First steps in global Lie theory: wild Riemann surfaces, their character varieties and topological symplectic structures

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- see also short survey arxiv: 1703 for more background
- course notes: ~/cours23/

Geometrically, what are the six Painlevé equations* trying to tell us?

* Picard, Painlevé, R. Fuchs, Gambier

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Henri Poincaré:

« Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres ; l'œuvre de Paul Painlevé est une île originale et splendide dans l'océan voisin »

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Geometrically, what is the Riemann–Hilbert–Birkhoff correspondence* trying to tell us?

* Stokes, Birkhoff, Malgrange, Sibuya, Jurkat, Deligne, Écalle, Martinet, Ramis, ...

$G = GL_n(\mathbb{C})$ (or any other complex reductive group)

Riemann surface Σ $\rightsquigarrow$ character variety

$$\mathcal{M}_B = \mathcal{R}/G$$

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, b), G)$$

representation variety

wild Riemann surface $\underset{\sim}{\Sigma}$ $\rightsquigarrow$ wild character variety

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wild representation variety

Thm (B.-Yamakawa)

$\mathcal{M}_B$ is a.g. Poisson variety, points are the reductive Stokes representations, any admissible deformation of $\underset{\sim}{\Sigma}$ $\Rightarrow$ local system of Poisson varieties

Lie theory

$$\mathfrak{g} \longrightarrow G$$

$$X \longmapsto \exp(X)$$

Connection $\quad \dfrac{X}{2\pi i}\dfrac{dz}{z} \longmapsto$ monodromy

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Global Lie theory

Connection $\quad \left(\sum_{i=1}^{m}\sum_{j=1}^{r_i}\frac{A_{ij}}{(z-a_i)^j}\right)dz \longmapsto$ monodromy & Stokes data

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moduli spaces: $\quad \mathcal{M}^* \longrightarrow \mathcal{M}_B$ wild character variety

(same dimension)

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moduli spaces: $\qquad \mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B$ wild character variety (same dimension)

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$$\mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B \quad \text{wild character variety (same dimension)}$$

$$\mathcal{M}_{DR} \xrightarrow{\cong} \mathfrak{M} \quad \text{wild harmonic bundles (2d self-duality)}$$

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moduli spaces:

$$\begin{tikzcd}
\mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[d, leftrightarrow, "\cong", "\text{wild nonabelian Hodge}"'] \arrow[dr, "\cong"] & \mathcal{M}_B \\
& \mathcal{M}_{Dol} \arrow[r, "\cong"] & \mathfrak{M}
\end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

$\mathcal{M}_{Dol}$: meromorphic Higgs bundles

$\mathfrak{M}$: wild harmonic bundles (2d self-duality)

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wild character variety (same dimension)

wild harmonic bundles (2d self-duality)

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"Nonabelian Hodge space"

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meromorphic Higgs bundles

wild harmonic bundles (2d self-duality)

"Nonabelian Hodge space"

Classify via diagrams? (e.g. sometimes $\mathcal{M}^*$ is a quiver variety)

Much of the story can be summarised in the (slightly oversimplified) diagram:

<u>Dolbeault</u>		<u>De Rham</u>		<u>Betti</u>
Rational Lax matrices L $\dot{L}=[P,L]$		Rational diff. op.s $\frac{d}{dz}-B$		Stokes and monodromy data
$\bigcap$		$\bigcap$		$\bigcap$
Mero. Higgs bundles (E,Φ)	$\xleftrightarrow{\text{wnAbH}}$	Mero. connections (E,∇)	$\xleftrightarrow{\text{RHB}}$	Stokes local systems
$\bigcup$		$\bigcup$		$\bigcup$
Holom. Higgs bundles (E,Φ)	$\xleftrightarrow{\text{nAbH}}$	Holom. connections (E,∇)	$\xleftrightarrow{\text{RH}}$	Local systems/ π_1-rep.s

$$
\begin{tikzcd}[column sep=huge, row sep=large]
 & \mathfrak{M} \arrow[dl, bend right, "1"', "\cong"] \arrow[d, "2"', "\cong"] \arrow[dr, dashed, bend left, "\cong"] & \\
\mathcal{M}_{\mathrm{Dol}} \arrow[r, dotted, no head, "\mathrm{wnAbH}", "\cong"'] & \mathcal{M}_{\mathrm{DR}} \arrow[r, "\mathrm{RHB}", "\cong"'] & \mathcal{M}_{\mathrm{B}}
\end{tikzcd}
$$

FIGURE 2. Nonabelian Hodge space $\mathfrak{M}$, with three preferred algebraic structures.

En définitive, le problème posé au n° 4 exige l'intégration du système aux dérivées partielles (19), (20), ou, si l'on veut, avec les notations ordinaires:

$$(A_t)\qquad \begin{cases} \dfrac{\partial A^j_{hk}}{\partial t_i} = \sum\limits_{l=1}^{m} \dfrac{A^i_{hl}A^j_{lk} - A^j_{hl}A^i_{lk}}{t_j - t_i} & (j \neq i), \\ & (h,\ k = 1, \ldots, m), \\ \sum\limits_{j=1}^{n+2} \dfrac{\partial A^j_{hk}}{\partial t_i} = 0. \end{cases}$$

C'est le système découvert par M. SCHLESINGER; il est complètement intégrable [car il a été déduit, moyennant une transformation (18) du système complètement intégrable (11)]; et sa solution dépend de $m^2(n+2)$ constantes arbitraires.

10. Je vais former maintenant le système dont l'intégration fait l'objet de ce Mémoire. Dans (A_t) remplaçons t_i par $\alpha_i + \varepsilon t_i$ $(i = 1, 2, \ldots, n)$; convenons toujours de prendre $\alpha_{n+1} = 0$, $\alpha_{n+2} = 1$, et remplaçons A^i par $\varepsilon^{-1}A^i$ $(i = 1, \ldots, n+2)$. Puis faisons tendre ε vers 0. A la limite, le système (A_t) deviendra

$$(A_\alpha)\qquad \begin{cases} \dfrac{\partial A^j_{hk}}{\partial t_i} = \sum\limits_{l=1}^{m} \dfrac{A^i_{hl}A^j_{lk} - A^j_{hl}A^i_{lk}}{\alpha_j - \alpha_i} & (j \neq i), \\ \sum\limits_{j=1}^{n+2} \dfrac{\partial A^j_{hk}}{\partial t_i} = 0 & (h,\ k = 1, \ldots, m). \end{cases}$$

D'après la façon même dont on l'a déduit de (A_t), le système (A_α) est le *simplifié* de

p.163 of: R. Garnier, Sur une classe de systèmes différentiels Abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo 43 (1919), 155–191.

$$a_{hk} = \sum_{i=1}^{n+3} \frac{A^i_{hk}}{x - \alpha_i};$$

et nous poserons

$$a_{hk} \equiv \frac{b_{hk}(x)}{\varphi(x)}$$

de sorte que $b_{hk}(x)$ sera un polynome en x, de degré $n + 1$ au plus, le coefficient de x^{n+1} étant indépendant des t_i, en vertu des équations $(A_\alpha)_2$.

Enfin, de toute l'analyse développée au Chapitre I nous ne retiendrons, pour l'intégration de (A_α), qu'une seule formule, à la vérité d'importance capitale: à savoir la *simplifiée* de (11) pour $j = 0$; cette simplifiée s'écrit avec les notations actuelles:

$$\frac{\partial b}{\partial t_i} \div \frac{A^i b - b A^i}{x - \alpha_i} \qquad (i = 1, \ldots, n). \tag{21}$$

Cela étant, je vais établir deux propositions préliminaires sur lesquelles repose toute l'intégration du système (A_α).

12. Théorème fondamental (I). — *Si dans la relation* [12])

$$f(x, y) \equiv \begin{vmatrix} b_{11}(x) + y & b_{12}(x) & \ldots & b_{1m}(x) \\ b_{21}(x) & b_{22}(x) + y & \ldots & b_{2m}(x) \\ \ldots & \ldots & \ldots & \ldots \\ b_{m1}(x) & b_{m2}(x) & \ldots & b_{mm}(x) + y \end{vmatrix} = 0 \tag{22}$$

on remplace les A^i_{hk} qui figurent dans b_{hk} par des intégrales de (A_α), la courbe algébrique $f(x, y) = 0$ ainsi obtenue a tous ses coefficients indépendants des t_i.

p.164 of: R. Garnier, Sur une classe de systèmes différentiels Abéliens déduits de la théorie des équations linéaires, Rend. Circ. Mat. Palermo 43 (1919), 155–191.

THEOREM 1. *There is a one-to-one correspondence between*

(i) *a polynomial* $A = \sum_{s=0}^{\nu} A_s h^s$ *with matrix coefficients (modulo conjugation by complex diagonal matrices), having the propertie.* $A_\nu = \operatorname{diag}(a_1, ..., a_n)$, $a_i \in \mathbb{C}^*$, $\prod_{i<j} (a_i - a_j) \neq 0$, *and* $(A_{\nu-1})_{1,k} \neq 0$ $(k \neq 1)$; *moreover* A *has in the limit* $h \to 0$ *distinct eigenvectors all not perpendicular*[11] *to* e^k *for some* k.

(ii) *a curve* X *of genus* $g = (n(n-1)/2)\nu - n + 1$ *with* $2n$ *distinct points* $P_1, ..., P_n, Q_1, ..., Q_n$ *and a general positive divisor* $\mathcal{D}$ *on* X *of degree* g *not containing any of the points* P_i *or* Q_i; *the points above have the following properties: for some meromorphic functions* h *and* z *on* X

$$(h) = -\sum_1^n P_i + \sum_1^n Q_i$$

and

$$(z) = -\nu \sum_1^n P_i + n\nu \text{ zeros, distinct from the } P_i .$$

Moreover any polynomial function $u = P(z, h, h^{-1})$ *on* X *leads to an isospectral deformation of* A

$$\dot{A} = [A, P(A, h, h^{-1})_+],$$

where $P(A, h, h^{-1})_+$ *denotes the polynomial part (in* h*) of* $P(A, h, h^{-1})$. *The flow above is a linear flow on* $\operatorname{Jac}(X)$ *defined by*

$$\sum_{i=1}^{g} \int_{\nu_i}^{\nu_i(t)} \omega = \sum_{l=1}^{n} \operatorname{Res}_{P_l}(\omega u) t.$$

In particular the flows (*cf. I* (4.43))

$$\dot{A} = [A, (f'(Ah^{-j})h^{k-j})_+]$$

are linear flows on $\operatorname{Jac}(X)$; *they are equivalent to one of the polynomial flows above.*

Proof. As a first step, the curve X, defined by the algebraic equation

$$Q(z, h) \equiv \det(A - zI),$$

is shown to have the properties stated in (ii). For z, h large,

$$Q(z, h) = \prod_1^n (a_i h^\nu - z) + C_1 h^{n\nu-1} + C_2 z^{n-1} + \text{lower-order terms},$$

[11] Dubrovin *et al.* [6] have considered similar matrix polynomials.

Adler–Van Moerbeke 1980

Commun. math. Phys. 76, 65–116 (1980)

Communications in Mathematical Physics

Monodromy- and Spectrum-Preserving Deformations I

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Abstract. A method for solving certain nonlinear ordinary and partial differential equations is developed. The central idea is to study monodromy preserving deformations of linear ordinary differential equations with regular and irregular singular points. The connections with isospectral deformations and with classical and recent work on monodromy preserving deformations are discussed. Specific new results include the reduction of the general initial

Simplified Schlesinger's Systems.

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(ricevuto il 10 Settembre 1979)

It is known that classical Painlevé transcendents I-II in the limit $t \to a + \varepsilon t_1 : \varepsilon \to 0$ turn to be Weierstrass or Jacobi elliptic functions (1). GARNIER (2) made a careful analysis of more general transcendents like Painlevé I-VI and their generalizations. It is natural to presume that the most general isomonodromy deformation equations —Schlesinger's equations (3)—in the limit $t_i \to a_i + \varepsilon t_{1i} : \varepsilon \to 0$ are reduced to some classical completely integrable systems.

Very good connections

$\sim$ models in Biquard–B. 2004

(cf. exposition in $\begin{cases} \text{arXiv:1203.6607} \\ \text{arXiv:1703.10376} \end{cases}$)

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(*\underset{\sim}{a})$ meromorphic connection

such that ...

Very good connections

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(cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

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$\mathcal{F}$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic connection

such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $\nabla = d - A$, $\quad A = dQ + \Lambda \frac{dz}{z}$ + holomorphic terms

  $Q = \sum_1^k \frac{A_i}{z^i}$, A_i diagonal matrices (irregular type)

  $\Lambda \in \mathfrak{h}$ preserves $\mathcal{F}_a$, $\mathfrak{h} = \mathrm{Lie}(H)$, $H = C_G(Q)$

["Good" if some local cyclic pullback is very good (twisted case)]

$\rightsquigarrow \mathcal{M}_{DR}$ moduli of stable connections, $\underset{\sim}{Q}$, $\mathrm{Gr}(\Lambda)$, parabolic weights fixed

Very good Higgs bundles

~ models in Biquard–B. 2004 (cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

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$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\underline{\Phi} : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic Higgs field

such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $$\underline{\Phi} = dQ + \Lambda \frac{dz}{z} + \text{holomorphic terms}$$

$$Q = \sum_1^k \frac{A_i}{z^i}, \quad A_i \text{ diagonal matrices}$$ (irregular type)

$$\Lambda \in \mathfrak{h} \text{ preserves } \mathcal{F}_a, \quad \mathfrak{h} = \mathrm{Lie}(H),\ H = C_G(Q)$$

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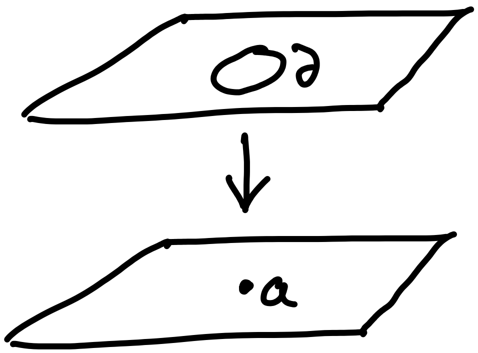
$\rightsquigarrow \mathcal{M}_{\mathrm{Dol}}$ moduli of stable Higgs bundles, $\underset{\sim}{Q}$, $\mathrm{Gr}(\Lambda)$, parabolic weights fixed

General choices / boundary data (twisted case) [Betti weights zero]

Fact $\exists$ covering $\mathcal{I} \to \partial$ such that:

$$\left\{\begin{array}{c}\text{connections on formal}\\ \text{punctured disk}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\mathcal{I}\text{-graded local systems}\\ \text{of vector spaces}\end{array}\right\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Jurkat, Deligne]



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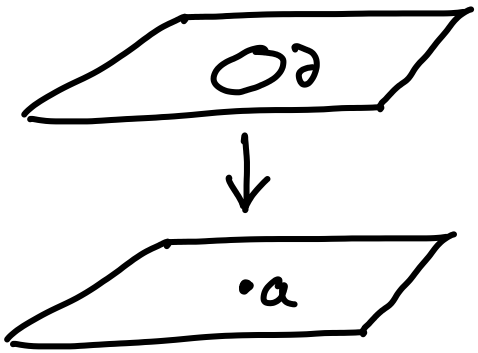
[Fabry, Hukuhara, Turrittin, Levelt, Jurkat, Deligne]



function on sector: $q = \sum_{i>0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

$\Rightarrow$ Stokes circle $\langle q \rangle$ (Riemann surface / Galois orbit)

$$\begin{matrix}\langle q \rangle \\ \downarrow \\ \partial\end{matrix}$$

$$\boxed{\begin{matrix}\tilde{I} = \bigcup \langle q \rangle \\ \downarrow \quad \text{exponential local system} \\ \partial\end{matrix}}$$

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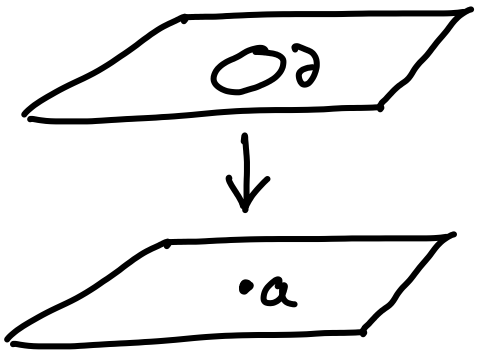

function on sector: $q = \sum_{i\geq 0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

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$$\langle q \rangle \to \partial$$

$$\boxed{\begin{array}{l} \mathcal{I} = \bigcup \langle q \rangle \\ \downarrow \qquad \text{exponential local system} \\ \partial \end{array}}$$

$\mathcal{I}$-graded local system $V \to \partial$ of vector spaces

$\iff$ local system $V \to \mathcal{I}$ with compact support

i.e. $V \to I$, $I \subset \mathcal{I}$ finite subcover

$\Rightarrow$ Irregular class $\Theta = n_1\langle q_1 \rangle + \dots + n_m \langle q_m \rangle$ $\quad n_i = \operatorname{rk} V|_{\langle q_i \rangle}$

\+ monodromy classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$

In simple examples this growth/decay can be easily visualised in the Stokes diagram, as in the example of $q = x^{17}$ in Figure 5, where the singularity is at $a = \infty$ (so $z = x^{-1}$ is a local coordinate vanishing at a). For example we see on the positive real axis that the function $\exp(x^{17})$ has maximal growth there, and there are 16 other evenly spaced directions of maximal growth, interlaced with 17 directions of maximal decay, the first at $\arg(x) = \pi/17$.

