Algebraic Poisson structures in global Lie theory

Philip Boalch, IMJ-PRG & CNRS Paris

- see also short survey arxiv: 1703 for more background
- course notes: ~/cours23/

Geometrically, what are the six Painlevé equations* trying to tell us?

∗ Picard, Painlevé, R. Fuchs, Gambier

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Henri Poincaré:

« Les Mathématiques constituent un continent solidement agencé, dont tous les pays sont bien reliés les uns aux autres; l'œuvre de Paul Painlevé est une île originale et splendide dans l'océan voisin »

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Geometrically, what is the Riemann–Hilbert–Birkhoff correspondence* trying to tell us?

* Stokes, Birkhoff, Malgrange, Sibuya, Jurkat, Deligne, Écalle, Martinet, Ramis, ...

$G = GL_n(\mathbb{C})$ (or any other complex reductive group)

Riemann surface Σ $\rightsquigarrow$ character variety

$$\mathcal{M}_B = \mathcal{R}/G$$

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, b), G)$$

representation variety

wild Riemann surface $\underset{\sim}{\Sigma}$ $\rightsquigarrow$ wild character variety

$$\mathcal{M}_B = \mathcal{R}/\underset{\sim}{H}$$

$$\mathcal{R} = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$$

wild representation variety

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wild representation variety

Thm (B.-Yamakawa)

$\mathcal{M}_B$ is alg. Poisson variety, points are the reductive Stokes representations, any admissible deformation of $\underset{\sim}{\Sigma}$ $\Rightarrow$ local system of Poisson varieties

More precise references:

• Irregular Atiyah–Bott $\int_\Sigma \mathrm{Tr}(\alpha \wedge \beta)$:

[1] P.B., *Symplectic manifolds and isomonodromic deformations*, Adv. in Math. **163** (2001), 137–205. (Oxford thesis 1999, ICM poster 1998)

• Hyperkähler upgrade of [1]—new complete hyperkähler manifolds, beyond instantons:

[2] O.Biquard and P.B., *Wild non-abelian Hodge theory on curves*, Compositio Math. **140** (2004), no. 1, 179–204. (arXiv:math/0111098, 2001)

• Purely algebraic construction of the topological symplectic/Poisson structures, via complex quasi-Hamiltonian geometry:

[3] P.B., *Quasi-Hamiltonian geometry of meromorphic connections*, Duke Math. J. **139** (2007), no. 2, 369–405, (arXiv:math/0203161, 2002).
[4] P.B., *Through the analytic halo: Fission via irregular singularities*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 7, 2669–2684, Volume in honour of B. Malgrange.
[5] P.B., *Geometry and braiding of Stokes data; Fission and wild character varieties*, Annals of Math. **179** (2014), 301–365.
[6] P.B. and D. Yamakawa, *Twisted wild character varieties*, arXiv:1512.08091, 2015.

Lie theory

$$\mathfrak{g} \longrightarrow G$$

$$X \longmapsto \exp(X)$$

Connection $\quad \dfrac{X}{2\pi i}\dfrac{dz}{z} \longmapsto$ monodromy

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Global Lie theory

Connection $\quad \left(\sum_{i=1}^{m} \sum_{j=1}^{r_i} \dfrac{A_{ij}}{(z-a_i)^j} \right) dz \longmapsto$ monodromy & Stokes data

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moduli spaces: $\quad \mathcal{M}^* \longrightarrow \mathcal{M}_B$ wild character variety (same dimension)

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moduli spaces: $\quad \mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \xrightarrow{\cong} \mathcal{M}_B \quad$ wild character variety (same dimension)

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moduli spaces:

$$\begin{tikzcd} \mathcal{M}^* \arrow[r, hook] & \mathcal{M}_{DR} \arrow[r, "\cong"] \arrow[dr, "\cong"] & \mathcal{M}_B \\ & & \mathfrak{M} \end{tikzcd}$$

$\mathcal{M}_B$: wild character variety (same dimension)

$\mathfrak{M}$: wild harmonic bundles (2d self-duality)

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$$\begin{tikzcd}
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& \mathcal{M}_{Dol} \arrow[r, "\cong"] & \mathfrak{M}
\end{tikzcd}$$

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wild character variety (same dimension)

meromorphic Higgs bundles

wild harmonic bundles (2d self-duality)

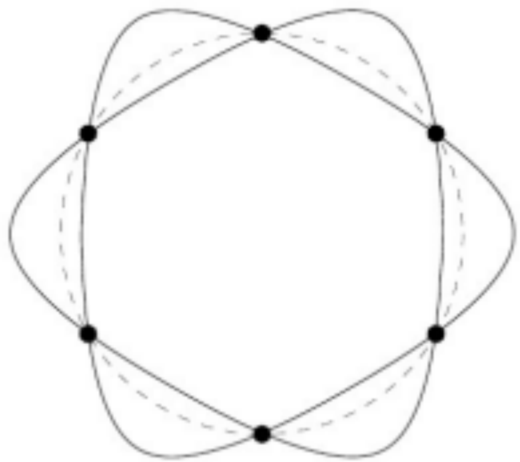
"Nonabelian Hodge space"

Classify via diagrams? (e.g. sometimes $\mathcal{M}^*$ is a quiver variety)

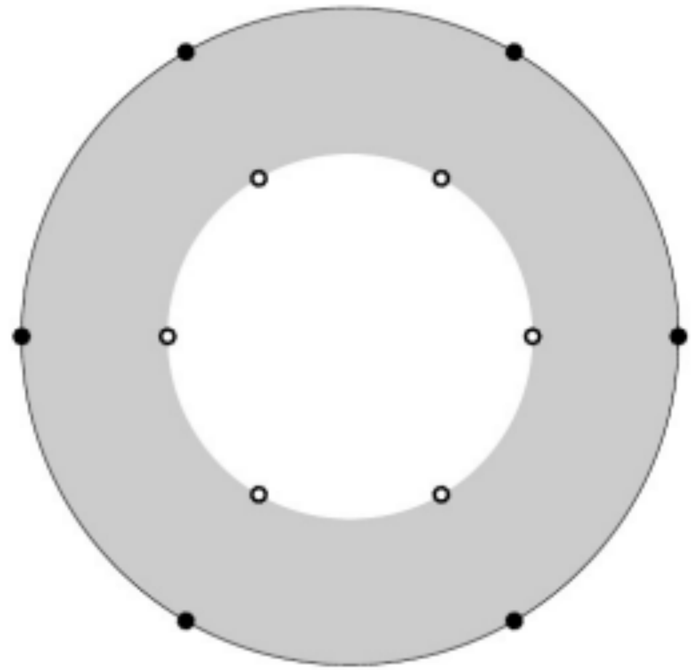
Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

solutions involve e^Q

plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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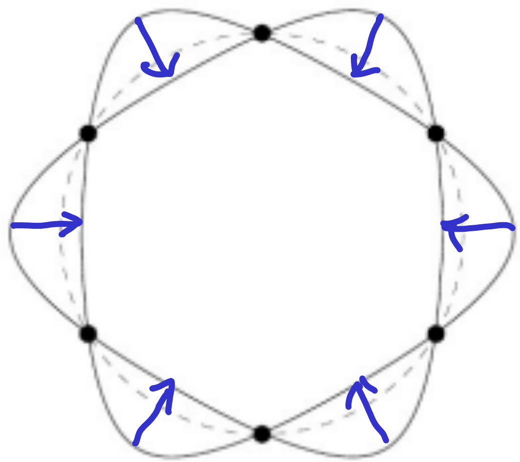

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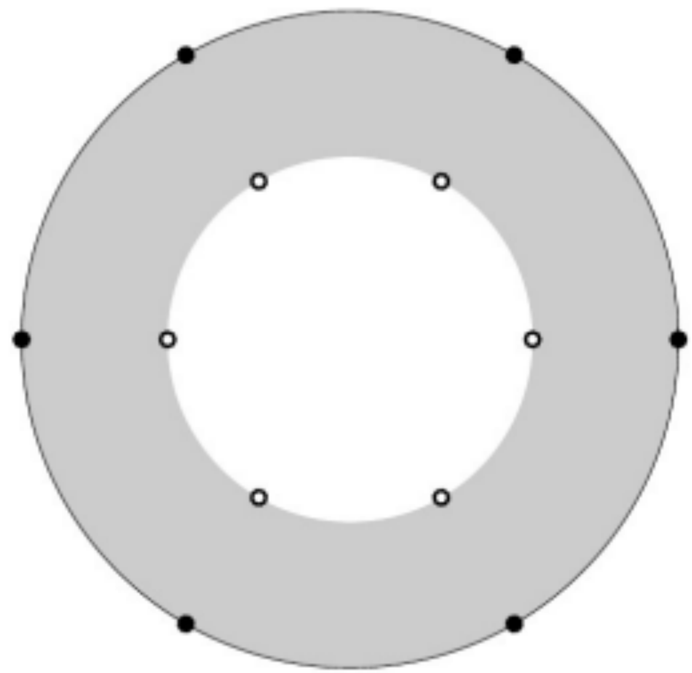
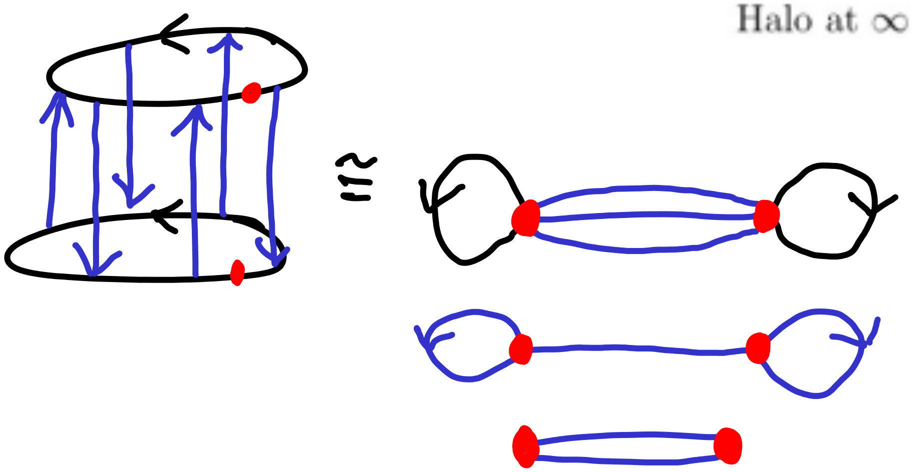


Halo at ∞ with singular directions



$\mu_G = 1$

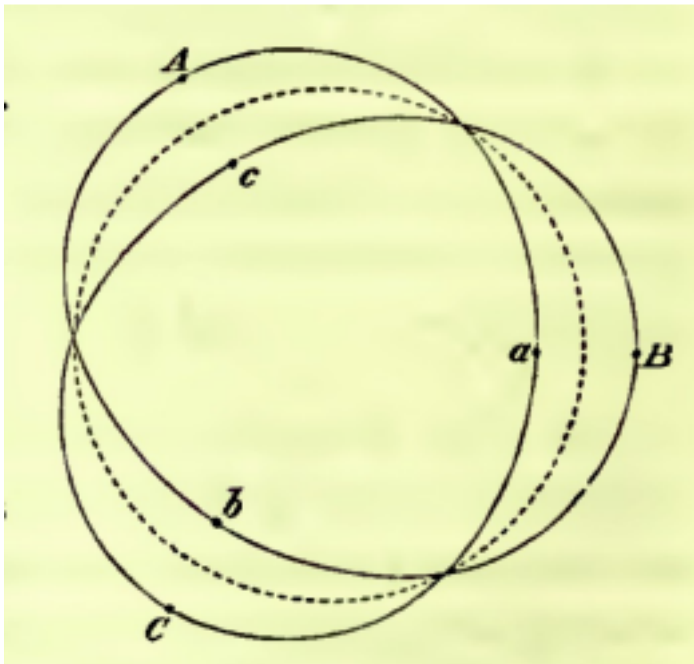
$h S_6 S_5 \dots S_1 = 1$

2x2 matrix relation

result: $\hat{A}_1$

Airy equation (Stokes 1857)

solutions involve $\exp(x^{3/2})$



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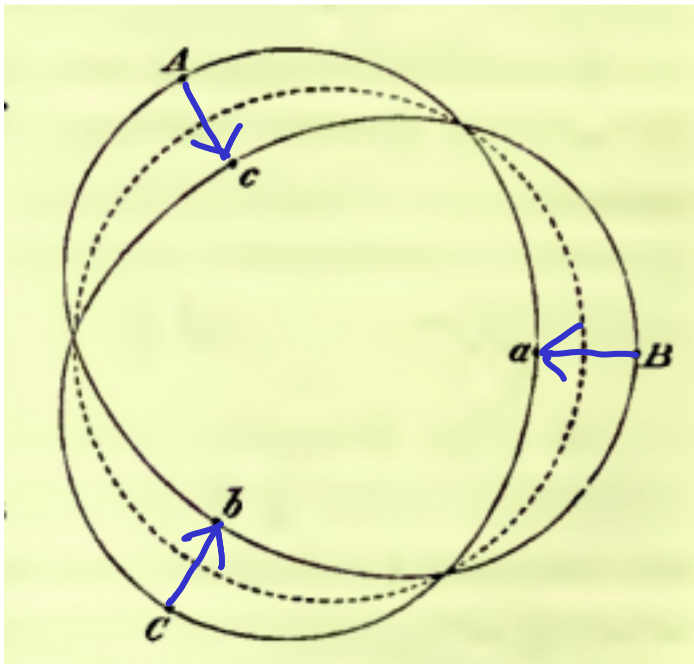
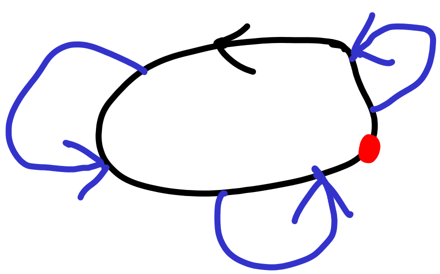
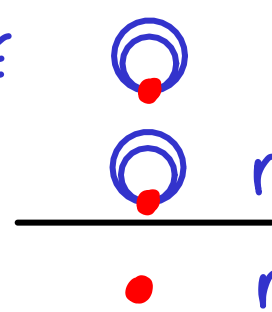


$$\mu_G = 1$$

$$\underbrace{h}_{\begin{pmatrix} 0 & * \\ 1 & 0 \end{pmatrix}} \; \underbrace{S_3}_{u_+} \; \underbrace{S_2}_{u_-} \; \underbrace{S_1}_{u_+} = 1$$



$\cong$

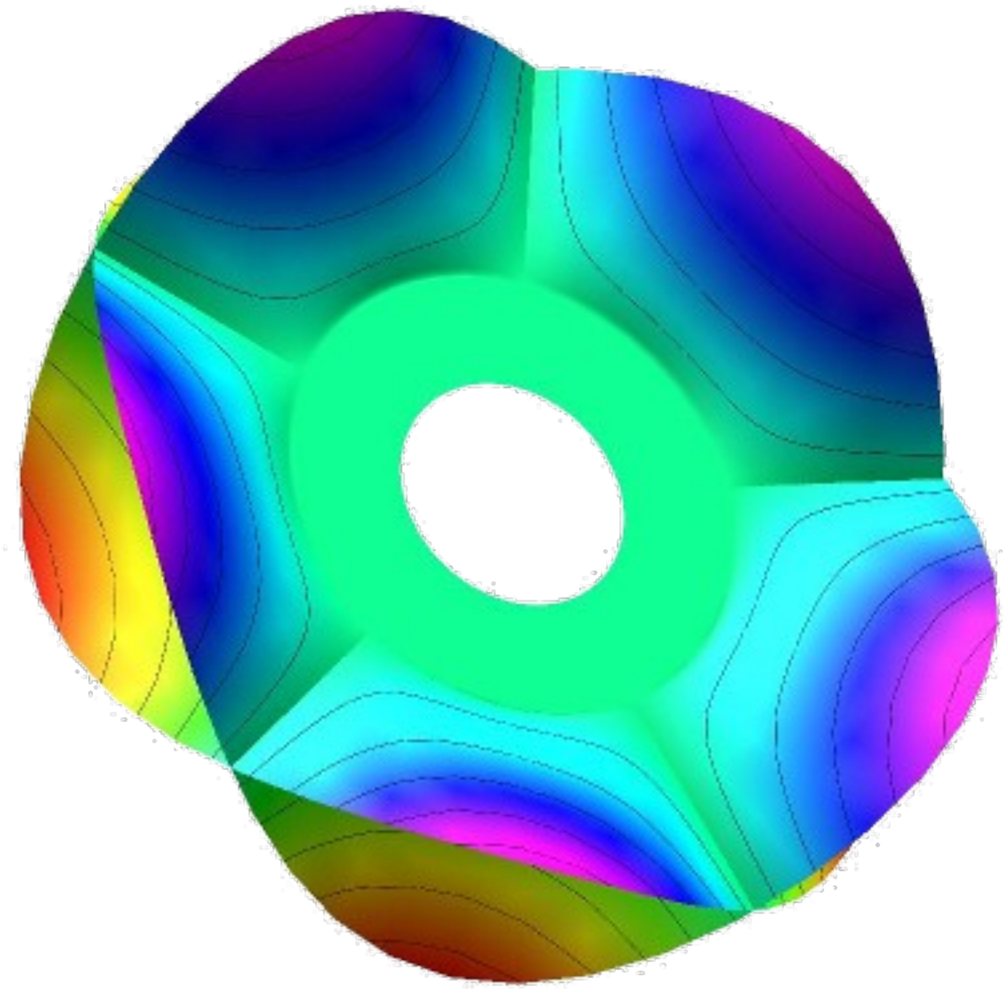


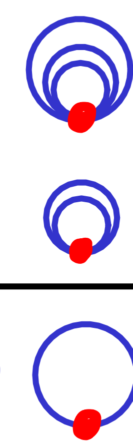
relations

resulting diagram

Painlevé 1

$\exp(x^{5/2})$





relations

resulting diagram

Much of the story can be summarised in the (slightly oversimplified) diagram:

$$
\begin{array}{ccccc}
\underline{\text{Dolbeault}} & & \underline{\text{De Rham}} & & \underline{\text{Betti}} \\
\begin{array}{c}\text{Rational Lax matrices } L \\ \dot{L} = [P, L]\end{array} & & \begin{array}{c}\text{Rational diff. op.s} \\ \frac{d}{dz} - B\end{array} & & \begin{array}{c}\text{Stokes and} \\ \text{monodromy data}\end{array} \\
\cap & & \cap & & \cap \\
\begin{array}{c}\text{Mero. Higgs bundles} \\ (E, \Phi)\end{array} & \overset{\text{wnAbH}}{\longleftrightarrow} & \begin{array}{c}\text{Mero. connections} \\ (E, \nabla)\end{array} & \overset{\text{RHB}}{\longleftrightarrow} & \begin{array}{c}\text{Stokes local} \\ \text{systems}\end{array} \\
\cup & & \cup & & \cup \\
\begin{array}{c}\text{Holom. Higgs bundles} \\ (E, \Phi)\end{array} & \overset{\text{nAbH}}{\longleftrightarrow} & \begin{array}{c}\text{Holom. connections} \\ (E, \nabla)\end{array} & \overset{\text{RH}}{\longleftrightarrow} & \begin{array}{c}\text{Local systems/} \\ \pi_1\text{-rep.s}\end{array}
\end{array}
$$

Very good connections

~ models in Biquard–B. 2004

(cf. exposition in $\left\{\begin{array}{l}\text{arXiv:1203.6607} \\ \text{arXiv:1703.10376}\end{array}\right.$)

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\ a \in \underset{\sim}{a}$)

$\nabla : V \to V \otimes \Omega^1(*\underset{\sim}{a})$ meromorphic connection

such that ...

