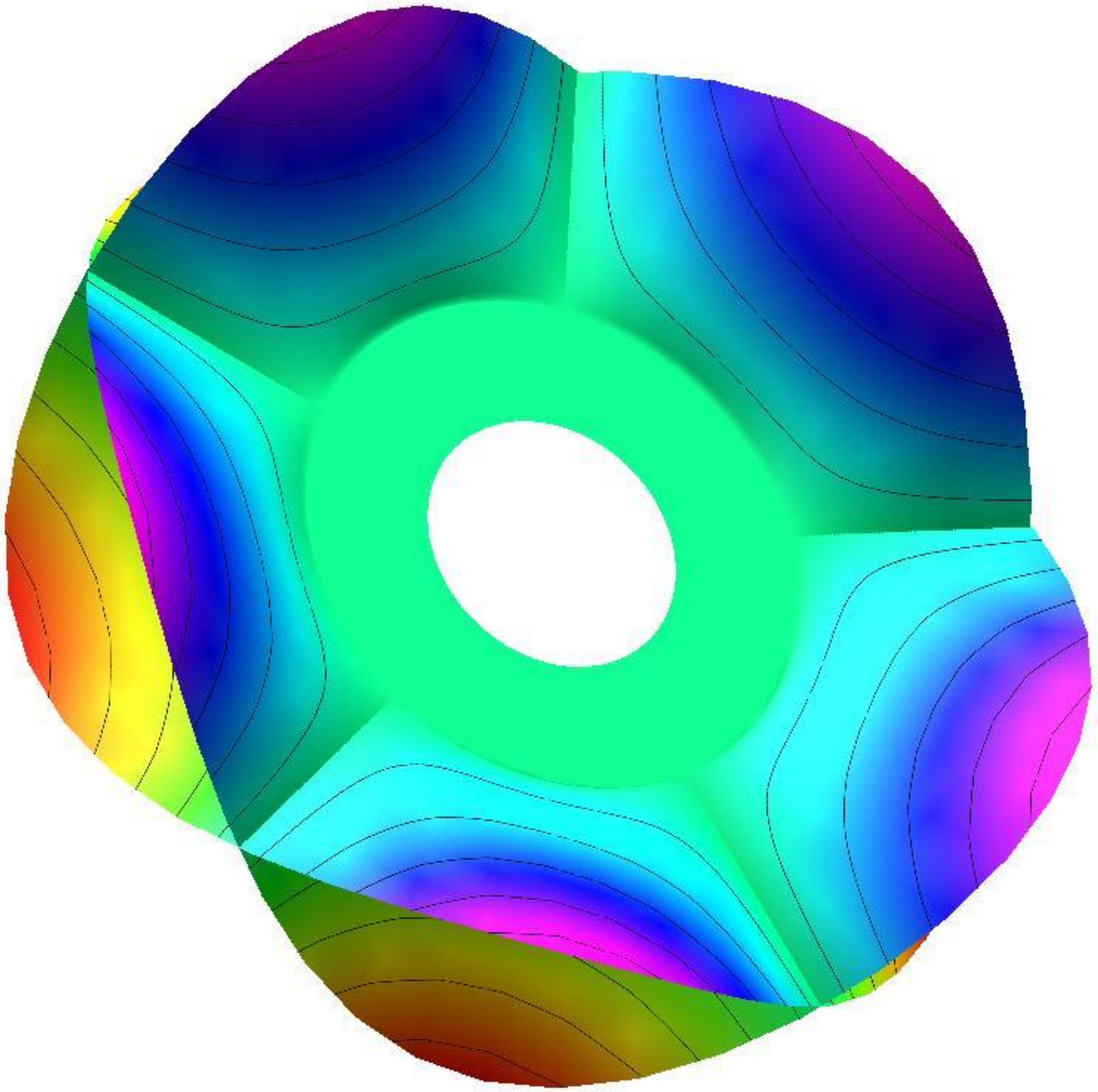
Nonabelian Hodge Spaces and nonlinear representation theory



P. Boalch, CNRS Orsay

(new parts are joint with
D. Yamakawa and/or R. Paluba)

Σ smooth algebraic curve $/\mathbb{C}$

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connections
on vector bundles $/\Sigma$ $\xleftrightarrow{RH}$ π_1 rep.s $\rightsquigarrow$ symplectic manifolds
with regular singularities

"character varieties"
(Atiyah-Bott / Goldman)

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$$\begin{tikzcd}
\begin{array}{c}\text{connections}\\ \text{on vector bundles }/\Sigma\\ \text{with regular singularities}\end{array} \arrow[r, leftrightarrow, "RH"] \arrow[d, phantom, "\cap" description] & \pi_1 \text{ rep.s} \arrow[r, rightsquigarrow] \arrow[d, phantom, "\cap" description] & \begin{array}{c}\text{symplectic manifolds}\\ \text{``character varieties''}\\ \text{(Atiyah-Bott / Goldman)}\end{array} \\
\begin{array}{c}\text{connections}\\ \text{on vector bundles}/\Sigma\end{array} \arrow[r, leftrightarrow, "RHB"] & \begin{array}{c}\text{Stokes \&}\\ \text{monodromy data}\end{array} \arrow[r, rightsquigarrow] & \begin{array}{c}\text{symplectic manifolds}\\ \text{``wild character varieties''}\\ \text{(B. '99 - '14, B.-Yamakawa '15)}\end{array}
\end{tikzcd}$$

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$\cap$ $\cap$

connections on vector bundles$/\Sigma$ $\xleftrightarrow{RHB}$ Stokes & monodromy data $\rightsquigarrow$ symplectic manifolds "wild character varieties" (B. '99 - '14, B.-Yamakawa '15)

- Hitchin 1987: complex character var.s are hyperkahler
- Biquard-B. '01: wild ———

$\}$ $\rightsquigarrow$ they admit special lagrangian fibrations

The Lax project

Try to classify integrable systems with nice properties

- finite dimensional complex algebraic completely integrable Hamiltonian system (M, X)
- admits a good Lax representation (any genus)

upto isomorphism (isogeny, deformation, ...)

Then look at different representations of each one

The Lax project

E.g. Look at isospectral deformations of rational matrix

$$A(z)$$

$$\mathcal{X} = \det(A(z) - \lambda) \quad \rightsquigarrow \text{ spectral curve}$$

$$\mathcal{M}^* = \{ A \mid \text{orbits of polar parts fixed} \} / G \quad \text{symplectic}$$

- lots of examples of such integrable systems

Jacobi, Garnier, ...

The Lax project

Hitchin systems (fix $G = GL_n(\mathbb{C})$, Σ compact Riemann surface)

$$\begin{tikzcd}
T^*\mathrm{Bun}_G \arrow[d, phantom, "\cap"] & = \{(V, \Phi) \mid V \text{ stable}, \Phi \in H^0(\mathrm{End}\, V \otimes \Omega^1)\}/_{\text{iso.}} \\
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(Higgs bundles)

The Lax project

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①

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(Higgs bundles)

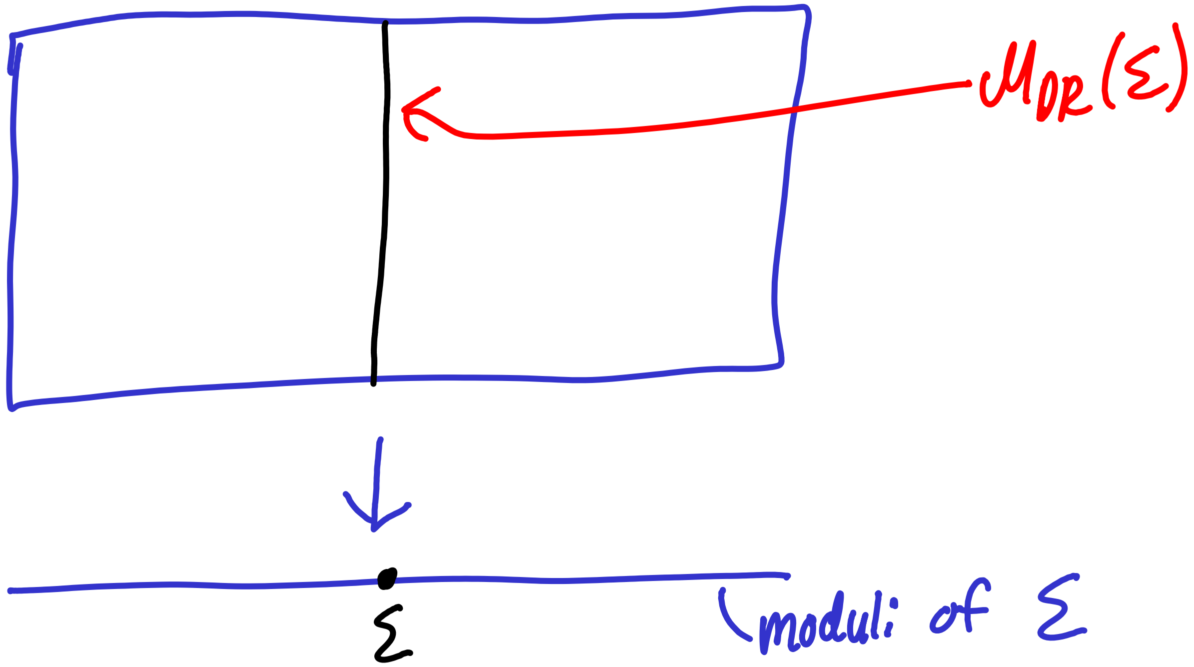
② Hyperkahler:

$$\mathcal{M}_{Dol} \overset{\substack{\text{nonabelian}\\ \text{Hodge}}}{\cong} \mathcal{M}_{DR} \overset{RH}{\cong} \mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

Higgs Connections Character variety

The Lax project

Vary $\Sigma \rightsquigarrow$ isomonodromy connection on spaces of connections



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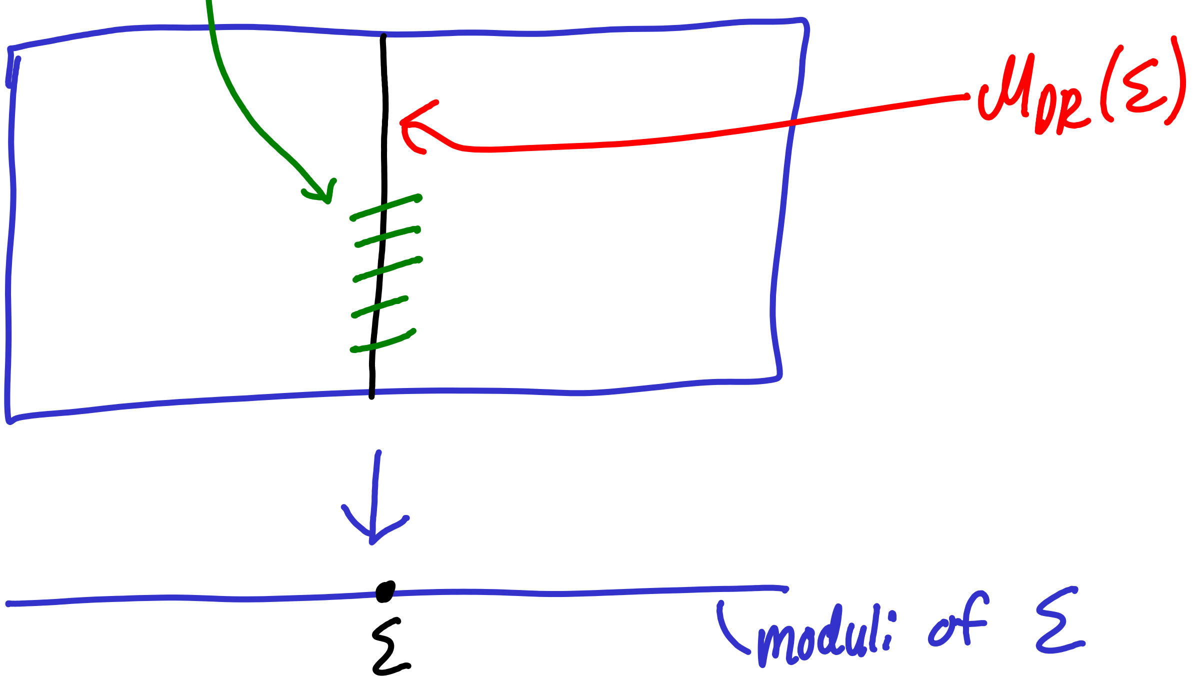

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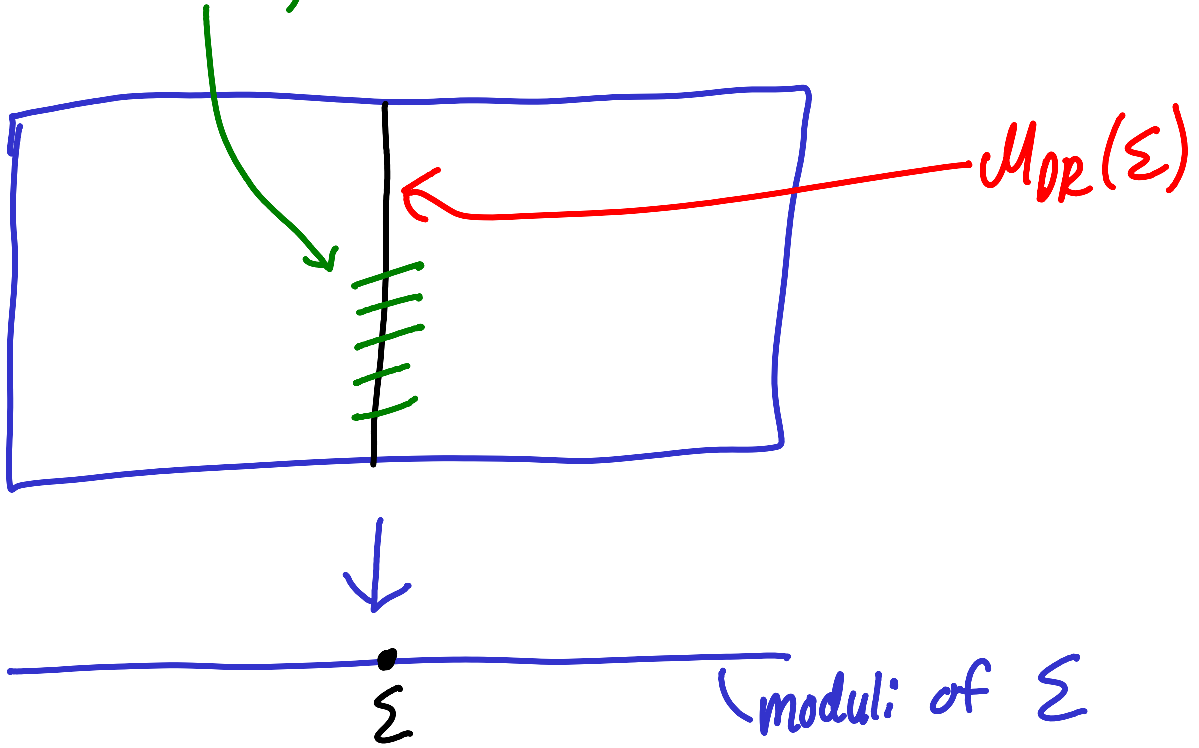

- Classify both ACIHS & isomonodromy systems at same time

(i.e. classify hyperkahler manifolds with such extra structure)

The Lax project

Definition

A "nonabelian Hodge space" is a hyperkahler manifold $\mathcal{M}$ with three preferred algebraic structures $\mathcal{M}_{Dol}$, $\mathcal{M}_{DR}$, $\mathcal{M}_B$ such that $\mathcal{M}_{Dol}$ is an integrable system with a Lax representation

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The Lax project

Back to rational matrices:

- $A(z)\,dz$ is a meromorphic Higgs field (V trivial)
- $d - A(z)\,dz$ is a meromorphic connection (V trivial)

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Theorem Moduli spaces of meromorphic Higgs bundles often have such structure

- Nitsure, Bottacin, Markman ~'95 ACIHS in Poisson sense
- PB. '99 Symplectic forms on $\mathcal{M}_{DR} \cong \mathcal{M}_B$ (mero. Atiyah-Bott/Goldman)
- Biquard-B. '01 Hyperkahler structure
- Algebraic approach to symplectic forms: Woodhouse '00, Krichever '01, B.'02, 09, 11, B.-Yamakawa '15

The Lax project

$$\underset{\text{mero. Higgs}}{\mathcal{M}_{Dol}} \overset{\substack{\text{wild}\\ \text{nonabelian Hodge}}}{\cong} \underset{\text{mero. Connections}}{\mathcal{M}_{DR}} \overset{\text{RHB}}{\cong} \underset{\text{wild character variety}}{\mathcal{M}_B} = \{\text{monodromy \& Stokes data}\}$$

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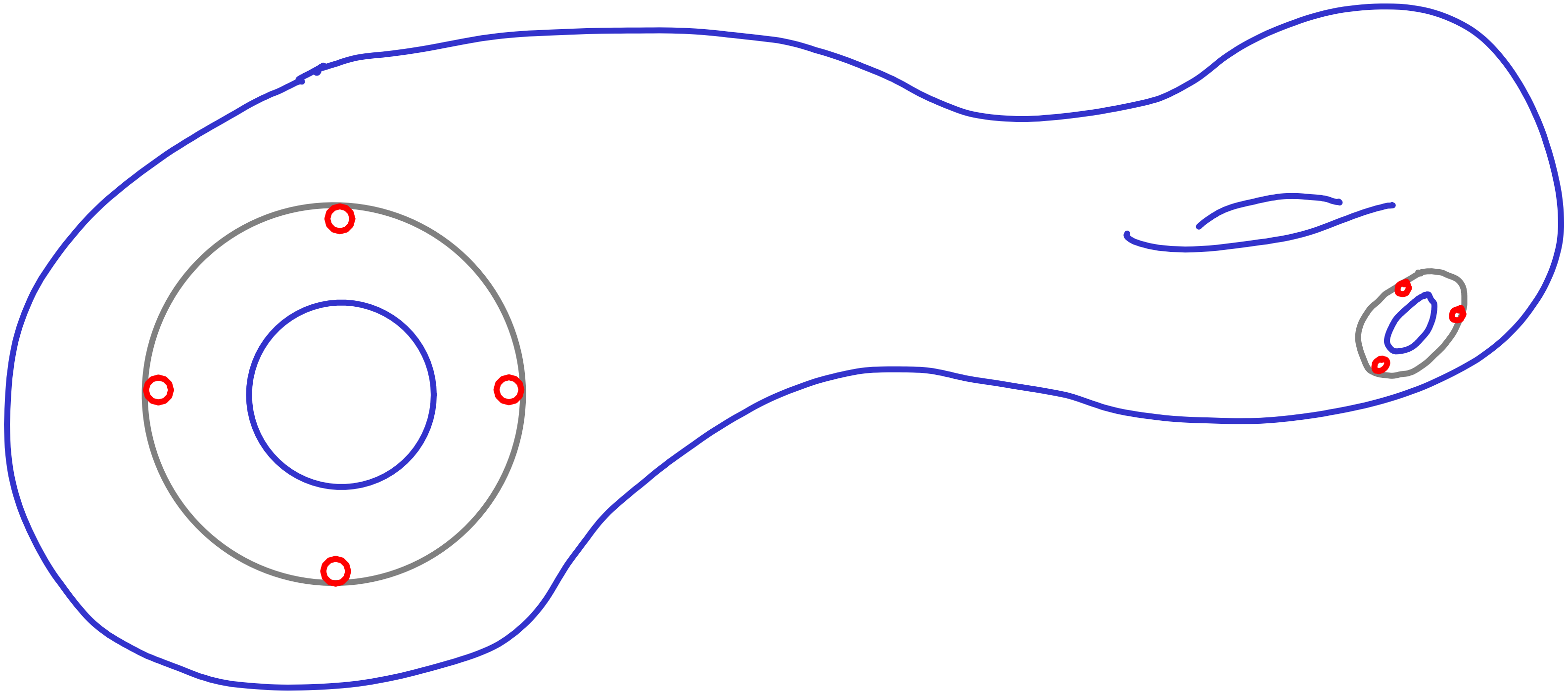
mero. Higgs mero. Connections wild character variety



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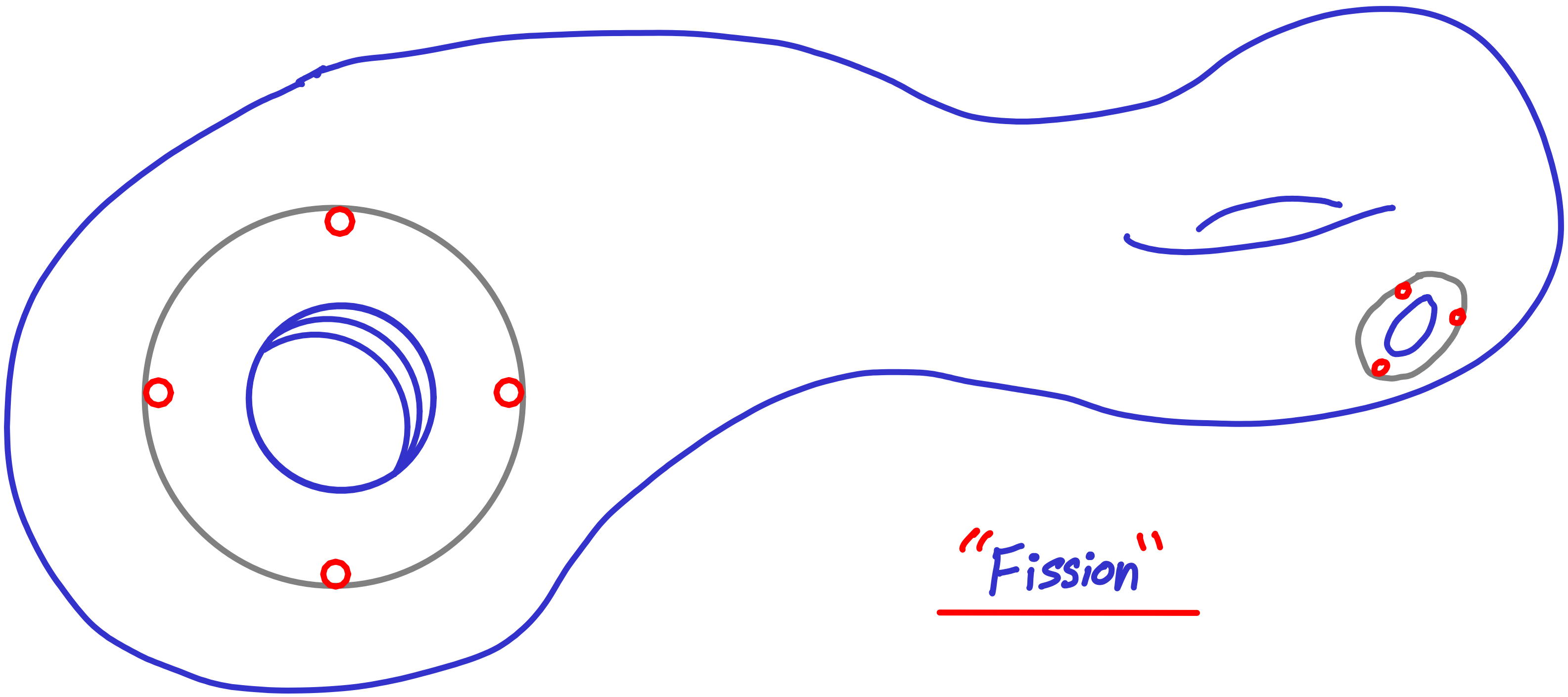

