Some explicit solutions to the Riemann–Hilbert problem

Philip Boalch

École Normale Supérieure
45 rue d'Ulm, 75005 Paris, France
email: `boalch@dma.ens.fr`

Dedicated to Andrey Bolibruch

Abstract. Explicit solutions to the Riemann–Hilbert problem will be found realising some irreducible non-rigid local systems. The relation to isomonodromy and the sixth Painlevé equation will be described.

Contents

1 Introduction

Unfortunately, to say that a particular Riemann–Hilbert problem is "solvable", one usually means that *there exists a solution* rather than that one is actually able to solve the problem explicitly.

   In this article we will confront the problem of explicitly solving the Riemann–Hilbert problem directly, for irreducible representations (so we already know the problem is "solvable"). We will describe how one soon becomes embroiled in isomonodromic deformation equations, from which it is easy to see the difficulty: in the

simplest non-trivial case the isomonodromy equations reduce to the sixth Painlevé equation P_{VI} and one knows that generic solutions of P_{VI} cannot be written explicitly in terms of classical special functions.

However there are some explicit solutions to P_{VI} and our aim will be to write down some new solutions controlling isomonodromic deformations of non-rigid rank two Fuchsian systems on the four-punctured sphere. The cases we will study here will have monodromy group equal to either the binary tetrahedral or octahedral group (the icosahedral case having been studied in [5]), or to one of the triangle groups Δ_{237} or Δ_{238}.

Previously a tetrahedral and an octahedral solution of P_{VI} have been constructed by Hitchin [13] and (up to equivalence) independently by Dubrovin [9]. Moreover with hindsight we see there are three other such solutions in the work of Andreev and Kitaev [1], [18]. Here we will classify all such solutions and find an explicit solution in each of the new cases that appear.

Amongst the solutions which look to be new (i.e. to the best of the author's knowledge have not previously appeared) there are five octahedral solutions including one of genus one, and two 18 branch genus one solutions with monodromy group Δ_{237}. The largest octahedral solution has sixteen branches which is (currently) the largest known genus zero solution (those with more branches in [5] having higher genus) and we will show it is equivalent to a solution with monodromy group Δ_{238}.

The results of sections 3 and 4 will be of particular interest to people interested in constructing linear differential equations with algebraic solutions (cf. e.g. [17], [3], [27], [4]). Indeed Tables 1 and 3 may be interpreted as the analogue for rank two Fuchsian systems with four poles on $\mathbb{P}^1$, of the tetrahedral and octahedral parts of Schwarz's famous list [25] of hypergeometric equations with algebraic bases of solutions.

2 From Riemann–Hilbert to Painlevé

Consider a logarithmic connection ∇ on the trivial rank n complex vector bundle over the Riemann sphere with singularities at points $a_1, \ldots, a_m$. Choosing a coordinate z on the sphere (in which $a_m = \infty$ say), this amounts to giving the Fuchsian system of differential equations $\nabla_{d/dz}$ which will have the form:

$$\frac{d}{dz} - A(z); \quad A(z) = \sum_{i=1}^{m-1} \frac{A_i}{z - a_i} \tag{1}$$

for complex $n \times n$ matrices A_i. The original Riemann–Hilbert map is the map which takes such a Fuchsian system to its monodromy data: restricting ∇ to the punctured sphere

$$\mathbb{P}^* := \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$$

yields a nonsingular holomorphic connection and taking its monodromy yields a representation

$$\rho \in \operatorname{Hom}(\pi_1(\mathbb{P}^*), G)$$

where $G = \operatorname{GL}_n(\mathbb{C})$. The Riemann–Hilbert problem is the following: given $a_1, \ldots a_m$ and ρ can we find such a connection ∇ with monodromy equal to ρ?

Upon choosing simple loops γ_i in $\mathbb{P}^*$ around a_i generating $\pi_1(\mathbb{P}^*)$ and such that $\gamma_m \circ \cdots \circ \gamma_1$ is contractible one sees that for each m-tuple of points $\boldsymbol{a} = (a_1, \ldots, a_m)$ the Riemann–Hilbert map amounts to a map between the following spaces:

$$\left\{ (A_1, \ldots, A_m) \mid \sum A_i = 0 \right\} \xrightarrow{\ \text{RH}_{\boldsymbol{a}}\ } \left\{ (M_1, \ldots, M_m) \mid M_m \ldots M_1 = 1 \right\} \quad (2)$$

where $M_i = \rho(\gamma_i) \in G$. The Riemann–Hilbert problem then becomes: given a point $\boldsymbol{M} = (M_1, \ldots, M_m)$ on the RHS of (2), are there matrices $\boldsymbol{A} = (A_1, \ldots A_m)$ with $\sum A_i = 0$ on the LHS such that $\text{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$?

Remark 1. So far we have ignored the questions of choosing a basepoint for $\pi_1(\mathbb{P}^*)$ and the choice of basis of the fibre at the basepoint. However it is immediate that if we have a solution $\text{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$ (defined with respect to some choice of basepoint/basis) then conjugating the matrices A_i by some constant matrix $g \in G$ corresponds to conjugating the monodromy matrices M_i as well. Thus the Riemann–Hilbert problem is independent of the choice of basepoint/basis since these just move to conjugate representations.

Some fundamental work on the Riemann–Hilbert problem was done by Schlesinger [24]. He considered the question of constructing new Riemann–Hilbert solutions from a given solution $\text{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$, in two ways:

1) Schlesinger examined the fibres of the Riemann–Hilbert map and defined "Schlesinger transformations", which move $\boldsymbol{A}$ within the fibres (cf. also [16]). Roughly speaking generic fibres are discrete and correspond to certain integer shifts in the eigenvalues of the matrices A_i; geometrically these Schlesinger transformations amount to rational gauge transformations with singularities at the poles of the Fuchsian system.