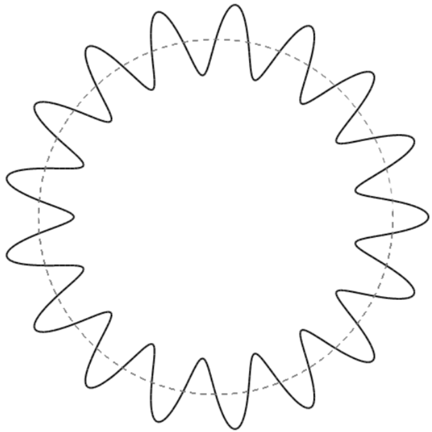


FIGURE 5. Stokes diagram for $\langle x^{17}\rangle$: the Stokes circle $\langle x^{17}\rangle$ is projected to the plane so as to indicate the growth/decay of $\exp(x^{17})$ near ∞.

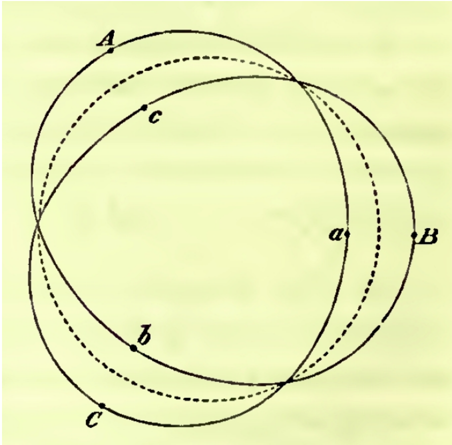
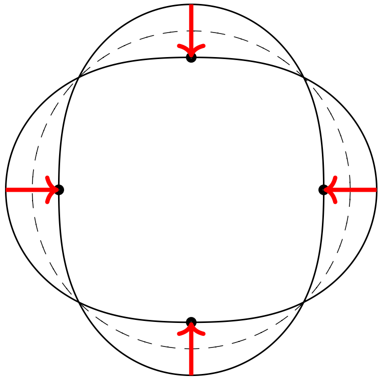


FIGURE 6. The Stokes diagram of $\langle 2x^{3/2}\rangle$, from Stokes' paper [?] on the Airy equation. The points a, b, c are the points of maximal decay.



Stokes diagram of the Weber equation, with Stokes arrows drawn.

There is a javascript program here:

https://webusers.imj-prg.fr/~philip.boalch/stokesdiagrams.html

to draw lots of other examples of Stokes diagrams, the Stokes diagrams of the "symmetric" or "hypotrochoid" irregular classes $I(a:b)$ (see the explanation in the box at the bottom there).[15] In brief $I(a:b)$ is the pull-back to the x-plane of the irregular class $\langle w^{1/b}\rangle$ under the map $w = x^a$. It has k Stokes circles where $k = (a, b)$ is the highest common factor. Explicitly:

$$I(a:b) = \bigsqcup_{i=0}^{k-1} \langle \varepsilon^i x^{a/b}\rangle \subset \mathcal{I}$$

where $\varepsilon = \exp(2\pi i/b)$. For example it is the irregular class at $x = \infty$ of the Molins–Turrittin equation $y^{(b)} = x^\nu y$, if $a = \nu + b$ [**?**, **?**]. Upto a constant $I(1:q+1)$ is also the irregular class at ∞ of the differential equation for the hypergeometric series ${}_0F_q$.

10.5. **Rank two examples.** The simplest rank two Stokes diagrams are collected in Figure 7. The left four are *rigid* in that their (symplectic) wild character varieties are dimension zero. They come from the ODEs of Clifford, Airy, Whittaker, Hermite–Weber. The next two, with 5 or 6 crossings, give the wild character varieties of Painlevé I and II.

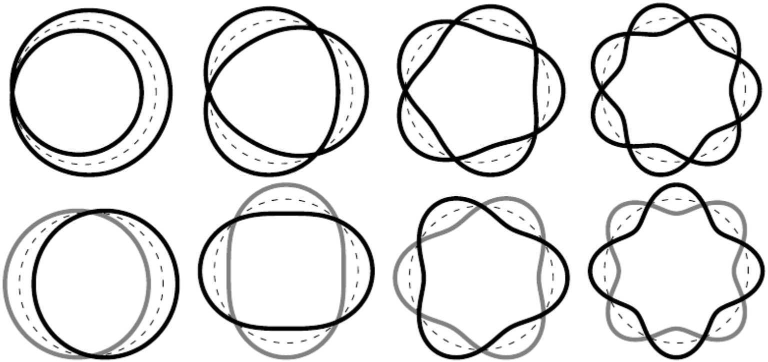


FIGURE 7. The simplest rank two Stokes diagrams $I(k:2), k = 1, 2, \ldots, 8$.

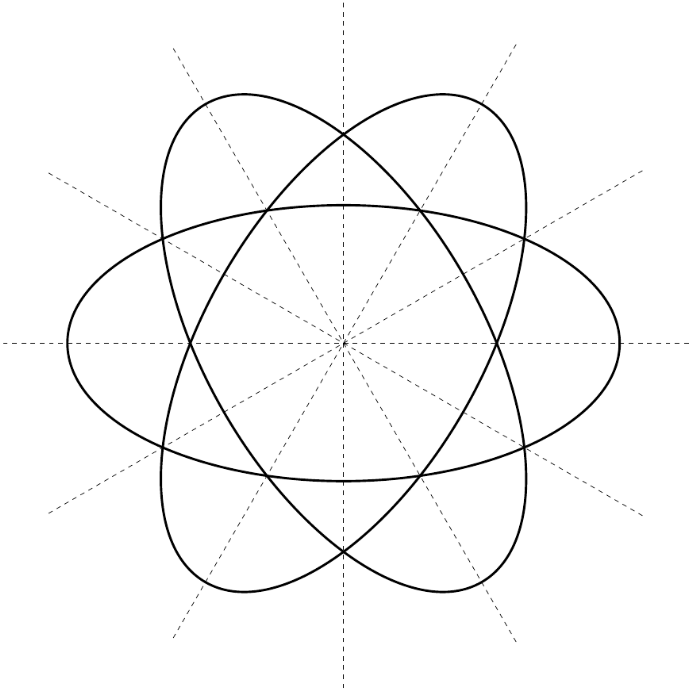


FIGURE 8. Example rank three Stokes diagram, $I(6:3)$.

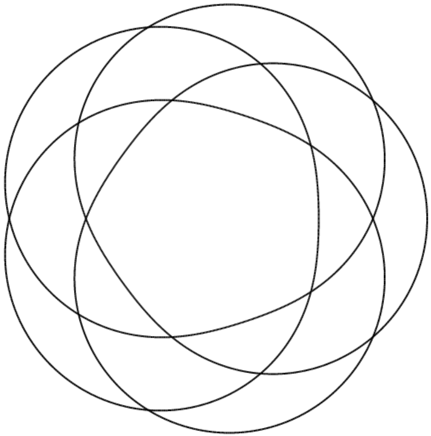


FIGURE 9. Stokes diagram at ∞ for the "hyperairy" equation $y^{(4)} = xy$

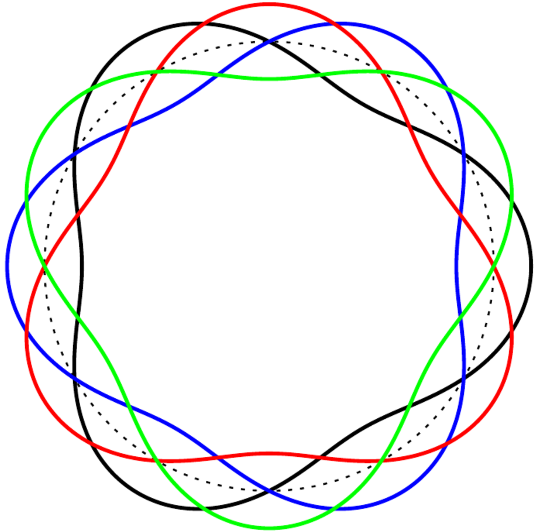


FIGURE 10. Another example rank four Stokes diagram, $I(12\!:\!4)$.

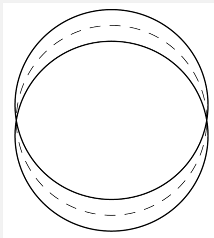
10.6. **Example Stokes diagrams: Bessel's equation.**
Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

where $\alpha \in \mathbb{C}$. This has a regular singularity at 0 and an irregular singularity at ∞. A short computation, or a glance at a book, shows that the irregular class $x = \infty$ is:

$$\Theta = \langle ix \rangle + \langle -ix \rangle$$

and that α determines the local monodromy eigenvalues at 0. In particular the singular directions are the two halves of the imaginary axis.



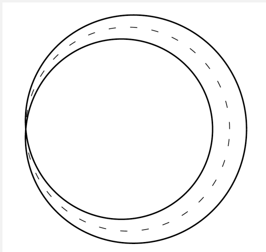
10.7. **Example Stokes diagrams: Bessel–Clifford equation.**
The Bessel–Clifford equation (also known as the confluent hypergeometric limit equation, Kummer's second equation, or the ${}_0F_1$ equation) is:

$$xy'' + ay' = y. \tag{10.1}$$

If f is any solution of this, then $x^{a-1} \cdot f(-x^2/4)$ solves the Bessel equation with parameter $\alpha = a - 1$. The irregular class at $x = \infty$ is

$$\langle 2x^{1/2} \rangle$$

and (if $a \notin \mathbb{Z}$) the monodromy around 0 has eigenvalues $1, \exp(-2\pi i a)$.



9.5. **Wild Riemann surfaces.** The irregular class makes up the basic "new modular parameters" that occur for irregular connections, behaving just like the modulus of the underlying Riemann surface and the location of the marked points $\mathbf{a}$.

In particular it behaves completely differently to the formal residue Λ.

This motivates the following definition:

Definition 9.5. *A rank n wild Riemann surface is a triple $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ where Σ is a Riemann surface, $\mathbf{a} \subset \Sigma$ is a finite subset and $\Theta = \{\Theta_a \bigm| a \in \mathbf{a}\}$ is the data of a rank n irregular class at each point $a \in \mathbf{a}$.*

Here we are mainly interested in the case where Σ is compact. We will define the character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ of any such wild Riemann surface, show that it is Poisson and forms a local system of varieties under any admissible deformation of $\mathbf{\Sigma}$.

Of course if all the irregular classes are trivial then $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ just amounts to choosing a Riemann surface with some marked points, and then $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ will be the usual (tame) character variety defined previously $\cong \mathrm{Hom}(\pi_1(\Sigma^\circ, b), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$.

Notes: This definition is from [B2014] Defn 8.1, Rmk 10.6, [BY2015] §4. There are several minor variations that we won't worry about here, but are sometimes useful: One can work with irregular types instead of irregular classes (which were called "bare irregular types" in [B2014] Rmk 10.6); this is analogous to whether or not we order the points $\mathbf{a}$. Also one can work with smooth complex algebraic curves instead of Riemann surfaces (which doesn't make much difference in the compact case); the terms "irregular curve" or "wild curve" are sometimes used to replace the term "wild Riemann surface" in the algebraic case. Op. cit. give the definition for any complex reductive group, not just $\mathrm{GL}_n(\mathbb{C})$.

Séminaire BOURBAKI
(Mai 1958)

MODULES DES SURFACES DE RIEMANN

par André WEIL

Par la combinaison des idées (récentes) de KODAIRA et SPENCER sur la variation des structures complexes avec les idées (anciennes) de TEICHMÜLLER sur le problème des modules, la théorie a fait dernièrement quelques progrès qu'on se propose d'exposer ici.

Soit T_o une surface orientée compacte de genre g, donnée une fois pour toutes. Par une surface de Riemann de genre g, on entend, comme d'habitude, une variété complexe compacte de dimension complexe 1, de genre g, munie de son orientation naturelle. Par une surface de Teichmüller de genre g, on entendra une surface de Riemann S de genre g, munie de plus d'une classe (au sens de l'homotopie) d'applications de T_o dans S, classe dont on suppose qu'elle contient au moins un homéomorphisme conservant l'orientation ; c'est là une structure (plus "riche" que celle de structure de surface de Riemann). Si π^o désigne le

Il est utile de définir une notion intermédiaire entre celle de surface de Riemann et celle de surface de Teichmüller : on l'obtient en se donnant les images des A_i^o, non dans $\pi(S)$, mais dans $H_1(S)$; la donnée de ces images sur la surface de Riemann S détermine ce qu'on appellera une "surface de Torelli". Au

Nonabelian Hodge theory on wild Riemann surfaces

Let $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ be a rank n wild Riemann surface whose underlying Riemann surface Σ is compact. Choose some residue data $\mathbf{R}$ for $\mathbf{\Sigma}$ of (global) degree zero. Recall that a "connection on $\mathbf{\Sigma}$" means a good meromorphic connection on a parabolic vector bundle on Σ with poles/parabolic filtrations at $\mathbf{a}$, and irregular class Θ_a at each point $a \in \mathbf{a}$. Similarly for Higgs bundles on $\mathbf{\Sigma}$.

Let $\mathcal{M}_{\mathrm{DR}}(\mathbf{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable connections on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$. Similarly let $\mathcal{M}_{\mathrm{Dol}}(\mathbf{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable Higgs bundles on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$. We suppose that the boundary data is chosen so they are not empty.

Theorem 1.1 (Biquard–B. 2004)**.** *There is a hyperkähler manifold $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ (equipped with a family of complex structures parameterised by $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$) that is a moduli space of irreducible wild harmonic bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ with boundary conditions determined by $\mathbf{\Sigma}, \mathbf{R}$ such that:*
1) In the complex structure determined by $1 \in \mathbb{P}^1$ the space $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{DR}}(\mathbf{\Sigma}, \mathbf{R})$ of stable good meromorphic connections,
2) In the complex structure determined by $0 \in \mathbb{P}^1$ the space $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{Dol}}(\mathbf{\Sigma}, \mathbf{R})$ of stable good meromorphic Higgs bundles,
3) If the residue data $\mathbf{R}$ is semisimple and there are no strictly semistable connections on $\mathbf{\Sigma}$ with residue data $\mathbf{R}$, then the hyperkähler metric on $\mathfrak{M}(\mathbf{\Sigma}, \mathbf{R})$ is complete.

The boundary data is related by the following table:

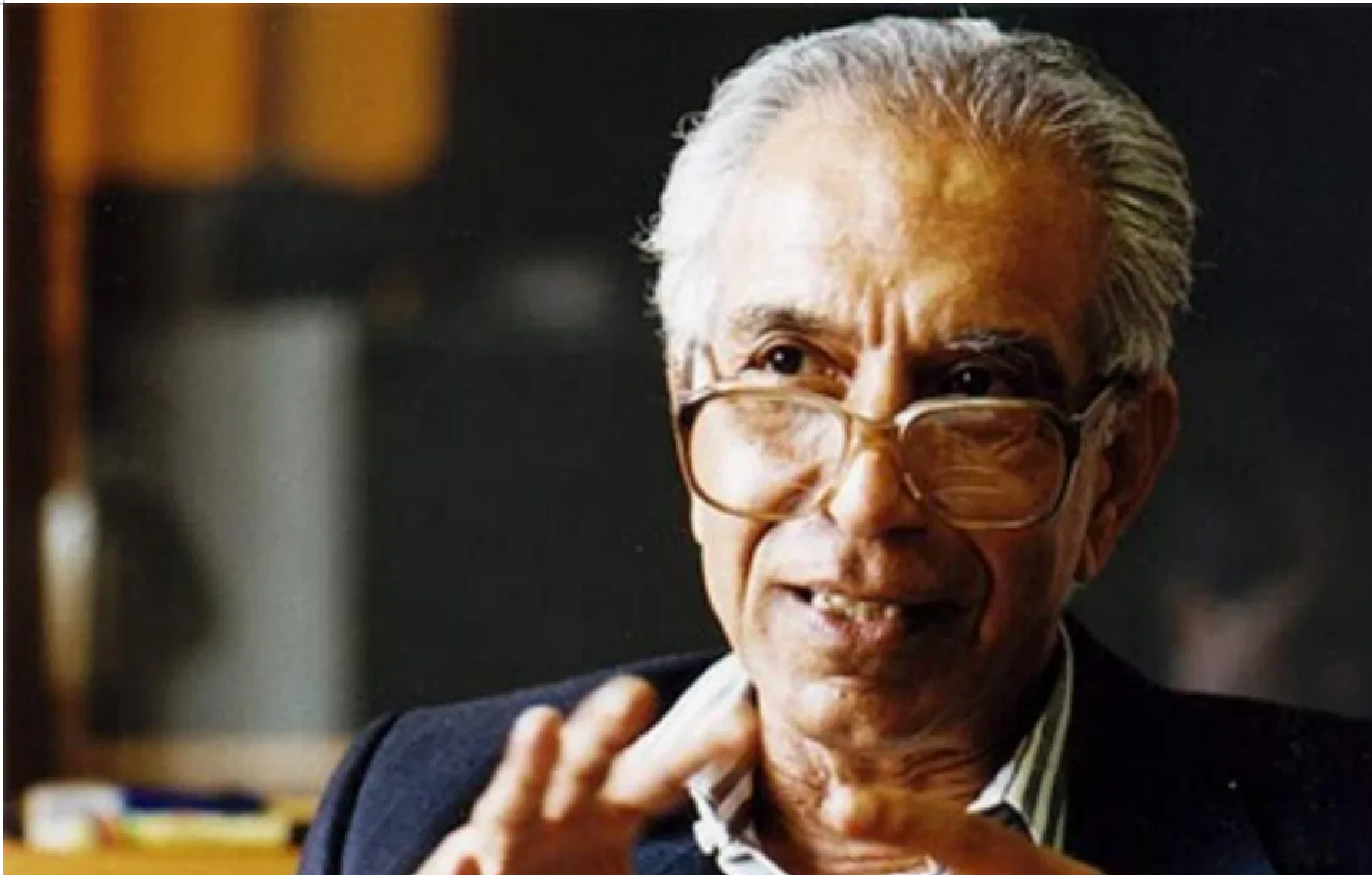
	Dolbeault	De Rham	Betti
weights $\in [0,1), [0,1), \mathbb{R}$	$\lceil\tau\rceil - \tau$	θ	$\phi = \theta + \tau$
eigenvalues $\in \mathbb{C}, \mathbb{C}, \mathbb{C}^*$	$\frac{1}{2}(\phi + \sigma)$	$\lambda = \tau + \sigma$	$\mu = \exp(2\pi i \lambda)$
exponential factors	$\frac{1}{2}q$	q	$\langle q \rangle$

• In tame case ($q = 0$) most of this is due to Konno 1993 and Nakajima 1996 (using Biquard's weighted Sobolev space approach), strengthening Simpson's 1990 tame bijective correspondence in to a diffeomorphism. Even then the completeness statement (beyond the finite energy "strongly parabolic" setting in Konno's paper) is new.