Very good connections

$\sim$ models in Biquard–B. 2004
(cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

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such that have local bases (at each $a \in \underset{\sim}{a}$) splitting $\mathcal{F}_a$ such that:

- $\nabla = d - A$, $\quad A = dQ + \Lambda \dfrac{dz}{z}$ + holomorphic terms

  $Q = \sum_1^k \dfrac{A_i}{z^i}$, A_i diagonal matrices (irregular type)

  $\Lambda \in \mathfrak{h}$ preserves $\mathcal{F}_a$, $\mathfrak{h} = \mathrm{Lie}(H)$, $H = C_G(Q)$

["Good" if some local cyclic pullback is very good (twisted case)]

$\rightsquigarrow \mathcal{M}_{DR}$ moduli of stable connections, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

Very good Higgs bundles

$\sim$ models in Biquard–B. 2004
(cf. exposition in {arXiv:1203.6607, arXiv:1703.10376})

Σ compact Riemann surface, $\underset{\sim}{a} \subset \Sigma$ finite subset

$V \to \Sigma$ holomorphic vector bundle

$\exists$ parabolic filtrations (in V_a $\forall\, a \in \underset{\sim}{a}$)

$\Phi : V \to V \otimes \Omega^1(* \underset{\sim}{a})$ meromorphic Higgs field

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- $$\Phi = dQ + \Lambda \frac{dz}{z} + \text{holomorphic terms}$$

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$$\Lambda \in \mathfrak{h} \text{ preserves } \mathcal{F}_a, \quad \mathfrak{h} = \mathrm{Lie}(H),\ H = C_G(Q)$$

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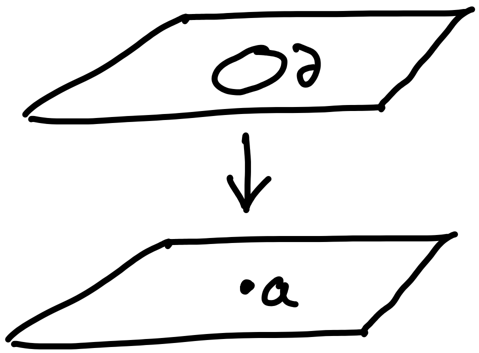
$\rightsquigarrow \mathcal{M}_{DoL}$ moduli of stable Higgs bundles, $\underset{\sim}{Q}$, $Gr(\Lambda)$, parabolic weights fixed

General choices / boundary data (twisted case) [Betti weights zero]

Fact $\exists$ covering $\mathcal{I} \to \partial$ such that:

$$\left\{\begin{array}{c}\text{connections on formal}\\ \text{punctured disk}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\mathcal{I}\text{-graded local systems}\\ \text{of vector spaces}\end{array}\right\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Jurkat, Deligne]

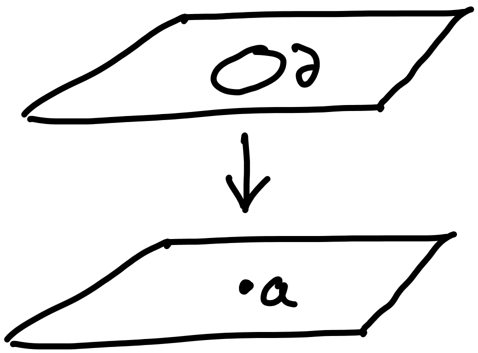


General choices / boundary data (twisted case) [Betti weights zero]

Fact $\exists$ covering $\tilde{I} \to \partial$ such that:

$$\left\{\text{connections on formal punctured disk}\right\} \iff \left\{\tilde{I}\text{-graded local systems of vector spaces}\right\}$$

[Fabry, Hukuhara, Turrittin, Levelt, Jurkat, Deligne]



function on sector: $q = \sum_{i>0} a_i z^{-i/r}$ $(r \in \mathbb{N})$

$\Rightarrow$ Stokes circle $\langle q \rangle$ (Riemann surface / Galois orbit)

$$\begin{array}{c} \langle q \rangle \\ \downarrow \\ \partial \end{array}$$

$$\boxed{\begin{array}{c} \tilde{I} = \bigcup \langle q \rangle \\ \downarrow \\ \partial \end{array} \quad \text{exponential local system}}$$

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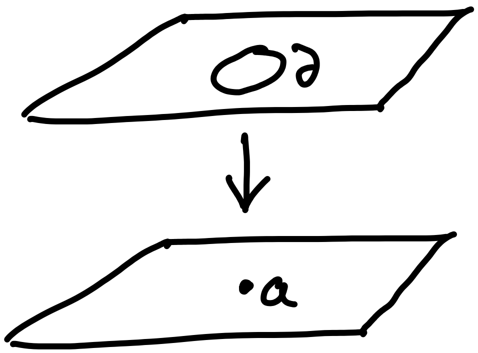

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$\mathcal{I}$-graded local system $V \to \partial$ of vector spaces

$\Leftrightarrow$ local system $V \to \mathcal{I}$ with compact support

i.e. $V \to I$, $I \subset \mathcal{I}$ finite subcover

$\Rightarrow$ Irregular class $\Theta = n_1\langle q_1\rangle + \dots + n_m \langle q_m \rangle$ $\quad n_i = \mathrm{rk}\, V|_{\langle q_i\rangle}$

$+$ monodromy classes $\quad \mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$

In simple examples this growth/decay can be easily visualised in the Stokes diagram, as in the example of $q = x^{17}$ in Figure 5, where the singularity is at $a = \infty$ (so $z = x^{-1}$ is a local coordinate vanishing at a). For example we see on the positive real axis that the function $\exp(x^{17})$ has maximal growth there, and there are 16 other evenly spaced directions of maximal growth, interlaced with 17 directions of maximal decay, the first at $\arg(x) = \pi/17$.

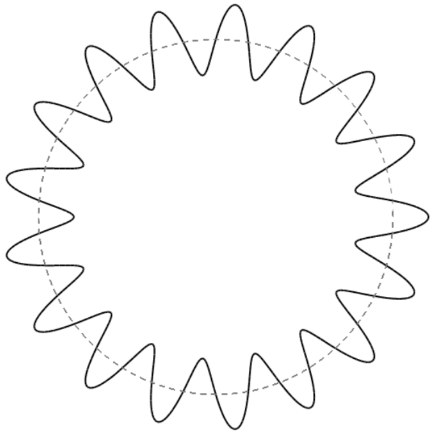


FIGURE 5. Stokes diagram for $\langle x^{17}\rangle$: the Stokes circle $\langle x^{17}\rangle$ is projected to the plane so as to indicate the growth/decay of $\exp(x^{17})$ near ∞.

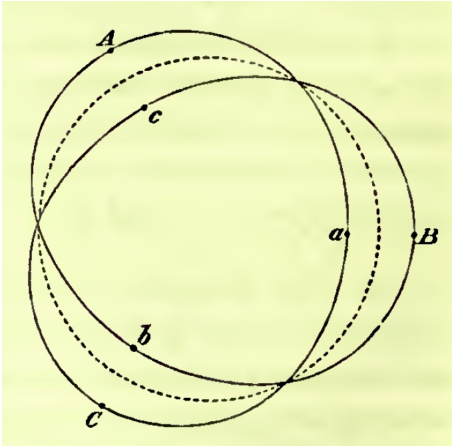
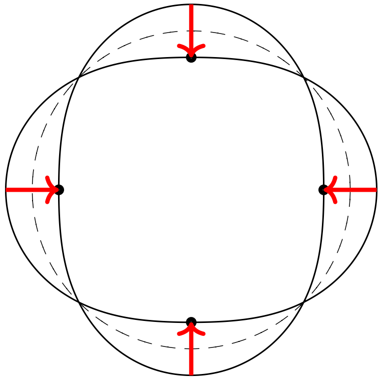


FIGURE 6. The Stokes diagram of $\langle 2x^{3/2} \rangle$, from Stokes' paper [?] on the Airy equation. The points a, b, c are the points of maximal decay.



Stokes diagram of the Weber equation, with Stokes arrows drawn.

There is a javascript program here:

https://webusers.imj-prg.fr/~philip.boalch/stokesdiagrams.html

to draw lots of other examples of Stokes diagrams, the Stokes diagrams of the "symmetric" or "hypotrochoid" irregular classes $I(a:b)$ (see the explanation in the box at the bottom there).[15] In brief $I(a:b)$ is the pull-back to the x-plane of the irregular class $\langle w^{1/b}\rangle$ under the map $w = x^a$. It has k Stokes circles where $k = (a, b)$ is the highest common factor. Explicitly:

$$I(a:b) = \bigsqcup_{i=0}^{k-1} \langle \varepsilon^i x^{a/b} \rangle \subset \mathcal{I}$$

where $\varepsilon = \exp(2\pi i/b)$. For example it is the irregular class at $x = \infty$ of the Molins–Turrittin equation $y^{(b)} = x^\nu y$, if $a = \nu + b$ [**?**, **?**]. Upto a constant $I(1:q+1)$ is also the irregular class at ∞ of the differential equation for the hypergeometric series ${}_0F_q$.

10.5. **Rank two examples.** The simplest rank two Stokes diagrams are collected in Figure 7. The left four are *rigid* in that their (symplectic) wild character varieties are dimension zero. They come from the ODEs of Clifford, Airy, Whittaker, Hermite–Weber. The next two, with 5 or 6 crossings, give the wild character varieties of Painlevé I and II.

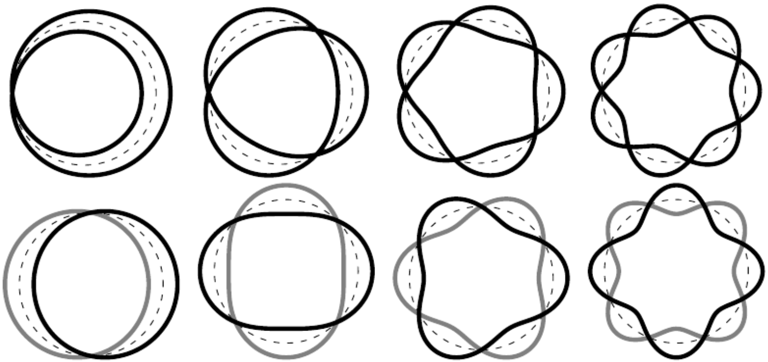


FIGURE 7. The simplest rank two Stokes diagrams $I(k:2), k = 1, 2, \ldots, 8$.

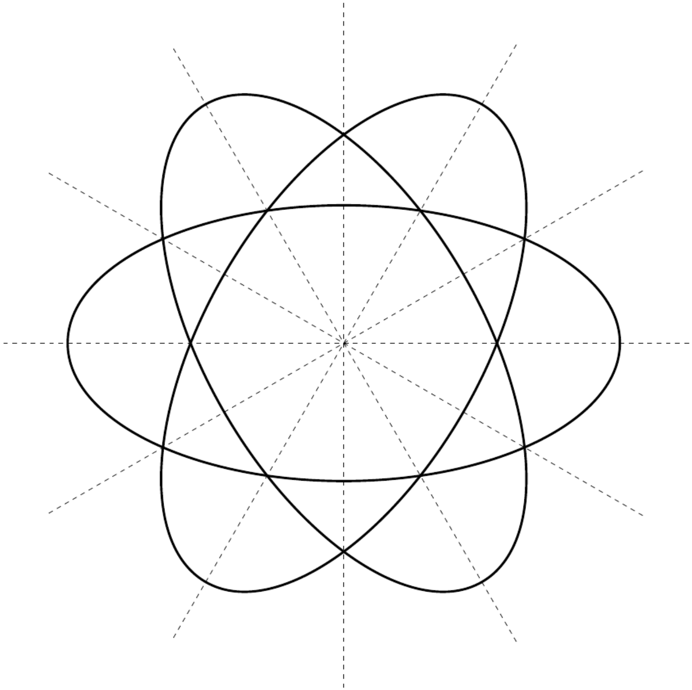


FIGURE 8. Example rank three Stokes diagram, $I(6\!:\!3)$.

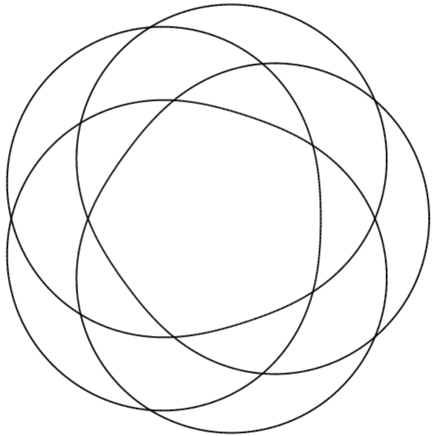


FIGURE 9. Stokes diagram at ∞ for the "hyperairy" equation $y^{(4)} = xy$

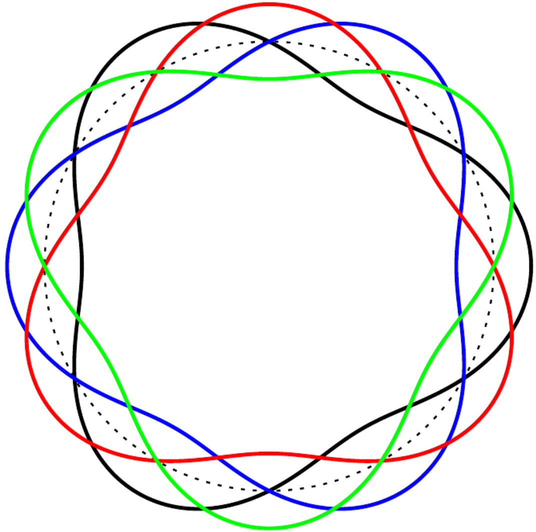


FIGURE 10. Another example rank four Stokes diagram, $I(12:4)$.

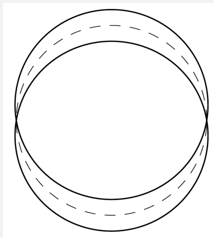
10.6. **Example Stokes diagrams: Bessel's equation.**
Bessel's differential equation is

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

where $\alpha \in \mathbb{C}$. This has a regular singularity at 0 and an irregular singularity at ∞. A short computation, or a glance at a book, shows that the irregular class $x = \infty$ is:

$$\Theta = \langle ix \rangle + \langle -ix \rangle$$

and that α determines the local monodromy eigenvalues at 0. In particular the singular directions are the two halves of the imaginary axis.



10.7. **Example Stokes diagrams: Bessel–Clifford equation.**

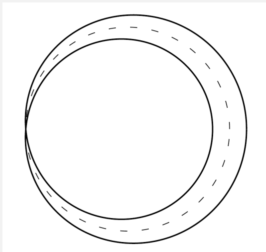
The Bessel–Clifford equation (also known as the confluent hypergeometric limit equation, Kummer's second equation, or the ${}_0F_1$ equation) is:

$$xy'' + ay' = y. \tag{10.1}$$

If f is any solution of this, then $x^{a-1} \cdot f(-x^2/4)$ solves the Bessel equation with parameter $\alpha = a - 1$. The irregular class at $x = \infty$ is

$$\langle 2x^{1/2} \rangle$$

and (if $a \notin \mathbb{Z}$) the monodromy around 0 has eigenvalues $1, \exp(-2\pi i a)$.



9.5. **Wild Riemann surfaces.** The irregular class makes up the basic "new modular parameters" that occur for irregular connections, behaving just like the modulus of the underlying Riemann surface and the location of the marked points $\mathbf{a}$.

In particular it behaves completely differently to the formal residue Λ.

This motivates the following definition:

Definition 9.5. *A rank n wild Riemann surface is a triple $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ where Σ is a Riemann surface, $\mathbf{a} \subset \Sigma$ is a finite subset and $\Theta = \{\Theta_a \mid a \in \mathbf{a}\}$ is the data of a rank n irregular class at each point $a \in \mathbf{a}$.*

Here we are mainly interested in the case where Σ is compact. We will define the character variety $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ of any such wild Riemann surface, show that it is Poisson and forms a local system of varieties under any admissible deformation of $\mathbf{\Sigma}$.

Of course if all the irregular classes are trivial then $\mathbf{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ just amounts to choosing a Riemann surface with some marked points, and then $\mathcal{M}_{\mathrm{B}}(\mathbf{\Sigma})$ will be the usual (tame) character variety defined previously $\cong \mathrm{Hom}(\pi_1(\Sigma^\circ, b), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$.

Notes: This definition is from [B2014] Defn 8.1, Rmk 10.6, [BY2015] §4. There are several minor variations that we won't worry about here, but are sometimes useful: One can work with irregular types instead of irregular classes (which were called "bare irregular types" in [B2014] Rmk 10.6); this is analogous to whether or not we order the points $\mathbf{a}$. Also one can work with smooth complex algebraic curves instead of Riemann surfaces (which doesn't make much difference in the compact case); the terms "irregular curve" or "wild curve" are sometimes used to replace the term "wild Riemann surface" in the algebraic case. Op. cit. give the definition for any complex reductive group, not just $\mathrm{GL}_n(\mathbb{C})$.

Séminaire BOURBAKI
(Mai 1958)

MODULES DES SURFACES DE RIEMANN

par André WEIL

Par la combinaison des idées (récentes) de KODAIRA et SPENCER sur la variation des structures complexes avec les idées (anciennes) de TEICHMÜLLER sur le problème des modules, la théorie a fait dernièrement quelques progrès qu'on se propose d'exposer ici.

Soit T_o une surface orientée compacte de genre g, donnée une fois pour toutes. Par une surface de Riemann de genre g, on entend, comme d'habitude, une variété complexe compacte de dimension complexe 1, de genre g, munie de son orientation naturelle. Par une <u>surface de Teichmüller</u> de genre g, on entendra une surface de Riemann S de genre g, munie de plus d'une classe (au sens de l'homotopie) d'applications de T_o dans S, classe dont on suppose qu'elle contient au moins un homéomorphisme conservant l'orientation ; c'est là une structure (plus "riche" que celle de structure de surface de Riemann). Si π^o désigne le

Il est utile de définir une notion intermédiaire entre celle de surface de Riemann et celle de surface de Teichmüller : on l'obtient en se donnant les images des A_i^o, non dans $\pi(S)$, mais dans $H_1(S)$; la donnée de ces images sur la surface de Riemann S détermine ce qu'on appellera une "surface de Torelli". Au

Nonabelian Hodge theory on wild Riemann surfaces

Let $\boldsymbol{\Sigma} = (\Sigma, \mathbf{a}, \Theta)$ be a rank n wild Riemann surface whose underlying Riemann surface Σ is compact. Choose some residue data $\mathbf{R}$ for $\boldsymbol{\Sigma}$ of (global) degree zero. Recall that a "connection on $\boldsymbol{\Sigma}$" means a good meromorphic connection on a parabolic vector bundle on Σ with poles/parabolic filtrations at $\mathbf{a}$, and irregular class Θ_a at each point $a \in \mathbf{a}$. Similarly for Higgs bundles on $\boldsymbol{\Sigma}$.