"Fission"

Example	Higgs Integrable system	Connections Isomonodromy system	Monodromy/ Stokes
Φ	$\mathcal{M}_{Dol}$	$\mathcal{M}_{DR}$	$\mathcal{M}_B$
$(A_1 + A_2 z)\frac{dz}{z}$	Manakov	Dual Schlesinger	G^*

Example

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Duality:

$$A + P(z-B)^{-1}Q \quad \longleftrightarrow \quad B + Q(z-A)^{-1}P \qquad \text{(up to signs)}$$

AHH, Harnad

Fourier-Laplace

Example

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$$\begin{tikzcd}
(A_1 + A_2 z)\frac{dz}{z} \arrow[dd, bend right, "sl_3"'] & \\
\sum \frac{A_i}{z-a_i} dz \arrow[d, "4\text{poles } gl_2"] & \\
\text{Painlevé 6} &
\end{tikzcd}$$

$\mathcal{M}_B \cong$ Fricke-Klein-Vogt surface

$$xyz + x^2 + y^2 + z^2 + ax + by + cz = d$$

(Hyperkähler four manifold)

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$\mathfrak{sl}_3$: $(A_1 + A_2 z)\frac{dz}{z} \longrightarrow$ Painlevé 6

4 poles $\mathfrak{gl}_2$: $\sum \frac{A_i}{z-a_i} dz \longrightarrow$ Painlevé 6

Painlevé 6 $\quad \mathcal{M}_B \cong$ Fricke-Klein-Vogt surface

$$xyz + x^2+y^2+z^2 + ax+by+cz = d$$

$$\cong \quad \mathcal{C}/\!/T \ , \quad \mathcal{C} \subset SL_3^* \ , \quad \dim \ 6-2\cdot 2 = 2$$

$$\cong \quad \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3 \times \mathcal{C}_4 /\!/ GL_2 \ , \quad \dim \ 4\cdot 2 - 2\cdot 3 = 2$$

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$sl_3 \downarrow \qquad \downarrow$ 4 poles gl_2

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$\cong \quad \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}_\infty /\!/ G_2 \qquad \dim \ 3\cdot 6 + 12 - 2\cdot 14 = 2 \quad (a=b=c)$

└ G_2 representation of Painlevé VI (B.– Paluba, JAG '16)

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$\mathcal{M}_B \cong$ Flaschka-Newell surface

$$xyz + x + y + z = b - b^{-1} \qquad b \in \mathbb{C}^*$$

(New hyperkahler 4manifold, via Biquard-B. '01)

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⋮			

Dynkin diagrams

Okamoto ('80s):

P_6 has D_4 affine Weyl group symmetry

P_2 – A_1 ————————

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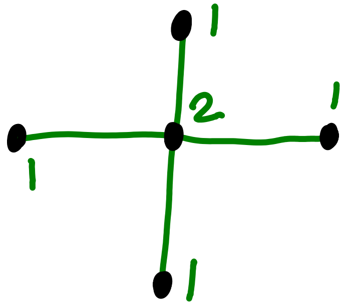
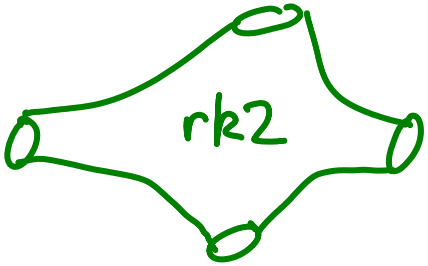
P_6





$\mathcal{M}^* \cong D_4$ ALE space / quiver variety $\hookrightarrow \mathcal{M}_{DR} \cong \mathcal{M}_B$

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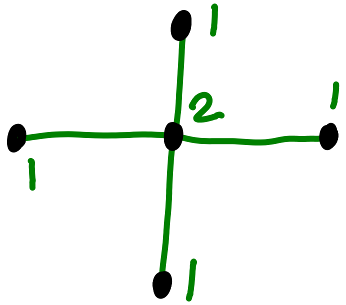

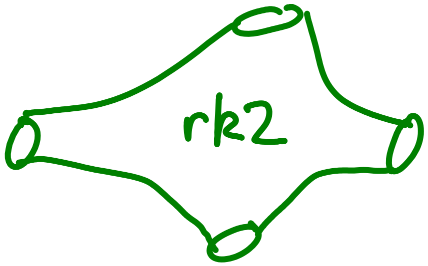

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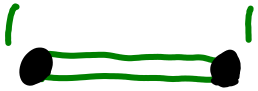
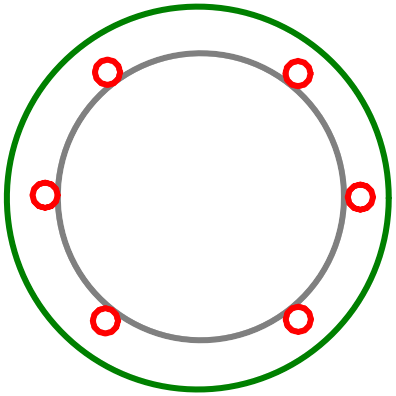
P_2





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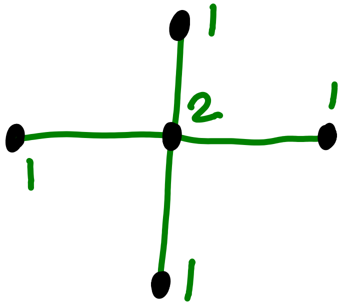

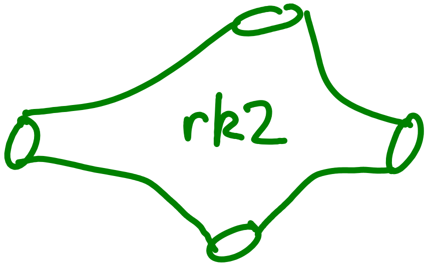

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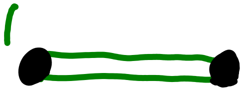

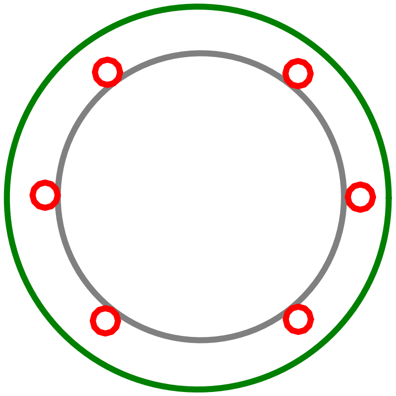

$\mathcal{M}^* \cong A_1$ ALE space / Eguchi-Hanson $\hookrightarrow \mathcal{M}_{DR} \cong \mathcal{M}_B$

(Ex.3, 0706.2634)

Spaces from graphs/quivers

$\Gamma =$ 

$I = \{\text{nodes}(\Gamma)\}$

Spaces from graphs/quivers

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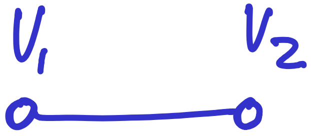

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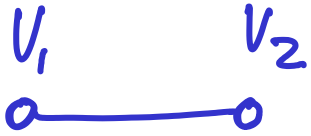

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$V = V_1 \oplus V_2$ (I graded complex vector space)

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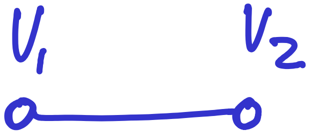
$\Gamma =$ 

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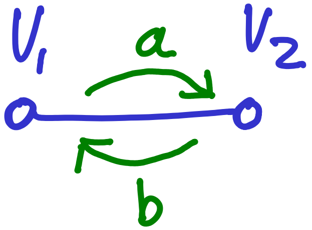

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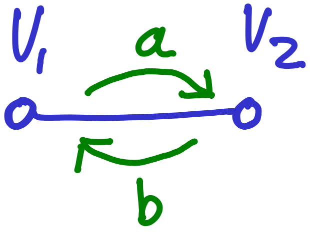

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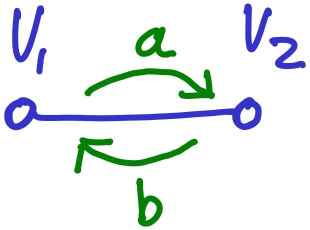

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$H := GL(V_1) \times GL(V_2)$ acts on $\mathrm{Rep}(\Gamma, V)$

with moment map $\mu(a,b) = (ab, -ba)$

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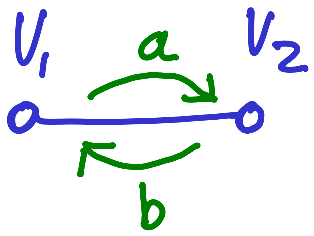

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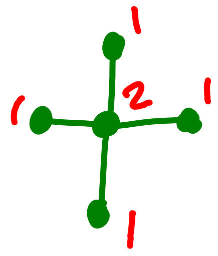
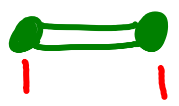
Additive/Nakajima quiver variety : $\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H = \mu^{-1}(\lambda)/H \quad (\lambda \in \mathbb{C}^I \subset \mathrm{Lie}(H)^*)$

Spaces from graphs/quivers



Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then $\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ is cx dim^{n} 2



$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

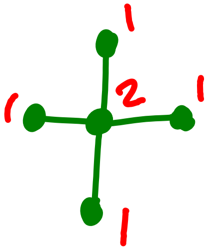
$$\cong T^{*}\mathrm{Hom}(V_1, V_2)$$ (symplectic)

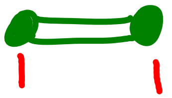
$H := GL(V_1) \times GL(V_2)$ acts on $\mathrm{Rep}(\Gamma, V)$ with moment map $\mu(a,b) = (ab, -ba)$

Additive/Nakajima quiver variety: $\mathrm{Rep}(\Gamma, V) \underset{\lambda}{/\!\!/} H = \mu^{-1}(\lambda)/H \quad \left(\lambda \in \mathbb{C}^{I} \subset \mathrm{Lie}(H)^{*}\right)$

Spaces from graphs/quivers

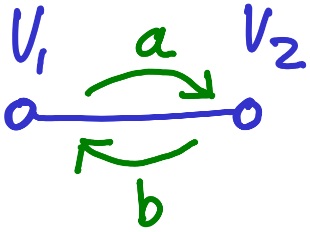
Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then $\mathrm{Rep}(\Gamma, V) /\!/_{\lambda} H$ is cx dim^n 2





Multiplicative version

$\Gamma =$



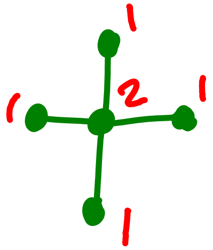
$$\begin{array}{c} \mathrm{Rep}^*(\Gamma, V) = \{(a,b) \mid 1+ab \text{ invertible}\} \\ \cap \\ \mathrm{Rep}(\Gamma, V) \end{array}$$

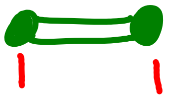
"invertible representations"

Spaces from graphs/quivers

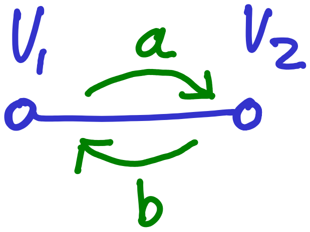
Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then

$\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ is cx dim^n 2





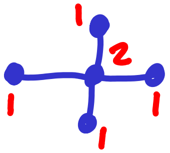
Multiplicative version

$\Gamma =$ 

$$\begin{array}{l} \mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \} \\ \qquad \cap \\ \mathrm{Rep}(\Gamma, V) \end{array}$$

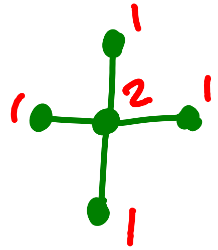
"invertible representations"

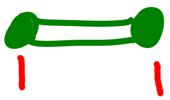
Thm (VandenBergh '04) $\mathrm{Rep}^*(\Gamma,V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = \left(1+ab, (1+ba)^{-1}\right) \in H$

E.g. Mult. Quiver Var.() $\cong \{ xyz + x^2+y^2+z^2 = ax+by+cz+d \}$

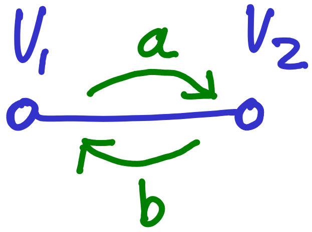
Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then $\mathrm{Rep}(\Gamma, V) /\!\!/_\lambda H$ is cx dim^n 2





Multiplicative version

$\Gamma =$ 

$\mathcal{B}(V_1, V_2) :=$

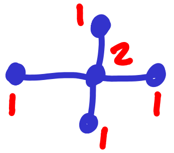
$$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

"invertible representations"

Thm (VandenBergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

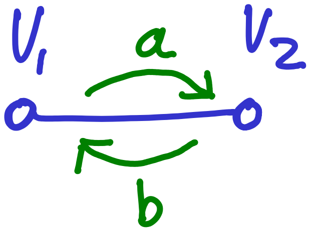
E.g. Mult. Quiver Var.() $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

Qn Suppose $\Gamma =$  or  etc

then what is $\mathrm{Rep}^*(\Gamma, V)$?

Multiplicative version

$\Gamma =$



$\mathcal{B}(V_1, V_2) :=$

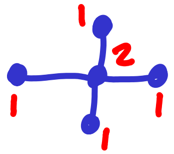
$$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

"invertible representations"

$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

Thm (Van den Bergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

E.g. Mult. Quiver Var. () $\cong \{xyz + x^2+y^2+z^2 = ax+by+cz+d\}$

SPECIMEN
ALGORITHMI SINGVLARIS.

Auctore
L. EVLERO.

1.

Conſideratio fractionum continuarum, quarum vſum vberrimum per totam Analyſin iam aliquoties oſtendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita eſt comparata, vt ſingularem algorithmum requirat. Cum igitur ſumma Analyſeos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) = a$$

$$(a,b) = ab+1$$

$$(a,b,c) = abc+c+a$$

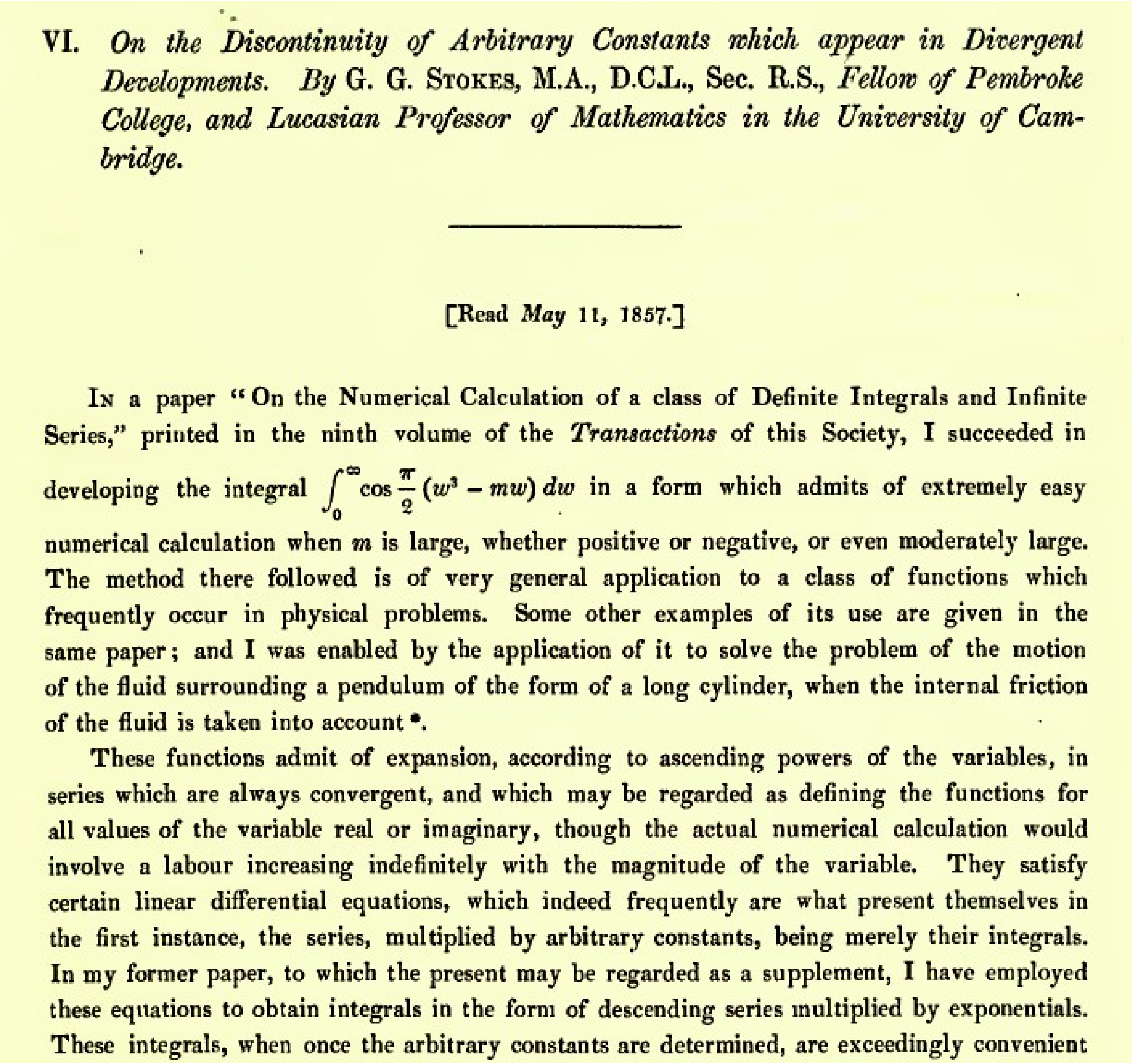
$$(a,b,c,d) = abcd+cd+ad+ab+1$$

$$(a,b,c,d,e) = abcde+cde+ade+abe+abc+e+c+a$$

etc.

"Euler's continuant polynomials"

G. G. Stokes 1857



VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. Stokes, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

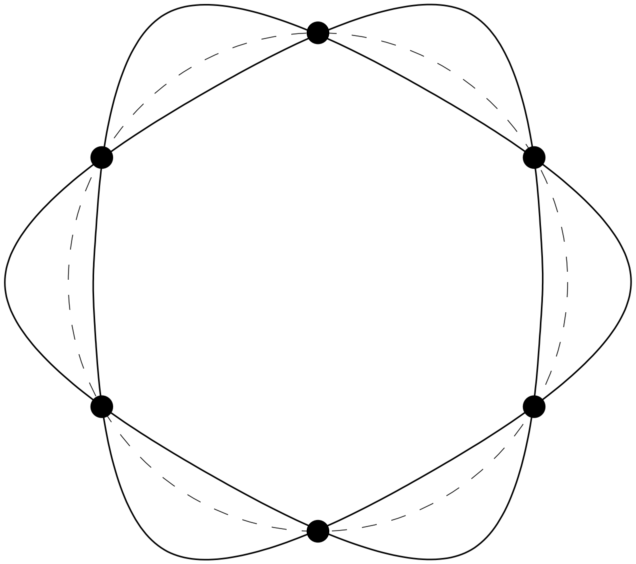
[Read *May* 11, 1857.]

In a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account [*].