• In the wild case the construction of harmonic bundles from irreducible irregular connections on meromorphic bundles (i.e. Betti weights zero) was established earlier by Sabbah 1999.

• In the nonsingular/compact case ($q = 0 = \lambda = \theta$) it is due to Hitchin, Donaldson, Corlette, Simpson, (Fujiki, Diederich–Ohsawa).

• If also the Higgs field is zero this gives the Narasimhan–Seshadri theorem.



Actes, Congrès intern. Math., 1970. Tome 2, p. 199 à 201.

GEOMETRY OF MODULI SPACES OF VECTOR BUNDLES

by M. S. NARASIMHAN

Canonical hermitian metrics

We now introduce a hermitian metric on M. To do this it is sufficient to introduce a positive definite hermitian form on $H^1(X, W(\mathrm{Ad}\rho))$. Since $W(\mathrm{Ad}\rho)$ is given by a local system, the operator of exterior differentiation, d, is well defined on C^∞ differential forms with values in $W(\mathrm{Ad}\rho)$. Let $T(\rho)$ denote the space of d-closed C^∞ forms of type $(0, 1)$ with coefficient in $W(\mathrm{Ad}\rho)$. Then $T(\rho)$ is canonically isomorphic to $H^1(X, W(\mathrm{Ad}\rho))$. So it suffices to introduce a positive definite hermitian form on $T(\rho)$. If $\omega \in T(\rho)$, let $\omega^\#$ denote the $(1, 0)$ form with coefficients in $W(\mathrm{Ad}\rho)$ obtained by using the conjugation $A \mapsto A^*$ in $\mathfrak{gl}(n, C)$. (A^* denotes the conjugate transpose of A. Locally, if $\omega = A(z)\,d\bar{z}$, $\omega^\# = A^*(z)\,dz$). Define the hermitian scalar product in $T(\rho)$ by

$$(\omega_1, \omega_2) = \frac{1}{i}\int_X \mathrm{Trace}\,(\omega_1, \omega_2^\#),\ \omega_1, \omega_2 \in T(\rho),$$

• Cecotti–Vafa (1993), Dubrovin (1994) embedded some wall-crossing problems into irregular isomonodromy:

Braiding of some BPS states $\subset$ Braiding of Stokes data

This leads to the idea of

“Pure wall-crossing”

—i.e. to better understand and generalise **Irregular Isomonodromy**, beyond the generic case for $\mathrm{GL}_n(\mathbb{C})$, not focusing on specific applications.

—get many new natural nonlinear local systems of varieties (complete flat nonlinear Ehresmann connections).

Dubrovin's example: braiding of BPS states

B. Dubrovin's 1995 paper "Geometry of topological field theories" contained an inspiring example. In brief he looked at $n \times n$ operators of the form

$$\Lambda = \frac{d}{dz} - U - \frac{1}{z}V(u)$$

where U is a diagonal matrix with entries $u = (u_1, \dots, u_n), u_i \neq u_j$ and V was a skew-symmetric complex matrix. Viewed as a connection this has an irregular singularity at ∞, and Dubrovin defined its *Stokes matrix* S (a complex upper triangular unipotent matrix). He used them to classify *massive Frobenius manifolds* (a certain axiomatisation of certain 2d topological field theories).

On one hand, in earlier work of Cecotti–Vafa the matrix entries of S were integers counting BPS states (solitons between vacua), and a natural braid group action was defined on the space of matrices S. On the other hand Dubrovin found a braid group invariant Poisson structure in the case $n = 3$, where

$$S = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

that "looked to be new". Here are some key excerpts of his paper:

Theorem 3.2. *There exists a local one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{Massive Frobenius manifolds} \\ \text{modulo transformations (B.2)} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Stokes matrices of differential} \\ \text{operators } \Lambda \text{ modulo transformations (3.164)} \end{array} \right\}.$$

Definition 3.5. The Stokes matrix S of the operator (3.120) considered modulo the transformations (3.164) will be called *Stokes matrix of the Frobenius manifold.*

Remark 3.10. In the paper [29] Cecotti and Vafa found a physical interpretation of the matrix entries S_{ij} for a Landau - Ginsburg TFT as the algebraic numbers of solitons propagating between classical vacua. In this interpretation S always is an integer-valued matrix. Due to (3.134) they arrive thus at the problem of classification of integral matrices S such that all the eigenvalues of $S^T S^{-1}$ are unimodular. This is the main starting point in the programme of classification of $N = 2$ superconformal theories proposed in [29].

It is interesting that *the same* Stokes matrix appears, according to [29], in the Riemann - Hilbert problem of [51] specifying the Zamolodchikov (or $t\,t^*$) hermitean metric on these Frobenius manifolds.

Example F.2. For $n = 3$ we put $s_{12} = x$, $s_{13} = y$, $s_{23} = z$. The transformations of the braid group act as follows:

$$\sigma_1 : (x, y, z) \mapsto (-x, z - xy, y), \tag{F.18a}$$

$$\sigma_2 : (x, y, z) \mapsto (y - xz, x, -z). \tag{F.18b}$$

These preserve the polynomial

$$x^2 + y^2 + z^2 - xyz. \tag{F.19}$$

Indeed, the characteristic equation of the matrix $S^T S^{-1}$ has the form

$$(\lambda - 1)[\lambda^2 + (x^2 + y^2 + z^2 - xyz - 2)\lambda + 1] = 0. \tag{F.20}$$

The action of the group B_3 (in fact, this can be reduced to the action of $PSL(2, \mathbf{Z})$) admits also an invariant Poisson bracket

$$\begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x\,. \\ \{z, x\} &= zx - 2y \end{aligned} \tag{F.21}$$

The polynomial (F.19) is the Casimir of the Poisson bracket. Thus an invariant symplectic structure is induced on the level surfaces

$$x^2 + y^2 + z^2 - xyz = \text{const.}$$

A B_n-invariant Poisson bracket exists also on the space of Stokes matrices of the order n. But it has more complicated structure.

For integer x, y, z this action on the invariant surface $x^2 + y^2 + z^2 = xyz$ was discussed first by Markoff in 1876 in the theory of Diophantine approximations [27]. The general

action (F.13b), (F.14) (still on integer valued matrices) appeared also in the theory of exceptional vector bundles over projective spaces [128]. Essentially it was also found from physical considerations in [29] (again for integer matrices S) describing "braiding of Landau - Ginsburg superpotential". The invariant Poisson structure (F.21) looks to be new.

These surfaces $x^2 + y^2 + z^2 - xyz = \text{const.}$ are of course easily seen to be isomorphic to Fricke–Klein–Vogt surfaces with the same braid group action, but Dubrovin's example was inspiring since the braid group action is not arising from the motion of the poles of the operator Λ (which has only two poles, at $0, \infty$). Rather the braid group action came from the motion of the matrix U, the leading coefficient of the leading term Udz of the connection, with a pole of order 2 at $z = \infty$, the irregular part of the connection.

This led to the realisation that there is a whole new paradigm for (Poisson) braid group actions on spaces of monodromy data: if you include irregular connections there is a new type of braiding that is possible, where the space of deformation parameters is related to the structure group, not just the pole positions (and the moduli of Riemann surface). This idea led to the natural appearance of G-braid groups in 2d gauge theory [?], the new topological symplectic structures on Stokes data in general [?, ?, ?, ?, ?], and the wild nonabelian Hodge correspondence on curves [?], that we want to describe in detail. Some aspects of such braiding had been studied earlier (by Garnier, Malgrange, Jimbo–Miwa–Ueno) for generic connections on vector bundles.

24. *Monodromy Preserving Deformation of Linear Differential Equations with Irregular Singular Points*

By Kimio UENO
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1980)

§ 1. **Introduction.** The purpose of the present article is to study the monodromy preserving deformation of linear ordinary differential equations with irregular singular points.

The theory of monodromy preserving deformation originates in the classical works of continental mathematicians in the beginning of this century (L. Schlesinger [1], R. Fuchs [2], R. Garnier [3]). In particular they revealed that the Painlevé equations are nothing other than the deformation equations for appropriate linear differential equations. Their works, however, had somehow been forgotten until interest is aroused quite recently both from mathematical side (K. Aomoto [4], K. Okamoto [5], B. Klares [6]) and from physico-mathematical side (J. Myers [7], Wu *et al.* [8], Sato *et al.* [9], Ablowitz *et al.* [10], Flaschka-Newell [11]). Since most of the problems appearing in applied mathematics or mathematical physics have irregular singular points, it seems important, not only from theoretical viewpoints but also for applications, to establish a general theory that will cover the cases admitting irregular as well as regular singularities.

K. Ueno's first note $\rightsquigarrow$ isomonodromy of generic irregular connections, Jimbo–Miwa–Ueno 1981.

—used developments in Stokes data (Sibuya, Wasow, Balser–Jurkat–Lutz) to cross the walls in Birkhoff's 1913 approach.

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Symplectic Manifolds and Isomonodromic Deformations

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We study moduli spaces of meromorphic connections (with arbitrary order poles) over Riemann surfaces together with the corresponding spaces of monodromy data (involving Stokes matrices). Natural symplectic structures are found and described both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah–Bott approach). This enables us to give an intrinsic symplectic description of the isomonodromic deformation equations of Jimbo, Miwa and Ueno, thereby putting the existing results for the six Painlevé equations and Schlesinger's equations into a uniform framework.

Although apparently not mentioned in the literature, a useful perspective (explained in Section 7) has been to interpret the paper [40] of Jimbo, Miwa and Ueno, as stating that the Gauss–Manin connection in non-Abelian cohomology (in the sense of Simpson [64]) generalises to the case of *meromorphic* connections. This offers a fantastic guide for future generalisation.

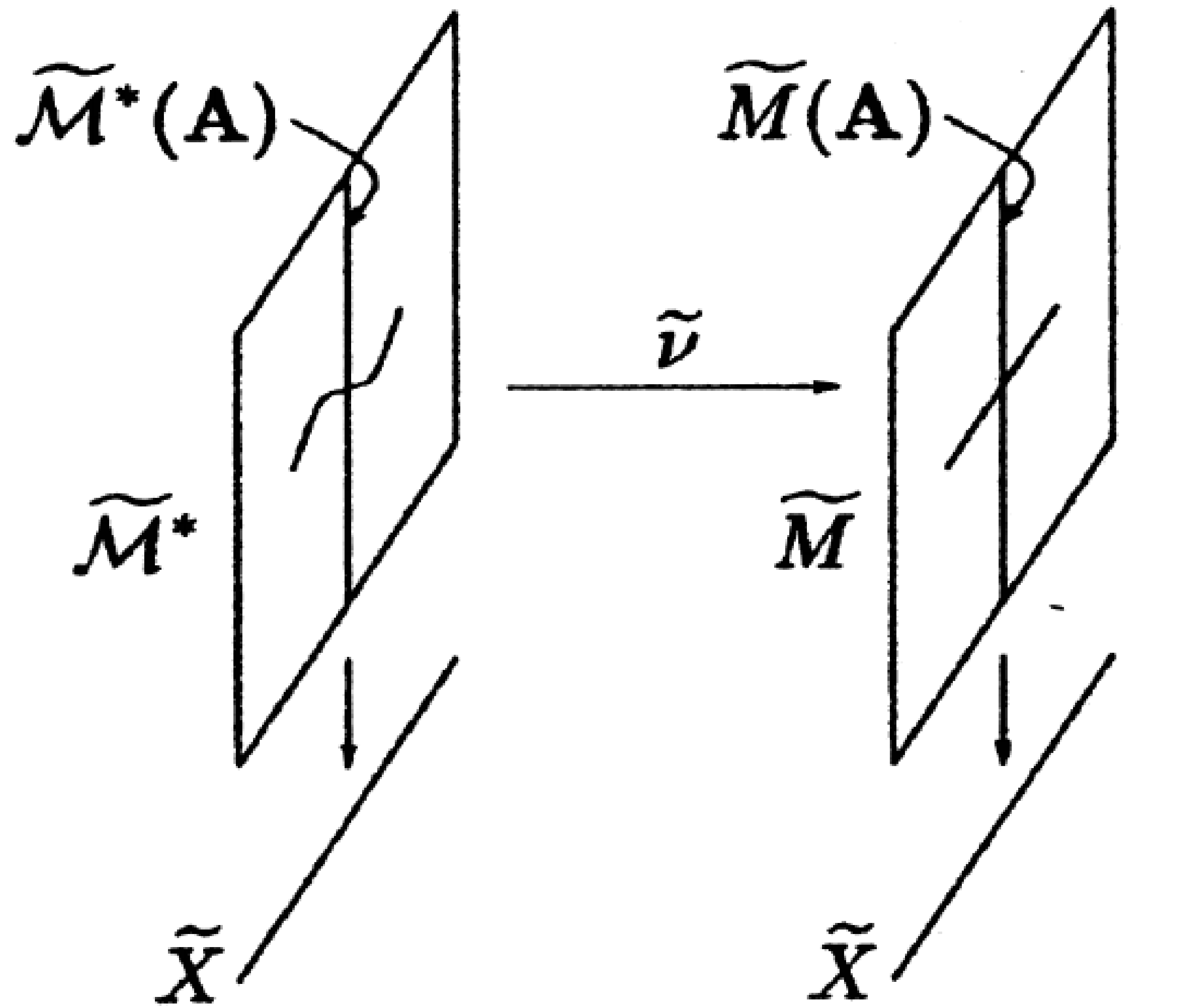

FIG. 1. Isomonodromic Deformations.

6. THE MONODROMY MAP IS SYMPLECTIC

Most of the story so far can be summarised in the commutative diagram:

$$\begin{tikzcd}
 & \widetilde{\mathscr{M}}(\mathbf{A}) \arrow[r, "\cong"] & \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathscr{G}_1 \arrow[d, "\cong"] \\
 & \bigcup & \\
\widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!/ G \cong & \widetilde{\mathscr{M}}^*(\mathbf{A}) \arrow[r, "\tilde{\nu}"] & \widetilde{M}_0(\mathbf{A}).
\end{tikzcd} \tag{29}$$

Before proving this we deduce what the monodromy data corresponds to in the meromorphic world:

COROLLARY 4.9. *Taking monodromy data induces bijections*

$$\widetilde{\mathcal{M}}(\mathbf{A}) \cong \widetilde{M}_0(\mathbf{A}) \qquad \textit{and} \qquad \mathcal{M}(\mathbf{A}) \cong M(\mathbf{A})$$

between the spaces of meromorphic connections on degree zero bundles and the corresponding spaces of monodromy data. In particular $\widetilde{\mathcal{M}}(\mathbf{A})$ *inherits the structure of a complex manifold from* $\widetilde{M}_0(\mathbf{A})$.

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Inventiones mathematicae

Stokes matrices, Poisson Lie groups and Frobenius manifolds

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1. Introduction

The purpose of this paper is to point out and then draw some consequences of the fact that the Poisson Lie group G^* dual to $G = GL_n(\mathbb{C})$ may be identified with a certain moduli space of meromorphic connections over the unit disc having an irregular singularity at the origin. (G^* will be fully described in Sect. 2.)

The key feature of this point of view is that there is a holomorphic map

$$\nu : \mathfrak{g}^* \longrightarrow G^*$$

from the dual of the Lie algebra to the group G^*, for each choice of diagonal matrix A_0 with distinct eigenvalues—the 'irregular type'. This map is essentially the Riemann-Hilbert map or de Rham morphism for such connections (we will call it the 'monodromy map'); it is generically a local analytic isomorphism. The main result is:

Theorem 1. *The monodromy map ν is a Poisson map for each choice of irregular type, where $\mathfrak{g}^*$ has its standard complex Poisson structure and G^* has its canonical complex Poisson Lie group structure, but scaled by a factor of $2\pi i$.*

IMRN International Mathematics Research Notices
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G-**Bundles, Isomonodromy, and Quantum Weyl Groups**

Philip P. Boalch

1 Introduction

It is now twenty years since Jimbo, Miwa, and Ueno [23] generalized Schlesinger's equations (governing isomonodromic deformations of logarithmic connections on vector bundles over the Riemann sphere) to the case of connections with arbitrary order poles. An interesting feature was that new deformation parameters arose: one may vary the *irregular type* of the connections at each pole of order two or more (irregular pole), as well as the pole positions. Indeed, for each irregular pole the fundamental group of the space of deformation parameters was multiplied by a factor of

$$P_n = \pi_1(\mathbb{C}^n \setminus \text{diagonals}), \tag{1.1}$$

where n is the rank of the vector bundles. (This factor arose because the connections must be *generic*; the leading term at each irregular pole must have distinct eigenvalues.)