Let $\mathcal{M}_{\mathrm{DR}}(\boldsymbol{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable connections on $\boldsymbol{\Sigma}$ with residue data $\mathbf{R}$. Similarly let $\mathcal{M}_{\mathrm{Dol}}(\boldsymbol{\Sigma}, \mathbf{R})$ be the holomorphic moduli space of stable Higgs bundles on $\boldsymbol{\Sigma}$ with residue data $\mathbf{R}$. We suppose that the boundary data is chosen so they are not empty.

Theorem 1.1 (Biquard–B. 2004)**.** *There is a hyperkähler manifold $\mathfrak{M}(\boldsymbol{\Sigma}, \mathbf{R})$ (equipped with a family of complex structures parameterised by $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$) that is a moduli space of irreducible wild harmonic bundles on $\Sigma^\circ = \Sigma \setminus \mathbf{a}$ with boundary conditions determined by $\boldsymbol{\Sigma}, \mathbf{R}$ such that:*
1) In the complex structure determined by $1 \in \mathbb{P}^1$ the space $\mathfrak{M}(\boldsymbol{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{DR}}(\boldsymbol{\Sigma}, \mathbf{R})$ of stable good meromorphic connections,
2) In the complex structure determined by $0 \in \mathbb{P}^1$ the space $\mathfrak{M}(\boldsymbol{\Sigma}, \mathbf{R})$ is isomorphic as a complex manifold to the moduli space $\mathcal{M}_{\mathrm{Dol}}(\boldsymbol{\Sigma}, \mathbf{R})$ of stable good meromorphic Higgs bundles,
3) If the residue data $\mathbf{R}$ is semisimple and there are no strictly semistable connections on $\boldsymbol{\Sigma}$ with residue data $\mathbf{R}$, then the hyperkähler metric on $\mathfrak{M}(\boldsymbol{\Sigma}, \mathbf{R})$ is complete.

The boundary data is related by the following table:

	Dolbeault	De Rham	Betti
weights $\in [0,1), [0,1), \mathbb{R}$	$\lceil\tau\rceil - \tau$	θ	$\phi = \theta + \tau$
eigenvalues $\in \mathbb{C}, \mathbb{C}, \mathbb{C}^*$	$\frac{1}{2}(\phi + \sigma)$	$\lambda = \tau + \sigma$	$\mu = \exp(2\pi i\lambda)$
exponential factors	$\frac{1}{2}q$	q	$\langle q\rangle$

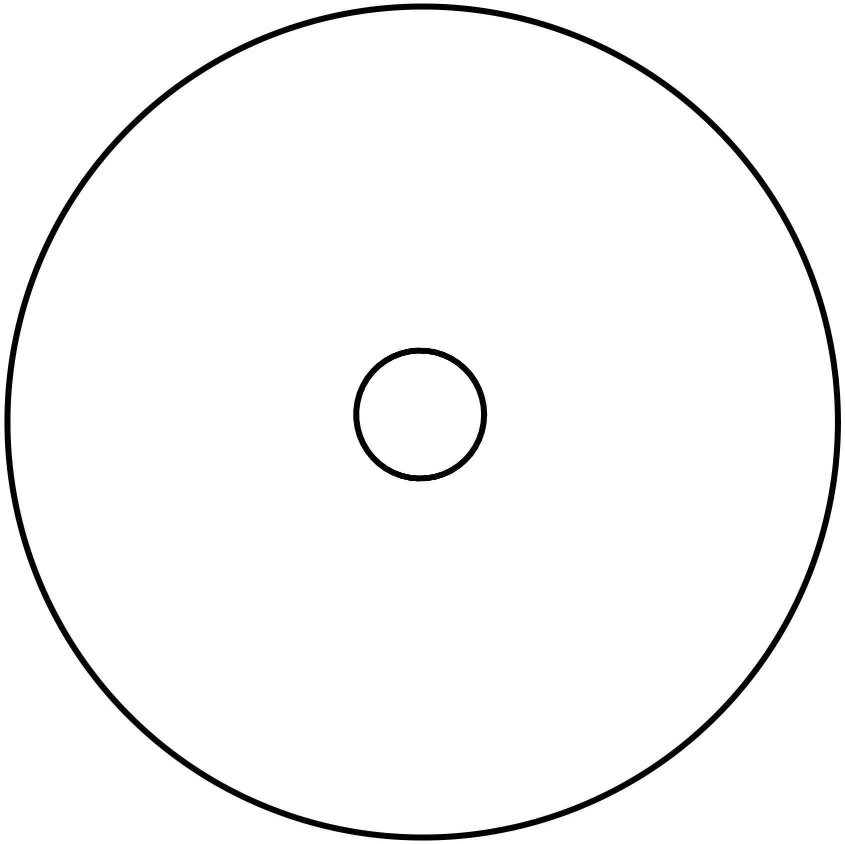
• In tame case ($q = 0$) most of this is due to Konno 1993 and Nakajima 1996 (using Biquard's weighted Sobolev space approach), strengthening Simpson's 1990 tame bijective correspondence in to a diffeomorphism. Even then the completeness statement (beyond the finite energy "strongly parabolic" setting in Konno's paper) is new.

• In the wild case the construction of harmonic bundles from irreducible irregular connections on meromorphic bundles (i.e. Betti weights zero) was established earlier by Sabbah 1999.

• In the nonsingular/compact case ($q = 0 = \lambda = \theta$) it is due to Hitchin, Donaldson, Corlette, Simpson, (Fujiki, Diederich–Ohsawa).

• If also the Higgs field is zero this gives the Narasimhan–Seshadri theorem.

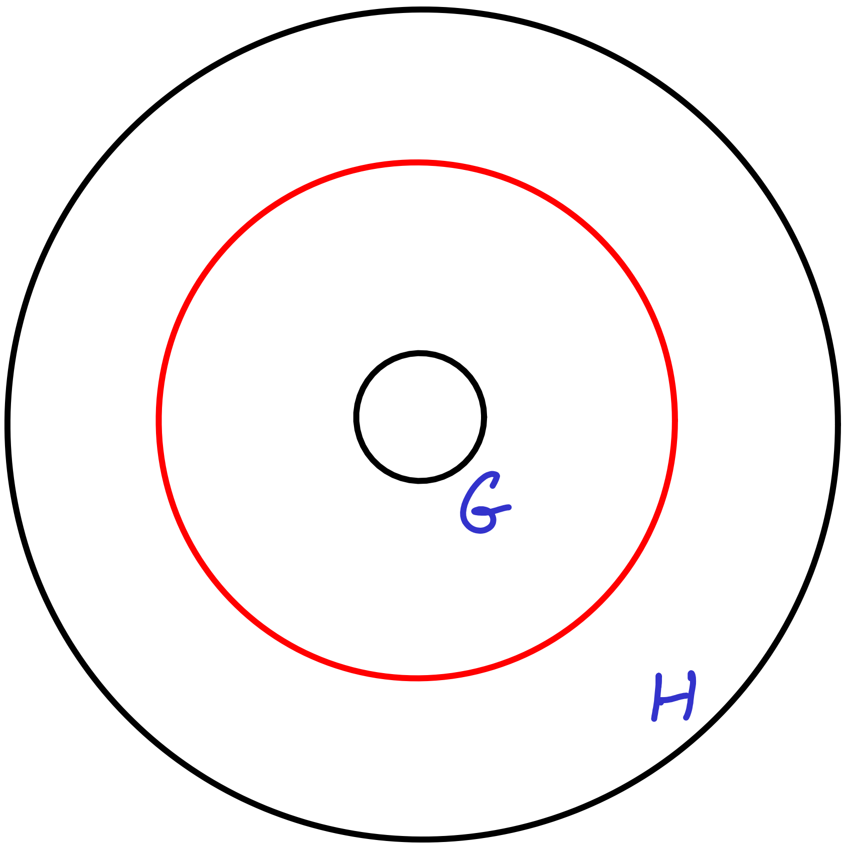
Fission spaces



Fission spaces

$$V = \bigoplus_{i \in I} V_i \qquad I \text{ graded vector space}$$

$$G = GL(V) \supset H = GrAut(V) \cong \prod GL(V_i)$$

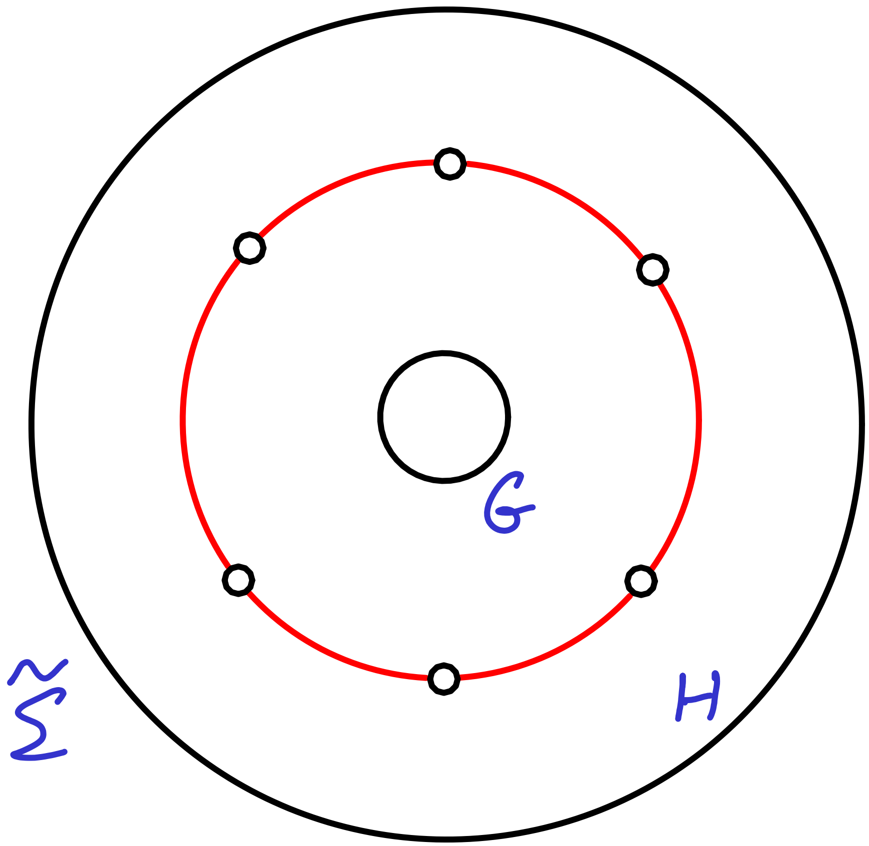


Fission spaces

$V = \bigoplus_{i \in I} V_i$ I graded vector space

$G = GL(V) \supset H = \mathrm{GrAut}(V) \cong \prod GL(V_i)$

- $2k$ tangential punctures $\circ$



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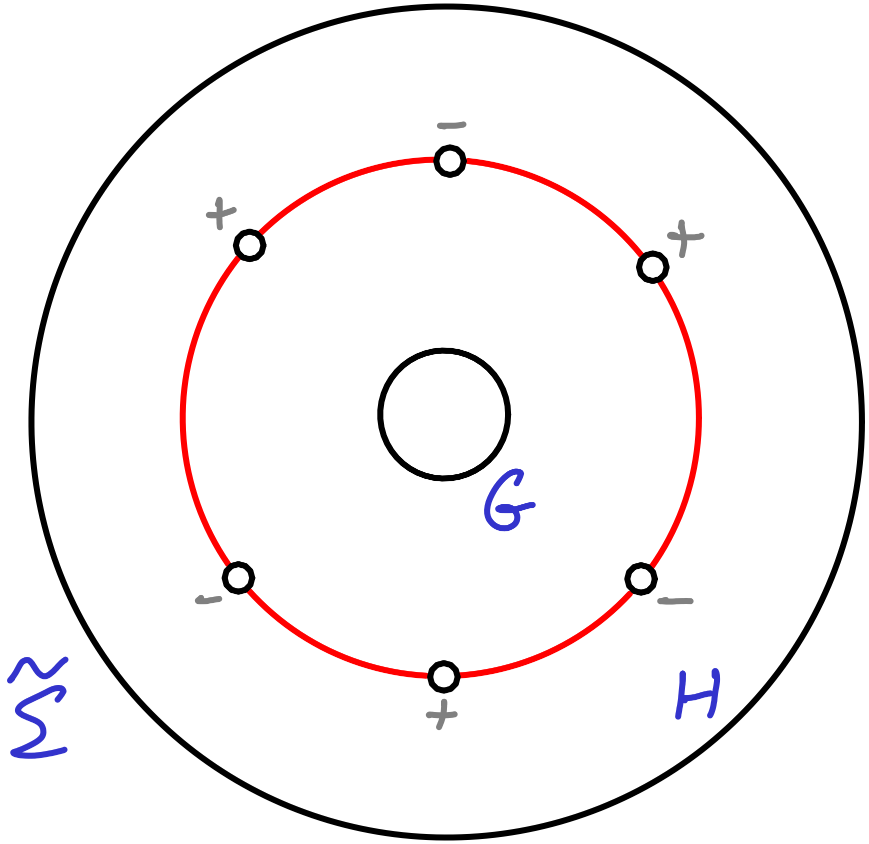

- $2k$ tangential punctures ∘
- $U_+, U_- = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix} \subset G$ (Stokes groups)

Fission spaces

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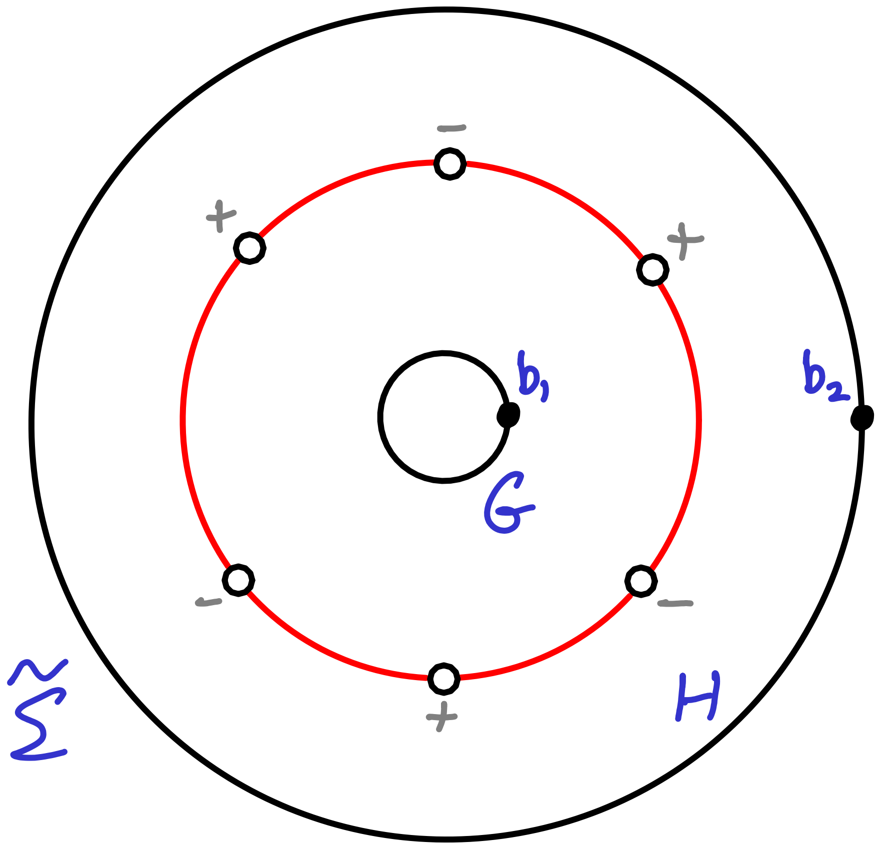

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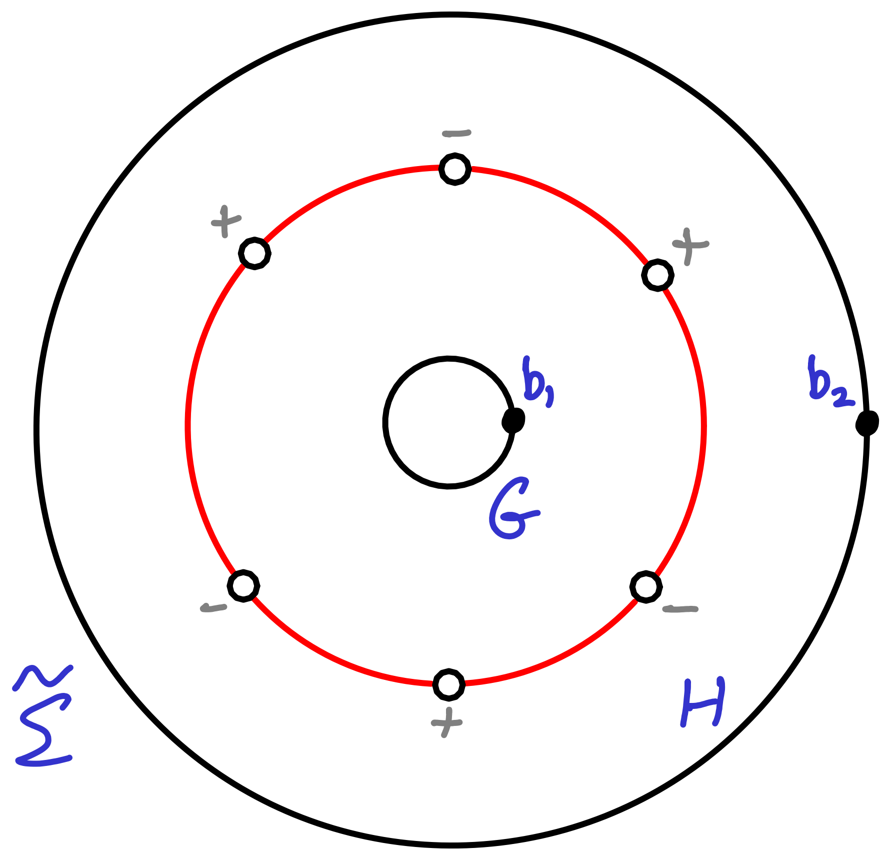

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- $\mathcal{A} = {}_G\mathcal{A}^k_H = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$$\cong G \times H \times (U_+ \times U_-)^k$$

