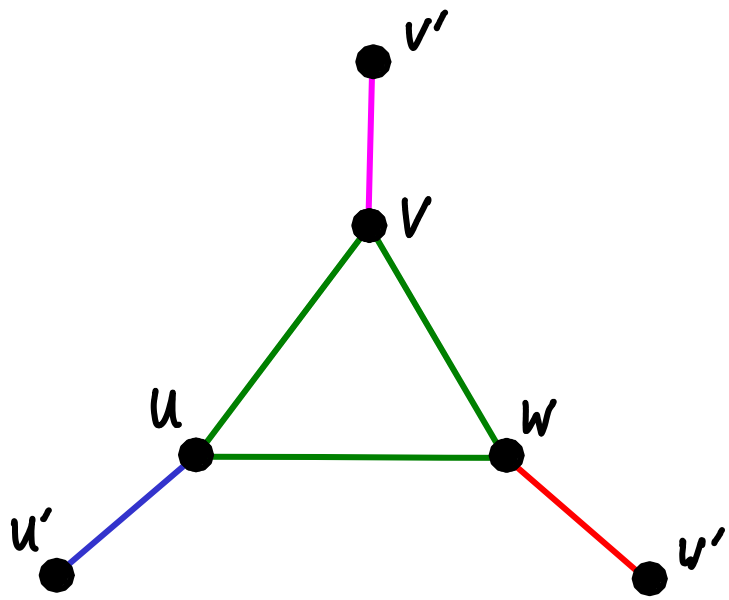


Pole order(s)	Rank
3	$\dim(U \oplus V \oplus W)$



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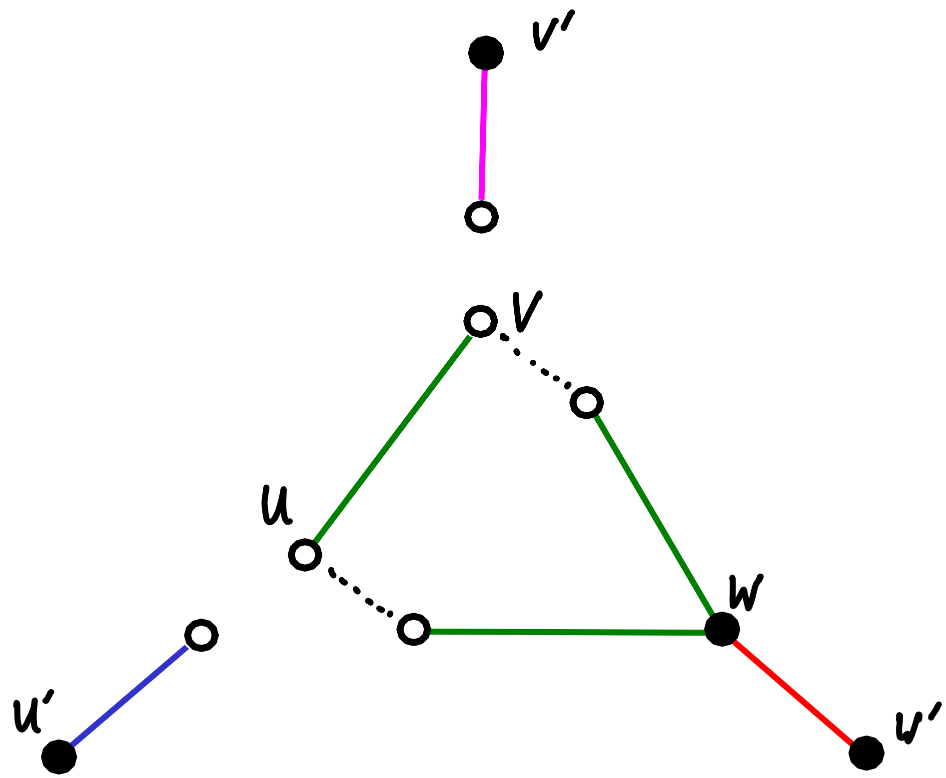

