WILD CHARACTER VARIETIES

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Wild character varieties are moduli spaces of monodromy data of connections on bundles on smooth algebraic curves. They generalize the tame character varieties, which are moduli spaces of monodromy data of regular singular connections, i.e. spaces of representations of the fundamental group. The wild character varieties were shown to admit holomorphic symplectic structures in [B99, B01a] and to admit (complete) hyperkähler metrics in [BB04]. (Note that the terminology "wild character variety" is more recent however.) This hyperkähler property implies they admit a family of complex structures. In their natural algebraic structure they are affine varieties (at least if the Betti weights are trivial), but in another complex structure they are algebraically completely integrable Hamiltonian systems, fibred by Lagrangian abelian varieties (meromorphic Higgs bundles/Hitchin systems). Thus, by hyperkähler rotation, the wild character varieties themselves admit natural special Lagrangian torus fibrations (used for example in Witten's approach [Wit08] to ramified geometric Langlands).

This motivates the study of the wild character varieties from an algebraic perspective: they are basic examples of "non-perturbative" or "multiplicative" symplectic varieties, that cannot be constructed from finite dimensional cotangent bundles or coadjoint orbits by symplectic reduction. On the other hand they may be constructed algebraically as finite dimensional *multiplicative* symplectic quotients (in the framework of group valued moment maps [AMM98]), via the operations of fusion and fission (see [B02b, B07, B09, B14a, B14b]).

A simple example was shown to underlie the Drinfeld–Jimbo quantum group in [B01b] (as conjectured in [B99, B01a]) and further it was shown in [B02a] that Lusztig's symmetries (a.k.a. Soibelman, Kirillov–Reshetikhin's quantum Weyl group) are the quantization of a simple example of a wild mapping class group action on a wild character variety. This example is a simple generalization of the space of Stokes data appearing in Dubrovin's work [Dub95] on semisimple Frobenius manifolds, in turn closely related to the Stokes data in the tt^* story of Cecotti–Vafa [CV93, Dub93], where the entries of the Stokes matrix are counts of BPS states. This example provided the original motivation for the general study of wild mapping class group actions on wild character varieties [B02a, B14a] (the simplest examples of which are Poisson braid group actions on Stokes data).

The purpose of this talk is to describe this circle of ideas, make precise the notion of "non-perturbative symplectic manifold" and discuss recent progress.

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