$\cong \{\text{Stokes local systems framed at } b_1, b_2\} / \text{iso.}$

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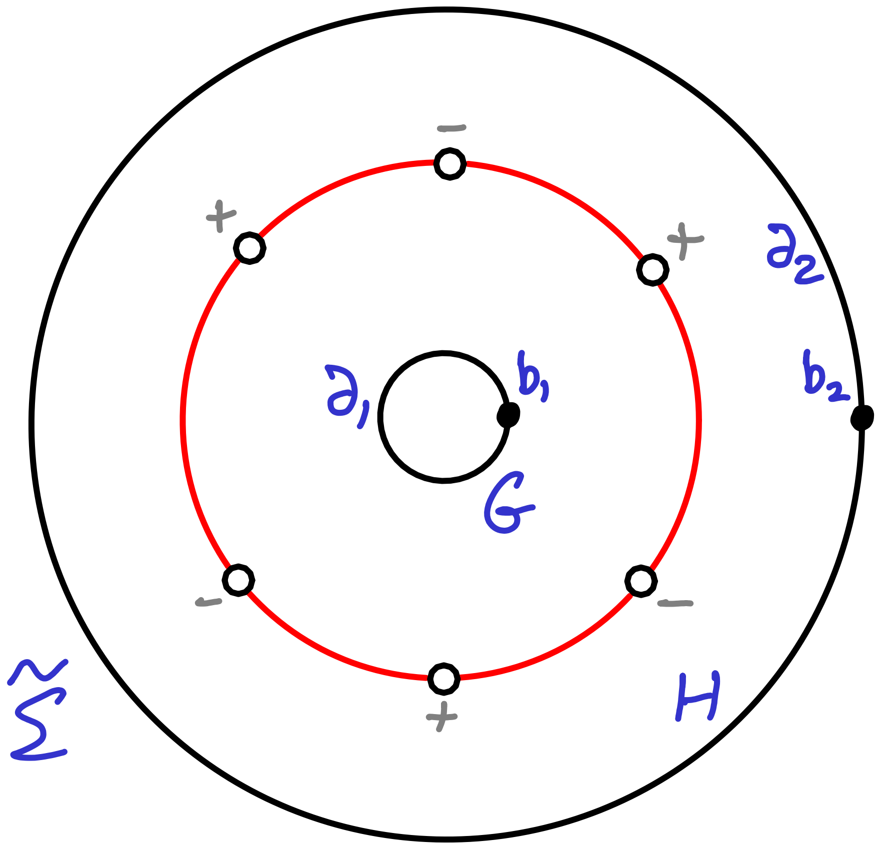

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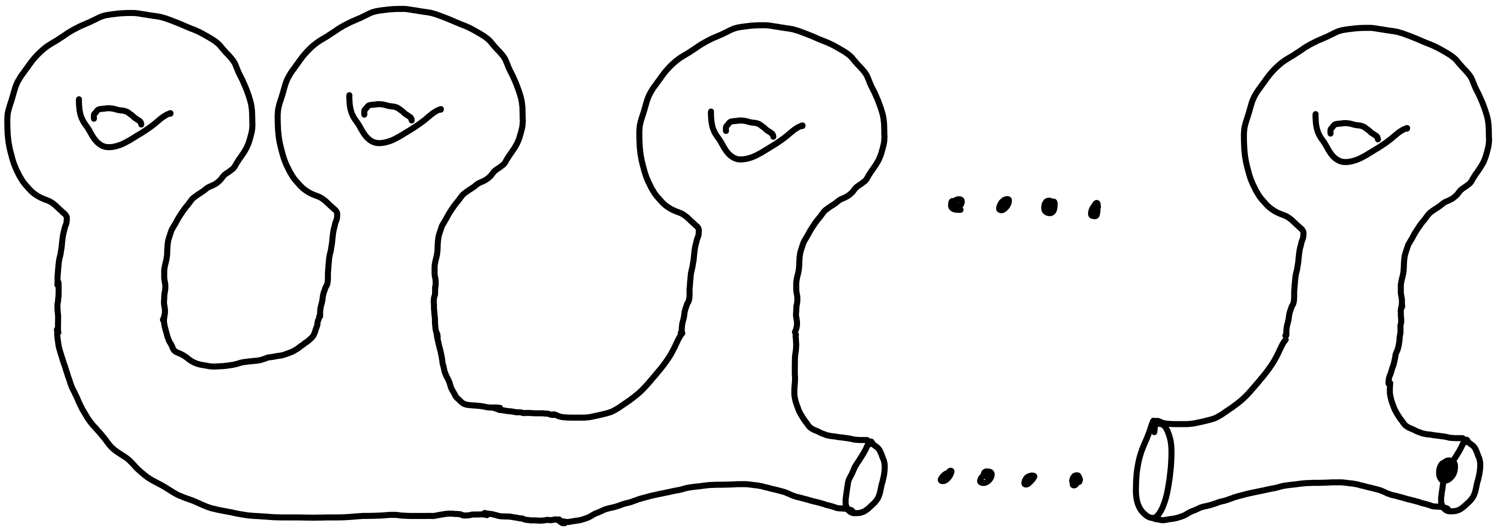
Thm $\mathcal{A}$ is a quasi-Hamiltonian $G \times H$ space with moment map $\mu : \mathcal{A} \to G \times H$, $\quad \mu(\rho) = (\rho(\partial_1), \rho(\partial_2))$

(2002 $H = T$ (any G), 2009 any H, G ($k = 1$), 2011 in general)

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $R = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$$\cong G^{2g}, \quad \mu = [A_1, B_1] \cdots [A_g, B_g] : R \to G$$

$[a,b] = aba^{-1}b^{-1}$

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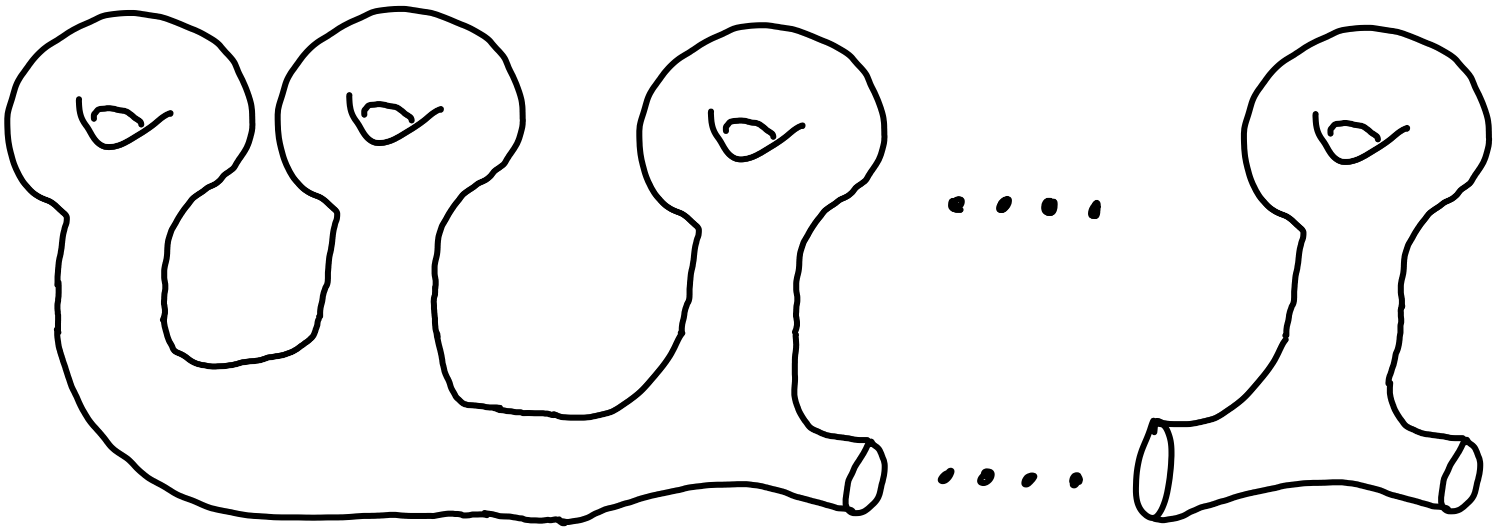

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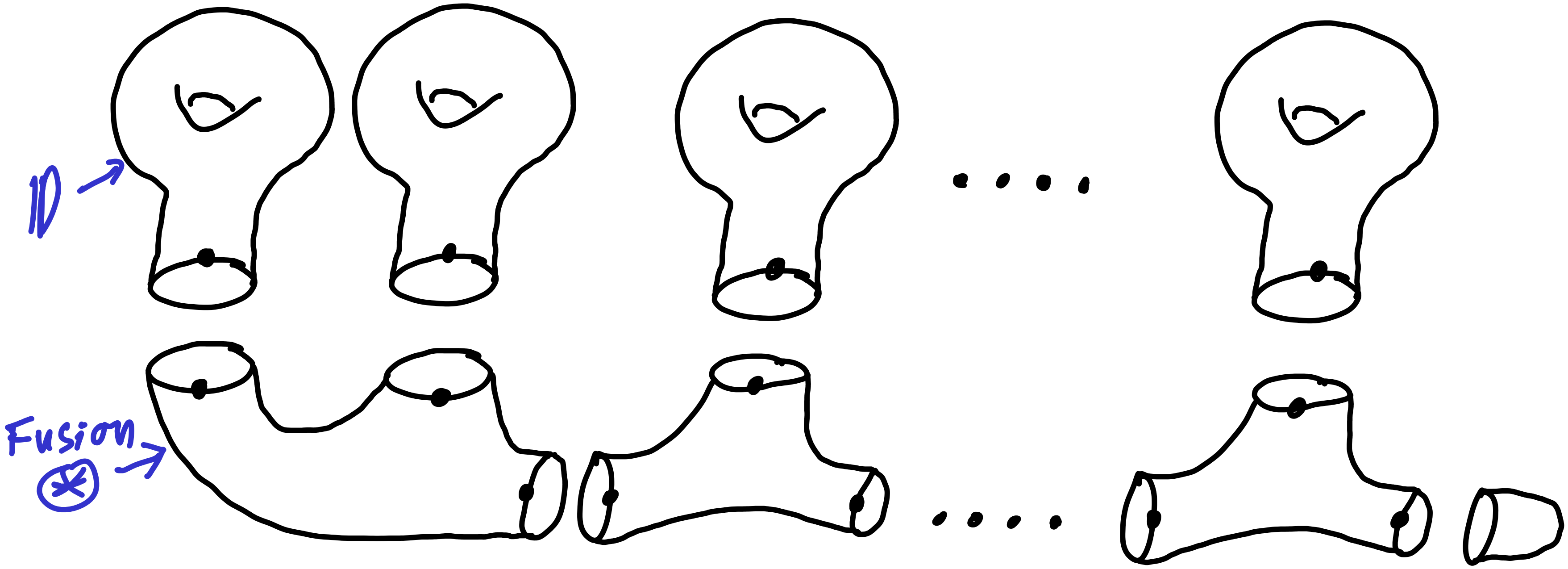
Cor.
- $\mathcal{M}_B = R/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$

E.g. $\mathcal{M}_B(\Sigma_g) = R /\!/ G = \mu^{-1}(1)/G = \{A, B \in G^{2g} \mid \prod [A_i, B_i] = 1\}/G$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



Thm. $\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma_{g,1}), G)$ is a quasi-Hamiltonian G-space

$$\cong G^{2g}, \quad \mu = [A_1, B_1] \cdots [A_g, B_g] : \mathcal{R} \to G$$

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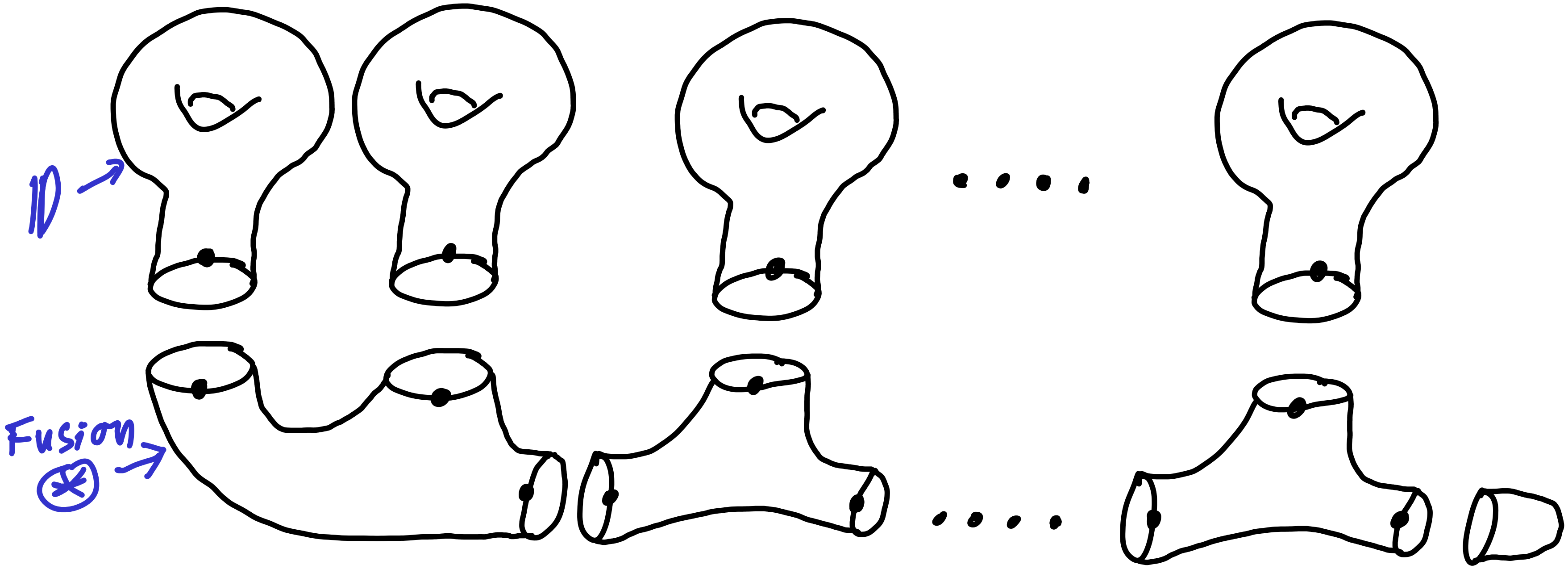
Cor.
- $\mathcal{M}_B = \mathcal{R}/G$ is a Poisson variety
- The symplectic leaves are $\mathcal{M}_B(\mathcal{C}) = \mu^{-1}(\mathcal{C})/G$ for conjugacy classes $\mathcal{C} \subset G$
- Can fuse simple pieces: $\mathcal{R} = \mathbb{D} \circledast \cdots \circledast \mathbb{D}$, $\mathbb{D} = \mathcal{R}(\Sigma_{1,1})$

Tame character varieties (after Alekseev-Malkin-Meinrenken 1998)



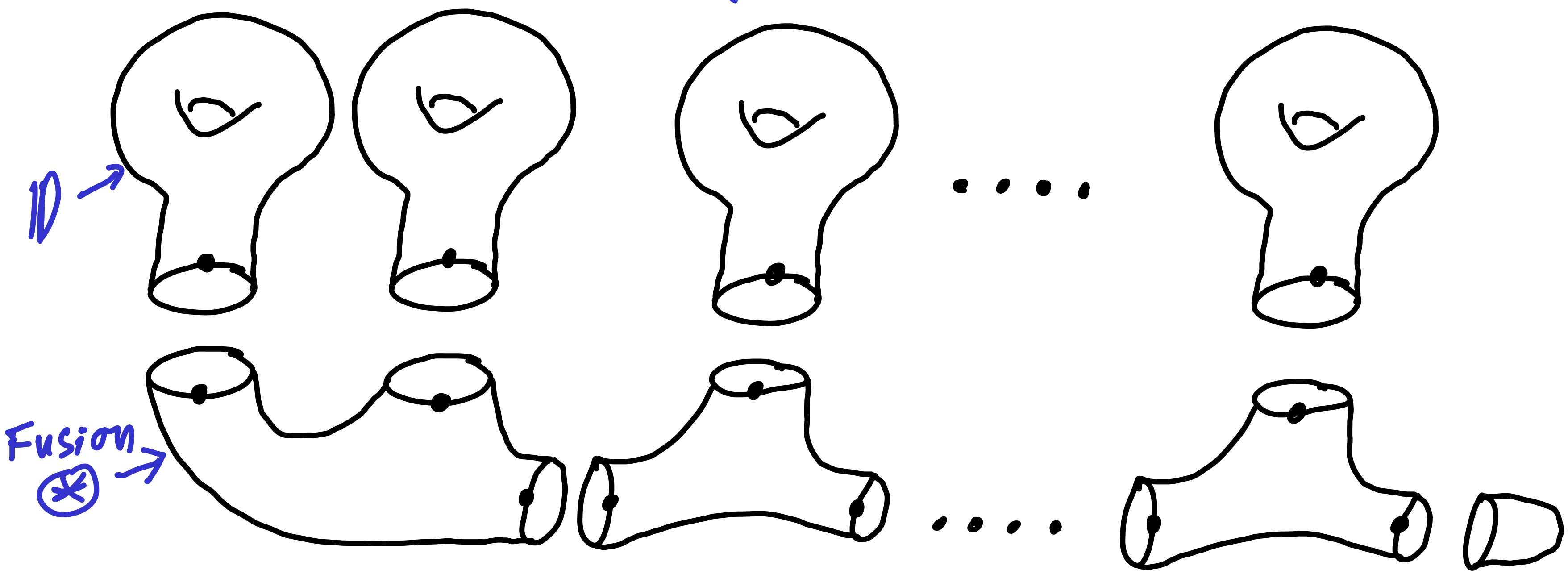
Toolbox: • $\mathbb{D} = \mathcal{R}(\Sigma_{1,1}) \cong G \times G$, • $\mathcal{C} \subset G$

• $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\boxed{\quad}) \cong G \times G$ "double"

• $\circledast$ fusion , • $\oslash$ reduction $(/\!/G)$

$$\mathcal{M}_B(\underset{\sim}{\mathcal{C}}) = \mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!/ G$$

Tame character varieties (after Alekseev–Malkin–Meinrenken 1998)



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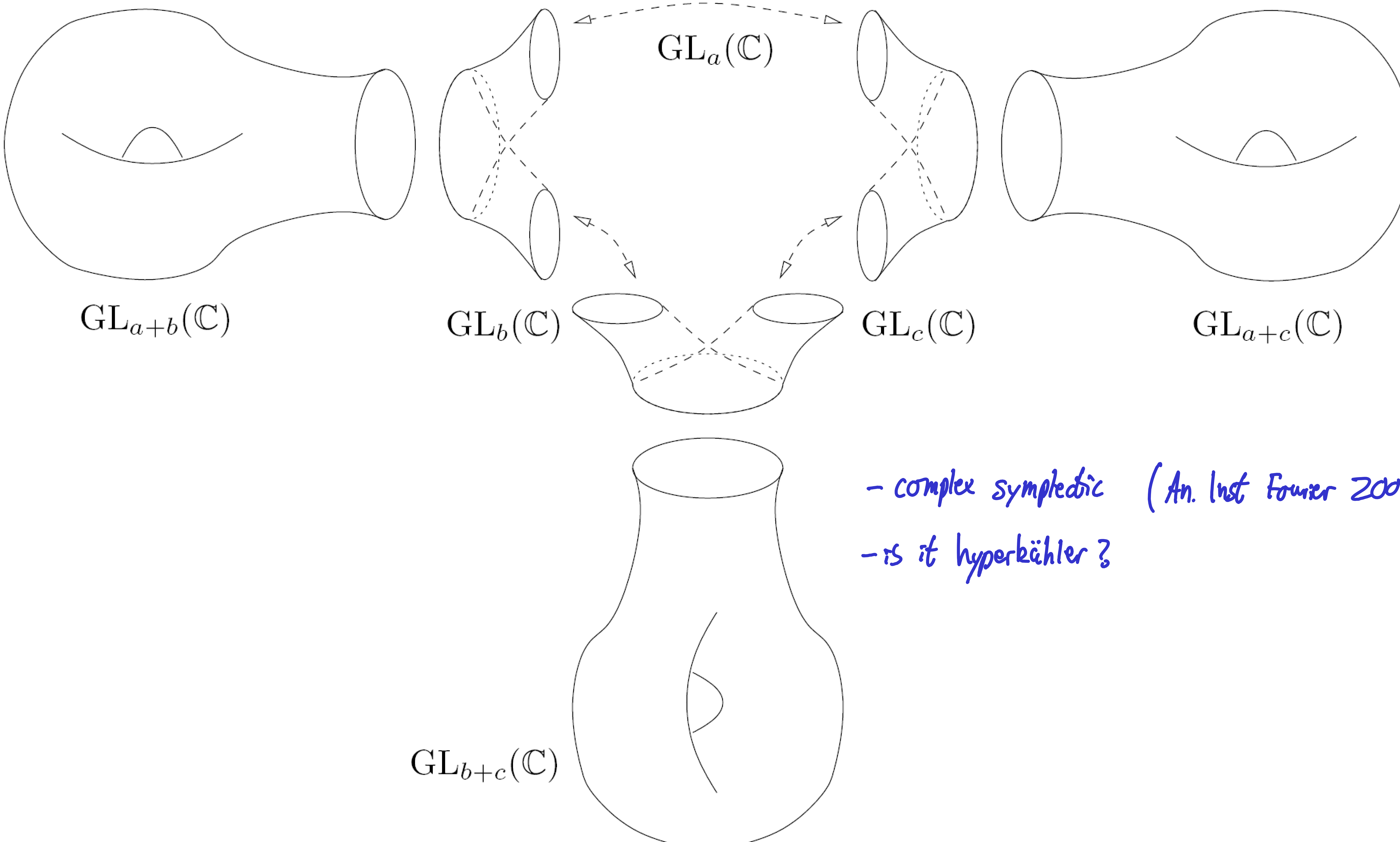
• $D = \mathcal{R}(\Sigma_{0,2}) = \mathcal{R}(\ \ ) \cong G \times G$ "double"

• $\circledast$ fusion, • $\oslash$ reduction $(/\!\!/ G)$

Now add fission spaces $\mathcal{A} = {}_G\mathcal{A}_H^k \quad \forall\, G, H, k$

$\Rightarrow$ lots of new algebraic symplectic/Poisson varieties

"fission varieties" $\supsetneq$ (untwisted) wild character varieties



- complex symplectic (An. Inst Fourier 2009)
- is it hyperkähler?