2) Schlesinger also found how the matrices $\boldsymbol{A}$ can be varied as one moves the pole positions $\boldsymbol{a}$ in order to realise the same monodromy data $\boldsymbol{M}$. (Locally – for small deformations of $\boldsymbol{a}$ – this makes sense as one can use the same loops generating $\pi_1(\mathbb{P}^*)$; globally one should drag the loops around with the points $\boldsymbol{a}$, so on returning $\boldsymbol{a}$ to their initial configuration ρ may have changed by the action of the mapping class group of the m-pointed sphere.) He discovered that if the matrices A_i satisfy the following nonlinear differential equations, now known as the Schlesinger equations, then locally the monodromy data is preserved (up to overall conjugation):

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if } i \neq j, \text{ and} \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}. \quad (3)$$

In the generic case such an "isomonodromic deformation" necessarily satisfies these equations (up to conjugation). This gives a hint at the difficulty of the Riemann–

Hilbert problem: even if one knows a solution for some configuration of pole positions, one must integrate some nonlinear differential equations to obtain solutions for a deformed configuration.

This also gives a hint at how one might find some interesting solutions to the Riemann–Hilbert problem. Namely since one can move the pole positions one may consider degenerations into systems with fewer poles (for which the problem should be easier). Using solutions to these degenerate Riemann–Hilbert problems one can get asymptotics for the original solution to the Schlesinger equations and in good circumstances this enables computation of the solution. This is in effect what we will do below (using the analysis of the degenerations in [23], part II, and [15]).

Suppose we fix an irreducible representation $\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}^*), G)$. Let $\mathcal{C}_i \subset G$ be the conjugacy class containing $M_i = \rho(\gamma_i)$ which we will suppose for simplicity is regular semisimple, although this is not strictly necessary. (We are thus considering "generic" representations.)

Since ρ is irreducible we know [2] there exists some Riemann–Hilbert solution $\mathrm{RH}_{\boldsymbol{a}}(A) = M$. Let $\mathcal{O}_i \subset \mathfrak{g}$ be the adjoint orbit of A_i (in the Lie algebra of $n \times n$ complex matrices). By genericity we know $\exp(2\pi\sqrt{-1}\mathcal{O}_i) = \mathcal{C}_i$. Indeed if in the Riemann–Hilbert map we restrict to $A_i \in \mathcal{O}_i$ then one has $M_i \in \mathcal{C}_i$. Also, as mentioned above, the map is equivariant under diagonal conjugation and so there is a "reduced Riemann–Hilbert map":

$$\boldsymbol{\mathcal{O}} := \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G \xrightarrow{\nu_a} \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G =: \boldsymbol{\mathcal{C}} \qquad (4)$$

where the space $\boldsymbol{\mathcal{O}}$ is the quotient of $\{(A_1, \ldots, A_m) \mid A_i \in \mathcal{O}_i, \sum A_i = 0\}$ by overall conjugation by G and $\boldsymbol{\mathcal{C}}$ is the quotient of $\{(M_1, \ldots, M_m) \mid M_i \in \mathcal{C}_i, M_m \ldots M_1 = 1\}$ by overall conjugation by G. Generally this map ν_a is an injective holomorphic symplectic map between complex symplectic manifolds of the same dimension.

The simplest case is when the representation is rigid, i.e. when the expected dimensions of both sides of (4) is zero. Then one knows the RHS of (4) consists of precisely one point and the LHS (at most) one point.

Our basic strategy is to look at the next simplest case, with the aim of degenerating into the rigid case. Since the spaces are symplectic, this corresponds to complex dimension two, i.e. both sides of (4) are complex surfaces.

The principal example of such "minimally non-rigid" systems occurs if we look at rank two systems with four poles on the sphere (i.e. $n = 2, m = 4$). Without loss of generality (by tensoring by logarithmic connections on line-bundles) one can work with $G = \mathrm{SL}_2(\mathbb{C})$ rather than $\mathrm{GL}_2(\mathbb{C})$ and, using automorphisms of the sphere we can fix three of the poles at $0, 1, \infty$ and label the remaining pole position t. Thus we are considering systems of the form:

$$\frac{d}{dz} - \left(\frac{A_1}{z} + \frac{A_2}{z - t} + \frac{A_3}{z - 1} \right), \quad A_i \in \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C}). \qquad (5)$$

By convention we denote the eigenvalues of A_i by $\pm\theta_i/2$ for $i = 1, 2, 3, 4$. Schlesinger's equations imply that the residue $A_4 = -\sum_1^3 A_i$ at infinity remains fixed; we

will conjugate the system so that $A_4 = \frac{1}{2}\operatorname{diag}(\theta_4, -\theta_4)$. The remaining conjugation freedom is then just conjugation by the one-dimensional torus $T := \operatorname{diag}(a, 1/a)$, $a \in \mathbb{C}^*$; the space of such systems is then three dimensional (quotienting by T yields the surface $\mathcal{O}$).

Following [16] (pp. 443–446) one may choose certain coordinates x, y, k on this space of systems and write down what Schlesinger's equations become. One obtains a pair of coupled first-order nonlinear differential equations in x, y (not dependent on k) and an equation for k of the form $\frac{dk}{dt} = f(y, t)k$. The coordinate k corresponds to the torus action, which we can forget about since we are happy to consider Fuchsian systems up to conjugation. Eliminating x from the coupled system yields the sixth Painlevé equation

$$
\frac{d^2 y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}
$$
$$
+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
$$

where the constants α, β, γ, δ are related to the θ-parameters as follows:

$$
\alpha = (\theta_4 - 1)^2/2, \quad \beta = -\theta_1^2/2, \quad \gamma = \theta_3^2/2, \quad \delta = (1 - \theta_2^2)/2. \tag{6}
$$

Since we will want to go back from a solution of P_{VI} to an explicit isomonodromic family of Fuchsian systems, we will give the explicit formulae for the matrix entries of the system in terms of y, y', in Appendix A.