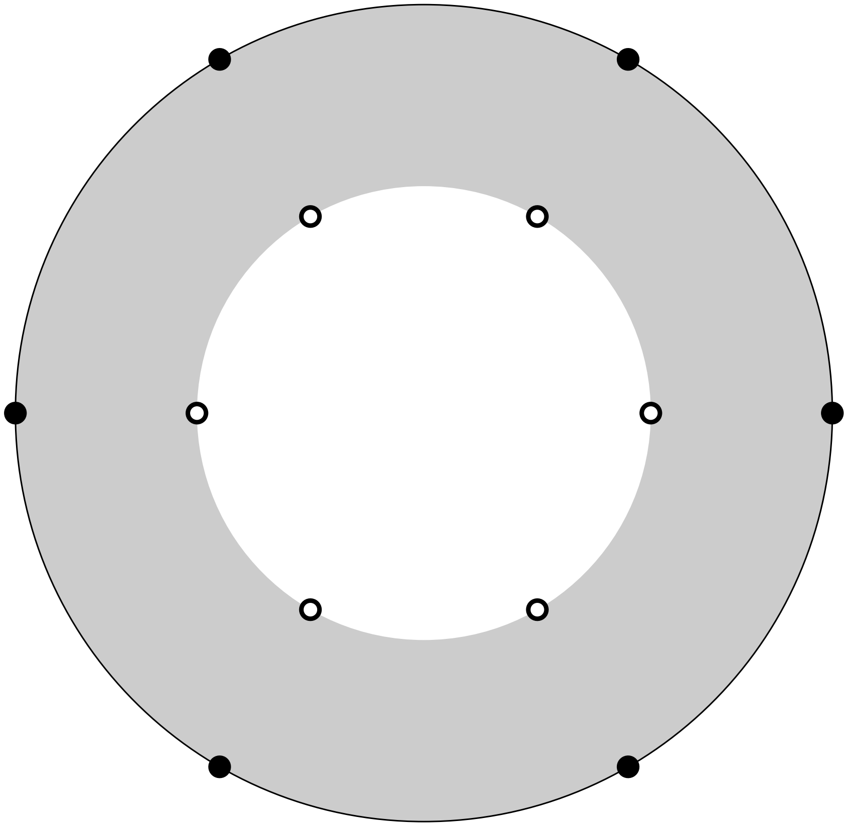
These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



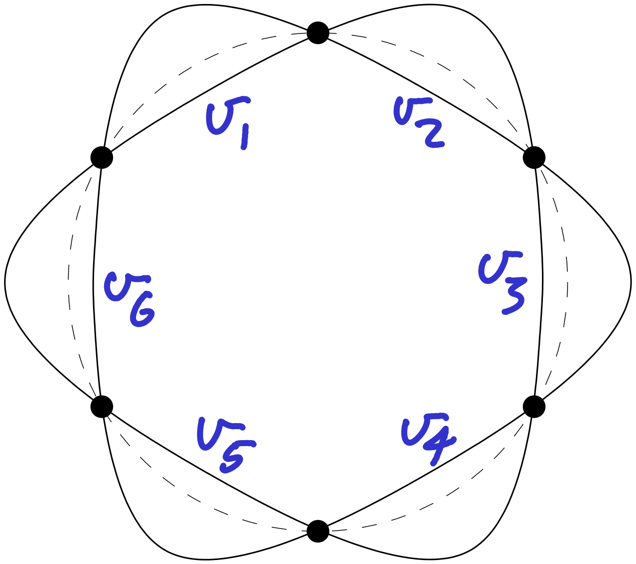
Stokes diagram with Stokes directions



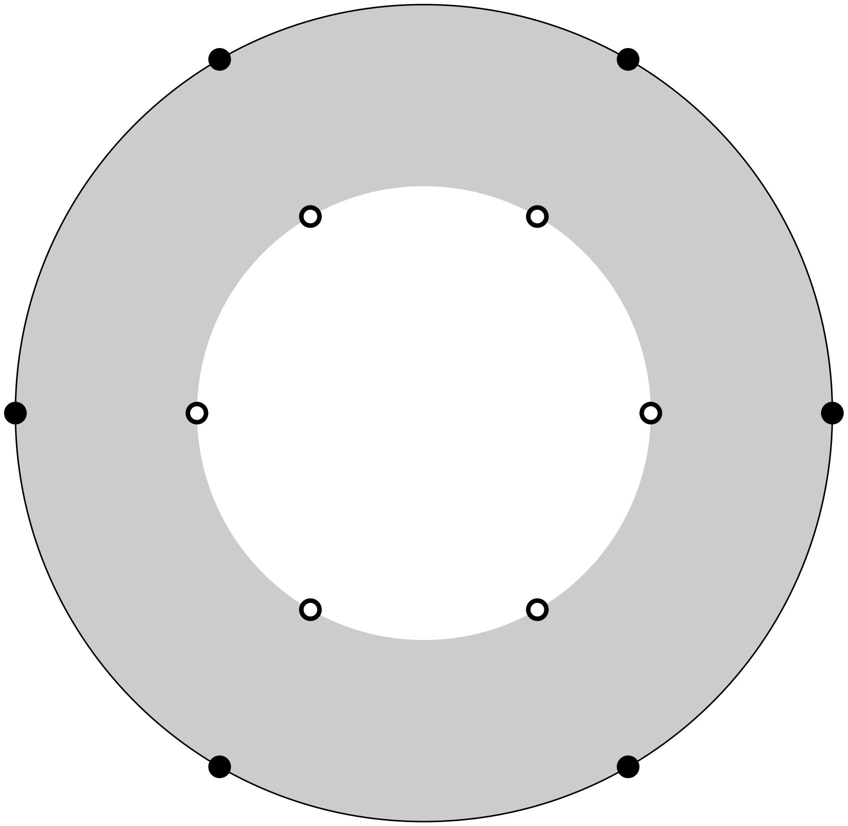
Halo at ∞ with singular directions

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions

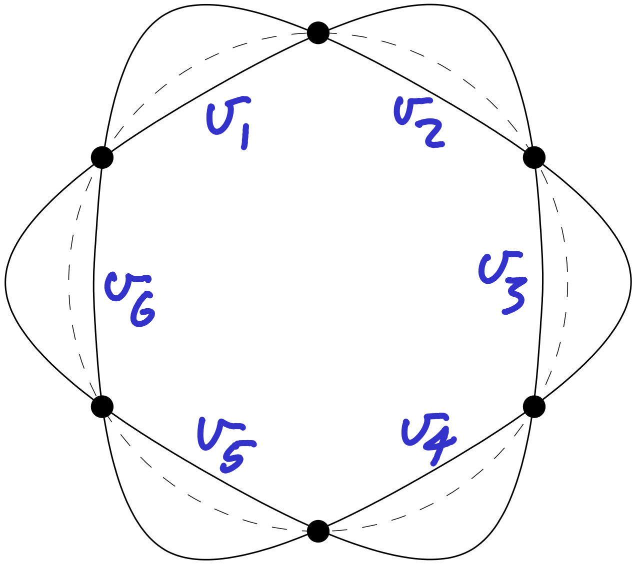


Halo at ∞ with singular directions

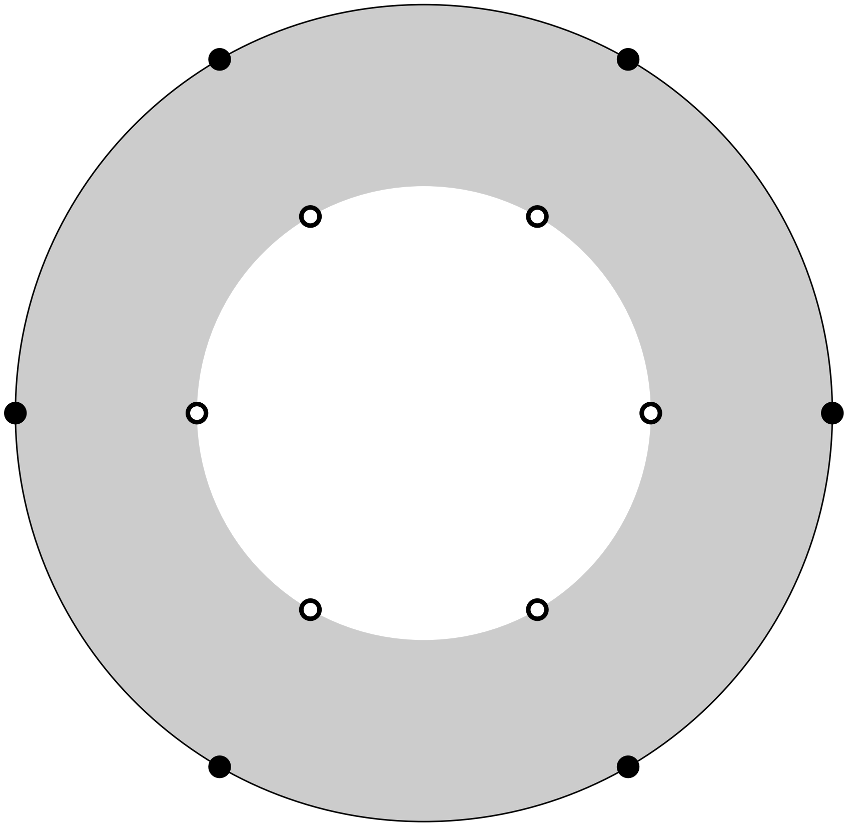
Subdominant solutions $v_i \nparallel v_{i+1}$

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions

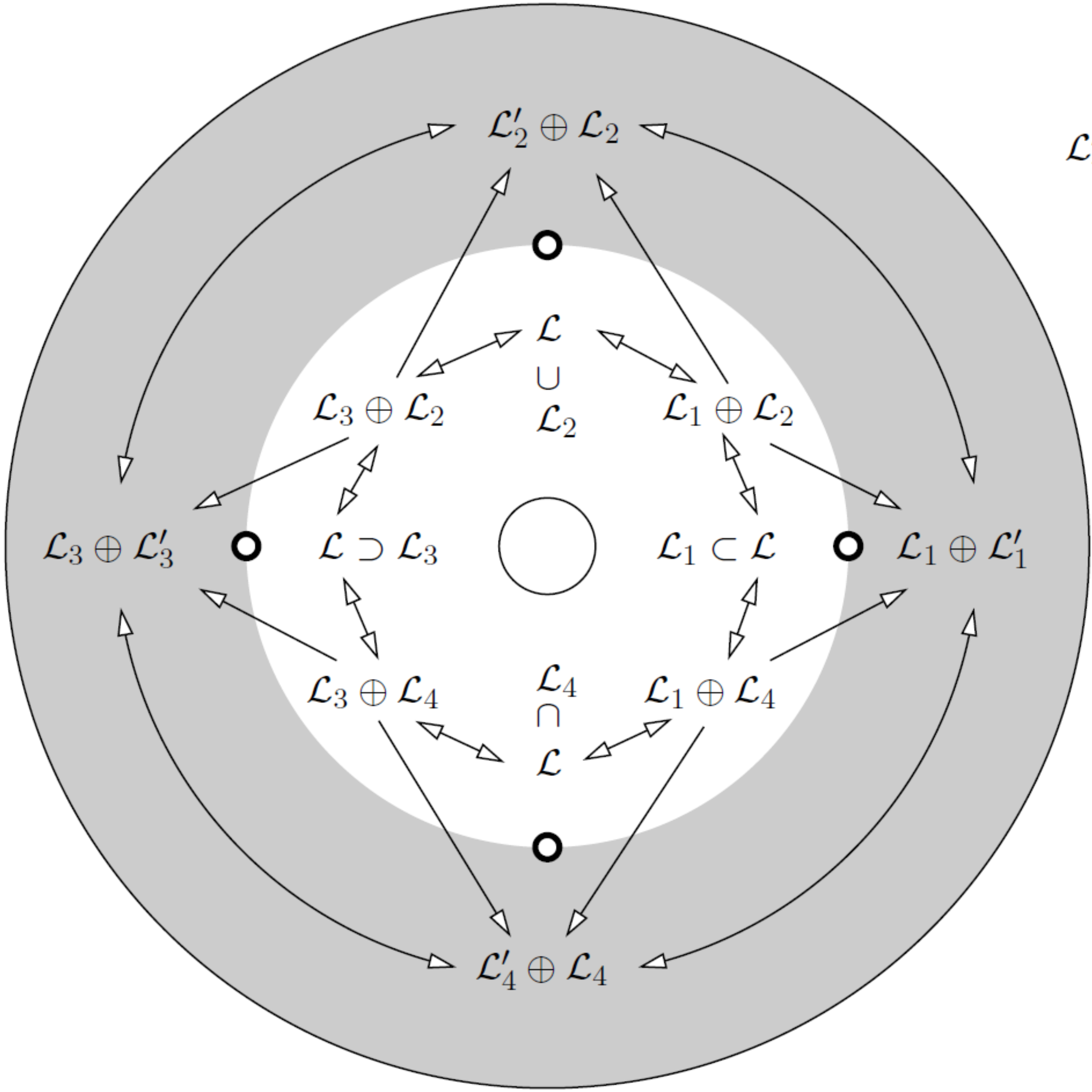


Halo at ∞ with singular directions

Subdominant solutions $\quad v_i \nparallel v_{i+1}$

$$\mathcal{M}_B \cong \{ xyz + x+y+z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \ldots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \bmod 6 \\ \dfrac{(p_1-p_2)(p_3-p_4)(p_5-p_6)}{(p_2-p_3)(p_4-p_5)(p_6-p_1)} = b^2 \end{array} \right\} \Big/ PSL_2(\mathbb{C})$$

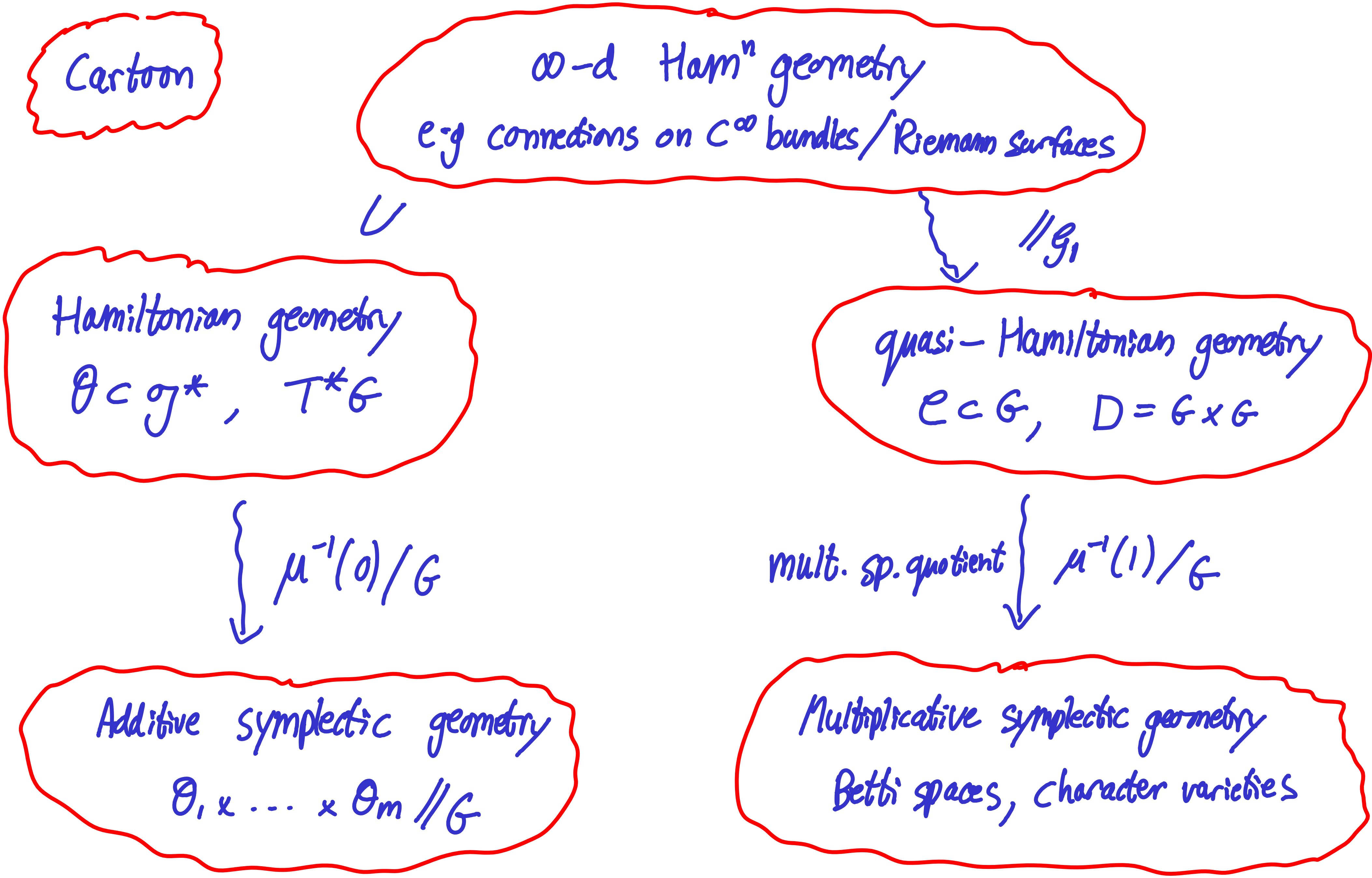


$$\mathcal{L}'_i := \mathcal{L}/\mathcal{L}_i$$

FIGURE 3. Stokes local system from Stokes structure

(1501.00930v4)

Cartoon



∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/\mathcal{G}_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$\mu^{-1}(0)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

quasi-Hamiltonian geometry

$\mathcal{C} \subset G$, $D = G \times G$

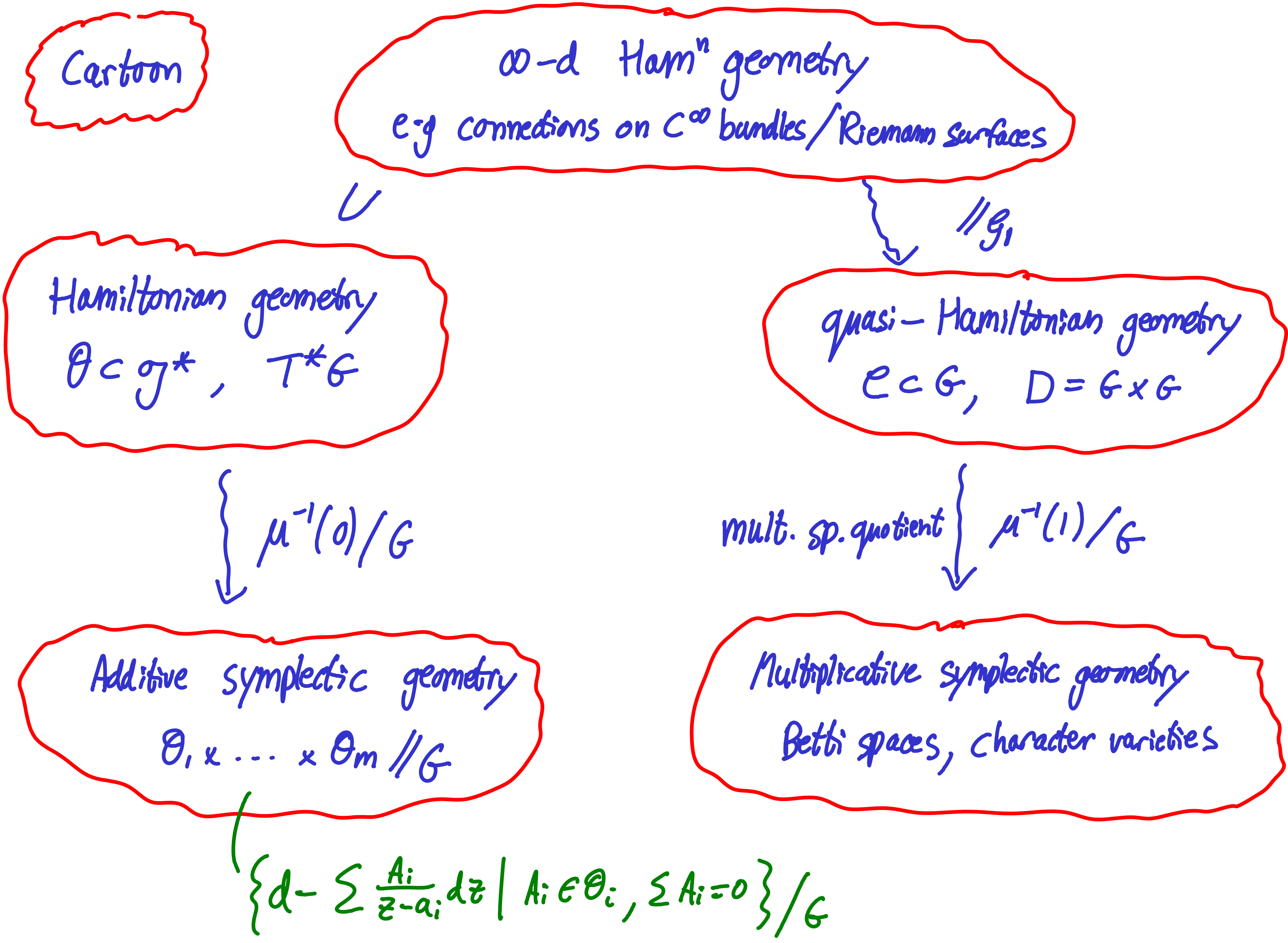
mult. sp. quotient

$\mu^{-1}(1)/G$

Multiplicative symplectic geometry

Betti spaces, character varieties

Cartoon



∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*, \quad T^*G$

$/\!/ \mathcal{G}_1$

quasi-Hamiltonian geometry

$\mathcal{C} \subset G, \quad D = G \times G$

$\mu^{-1}(0)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

mult. sp. quotient

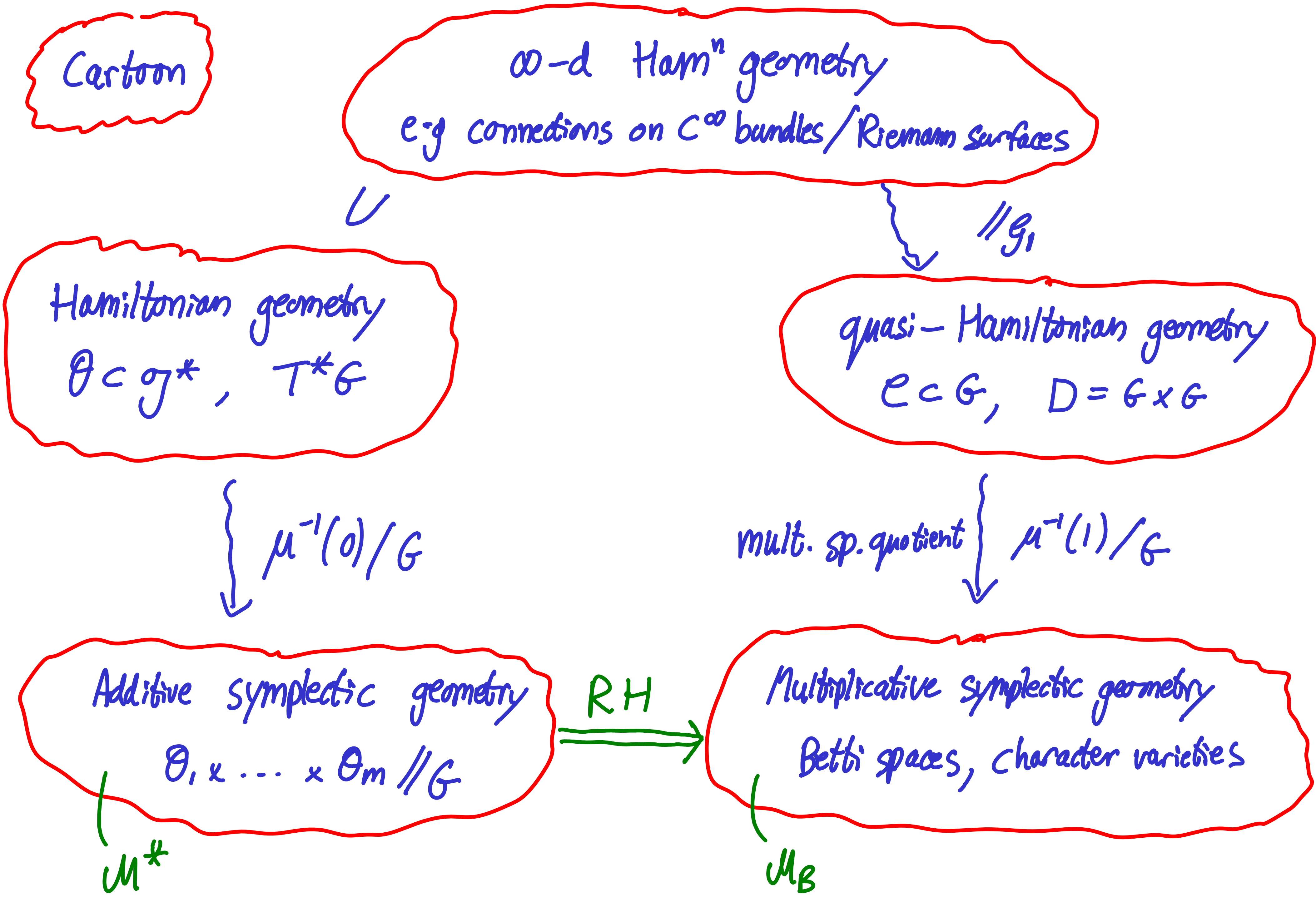
$\mu^{-1}(1)/G$

Multiplicative symplectic geometry

Betti spaces, character varieties

$$\left\{ d - \sum \frac{A_i}{z-a_i} dz \;\middle|\; A_i \in \mathcal{O}_i, \; \sum A_i = 0 \right\} / G$$