Wild character varieties

E.g. Birkhoff 1913 wrote presentations in generic setting:

$$(C_1^{-1} h_1 S_{2k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{2k_m}^{(m)} \cdots S_1^{(m)} C_m) = 1$$

(see Jimbo–Miwa–Ueno 1981 equation 2.46)

Wild character varieties

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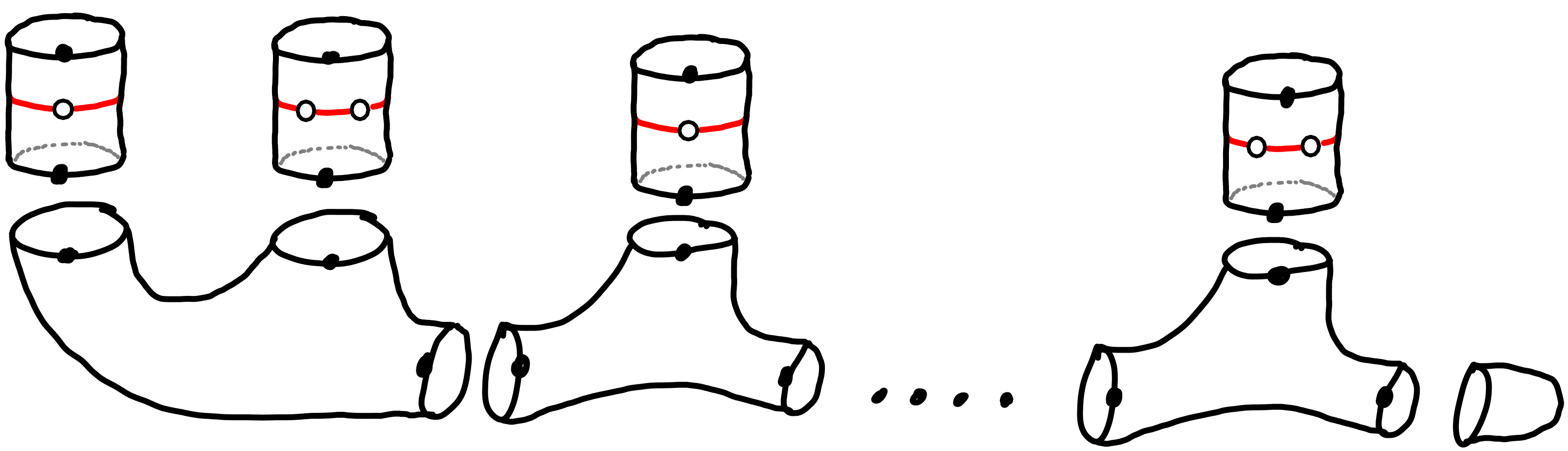
(see Jimbo-Miwa-Ueno 1981 equation 2.46)

$$\mathcal{R} = {}_G\mathcal{A}_T^{k_1} \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_2} \underset{G}{\circledast} \cdots \underset{G}{\circledast} {}_G\mathcal{A}_T^{k_m} \xrightarrow{\mu} T^m \times G$$



<u>Thm</u> Reductions with fixed $h_i \in T$ are symplectic

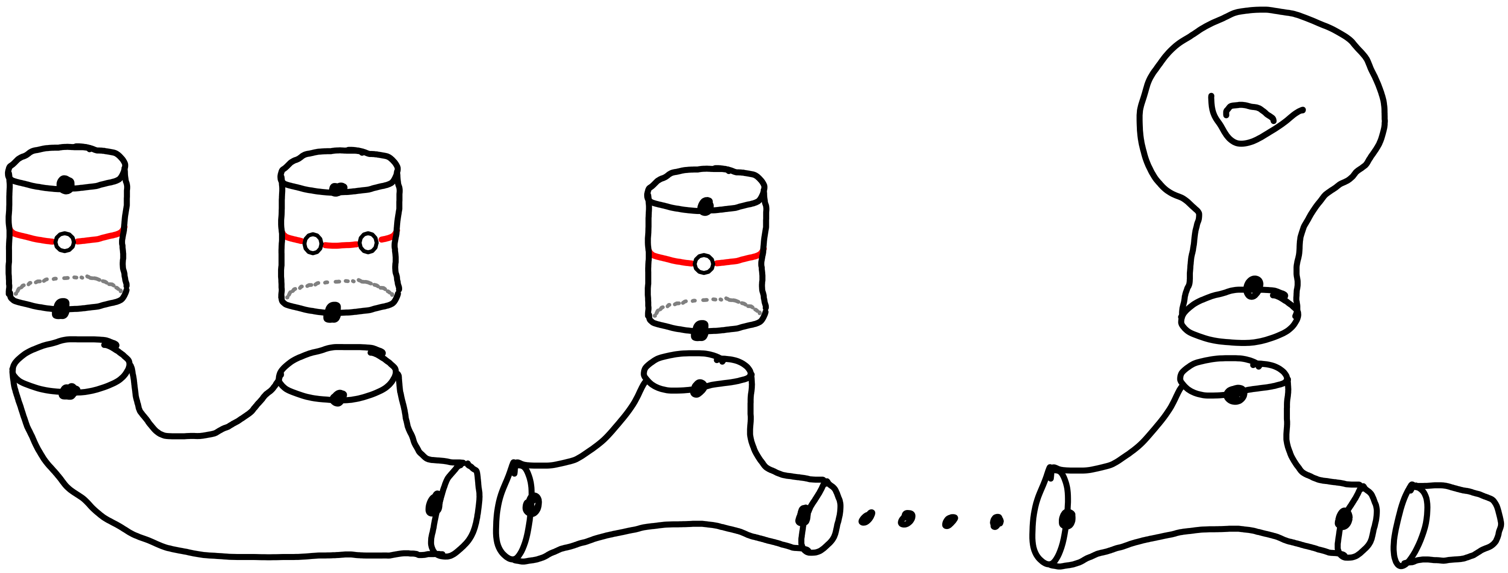
(Adv. Math. 2001 "irreg. Atiyah Bott", algebraic quasi-Hamiltonian approach 2002)

Wild character varieties

Similarly in general (~ any alg. connections on twisted G-bundles)

$$(C_1^{-1} h_1 S_{k_1}^{(1)} \cdots S_1^{(1)} C_1) \cdots (C_m^{-1} h_m S_{k_m}^{(m)} \cdots S_1^{(m)} C_m) \overrightarrow{\prod_1^g} A_i B_i A_i^{-1} B_i^{-1} = 1$$

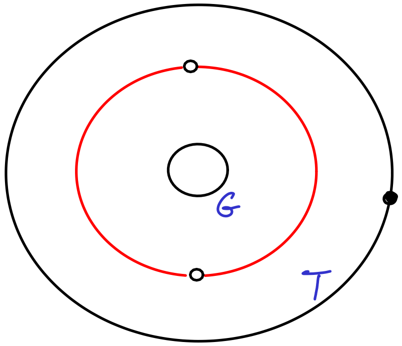
$$\mathrm{THom}_{\mathbb{S}}(\Pi, G) = \mathcal{A}_1 \circledast \cdots \circledast \mathcal{A}_m \circledast \mathbb{D}^{\circledast g} /\!/ G$$



Thm Wild character variety $\mathcal{M}_B = \mathrm{THom}_{\mathbb{S}}(\Pi, G)/\underset{\sim}{H}$ is a Poisson variety with symplectic leaves got by fixing (twisted) conjugacy classes of formal monodromy

... An. Inst. Fourier '09, arXiv:1111.6228, arXiv:1512.08091 (with D. Yamakawa)

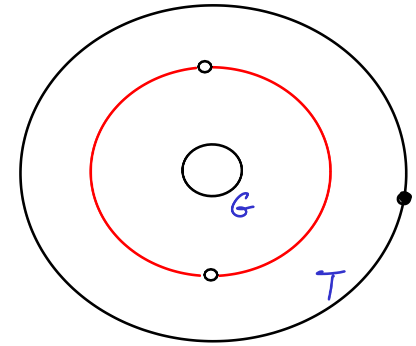
Wild character varieties



E.g. $\quad {}_G\mathcal{A}^1_T/G \cong T \times U_+ \times U_-$

is thus a nonlinear Poisson variety (with Hamiltonian T-action)

Wild character varieties



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Thm (Drinfeld / Semenov Tian Shansky, DeConcini Procesi 1993)

$U_q(\mathfrak{g})$ quantizes a Poisson variety $G^* \cong T \times U_+ \times U_-$

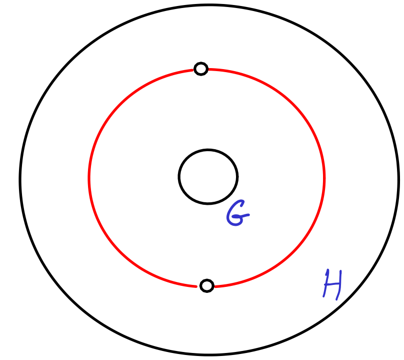
Thm (PB Invent. Math 2001)

$G^* \cong {}_G\mathcal{A}^1_T/G$ as a Poisson variety

Cor. The Drinfeld-Jimbo quantum group is modular

(comes from moduli of connections on curves)

Wild character varieties



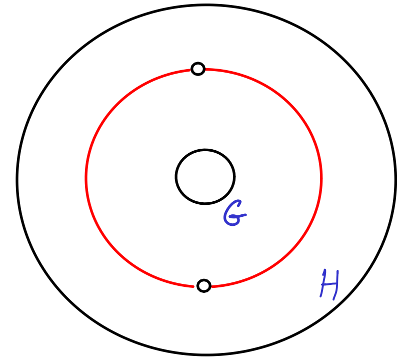
E.g. $\quad {}_G\mathcal{A}^1_H \big/ G\times H \cong (H \times U_+ \times U_-)/H$

is an algebraic Poisson variety with symplectic leaves

$$\mathcal{M}_B(\mathcal{C}, \check{\mathcal{C}}) = \{ h, S_1, S_2 \mid h \in \check{\mathcal{C}},\ hS_1S_2 \in \mathcal{C} \}/H$$

for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

Wild character varieties



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for conjugacy classes $\check{\mathcal{C}} \subset H$, $\mathcal{C} \subset G$

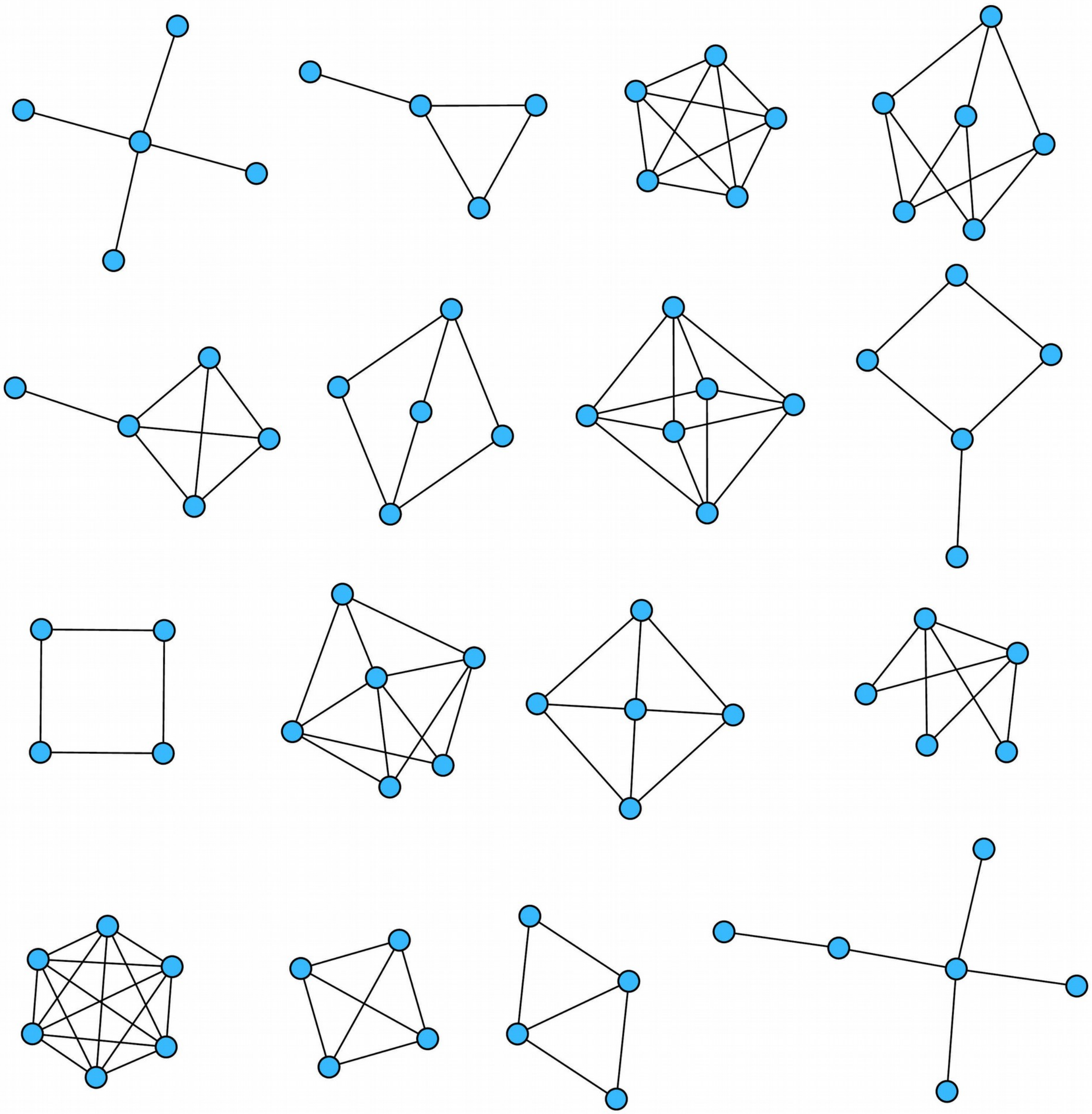
Thm (Fourier-Laplace, Malgrange 1991)

This class of varieties $\cong$ all tame genus zero character varieties

Thm — symplectic structures match too (PB arXiv 1307)

— and the hyperkähler metrics (Sz. Szabo arXiv 1407)

$\rightsquigarrow$ notion of "representations" of abstract moduli space



Sakai's question

Plato to Painlevé (McKay – Harnad) c.f. PB 0706.2634 Exercise 3

Sakai's question

Plato to Painlevé (McKay-Harnad) c.f. PB 0706.2634 Exercise 3

groups: Tetra. Octa. Icosa. $\subset$ $SO_3(\mathbb{R})$

Plato to Painlevé (McKay–Harnad) c.f. PB 0706.2634

Sakai's question

Exercise 3

groups: Tetra. Octa. Icosa. $\subset$ $SO_3(\mathbb{R})$

$$\uparrow$$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset$ $SU_2 \subset SL_2(\mathbb{C})$

Sakai's question

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$$\uparrow \qquad \uparrow \qquad \uparrow$$

resolve: X_T X_O X_I

Sakai's question

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$\uparrow$ $\uparrow$ $\uparrow$

resolve: $\tilde{X}_T$ $\tilde{X}_O$ $\tilde{X}_I$

+deform

$\downarrow$ $\downarrow$ $\downarrow$

$\mathbb{C}^6$ $\mathbb{C}^7$ $\mathbb{C}^8$

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$$\uparrow \qquad \uparrow \qquad \uparrow$$

resolve: $\tilde{X}_T$ $\tilde{X}_O$ $\tilde{X}_I$

+deform

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$\mathbb{C}^6$ $\mathbb{C}^7$ $\mathbb{C}^8$

$$\circlearrowleft \qquad \circlearrowleft \qquad \circlearrowleft$$

Weyl groups: $W(E_6)$ $W(E_7)$ $W(E_8)$

Sakai's question

Plato to Painlevé (McKay – Harnad) c.f. PB 0706.2634 Exercise 3

groups: Tetra. Octa. Icosa. $\subset SO_3(\mathbb{R})$

$$\uparrow$$

binary groups: $\tilde{T}$ $\tilde{O}$ $\tilde{I}$ $\subset SU_2 \subset SL_2(\mathbb{C})$

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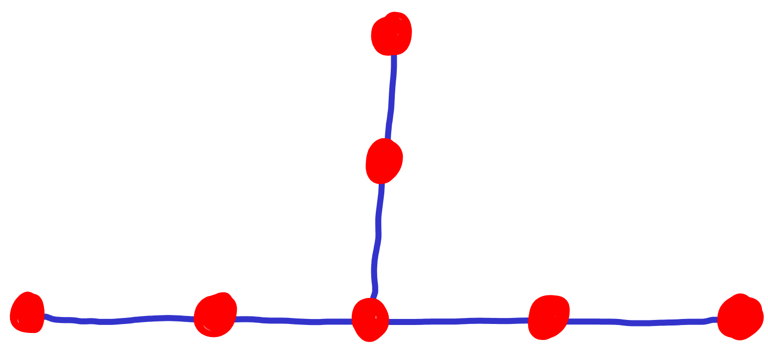
Weyl groups: $W(E_6)$ $W(E_7)$ $W(E_8)$