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G-Bundles, Isomonodromy, and Quantum Weyl Groups

Philip P. Boalch

In more detail, in this "simplest case" the fundamental group of the space of deformation parameters is the generalized pure braid group associated to $\mathfrak{g} = \mathrm{Lie}(G)$:

$$P_{\mathfrak{g}} = \pi_1(\mathfrak{t}_{\mathrm{reg}}), \tag{1.2}$$

where $\mathfrak{t}_{\mathrm{reg}}$ is the regular subset of a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. By considering isomonodromic deformations, one obtains a nonlinear (Poisson) action of $P_{\mathfrak{g}}$ as follows (this is purely geometrical—as explained in [9] the author likes to think of isomonodromy as a natural analogue of the Gauss-Manin connection in non-abelian cohomology): there is a

For $k = 2$ (which will be prominent in Section 3), $\mathbb{A}$ simply consists of the directions from 0 to $\alpha(A_0)$ for all $\alpha \in \mathcal{R}$. (In general $\mathbb{A}$ is just the inverse image under the $k-1$ fold covering map $z \to z^{k-1}$ of the directions to the points of the set $\langle A_0, \mathcal{R}\rangle \subset \mathbb{C}^*$.) Clearly, $\mathbb{A}$ has $\pi/(k-1)$ rotational symmetry and so $l := \#\mathbb{A}/(2k-2)$ is an integer. We

- The *roots* $\mathcal{R}(d)$ of d are the roots $\alpha \in \mathcal{R}$ *supporting* d:

$$\mathcal{R}(d) := \{\alpha \in \mathcal{R} \mid (\alpha \circ q)(z) \in \mathbb{R}_{<0} \text{ for } z \text{ along } d\}. \tag{2.3}$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
GL_n other G

$$Q = -A/z$$ (A diagonal with distinct eigenvalues)

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

GL_n (Balser-Jurkat-Lutz '79), other G (P.B. '01)

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
GL_n other G

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

$G = K_{\mathbb{C}}$ complex reductive group (e.g. $GL_n(\mathbb{C})$)

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
GL_n other G

$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

$G = K_{\mathbb{C}}$ complex reductive group (e.g. $GL_n(\mathbb{C})$)

$T \subset G$ maximal torus

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
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$$Q = -A/z \qquad \left(A \in \mathfrak{t}_{reg}\right)$$

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$T \subset G$ maximal torus

$\mathfrak{g} = \mathrm{Lie}(G)$

$\mathfrak{t} = \mathrm{Lie}(T)$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
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$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

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$T \subset G$ maximal torus

$$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

$\mathfrak{t} = \mathrm{Lie}(T)$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)
GL_n other G

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

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$$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

root spaces $\cong \mathbb{C}$

roots $R \subset \mathfrak{t}^*$

$\mathfrak{t} = \mathrm{Lie}(T)$

Example of definition of Stokes groups (Balser–Jurkat–Lutz '79 GL_n, P.B. '01 other G)

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$G = K_{\mathbb{C}}$ complex reductive group (e.g. $GL_n(\mathbb{C})$)

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$\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$

root spaces $\cong \mathbb{C}$

roots $R \subset \mathfrak{t}^*$

$\mathfrak{t} = \mathrm{Lie}(T)$

$$\mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} \mid [X, Y] = \alpha(X) Y \quad \forall X \in \mathfrak{t} \}$$

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79, P.B. '01)

Balser-Jurkat-Lutz: GL_n P.B.: other G

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad , \quad R \subset \mathfrak{t}^*$$

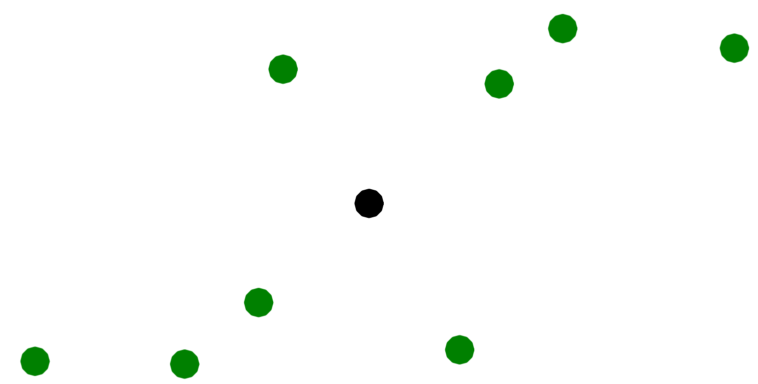
z

Example of definition of Stokes groups (Balser-Jurkat-Lutz '79 (GL_n), P.B. '01 (other G))

$$Q = -A/z \qquad (A \in \mathfrak{t}_{reg})$$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \, , \qquad R \subset \mathfrak{t}^*$$

z



Plot $\langle R, A \rangle \subset \mathbb{C}^*$

Example of definition of Stokes groups (Balser–Jurkat–Lutz '79, P.B. '01)
GL_n other G

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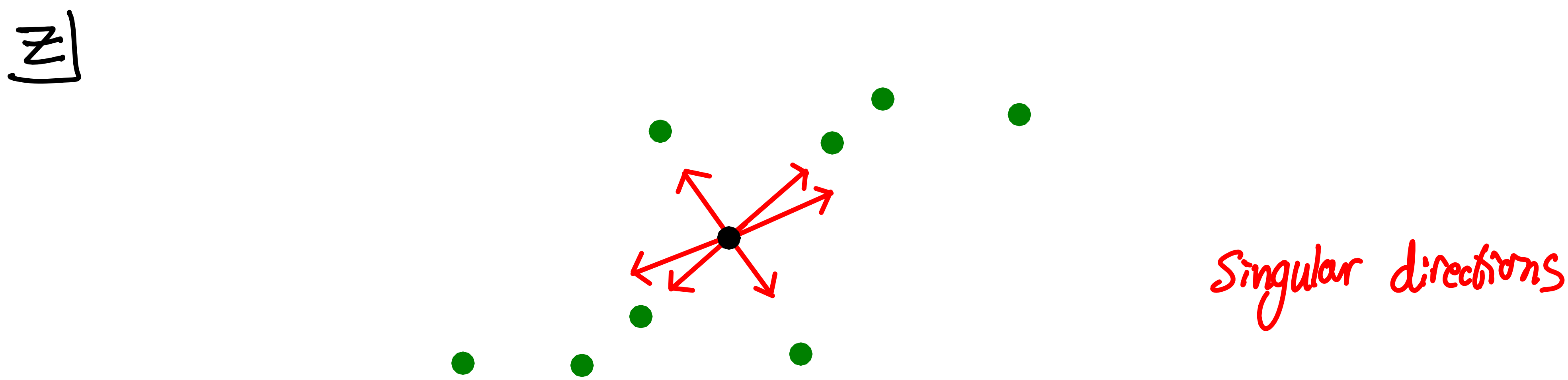

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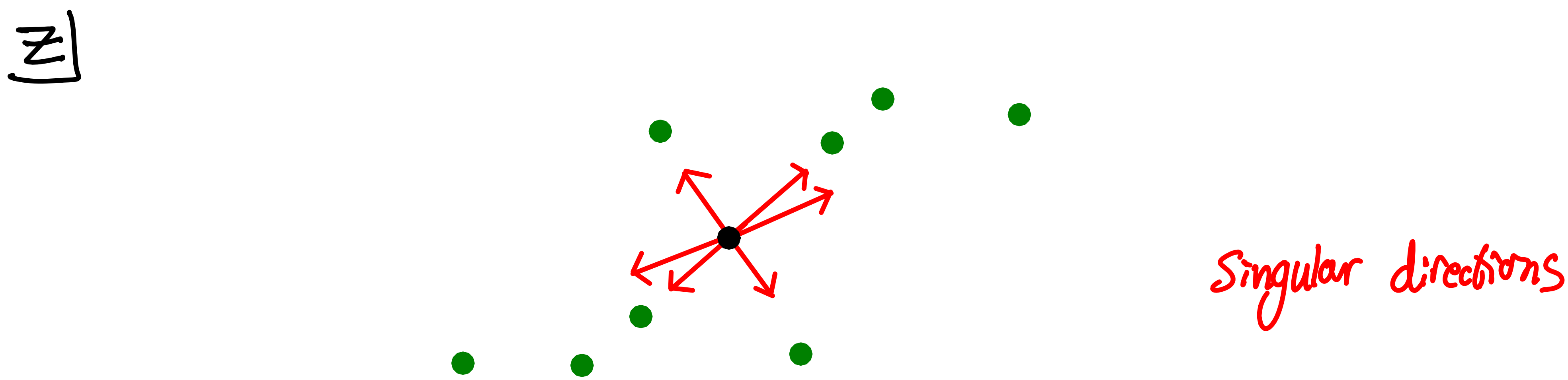

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

$$\mathrm{Sto}_d = \prod_{\{\alpha \,|\, \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

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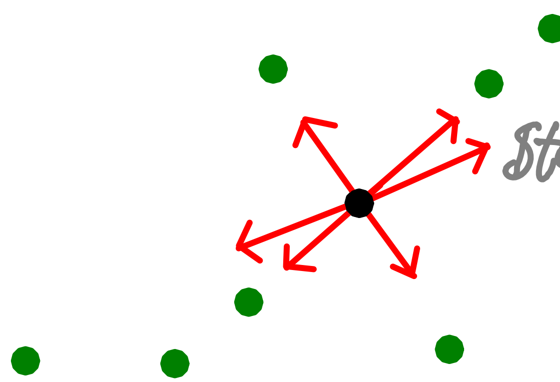

$\mathrm{Sto}_{d_1} = \exp(\mathfrak{g}_{\alpha_1})$

Singular directions

Plot $\langle R, A \rangle \subset \mathbb{C}^*$

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Example of definition of Stokes groups (Balser–Jurkat–Lutz '79 (GL_n), P.B. '01 (other G))

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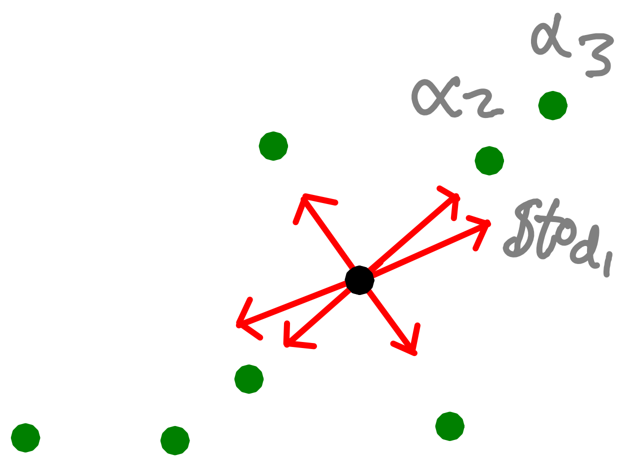
$$Sto_{d_2} = \exp(\mathfrak{g}_{\alpha_2})\exp(\mathfrak{g}_{\alpha_3})$$

z



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GL_n other G

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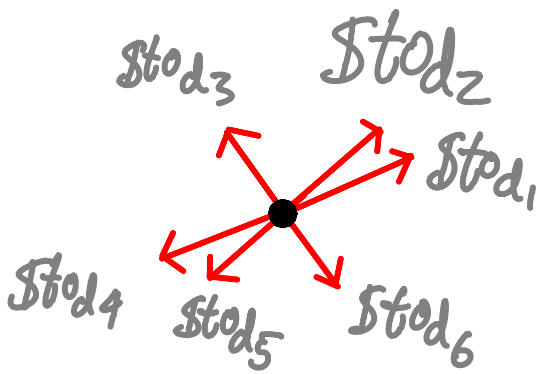

Stokes groups

singular directions

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Example of definition of Stokes groups (Balser–Jurkat–Lutz '79 (GL_n), P.B. '01 (other G))

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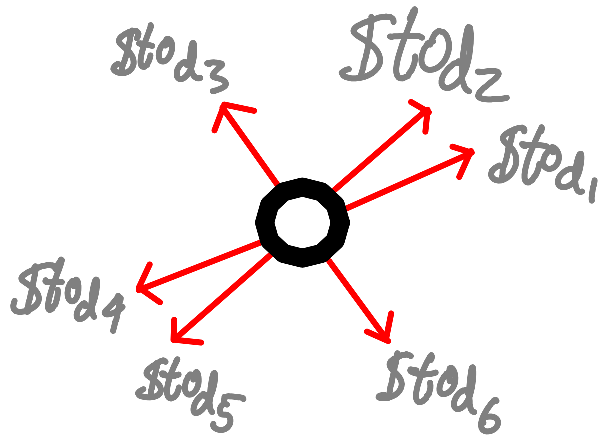

Stokes groups

Singular directions

Real blow-up

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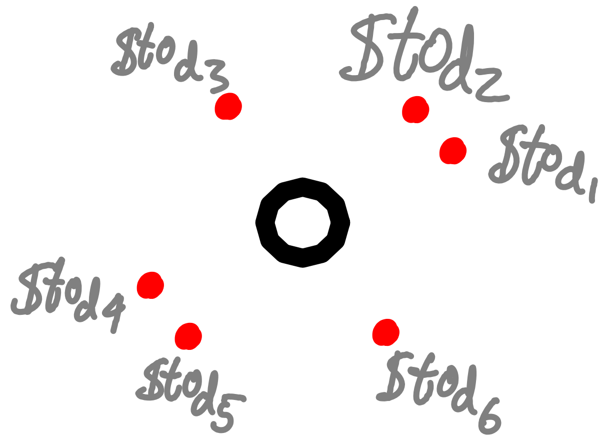

$\widetilde{\Sigma}$

Stokes groups

extra punctures

Real blow-up

$$\text{Plot } \langle R, A \rangle \subset \mathbb{C}^*$$

$$\mathbb{S}to_d = \prod_{\{\alpha \mid \alpha(A) \in d\}} \exp(\mathfrak{g}_\alpha) \subset G$$

Example of definition of Stokes groups (Balser–Jurkat–Lutz '79, P.B. '01)
GL_n other G

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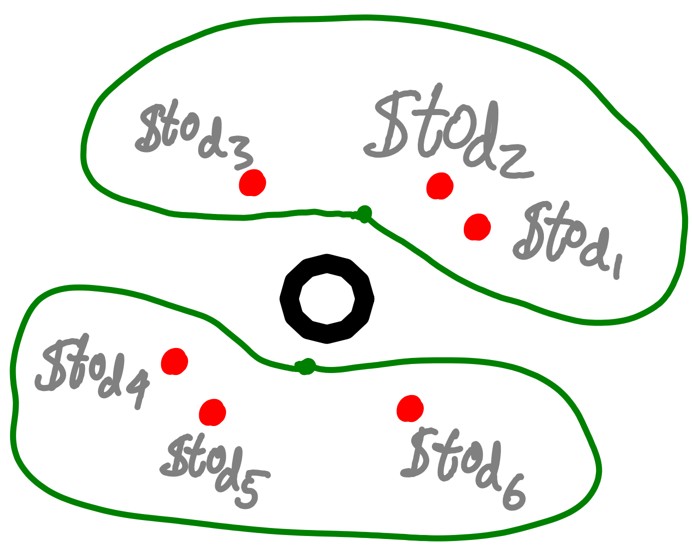
$z|$

Half-periods ⟹ unipotent radicals of Borels



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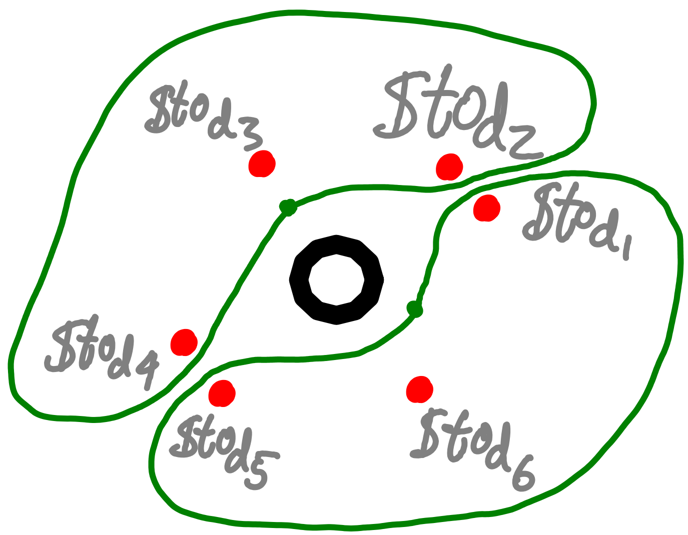

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GL_n $\qquad$ other G

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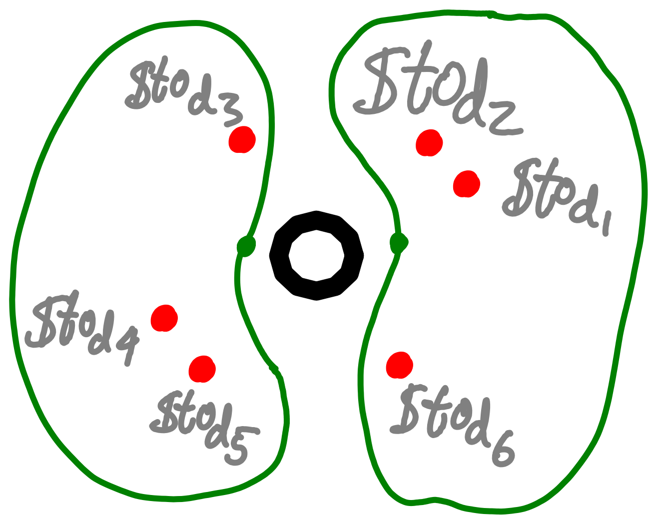
z