Cartoon



∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

U

$//\mathcal{G}_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

quasi-Hamiltonian geometry

$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient

$\mu^{-1}(1)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m // G$

$\mathcal{M}^*$

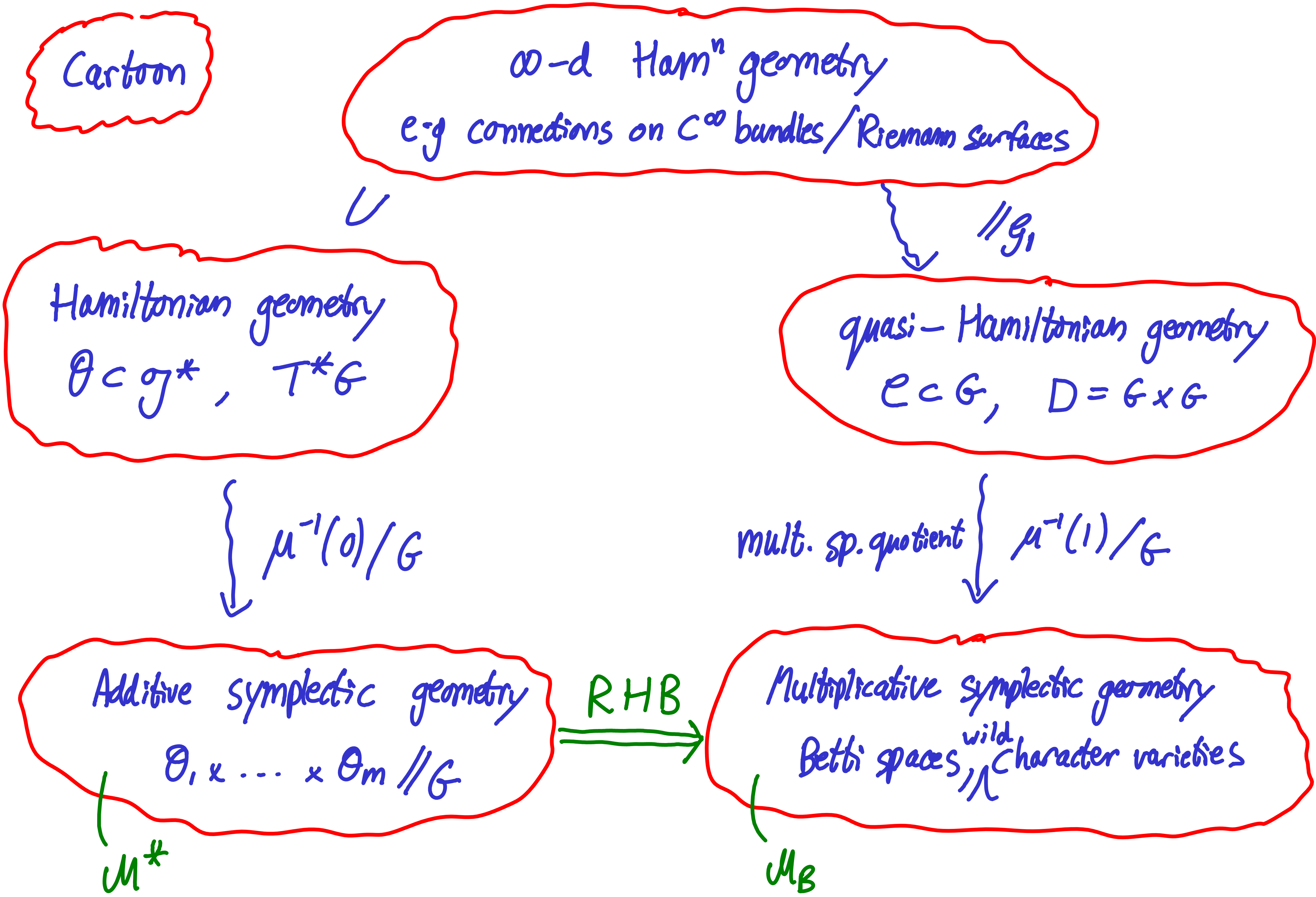
RH

$\Longrightarrow$

Multiplicative symplectic geometry

Betti spaces, character varieties

$\mathcal{M}_B$



Cartoon

∞-d Ham^n geometry

e.g connections on C^∞ bundles/Riemann surfaces

$\cup$

$/\!/\mathcal{G}_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

quasi-Hamiltonian geometry

$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\mu^{-1}(1)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

$\mathcal{M}^*$

RHB $\Longrightarrow$

Multiplicative symplectic geometry

Betti spaces, wild character varieties

$\mathcal{M}_B$

Wild Character Varieties

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface $\Longrightarrow$ $\overset{\text{symplectic variety}}{\mathcal{M}_B} = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

$$\|\!\wr \ \mathrm{RH}$$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

$$\Big\| \wr \ \text{RH}$$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\} / \text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$

$\Sigma^0 = \Sigma \setminus \underset{\sim}{a}$

$\Longrightarrow$

Poisson variety

$$\mathcal{M}_B^{tame} = \mathrm{Hom}(\pi_1(\Sigma^0), G)/G$$

$$\Big\| \wr \ \text{RH}$$

$$\mathcal{M}_{DR}^{naive} = \left\{ \begin{array}{c} \text{Alg. connections on } G\text{-bundles on } \Sigma^0 \\ \text{with reg. sing.s} \end{array} \right\} \Big/ \text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B \underset{\text{RHB}}{\cong} \mathcal{M}_{DR}^{\text{naive}} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ\} / \text{isom}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \ldots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\cong$ RHB

$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B$$
$$\cong \text{RHB}$$
$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$$

Cartan subalg.

$$Q_i \in \mathfrak{t}_i \subset \mathfrak{g}((z_i))$$

Wild Character Varieties

Fix G $(\text{e.g } GL_n(\mathbb{C}))$

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B \; \wr\!\!\mid\!\!\mid \; \text{RHB}$$

$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Cartan subalg.

e.g. $\quad Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i)) \qquad \mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ $\Longrightarrow$ wild character variety

Σ compact Riemann Surface
with marked points
$\underset{\sim}{a} = (a_1, \dots, a_m)$
and irregular types
$\underset{\sim}{Q} = Q_1, \dots, Q_m$

$\Longrightarrow$ $\mathcal{M}_B$

$\wr\wr$ RHB

$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Cartan subalg.

e.g. $Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i))$ $\qquad$ $\mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ $\Longrightarrow$ wild character variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\wr\wr$ RHB

$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

$$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$$

- at least for trivial Betti weights

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ $\Rightarrow$ wild character variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Rightarrow$ $\mathcal{M}_B$

$\wr\wr$ RHB

$$\mathcal{M}_{DR}^{\text{naive}} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

$$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$$

- at least for trivial Betti weights
- in general include parahoric extensions/weights θ

① v.good: $\nabla \cong dQ + \Lambda(z)\frac{dz}{z}$

② good if v.good after some pullback $z = t^r$

$$\left\{ \begin{array}{l} Q \in \mathfrak{t}((z)) \\ \Lambda(z)\frac{dz}{z} \quad \theta\text{-logahoric} \\ \theta \in \mathfrak{t}_{\mathbb{R}} \end{array} \right.$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$

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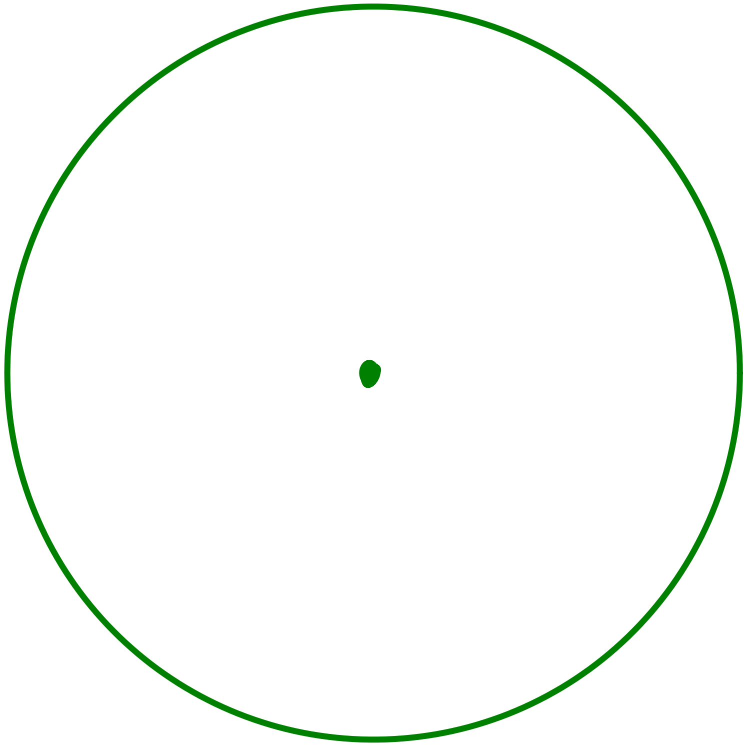

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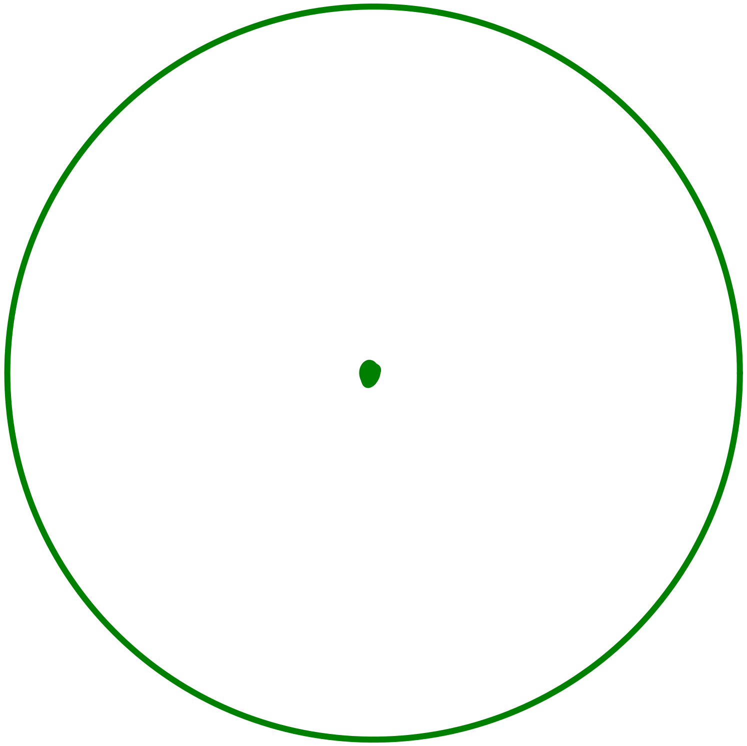

$Q \Rightarrow$

- centraliser group $C_G(Q)$ $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$

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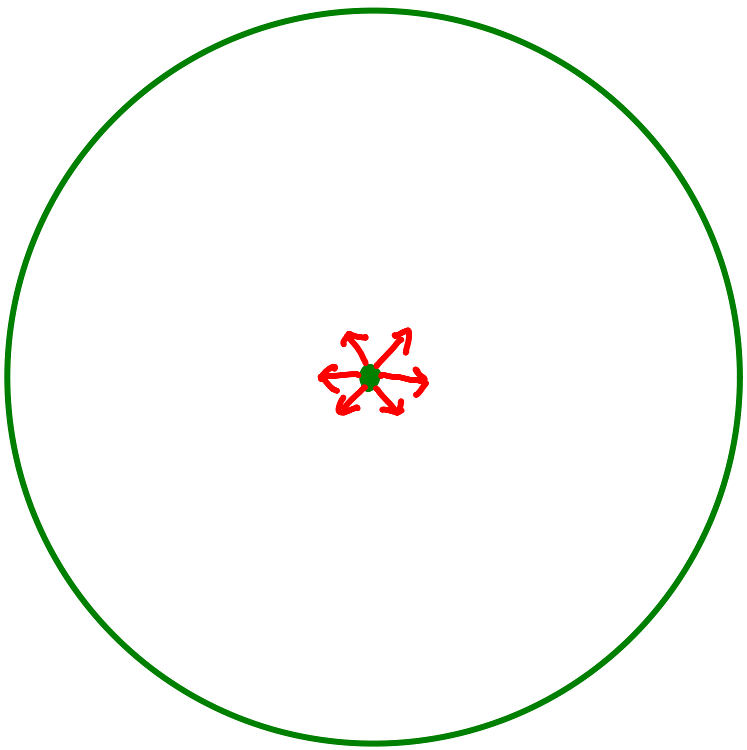

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- singular directions $\mathbb{A}$

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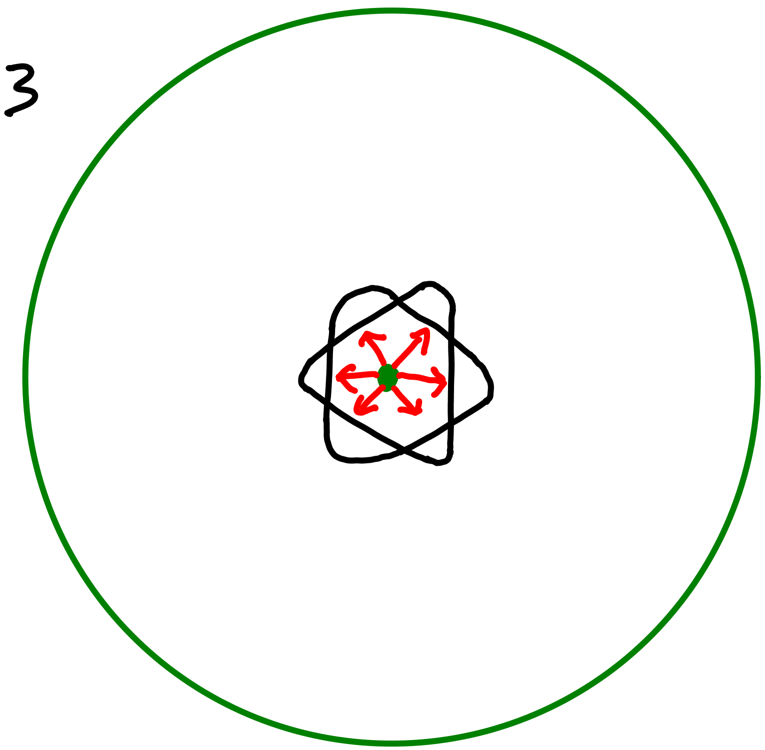
k=3



$Q \Rightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$

Solutions involve $\exp(Q)$

$Q = \operatorname{diag}(q_1, q_2)$

Stokes diagram: plot growth of $\exp(q_1)$, $\exp(q_2)$

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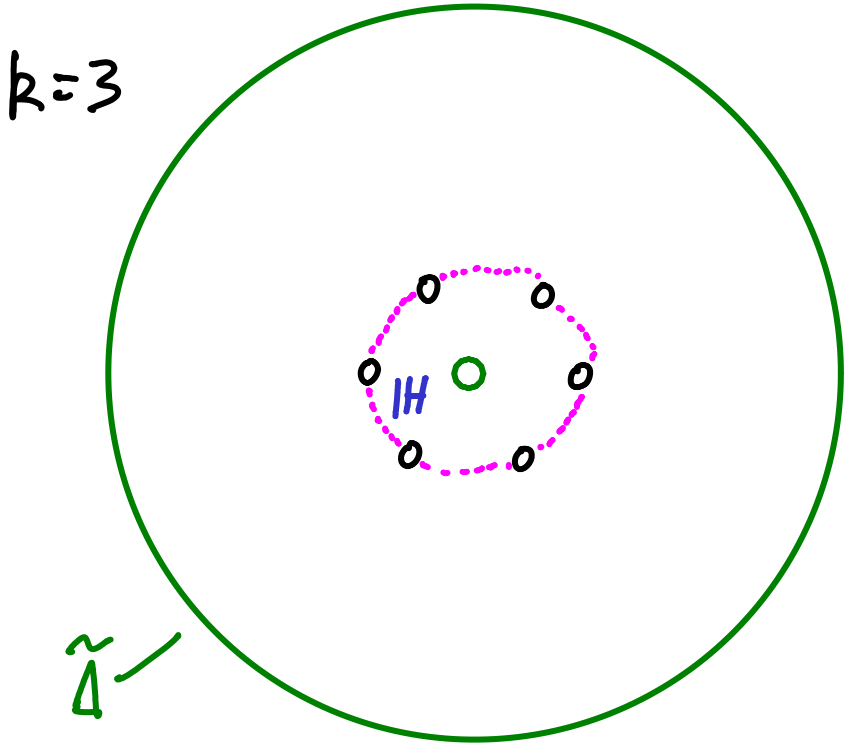

o $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

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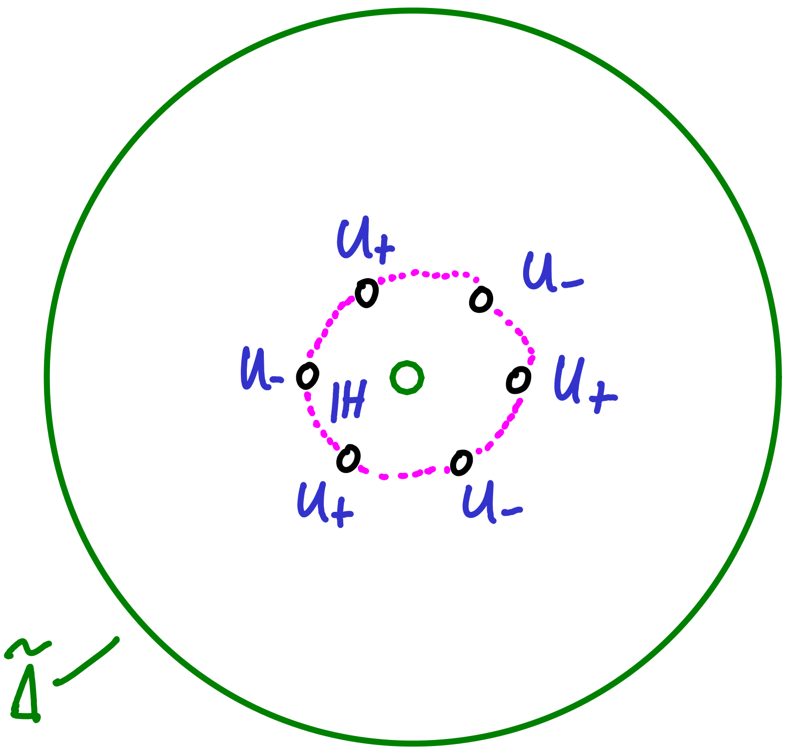

$Q \Rightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$
- Stokes groups $\mathbb{S}to_d \subset G \quad \forall d \in \mathbb{A}$

$\cong U_+$ or U_- here

$\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$

○ $\quad e(d)$ extra punctures

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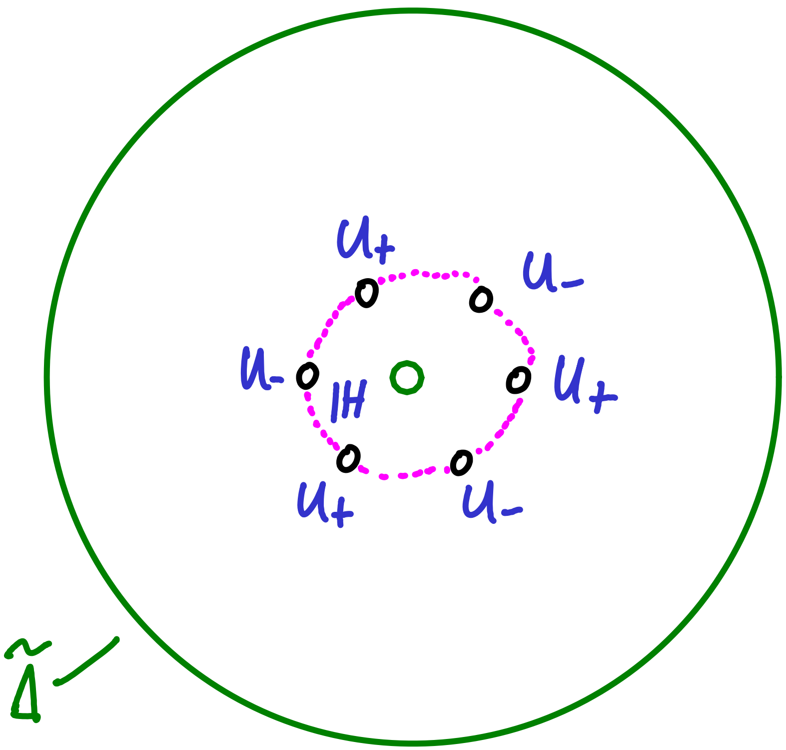

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Stokes local system:

- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in Sto_d

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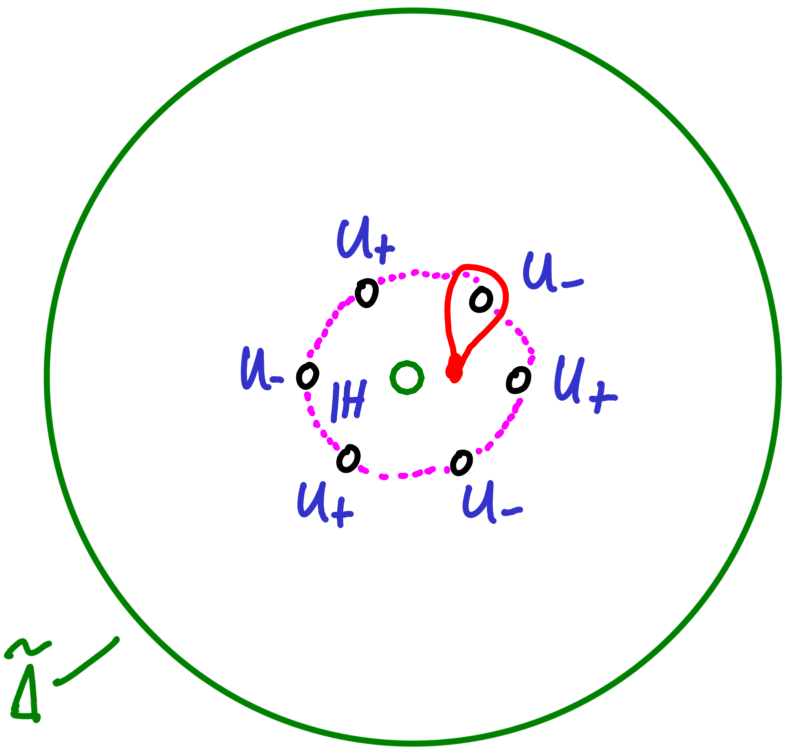