Now the bad news is that most solutions to P_{VI} cannot be written in terms of classical special functions. From Watanabe's work [28] one knows that either a solution is non-classical or it is a Riccati solution (corresponding to a reducible or rigid monodromy representation ρ) or the solution $y(t)$ is an algebraic function.

Since we are interested in explicit solutions corresponding to irreducible non-rigid representations, the only possibility is to seek algebraic solutions to P_{VI}, in other words solutions defined implicitly by equations of the form

$$
F(y, t) = 0
$$

for polynomials F in two variables. We can rephrase this more geometrically:

Definition 2. An algebraic solution of P_{VI} consists of a triple (Π, y, t) where Π is a compact (possibly singular) algebraic curve and y, t are rational functions on Π such that:

• $t \colon \Pi \to \mathbb{P}^1$ is a Belyi map (i.e. t expresses Π as a branched cover of $\mathbb{P}^1$ which only ramifies over 0, 1, ∞), and

• Using t as a local coordinate on Π away from ramification points, $y(t)$ should solve P_{VI}, for some value of the parameters α, β, γ, δ.

Indeed given an algebraic solution in the form $F(y, t) = 0$ one may take Π to be the closure in $\mathbb{P}^2$ of the affine plane curve defined by F. That t is a Belyi map on Π

follows from the Painlevé property of P_{VI}: solutions will only branch at $t = 0, 1, \infty$ and all other singularities are just poles. The reason we prefer this reformulation is that often the polynomial F is quite complicated and parameterisations of the plane curve defined by F are usually simpler to write down. (The polynomial F can be recovered as the minimal polynomial of y over $\mathbb{C}(t)$, since $\mathbb{C}(y, t)$ is a finite extension of $\mathbb{C}(t)$.)

We will say the solution curve Π is 'minimal' or an 'efficient parameterisation' if y generates the field of rational functions on Π, over $\mathbb{C}(t)$, so that y and t are not pulled back from another curve covered by Π (i.e. that Π is birational to the curve defined by F).

The main invariants of an algebraic solution are the genus of the (minimal) Painlevé curve Π and the degree of the corresponding Belyi map t (the number of branches the solution has over the t-line).

Now the basic question is: what representations ρ can we start with in order to obtain an algebraic solution to P_{VI}? Well, the solution must have only a finite number of branches and so we can start by looking for finite branching solutions, and hope to prove in each case that the solution is actually algebraic.

The important point is that one can read off the branching of the solution y as t moves around loops in the three-punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in terms of the corresponding linear representations ρ. One finds (cf. e.g. [5], section 4) that ρ transforms according to the natural action of the pure mapping class group (which is isomorphic to $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$) and thus to the free group on two-letters $\mathcal{F}_2$. Explicitly the generators w_1, w_2 of $\mathcal{F}_2$ act on the monodromy matrices $\boldsymbol{M}$ via $w_i = \omega_i^2$ where ω_i fixes M_j for $j \neq i, i + 1$, $(1 \leqslant j \leqslant 4)$ and

$$\omega_i(M_i, \ M_{i+1}) = (M_{i+1}, \ M_{i+1} M_i M_{i+1}^{-1}). \tag{7}$$

(Incidentally the geometric origins of this in the context of P_{VI} can be traced back at least to Malgrange's work [22] on the global properties of the Schlesinger equations.) The full classification of the representations ρ living in finite orbits of this action is still open, but there are some obvious ones: namely if ρ takes values in a finite subgroup of $SL_2(\mathbb{C})$ then the $\mathcal{F}_2$ orbit will clearly be finite.

Thus the program is to take such a finite subgroup $\Gamma \subset G$, and go through the possible representations $\rho\colon \pi_1(\mathbb{P}^*) \to \Gamma$ (whose image generates Γ say) and find the corresponding P_{VI} solutions. The two main problems to overcome in completing this program are:

1) There are lots of such representations (even up to conjugation), for example for the binary tetrahedral group from [11] one knows there are 520 conjugacy classes of triples of generators.

2) We still need to find the P_{VI} solution explicitly.

For 1) we proceed as in [5]: by using Okamoto's affine F_4 symmetry group of P_{VI} we can drastically reduce the number of classes that arise for each group. It is worth emphasising that upon applying an Okamoto transformation the monodromy group may well become infinite, and currently there are very few examples of algebraic

solutions to P$_{VI}$ which are not equivalent to (or simple deformations of) a solution with finite linear monodromy group (see the final remark of Section 5 below).

For 2) we use Jimbo's asymptotic formula (see [15] and the corrected version in [7], Theorem 4). By looking at the degeneration of the Fuchsian system into systems with only three poles (hypergeometric systems) and using explicit solutions of their Riemann–Hilbert problems, Jimbo found explicit formulae for the leading term in the asymptotic expansion of P$_{VI}$ solutions at zero. Using the P$_{VI}$ equation these leading terms determine the Puiseux expansions of each branch of the solution at zero and, taking sufficiently many terms, these enable us to find the solution completely if it is algebraic.

Philosophically the author views this work as an illustration of the utility of Jimbo's asymptotic formula. An alternative method of constructing solutions of Painlevé VI has been proposed by Kitaev (and Andreev) [20], [1], [18] who call it the "RS" method (see also Doran [8] for similar ideas, as well as Section 5 below and also [21] for closely related ideas of F. Klein). Kitaev [20] conjectures that all algebraic solutions arise in this way and, with Andreev, has found some solutions essentially by starting to enumerate all suitable rational maps along which a hypergeometric system may be pulled back.