Half-periods $\Rightarrow$ unipotent radicals of Borels



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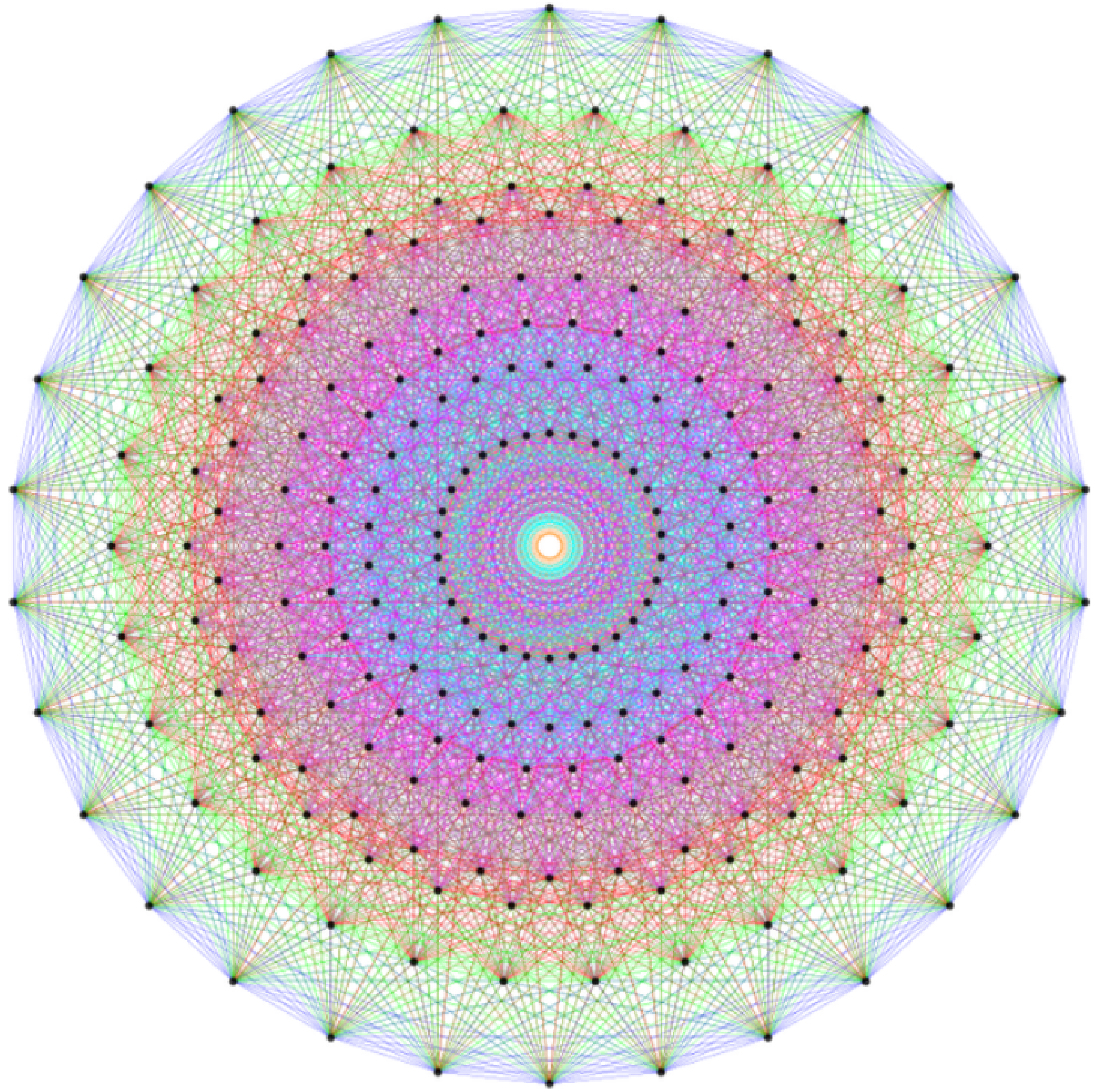


FIGURE 3. Coxeter projection of the E_8 roots; There are 60 rays with four roots; they give the 60 four-dimensional Stokes groups of the degree 30 cyclic pullback of the E_8 Frenkel–Gross connection

Annals of Mathematics **179** (2014), 1–65

Geometry and braiding of Stokes data; Fission and wild character varieties

By P. P. Boalch

To Robbie

Let $\mathbf{H} = H_1 \times \cdots \times H_m \subset G^m$. The main result is then:

Theorem 1.1. *The space* $\operatorname{Hom}_{\mathbb{S}}(\Pi, G)$ *of Stokes representations is both a smooth affine variety and a quasi-Hamiltonian* $\mathbf{H}$*-space.*

This implies that the quotient $\operatorname{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$, which classifies meromorphic connections with the given irregular types, inherits a Poisson structure, and its symplectic leaves are obtained by fixing a conjugacy class $\mathcal{C}_i \subset H_i$ for each $i = 1, \ldots, m$. We will also characterise the stable points of $\operatorname{Hom}_{\mathbb{S}}(\Pi, G)$

Let $\pi : \Sigma \to \mathbb{B}$ be an admissible family of irregular curves, and for any $p \in \mathbb{B}$, let M_p denote the Poisson variety $\operatorname{Hom}_{\mathbb{S}}(\Pi, G)/\mathbf{H}$ associated to the irregular curve Σ_p.

Theorem 10.2. *The varieties* M_p *assemble into a local system of Poisson varieties over* $\mathbb{B}$.

TWISTED WILD CHARACTER VARIETIES

PHILIP BOALCH AND DAISUKE YAMAKAWA

Abstract. We will construct twisted versions of the wild character varieties.

arXiv:1512.08091v1 [math.AG] 26 Dec 2015

1. Introduction

This article extends the algebraic construction of the wild character varieties [12, 14, 16] to the case of "twisted" Stokes local systems. There are two, closely related, types of twist that appear, both already mentioned in [16]. Firstly we will consider Stokes data for connections with twisted formal normal forms (often called the "ramified case"). A simple example where this type of twist occurs is the Airy equation, which was studied in Stokes' original article on the Stokes phenomenon [42]. Here the "twist" can be understood visually from the following diagram, drawn by Stokes:

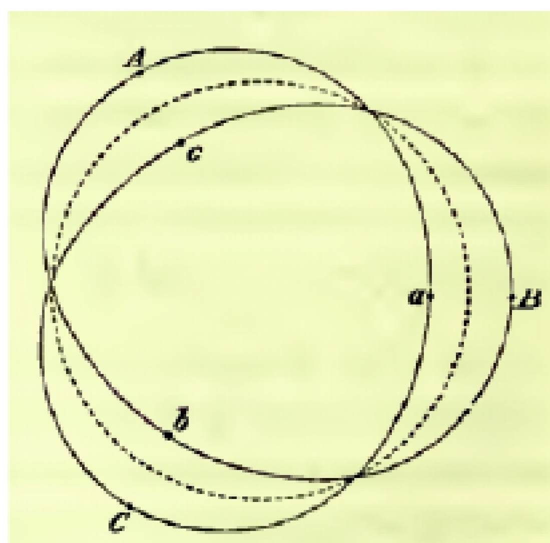
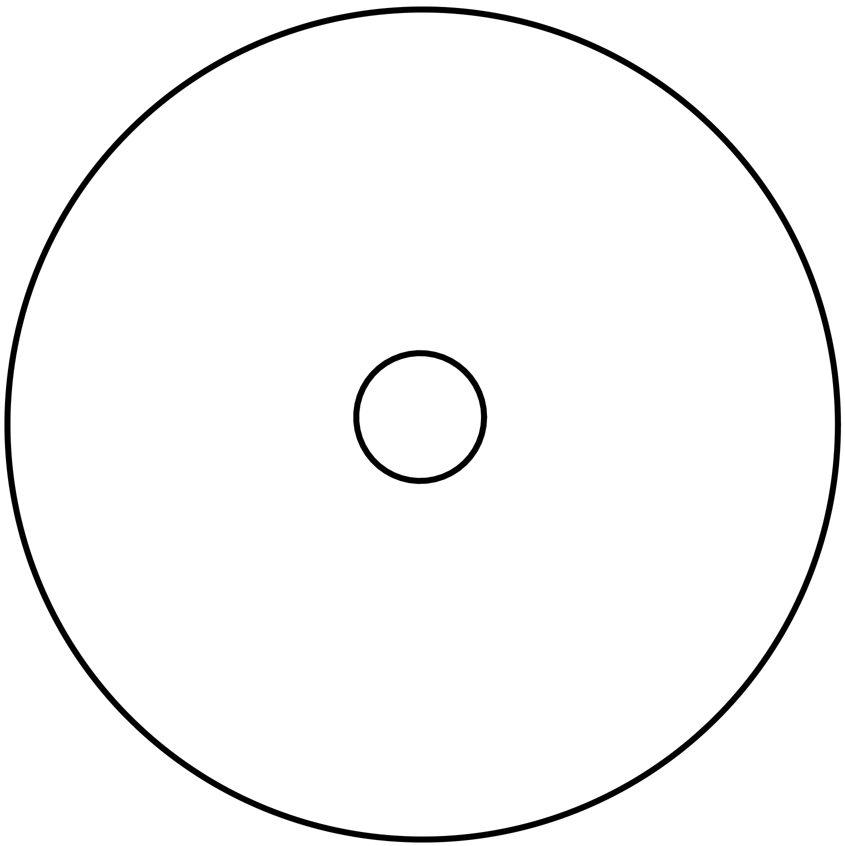


Figure 1. The Stokes diagram of the Airy equation, from [42] p.116.

Such twists can be recognised by the appearance of fractional powers of a local coordinate in the exponential factors of formal solutions, such as $\exp(\pm 2x^{3/2})$ in the

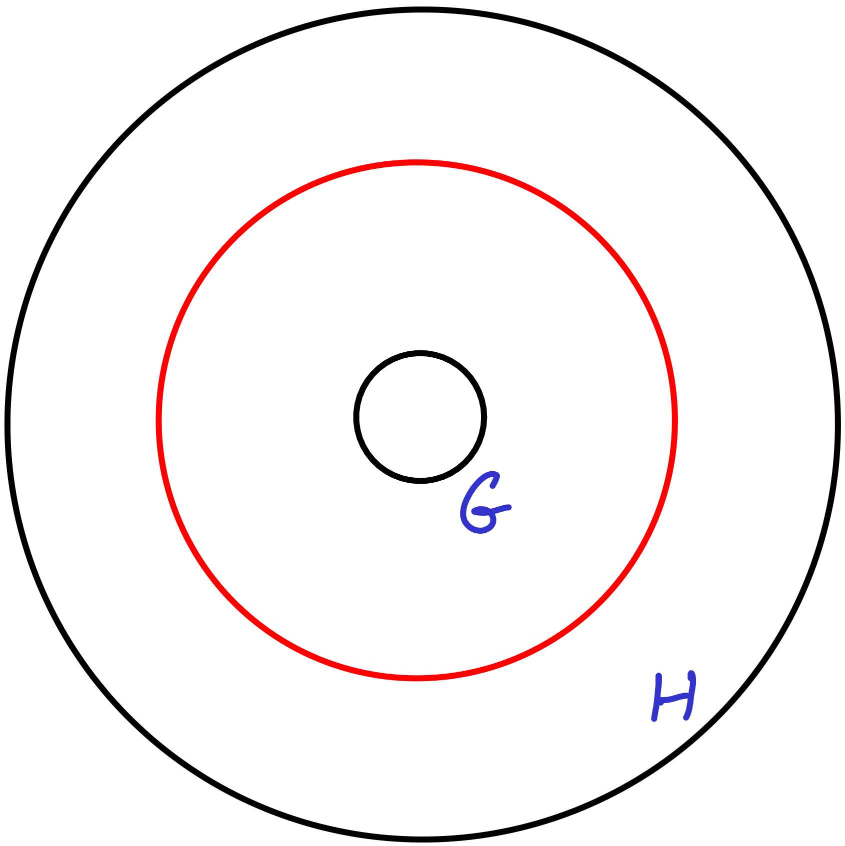
Fission spaces



Fission spaces

$V = \bigoplus_{i \in I} V_i$ $\quad$ I graded vector space

$G = GL(V) \supset H = GrAut(V) \cong \prod GL(V_i)$



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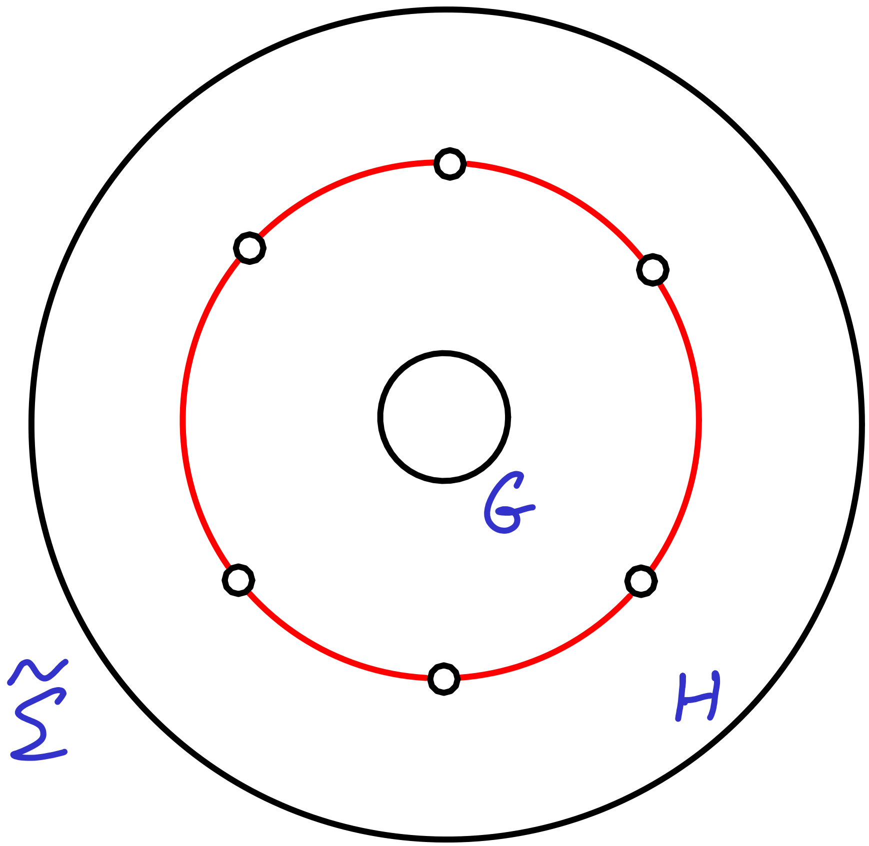
- $2k$ tangential punctures $\circ$



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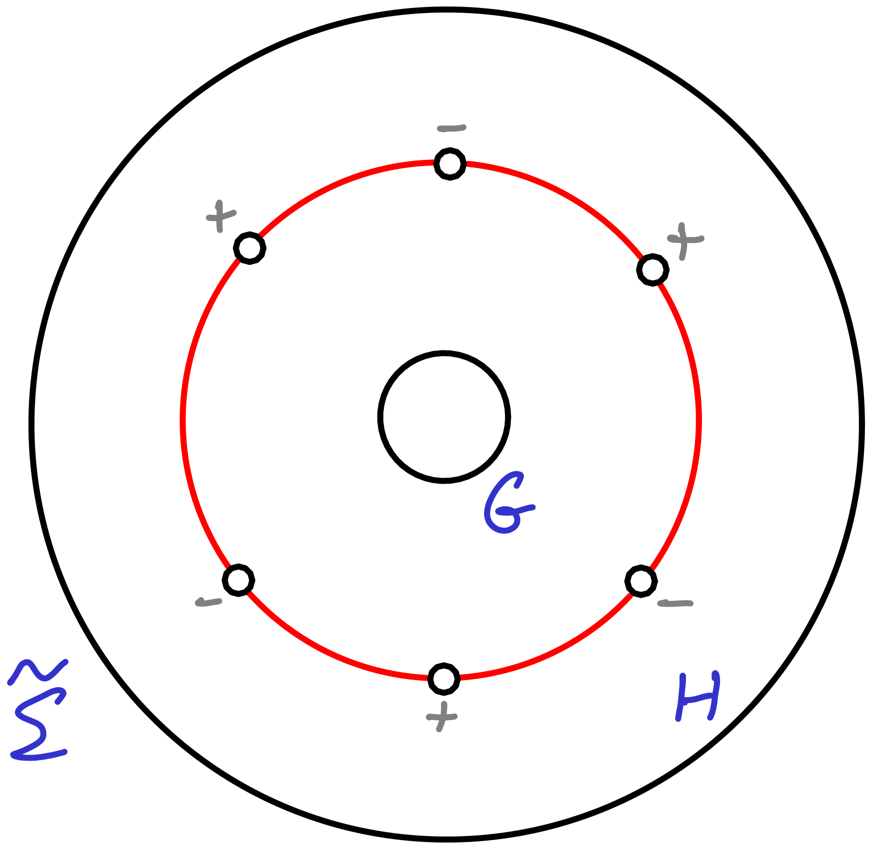

- 2k tangential punctures ∘
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)