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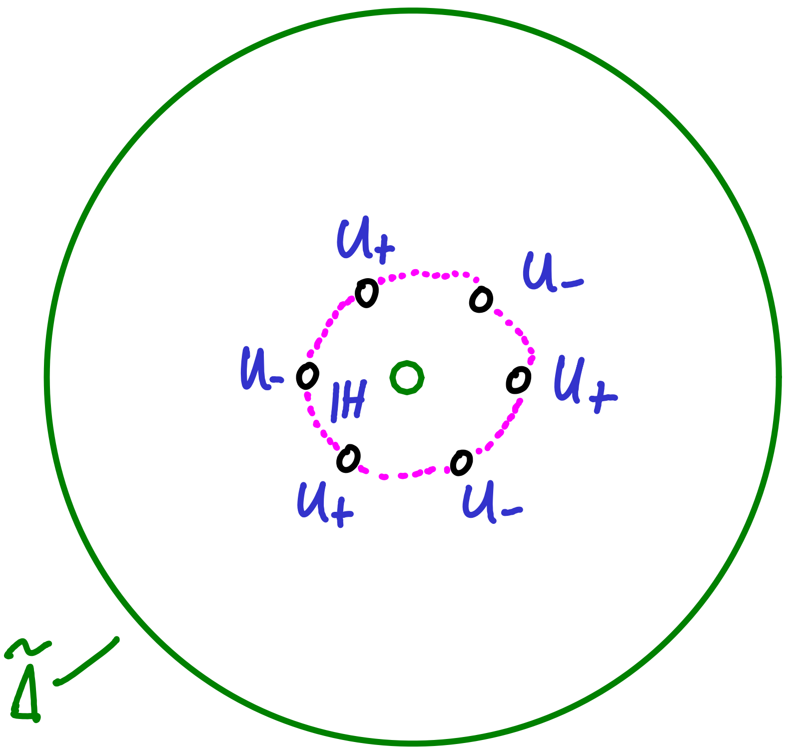

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Stokes local system:

- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}\mathrm{to}_d$

– Topological data that the multisummation approach to Stokes data gives

$$\left\{ \begin{array}{l} \text{Connections with} \\ \text{irreg. type } Q \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Stokes local} \\ \text{systems} \end{array} \right\}$$

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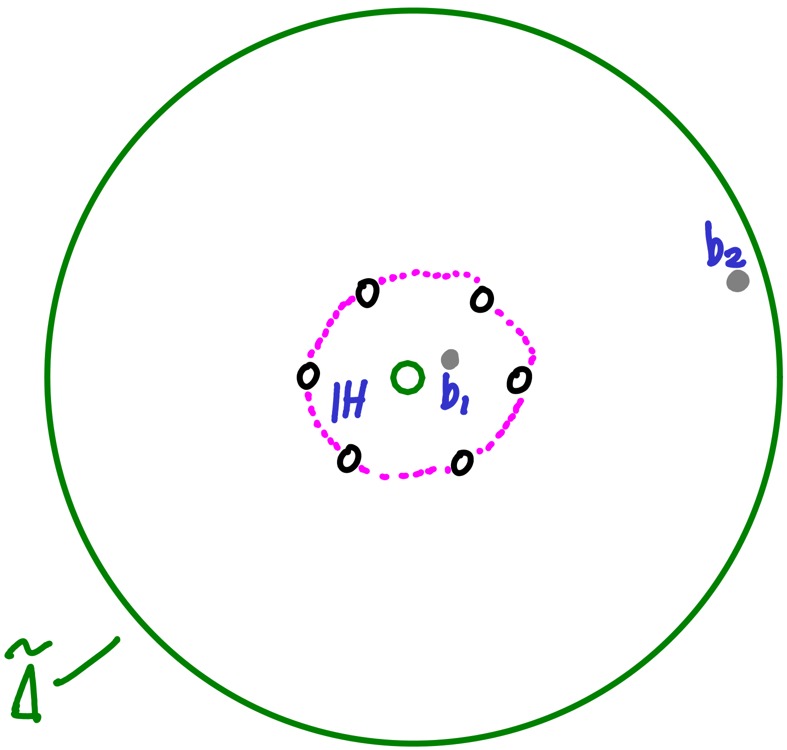
basepoints b_1, b_2



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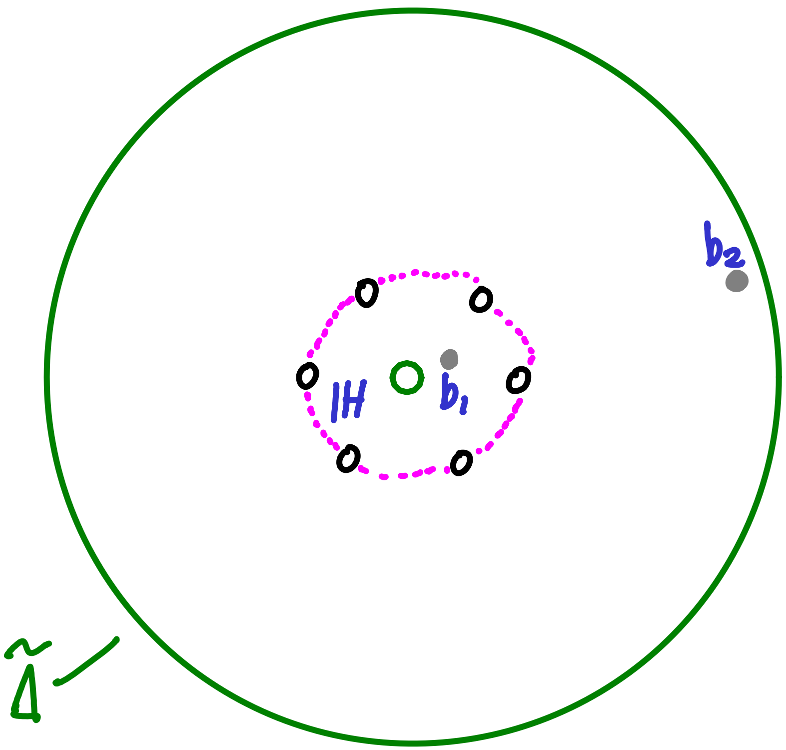

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$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

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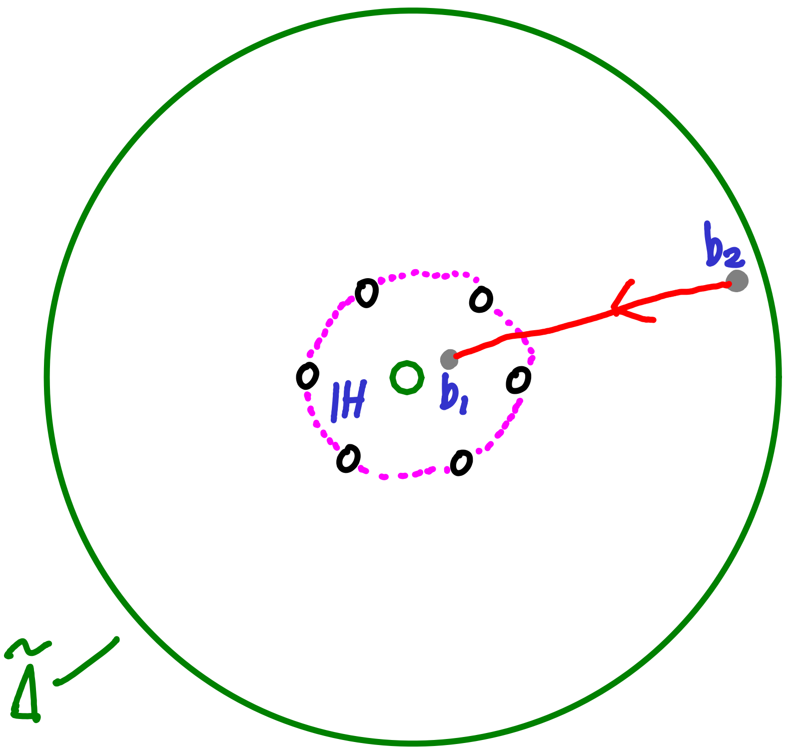

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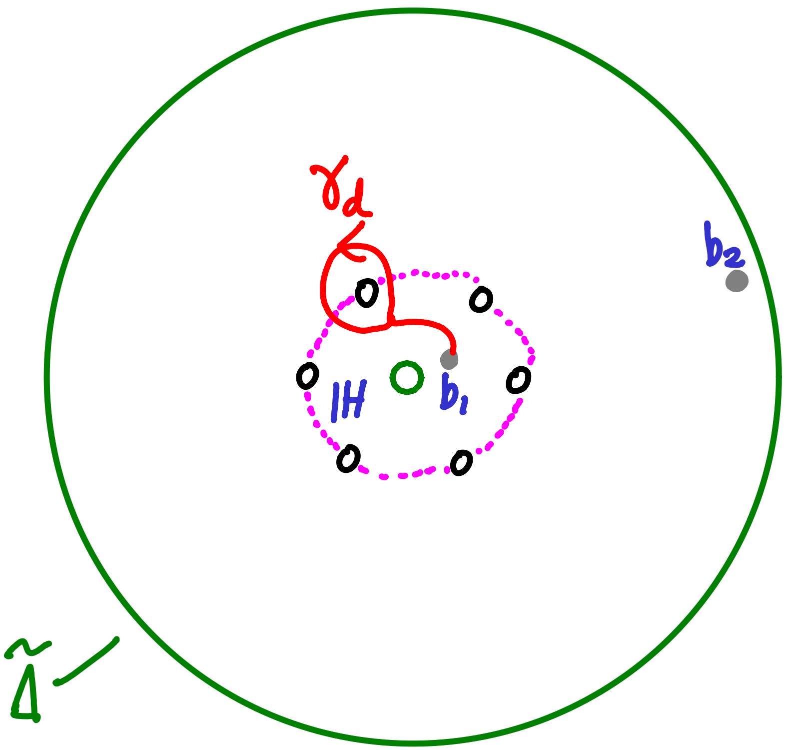

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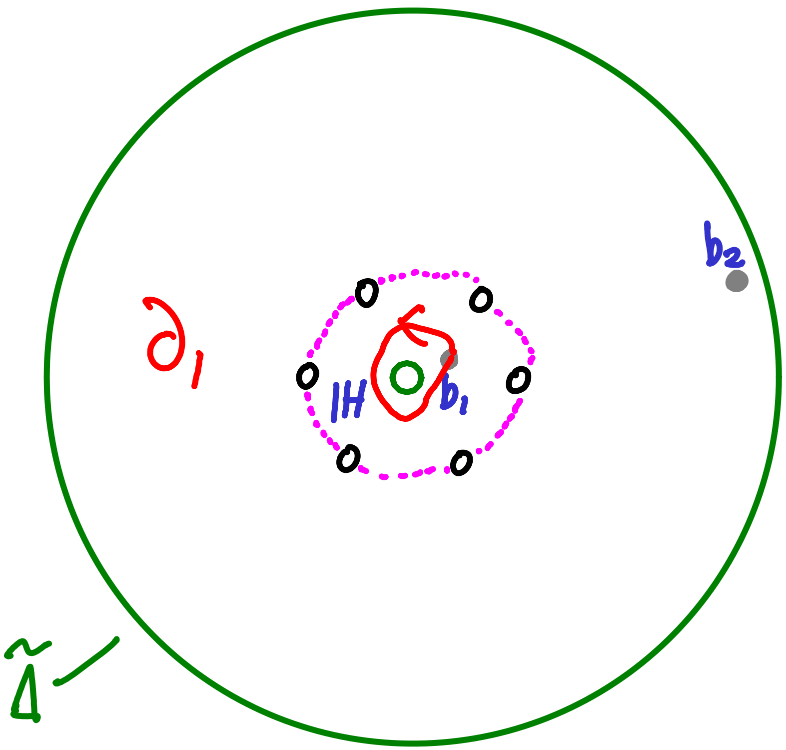

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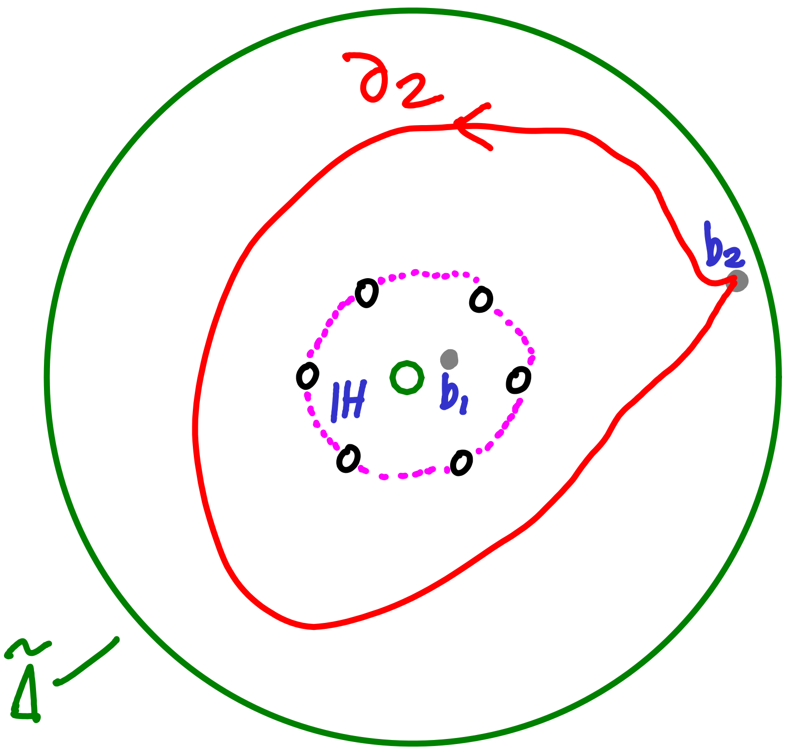

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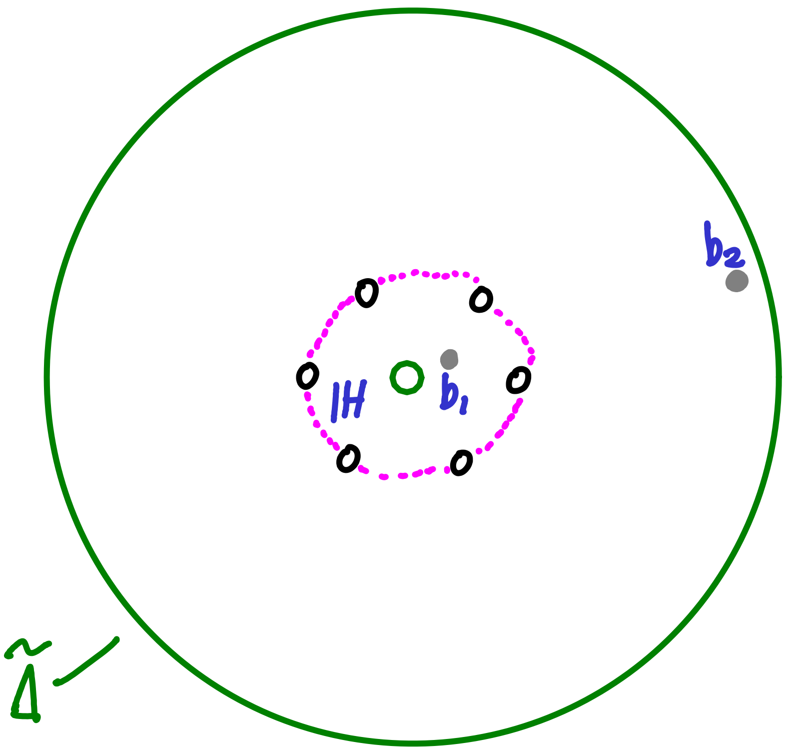

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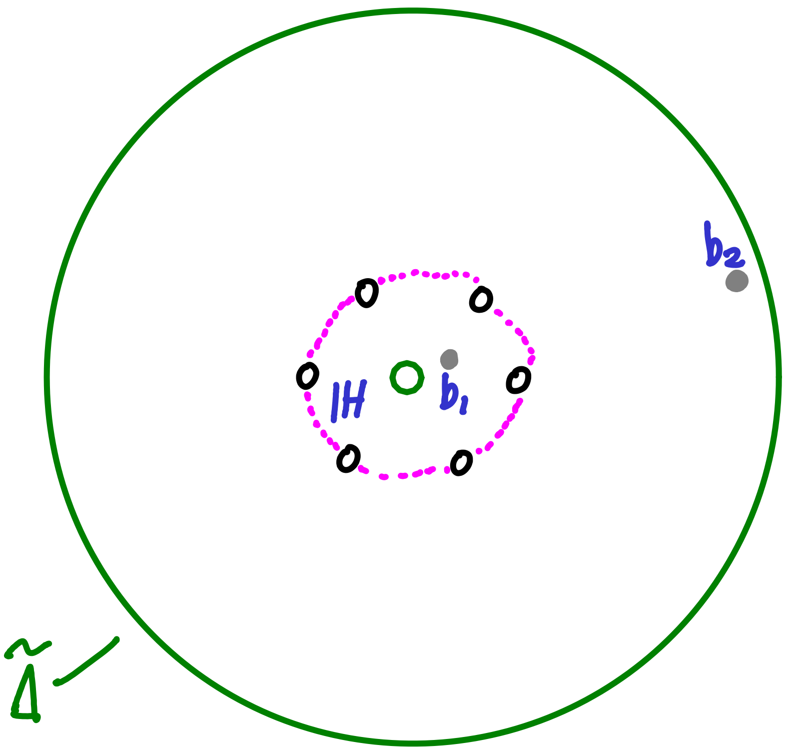

o $e(d)$ extra punctures

IH halo/annulus

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$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_g(\Pi, G)$$

$$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathbb{A} \end{array} \right\}$$

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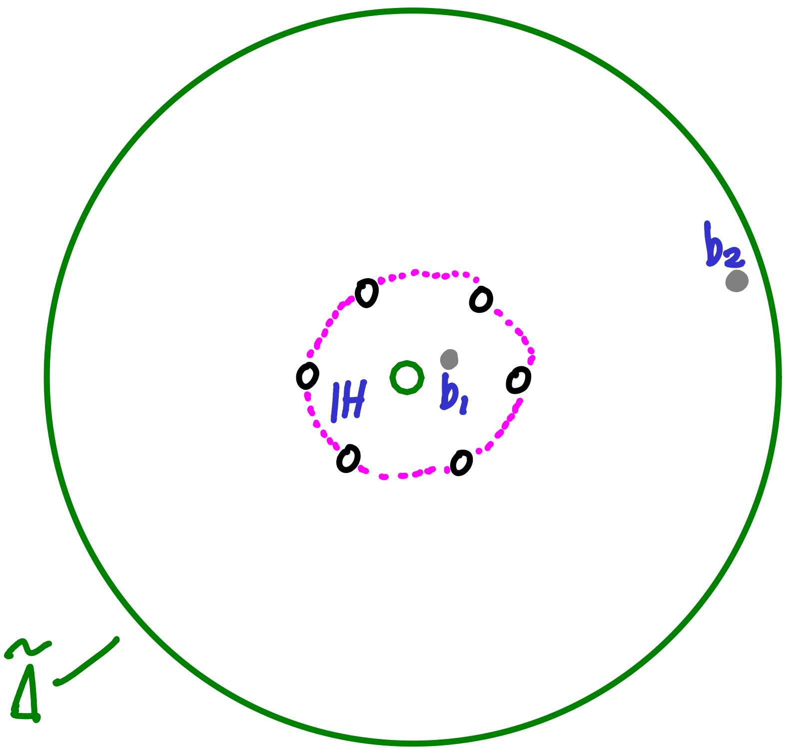

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Thm (arXiv 0203.*****)

$\tilde{\mathcal{M}}_B$ is a quasi-Hamiltonian $G \times H$ space

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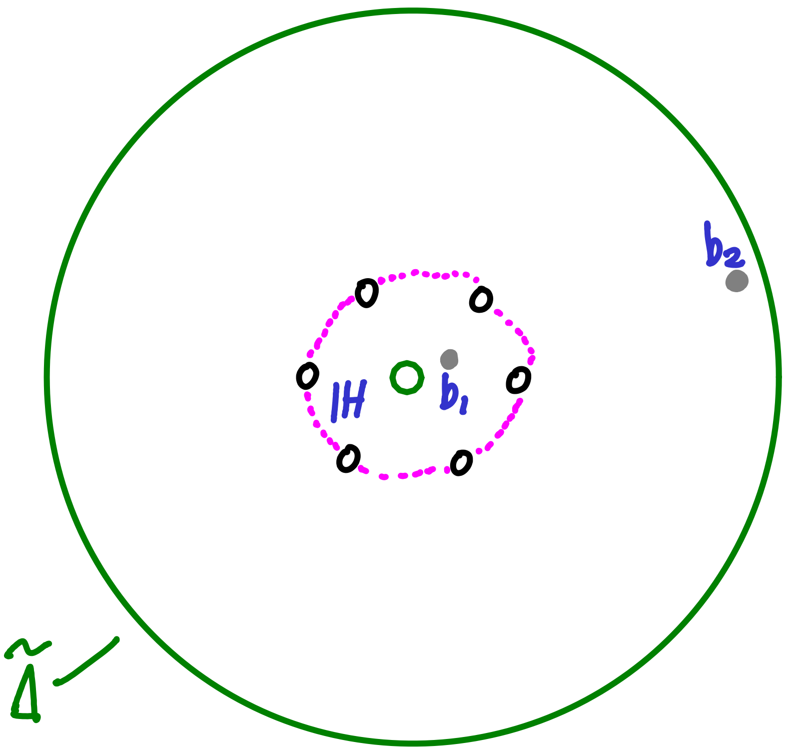

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$$\cong G \times (U_+ \times U_-)^k \times H$$

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Thm (arXiv 0203.****)

$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

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$\rotatebox{90}{$\in$}$

$(C, \underset{\sim}{S}, h)$ $\underset{\sim}{S} = (S_1, \ldots, S_{2k})$ $S_{odd/even} \in U_{+/-}$

Moment map $\mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, h^{-1}) \in G \times H$

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Cor. $\mathcal{B}(Q) := \mathcal{A}(Q) /\!/ G$ is a quasi-Hamiltonian H-space

$= \mu_G^{-1}(1) / G$ $\qquad = \widetilde{\mathcal{M}}_B((\mathbb{P}^1, 0, Q))$

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$$\cong \{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$$

Wild Character Varieties

Cor.

