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RIEMANN–HILBERT FOR TAME COMPLEX PARAHORIC CONNECTIONS

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Abstract. A local Riemann–Hilbert correspondence for tame meromorphic connections on a curve compatible with a parahoric level structure will be established. Special cases include logarithmic connections on G-bundles and on parabolic G-bundles. The corresponding Betti data involves pairs (M, P) consisting of the local monodromy $M \in G$ and a (weighted) parabolic subgroup $P \subset G$ such that $M \in P$, as in the multiplicative Brieskorn–Grothendieck–Springer resolution (extended to the parabolic case). The natural quasi-Hamiltonian structures that arise on such spaces of enriched monodromy data will also be constructed.

1. Introduction

The starting point of this paper was an attempt to extend to G-bundles the local classification of logarithmic connections on vector bundles on curves in terms of Levelt filtrations, where G is a connected complex reductive group. Namely, logarithmic connections on vector bundles are classified locally by triples (V, F, M) where V is a finite-dimensional complex vector space, F is a decreasing finite filtration of V indexed by $\mathbb{Z}$ and $M \in \mathrm{GL}(V)$ preserves the filtration F. If we forget the filtration then we obtain the local classification of regular singular connections, much studied, e.g. by Deligne [13] (in arbitrary dimensions)—they form a Tannakian category (cf. [17]) and the extension to G-bundles is then straightforward (they are classified by their monodromy $M \in G$ up to conjugation) although a direct approach is possible (see [2]).

Thus for general G we wish to describe the extra data needed to determine a logarithmic connection and establish the precise correspondence. Unfortunately, the category of triples (V, F, M) is not abelian, and so not Tannakian, and so it seems a direct approach is necessary (if it were Tannakian we could just take the space of homomorphisms from the corresponding group into G). The key point in the above classification of logarithmic connections is that one may choose a local holomorphic trivialization and a one-parameter subgroup $\varphi : \mathbb{C}^* \to \mathrm{GL}(V)$ such that if we view φ as a meromorphic gauge transformation, then in the resulting

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trivialization the connection takes the simple form

$$R\frac{dz}{z}$$

for some $R \in \mathrm{End}(V)$ with eigenvalues all having real parts in the interval $[0,1)$ (using a fixed local coordinate z). The resulting data is then (V, F, M) where $M = e^{2\pi i R}$ is the local monodromy and F is the filtration naturally associated to φ. The utility of the filtration is that if $g \in \mathrm{GL}(V)$ and $\psi = g\varphi g^{-1}$ is a conjugate one-parameter subgroup then the meromorphic group element $\varphi\psi^{-1}$ is holomorphic if and only if φ and ψ determine the *same* filtration, i.e. g preserves F. This is why the Levelt filtration (from [19, (2.2)]) gives a much cleaner approach than the naive viewpoint of directly recording the extra terms that may occur in the case of "resonant" connections.

For general G the notion of flag generalizes directly to the notion of parabolic subgroup, and one may in general attach a parabolic subgroup $P \subset G$ to a one-parameter subgroup (see, e.g. Mumford et al. [21, p. 55]). However, it is *not true*, even for $\mathrm{SL}_2(\mathbb{C})$, that every logarithmic connection may be put in the simple form $R\,dz/z$ with $R \in \mathfrak{g} = \mathrm{Lie}(G)$ via a suitable trivialization and a one-parameter subgroup (see [2, p .65]), and even if we did restrict ourselves to such connections a good analogue of the above normalization of the eigenvalues looks to be elusive. At first sight this is bad news since it means the direct analogue of the above $\mathrm{GL}_n(\mathbb{C})$ classification does not seem to hold, but it is also good news: the failure to reduce to the simple form corresponds directly to the fact that there are logarithmic connections whose monodromy M is not in the image of the exponential map, so we can hope for a more complete correspondence involving all possible monodromy conjugacy classes.

Whilst extending the nonabelian Hodge correspondence to open curves Simpson [25] gave an alternative approach, which he also applies to more general objects ("filtered tame $\mathcal{D}$-modules"), but still in the context of vector bundles. In the case of logarithmic connections this amounts to refining the Levelt filtration to take into account the exact rate of growth of solutions rather than its integer part as was effectively done above. It is this approach that we are able to extend to all complex reductive groups. Moreover, the final version of the correspondence (Theorem D) involves some new features which do not occur in the case of vector bundles. Also a surprisingly clean statement (Corollary E) is possible if we use Bruhat–Tits buildings.

Our motivation was to understand the spaces of monodromy-type data that occur in the extension of the nonabelian Hodge correspondence to the case of irregular connections on curves [23], [5], and its extension to arbitrary G. Using the quasi-Hamiltonian approach this problem may be broken up into pieces: understanding the Stokes data, and understanding what to do for regular singularities. Since it is possible to understand the Stokes data for arbitrary G (cf. [7], [8], [9]) we are left with the problem of extending Simpson's tame Riemann–Hilbert correspondence [25] to general G, which we will do here. At the end of the day this will give the algebraic "Betti" description of some complex manifolds supporting hyper-Kähler metrics (appearing in the nonabelian Hodge theory of curves).

Some motivation also came from trying to understand the recent work of Gukov and Witten [15], [16] on the tamely ramified geometric Langlands correspondence (in particular this justifies our desire to work uniformly with arbitrary complex reductive groups).

Results and further evolution

We will state three local classification results, of increasing complexity, since each may be of interest to different readers. In the case of logarithmic connections the statement is as follows. Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra corresponding to a maximal torus $T \subset G$, and let $\mathfrak{t}_\mathbb{R} = X_*(T) \otimes_\mathbb{Z} \mathbb{R}$ be the space of real cocharacters so that $\mathfrak{t} = \mathfrak{t}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$. Choose an element $\tau + \sigma \in \mathfrak{t}$ with real part $\tau \in \mathfrak{t}_\mathbb{R}$ and a nilpotent element $n \in \mathfrak{g}$ commuting with $\tau + \sigma$. Let $O \subset \mathfrak{g}$ be the adjoint orbit of $\tau + \sigma + n \in \mathfrak{g}$. Let $L \subset G$ be the centralizer of τ and let $P_\tau \subset G$ be the parabolic subgroup determined by τ (see Section 2), so that L is a Levi subgroup of P_τ. Let $\mathcal{C} \subset L$ be the conjugacy class containing the element $\exp(2\pi i(\tau + \sigma + n)) \in L$. Then $\mathcal{C}$ canonically determines a conjugacy class in the Levi factor of any parabolic subgroup of G conjugate to P_τ (see Lemma 1).

Theorem A (Logarithmic case). *There is a canonical bijection between isomorphism classes of germs of logarithmic connections on G-bundles with residue in O and conjugacy classes of pairs (M, P) with $P \subset G$ a parabolic subgroup conjugate to P_τ and $M \in P$ such that $\pi(M) \in \mathcal{C}$, where π is the natural projection from P onto its Levi factor.*

Note that if O is nonresonant (i.e. $\alpha(\tau+\sigma)$ is not a nonzero integer for any root α) then the condition $\pi(M) \in \mathcal{C}$ implies that M itself is conjugate to $\exp(2\pi i(\tau + \sigma + n))$.

At this point we investigated the spaces of enriched monodromy data that start to appear here from a quasi-Hamiltonian viewpoint. In the case of compact groups, when studying moduli space of flat connections on open Riemann surfaces, one fixes the conjugacy class of monodromy around each boundary component/puncture in order to obtain symplectic moduli spaces. As in [25] for $\mathrm{GL}_n(\mathbb{C})$ we now see this is not the most general thing that arises in the case of complex reductive groups: in general, one should fix the conjugacy class of the image in a Levi factor. In the quasi-Hamiltonian approach where one constructs spaces of (generalized) monodromy data by fusing together some basic pieces this corresponds to a "new piece", as follows.

Let $P_0 \subset G$ be a fixed parabolic subgroup with Levi factor L. Choose a conjugacy class $\mathcal{C} \subset L$ (as remarked above this canonically determines a conjugacy class in the Levi factor of any conjugate parabolic subgroup). Let $\mathbb{P} \cong G/P_0$ be the set of parabolic subgroups conjugate to P_0.

Theorem B. *The smooth variety $\widehat{\mathcal{C}}$ of pairs $(M, P) \in G \times \mathbb{P}$ such that $M \in P$ and $\pi(M) \in \mathcal{C}$ is a quasi-Hamiltonian G-space with moment map given by*

$$(M, P) \mapsto M \in G.$$

If P_0 is a Borel, these spaces appear in the multiplicative Brieskorn–Grothendieck–Springer resolution. If $P_0 = G$ then $\widehat{\mathcal{C}} = \mathcal{C}$. The additive analogue (on the Lie algebra level) of this is well known, when the resolution is the moment map in the usual sense (see [4, Theorem 2]). Some Poisson aspects of the multiplicative case are studied in [14], but the quasi-Hamiltonian (or quasi-Poisson) viewpoint looks to be more natural. The $\mathrm{GL}_n(\mathbb{C})$ case may be constructed differently via quivers (cf. [27]).

This enables us to construct lots of complex symplectic manifolds of "enriched monodromy data" of the form

$$\big(\mathbb{D} \circledast \cdots \circledast \mathbb{D} \circledast \widehat{\mathcal{C}}_1 \circledast \cdots \circledast \widehat{\mathcal{C}}_m\big)/\!\!/G,$$

where $\mathbb{D} \cong G\times G$ is the internally fused double, the $\mathcal{C}_i$ are conjugacy classes in Levi factors of various parabolic subgroups of G and "$/\!\!/$" denotes a quasi-Hamiltonian quotient (a quotient of a subvariety). The problem now is to try to interpret these spaces as spaces of meromorphic connections on Riemann surfaces (of genus equal to the number of factors of $\mathbb{D}$ appearing here). This almost immediately reduces to the local problem of interpreting the spaces $\widehat{\mathcal{C}}$—clearly only some of them arise in Theorem A since τ determines the parabolic subgroup P_τ and also arises in the choice of $\mathcal{C}$.

The next generalization is to consider logarithmic connections on parabolic bundles as follows. We will say an element $\theta \in \mathfrak{t}_\mathbb{R}$ is *small* if $\alpha(\theta) < 1$ for all roots α. Choose a small element θ and let $P_\theta \subset G$ be the corresponding parabolic subgroup. A (germ of a) parabolic bundle with weight θ is a G-bundle E on a disk together with a reduction of structure group[1] to P_θ at 0. A logarithmic connection on a parabolic G-bundle E is then a logarithmic connection whose residue preserves the parabolic structure. In local coordinates and trivialization this means the connection takes the form

$$A = \left(\sum_{i\ge 0} A_i z^i\right)\frac{dz}{z},$$

with $A_i \in \mathfrak{g}$ and the reduction determines a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and the compatibility condition means $A_0 \in \mathfrak{p}$.