Pole order(s)	Rank
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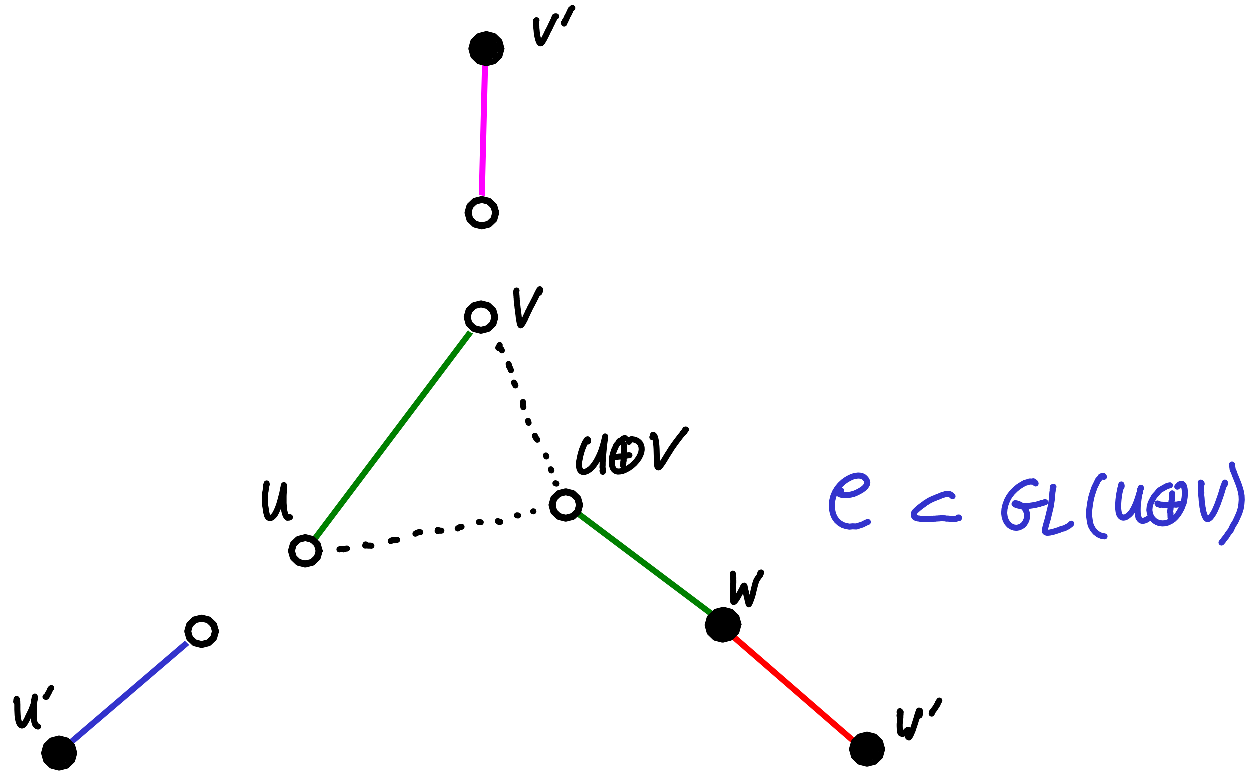

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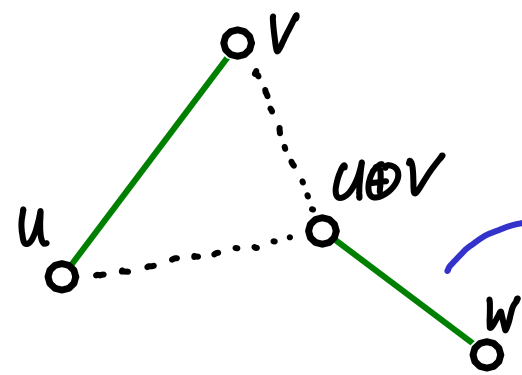

$B(u\oplus v, w)$

$$\mathcal{A}^2(u,v) \succ\!\underset{G}{\text{—}}\!\prec B(u\oplus v, w)$$

1-fission operator $= {}_H\mathcal{A}^2_G \cong H \times (U_+ \times U_-)^2 \times G$

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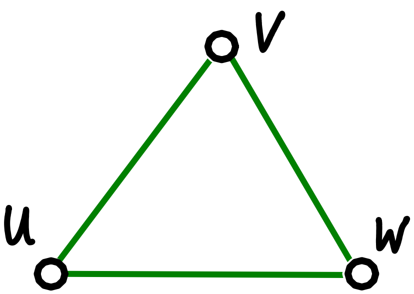
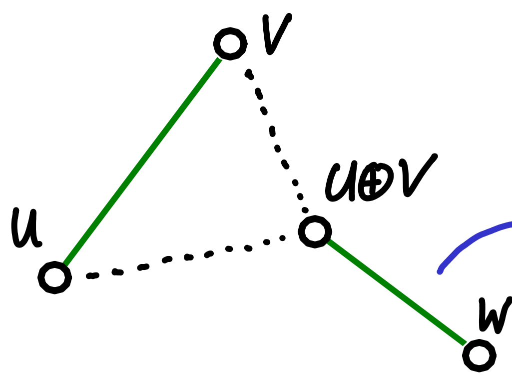
Thm (§7 1307.1033)



$\cong$



$\mathcal{B}(U \oplus V, W)$

$$\mathcal{B}(U,V,W) \cong \mathcal{A}^2(U,V) \circledast_G \mathcal{B}(U\oplus V, W)$$

(Alg. isom. of q-Ham^s spaces)

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