One of our original aims was to try to ascertain what algebraic P$_{VI}$ solutions are known, up to equivalence under Okamoto transformations and simple deformation (cf. e.g. [5], Remark 15). In other words the aim was to see how much is known of what might be called the "non-abelian Schwarz list", viewing P$_{VI}$ as the simplest non-abelian Gauss–Manin connection. The result is that, so far, all the algebraic solutions the author has seen have turned out to be related to a finite subgroup of $SL_2(\mathbb{C})$ or to the 237 triangle group (see Section 5 below).[1] As an illustrative example of what can happen consider solution 4.1.7.B of [1]: At first glance we see t is a degree 8 function of the parameter s and so one imagines a solution with 8 branches (and wonders if it is related to one of the eight-branch solutions of [5] or [12] or of Section 4 below). However one easily confirms that in fact

$$y = y_{21} = t + \frac{3\eta_\infty \sqrt{t(t-1)}}{\eta_\infty + 1}$$

so it really only has two branches (it was inefficiently parameterised). In turn one finds (for any value of the constant η_∞) this is equivalent to the well-known solution $y = \sqrt{t}$.

On the other hand Jimbo's formula gives us great control, in that we can often go directly from a linear representation ρ to the corresponding P$_{VI}$ solution. In particular the mapping class group orbit of ρ tells us a priori the number of branches (and lots more) that the solution will have. At some point the author realised (see the introduction to [5]) that there should be more solutions related to the symmetries of the Platonic solids than had already appeared; we have found it to be more efficient to

---

[1] One might be so bold as to conjecture that there are no others, simply because no others have yet been seen, in spite of the variety of approaches used.

first ascertain directly what solutions arise in this way, than for example to enumerate rational maps. (The author's understanding is that a theorem of Klein implies that the solutions of sections 3 and 4 below and of [5] will arise via rational pullbacks of a hypergeometric system, but it is not clear if the enumeration started in [1] would ever have found all the corresponding rational maps independently.)

3 The tetrahedral solutions

In this section we will classify the solutions to P_{VI} having linear monodromy group equal to the binary tetrahedral group $\Gamma \subset G = SL_2(\mathbb{C})$. The procedure is similar to that used in [5] for the icosahedral group.

First we examine (as in [5], section 2) the set S of G-conjugacy classes of triples of generators (M_1, M_2, M_3) of Γ (i.e. two triples are identified if they are related by conjugating by an element of G). (Equivalently this is the set of conjugacy classes of representations ρ of the fundamental group of the four-punctured sphere into Γ, once we choose a suitable set of generators.) From Hall's formulae [11] one knows there are 12480 triples of generators of Γ and dividing by 24 (the size of the image in $PSL_2(\mathbb{C})$ of the normaliser of Γ in G) we find that S has cardinality 520. Then we quotient S further by the relation of geometric equivalence (cf. [5], section 4): two representations are identified if they are related by the full mapping class group, or by the set of even sign changes of the four monodromy matrices M_i (with $M_4 = (M_3 M_2 M_1)^{-1}$). One finds there are precisely six such geometric equivalence classes, and by Lemma 9 of [5] this implies there are at most six solutions to P_{VI} with tetrahedral monodromy which are inequivalent under Okamoto's affine F_4 action.

On the other hand we can look at the set of θ-parameters corresponding to the representations in S. Since Okamoto transformations act by the standard $W_a(F_4)$ action on the space of parameters, it is easy to find the set of inequivalent parameters that arise from S, cf. [5], section 3. (Since they are real we can map them all into the closure of a chosen alcove.) We find there are exactly six sets of inequivalent parameters that arise and so there are at least six inequivalent tetrahedral solutions. Combining with the previous paragraph we thus see there are precisely six inequivalent tetrahedral solutions to P_{VI}.

Various data about the six classes and the corresponding P_{VI} solutions are listed in Tables 1 and 2. Table 2 lists a representative set of θ-parameters for each class together with numbers σ_{ij} which uniquely determine a triple M_1, M_2, M_3 in S (and thus the linear representation ρ) for that class with the given θ values, via the formula

$$\mathrm{Tr}(M_i M_j) = 2 \cos(\pi \sigma_{ij}).$$

The first two columns of Table 1 list the degree and genus of the P_{VI} solution. The column labelled "Walls" lists the number of affine F_4 reflection hyperplanes the parameters of the solution lie on. The type of the solution enables us to see at a glance

Table 1. Properties of the tetrahedral solutions.

	Degree	Genus	Walls	Type	Alcove Point	n	Group	Partitions
1	1	0	2	ab^2	35, 15, 15, 5	96	1	
2	1	0	3	b^3	30, 10, 10, 10	32	1	
3	2	0	3	b^4-	50, 10, 10, 10	48	S_2	1, 2
4	3	0	3	b^4+	40, 0, 0, 0	72	S_3	3, 2
5	4	0	2	ab^3	45, 5, 5, 5	128	A_4	3
6	6	0	3	a^2b^2	50, 10, 0, 0	144	A_4	$2^2, 3^2$

Table 2. Representative parameters for the tetrahedral solutions

	$(\theta_1, \theta_2, \theta_3, \theta_4)$	$(\sigma_{12}, \sigma_{23}, \sigma_{13})$
1	1/2, 0, 1/3, 1/3	1/2, 1/3, 1/3
2	1/3, 0, 1/3, 1/3	1/3, 1/3, 1/3
3	1/3, 2/3, 2/3, 2/3	1/2, 1/3, 1/2
4	2/3, 1/3, 1/3, 2/3	1/2, 1/3, 1/2
5	1/3, 1/3, 1/3, 1/2	1/3, 2/3, 1/3
6	1/2, 1/3, 1/3, 1/2	1/3, 1/2, 1/3

which class a given element of S lies in: Given $M_1, M_2, M_3, M_4 \in \Gamma$ their images in $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$ are real rotations and we write an "a" for each rotation by half of a turn, a "b" for each rotation by a third of a turn, and write nothing for each trivial rotation thus obtained. This distinguishes all classes except 3 and 4 which both correspond to four rotations by a third of a turn: each M_i thus has parameter $\theta_i = 1/3$ or $\theta_i = 2/3$. For class 3 there are always an odd number of each type of θ (1/3 or 2/3) so we write a minus, and for class 4 there are always an even number of each type, so we write a plus.