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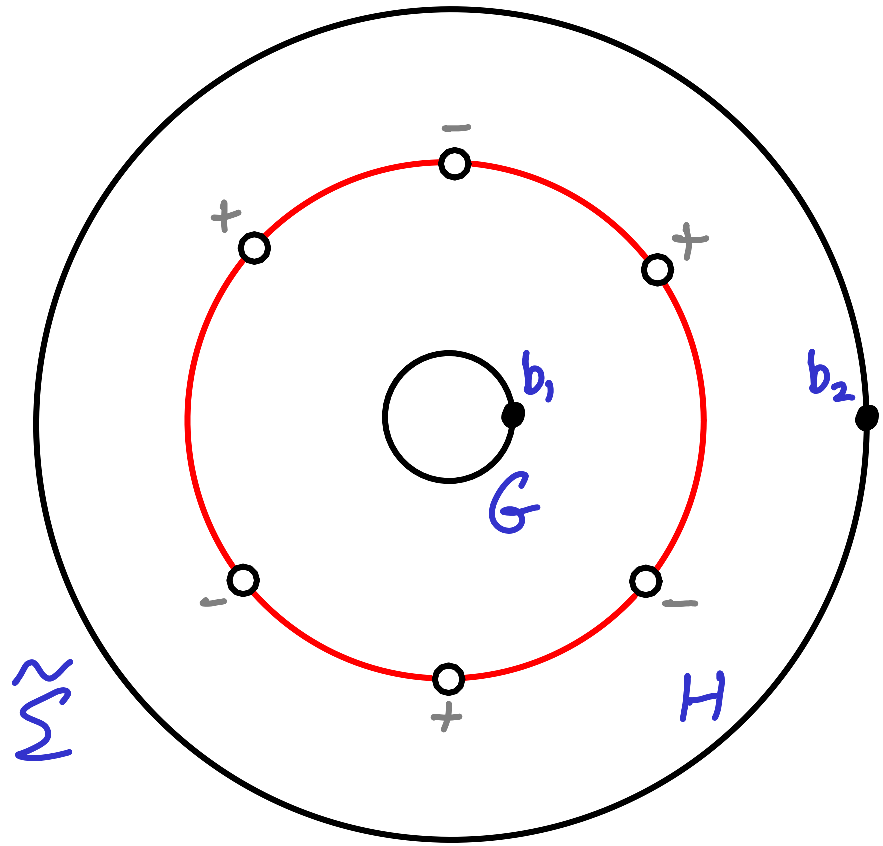

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- $\Pi = \Pi_1(\widetilde{\Sigma}, \{b_1, b_2\})$ (wild surface groupoid)

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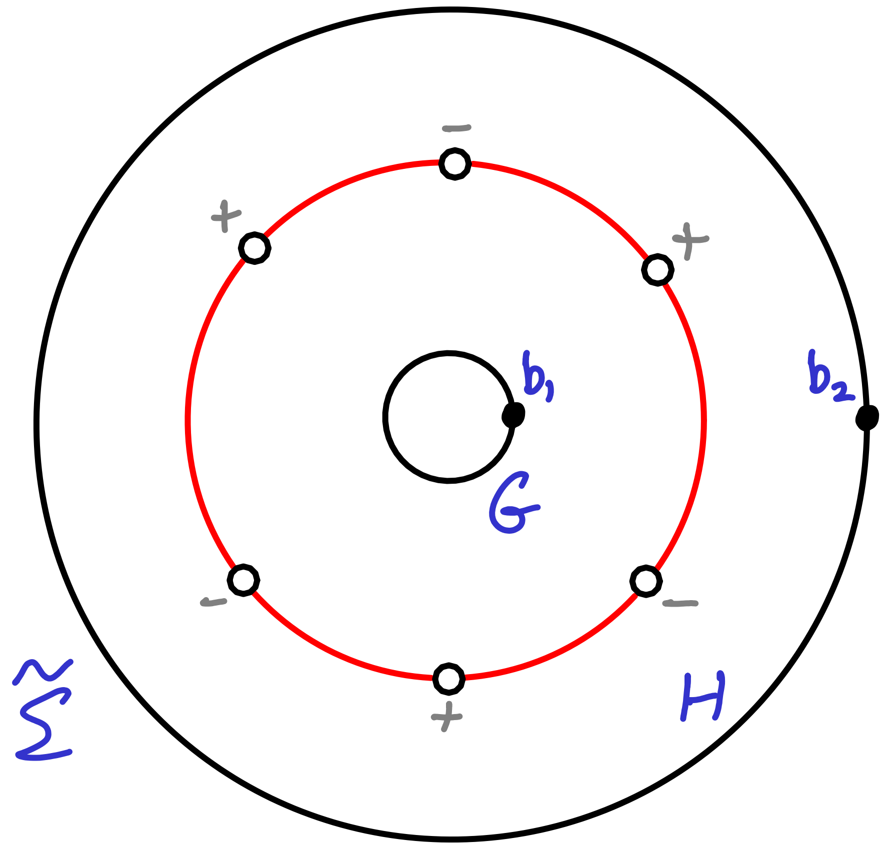

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- $\mathcal{A} = {}_G\mathcal{A}^k_H = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$\cong G \times H \times (U_+ \times U_-)^k$

$\cong \{\text{Stokes local systems framed at } b_1, b_2\} / \text{iso.}$

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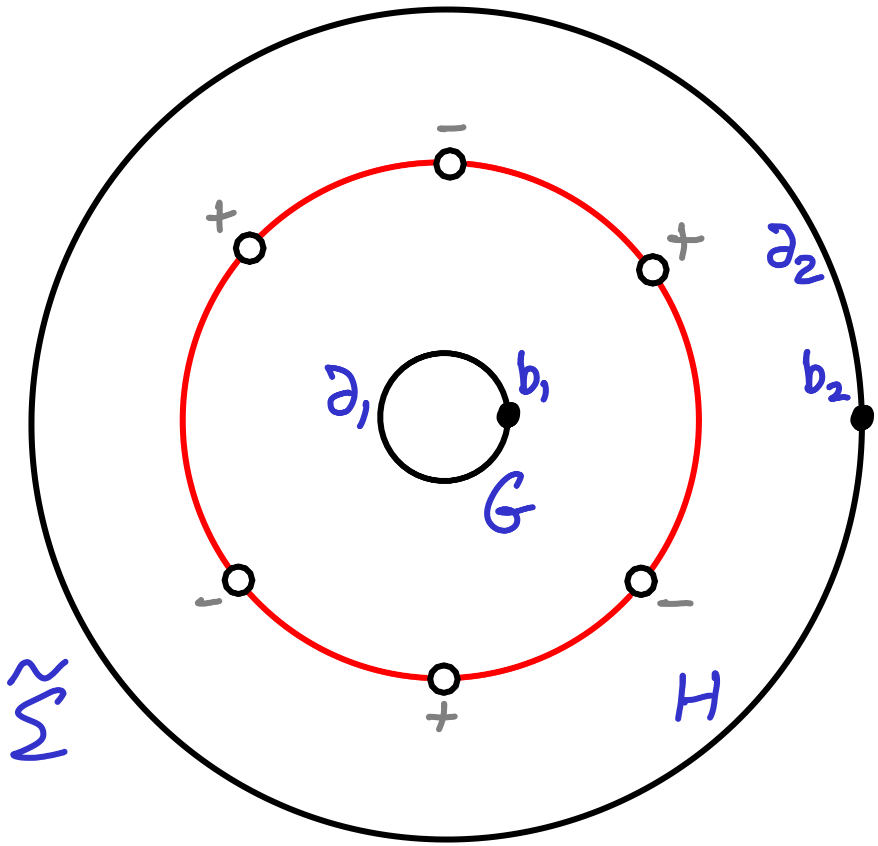

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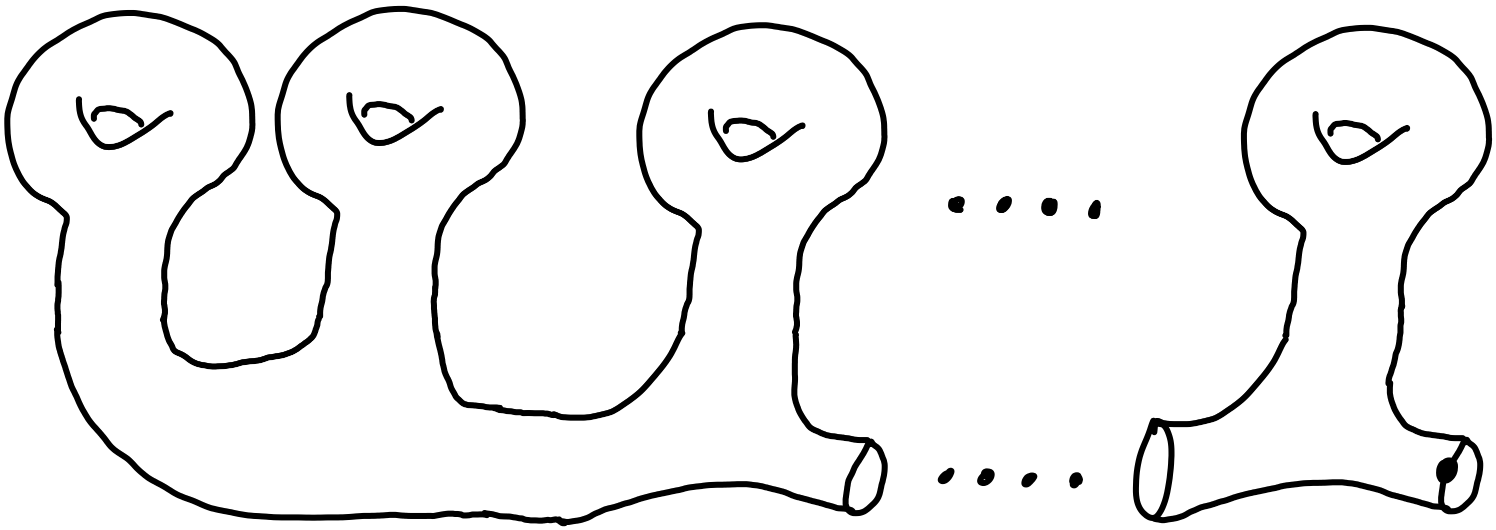
Thm $\mathcal{A}$ is a quasi-Hamiltonian $G \times H$ space with moment map $\mu : \mathcal{A} \to G \times H$, $\quad \mu(\rho) = (\rho(\partial_1), \rho(\partial_2))$

(2002 $H = T$ (any G), 2009 any H, G ($k=1$), 2011 in general)

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $R = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$\cong G^{2g}$, $\mu = [A_1, B_1] \cdots [A_g, B_g] : R \to G$

$[a,b] = aba^{-1}b^{-1}$

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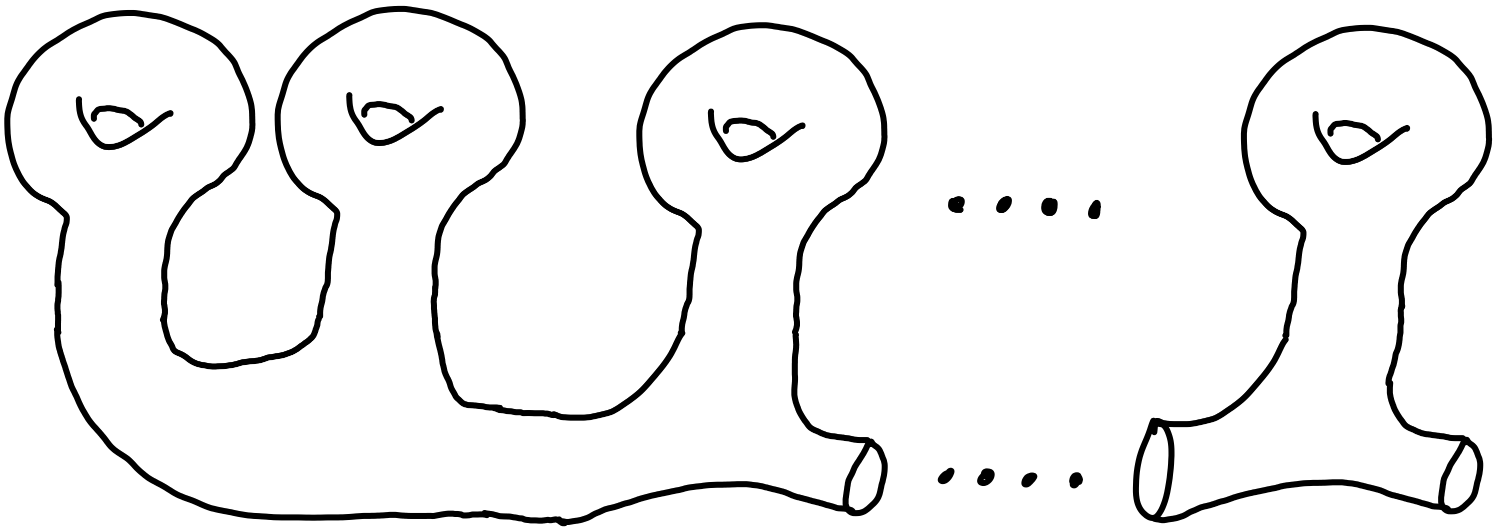

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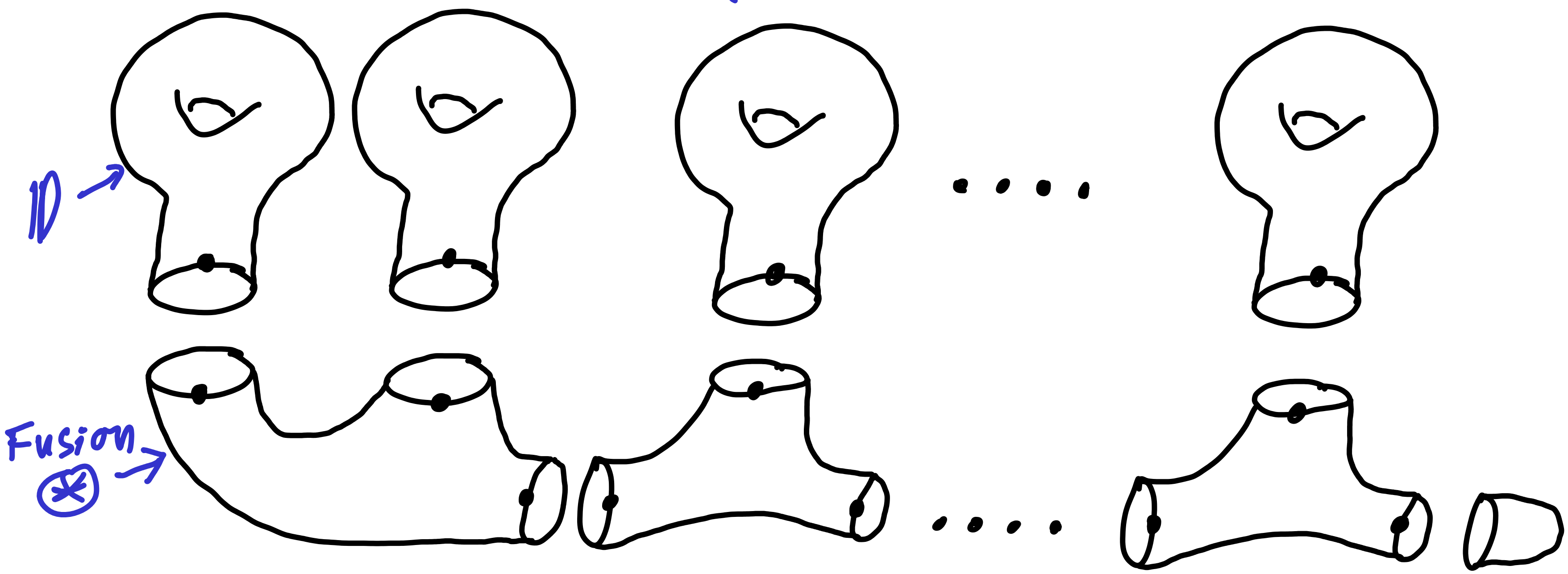
Cor.
- $\mathcal{M}_B = R/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$

E.g. $\mathcal{M}_B(\Sigma_g) = R/\!/G = \mu^{-1}(1)/G = \{A, B \in G^{2g} \mid \prod [A_i, B_i] = 1\}/G$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$\cong G^{2g}$, $\mu = [A_1, B_1] \cdots [A_g, B_g] : \mathcal{R} \to G$

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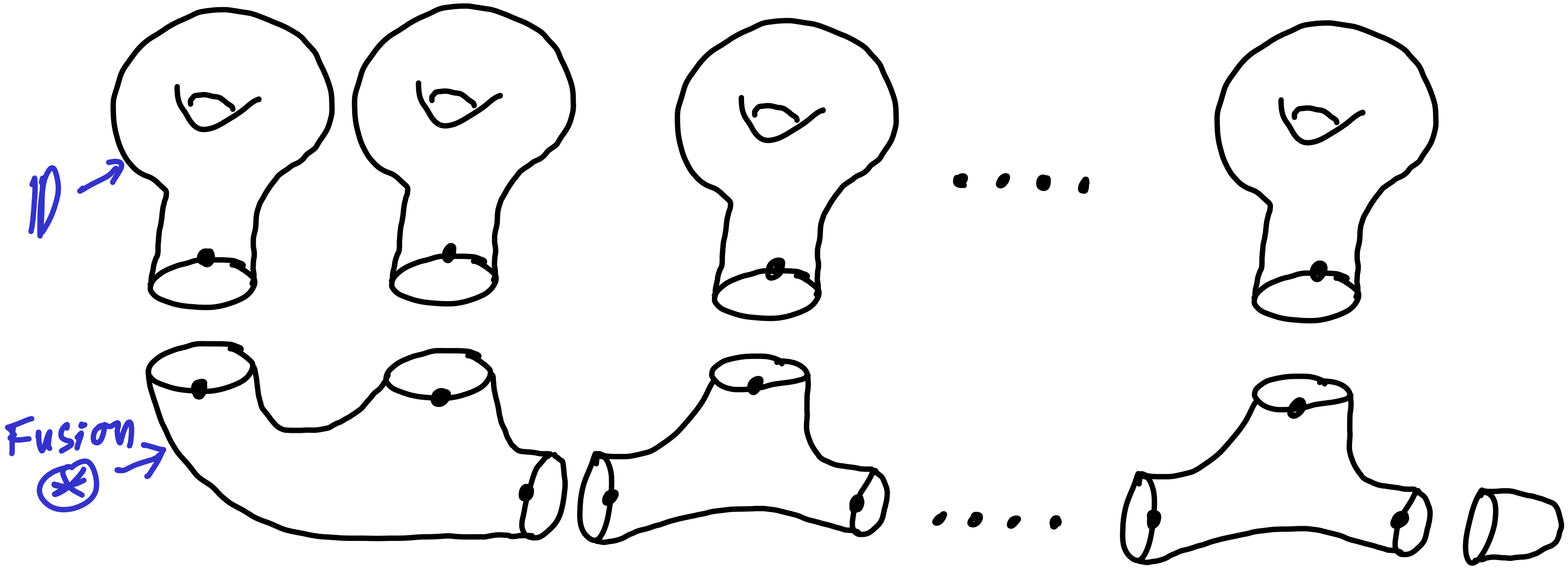
Cor.
- $\mathcal{M}_B = \mathcal{R}/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$
- Can fuse simple pieces: $\mathcal{R} = \mathbb{D} \circledast \cdots \circledast \mathbb{D}$, $\mathbb{D} = \mathcal{R}(\Sigma_{1,1})$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)