$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$ is a quasi-Hamiltonian H-space

Wild Character Varieties

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$$\cong \{ (S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \quad (\text{Gauss})$$

Wild Character Varieties

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$$\cong \{(S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\} \qquad (\text{Gauss})$$

E.g. $k=2$ $$\left(\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}\right)_{11} = 1 + ab$$

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E.g. $k=2$ $\quad \left(\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ b & 1\end{pmatrix}\right)_{11} = 1 + ab$

So $\mathcal{B}(Q) \cong \mathcal{B}(V)$ of Van den Bergh

$\mu = h^{-1} = (1+ab, (1+ba)^{-1})$

Wild Character Varieties

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$$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$$ is a quasi-Hamiltonian H-space

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Lemma

$$\left(\begin{pmatrix}1 & a_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_1 & 1\end{pmatrix} \cdots \begin{pmatrix}1 & a_r\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_r & 1\end{pmatrix}\right)_{11} = (a_1, b_1, \dots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

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$\cong \{(S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\}$ (Gauss)

$\cong \{\underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \dots, a_{k-1}, b_{k-1}) \neq 0\}$

$\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

Lemma

$$\left(\begin{pmatrix}1 & a_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & a_r\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_r & 1\end{pmatrix}\right)_{11} = (a_1, b_1, \dots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

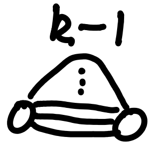
Cor.

$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$ is a quasi-Hamiltonian H-space

$$\cong \{(S_2, \dots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G\}$$

$$\cong \{(S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\} \qquad (\text{Gauss})$$

$$\cong \{\underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \dots, a_{k-1}, b_{k-1}) \neq 0\}$$

$=: \mathrm{Rep}^*(\Gamma, V)$ $\qquad \Gamma =$ 

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, $V = \mathbb{C} \oplus \mathbb{C}$

[Similarly for $V = V_1 \oplus V_2$ any dimension
(2009-2015) Γ any "fission graph"]

$$\mu(a_1, \dots, b_{k-1}) = ((a_1, b_1, \dots, a_{k-1}, b_{k-1}), (b_{k-1}, \dots, b_1, a_1)^{-1})$$

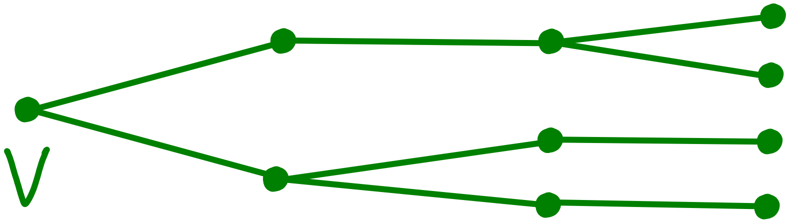
Fission graphs (arxiv 0806 appendix C)

$G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3:$



"fission tree"

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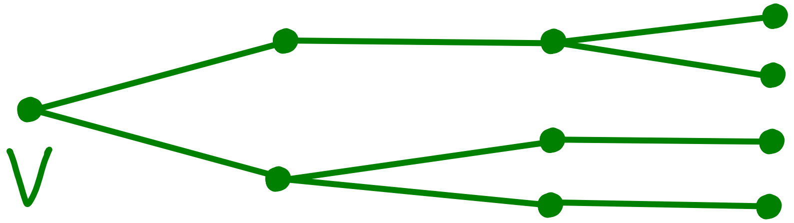

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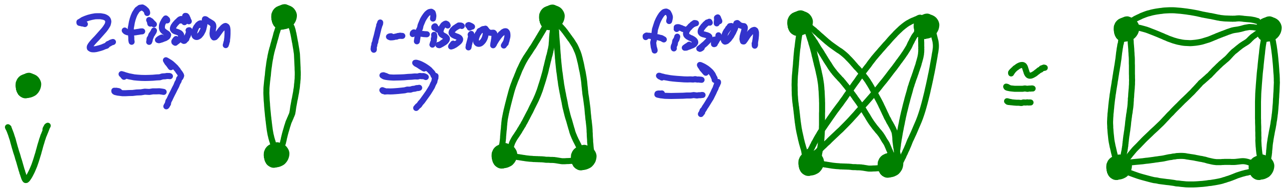
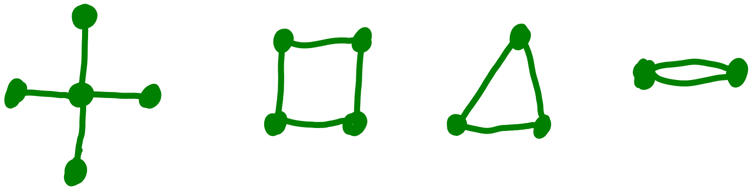
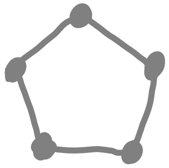


"fission graph" $\Gamma(Q)$

- $r=2$ get all complete k-partite graphs
- e.g.

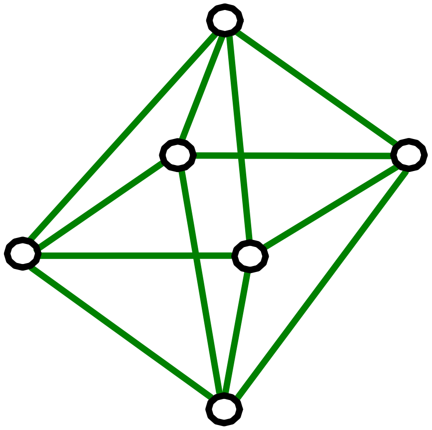


but not

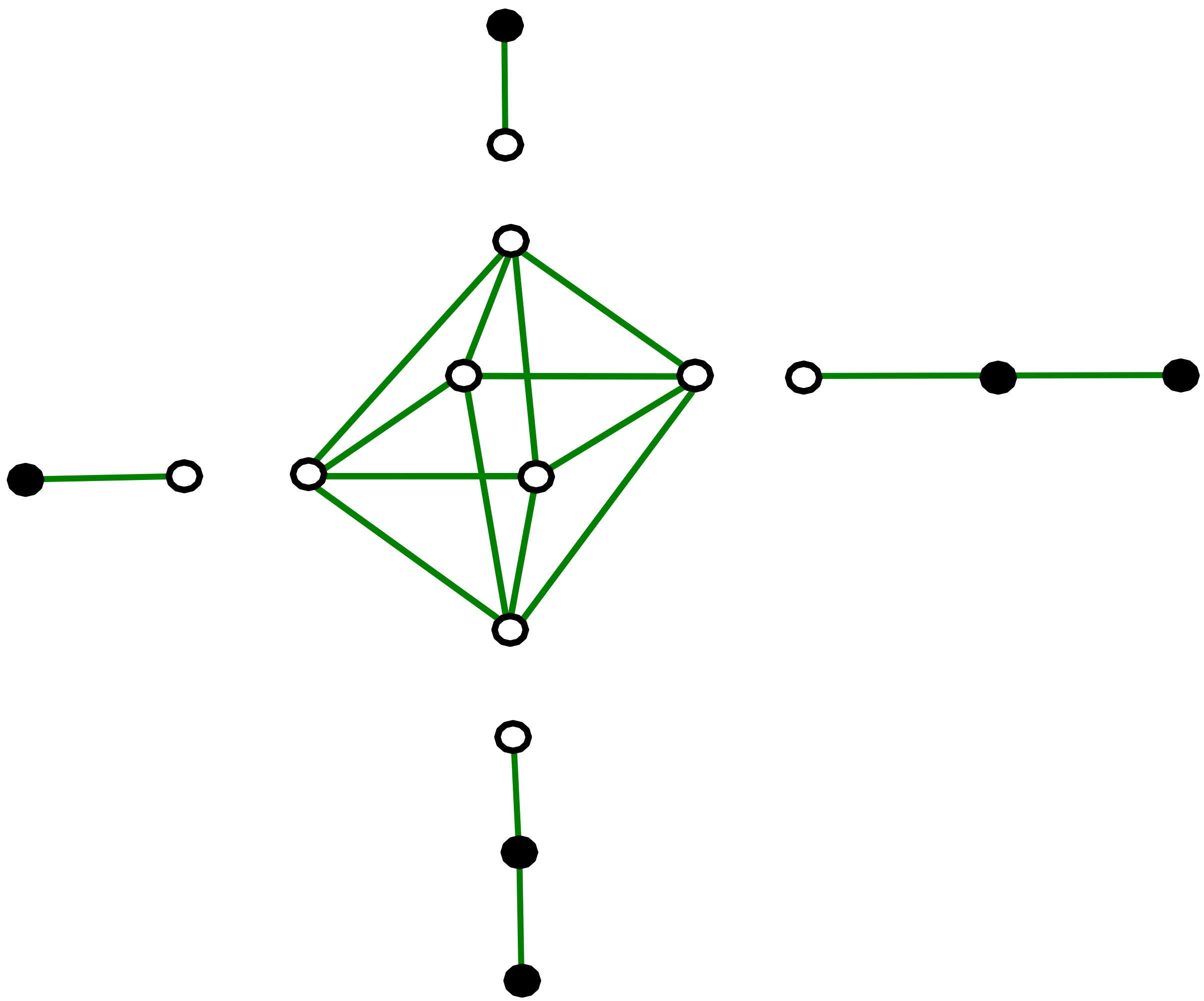


$Q = \text{diag}(q_1, \ldots, q_n) \Rightarrow$ nodes $= \{1, \ldots, n\}$, # edges $i \leftrightarrow j = \deg_w(q_i - q_j) - 1$

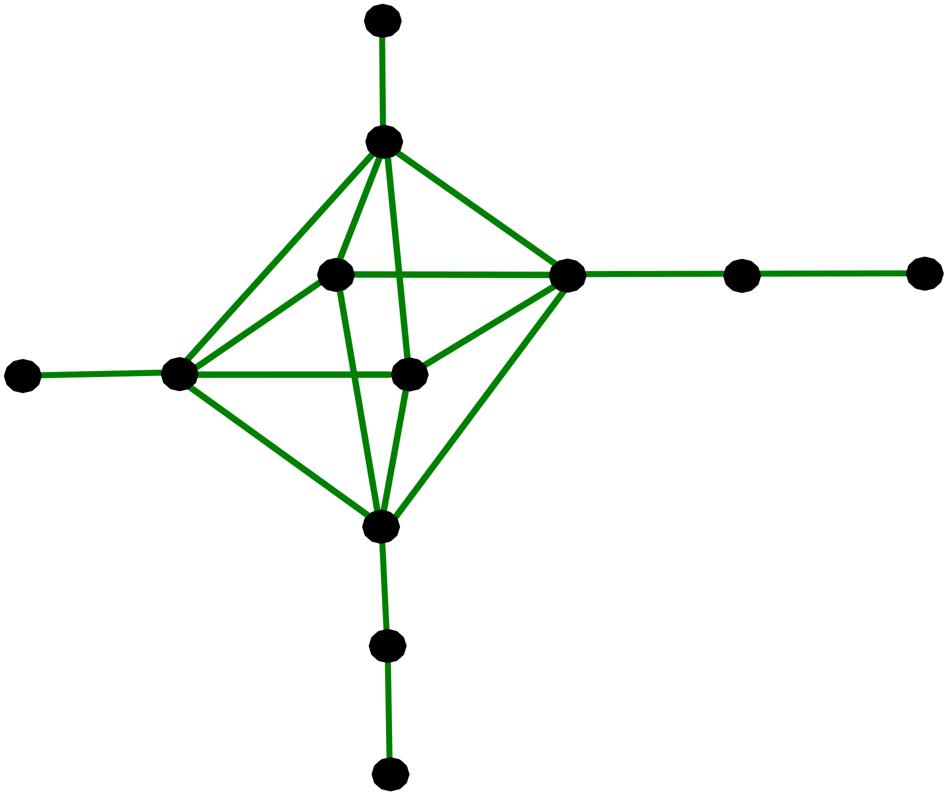
Fission graph



Fission graph + legs



Fission graph + legs = supernova graph



Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \widetilde{\mathcal{M}}_B /\!\!/_{(q_1,q_2)} H$$

$$= \mathrm{Rep}^*(\Gamma, V) /\!\!/_{(q_1,q_2)} H$$

$\Gamma =$ 

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"multiplicative quiver variety"

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E.g. $k=3$ (Painlevé 2 Betti space)

$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

(Flaschka-Newell surface)

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Also $\mathcal{M}^* \cong \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ "Nakajima/additive quiver variety"

(P.B 2008, Hiroe-Yamakawa 2013)

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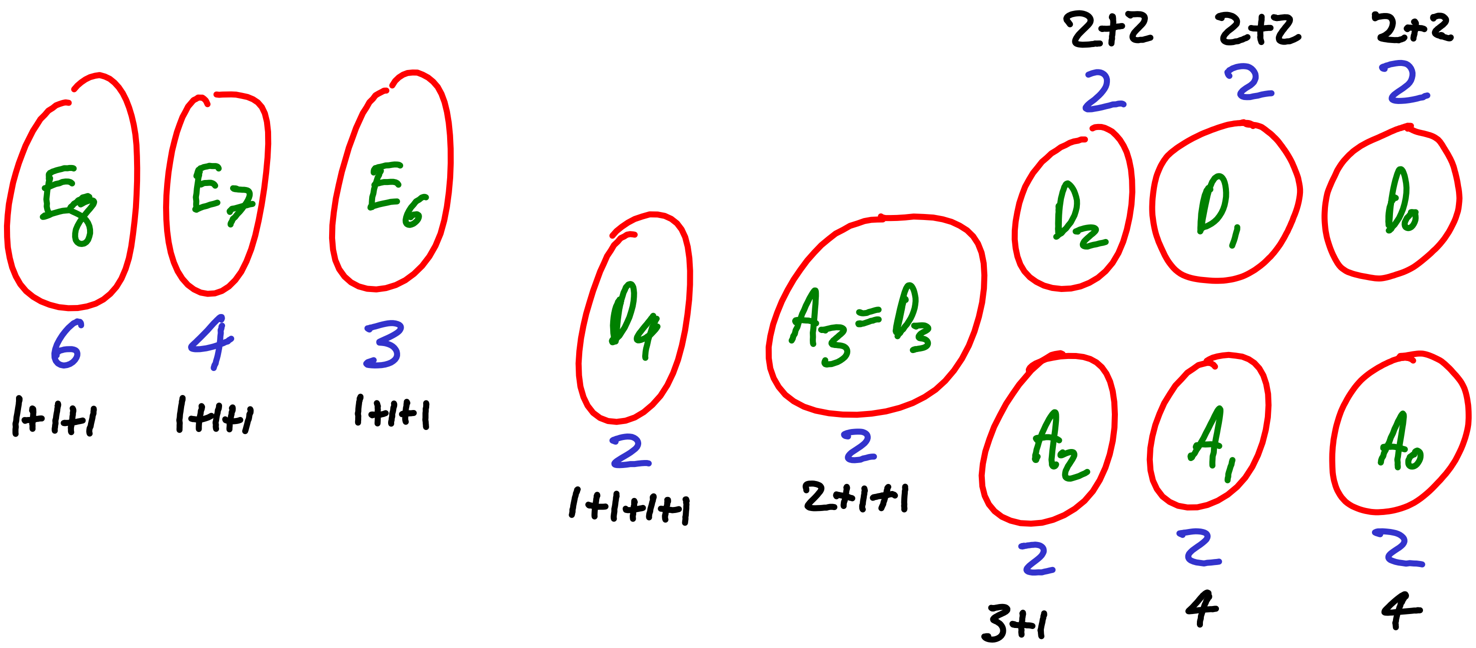
$$\begin{tikzcd}
\mathcal{M}^* \arrow[r, Rightarrow, "RHB"] \arrow[d, phantom, "\cong"] & \mathcal{M}_B \arrow[d, phantom, "\cong"] \\
\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H & \mathrm{Rep}^*(\Gamma, V) /\!\!/_{q} H
\end{tikzcd}$$

(Higgs, Hitchin, Hodge)

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "H3 surfaces"



affine Weyl group

minimal rank of bundles

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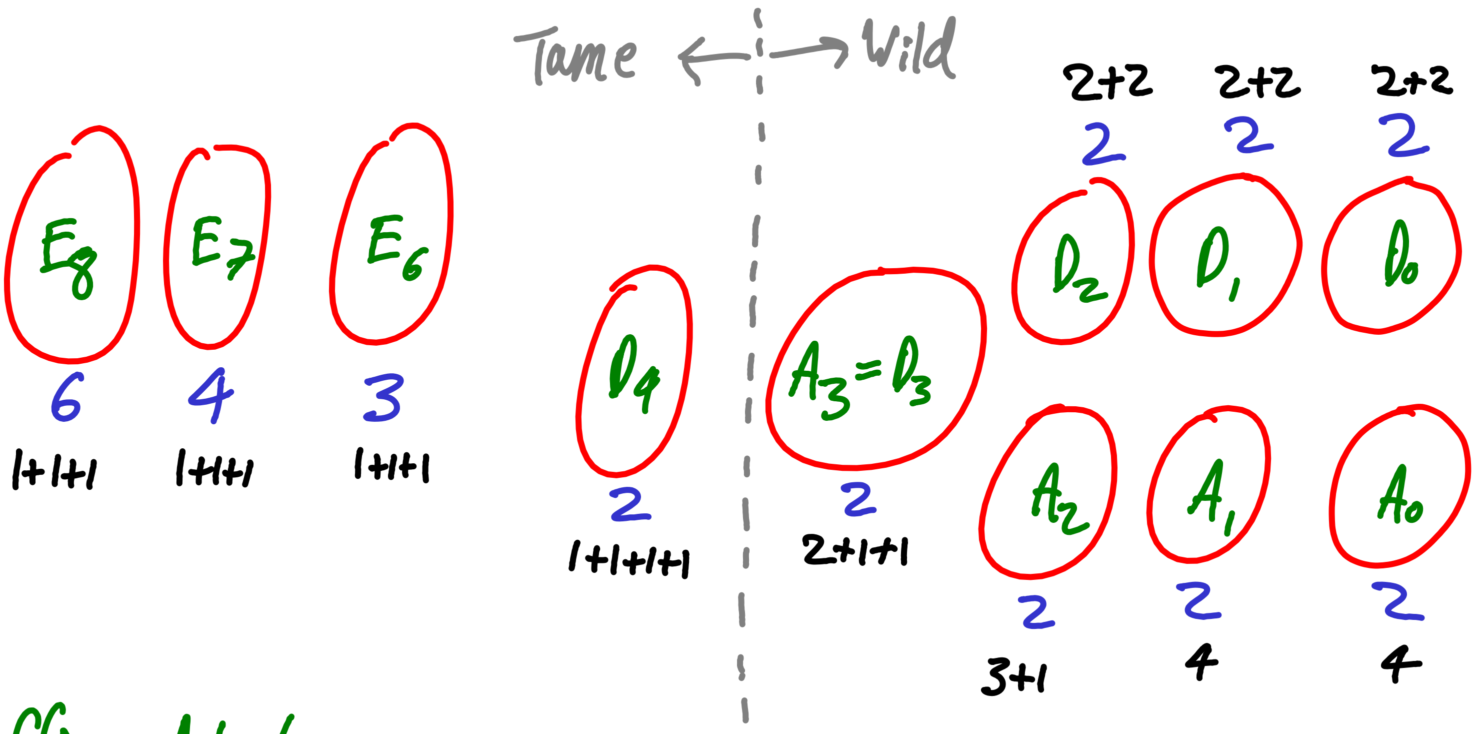

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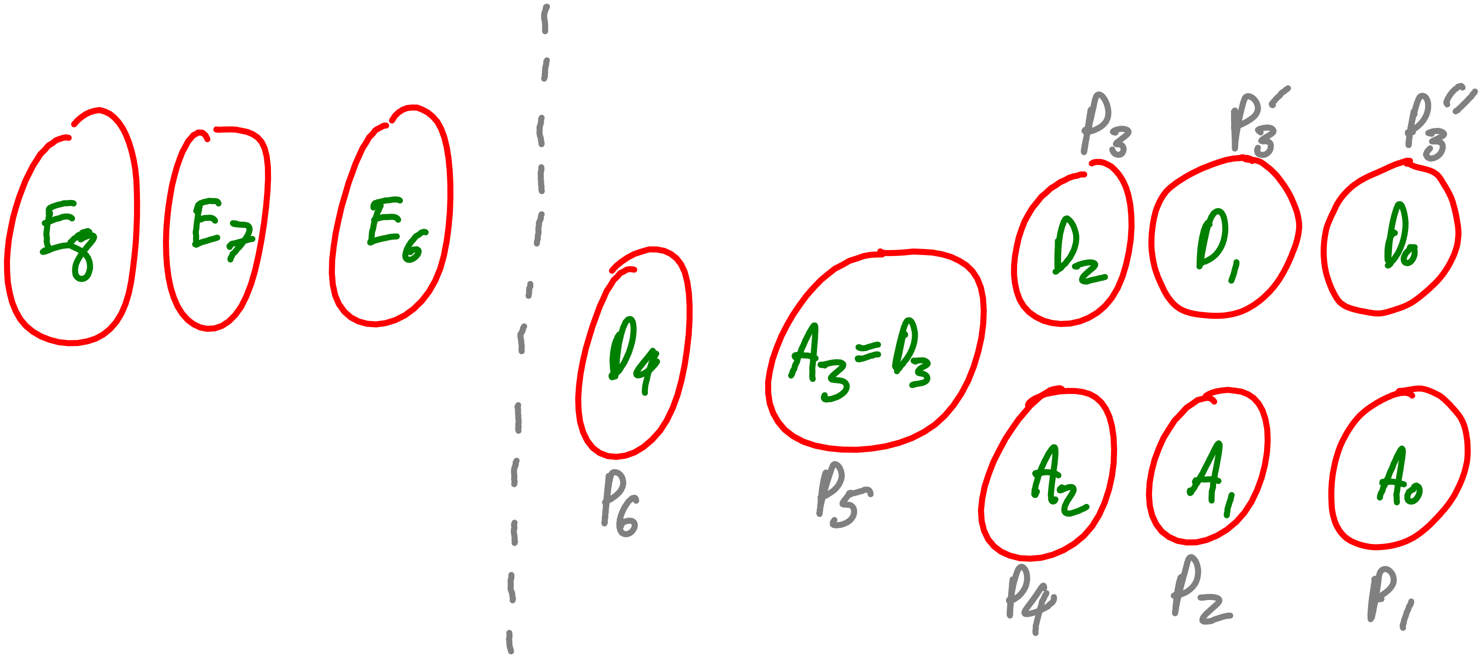