The parabolic correspondence is then as follows. Fix $\tau+\sigma+n \in \mathfrak{g}$ as above and suppose further that n commutes with θ. Let $H_\theta \subset G$ be the centralizer of θ (a Levi subgroup of P_θ) and let $O \subset \mathfrak{h}_\theta$ be the adjoint orbit of $\tau+\sigma+n \in \mathfrak{h}_\theta := \mathrm{Lie}(H_\theta)$. The orbit O canonically determines an adjoint orbit in the Levi factor $\mathfrak{h}$ of any parabolic subalgebra $\mathfrak{p}$ conjugate to $\mathrm{Lie}(P_\theta)$. We will say a parabolic connection "lies over O" if its residue (in $\mathfrak{p}$) projects to an element of $O \subset \mathfrak{h}$ under the canonical map $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$, quotienting by the nilradical. Now set

$$\phi = \tau + \theta \in \mathfrak{t}_\mathbb{R}$$

[1]This is a choice of a point of E_0/P_θ where $E_0 \cong G$ is the fibre of E at 0. Equivalently, it is the choice of a parabolic subgroup conjugate to P_θ in $G(E)_0 \cong G$, where $G(E)$ is the associated adjoint group bundle.

and let $P_\phi \subset G$ be the corresponding parabolic subgroup and let $L \subset P_\phi$ be the centralizer in G of ϕ (a Levi subgroup of P_ϕ). Then $\exp(2\pi i(\tau + \sigma + n))$ is in L and we let $\mathcal{C} \subset L$ be its conjugacy class.

Theorem C (Parabolic case). *Suppose that the centralizer in G of* $\exp(2\pi i\theta) \in G$ *is connected. Then there is a canonical bijection between isomorphism classes of germs of parabolic connections on G-bundles with weight θ and residue lying over O, and conjugacy classes of pairs (M, P) with $P \subset G$ a parabolic subgroup conjugate to P_ϕ and $M \in P$ such that $\pi(M) \in \mathcal{C}$, where π is the natural projection from P onto its Levi factor.*

This clearly captures many more of the spaces $\widehat{\mathcal{C}}$, and specializes to Theorem A if $\theta = 0$. But it is still not entirely satisfactory for several reasons. First, by definition $\mathcal{C} \subset L$ is always in the image of the exponential map (so we do not always get all possible classes). Second, Theorem C involves a connected centralizer condition—this holds automatically if the derived subgroup of G is simply connected (e.g. for $\mathrm{GL}_n(\mathbb{C})$ or for any simply connected semisimple group), but not always. For example, Theorem C does not apply to $\mathrm{PGL}_2(\mathbb{C})$ and $\theta = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}/2$. Third, we have restricted ourselves to *small* weights θ (such that $\alpha(\theta) < 1$ for all roots α).[2]

Somewhat miraculously all the problems disappear if we pass to the objects which naturally appear when we do not restrict to small weights and if we use their most natural groups of automorphisms. This is most simply described in local coordinates/trivializations. Given any $\theta \in \mathfrak{t}_{\mathbb{R}}$ we have a decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$ of the Lie algebra of G into the eigenspaces of ad_θ and we may consider the space of "tame parahoric" connections of the form

$$\mathcal{A}_\theta = \left\{ A = \Bigg(\sum_{i\in\mathbb{Z},\lambda\in\mathbb{R}} A_{i\lambda} z^i \Bigg) \frac{dz}{z} \;\Bigg|\; A_{i\lambda} \in \mathfrak{g}_\lambda \text{ and } i + \lambda \geq 0 \right\} \subset \mathfrak{g}(\!(z)\!)\, dz.$$

This is acted on (by gauge transformations) by the extended parahoric subgroup

$$\widehat{\mathcal{P}}_\theta = \{ g \in G(\!(z)\!) \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ along any ray} \}$$

where $z^\theta = \exp(\theta \log(z))$ (see Section 2). The main result is the classification of $\widehat{\mathcal{P}}_\theta$ orbits of such connections. That this is a nontrivial generalization is clear if we consider for example the case $G = E_8$: then there are 511 conjugacy classes of parahoric subgroups, of which only 256 arise in the parabolic case. To describe the classification we will first discuss the generalization of the notion of fixing the adjoint orbit of the residue.

Let $\widehat{H}_\theta \subset G$ be the centralizer of $\exp(2\pi i\theta)$ (which might be disconnected), and now set $\mathfrak{h}_\theta = \mathrm{Lie}(\widehat{H}_\theta)$, which agrees with the previous definition for small θ. The group $\widehat{H}_\theta$ is isomorphic to the "Levi" subgroup $\widehat{\mathcal{L}}_\theta = \{ z^{-\theta} h z^\theta \mid h \in \widehat{H}_\theta \}$

[2]Note for $\mathrm{GL}_n(\mathbb{C})$ one can always reduce to the case of small weights, every Levi subgroup has surjective exponential map, and the centralizer of any semisimple group element is connected.

of $\widehat{\mathcal{P}}_\theta$. The finite-dimensional weight zero piece $\mathcal{A}_\theta(0) = \{\sum A_{i,-i} z^i \, dz/z\}$ of $\mathcal{A}_\theta$ is acted on by $\widehat{\mathcal{L}}_\theta$ and the orbits correspond to adjoint orbits of $\widehat{H}_\theta$ (see Lemma 4). The generalization of fixing the adjoint orbit of the (Levi quotient of the) residue is to fix the adjoint orbit $O \subset \mathfrak{h}_\theta$ corresponding to the weight zero part of the connection. Notice that in general one now gets a richer class of subalgebras $\mathfrak{h}_\theta \subset \mathfrak{g}$: it is not necessarily the Levi factor of a parabolic (e.g. if $G = G_2$ one may obtain $\mathfrak{sl}_3(\mathbb{C}) \subset \mathfrak{g}$ which is still simple of rank 2). The full statement of the local correspondence is then as follows.

Fix elements $\theta, \tau \in \mathfrak{t}_\mathbb{R}$ and $\sigma \in \sqrt{-1}\mathfrak{t}_\mathbb{R}$ and set $\phi = \theta + \tau$. Choose a nilpotent element $n \in \mathfrak{h}_\theta \subset \mathfrak{g}$ commuting with ϕ and σ. (Thus there is a finite decomposition $n = \sum a_i$ with $[\tau, a_i] = i a_i = [a_i, \theta]$ for $i \in \mathbb{Z}$.) Let $O \subset \mathfrak{h}_\theta$ be the adjoint orbit of the element $\phi + \sigma + n \in \mathfrak{h}_\theta$. This corresponds to the element $(\tau + \sigma + \sum a_i z^i) \, dz/z \in \mathcal{A}_\theta(0)$. Let $L \subset P_\phi$ be the Levi subgroup as above, but define $\mathcal{C} \subset L$ to be the conjugacy class containing the element

$$\exp(2\pi i(\tau + \sigma)) \exp(2\pi i n) \in L.$$

Note that $\mathcal{C}$ is not necessarily an exponential conjugacy class, since n and τ might not commute—indeed the Jordan decomposition implies that all conjugacy classes arise in this way.

Theorem D (Parahoric case). *There is a canonical bijection between the $\widehat{\mathcal{P}}_\theta$ orbits of tame parahoric connections in $\mathcal{A}_\theta$ lying over O and conjugacy classes of pairs (M, P) with $P \subset G$ a parabolic subgroup conjugate to P_ϕ and $M \in P$ such that $\pi(M) \in \mathcal{C}$.*

This is the main result and specializes to Theorems A and C. Finally, by considering the space $\mathbb{B}(G)$ of weighted parabolic subgroups of G, and the space $\mathcal{B}(LG)$ of weighted parahoric subgroups of the local loop group $LG = G(\!(z)\!)$, it is possible to deduce the following statement, not involving orbit choices etc:

Corollary E. *There is a canonical bijection between LG orbits of tame parahoric connections and G orbits of enriched monodromy data:*

$$\{(A, p) \mid p \in \mathcal{B}(LG), A \in \mathcal{A}_p\}/LG \;\cong\; \{(M, b) \mid b \in \mathbb{B}(G), M \in P_b\}/G.$$

The layout of this paper is as follows. In Section 2 we give basic definitions—this is divided into three parts: reductive groups, loop groups and meromorphic connections. Section 3 then establishes the main correspondence (Theorem D). Next Section 4 is devoted to quasi-Hamiltonian geometry and establishes Theorem B. Finally, Section 5 discusses Bruhat–Tits buildings and weighted parahoric subgroups and deduces Corollary E. Some further directions are mentioned at the end.

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2. Basic definitions

2.1. Background on reductive groups

Let G be a connected complex reductive group. Let $T \subset G$ be a maximal torus and let $B \subset G$ be a Borel subgroup containing T. Write the Lie algebras as $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\mathcal{R} \subset \mathfrak{t}^*$ denote the set of roots and let $\Delta \subset \mathcal{R}$ denote the simple roots determined by B. We will identify the roots with characters of T whenever convenient. Let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the root space corresponding to $\alpha \in \mathcal{R}$ and let $U_\alpha \subset G$ denote the corresponding root group.

Let $X_*(T)$ denote the set of one-parameter subgroups $\varphi : \mathbb{C}^* \to T$ of T. Taking the derivative ($\varphi = z^\phi \mapsto \phi$) embeds $X_*(T)$ as a lattice in $\mathfrak{t}$, and we define $\mathfrak{t}_\mathbb{R} = X_*(T) \otimes_\mathbb{Z} \mathbb{R} \subset \mathfrak{t}$, so that $\mathfrak{t}$ is the complexification of the real vector space $\mathfrak{t}_\mathbb{R}$.

Recall the Jordan decompositions:

(1) $X \in \mathfrak{g}$ has a unique decomposition $X = X_s + X_n$ with X_s semisimple, X_n nilpotent and $[X_s, X_n] = 0$,

(2) $g \in G$ has a unique decomposition $g = g_s g_u$ with g_s semisimple, g_u unipotent and $g_s g_u = g_u g_s$.

An element of $X \in \mathfrak{g}$ will be said to have *real eigenvalues* if its adjoint orbit contains an element whose semisimple part is in $\mathfrak{t}_\mathbb{R}$. Said differently there are a finite number of *commuting* one-parameter subgroups λ_i such that $X_s = \sum a_i \, d\lambda_i$ for real numbers a_i.

Recall that the standard parabolic subgroups $P_I \subset G$ are the subgroups containing B. They are determined by subsets I of the nodes of the Dynkin diagram Δ. The Lie algebra of P_I is that of B plus the sum of the root spaces $\mathfrak{g}_{-\alpha}$ for positive roots α which are linear combinations of the elements of I. The parabolic subgroups $P \subset G$ may be characterized as the subgroups conjugate to a standard parabolic. The Levi factor of P is the quotient $L = P/U$ of P by the unipotent radical $U = \mathrm{Rad}_u(P)$ of P; it is again a connected complex reductive group. One can choose a lifting of L to a subgroup of P (and thus of G) and P is isomorphic to the semi-direct product of L and U. If $T \subset B \subset P$ then we have a preferred lift L with $T \subset L$, but in general there are many lifts, since we can conjugate the lift L by elements of P.