Kronheimer: (1989)

- smooth fibres are complete hyperkähler 4-folds
- construct in terms of affine Dynkin graph

E.g. E_6 case (hol. symplectic approach)



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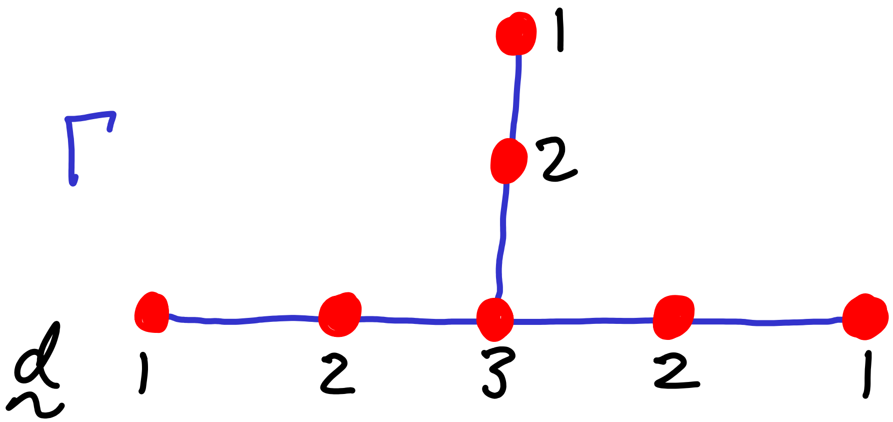
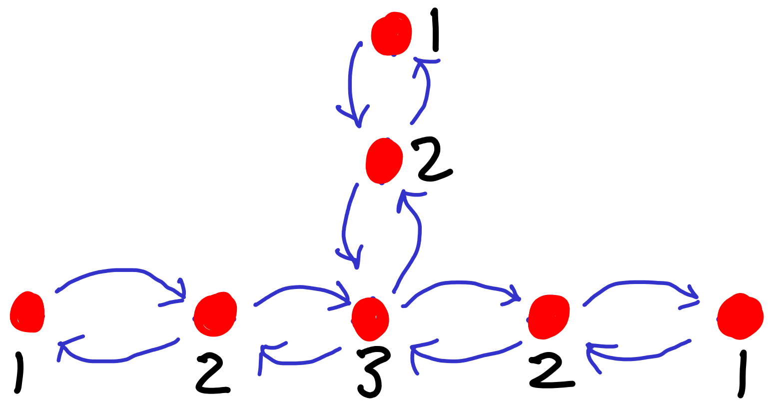


$$\mathbb{V} = \mathrm{Rep}(\Gamma, \mathbb{C}^{\underset{\sim}{d}})$$

E.g. E_6 case (hol. symplectic approach)

Γ



$\underset{\sim}{d}$

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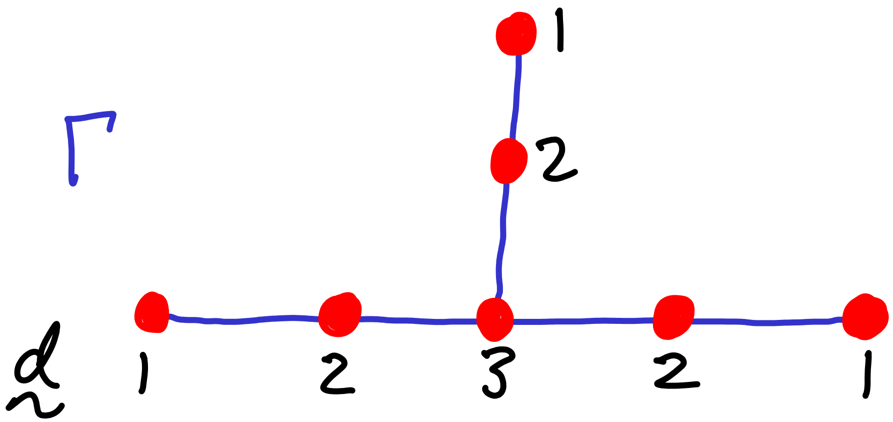

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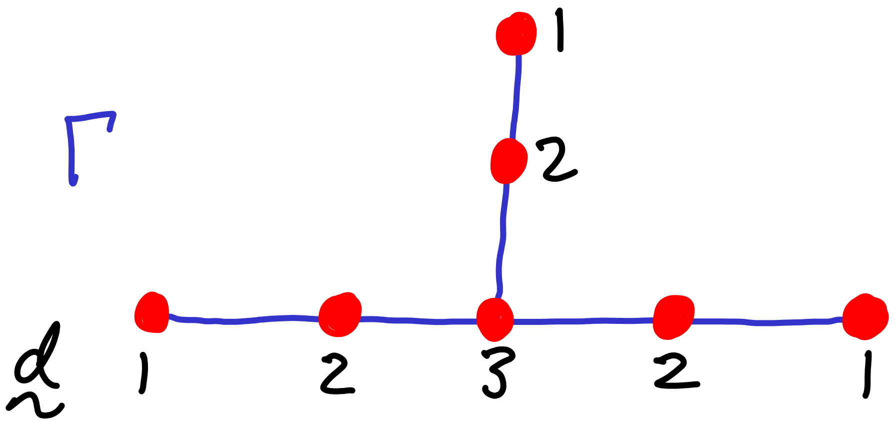

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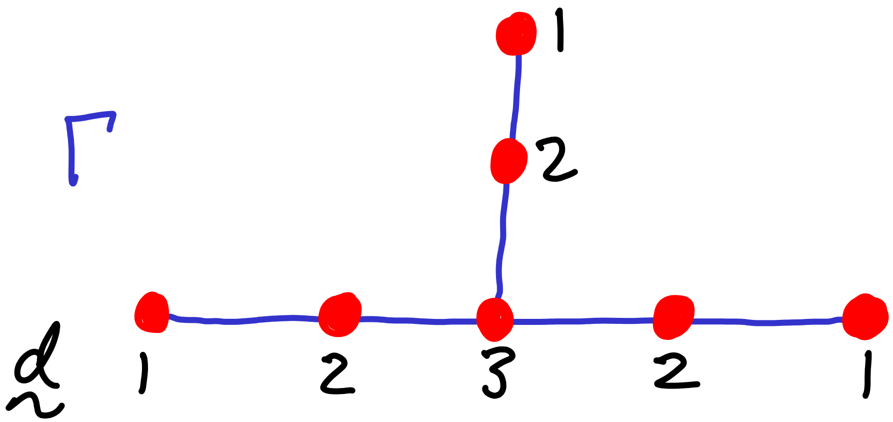

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$$N = NQV(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d}) = \mathbb{V} /\!/_{\underset{\sim}{\lambda}} G = \mu^{-1}(\underset{\sim}{\lambda}) / G$$

$$\underset{\sim}{\lambda} \in \mathrm{Lie}(G)^* \cong \prod \mathrm{End}(\mathbb{C}^{d_i}) \quad \text{central}$$

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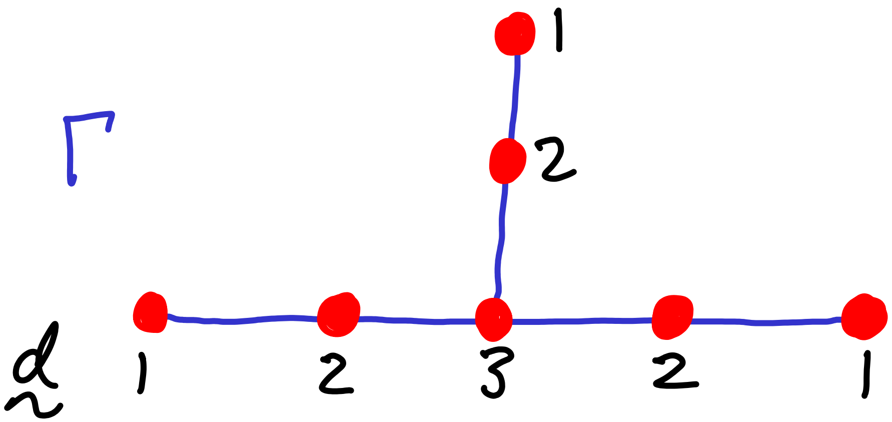

$$\boxed{\dim_{\mathbb{C}}(\mathcal{N}) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})}$$

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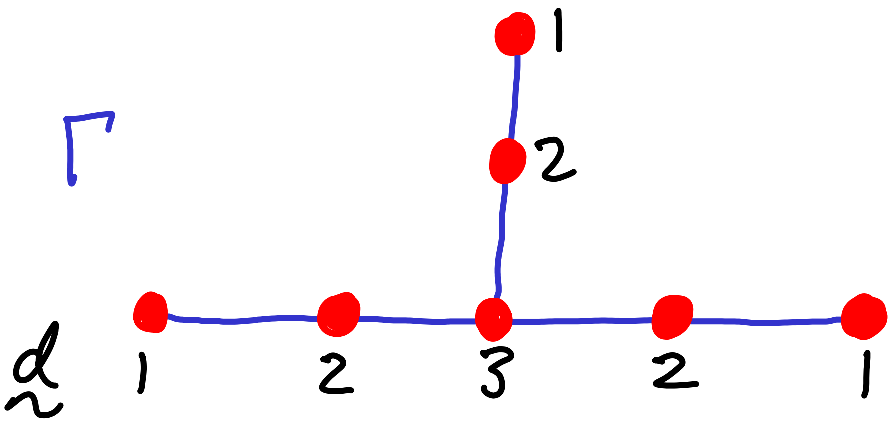

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$\mathcal{N}$ is modular $\cong$ moduli space of Fuchsian systems $\mathcal{M}^*$

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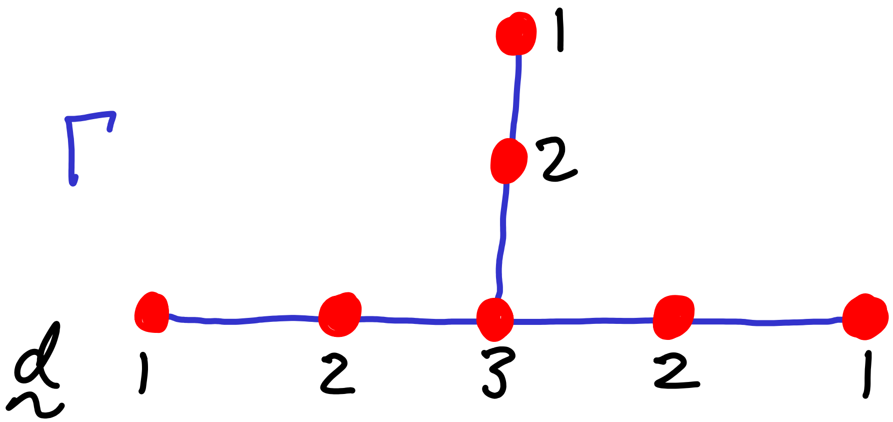

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$$\dim_{\mathbb{C}} = 6 + 6 + 6 - 2(9-1) = 2$$

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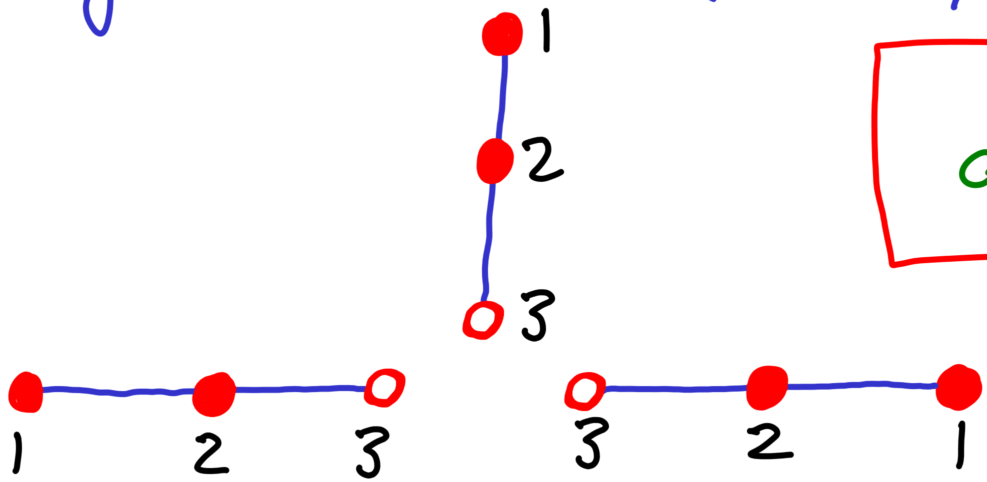

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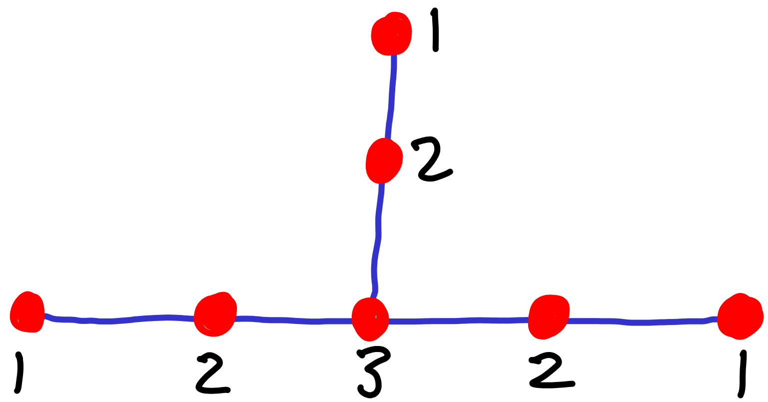

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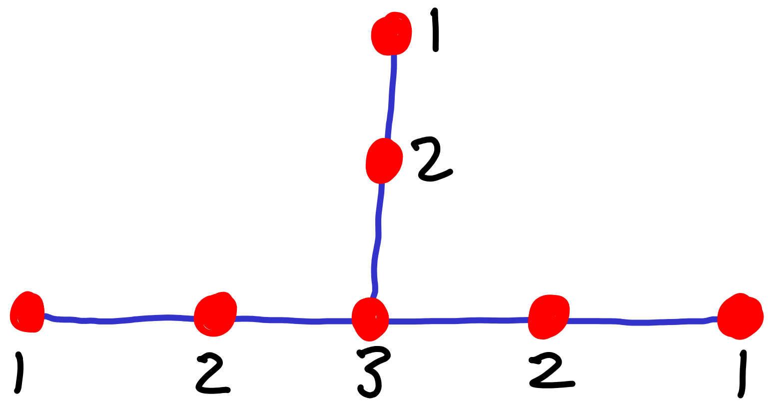

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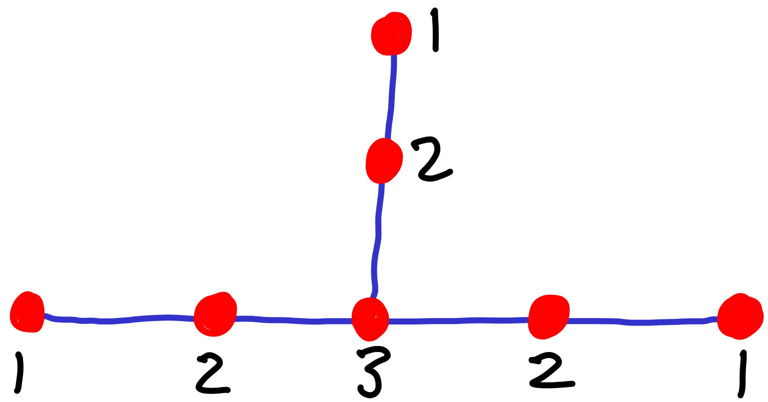
E.g. E_6 case (hol. symplectic approach)

Γ



$\underset{\sim}{d}$

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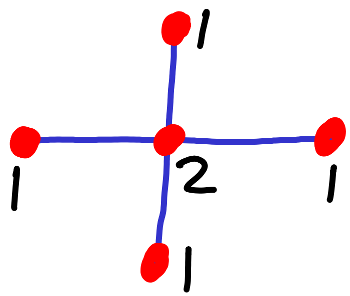
- NQV of any star-shaped Γ is modular (Kraft-Procesi, Nakajima, Crawley-Boevey)
- Get multiplicative version = character variety $\mathcal{M}_B \cong \mathcal{C}_1 \circledast \mathcal{C}_2 \circledast \mathcal{C}_3 /\!/ GL_3$

$$\mathcal{M}^* \subset \mathcal{M}_{DR} \xrightarrow{RH} \mathcal{M}_B \qquad \text{"Global Lie theory"}$$

$\exists$ one more star-shaped affine Dynkin graph:

$\hat{D}_4$



$\sim$ quaternion group $\subset SU_2$

$\{\pm 1, \pm i, \pm j, \pm k\}$

$W(D_4) \circlearrowleft \mathbb{C}^4$ "constants"

Rank 2 Fuchsian systems with 4 poles $\rightsquigarrow$ cross ratio $\in \mathcal{M}_{0,4}$

"modular parameters" / "times"

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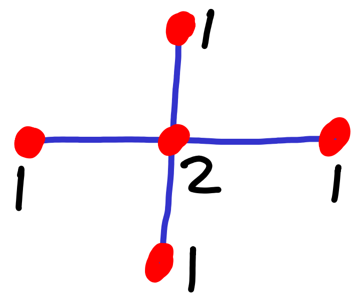

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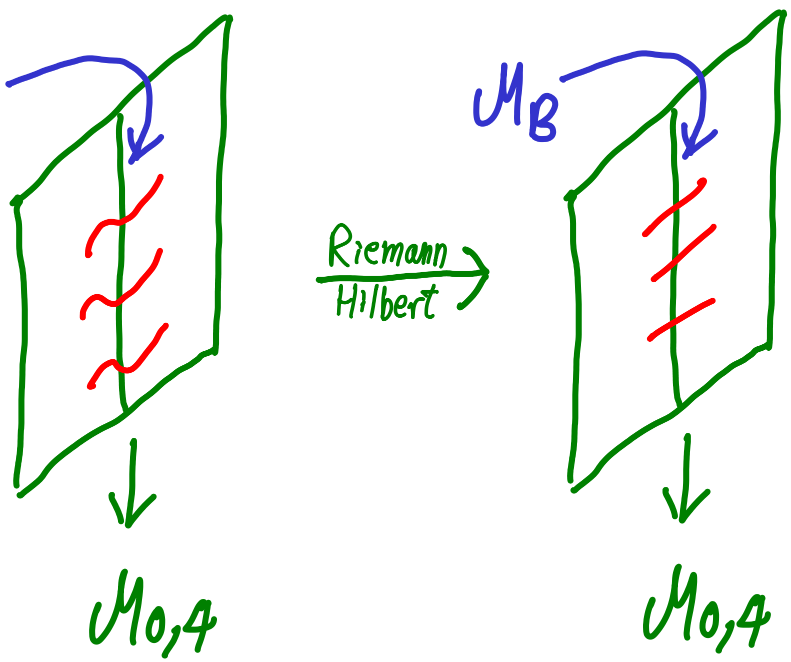
- Familiar from the Painlevé VI equation: (Richard Fuchs 1905)