Phase spaces for Painlevé differential equations

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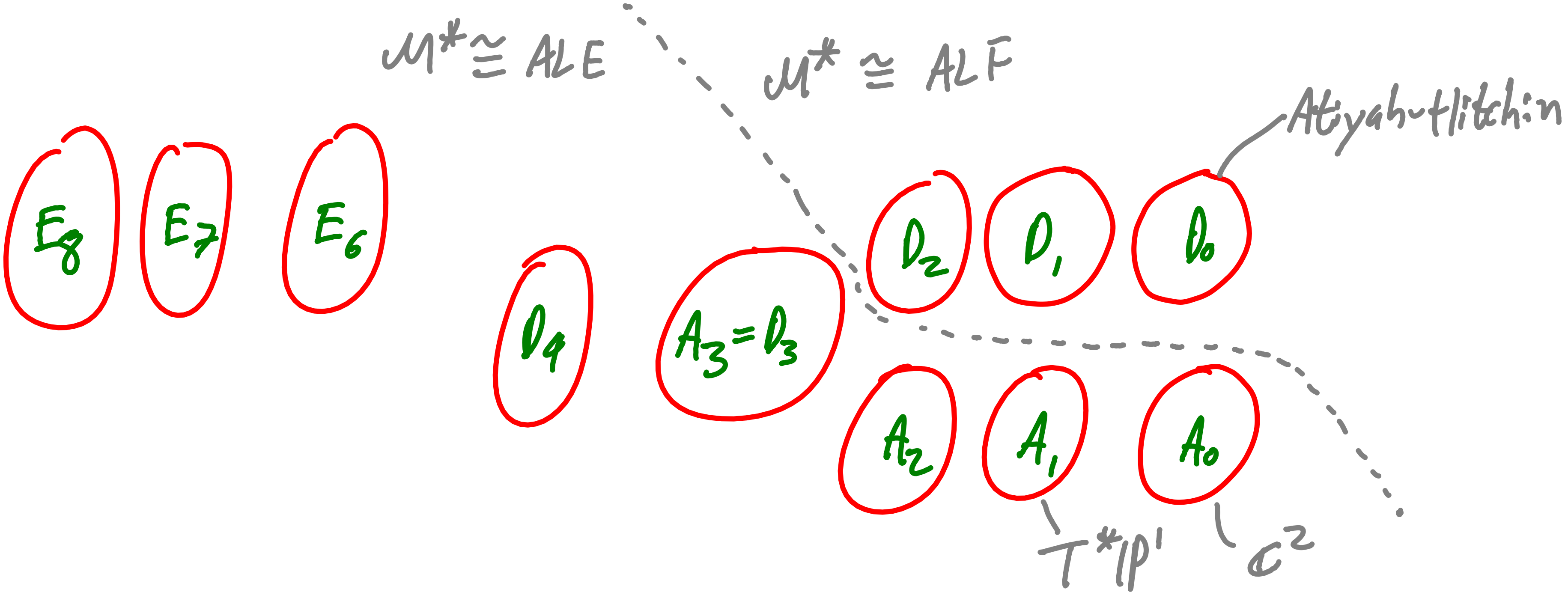
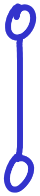


[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]

Summary



$$B_2 = B(v_1, v_2)$$

$$\mu \sim (a,b) = ab+1$$

Summary

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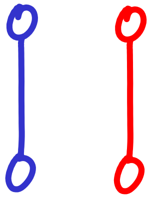

$$B_2 \times B_2$$

Summary

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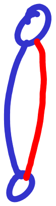

$$\mathcal{B}_2 \underset{H}{\otimes} \mathcal{B}_2$$

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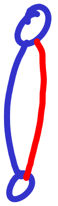

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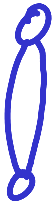

$\mathcal{B}_4$

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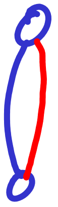

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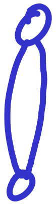

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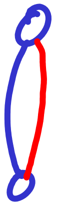

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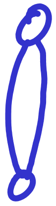

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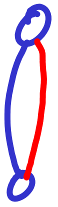

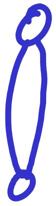

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$$\begin{tikzcd}
\mathcal{B}_2 \otimes_H \mathcal{B}_2 \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & \mathcal{B}_4
\end{tikzcd}$$

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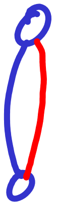

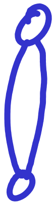

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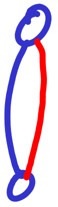
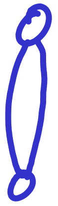
Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

Summary





$$\begin{tikzcd} \mathcal{B}_2 = \mathcal{B}(V_1,V_2) & \mathcal{B}_2 \circledast_H \mathcal{B}_2 \arrow[r, hook, shift left, "L"] \arrow[r, hook, shift right, "R"'] & \mathcal{B}_4 \end{tikzcd}$$

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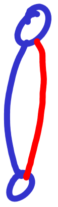

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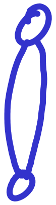

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& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

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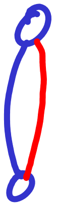

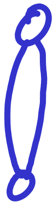

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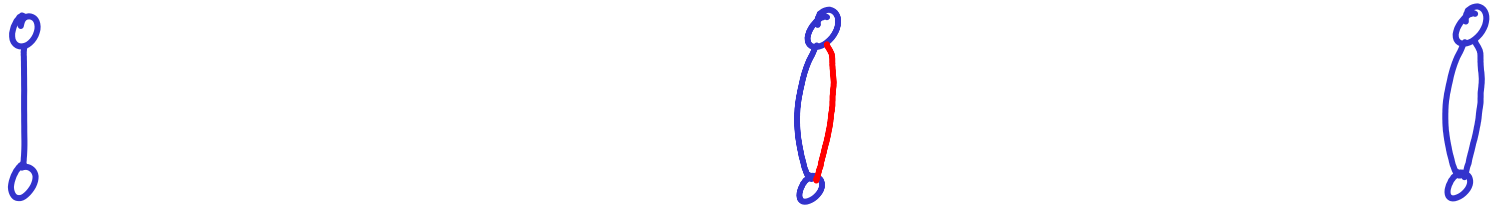

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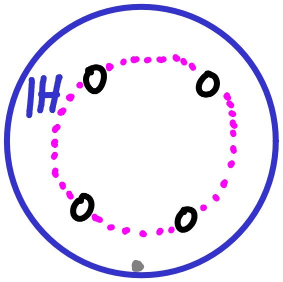
- follows since L & R form "free duplicial algebra" (Loday)
- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary



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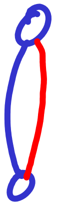

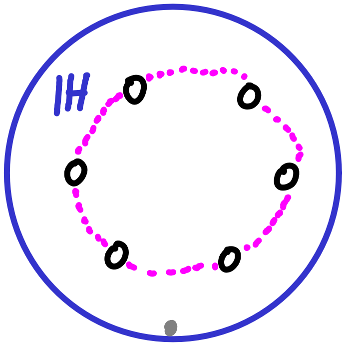

$$\begin{tikzcd} \mathcal{B}_2 \circledast_H \mathcal{B}_2 \arrow[r, shift left, "L"] \arrow[r, shift right, "R"'] & \mathcal{B}_4 \end{tikzcd}$$

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Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) $\rightsquigarrow 14$

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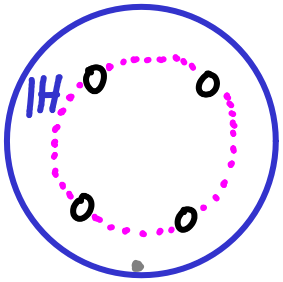

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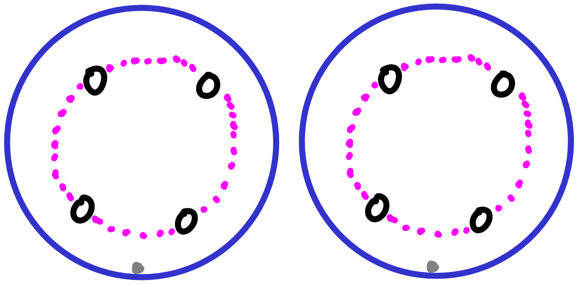

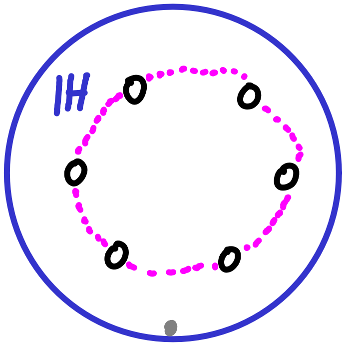

$\mathcal{B}_2 \times \mathcal{B}_2 \;\substack{\xrightarrow{L} \\ \xrightarrow[R]{}}\; \mathcal{B}_4$

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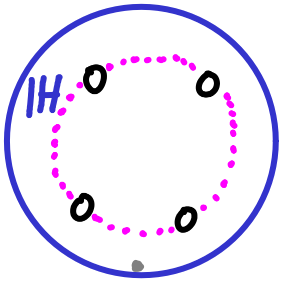

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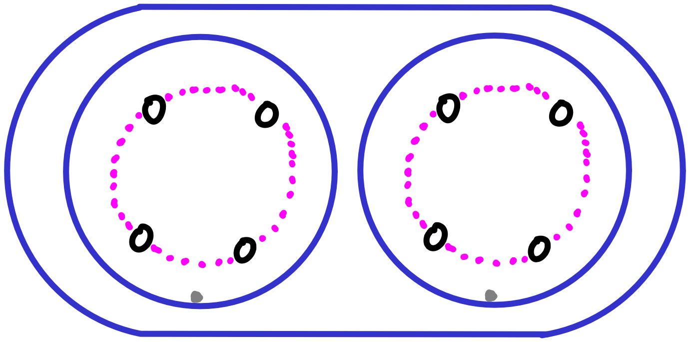

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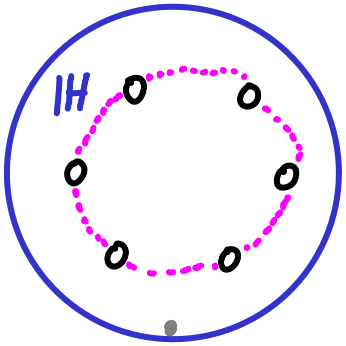

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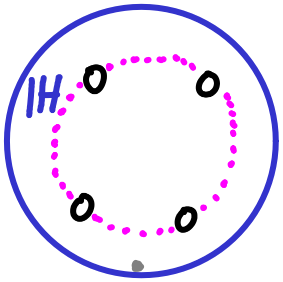

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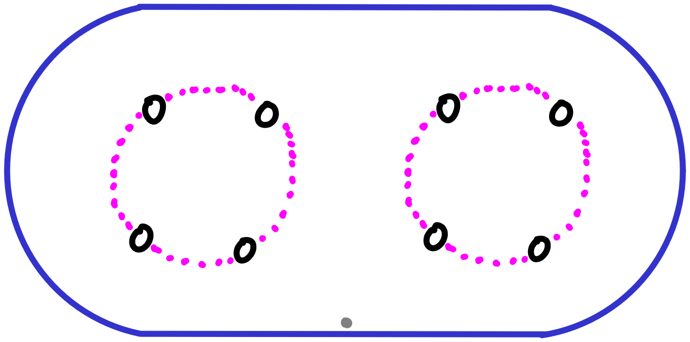

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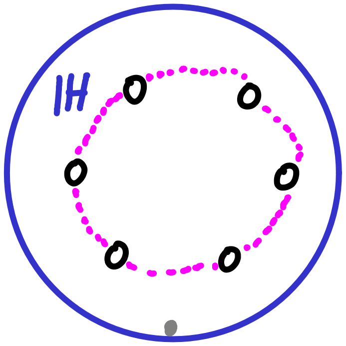

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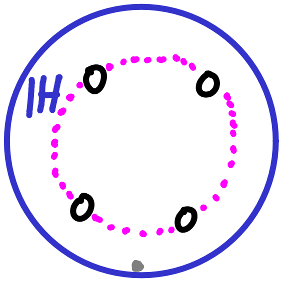

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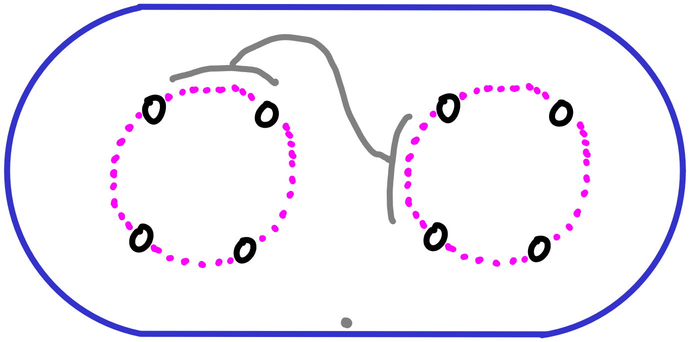

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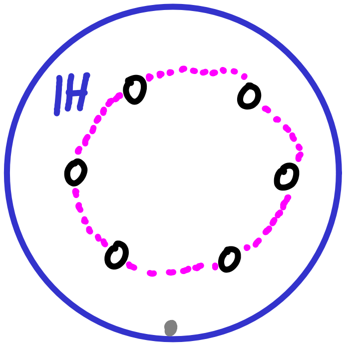

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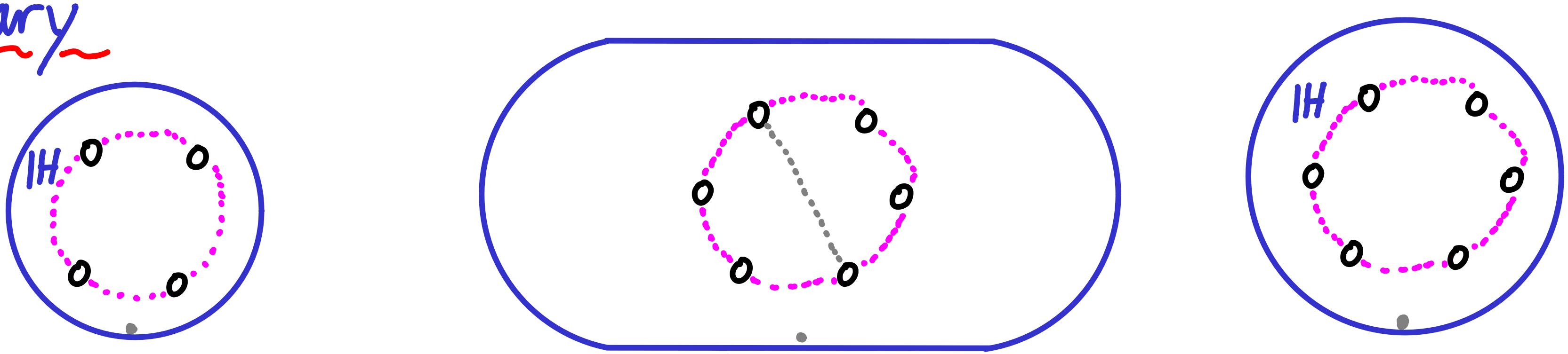

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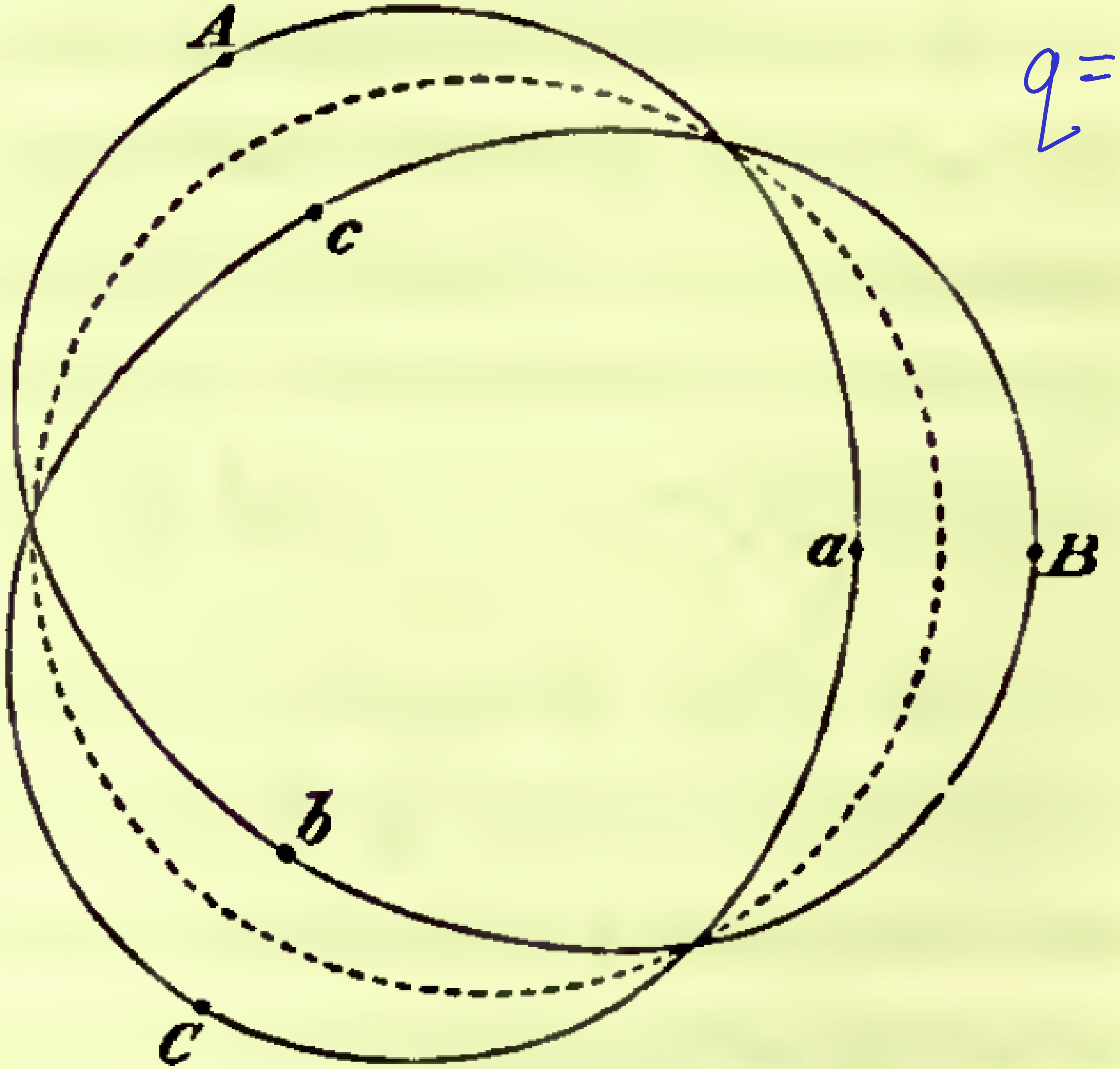
(Stokes 1857)

Fig. 1.

Stokes diagram of Airy equation

$$q = \pm 2 w^{3/2}$$



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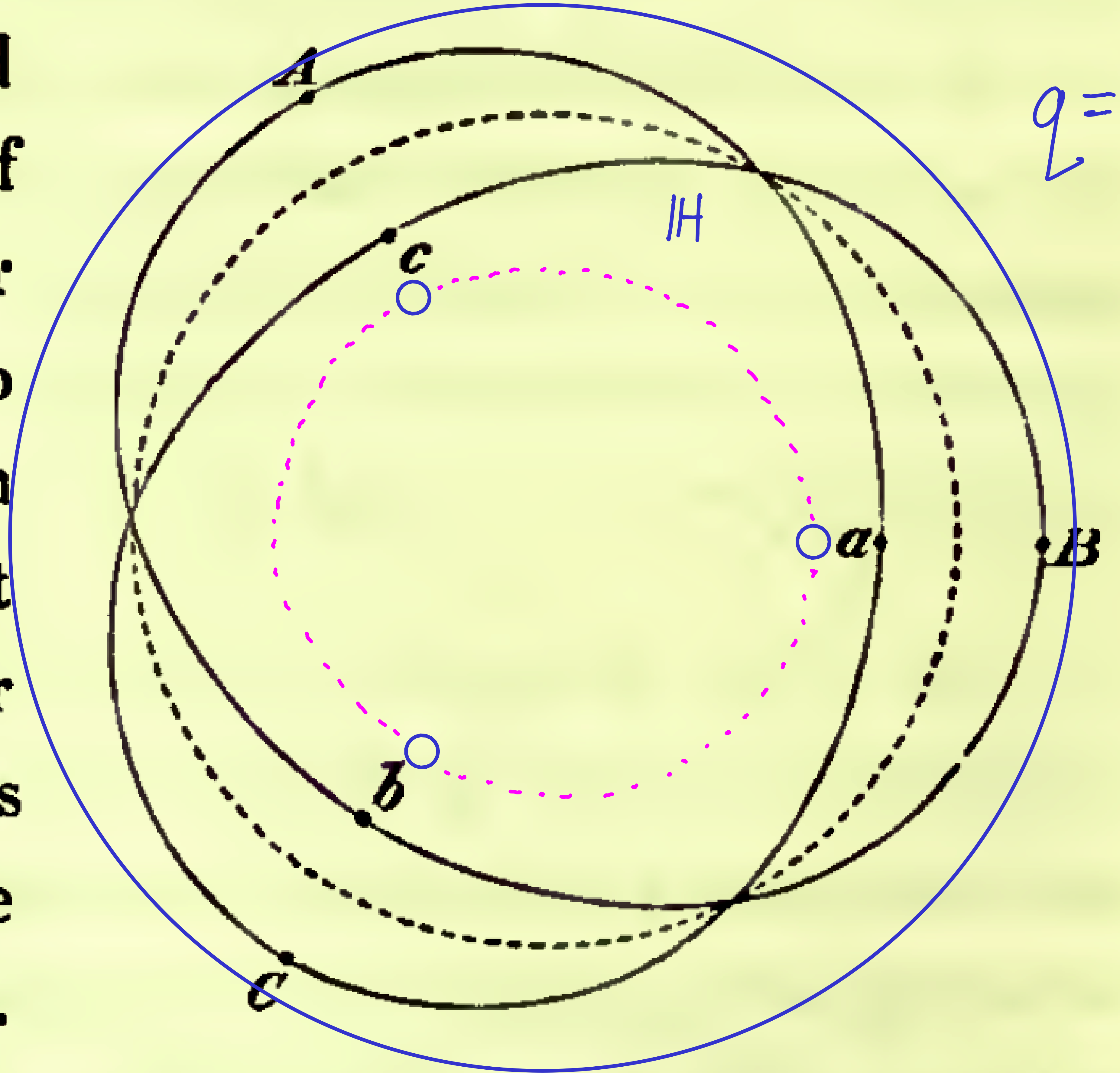