Any semisimple element $\theta \in \mathfrak{g}$ with real eigenvalues (and in particular any one-parameter subgroup) has an associated parabolic subgroup:

$$P_\theta = \{g \in G \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ along any ray}\} \subset G$$

where $z^\theta = \exp(\theta \log(z))$. Equivalently, P_θ is $L \cdot U \subset G$ where the Levi factor $L \subset G$ is the centralizer of θ and $U \subset G$ is the unipotent subgroup whose Lie algebra is the direct sum of the eigenspaces of $\mathrm{ad}_\theta \in \mathrm{End}(\mathfrak{g})$ with strictly positive eigenvalues. For one-parameter subgroups this notion is used by Mumford [21, p. 55]. If we choose θ (or T) such that $\theta \in \mathfrak{t}_\mathbb{R}$ then P_θ is the group generated by T and the root groups U_α such that $\alpha(\theta) \geq 0$. (If further θ is in the closed positive Weyl chamber then $P_\theta = P_I$ where $I = \{\alpha \in \Delta \mid \alpha(\theta) = 0\}$ is the set of walls containing θ.) Note that $P_{\mathrm{Ad}_h(\theta)} = hP_\theta h^{-1}$ for any $h \in G$.

Now let $P \subset G$ be a parabolic subgroup and let $\mathcal{C} \subset L$ be a conjugacy class in the Levi factor L of P.

Lemma 1. *The conjugacy class $\mathcal{C} \subset L$ uniquely determines a conjugacy class in the Levi factor of any parabolic subgroup of G conjugate to P.*

Proof. Given $l \in \mathcal{C} \subset L$ and $g \in G$, then glg^{-1} projects to an element $h = \pi(glg^{-1})$ of the Levi factor H of the parabolic $Q = gPg^{-1}$ (where $\pi : Q \to H := Q/\mathrm{Rad}_u(Q)$). The conjugacy class in H of h is uniquely determined. Since parabolics are their own normalizers ([11, 11.16]) Q determines g up to left multiplication by an element q of Q. Replacing g by qg only conjugates h by $\pi(q)$. Choosing a different $l \in \mathcal{C}$ corresponds to right multiplication of g by an element p of P—this does not change Q so by the above corresponds to conjugating h. □

Similarly, an adjoint orbit $O \subset \mathrm{Lie}(L)$ uniquely determines an adjoint orbit of the Lie algebra of the Levi factor of any conjugate parabolic. Similarly, also for coadjoint orbits in $\mathrm{Lie}(L)^*$.

Given a parabolic subgroup $P \subset G$, a *set of weights* for P is an element $[\theta]$ of the centre of the Lie algebra of the Levi factor L of P such that

(1) it is semisimple and has real eigenvalues; and
(2) given any lift of L to a subgroup of P the corresponding lift $\theta \in \mathfrak{p} \subset \mathfrak{g}$ of $[\theta]$ determines P, i.e. $P_\theta = P$.

A *weighted parabolic subgroup* is a parabolic subgroup P together with a set of weights for P. More concretely, $[\theta]$ is a (one-point) adjoint orbit of L and so corresponds uniquely to an adjoint orbit of the Levi factor of the standard parabolic P_I conjugate to P. Then $[\theta]$ just corresponds to a point θ' of the closed Weyl chamber such that $P_I = P_{\theta'}$. Thus if G is semisimple this amounts to choosing a strictly positive real number for each element of $\Delta \setminus I$.

Lemma 2. *A semisimple element $\theta \in \mathfrak{g}$ with real eigenvalues determines a set of weights $[\theta]$ for the associated parabolic subgroup $P_\theta \subset G$. In general there are many elements θ determining the same pair $(P_\theta, [\theta])$.*

Proof. Indeed, θ determines a Levi decomposition $P_\theta = LU$ (with L the centralizer of θ) and θ is in the Lie algebra of the centre of L, so determines a weight. (Less abstractly θ is conjugate to a unique element θ' of the closed Weyl chamber in $\mathfrak{t}_\mathbb{R}$.) Finally, it is clear that θ and $g\theta g^{-1}$ determine the same pair for any $g \in P_\theta$. □

Let $\mathbb{B}(G)$ denote the set of weighted parabolic subgroups of G. (This will be discussed in more detail in Section 5.)

2.2. Background on loop groups

Now we will consider the analogous definitions for the complex (local) loop group. We will work with the ring $\mathcal{O} = \mathbb{C}\{z\}$ of germs of holomorphic functions (equivalently, power series with radius of convergence > 0) and its field of fractions $\mathcal{K} = \mathbb{C}\{(z)\} = \mathbb{C}\{z\}[z^{-1}]$. (The proofs we will give also yield the analogous results for the completions $\widehat{\mathcal{O}} = \mathbb{C}[\![z]\!]$ and $\widehat{\mathcal{K}} = \mathbb{C}(\!(z)\!)$—in fact this case is slightly easier—for simplicity only the completed results were stated in the Introduction.) The convergent local loop group is $LG = G(\mathcal{K})$, the group of $\mathcal{K}$ points of the algebraic group G. The subgroups of LG analogous to parabolic subgroups of G are the *parahoric* subgroups of LG. (Unlike in the finite-dimensional case parahoric subgroups are not always self-normalizing.) A basic example of a parahoric subgroup is the subgroup $G(\mathcal{O})$ which arises as the group of germs of bundle automorphisms if we choose a local trivialization of a principal G-bundle. Similarly, the Iwahori subgroup

$$\mathcal{I} = \{g \in G(\mathcal{O}) \mid g(0) \in B\}$$

and its parabolic generalizations

$$\{g \in G(\mathcal{O}) \mid g(0) \in P\}$$

(where $P \subset G$ is a parabolic subgroup) arise if we consider parabolic G-bundles. These are also parahoric subgroups of LG but they do not exhaust all the possibilities. Indeed, if G is simple, conjugacy classes of parahoric subgroups of LG correspond to proper subsets of the nodes of the affine Dynkin diagram, whereas those above correspond to parabolic subgroups of G, i.e. to subsets of the usual Dynkin diagram. For example, if $G = E_8$ there are 511 conjugacy classes of parahoric subgroups of LG, of which only 256 arise from parabolic subgroups of G. On the other hand, if $G = \mathrm{GL}_n$ any parahoric subgroup is conjugate to a subgroup arising from a parabolic subgroup of G.

The general setup we will need for Theorem D is as follows. Given an element $\theta \in \mathfrak{t}_{\mathbb{R}}$ we will define an associated parahoric subgroup of the loop group. First, θ gives a grading of the Lie algebra $\mathfrak{g}$, namely it decomposes as

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{R}} \mathfrak{g}_\lambda$$

where $\mathfrak{g}_\lambda$ is the λ eigenspace of ad_θ. Then for any integer i we may define subspaces

$$\mathfrak{g}(i) = \bigoplus_{\lambda \geq -i} \mathfrak{g}_\lambda \subset \mathfrak{g} \qquad \text{and} \qquad \mathfrak{n}(i) = \bigoplus_{\lambda > -i} \mathfrak{g}_\lambda \subset \mathfrak{g}$$

so in particular $\mathfrak{g}(0) = \mathfrak{p}_\theta$ is the Lie algebra of the parabolic associated to θ, and $\mathfrak{n}(0)$ is its nilradical (and $\mathfrak{g}_0$ is its Levi factor). To emphasize the dependence on θ we will sometimes write $\mathfrak{g}^\theta_\lambda = \mathfrak{g}_\lambda$ and $\mathfrak{g}^\theta(i) = \mathfrak{g}(i)$. Note that the subset

$$\wp_\theta := \left\{ X = \sum_{i \in \mathbb{Z}} X_i z^i \in \mathfrak{g}\{(z)\}) \;\middle|\; X_i \in \mathfrak{g}(i) \right\}$$

is a Lie subalgebra of $L\mathfrak{g} = \mathfrak{g}\{(z)\}$. Said differently θ determines a grading of the vector space $L\mathfrak{g}$, with finite-dimensional pieces

$$L\mathfrak{g}(r) = \left\{ \sum X_i z^i \in L\mathfrak{g} \mid X_i \in \mathfrak{g}_\lambda \text{ where } \lambda + i = r \right\}$$

for all $r \in \mathbb{R}$. Then $\wp_\theta$ is the subalgebra of $L\mathfrak{g}$ with weights $r \geq 0$. The weight zero piece will be a subalgebra which we will denote as

$$\mathfrak{l}_\theta := L\mathfrak{g}(0) = \{X = \sum X_i z^i \in \wp_\theta \mid X_i \in \mathfrak{g}_{-i}\}.$$

This is finite-dimensional and in fact reductive. We view $\mathfrak{l}_\theta$ as the Levi factor of $\wp_\theta$. By setting $z = 1$ there is an embedding

$$\iota : \mathfrak{l}_\theta \hookrightarrow \mathfrak{g}.$$

Let $\widehat{H}_\theta = C_G(e^{2\pi i\theta})$ be the centralizer in G of $e^{2\pi i\theta}$, and let $\mathfrak{h}_\theta \subset \mathfrak{g}$ be its Lie algebra. Then the image $\iota(\mathfrak{l}_\theta)$ is $\mathfrak{h}_\theta$. (Note that $\mathfrak{h}_\theta$ is not necessarily isomorphic to a Levi factor of a parabolic subalgebra of $\mathfrak{g}$—for example, for simple $\mathfrak{g}$, $\mathfrak{h}_\theta$ could be the Lie algebra determined by any proper subset of the nodes of the affine Dynkin diagram of $\mathfrak{g}$, so may be a proper semisimple subalgebra of the same rank, such as $\mathfrak{sl}_3 \subset \mathfrak{g}_2$, as in Borel–De Siebenthal theory.) More generally, we may consider the subgroup

$$\widehat{\mathcal{L}}_\theta = \{z^{-\theta} h z^{\theta} \mid h \in \widehat{H}_\theta\} \subset LG$$

of LG (this is indeed well defined since h commutes with the monodromy of $z^\theta = \exp(\theta \log(z))$). By setting $z = 1$ we see $\widehat{\mathcal{L}}_\theta$ is isomorphic to $\widehat{H}_\theta$, and ι is the corresponding map on the level of Lie algebras. Let H_θ denote the identity component of $\widehat{H}_\theta$ and let $\mathcal{L}_\theta \subset \widehat{\mathcal{L}}_\theta$ denote the corresponding subgroup of the loop group. Thus the Lie algebra of $\widehat{\mathcal{L}}_\theta$ and $\mathcal{L}_\theta$ is $\mathfrak{l}_\theta$.