$N \cong \mathcal{M}^*$ $\quad$ $\mathcal{M}_B$



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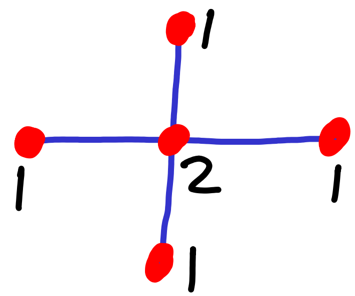

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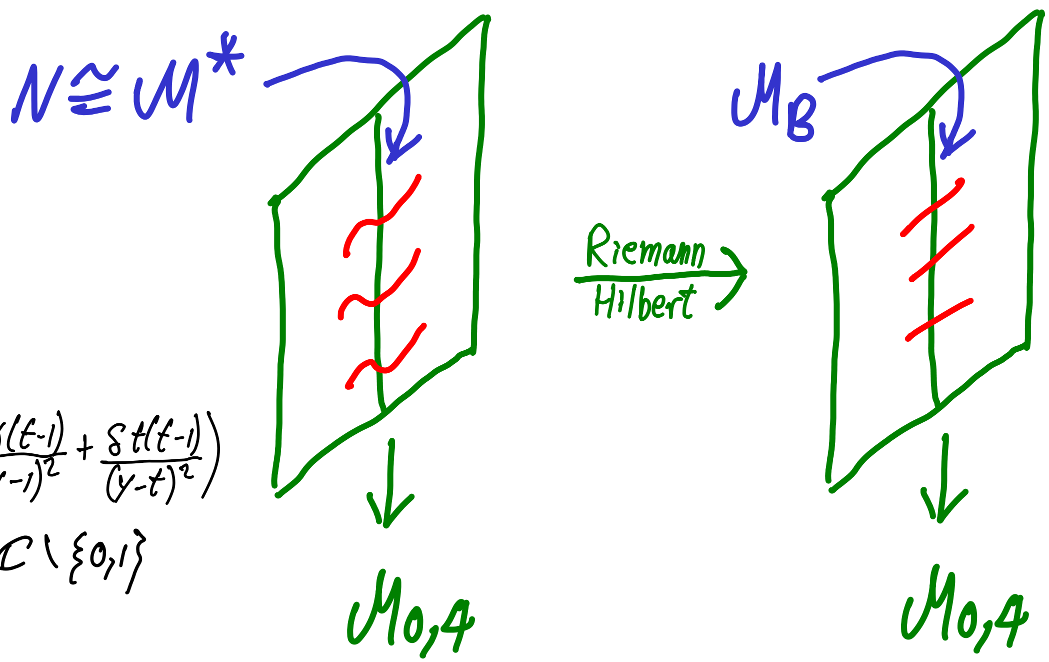
- Familiar from the Painlevé VI equation: (Richard Fuchs 1905)

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{(y')^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2}\right)$$

$\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $t \in \mathcal{M}_{0,4} \cong \mathbb{C} \setminus \{0,1\}$



$\mathcal{M}_B \cong$ Fricke-Klein-Vogt cubic surface

$$xyz + x^2 + y^2 + z^2 = ax + by + cz + d$$

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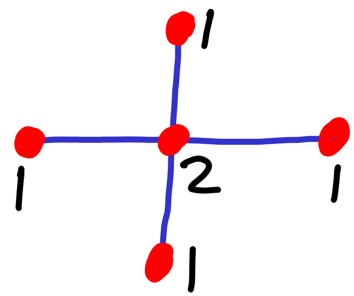
$\hat{D}_4$



~ quaternion group ⊂ SU_2

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Rank 2 Fuchsian systems with 4 poles ⇝ cross ratio $\in \mathcal{M}_{0,4}$

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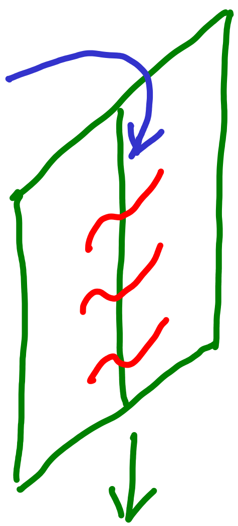
Okamoto 1987: affine Weyl group $W(\hat{D}_4) \circlearrowright \mathbb{C}^4$ relating $P_{\overline{VI}}$ equations

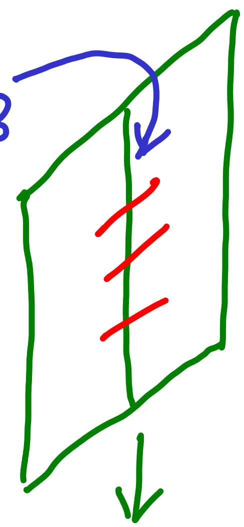
$N \cong \mathcal{M}^*$



$\mathcal{M}_{0,4}$

$\xrightarrow[\text{Hilbert}]{\text{Riemann}}$

$\mathcal{M}_B$



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THE PAINLEVÉ EQUATIONS AND THE DYNKIN DIAGRAMS

Kazuo Okamoto

Department of Mathematics
College of Arts and Sciences
University of Tokyo
Tokyo, Japan

1 Painlevé Systems

Let δ be a differential on $\mathbf{C}(t)$, i.e.

$$\delta = f(t)\frac{d}{dt},$$

$f(t)$ being a rational function in t, and

$$\mathrm{H}(t; q, p) \in \mathbf{C}[t, q, p],$$

a polynomial in three variables (t, q, p). We consider the Hamiltonian system of ordinary differential equations:

$$\begin{aligned} \delta q &= \frac{\partial \mathrm{H}}{\partial p}, \\ \delta p &= -\frac{\partial \mathrm{H}}{\partial q}, \end{aligned} \tag{1}$$

under the assumption that H is of the second degree with respect to p. Therefore, by

P_J	1	2	3	4	5	6
δ	$\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$\frac{d}{dt}$	$t\frac{d}{dt}$	$t(t-1)\frac{d}{dt}$
number of parameters	0	1	2	2	3	4
Affine Weyl Group	--	$\mathbf{A}_1$	$\mathbf{B}_2$	$\mathbf{A}_2$	$\mathbf{A}_3$	$\mathbf{D}_4$
Particular solutions	--	Airy	Bessel	Hermite-Weber	Confluent Hyper-geometric	Gauß' Hyper-geometric

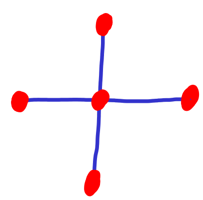
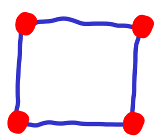
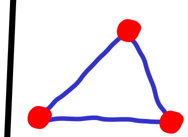
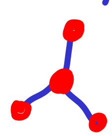
(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

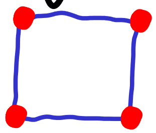
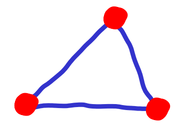
$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) & \text{(ALE space of type } \hat{A}_3, \hat{A}_2, \hat{A}_1\text{)} \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

(0706.2634) Exercise 3: works for Painlevé 5, 4, 2 too:

$$\begin{cases} \mathcal{M}^* \cong NQV(\Gamma) \\ \Gamma = \text{affine Dynkin graph of Okamoto symmetry group} \end{cases}$$

(ALE space of type $\hat{A}_3, \hat{A}_2, \hat{A}_1$)

Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, $rk\,2$)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram				?		?
Special solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$?	Weber ?	Bessel-Clifford ${}_0F_1$?	Airy ?	–

Questions ① What are the higher dimensional modular quiver varieties lying over    generalising the stars?

② What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \ncong NQV(\Gamma)\ \forall \Gamma$) & their higher dimensional analogues?

Painlevé equation	6	5	4	3	2	1
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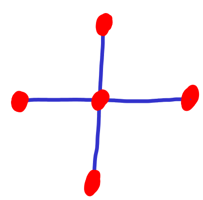
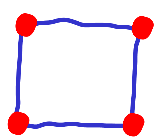
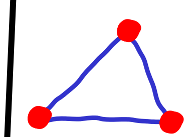
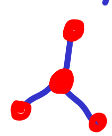
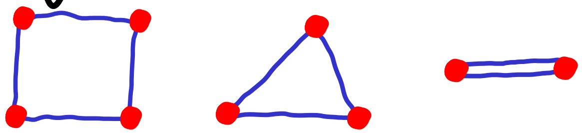
Diagram				?		?
Special solutions	Gauss ${}_2F_1$ 	Kummer ${}_1F_1$?	Weber ?	Bessel-Clifford ${}_0F_1$?	Airy ?	–

Questions 1 What are the higher dimensional modular quiver varieties lying over  generalising the stars?

2 What about Painlevé 1 & Painlevé 3 ($\mathcal{M}^*(P_3) \ncong NQV(\Gamma)\ \forall \Gamma$) & their higher dimensional analogues?

---

3 What is the 'deeper' analogue of $\mathcal{M}_{0,4}$ in general?

$\rightsquigarrow$ moduli of wild Riemann surfaces

4 What is the 'deeper' analogue of the nonlinear local system $\mathcal{M}_B \to \mathcal{M}_{0,4}$?

$\rightsquigarrow$ local system of wild character varieties over any admissible deformation of a wild Riemann surface

[P.B. Annals of Math. 2014]

Choices: • $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)

• eigenvalues/orbit of Λ, parabolic weights (constants)

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Qn ① Consider $(\mathbb{P}^1, \infty, Q)$ very good, 1 pole, wts zero

$\mathcal{M}^* \subset \mathcal{M}_{DR}$ where $V \to \mathbb{P}^1$ trivial

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$$\mathcal{M}^* \subset \mathcal{M}_{DR} \quad \text{where } V \to \mathbb{P}^1 \text{ trivial}$$

Quiver modularity theorem

{PB simply laced case + general conjecture
{Hiroe-Yamakawa general proof

$$\mathcal{M}^*(Q, \Lambda) \cong NQV(\Gamma, \underset{\sim}{\lambda}, \underset{\sim}{d})$$ for some Γ

Choices: • $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q})$ "wild Riemann surface" (modular parameters)

• eigenvalues/orbit of Λ, parabolic weights (constants)

Qn ① Consider $(\mathbb{P}^1, \infty, Q)$ very good, 1 pole, wts zero

$\mathcal{M}^* \subset \mathcal{M}_{DR}$ where $V \to \mathbb{P}^1$ trivial

Quiver modularity theorem {PB simply laced case + general conjecture; Hiroe-Yamakawa general proof}

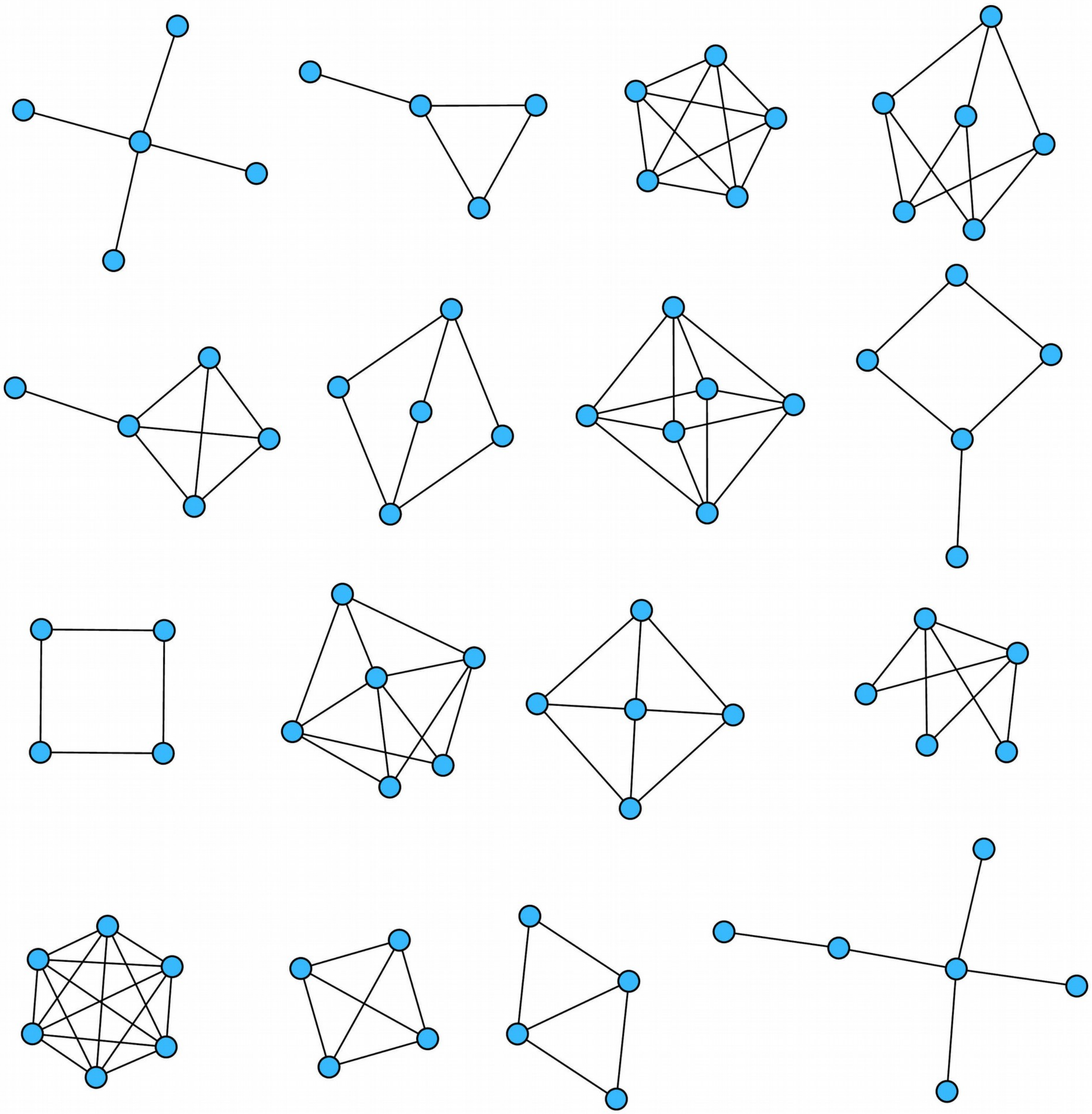
$$\mathcal{M}^*(Q, \Lambda) \cong NQV(\Gamma, \underset{\sim}{\delta}, \underset{\sim}{d})$$ for some Γ

"supernova graphs" (core + legs)

$$Q = \begin{pmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{pmatrix}, \quad q_i \in x\,\mathbb{C}[x]$$

core nodes $= \{q_i\}$, $\#\text{edges}(q_i, q_j) = \deg(q_i - q_j) - 1$

+ legs from $\Lambda \in \mathfrak{h} = \prod \mathfrak{gl}_{d_i}(\mathbb{C})$



<u>Idea</u> $\mathcal{M}^* \cong \mathcal{O} /\!/ G$

$$\begin{cases} dQ + \Lambda \frac{dz}{z} \in \mathcal{O} \subset \mathfrak{g}_k^* \\ G_k = GL_n(\mathbb{C}[z]/z^{k+1}) \end{cases}$$

$\cong H /\!/ \tilde{\mathcal{O}} /\!/ G$ "extended orbit" $\tilde{\mathcal{O}} \circlearrowleft G \times H$
Λ

$$1 \to B_k \to G_k \xrightarrow{ev} G \to 1$$

$\cong H /\!/ \mathcal{O}_B$
Λ

$\mathcal{O}_B \subset \mathfrak{b}_k^*$ Birkhoff orbit
B_k-coadjoint orbit of dQ

decoupling: $\boxed{\tilde{\mathcal{O}} \cong (T^*G) \times \mathcal{O}_B}$

<u>Thm</u> $\mathcal{O}_B \cong \mathbb{V}(\text{core graph})$ as a Hamiltonian H-space

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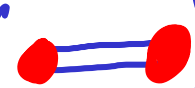
Thm $\mathcal{O}_B \cong \mathbb{W}(\text{core graph})$ as a Hamiltonian H-space

E.g. $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$, $\mathcal{O}_B \cong T^*\mathbb{C}^2 = \mathbb{W}($  $)$ (Painlevé II)

$\mathcal{M}^* \cong$ Eguchi-Hanson space ($\hat{A}_1$ ALE space) $T^*\mathbb{P}^1$, $\mathcal{O} \subset \mathfrak{sl}_2(\mathbb{C})$

Qn 2

Thm (B-Yamakawa 2020)

$\exists$ uniform way to define a diagram for any meromorphic connection on $\mathbb{P}^1$ with ≤ 1 irreg. singularity

- $\dim(\mathcal{M}_B) = 2 - (\underset{\sim}{d}, \underset{\sim}{d})$ form from Cartan matrix C of diagram
- Can have loops/edges of negative multiplicity

[Any moduli space on $\mathbb{P}^1$ has such a representation]

Qn 2

Thm (B-Yamakawa 2020, Douçot 2021)

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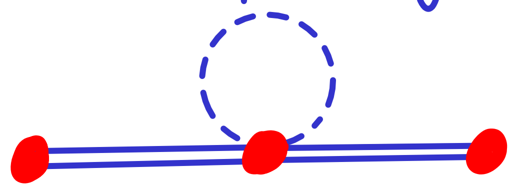
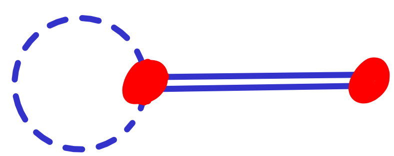
e.g. Painlevé III



$$C = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

$$\underset{\sim}{d} = (1, 1, 1)$$

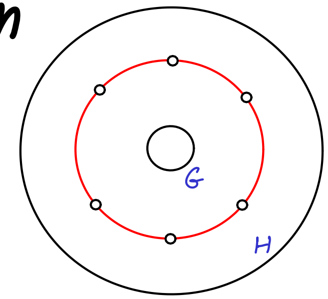
- $\sim_{\mathbb{Z}}$ Intersection form of $\mathcal{M}_{DR}$
- Weyl group $\cong W_{aff}(B_2)$ as Okamoto had
- special solutions (Bessel-Clifford)

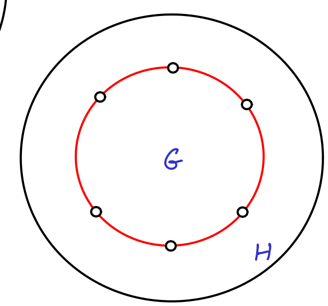


Qn 2 Idea: pass to wild character variety / & use general presentations of them

$$\mathcal{M}^* \hookrightarrow \mathcal{M}_{DR} \overset{RHB}{\cong} \mathcal{M}_B$$ wild character variety

	Hamiltonian	quasi-Hamiltonian
$G \times H$-spaces:	$\tilde{\mathcal{O}}$	$\mathcal{A}$
H-spaces:	$\mathcal{O}_B$	$\mathcal{B} = \mathcal{A}/\!/G$
G-spaces:	$\mathcal{O}$	$\mathcal{C} = \mathcal{A}/\!/_{\gamma} H$





"deeper conjugacy classes" ($\gamma = e^{2\pi i \Lambda}$)

can do all this side in general twisted case (B-Y 2015)
+ looks like quiver rep. for GL_n