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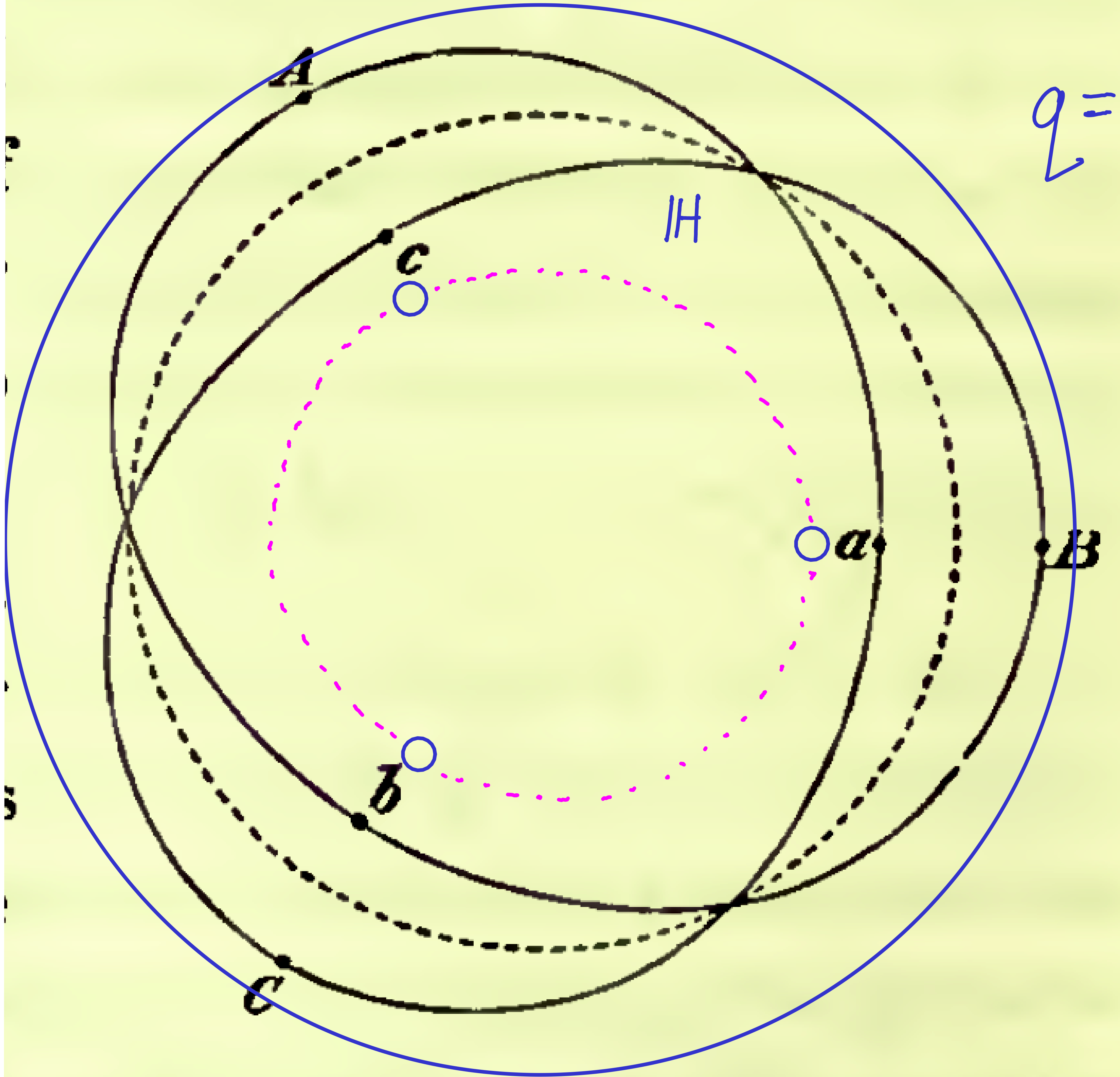

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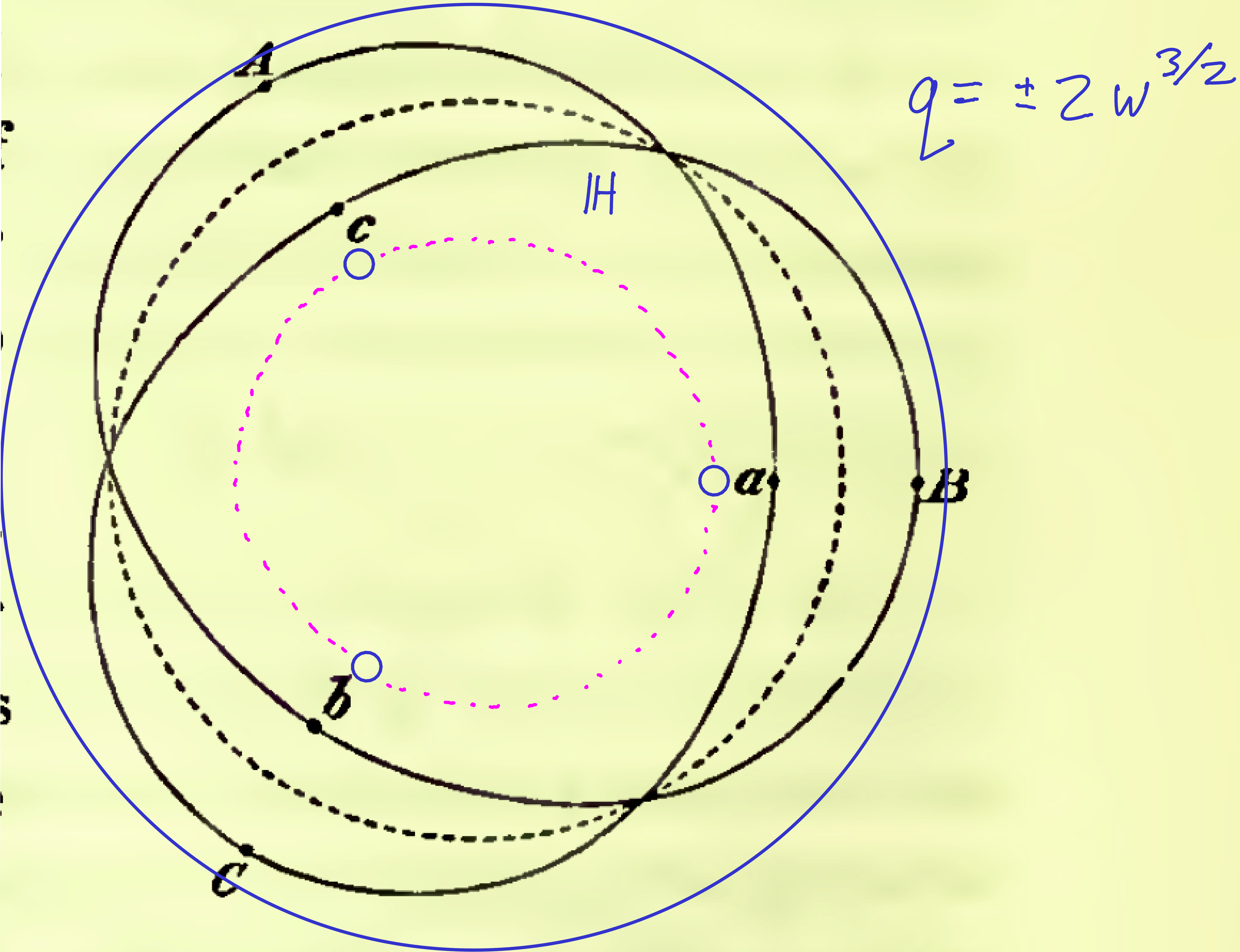

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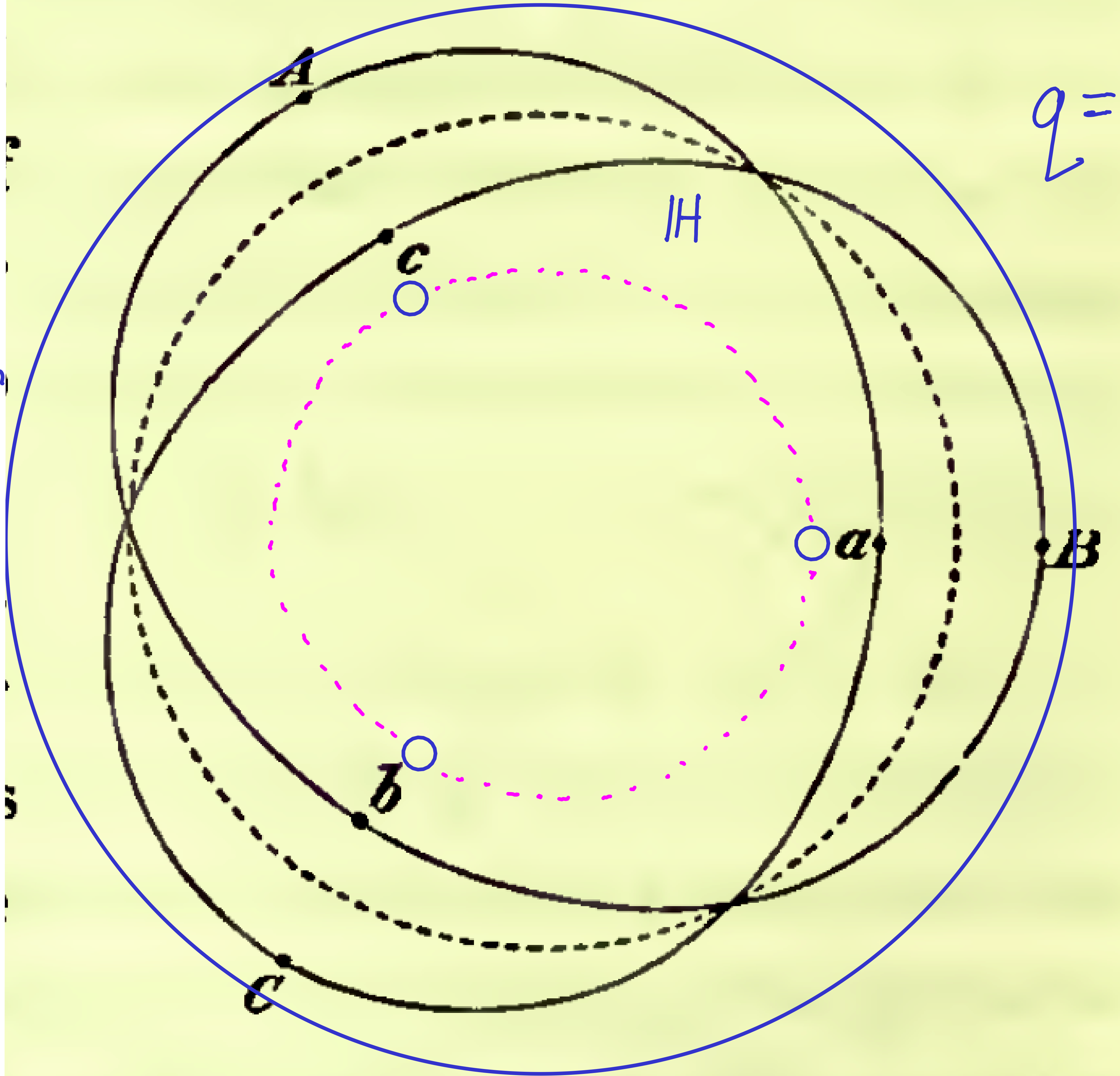

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Can now glue these Airy triangles $(\mathcal{B}_1)$ as before, so clearly

factorisations $\Longleftrightarrow$ triangulations

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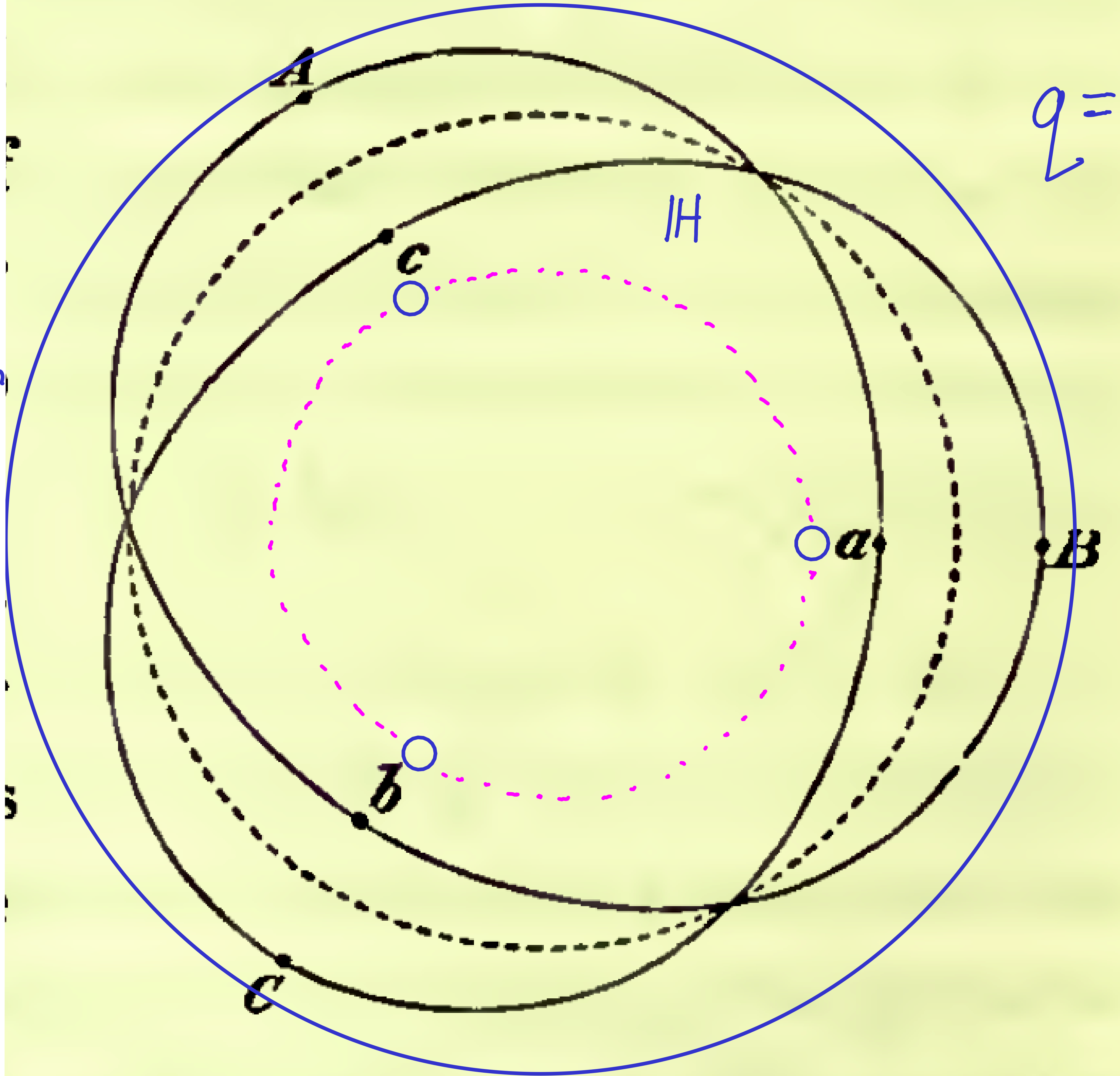

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If $\dim(V_1)=1$ this is familiar from complex WKB, but now see how to glue the triangles via QH fusion

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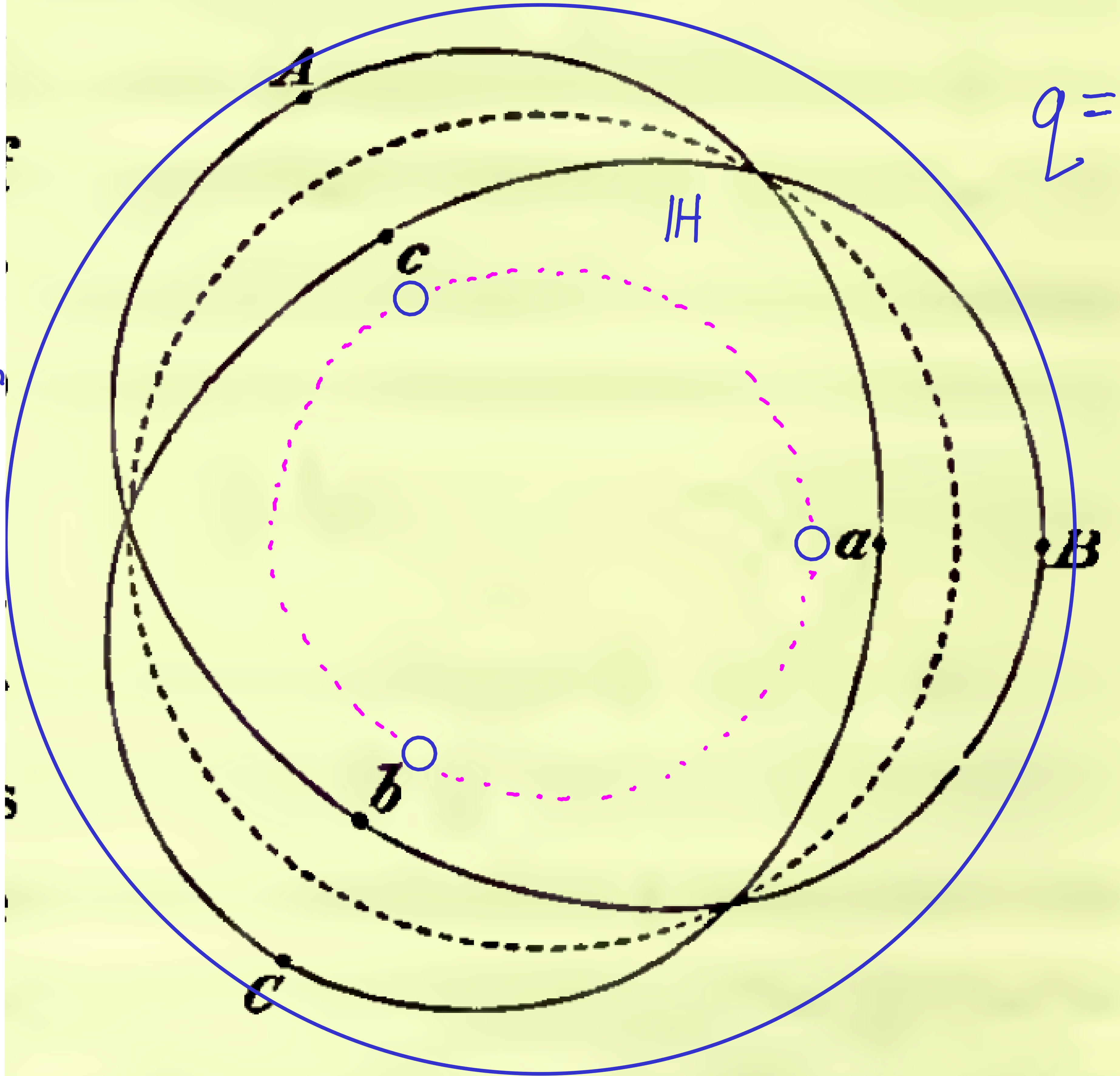

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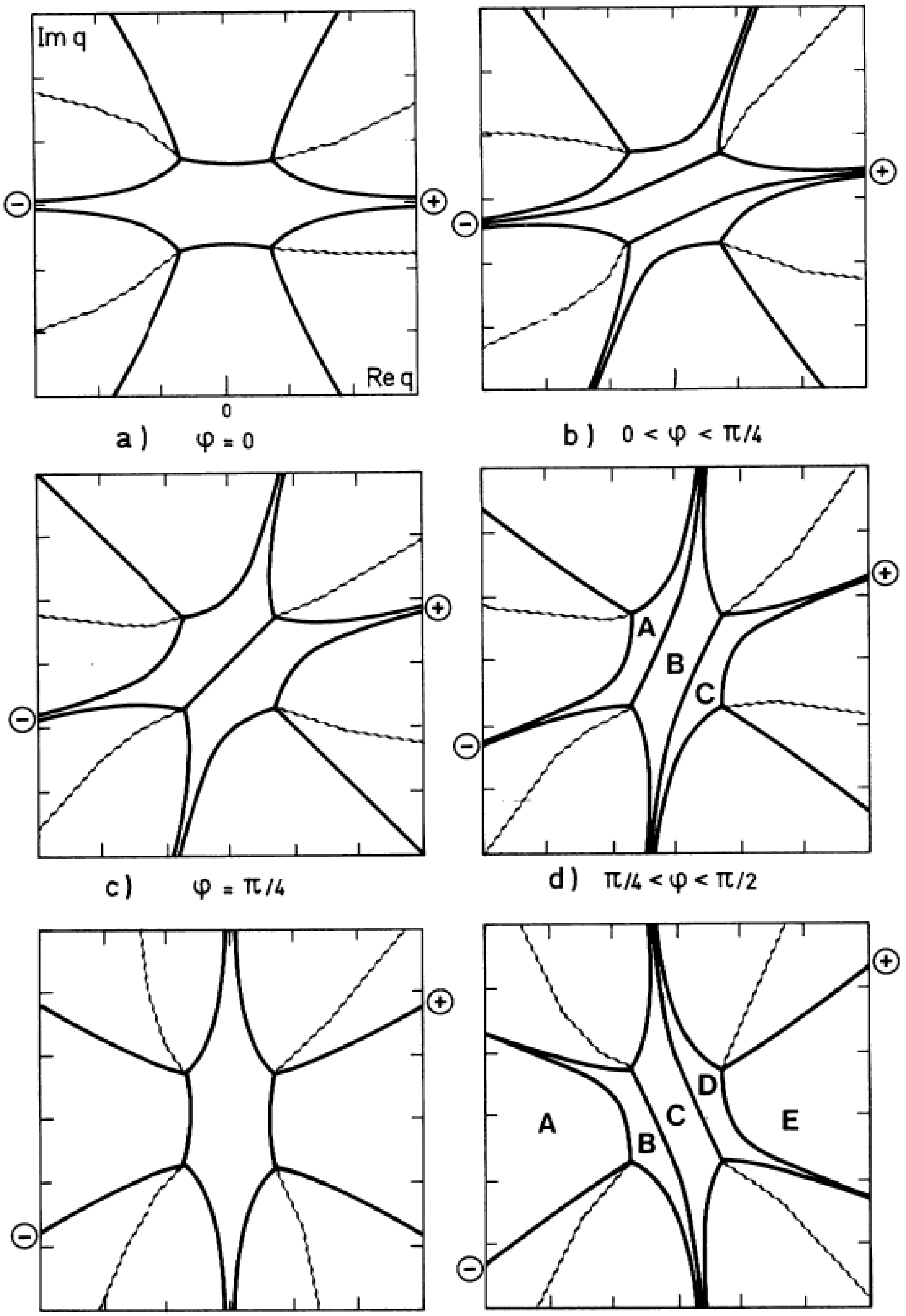
Voros '83



a) φ = 0

b) 0 < φ < π/4

c) φ = π/4

d) π/4 < φ < π/2

e) φ = π/2

f) π/2 < φ < 3π/4

FIG. 19.

— Stokes lines.

∿ Cuts.