The extended parahoric subgroup determined by θ is the subgroup

$$\widehat{\mathcal{P}}_\theta = \{g \in LG \mid z^\theta g z^{-\theta} \text{ has a limit as } z \to 0 \text{ along any ray}\}.$$

This definition is perhaps best understood by thinking in terms of a faithful representation, whence θ is a diagonal matrix and we can see explicitly what the condition means in terms of matrix entries. Alternatively, one can work with the Bruhat decomposition, and show that $\widehat{\mathcal{P}}_\theta$ is generated by:

(1) elements of $\widehat{\mathcal{L}}_\theta$;
(2) elements of the form $\exp(Xz^i)$ with $X \in \mathfrak{g}_\alpha$ such that $\alpha(\theta) + i > 0$ (or $X \in \mathfrak{t}$ and $i > 0$); and
(3) elements of the form $\exp(Y(z))$ with $Y \in z^N\mathfrak{g}\{z\}$ with N a sufficiently large integer (so that $Y \in \wp_\theta$).

Heuristically, the Lie algebra of $\widehat{\mathcal{P}}_\theta$ is $\wp_\theta$. This has Levi subgroup $\widehat{\mathcal{L}}_\theta$ and pro-unipotent radical

$$\mathcal{U}_\theta = \{g \in LG \mid z^\theta g z^{-\theta} \text{ tends to } 1 \text{ as } z \to 0 \text{ along any ray}\}$$

(which has Lie algebra the part of $L\mathfrak{g}$ of weight > 0, and is generated by elements just of type (2) and (3) above). The group $\widehat{\mathcal{P}}_\theta$ is the semidirect product of $\widehat{\mathcal{L}}_\theta$ and $\mathcal{U}_\theta$.

The parahoric subgroup associated to θ is the group generated by $\mathcal{U}_\theta$ and the connected group $\mathcal{L}_\theta$:

$$\mathcal{P}_\theta = \mathcal{L}_\theta \cdot \mathcal{U}_\theta \subset \widehat{\mathcal{P}}_\theta.$$

This is a normal subgroup of $\widehat{\mathcal{P}}_\theta$ and the quotient $\widehat{\mathcal{P}}_\theta/\mathcal{P}_\theta \cong \widehat{H}_\theta/H_\theta$ is finite.

2.3. Germs of meromorphic connections

Choose $\theta \in \mathfrak{t}_{\mathbb{R}}$ and let $\mathcal{P}_\theta$ be the corresponding parahoric subgroup with Lie algebra $\wp_\theta$. Then we may consider the space $\mathcal{A} = \mathfrak{g}(\mathcal{K})\,dz$ of meromorphic connections (on the trivial G-bundle over the disk) and the subspace

$$\mathcal{A}_\theta = \wp_\theta \frac{dz}{z}.$$

Thus if $\theta = 0$ this is just the space of logarithmic connections. If θ is small, these are the logarithmic connections with residue in the Lie algebra $\mathfrak{p}_\theta$ of P_θ, as occurs in the case of parabolic bundles. (Parabolic G-bundles are studied, e.g. in [26], in the case where G is simple and simply connected, and the weights are small and rational.) In general elements of $\mathcal{A}_\theta$ will have poles of order greater than 1, but we will see in the course of the proof of Theorem 6 below that they always have regular singularities: fundamental solutions have at most polynomial growth at zero. They should perhaps be viewed as the right notion of "logarithmic parahoric connections" (as the pole is of order 1 greater than that permitted by the parahoric structure) but this term is cumbersome and possibly confusing. We will call them tame parahoric connections (although perhaps *"logahoric"* is simplest).

Lemma 3. *The natural (gauge) action of $\widehat{\mathcal{P}}_\theta$ on $\mathcal{A}$ preserves $\mathcal{A}_\theta$.*

Proof. Given $g \in LG$ and a connection $A \in \mathcal{A}$, the gauge action of g on A is $g[A] := \mathrm{Ad}_g(A) + (dg)g^{-1}$ where for any $g \in LG$ we define the $\mathfrak{g}$-valued meromorphic 1-form (on a neighbourhood of $0 \in \mathbb{C}$)

$$(dg)g^{-1} := g^*(\overline{\Theta}),$$

where $\overline{\Theta} \in \Omega^1(G, \mathfrak{g})$ is the right-invariant Maurer–Cartan form on G. (The sign conventions used here are as in [7].) Thus it is sufficient to check that $(dg)g^{-1} \in \mathcal{A}_\theta$ for any $g \in \widehat{\mathcal{P}}_\theta$. First, if $g = \exp(Xz^i)$ for $X \in \mathfrak{g}_\alpha$ with $\alpha(\theta) + i \geq 0$ then $(dg)g^{-1} = (iXz^i)\,dz/z \in \mathcal{A}_\theta$. Second, if $g = \exp(X(z))$ with $X \in z^N\mathfrak{g}\{z\}$ for N a sufficiently large integer again we have $(dg)g^{-1} \in \mathcal{A}_\theta$. Such elements generate $\mathcal{P}_\theta$ so it follows that $\mathcal{P}_\theta$ preserves $\mathcal{A}_\theta$ (since $d(gh)(gh)^{-1} = \mathrm{Ad}_g(dhh^{-1}) + dgg^{-1}$). Finally we must check $\widehat{\mathcal{L}}_\theta$ preserves $\mathcal{A}_\theta$ (since $\widehat{\mathcal{P}}_\theta$ is generated by this and $\mathcal{P}_\theta$). But if $g = z^{-\theta}hz^\theta \in \widehat{\mathcal{L}}_\theta$ then one finds $dgg^{-1} = (\mathrm{Ad}_g(\theta) - \theta)\,dz/z$ and this will be in $\mathcal{A}_\theta$ if $\mathrm{Ad}_{z^\theta}(\mathrm{Ad}_g(\theta) - \theta)$ has a limit as $z \to 0$ along any ray. But it does have a limit, since it is constant and equals $\mathrm{Ad}_h(\theta) - \theta$. □

Note that the gauge action may also be interpreted as the (level 1) coadjoint action of a central extension of LG, although we will not need this interpretation here. A closer examination of the action of $\widehat{\mathcal{L}}_\theta$ on the weight zero piece $\mathcal{A}_\theta(0) = L\mathfrak{g}(0)\,dz/z$ of $\mathcal{A}_\theta$ yields the following.

Lemma 4. *The map $A \mapsto z^\theta[A]$ is well defined on the weight zero piece $\mathcal{A}_\theta(0)$ of $\mathcal{A}_\theta$ and provides an isomorphism*

$$\mathcal{A}_\theta(0) \cong \mathfrak{h}_\theta \frac{dz}{z} \subset \mathfrak{g}\frac{dz}{z},$$

which is equivariant with respect to the gauge action of $\widehat{\mathcal{L}}_\theta$ on $\mathcal{A}_\theta(0)$ and the adjoint action of $\widehat{H}_\theta$ on $\mathfrak{h}_\theta$.

Proof. $\mathcal{A}_\theta(0)$ is just the set of elements $A = \sum A_i z^i\, dz/z$ with $A_i \in \mathfrak{g}$ in the $-i$ eigenspace of ad_θ. Thus $z^\theta[A] = (\theta + \sum A_i)\, dz/z \in \mathfrak{g}\, dz/z$, and $B := \theta + \sum A_i$ is just an arbitrary element of $\mathfrak{h}_\theta$ (i.e. an element with components only in the integer eigenspaces of ad_θ). Clearly if $h \in \widehat{H}_\theta$ and $g = z^{-\theta} h z^\theta \in \widehat{\mathcal{L}}_\theta \subset LG$ is the corresponding element of the loop group then

$$z^\theta\,[g[A]] = h[B\, dz/z] = \mathrm{Ad}_h(B)\frac{dz}{z}$$

so we have the desired equivariance. □

3. Main correspondence

Having now covered the background definitions we can move on to the main result. Fix elements $\theta, \tau \in \mathfrak{t}_\mathbb{R}$ and $\sigma \in \sqrt{-1}\mathfrak{t}_\mathbb{R}$ and set $\phi = \theta + \tau$. Choose a nilpotent element $n \in \mathfrak{g}$ commuting with ϕ and σ and such that $\mathrm{Ad}_t(n) = n$ where $t := \exp(2\pi i\tau) \in G$. (This means there is a (finite) decomposition $n = \sum a_i$ with $[\tau, a_i] = i a_i$ for $i \in \mathbb{Z}$.)

As above θ determines a space $\mathcal{A}_\theta$ of θ parahoric connections and an extended parahoric subgroup $\widehat{\mathcal{P}}_\theta \subset LG$ with Levi subgroup $\widehat{\mathcal{L}}_\theta$. Moreover θ determines an isomorphism $\widehat{\mathcal{L}}_\theta \cong \widehat{H}_\theta := C_G(e^{2\pi i\theta}) \subset G$. The corresponding Lie algebras are denoted $\mathfrak{l}_\theta \cong \mathfrak{h}_\theta \subset \mathfrak{g}$.

Let $O \subset \mathfrak{h}_\theta$ be the adjoint orbit (under the possibly disconnected group $\widehat{H}_\theta$) of the element

$$\phi + \sigma + n \in \mathfrak{h}_\theta.$$

This corresponds to the $\widehat{\mathcal{L}}_\theta$ orbit in $\mathcal{A}_\theta(0)$ containing the element

$$\left(\tau + \sigma + \sum a_i z^i\right)\frac{dz}{z}. \tag{1}$$

Also ϕ determines a parabolic subgroup $P_\phi \subset G$ and a weight $[\phi]$ for P_ϕ. Let L be the centralizer of ϕ (a Levi subgroup of P_ϕ). By construction τ, σ and n commute with ϕ, so are in the Lie algebra of L. Then we define $\mathcal{C} \subset L$ to be the conjugacy class containing the element

$$\exp(2\pi i(\tau + \sigma))\exp(2\pi i n) \in L.$$

Note that $\mathcal{C}$ is not necessarily an exponential conjugacy class, since n and τ in general do not commute. (This was one of our motivations for considering more general objects than logarithmic or parabolic connections.)