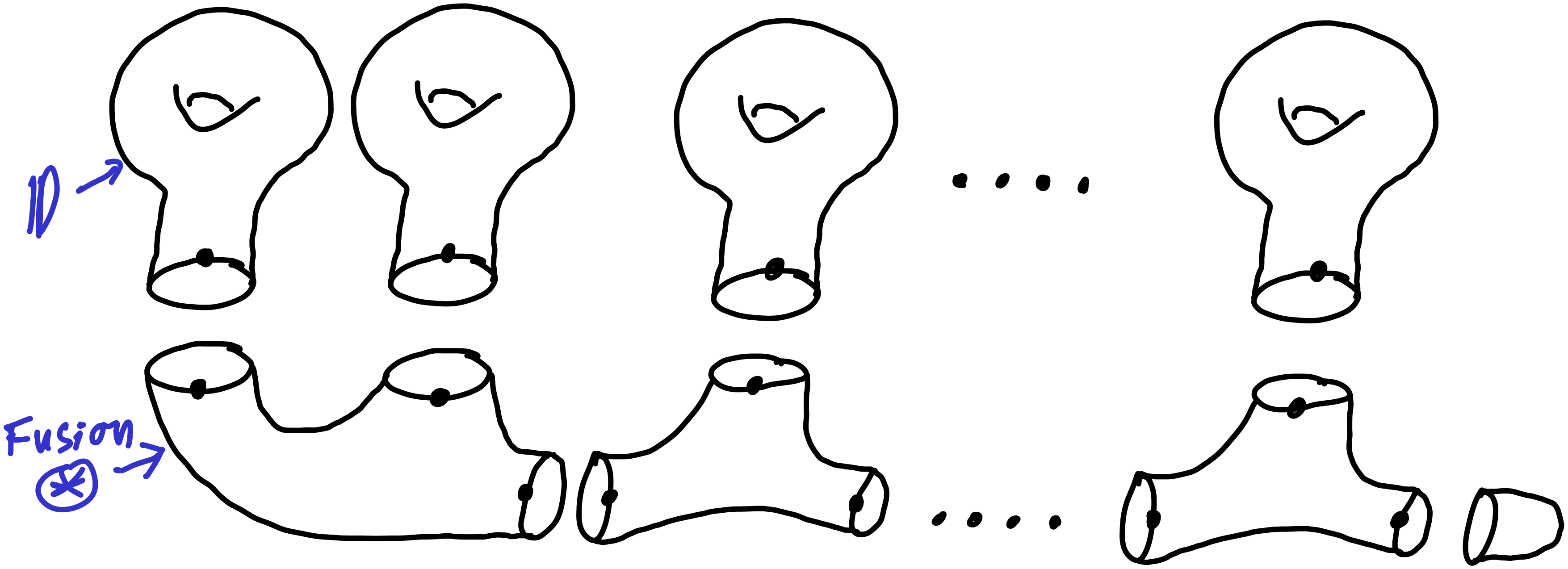


Toolbox:

- $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$
- $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\quad) \cong G \times G$ "double"
- $\circledast$ fusion , • ⦶ reduction $(/\!/G)$

$$\mathcal{M}_B(\underset{\sim}{\mathcal{C}}) = \mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Toolbox: • $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$

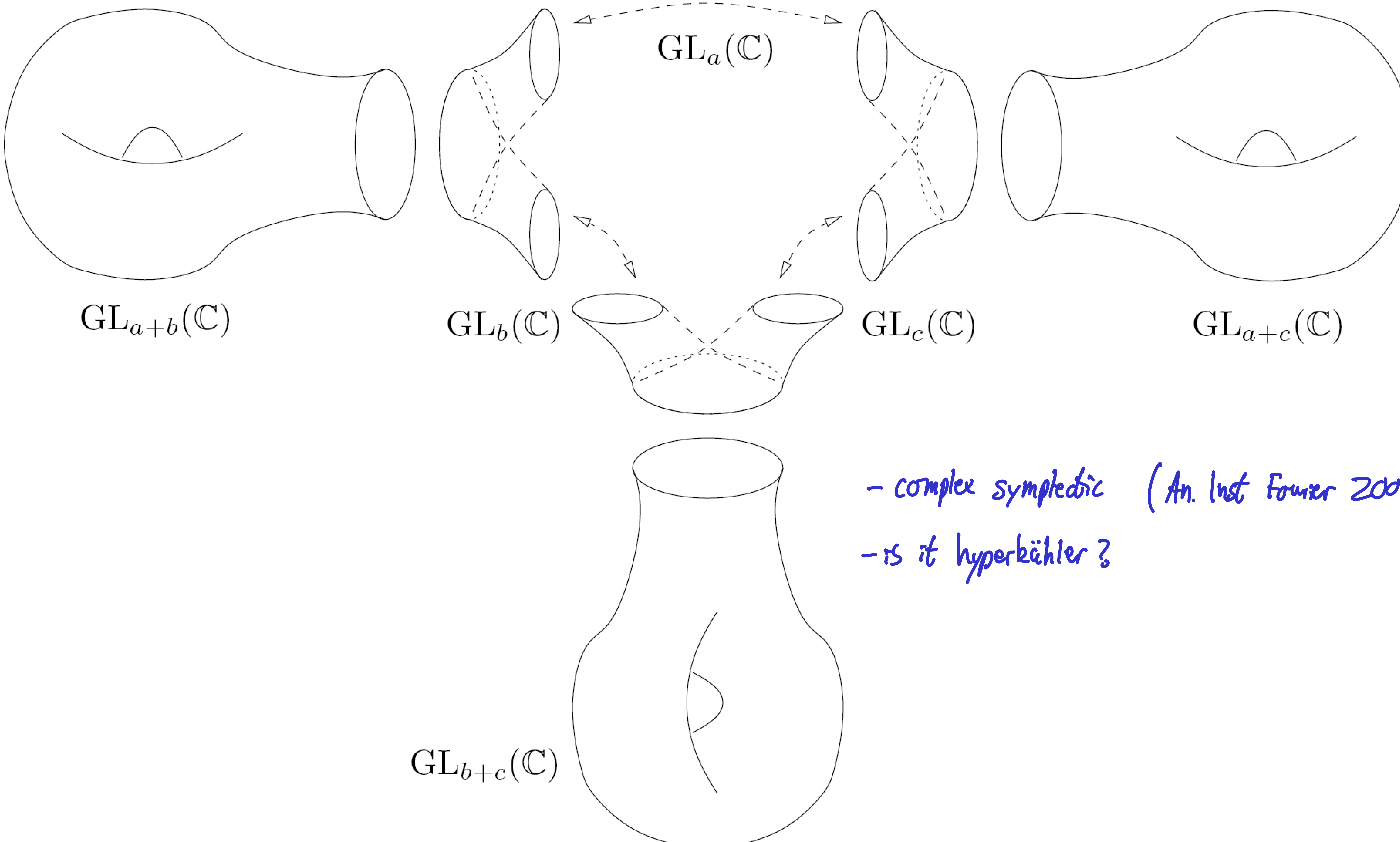
• $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\text{cylinder}) \cong G \times G$ "double"

• $\circledast$ fusion, • $\oslash$ reduction $(/\!/G)$

Now add fission spaces $\mathcal{A} = {}_G\mathcal{A}^k_H \quad \forall\, G, H, k$

$\Rightarrow$ lots of new algebraic symplectic/Poisson varieties

"fission varieties" $\supsetneq$ (untwisted) wild character varieties



- complex symplectic (An. Inst Fourier 2009)
- is it hyperkähler?

Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$(C_1^{-1} h_1 S_{2k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{2k_m}^{(m)} \cdots S_1^{(m)} C_m) = 1$$

(see Jimbo-Miwa-Ueno 1981 equation 2.46)

Wild character varieties

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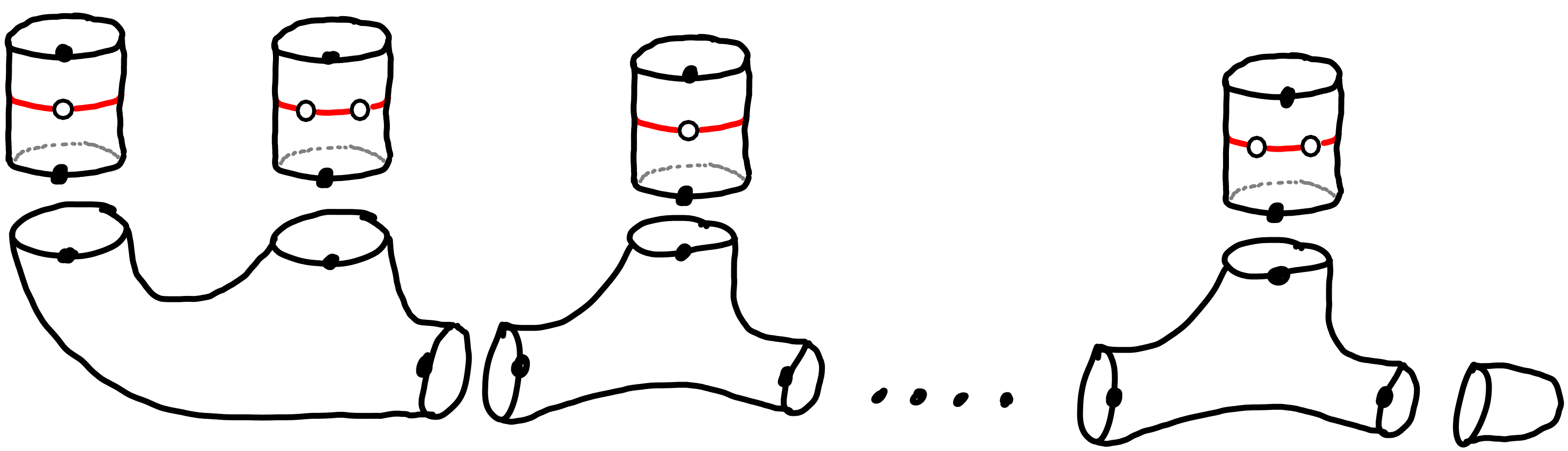
(see Jimbo-Miwa-Ueno 1981 equation 2.46)

$$\mathcal{R} = {}_G\mathcal{A}_T^{k_1} \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_2} \underset{G}{\circledast} \cdots \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_m} \xrightarrow{\mu} T^m \times G$$



Thm Reductions with fixed $h_i \in T$ are symplectic

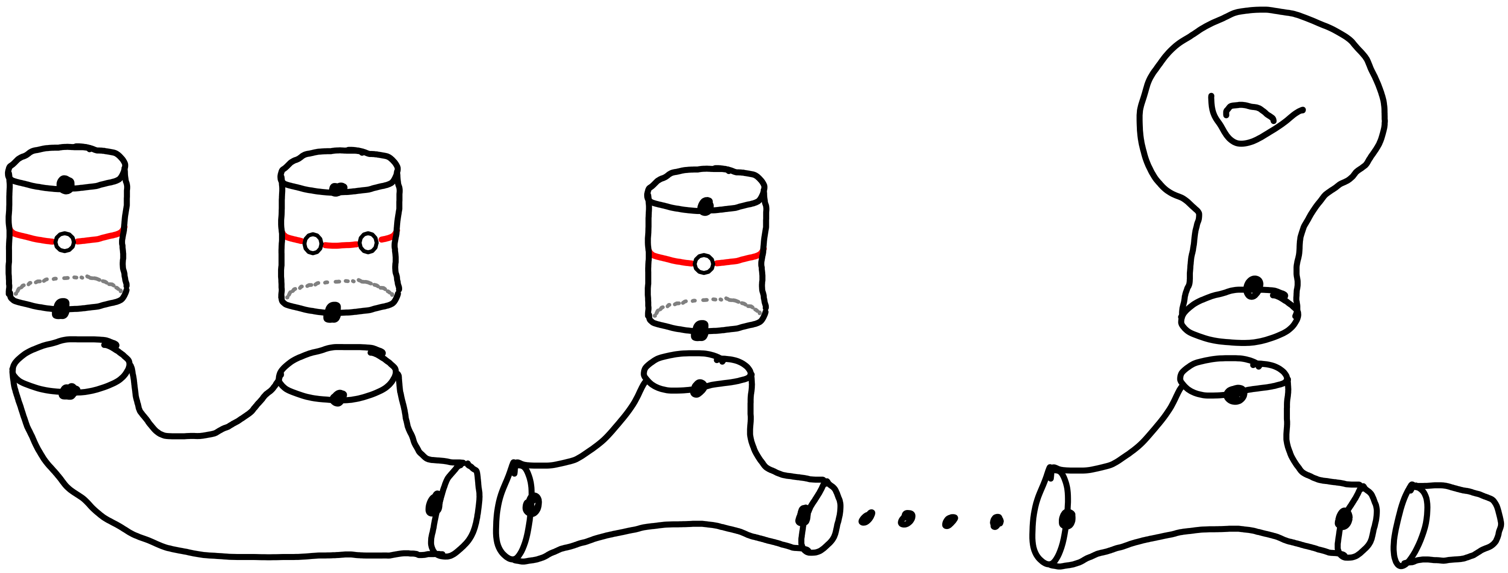
(Adv. Math. 2001 "irreg. Atiyah Bott", algebraic quasi-Hamiltonian approach 2002)

Wild character varieties

Similarly in general ($\sim$ any alg. connections on twisted G-bundles)

$$(C_1^{-1} h_1 S_{k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{k_m}^{(m)} \cdots S_1^{(m)} C_m) \overrightarrow{\prod_1^g} A_i B_i A_i^{-1} B_i^{-1} = 1$$

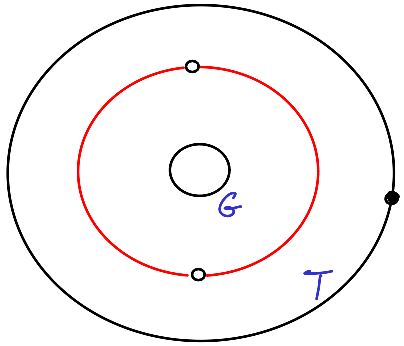
$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) = \mathcal{A}_1 \circledast \cdots \circledast \mathcal{A}_m \circledast \mathbb{D}^{\circledast g} /\!/ G$$



<u>Thm</u> Wild character variety $\mathcal{M}_B = \mathrm{THom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ is a Poisson variety with symplectic leaves got by fixing (twisted) conjugacy classes of formal monodromy

... An. Inst. Fourier '09, arXiv:1111.6228, arXiv:1512.08091 (with D. Yamakawa)

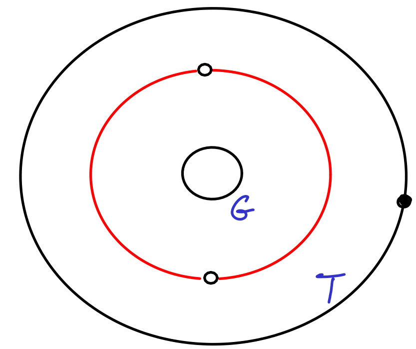
Wild character varieties



E.g. $\quad {}_G\mathcal{A}^1_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Wild character varieties



E.g. $${}_G\mathcal{A}^1_T/G \cong T \times U_+ \times U_-$$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Thm (Drinfeld/Semenov Tian Shansky, DeConcini Procesi 1993)

$U_q(\mathfrak{g})$ quantizes a Poisson variety $G^* \cong T \times U_+ \times U_-$

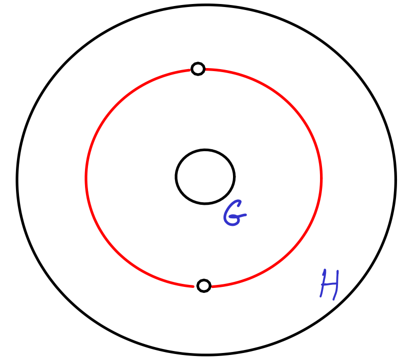
Thm (PB Invent. Math 2001)

$G^* \cong {}_G\mathcal{A}^1_T/G$ as a Poisson variety

Cor. The Drinfeld-Jimbo quantum group is modular

(comes from moduli of connections on curves)

Wild character varieties



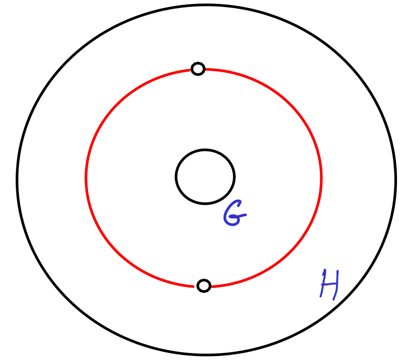
E.g. $${}_G\mathcal{A}^1_H \,/\, G\times H \cong (H \times U_+ \times U_-)/H$$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}},\ hS_1S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Wild character varieties