- points of maximal decay $\partial \subset \mathcal{I}$

  $\partial(q) \subset \langle q \rangle$ where e^q max decay

- Irregularity: $\mathrm{Irr}(q) = \# \partial(q)$

  $\mathrm{Irr}(\sum n_i \langle q_i \rangle) = \sum n_i \mathrm{Irr}(q_i)$

- Ramification: $\mathrm{Ram}(q) = \deg \pi : \langle q \rangle \longrightarrow \partial$ (min r)

Choose $\Theta = \sum n_i \langle q_i \rangle$, $\quad \mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, at $\infty \in \mathbb{P}^1$

wild Riemann surface $(\mathbb{P}^1, \infty, \Theta)$

- points of maximal decay $\partial \subset \mathcal{I}$

$\partial(q) \subset \langle q \rangle$ where e^q max decay

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Choose $\Theta = \sum n_i \langle q_i \rangle$, $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$, at $\infty \in \mathbb{P}^1$

Core diagram: nodes $\sim \{\langle q_i \rangle\}$

$\#$ arrows $\langle q_i \rangle \to \langle q_j \rangle = B_{ij} := \begin{cases} A_{ij} - \beta_i \beta_j & i \neq j \\ A_{ii} - \beta_i^2 + 1 & i = j \end{cases}$

$A_{ij} := \mathrm{Irr}(\mathrm{Hom}(\langle q_i \rangle, \langle q_j \rangle))$, $\beta_i = \mathrm{Ram}(q_i)$

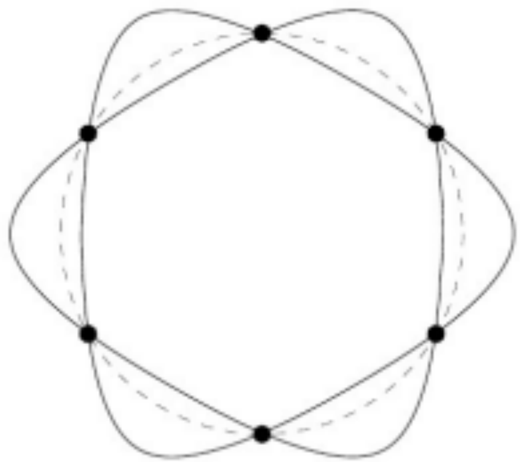
(symmetrized) Cartan matrix: $\boxed{C = 2 - B}$

Then glue on legs from classes $\mathcal{C}_i \subset GL_{n_i}(\mathbb{C})$ as before

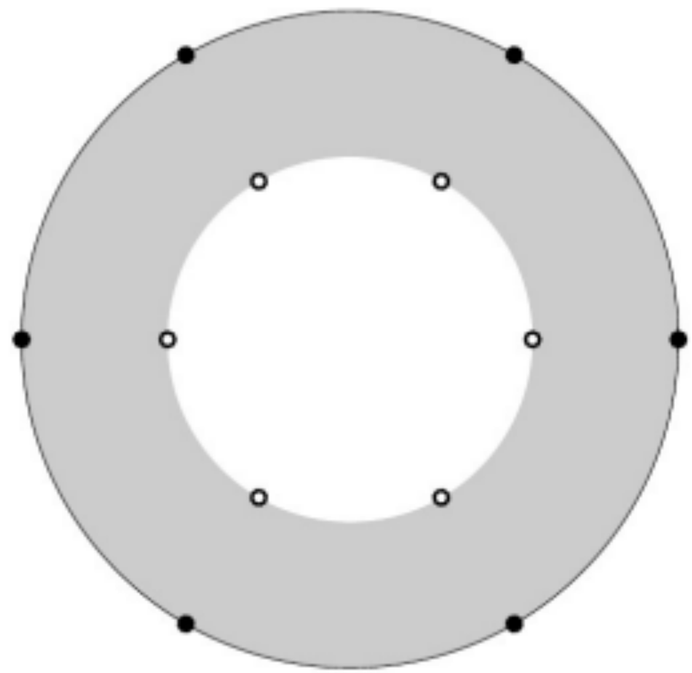
Painlevé II: $Q = \begin{pmatrix} x^3 & \\ & -x^3 \end{pmatrix}$

solutions involve e^Q

plot growth/decay of $\exp(x^3)$, $\exp(-x^3)$:



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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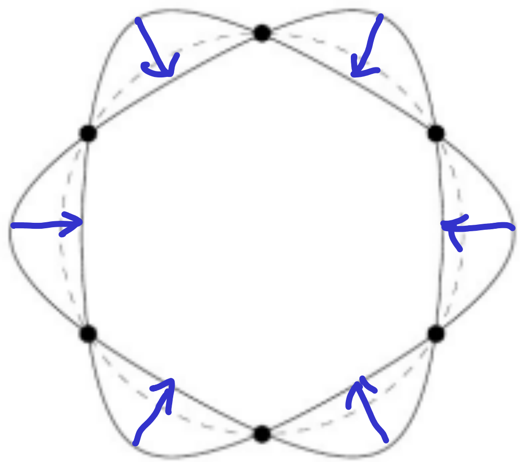

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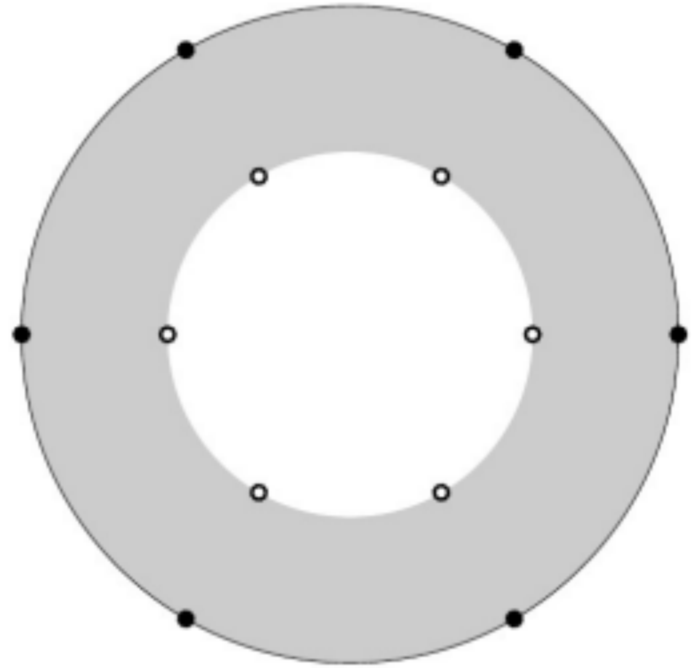
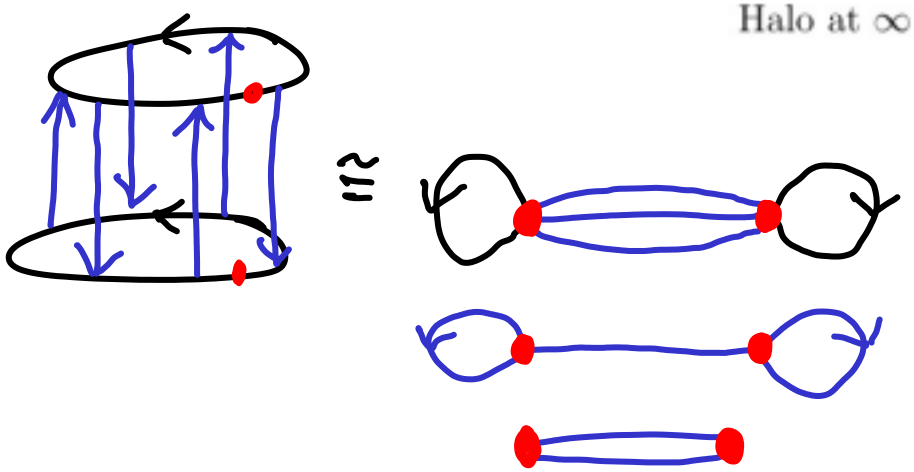


Halo at ∞ with singular directions



$\mu_G = 1$

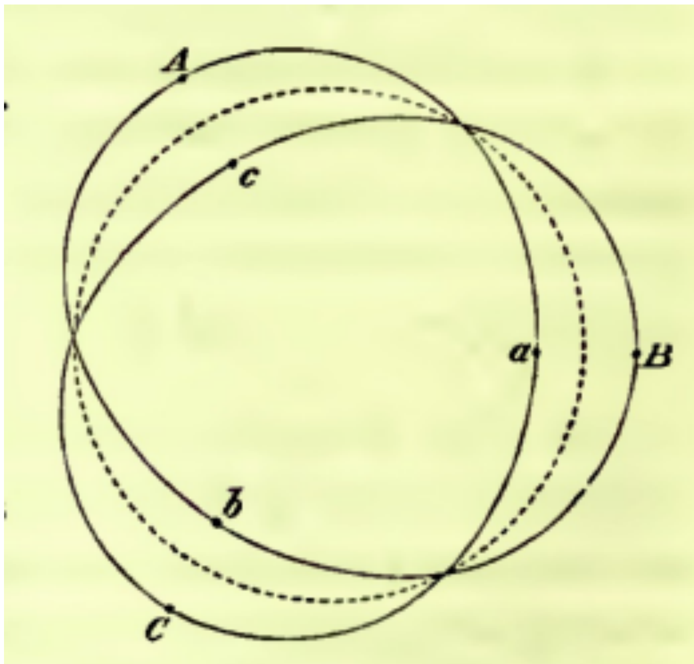
$h S_6 S_5 \cdots S_1 = 1$

2x2 matrix relation

result: $\hat{A}_1$

Airy equation (Stokes 1857)

solutions involve $\exp(x^{3/2})$



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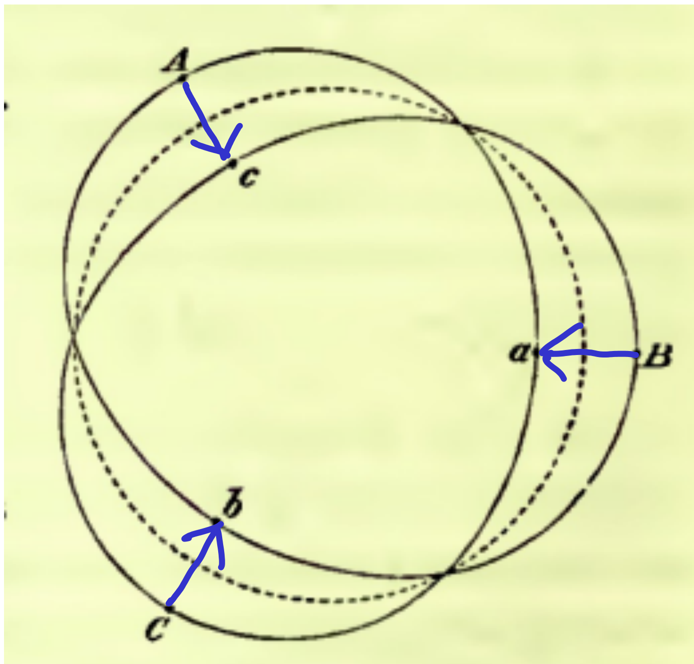
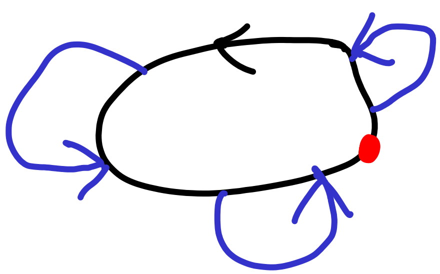
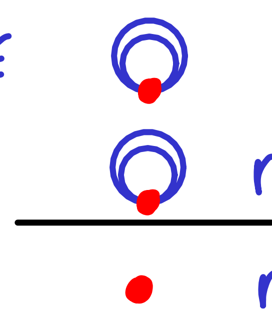


$$\mu_G = 1$$

$$\underbrace{h}_{\begin{pmatrix} 0 & * \\ 1 & 0 \end{pmatrix}} \underbrace{S_3}_{u_+} \underbrace{S_2}_{u_-} \underbrace{S_1}_{u_+} = 1$$



$\cong$

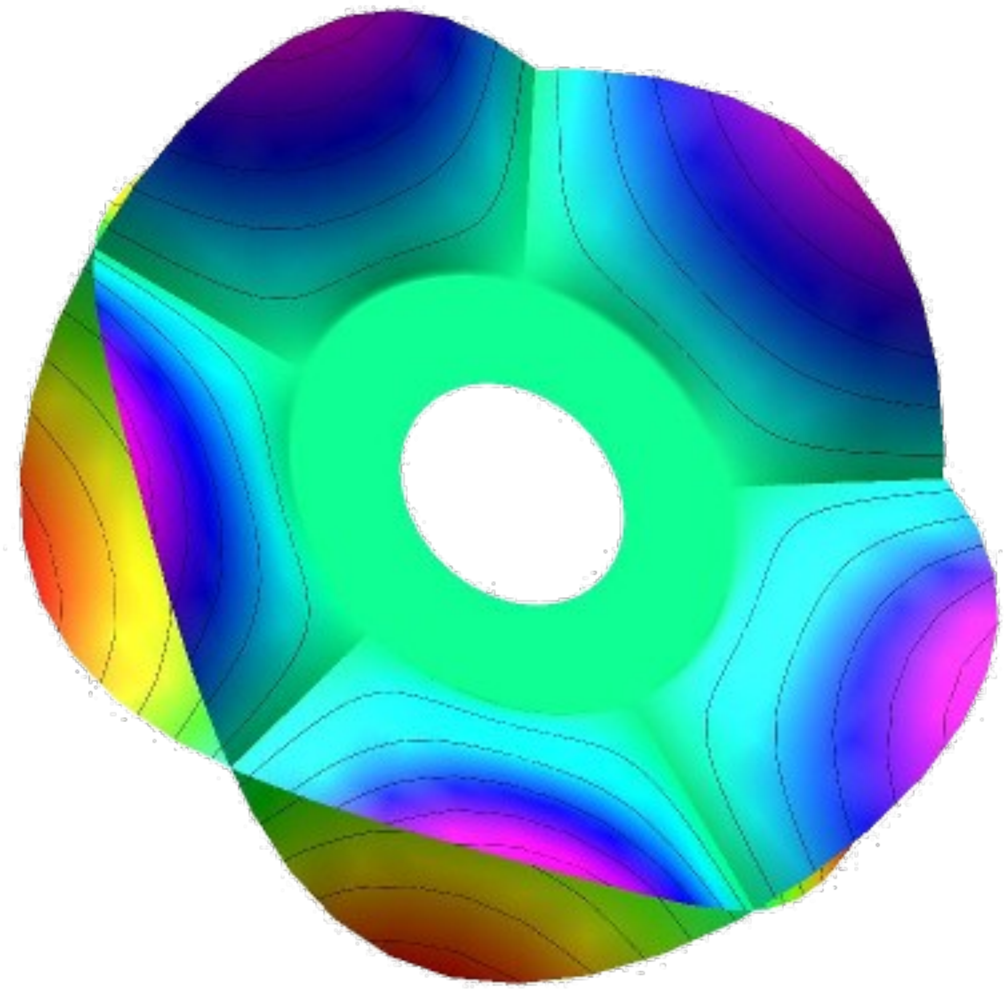


relations

resulting diagram

Painlevé 1

$\exp(x^{5/2})$



relations

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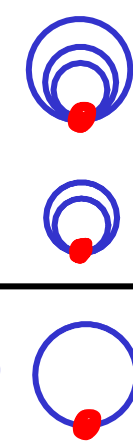
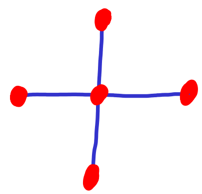
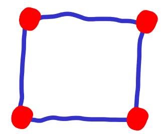
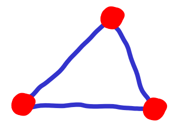
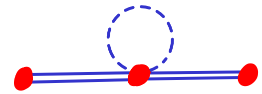
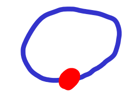
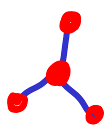
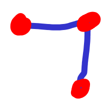
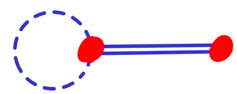


Painlevé equation	6	5	4	3	2	1
pole orders ($g=0$, rk 2)	1111	211	31	$22/11\tilde{2}$	$4/1\tilde{3}$	$\tilde{4}$
# constants	4	3	2	2	1	0
Diagram						
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$ 	Weber 	Bessel-Clifford $_0F_1$ 	Airy 	–

Painlevé equation	6	5	4	3	2	1
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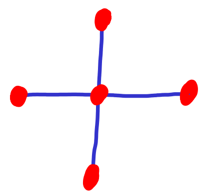
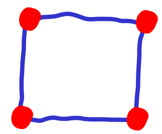
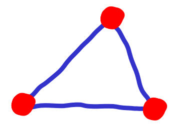
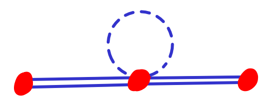
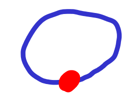
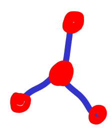
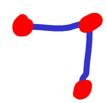
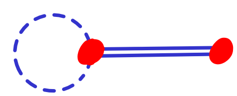
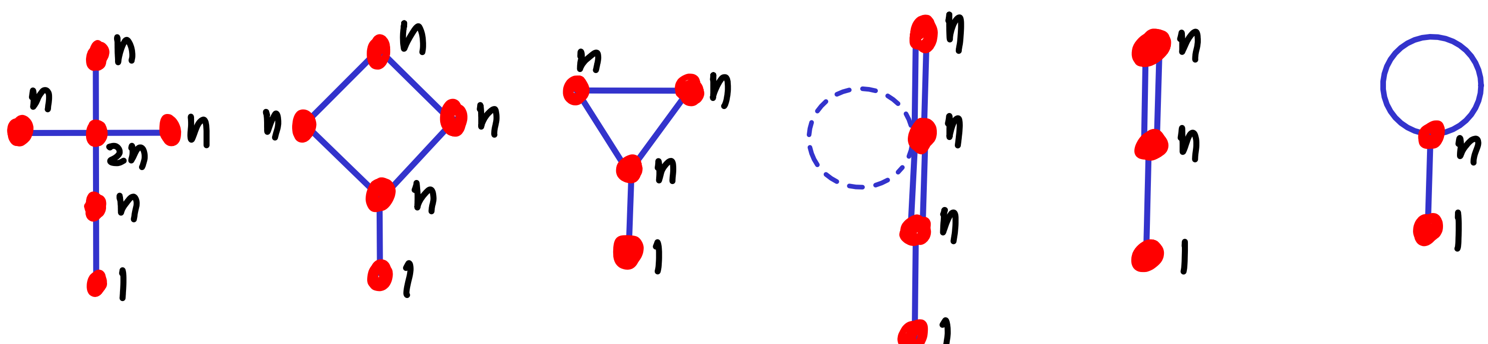
Diagram						
Special solutions	Gauss $_2F_1$ 	Kummer $_1F_1$ 	Weber 	Bessel-Clifford $_0F_1$ 	Airy 	–

Higher Painlevé spaces:



dim $2n$, conjecturally $\cong \mathrm{Hilb}^n(2d\,\mathcal{M}_B)$ (known by Groechenig in tame case)