Lemma 5. *The triple $([\phi], P_\phi, \mathcal{C})$ is uniquely determined up to conjugacy by θ and the orbit $O \subset \mathfrak{h}_\theta$. Moreover, any such triple $([\phi], P_\phi, \mathcal{C})$ arises in this way (upto conjugacy).*

Proof. Any other element of O will be of the form $\mathrm{Ad}_g(\phi + \sigma + n)$ with $g \in \widehat{H}_\theta$. Suppose we choose g so that the semisimple part $\mathrm{Ad}_g(\phi + \sigma)$ is in $\mathfrak{t}$. This yields new choices $\phi' = \mathrm{Ad}_g\phi, \tau' = \phi' - \theta, \sigma' = \mathrm{Ad}_g\sigma, n' = \mathrm{Ad}_g n$ and we should check that $([\phi'], P_{\phi'}, \mathcal{C}')$ is conjugate to $([\phi], P_\phi, \mathcal{C})$. But this follows from the fact that $\exp(2\pi i\tau') = g\exp(2\pi i\tau)g^{-1}$, as g commutes with $\exp(2\pi i\theta)$. The fact that all such triples arise follows immediately from the multiplicative Jordan decomposition. □

Note that, given $\phi = \theta+\tau$ and σ, the precise correspondence between the adjoint orbits $O \subset \mathfrak{h}_\theta$ and the conjugacy classes $\mathcal{C} \subset L = C_G(\phi)$ rests on the identification

$$\begin{aligned}\{X \in \mathfrak{h}_\theta \mid [X,\phi] = [X,\sigma] = 0\} &= \{X \in \mathfrak{g} \mid [X,\phi] = [X,\sigma] = 0, \mathrm{Ad}_{e^{2\pi i\theta}}X = X\}\\ &= \{X \in \mathfrak{g} \mid [X,\phi] = [X,\sigma] = 0, \mathrm{Ad}_{e^{2\pi i\tau}}X = X\}\\ &= \{X \in \mathrm{Lie}(L) \mid [X,\sigma] = 0, \mathrm{Ad}_{e^{2\pi i\tau}}X = X\}\end{aligned}$$

since n is a nilpotent element of this (reductive) Lie algebra.

We will say that a connection $A \in \mathcal{A}_\theta$ "lies over O" if its weight zero component is in the $\widehat{\mathcal{L}}_\theta$ orbit corresponding to O. Similarly, if $P \subset G$ is a parabolic subgroup conjugate to P_ϕ, we will say $M \in P$ "lies over $\mathcal{C}$" if $\pi(M) \in \mathcal{C}$, where π is the canonical projection from P onto its Levi factor (and we transfer $\mathcal{C}$ from $L \subset P_\phi$ as in Lemma 1).

The main statement (Theorem D of the Introduction) is then:

Theorem 6. *There is a canonical bijection between the $\widehat{\mathcal{P}}_\theta$ orbits of tame parahoric connections in $\mathcal{A}_\theta$ lying over O and conjugacy classes of pairs (M,P) with $P \subset G$ a parabolic subgroup conjugate to P_ϕ and $M \in P$ an element lying over $\mathcal{C}$.*

Proof. To start we will explain how to put such connections in a simpler form. Suppose $A \in \mathcal{A}_\theta$ lies over O. First we may do a gauge transform by an element of $\widehat{\mathcal{L}}_\theta$ so the weight zero component of A equals $A(0) := (\tau+\sigma+\sum a_i z^i)\,dz/z$. Then we claim we can do a gauge transformation by an element g of $\mathcal{U}_\theta$ such that $g[A]$ is normalized in the following way:

$$g[A] = \left(\tau + \sigma + \sum_{i\in\mathbb{Z}} A_i z^i\right)\frac{dz}{z} \tag{2}$$

with each $A_i \in \mathfrak{g}(i)$ and

$$[\tau, A_i] = iA_i, \qquad [\sigma, A_i] = 0,$$

for all $i \in \mathbb{Z}$ (and a_i is the component of A_i in the $-i$ eigenspace of ad_θ). This implies that only finitely many of the A_i are nonzero.

To prove the claim we extend the usual argument in the logarithmic case (cf. [2]) as follows. Let $0 = r_0 < r_1 < \cdots$ be the sequence of positive real numbers such that $L\mathfrak{g}(r_i) \neq 0$. Suppose inductively that the piece of A in $L\mathfrak{g}(r_i)\,dz/z$ has been normalized for $0 \le i < k$. Then we claim we may choose $X(k) \in L\mathfrak{g}(r_k)$ so that the piece of $g_k[A]$ in $L\mathfrak{g}(r_i)\,dz/z$ is normalized for $0 \le i \le k$, where $g_k = \exp(X(k)) \in \mathcal{U}_\theta$. Indeed, $g_k[A]$ will equal A up to $L\mathfrak{g}(r_{k-1})\,dz/z$ and will have subsequent coefficient

$$A(k) + [X(k), A(0)] + z\frac{d}{dz}X(k) \in L\mathfrak{g}(r_k), \tag{3}$$

where $A = \sum_{i\geq 0} A(i)\,dz/z$ with $A(i) \in L\mathfrak{g}(r_i)$. Thus ideally we would like to choose $X(k)$ such that this was zero, i.e. $\big(\mathrm{ad}_{A(0)} - z\,d/dz\big)\,X(k) = A(k)$. This is not always possible, but we can make the difference small, as follows. Note that $\mathrm{ad}_{A(0)}$ restricts to a linear operator on the finite-dimensional vector space $L\mathfrak{g}(r_k)$ and so we may decompose $L\mathfrak{g}(r_k)$ into its generalized eigenspaces. Since $\tau + \sigma$ is the semisimple part of $A(0)$, these generalized eigenspaces are just the eigenspaces of the semisimple operator $\mathrm{ad}_{\tau+\sigma}$. On the other hand, we also have $z\,d/dz \in \mathrm{End}(L\mathfrak{g}(r_k))$ preserving this eigenspace decomposition (as it commutes with $\mathrm{ad}_{\tau+\sigma}$) and having only integral eigenvalues (mapping xz^i to ixz^i). Thus if $a_{i\mu}z^i$ (resp. $x_{i\mu}z^i$) is the component of $A(k)$ (resp. $X(k)$) in the μ-eigenspace of $\mathrm{ad}_{\tau+\sigma}$ (and the i-eigenspace of $z\,d/dz$) then we may define $X(k)$ by setting $x_{i\mu} = 0$ if $\mu = i$ and

$$x_{i\mu}z^i = \left(\mathrm{ad}_{A(0)} - z\frac{d}{dz}\right)^{-1}(a_{i\mu}z^i)$$

if $i \neq \mu$, since then the operator $\mathrm{ad}_{A(0)} - z\,d/dz$ will be invertible on the corresponding joint eigenspace. If $X(k)$ is defined in this way we thus find that the next coefficient (3) is the sum of the components of $A(k)$ with $i = \mu$, i.e. $\sum_i a_{ii}z^i \in L\mathfrak{g}(r_k)$ (noting that θ commutes with $\tau+\sigma$ and d/dz). But this just means it is normalized: $[\tau, a_{ii}] = ia_{ii}, [\sigma, a_{ii}] = 0$. Thus inductively we may construct a formal transformation $g = \cdots g_3g_2g_1$ in the completion of $\mathcal{U}_\theta$ converting A into normal form. To conclude that this transformation is actually convergent we need to check that

Lemma 7. *Any connection $A \in \mathcal{A}_\theta$ is regular singular.*

Proof. Choose a faithful representation of G and work in this representation. Thus τ is a real diagonal matrix and we may choose a diagonal matrix λ with integral eigenvalues such that the diagonal entries of $\tau - \lambda$ are in $[0, 1)$. Let $\varphi = z^\lambda$ be the corresponding one-parameter subgroup. We may then choose k sufficiently large such that the convergent meromorphic gauge transformation $\varphi^{-1}g_k\cdots g_2g_1$ converts A into a logarithmic connection. □

Thus both the original connection and the resulting connection are convergent connections with regular singularities, and it follows that g is actually convergent and in $\mathcal{U}_\theta$ (in effect we constructed an $\widehat{\mathcal{O}}$ point of a group scheme and then deduced it is actually an $\mathcal{O}$ point, i.e. in $\mathcal{U}_\theta$).

Having completed the normalization now set $N = \sum A_i$, $R = \sigma + N$, $M_s = \exp(2\pi i(\tau+\sigma)), M_u = \exp(2\pi i N), M = M_sM_u \in G$.

Then M_sM_u is the Jordan decomposition of M and the connection A has monodromy M. Indeed A (after normalization) equals $z^\tau[R\,dz/z]$, which has fundamental solution $z^\tau z^R$. By construction $M \in P_\phi$, so we have attached a pair (M, P) to the original data (with $P = P_\phi$). If L is the Levi factor of P then the image of M in L is

$$\pi(M) = \exp(2\pi\sqrt{-1}(\tau+\sigma))\exp\left(2\pi\sqrt{-1}\sum a_i\right),$$

where a_i is the component of A_i in the $-i$ eigenspace of ad_θ (the component commuting with ϕ). It follows that $\pi(M) \in \mathcal{C}$.

Surjectivity. To give the inverse construction we proceed as follows. Suppose we have (M,P) with $M \in P$, P of type ϕ and $\pi(M) \in \mathcal{C} \subset L(P)$. Then we may conjugate by G so that $P = P_\phi$ and $M_s = \exp(2\pi i(\tau+\sigma)) \in T$, since $\pi(M) \in \mathcal{C}$. (Here M_s is the semisimple part of M.) Then we may write $M = \exp(2\pi i\tau)\exp(2\pi i R)$ for a unique element $R = \sigma + N \in \mathfrak{g}$ with N nilpotent (and τ, σ as fixed above). Moreover, N commutes with σ and $\mathrm{Ad}_t N = N$ where $t := \exp(2\pi i\tau) \in T$, but N does not necessarily commute with τ itself. This implies that there is a unique decomposition $N = \sum_{i\in\mathbb{Z}} A_i$ with $A_i \in \mathfrak{g}$ such that $[\tau, A_i] = iA_i$. (If N has components in any other eigenspace of ad_τ then one will not have $\mathrm{Ad}_t(N) = N$.) On the other hand, since $M \in P_\phi$ we have $N \in \mathfrak{p}_\phi$ and so N only has components in the positive weight spaces of ϕ. Now since $\theta = \phi - \tau$ the connection $A := z^\tau[R\,dz/z] = (\tau + \sigma + \sum A_i z^i)\,dz/z$ is in $\mathcal{A}_\theta$.

The component of A in the weight zero component $\mathcal{A}_\theta(0)$ is $(\tau+\sigma+\sum a_i z^i)\,dz/z$ where a_i is the component of A_i commuting with ϕ (i.e. weight $-i$ for θ). This is determined by $\mathcal{C}$ and lies over O by construction. Thus (M,P) is the data attached to the connection A.

The main lemma we need for the rest of the proof is the following.