Finally the rest of Table 1 lists the corresponding alcove point (scaled by 60), the number n of elements of S belonging to each class, the monodromy group of the cover $t : \Pi \to \mathbb{P}^1$ and the unordered collection of sets of ramification indices of this cover over $t = 0, 1, \infty$ (repeating the last set of indices until three are obtained). Thus for example each solution corresponding to row 6 has indices $(3, 3)$ over two points amongst $\{0, 1, \infty\}$ and indices $(1, 1, 2, 2)$ over the third.

All of the tetrahedral solutions have genus zero so we may take Π to be $\mathbb{P}^1$ with parameter s and write the solutions as functions of s. As in the icosahedral case the solutions with at most 4 branches are closely related to known solutions. For classes 1 and 2 one of the monodromy matrices is projectively trivial and so these rows correspond to pairs of generators of the tetrahedral group, i.e. to the two tetrahedral

entries on Schwarz's list of algebraic hypergeometric functions. The corresponding
P_{VI} solutions are both just $y = t$ with the parameters as listed in Table 2. As in [5] one
finds class 3 contains the solution $y = \pm\sqrt{t}$ (with the parameters as listed in Table 2).
Class 4 contains the tetrahedral solution

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \quad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)} \tag{8}$$

on p. 592 of [13] (with the parameters as listed in Table 2) and is equivalent to a
solution found independently by Dubrovin [9] (E.31). Also class 5 contains a simple
deformation of the four-branch dihedral solution in section 6.1 of [12]:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \quad t = \frac{s^3(s+2)}{2s+1}, \tag{9}$$

that is, this solution is tetrahedral if we use the parameters in Table 2, rather than the
parameters $(1/2, 1/2, 1/2, 1/2)$ for which it is dihedral.

Thus we are left with one solution, corresponding to row 6. Using Jimbo's asymp-
totic formula to compute the Puiseux expansions etc. (as in [7], section 5, especially
p. 193) we find the following solution in this class:

Tetrahedral solution 6, 6 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/3, 1/3, 1/2)$:

$$y = -\frac{s\,(s+1)\,(s-3)^2}{3\,(s+3)\,(s-1)^2}, \quad t = -\left(\frac{(s+1)\,(s-3)}{(s-1)\,(s+3)}\right)^3$$

(We have recently learnt that this is equivalent to solution 4.1.1A in [1].) It is now easy
to write down the explicit isomonodromic family of Fuchsian systems in this case,
thereby solving the Riemann–Hilbert problem for this class of representations ρ, for
an arbitrary configuration of the four pole positions (up to automorphisms of $\mathbb{P}^1$). (We
will leave for the reader the analogous substitutions for the other solutions below.)
Using the formulae in Appendix A one finds the family of systems parameterised by
$s \in \mathbb{P}^1$ is (up to overall conjugation):

$$\frac{d}{dz} - \left(\frac{A_1}{z} + \frac{A_2}{z - t(s)} + \frac{A_3}{z - 1}\right)$$

where

$$A_1 = \begin{pmatrix} (s^2+3)(s^6-51s^4+99s^2-81) & 4s(s^4-9) \\ 4(5s^6-75s^4+135s^2-81)s(s^4-9) & -(s^2+3)(s^6-51s^4+99s^2-81) \end{pmatrix} \Big/ \Delta$$

$$A_2 = \begin{pmatrix} 4(s+3)(s-1)^2s^2(s^3-s^2+3s+9) & -2(s+3)(s-1)^2(s^2+2s+3) \\ -2(s+3)(s-1)^2(s^3-3s^2-9s-9)(5s^5-5s^4-45s-27) & -4s^2(s+3)(s-1)^2(s^3-s^2+3s+9) \end{pmatrix} \Big/ \Delta$$

$$A_3 = \mathrm{diag}(-\theta_4, \theta_4)/2 - A_1 - A_2, \quad \Delta = -36(s^2+3)(s^2-1)^2(s^2-9).$$

Note that if the denominator Δ is zero then $t \in \{0, 1, \infty\}$ since

$$1 - t = 2\frac{(s^2+3)^2(s^2-3)}{(s+3)^3(s-1)^3}.$$

Thus the system is well defined for all s in $t^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ except possibly at $s = \infty$ (where $t = -1$). However writing $s = 1/s'$ it is easy to conjugate the system to one well-defined also at $s = \infty$. Thus one never encounters configurations requiring a nontrivial bundle; the Malgrange divisor is trivial in this situation (in spite of the fact the solution y does have a pole at $s = \infty$); indeed one knows the corresponding τ function (whose zeros lying over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ correspond to nontrivial bundles) satisfies:

$$d\log(\tau) = \mathrm{Tr}\left(A_2\left(\frac{A_1}{t} + \frac{A_3}{t-1}\right)\right) dt$$

$$= -\frac{s^6 + 6\,s^5 + 3\,s^4 - 8\,s^3 - 9\,s^2 - 54\,s - 27}{3\,(s^4 - 9)(s^2 - 1)(s^2 - 9)}\,ds$$

which is nonsingular for all $s \in t^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$.