E.g. $\quad {}_G\mathcal{A}'_H /_{G\times H} \cong (H \times U_+ \times U_-)/H$

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for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Thm (Fourier-Laplace, Malgrange 1991)

This class of varieties $\cong$ all tame genus zero character varieties

Thm — symplectic structures match too (PB arXiv 1307)

— and the hyperkähler metrics (Sz. Szabo arXiv 1407)

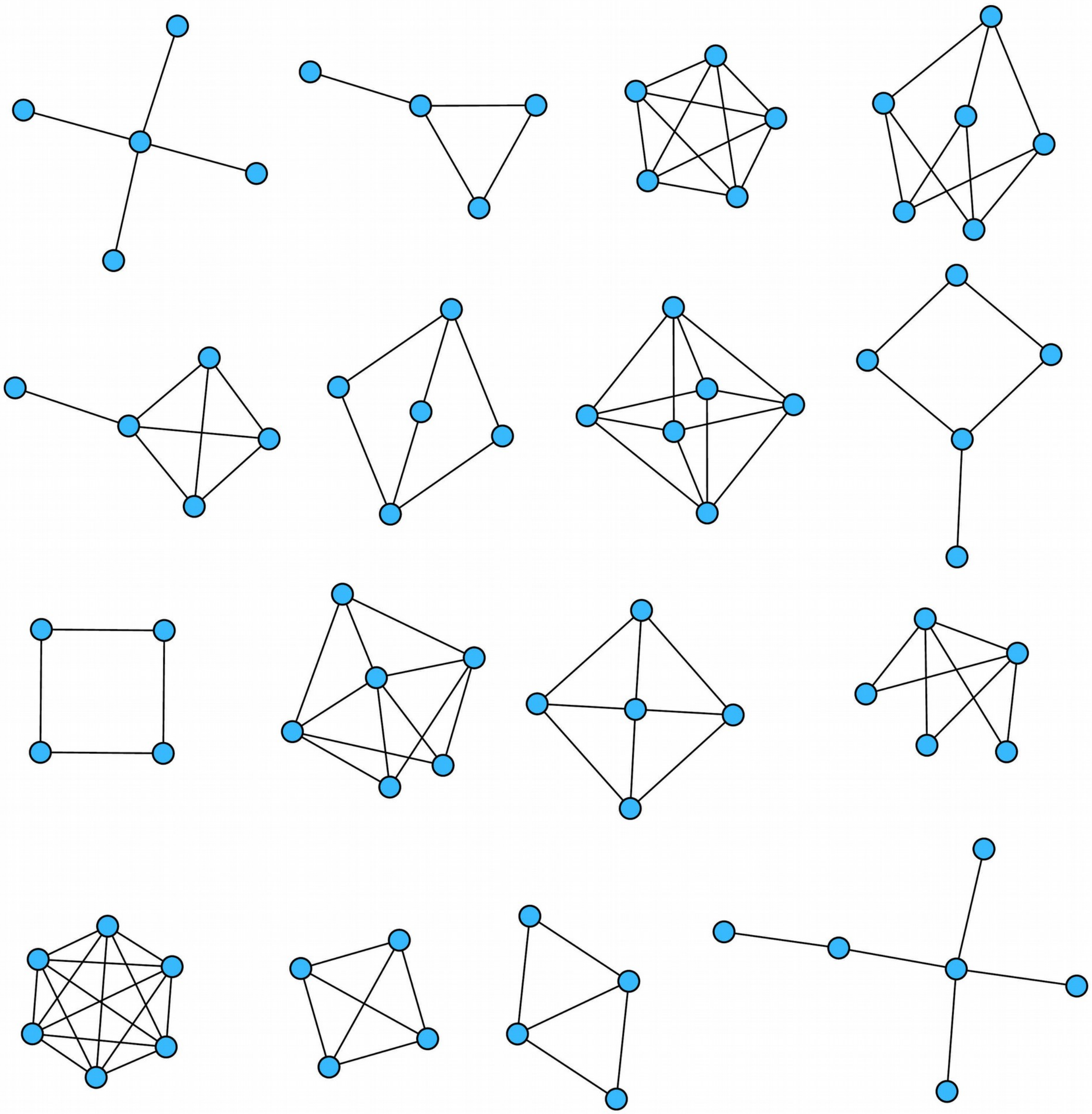
$\rightsquigarrow$ notion of "representations" of abstract moduli space

Plato to Painlevé

(McKay – Harnad) c.f. PB 0706.2634

Sakai's question

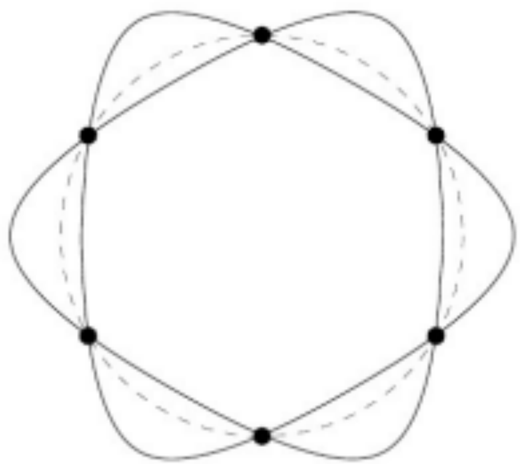
Exercise 3



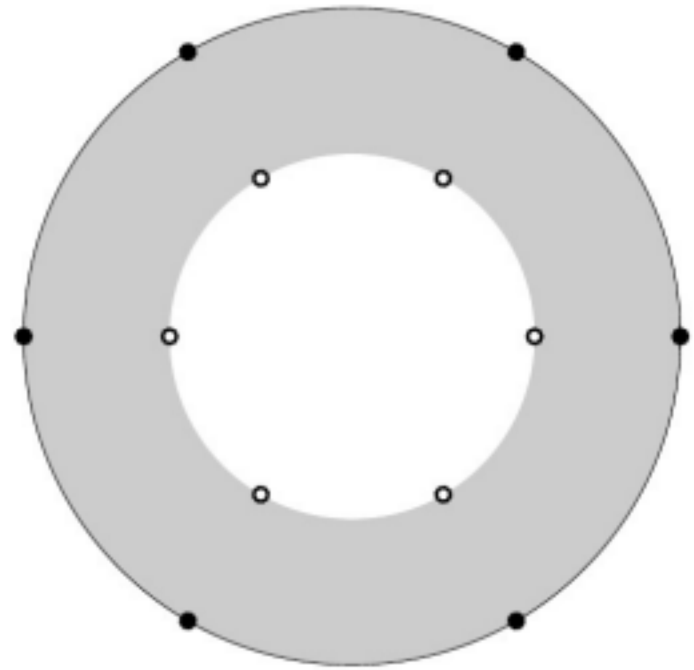
Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

solutions involve e^Q

plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

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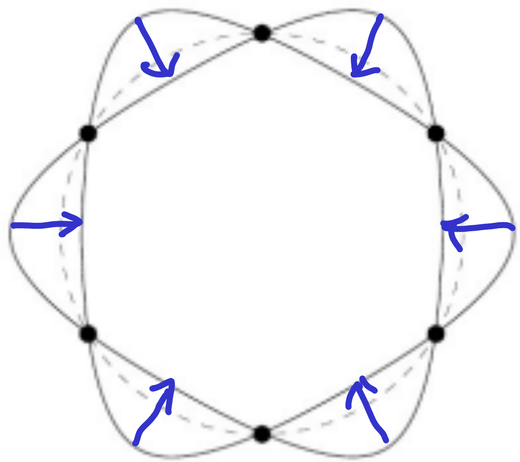
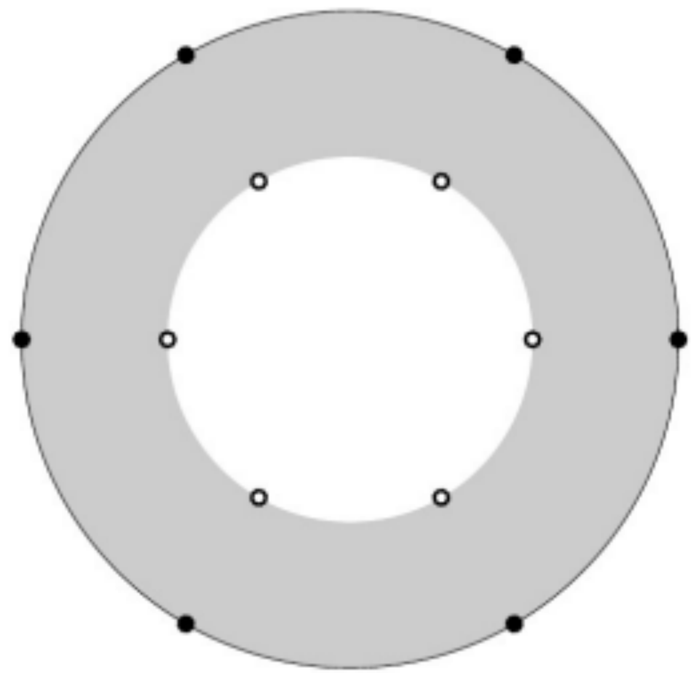
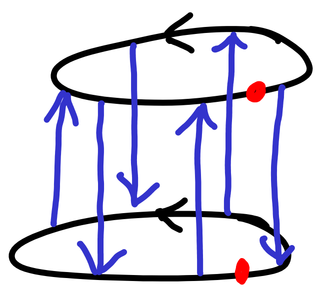
plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



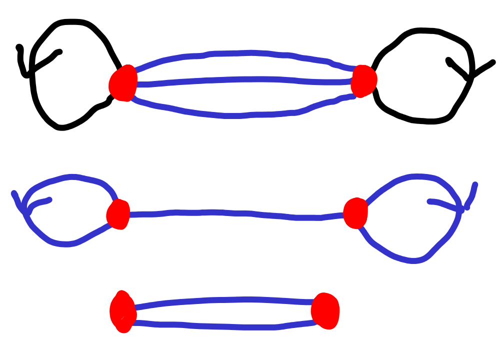
Stokes diagram with Stokes directions



Halo at ∞ with singular directions



$\simeq$



$\mu_G = 1$

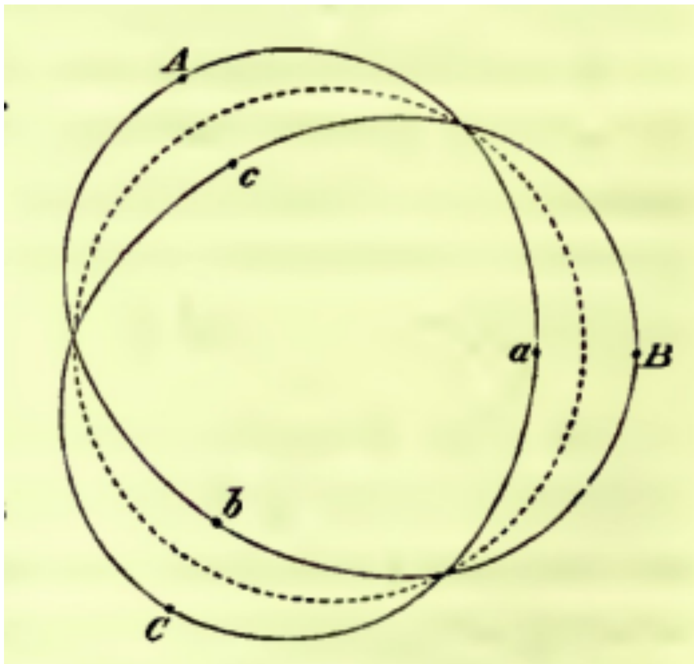
$h S_6 S_5 \cdots S_1 = 1$

2x2 matrix relation

result: $\hat{A}_1$

Airy equation (Stokes 1857)

solutions involve $\exp(x^{3/2})$



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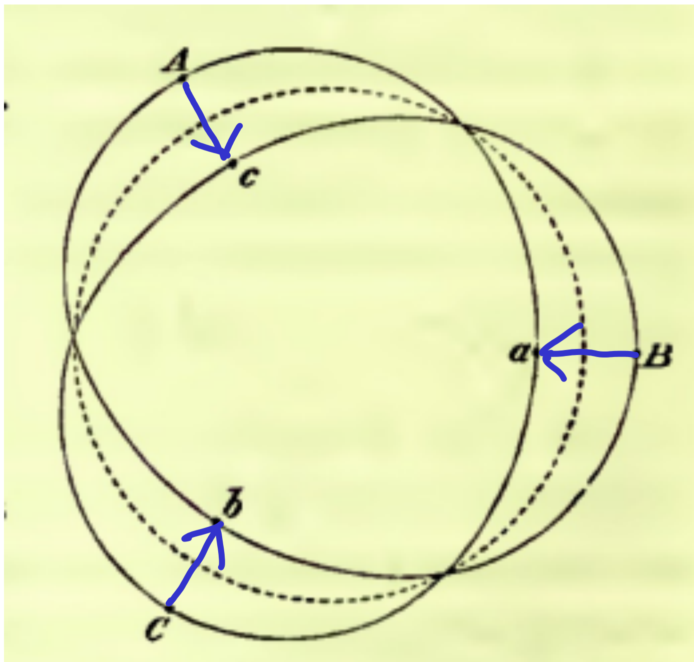
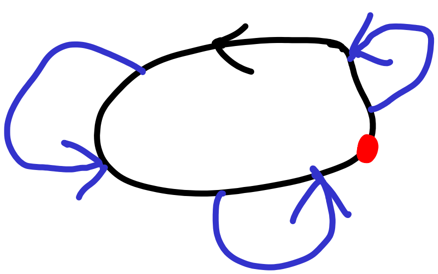
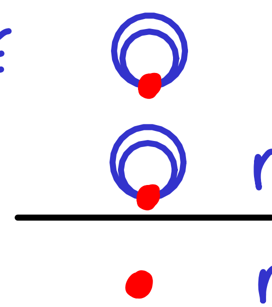


$$\mu_G = 1$$

$$\underset{\begin{pmatrix} 0 & * \\ 1 & 0 \end{pmatrix}}{h}\ \underset{u_+}{S_3}\ \underset{u_-}{S_2}\ \underset{u_+}{S_1} = 1$$



$\cong$

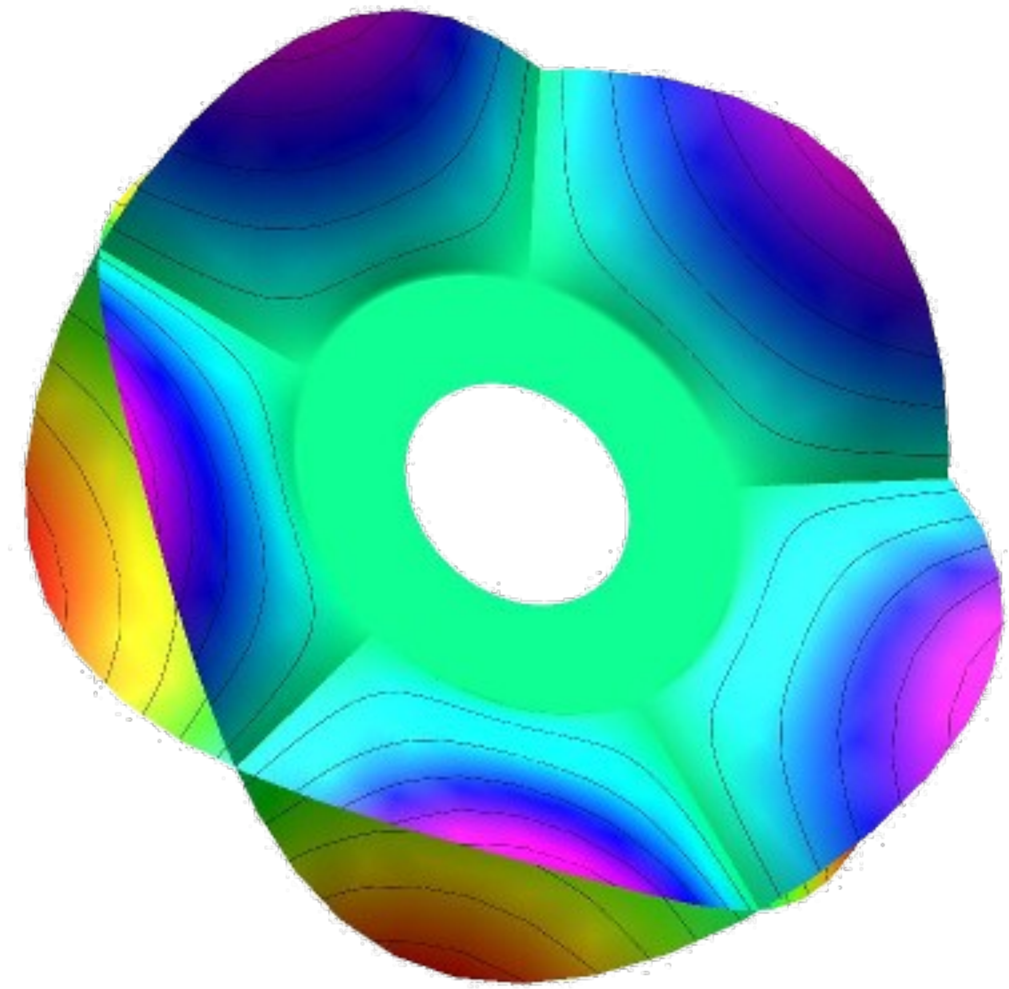


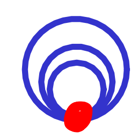
relations

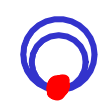
resulting diagram

Painlevé 1

$$\exp(x^{5/2})$$





relations 

resulting diagram