Lemma 8. *Suppose $C \in G$. Then $z^\tau C z^{-\tau}$ is in $\widehat{\mathcal{P}}_\theta$ if and only if:*

(a) *$C \in C_G(t)$ where $t = e^{2\pi i\tau}$; and*
(b) *$C \in P_\phi$.*

Proof. Condition (a) holds if and only if $p := z^\tau C z^{-\tau}$ is in LG (since that is the condition for it to have no monodromy). Then by definition $p \in \widehat{\mathcal{P}}_\theta$ if and only if $z^\theta p z^{-\theta}$ has a limit as $z \to 0$ along any ray, i.e. if $z^\phi C z^{-\phi}$ has a limit as $z \to 0$. But this is just the condition for $C \in P_\phi$. □

Well-defined on orbits. Next we will check that if two connections are in the same orbit then their data (M,P) are conjugate. Recall we have fixed $\theta, \tau \in \mathfrak{t}_\mathbb{R}$ and set $\phi = \tau + \theta$. Suppose $A, B \in \mathcal{A}_\theta$ are related by $g \in \widehat{\mathcal{P}}_\theta$. Without loss of generality we may assume A, B are both normalized. Thus they have fundamental solutions $\Phi_A = z^\tau z^R$ and $\Phi_B = z^\tau z^{R_1}$. Their monodromies are $M(A) = te^{2\pi i R}$ and $M(B) = te^{2\pi i R_1}$ (both in P_ϕ) where $t := e^{2\pi i\tau}$. The hypothesis means that $\Phi_A = g\Phi_B C$ for some $C \in G$. This implies $M(A) = C^{-1}M(B)C$, so the monodromies are conjugate, but we must show that $C \in P_\phi$. It follows (from $M(A) = C^{-1}M(B)C$) that C commutes with t and that $R_1 = \mathrm{Ad}_C(R)$. Thus the identity $\Phi_A = g\Phi_B C$ simplifies to $z^\tau = g(z)z^\tau C$, so that $z^\tau C z^{-\tau} = g^{-1} \in \widehat{\mathcal{P}}_\theta$. Thus by Lemma 8, $C \in P_\phi$ as desired.

Injectivity. Now suppose $A, B \in \mathcal{A}_\theta$ both lie over O and yield data conjugate to (M,P). We will show they are gauge equivalent by an element of $\widehat{\mathcal{P}}_\theta$. Without loss of generality we may assume $P = P_\phi$. Thus they have fundamental solutions $\Phi_A = f(z)z^\tau z^R$ and $\Phi_B = h(z)z^\tau z^{R_1}$, respectively, for some $f, h \in \widehat{\mathcal{P}}_\theta$, and they have monodromy $M(A) = te^{2\pi i R}$ and $M(B) = te^{2\pi i R_1}$ where $t := e^{2\pi i\tau}$. By assumption $M(B) = CM(A)C^{-1}$ for some $C \in P_\phi = N_G(P_\phi)$. Thus $M(B) = e^{2\pi i\tau_1}e^{2\pi i\mathrm{Ad}_C(R)}$ where $\tau_1 = \mathrm{Ad}_C(\tau)$. Comparing the semisimple parts of the two expressions for $M(B)$ we deduce C commutes with t (so that $t = e^{2\pi i\tau_1}$) and also that $\mathrm{Ad}_C(\sigma) = \sigma$. Using, for example, the Iwasawa decomposition it follows that $R_1 = \mathrm{Ad}_C(R)$. Thus B also has fundamental solution $h(z)z^\tau C z^R = \Phi_B C$. Thus it is sufficient to prove that

$$p := z^\tau C z^{-\tau} \quad \text{is in } \widehat{\mathcal{P}}_\theta$$

(since then $h(z)z^\tau C z^R \Phi_A^{-1} = hz^\tau C z^{-\tau} f^{-1}$ will be an element of $\widehat{\mathcal{P}}_\theta$ relating A and B). But by Lemma 8 this is now immediate. □

This establishes the main correspondence. For small weights θ this reduces to the parabolic statement in Theorem C (the connected centralizer condition ensures $\widehat{\mathcal{P}}_\theta = \mathcal{P}_\theta$; it is the group $\mathcal{P}_\theta$ that appears in the local moduli of parabolic bundles). For $\theta = 0$ one obtains the logarithmic statement (Theorem A).

Remark 1. Note it follows from the proof that the stabiliser in $\widehat{\mathcal{P}}_\theta$ of a connection in $\mathcal{A}_\theta$ is isomorphic to the centralizer in P_ϕ of the monodromy M. Indeed, for a connection in normal form, this correspondence is given by $C \in C_{P_\phi}(M) \leftrightarrow z^\tau C z^{-\tau}$, and in general one conjugates by any transformation putting the connection in normal form.

Remark 2. Analogously to [25] one may define the notion of a "filtered G-local system" on a smooth punctured Riemann surface U to be a G-local system $\mathbb{L}$ on U together with (on a small punctured disk Δ_i around the ith puncture, for each i) a P-local system $\mathbb{L}_i$ (for some weighted parabolic $P \subset G$) such that the restriction of $\mathbb{L}$ to Δ_i is the G-local system $\mathbb{L}_i \times_P G$ associated to $\mathbb{L}_i$. If U is a punctured disk, and we choose a basepoint in U, then specifying a filtered G-local system is the same as specifying the data $(M, P, [\phi])$ in our correspondence (so the correspondence could be restated more intrinsically in terms of filtered G-local systems).

Remark 3. Another motivation for studying such "enriched" (or "exact") Riemann–Hilbert correspondences is related to *isomonodromic deformations*. For example, such "monodromy preserving" deformations of a nonresonant logarithmic connection $A = \sum A_i dz/(z - a_i)$ on the trivial bundle on $\mathbb{P}^1$ are governed by Schlesinger's equations: $dA_i = -\sum_{j \neq i}[A_i, A_j]\, d\log(a_i - a_j)$. These are the deformations which preserve the conjugacy class of the monodromy representation of A. Of course, these equations make sense for any residues, and one may ask what exactly is preserved by Schlesinger's equations in the resonant case?[3] The answer (which is clear from [20], or may be extracted from [10]) is that the monodromy representation and the filtrations are preserved (up to overall conjugacy). This now extends immediately to arbitrary G. Such "resonant" deformations are important since, for example, soliton solutions arise as such (when one has a further irregular singularity at infinity).

[3] Recall that a logarithmic connection on a G-bundle E is *resonant* if the residue of the induced connection on the associated vector bundle $\mathrm{Ad}(E)$ has an eigenvalue in $\mathbb{Z} \setminus \{0\}$.

4. Quasi-Hamiltonian spaces

Fix a connected complex reductive group G and a parabolic subgroup $P_0 \subset G$. Let $\mathcal{C} \subset L$ be a conjugacy class of the Levi factor L of P_0. Let $\mathbb{P} \cong G/P_0$ be the variety of parabolic subgroups of G conjugate to P_0.

The aim of this section is to prove Theorem B, that the set $\widehat{\mathcal{C}}$ of pairs $(g, P) \in G \times \mathbb{P}$ with $g \in P$ and $\pi(g) \in \mathcal{C}$, is a quasi-Hamiltonian G space with G-valued moment map given by $(g, P) \mapsto g$.

Recall (cf. [1]) that a complex manifold M is a *complex quasi-Hamiltonian G-space* if there is an action of G on M, a G-equivariant map $\mu : M \to G$ (where G acts on itself by conjugation) and a G-invariant holomorphic 2-form $\omega \in \Omega^2(M)$ such that:

(QH1) $d\omega = \mu^*(\eta)$.
(QH2) For all $X \in \mathfrak{g}$, $\omega(v_X, \cdot\,) = \frac{1}{2}\mu^*(\Theta + \overline{\Theta}, X) \in \Omega^1(M)$.
(QH3) At each point $m \in M$: $\ker \omega_m \cap \ker d\mu = \{0\} \subset T_m M$.

Here we have chosen a symmetric nondegenerate invariant bilinear form $(\ ,\ ) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, the Maurer–Cartan forms on G are denoted $\Theta, \overline{\Theta} \in \Omega^1(G, \mathfrak{g})$, respectively (so in any representation $\Theta = g^{-1}dg, \overline{\Theta} = (dg)g^{-1}$), and the canonical bi-invariant 3-form on G is $\eta := \frac{1}{6}([\Theta, \Theta], \Theta)$. Moreover, if G acts on M, v_X is the fundamental vector field of $X \in \mathfrak{g}$; it is minus the tangent to the flow (so that the map $\mathfrak{g} \to \mathrm{Vect}_M; X \mapsto v_X$ is a Lie algebra homomorphism).

First we note that $\widehat{\mathcal{C}}$ is a complex manifold, in fact a smooth algebraic variety. Let $\widetilde{G}$ denote the subvariety of $G \times \mathbb{P}$ of pairs (g, P) with $g \in P$ (this is the multiplicative Brieskorn–Grothendieck space if P_0 is a Borel). There is a surjective map

$$\mathrm{pr} : G \times P_0 \to \widetilde{G}, \qquad (C, p) \mapsto (g, P) = (C^{-1}pC, C^{-1}P_0C),$$

whose fibres are precisely the orbits of a free action of P_0: explicitly $q \in P_0$ acts on $G \times P_0$ as $q(C, p) = (qC, qpq^{-1})$. Now choose a Levi decomposition $P_0 = LU$ of P_0 so that U is the unipotent radical of P_0 and $L \cong P_0/U$. Consider the (locally closed) subvariety $\mathcal{C}U \subset LU$ of P_0. Since $\mathcal{C}$ is a conjugacy class of L and P_0 acts on L via the projection $\pi : P_0 \to L$, the conjugation action of P_0 on itself preserves $\mathcal{C}U$. Then pr restricts to $G \times \mathcal{C}U$ and its image is $\widehat{\mathcal{C}}$, so that

$$\widehat{\mathcal{C}} \cong G \times_{P_0} \mathcal{C}U$$

and we deduce $\widehat{\mathcal{C}}$ is a smooth complex algebraic variety. (Note that $\mathcal{C}$ has a natural algebraic structure as a quotient of L, but will not be affine unless it is a semisimple conjugacy class.)

Rather than prove directly that $\widehat{\mathcal{C}}$ is quasi-Hamiltonian we will use the well known fact that $\mathcal{C}$ is a quasi-Hamiltonian L-space and obtain $\widehat{\mathcal{C}}$ by reduction from a quasi-Hamiltonian $G \times L$ space, as follows.

Recall that P_0 acts freely on $G \times P_0$. Let $\mathbb{M}$ denote the quotient $G \times_U P_0 = (G \times P_0)/U$ by the subgroup $U \subset P_0$. Thus $\mathbb{M}$ has a residual action of $L \cong P_0/U$ and also has a commuting action of G, from the action $g(C, p) = (Cg^{-1}, p)$ on $G \times P_0$. Moreover, there is a map $\mu : \mathbb{M} \to G \times L$ induced from the U invariant map

$$\widehat{\mu} : G \times P_0 \to G \times L, \qquad (C, p) \mapsto (C^{-1}pC, \pi(p)^{-1}),$$

where $\pi : P_0 \to L$ is the canonical projection.