Remark 3. Sometimes one is interested in Fuchsian equations with given monodromy, rather than systems. To obtain these one may choose a cyclic vector, or more simply substitute the P_{VI} solution into the standard formulae for the isomonodromic family of Fuchsian equations. In the present case one obtains the equation:

$$\frac{d^2}{dz^2} + a_1\frac{d}{dz} + a_2$$

where a_1 and $z(z-1)a_2$ are respectively:

$$\frac{1}{2z} + \frac{2}{3\,(z-1)} + \frac{2}{3(z - t(s))} - \frac{1}{z - y(s)},$$

$$\frac{7s^6 - 6s^5 + 3s^4 + 4s^3 - 63s^2 - 54s - 27}{18(s+3)^3(s-1)^3(z - t(s))} - \frac{(s^2 - 2s + 3)s^2}{9(s+3)(s-1)^2(z - y(s))} - \frac{1}{18}.$$

For generic s this is a Fuchsian equation with non-apparent singularities at $z = 0, 1, t, \infty$ and an apparent singularity at $z = y$, realising the given (projective) monodromy representation. In special cases (when $y = 0, 1, t, \infty$) it will have just the four non-apparent singularities (and will thus be a so-called "Heun equation"). For example specialising to $s = 0$ one finds $y = 0, t = -1$ and the equation becomes that with

$$a_1 = -\frac{1}{2z} + \frac{2}{3\,(z-1)} + \frac{2}{3(z+1)}, \quad a_2 = -\frac{1}{18\,(z-1)\,(z+1)}$$

which is a Heun equation whose projective monodromy representation is that specified by row 6 of Table 2.

Remark 4. At the editor's request we will explain how one may verify directly that these P_{VI} solutions actually do correspond to Fuchsian systems with linear monodromy representations as specified by Table 2. For the rigid cases, rows 1 and 2, this is immediate, by rigidity. For the others, first one may check that the solutions actually do solve P_{VI}. This can be done directly (by computing the derivatives of y with

respect to t and substituting into the P_{VI} equation).[2] Having done this we know the formulae of Appendix A do indeed give an isomonodromic family of Fuchsian systems. To see it has the monodromy representation specified by Table 2 we first compute the Puiseux expansions at 0 of each branch of the function $y(t)$ (only the leading terms will be needed). On the other hand, Jimbo's asymptotic formula (in the form in [7], Theorem 4) computes the leading term in the asymptotic expansion of the P_{VI} solution corresponding to the given monodromy representation ρ (the leading term is of the form at^b where a and b are explicit functions of $\theta_1, \theta_2, \theta_3, \theta_4, \sigma_{12}, \sigma_{23}, \sigma_{13}$). Then it is sufficient to check this leading term equals one of the leading terms of the Puiseux expansions of $y(t)$. The logic is that, in the cases at hand, the leading term determines the whole Puiseux expansion (using the recursion determined by the P_{VI} equation) and this is convergent so determines the solution locally, and thus globally by analytic continuation. (For the solutions we construct here this is automatic since we constructed the solution starting with the results of Jimbo's formula.)

Some simpler, but not entirely conclusive, checks are as follows:

1) Compare the monodromy of the Belyi map t with the $\mathcal{F}_2$ action (7) on the conjugacy class of the representation ρ (if we didn't know better it would appear as a miracle that the solution, constructed out of just the Puiseux expansion at 0, turns out to have the right branching at 1 and ∞ too).

2) Compute directly the Galois group of one of the Fuchsian systems in the isomonodromic family. (Together with the exponents this goes a long way to pinning down the monodromy representation.) There are various ways to do this, one of which is to convert the system into an equation (e.g. via a cyclic vector) and use the facility on Manuel Bronstein's webpage: http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html (This requires finding a suitable rational point on the Painlevé curve, which, if possible, is easy in the genus zero cases, and not too difficult using Magma in the genus one cases.)

4 The octahedral solutions

For the octahedral group we do better and find more new solutions. In this case, by [11] or direct computation, S has size 3360, which reduces to just thirteen classes under either geometric or parameter equivalence. Thus there are exactly thirteen octahedral solutions to P_{VI}, up to equivalence under Okamoto's affine F_4 action.

Data about these classes are listed in Tables 3 and 4. In this case the type of the solution may contain the symbol "g" which indicates that one of the corresponding rotations in SO_3 is a rotation by a quarter of a turn. Also, in some cases rather than list the monodromy group of the cover $t \colon \Pi \to \mathbb{P}^1$ we just give its size.

---

[2] To aid the reader interested in examining the solutions of this article (and to help avoid typographical errors) a Maple text file of the solutions has been included with the source file of the preprint version on the math arxiv

Table 3. Properties of the octahedral solutions.

	Deg.	Genus	Walls	Type	Alcove Point	n	Group	Partitions
1	1	0	1	abg	(65, 35, 25, 5)/2	192	1	
2	1	0	2	bg^2	25, 10, 10, 5	96	1	
3	2	0	2	b^2g^2	45, 15, 10, 10	96	S_2	1, 2
4	3	0	1	abg^2	40, 10, 5, 5	288	S_3	3, 2
5	4	0	2	ag^3	(75, 15, 15, 15)/2	128	A_4	3
6	4	0	3	g^4	30, 0, 0, 0	32	A_4	3
7	6	0	2	a^2bg	(95, 25, 5, 5)/2	576	24	$2^2, 3^2, 2\,4$
8	6	0	2	b^2g^2	35, 5, 0, 0	288	36	3, 2 4
9	8	0	1	ab^2g	(85, 15, 15, 5)/2	768	576	$2^2\,3, 2^2\,4$
10	8	0	3	a^2g^2	45, 15, 0, 0	192	192	$3^2, 2\,3^2$
11	12	0	3	a^2b^2	50, 10, 0, 0	288	576	$2^2\,3^2, 2^2\,4^2$
12	12	1	3	a^3b	55, 5, 5, 5	288	96	$3^4, 2^2\,4^2$
13	16	0	3	a^3g	(105, 15, 15, 15)/2	128	3072	$2^2\,3^4$