Theorem 9. *The space* $\mathbb{M}$ *is a quasi-Hamiltonian* $G\times L$ *space with moment map* μ *and* 2*-form* ω *determined by the condition*

$$\mathrm{pr}^*(\omega) = \tfrac{1}{2}\left(\overline{\gamma}, \mathrm{Ad}_p\,\overline{\gamma}\right) + \tfrac{1}{2}\left(\overline{\gamma}, \mathcal{P} + \overline{\mathcal{P}}\right) \in \Omega^2(G\times P_0),$$

where pr *is the projection* $G\times P_0 \to \mathbb{M}$ *and* $\overline{\gamma} = C^*(\overline{\Theta})$, $\mathcal{P} = p^*(\Theta), \overline{\mathcal{P}} = p^*(\overline{\Theta})$.

To deduce Theorem B from this we may perform the fusion $\mathbb{M} \circledast_L \mathcal{C}$ with the conjugacy class $\mathcal{C} \subset L$, and then perform the quasi-Hamiltonian reduction by the free action of L. The result $(\mathbb{M} \circledast_L \mathcal{C})/\!\!/L$ may be identified immediately with $\widehat{\mathcal{C}}$.

In the special case $P_0 = L = G$, the space $\mathbb{M}$ is just the double $D(G) \cong G\times G$ of [1]. In general, $\dim \mathbb{M} = 2\dim P_0$ and the 2-form ω on $\mathbb{M}$ may be derived from the 2-form ω_D on $D(G)$: one finds that the restriction of ω_D to $G\times P_0$ (via the inclusion $P_0 \subset G$) is basic for the U action and descends to the 2-form ω on $\mathbb{M}$. That the result is again quasi-Hamiltonian requires proof of course.

Remark 4. In the first instance the 2-form ω was arrived at by actually computing what arose from the Hamiltonian loop group spaces related to resonant logarithmic connections on a disk, similarly to Section 4 of [8]. This computation led to the 2-form from the double. Note that in the case of $G = \mathrm{GL}_n(\mathbb{C})$ the quasi-Hamiltonian spaces $\widehat{\mathcal{C}}$ may be constructed differently, in terms of quivers (see [27]), although even for GL_n the spaces $\mathbb{M}$ do not seem to arise from quivers.

Remark 5. Notice also that there are certain parallels with the Stokes phenomenon; e.g. for the global moduli spaces, again one must fix a certain *union* of local gauge orbits to fix a symplectic leaf (in [6] this union arose by fixing the *formal* gauge orbits). Also the spaces $\mathbb{M}$ may be viewed as a tame analogue of the fission spaces of [9] (and again one may glue on more complicated spaces, not just conjugacy classes).

Proof of Theorem 9. Write $\widehat{\omega} = \mathrm{pr}^*(\omega)$. Let U_- denote the unipotent radical of the parabolic opposite to P_0 (so that U_-P_0 is open in G). Condition (QH1) may be deduced directly from the result of [9] that ${}_G\mathcal{A}_L := G\times L\times U_-\times U$ is a quasi-Hamiltonian $G\times L$ space with moment map $\mu_{\mathcal{A}} : (C, h, u_-, u) \mapsto (C^{-1}pC, h^{-1}) \in G\times L$ where $p = u_-^{-1}hu$. Then we can consider the embedding

$$\iota : G\times P_0 \hookrightarrow {}_G\mathcal{A}_L, \quad (C, p) \mapsto (C, h, 1, u),$$

defined via the Levi decomposition $p = hu \in P_0$. Thus $\widehat{\mu} = \mu_{\mathcal{A}} \circ \iota$ and moreover $\iota^*\Omega = \widehat{\omega}$ where Ω is the quasi-Hamiltonian 2-form on ${}_G\mathcal{A}_L$ from [9]. Then (QH1) follows immediately:

$$\mathrm{pr}^* d\omega = d\mathrm{pr}^*\omega = d\iota^*\Omega = \iota^* d\Omega = \iota^*\mu_{\mathcal{A}}^*\eta = \widehat{\mu}^*\eta = \mathrm{pr}^*\mu^*\eta$$

so that $d\omega = \mu^*\eta$ since pr is surjective on tangent vectors. Condition (QH2) is straightforward and left as an exercise. (QH3) is trickier and we proceed as follows. It is sufficient to show that at each point $m \in M := G\times P_0$ the subspace $\ker\widehat{\omega} \cap \ker d\widehat{\mu}$ of the tangent space T_mM is contained in the space of tangents to the

U action. Thus choose $X \in T_m M$ and suppose that $X \in \ker(\widehat{\omega}) \cap \ker(d\widehat{\mu})$. Write $\widehat{\mu} = (\mu_G, \mu_L)$ for the components of the moment map. Since X is in the kernel of $d\mu_L$ we have $\eta' = 0$ (here primes denote derivatives along X, so $\eta' := \langle h^*(\Theta_L), X\rangle$, where Θ_L is the Maurer–Cartan form on L). Moreover, X being in the kernel of $d\mu_G$ amounts to the condition $\overline{\gamma}' + \mathcal{P}' = p^{-1}\overline{\gamma}' p$. Since $p = hu$ (and $\eta' = 0$) this becomes

$$\overline{\gamma}' + \mathcal{U}' = p^{-1}\overline{\gamma}' p. \tag{4}$$

(In general, here the adjoint action of $g \in G$ on $X \in \mathfrak{g}$ will be denoted by $gXg^{-1} := \mathrm{Ad}_g X$.) Now we choose an arbitrary tangent vector $Y \in T_m M$ and denote derivatives along Y by dots, so, e.g. $\dot{\mathcal{P}} = \langle Y, \mathcal{P}_m\rangle \in \mathrm{Lie}(P_0)$. We then compute

$$2\widehat{\omega}(X,Y) = 2\big(\overline{\gamma}', \dot{\mathcal{U}}\big) + \big(u\overline{\gamma}' u^{-1} + h^{-1}\overline{\gamma}' h, \dot{\eta}\big).$$

This should be zero for all Y; observe that each term on the right is really an independent condition on X. From the first term we deduce the component of $\overline{\gamma}'$ in $\mathrm{Lie}(U_-)$ is zero. The second term implies the $\mathrm{Lie}(L)$ component of $\overline{\gamma}'$ is also zero. Thus we find that $\overline{\gamma}' \in \mathrm{Lie}(U)$, and we know $\eta' = 0$ and equation (4) holds. But these three conditions characterize[4] the tangents to the U orbits on $G \times P_0$, so (QH3) follows. □

Remark 6. In particular it follows that all the spaces $\widehat{\mathcal{C}}$ arise as certain moduli spaces of framed connections on a disk. The precise statement is as follows. Let Δ be a closed disk in the z plane centred at zero. Replace $LG, \mathcal{A}_\theta, \widehat{\mathcal{P}}_\theta$ by their analogues defined on all of Δ (rather than just germs at zero). So, e.g. now $LG = G(R)$ where R is the ring of meromorphic functions of Δ, having poles only at zero. Choose a point q on the boundary of Δ and let $\widehat{\mathcal{P}}^1_\theta$ be the subgroup of $\widehat{\mathcal{P}}_\theta$ of elements taking the value $1 \in G$ at q. Also let $\mathcal{A}_\theta(O)$ denote the subset of $\mathcal{A}_\theta$ of elements lying over a fixed orbit O as in Theorem 6 (and suppose $\mathcal{C}, \phi$ are as defined there too). Then

$$\widehat{\mathcal{C}} \cong \mathcal{A}_\theta(O)/\widehat{\mathcal{P}}^1_\theta,$$

i.e. $\widehat{\mathcal{C}}$ is isomorphic to the space of connections on Δ lying over O with a framing at q. Moreover, the residual action of $\widehat{\mathcal{P}}_\theta/\widehat{\mathcal{P}}^1_\theta \cong G$ corresponds to the G action on $\widehat{\mathcal{C}}$.

5. Cleaner statement

A cleaner Riemann–Hilbert statement arises if we also allow the weight θ to vary in the correspondence, but for this we need to define the notion of a weighted parahoric subgroup, analogous to the notion of a weighted parabolic subgroup. This leads directly to the definition of the Bruhat–Tits building.

First, define the partly extended affine Weyl group to be $\widehat{W} = N(\mathcal{K})/T(\mathcal{O}) \cong W \ltimes X_*(T)$ where $X_*(T)$ is the cocharacter lattice, which we think of either as

[4]To see this choose $X \in \mathrm{Lie}(U)$ and consider the flow $(C(t), p(t)) = \exp(Xt)\cdot(C,p)$. Thus (differentiating with respect to t) $\overline{\gamma}' = C'C^{-1} = X \in \mathrm{Lie}(U)$ and, similarly, $\mathcal{P}' = p^{-1}Xp - X$. But since $p = hu$ we see h is constant and $\mathcal{U}' = \mathcal{P}'$, so (4) follows.

the set of 1 parameter subgroups of T, or as the kernel of $\exp(2\pi i\,\cdot) : \mathfrak{t} \to T$ (an element λ of this kernel corresponds to the one-parameter subgroup $\varphi = z^\lambda$). Here $N \subset G$ is the normalizer of T in G and $W = N/T$ is the finite Weyl group, which acts naturally on $\mathfrak{t}_\mathbb{R}$ (via the adjoint action of N). Note that $X_*(T) \cong T(\mathcal{K})/T(\mathcal{O})$ and by convention $X_*(T)$ acts on $\mathfrak{t}_\mathbb{R}$ via $z^\lambda \cdot \theta = \theta - \lambda$ (this is a standard convention, but beware it agrees with our conventions concerning gauge transformations only if we identify θ with minus the residue of the connection $-\theta\, dz/z$.) These two actions combine to give an action of $\widehat{W}$ on $\mathfrak{t}_\mathbb{R}$.

Definition 1. A *weighted parahoric subgroup* of LG is an equivalence class of elements $(g, \theta) \in LG \times \mathfrak{t}_\mathbb{R}$ where

$$(g, \theta) \sim (g', \theta')$$

if $\theta' = w\theta$ for some $w \in \widehat{W}$ and $g^{-1}g'\widehat{w} \in \widehat{\mathcal{P}}_\theta$ for some lift $\widehat{w}$ of w to $N(\mathcal{K}) \subset LG$.

This is the standard definition of the (extended) Bruhat–Tits building $\mathcal{B}(LG) = (LG \times \mathfrak{t}_\mathbb{R})/\sim$ of LG [12, p. 170]. Thus we are saying a weighted parahoric is a *point* of the building. (It seems one usually views the building as a simplicial complex and rarely regards its points in this sense.) Note that LG acts naturally on $\mathcal{B}(LG)$ via left multiplication on LG.