The octahedral solutions with at most 4 branches correspond to the following known solutions. As in [5] one finds: The first two classes correspond to the octahedral entries on Schwarz's list of algebraic hypergeometric functions (and the P_{VI} solution is $y = t$ with the parameters indicated in Table 4). Solution 3 is $y = \pm\sqrt{t}$ with the parameters listed in Table 4, solution 4 has 3 branches and is a simple deformation of the 3-branch tetrahedral solution above (namely it is the solution in equation (8), but with the parameters given in Table 4), solution 5 is a simple deformation of the 4-branch dihedral solution (namely it is the solution in equation (9), but with the parameters given in Table 4), and solution 6 is the 4-branch octahedral solution

$$y = \frac{(s-1)^2}{s(s-2)}, \quad t = \frac{(s+1)(s-1)^3}{s^3(s-2)} \tag{10}$$

on p. 588 of [13], with the parameters as in Table 4, which is equivalent to a solution found independently by Dubrovin [9] (E.29).

For the remaining 7 solutions, rows 7–13, we will construct an explicit solution in each class using Jimbo's asymptotic formula. More computational details appear in Appendix C. (We have recently learnt that solutions 8 and 10 are equivalent to those of [18], 3.3.3 top of p. 22, and 3.3.5 bottom of p. 23, respectively.) The formulae

Philip Boalch

Table 4. Representative parameters for the octahedral solutions

	$(\theta_1, \theta_2, \theta_3, \theta_4)$	$(\sigma_{12}, \sigma_{23}, \sigma_{13})$
1	1/2, 0, 1/3, 1/4	1/2, 1/3, 1/4
2	1/3, 0, 1/4, 1/4	1/3, 1/4, 1/4
3	1/3, 1/4, 1/4, 2/3	1/2, 1/2, 1/2
4	1/2, 1/4, 1/4, 2/3	1/2, 1/3, 3/4
5	1/4, 1/4, 1/4, 1/2	1/3, 1/2, 1/3
6	1/4, 1/4, 1/4, 1/4	1/3, 0, 1/3
7	1/2, 1/2, 1/4, 2/3	1/2, 1/2, 1/3
8	1/3, 3/4, 1/3, 3/4	1/2, 3/4, 1/3
9	1/3, 1/4, 1/2, 2/3	1/2, 2/3, 3/4
10	1/2, 1/4, 1/2, 3/4	2/3, 2/3, 1
11	1/3, 1/2, 1/2, 2/3	1/2, 1/2, 1/4
12	1/2, 1/2, 1/2, 2/3	1/2, 1/4, 2/3
13	1/2, 1/2, 1/2, 3/4	1/2, 2/3, 1/3

obtained are as follows:

Octahedral solution 7, 6 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/4, 2/3)$:

$$y = \frac{9s(2\,s^3 - 3\,s + 4)}{4(s+1)(s-1)^2(2\,s^2 + 6\,s + 1)}, \quad t = \frac{27s^2}{4(s^2 - 1)^3}$$

Octahedral solution 8, 6 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 3/4, 1/3, 3/4)$:

$$y = \frac{(2\,s^2 - 1)(3\,s - 1)}{2s(2\,s^2 + 2\,s - 1)(s - 1)}, \quad t = -\frac{(3\,s - 1)^2}{8(2\,s^2 + 2\,s - 1)(s - 1)s^3}$$

Octahedral solution 9, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/4, 1/2, 2/3)$:

$$y = \frac{s^3(2\,s^2 - 4\,s + 3)(s^2 - 2\,s + 2)}{(2\,s^2 - 2\,s + 1)(3\,s^2 - 4\,s + 2)}, \quad t = \left(\frac{s^2(2\,s^2 - 4\,s + 3)}{3\,s^2 - 4\,s + 2}\right)^2$$

Octahedral solution 10, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/4, 1/2, 3/4)$:

$$y = \frac{32s(s+1)(5\,s^2 + 6\,s - 3)}{(s^2 + 2\,s + 5)(3\,s^2 + 2\,s + 3)^2}, \quad t = \frac{1024s^3(s+1)^2}{(s^2 + 6\,s + 1)(3\,s^2 + 2\,s + 3)^3}$$

Octahedral solution 11, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/2, 1/2, 2/3)$:

$$y = \frac{(s+1)(7\,s^4 + 16\,s^3 + 4\,s^2 - 4)\,r}{s^3(s-2)(s^4 - 4\,s^2 + 32\,s - 28)}, \quad t = \left(\frac{(s+1)^2\,r}{(s-2)^2 s^2}\right)^2$$

where $r = 4(3\,s^2 - 4\,s + 2)/(s^2 + 4\,s + 6)$.

The next solution, number 12, has genus one. In this case we take Π to be the elliptic curve

$$u^2 = (2\,s + 1)(9\,s^2 + 2\,s + 1).$$

As functions on this curve the solution is:

Octahedral solution 12, genus one, 12 branches, $\theta = (1/2, 1/2, 1/2, 2/3)$:

$$y = \frac{1}{2} + \frac{45\,s^6 + 20\,s^5 + 95\,s^4 + 92\,s^3 + 39\,s^2 - 3}{4\,(5\,s^2 + 1)(s + 1)^2 u}$$

$$t = \frac{1}{2} + \frac{s(2\,s + 1)^2(27\,s^4 + 28\,s^3 + 26\,s^2 + 12\,s + 3)}{(s + 1)^3 u^3}$$

Finally the last solution, number 13, has 16 branches and genus zero. This is possibly the highest degree genus zero solution amongst all algebraic solutions of P_{VI}. It is also special since it has no real branches. Thus necessarily the parameterisation of the solution is not defined over $\mathbb{Q}$ although the solution curve $F(y, t) = 0$ itself has $\mathbb{Q}$ coefficients, as does the Belyi map t.