Lemma 10. *A weighted parahoric $p \in \mathcal{B}(LG)$ canonically determines a parahoric subgroup $\mathcal{P}_p \subset LG$ and a space of connections $\mathcal{A}_p \subset \mathcal{A}$.*

Proof. Suppose p is in the equivalence class of $(g, \theta) \in LG \times \mathfrak{t}_\mathbb{R}$, and $(g', \theta') \sim (g, \theta)$ with $\theta' = w\theta$. Choose a lift $\widehat{w}$ of w to $N(\mathcal{K}) = N(\mathbb{C}) \ltimes T(\mathcal{K}) \subset LG$. We may check directly that $\mathcal{P}_{\theta'} = \widehat{w}\mathcal{P}_\theta\widehat{w}^{-1}$ and that $\mathcal{A}_{\theta'} = \widehat{w}[\mathcal{A}_\theta]$. The first claim then follows since $\widehat{\mathcal{P}}_\theta$ normalizes $\mathcal{P}_\theta$: $\mathcal{P}_p := g\mathcal{P}_\theta g^{-1}$ is well defined. Second, we should check that $\mathcal{A}_p := g[\mathcal{A}_\theta]$ depends only on the equivalence class of p. But by Lemma 3 $g^{-1}g'\widehat{w}$ preserves $\mathcal{A}_\theta$ so $\mathcal{A}_p = g'[\mathcal{A}_{\theta'}]$. □

Remark 7. Note there is an embedding $\mathfrak{t}_\mathbb{R} \hookrightarrow \mathcal{B}(LG); \theta \mapsto [(1, \theta)]$ (whose image is the standard apartment) and one may then confirm (see Lemma 12) that $\widehat{\mathcal{P}}_\theta$ is exactly the stabilizer in LG of $\theta \in \mathcal{B}(LG)$. It follows in general that $\mathcal{P}_p$ is the identity component of $\mathrm{Stab}_{LG}(p)$.

Thus it makes sense to consider pairs (A, p) where $p \in \mathcal{B}(LG)$ is a weighted parahoric and $A \in \mathcal{A}_p$ is a compatible connection. It follows from the lemma that the loop group LG acts on the set of such pairs: $g(A, p) = (g[A], g(p))$.

The corresponding monodromy data consists of pairs $(M, b) \in G \times \mathbb{B}(G)$ with $M \in P_b$. Here $\mathbb{B}(G)$ is the space of weighted parabolic subgroups of G. A point of $\mathbb{B}(G)$ consists of a parabolic $P \subset G$ and a set of weights for P (as defined earlier). This can be rephrased to parallel the definition of $\mathcal{B}(LG)$ as follows.

Definition 2. A *weighted parabolic subgroup* of G is an equivalence class of elements $(g, \theta) \in G \times \mathfrak{t}_\mathbb{R}$ where $(g, \theta) \sim (g', \theta')$ if $\theta' = w\theta$ for some $w \in W$ in the Weyl group and $gP_\theta g^{-1} = g'P_{\theta'}(g')^{-1} \subset G$ (i.e. $g^{-1}g'\widehat{w} \in P_\theta$ for some lift $\widehat{w} \in N(\mathbb{C})$ of w).

Thus we can define $\mathbb{B}(G) = (G \times \mathfrak{t}_\mathbb{R})/\sim$ and note that $b \in \mathbb{B}(G)$ determines a parabolic subgroup $P_b = gP_\theta g^{-1} \subset G$. (Beware this is not the spherical building of G, it is more like the cone over the spherical building; if we choose a maximal compact subgroup $K \subset G$ then one may identify $\mathbb{B}(G) \cong i\,\mathrm{Lie}(K) \subset \mathfrak{g}$.)[5] In any case, basically as a corollary of Theorem 6 we find:

Corollary 11. *There is a canonical bijection between LG orbits of tame parahoric connections and G orbits of enriched monodromy data:*

$$\{(A,p) \mid p \in \mathcal{B}(LG), A \in \mathcal{A}_p\}/LG \cong \{(M,b) \mid b \in \mathbb{B}(G), M \in P_b\}/G.$$

Proof. Given (A,p) we may act by LG to move p to a point θ of the standard apartment, and thus suppose $A \in \mathcal{A}_\theta$ and $p = \theta \in \mathfrak{t}_\mathbb{R}$. We may further assume A is in normal form. Then we may obtain data M, ϕ as usual, with $M \in P_\phi$, i.e. a point of the right-hand side, with $b = \phi$. We should check that the G-orbit of (M,b) only depends on the LG orbit of (A,p). Firstly this is clear if we only move (A,p) by an element of $\widehat{\mathcal{P}}_\theta$ (so $p = \theta$ does not move) by Lemma 5 and Theorem 6. Second, we should examine what happens if we act by an element g of $N(\mathcal{K}) = N(\mathbb{C}) \ltimes T(\mathcal{K})$ (since any other element of the LG orbit of (A,p) above the standard apartment will arise in this way). We may write $g = hz^\mu t$ with $h \in N(\mathbb{C}), \mu \in X_*(T), t \in T(\mathcal{O})$. Since $t \in \widehat{\mathcal{P}}_\theta$ we may assume $t = 1$ here. Set $A' = g[A], \theta' = g \cdot \theta = \mathrm{Ad}_h(\theta - \mu)$. It is straightforward to check that A' is again in normal form: indeed, suppose, $A = (\tau + \sigma + \sum A_{i\alpha} z^i)\, dz/z$ with $\alpha \in \mathcal{R} \cup \{0\}$ and $A_{i\alpha} \in \mathfrak{g}_\alpha$ (or in $\mathfrak{t}$ if $\alpha = 0$), then

$$hz^\mu[A] = \mathrm{Ad}_h\Big(\tau + \mu + \sigma + \sum A_{i\alpha} z^{i+\alpha(\mu)}\Big)\frac{dz}{z}.$$

The key point then is that $\tau' = \mathrm{Ad}_h(\tau + \mu)$ so that

$$\phi' = \tau' + \theta' = \mathrm{Ad}_h(\tau + \mu + \theta - \mu) = \mathrm{Ad}_h(\phi)$$

so that ϕ only moves via the *finite* Weyl group W. The corresponding fundamental solutions are of the form $z^\tau z^R$ and $z^{\tau'} z^{R'} = hz^{\tau+\mu} z^R h^{-1}$ so it is clear that the monodromies, etc. are related by the action of h. This shows the map from left to right is well defined. Surjectivity follows from Theorem 6. Injectivity also largely follows from Theorem 6, but it remains to check that orbits with inequivalent θ map to different points. But this follows from the fact that (M,b) determines the $\widehat{W}$ orbit of $\theta \in \mathfrak{t}_\mathbb{R}$—indeed suppose we act by G so that $M \in P_\phi$, with $\phi \in \mathfrak{t}_\mathbb{R}$ determined up to the action of W. Then let $d \in T$ be any element conjugate to the semisimple part of $\pi(M) \in L = C_G(\phi)$, so that $d = \exp(2\pi i(\tau + \sigma))$ with τ determined up to the addition of an element of $X_*(T)$. This yields one choice of $\theta = \phi - \tau$ and the others are determined by making different choices, i.e. via the action of $\widehat{W}$. □

[5] Similarly it seems one may identify $\mathcal{B}(LG)$ with a space of K-connections, although we will not use this viewpoint.

6. Other directions

First it looks to be possible to extend the nonabelian Hodge correspondence to the present context (i.e. the correspondence on a smooth algebraic curve Σ between such connections and Higgs bundles, under stability conditions). The correspondence of the parameters will be as in Simpson's table [25, p. 720]—basically the parameters are rotated, and this now generalizes directly. In our notation this table is:

	Dolbeault	De Rham	Betti
weights $\in \mathfrak{t}_{\mathbb{R}}$	$-\tau$	θ	$\phi = \tau + \theta$
"eigenvalues" $\in \mathfrak{t}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}, \mathrm{T}(\mathbb{C})$	$-\frac{1}{2}(\phi + \sigma)$	$-(\tau + \sigma)$	$\exp(2\pi i(\tau + \sigma))$

where the columns correspond to Higgs bundles, connections and monodromy data, respectively.[6] Observe for example that the eigenvalues of the Higgs field will only vary under the finite Weyl group, as expected. This global correspondence is probably best phrased in terms of torsors for the parahoric (Bruhat–Tits) group schemes $\mathcal{G} \to \Sigma$, such that locally $\mathcal{G}$ looks like a parahoric subgroup $\mathcal{P}$ of the local loop group and at all but finitely many points $\mathcal{G}$ looks like $G(\mathcal{O})$. Such torsors have been studied recently (in more generality, but not with connections or weights) in [22]. On the other hand, quasi-parahoric Higgs bundles (i.e. without the weights) have been studied algebraically recently by Yun [28], and, in effect, the local picture of such Higgs bundles was studied by Kazhdan and Lusztig [18] in 1988. (Corollary E is related to the De Rham and Betti analogues of this.) It might also be profitable (in the case of rational weights) to relate the parahoric viewpoint here to the "ramified" approach of Balaji et al. [3] (see also Seshadri [24]), although they have not considered the analogue of logarithmic connections it seems.

A. Extra proofs

Lemma 12. *For any $\theta \in \mathfrak{t}_{\mathbb{R}}$, the group $\widehat{\mathcal{P}}_\theta$ is the stabilizer in LG of $p = [(1, \theta)] \in \mathcal{B}(LG)$.*

Proof. Clearly $\widehat{\mathcal{P}}_\theta$ does stabilize p. Conversely if $g \in LG$ stabilizes p then $(g, \theta) \sim (1, \theta)$ so that $g^{-1}\widehat{w} \in \widehat{\mathcal{P}}_\theta$ for some $\widehat{w} \in N(\mathcal{K})$ such that $w(\theta) = \theta$ (where w is the image of $\widehat{w}$ in $\widehat{W}$). Thus it is sufficient to show that all such elements $\widehat{w}$ are in $\widehat{\mathcal{P}}_\theta$. Thus we should check that $z^\theta \widehat{w} z^{-\theta}$ has a limit as $z \to 0$ along any ray. We may write $\widehat{w} = hz^\lambda t$ with $h \in N(\mathbb{C}), \lambda \in X_*(T), t \in T(\mathcal{O})$. Thus

$$z^\theta \widehat{w} z^{-\theta} = z^\theta h z^{-\theta+\lambda} t = h z^{\mathrm{Ad}_h^{-1}(\theta)} z^{-\theta+\lambda} t.$$

But the condition that $w\theta = \theta$ means $\mathrm{Ad}_h(\theta - \lambda) = \theta$, so that $\mathrm{Ad}_h^{-1}(\theta) = \theta - \lambda$, and the above expression reduces to ht, which clearly has a limit as $z \to 0$. □

[6]Beware that we use the opposite conventions to [25] for connections on vector bundles ($d - A$ rather than $d + A$ in local trivializations)—this explains the sign in the middle of the bottom row, that does not appear elsewhere in the present paper.

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