Octahedral solution 13, 16 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 3/4)$:

$$y = -\frac{(1 + i)(s^2 - 1)(s^2 + 2\,is + 1)(s^2 - 2\,is + 1)^2 c}{4\,s(s^2 + i)(s^2 - i)^2(s^2 + (1 + i)s - i)d},$$

$$t = \frac{(s^2 - 1)^2(s^4 + 6\,s^2 + 1)^3}{32\,s^2(s^4 + 1)^3}$$

where c and d are respectively:

$$s^8 - (2 - 2i)\,s^7 - (6 + 2i)\,s^6 + (10 + 2i)\,s^5 + 4\,is^4 + (10 - 2i)\,s^3 + (6 - 2i)\,s^2 - (2 + 2i)\,s - 1,$$

$$s^6 - (3 + 3i)\,s^5 + 3\,is^4 + (4 - 4i)\,s^3 + 3\,s^2 + (3 + 3i)\,s + i.$$

Remark 5. The author has recently understood that an alternative way to construct some (but not all) of these tetrahedral and octahedral solutions would have been to use the quadratic transformations of [19]. For example tetrahedral solution 6 could have been obtained from tetrahedral solution 4 in this way (a fact that was apparently not noticed in [1]). It is debatable whether this would have been simpler for us than the direct method used here, given what had already been done in [7], [5]. (The quadratic transformations were crucial however to construct the higher genus icosahedral solutions, cf. [6].)

5 Infinite monodromy groups

In this final section we will give some examples of solutions corresponding to non-rigid representations ρ into some infinite subgroups of $G = \mathrm{SL}_2(\mathbb{C})$. The point is that the method we are using to construct P_{VI} solutions should work provided that the $\mathcal{F}_2$ orbit of ρ is finite, and above we just used the finiteness of the image of ρ as a convenient way of ensuring this.

Thus we are looking for representations ρ having finite $\mathcal{F}_2$ orbits, or more concretely, matrices $M_1, M_2, M_3 \in G$ having finite orbit under the action (7). (Such an $\mathcal{F}_2$ orbit, on conjugacy classes of representations, gives the permutation representation of the Belyi cover $t\colon \Pi \to \mathbb{P}^1$ for the corresponding P_{VI} solution. We would like to find interesting $\mathcal{F}_2$ orbits in order to find interesting P_{VI} solutions.) So far there appear to be four ways (apart from guessing) of finding representations ρ having finite $\mathcal{F}_2$ orbits.

Firstly one can just set the parameters to be sufficiently irrational in one of the families of solutions. (For example $y = \sqrt{t}$ is a solution provided $\theta_1 + \theta_4 = 1$, $\theta_2 = \theta_3$ amongst other possibilities.)

Secondly one can sometimes apply an Okamoto transformation to a known solution and change ρ into a representation having infinite image. For example if we take the 16 branch octahedral solution above and apply the Okamoto transformation corresponding to the central node of the extended D_4 Dynkin diagram, then we obtain a P_{VI} solution whose corresponding linear representation has image equal to the $(2, 3, 8)$ triangle group. To see this we recall [14], [7] that Okamoto's affine D_4 action does not change the quadratic functions $\mathrm{Tr}(M_i M_j) = 2\cos(\pi \sigma_{ij})$ of the monodromy data, only the θ-parameters. In this case the 16 branch octahedral solution has data

$$\theta = (1/2, 1/2, 1/2, 3/4), \quad \sigma = (1/2, 2/3, 1/3)$$

on one branch and the solution after applying the transformation has data

$$\theta = (3/8, 3/8, 3/8, 5/8), \quad \sigma = (1/2, 2/3, 1/3).$$

One may show that the image of the corresponding triple (M_1, M_2, M_3) in $\mathrm{PSL}_2(\mathbb{C})$ generates a $(2, 3, 8)$ triangle group (see appendix B). The corresponding solution to P_{VI} is given by the formula

$$y_{238}(s) = y(s) + \frac{2 - \sum_1^4 \theta_i}{2\, p(y, y', t)}$$

where y, t, θ_i are as for the 16 branch octahedral solution and p as in Appendix A. Explicitly one finds the solution is

2, 3, 8 solution, genus zero, 16 branches, $\theta = (3/8, 3/8, 3/8, 5/8)$:

$$y = -\frac{(1+i)(s^2 - 1)(s^2 + 2\,i\,s + 1)(s^2 - 2\,i\,s + 1)^2 d'}{8s\,(s^2 + i)(s^2 - i)^2 d}$$

with t and $d(s)$ as for the 16 branch octahedral solution, and $d'(s) = \overline{d(\bar{s})}$. In turn, via the formulae in Appendix A, this yields the explicit family of Fuchsian systems having projective monodromy group the $(2, 3, 8)$ triangle group.

Thirdly, the idea of [7] was to use a different realisation of P_{VI} as controlling isomonodromic deformations of certain 3×3 systems. The corresponding monodromy representations were subgroups of $GL_3(\mathbb{C})$ generated by complex reflections, and again one will obtain finite branching solutions by taking representations into a finite group generated by complex reflections. Applying this to the Klein complex reflection group led to an algebraic solution to P_{VI} with 7 branches. Moreover [7] described explicitly how to go between this 3×3 picture and the standard 2×2 picture used here, both on the level of systems and monodromy data. The upshot is that if we substitute the Klein solution into the formulae of appendix A below, then we obtain an isomonodromic family of 2×2 Fuchsian systems with monodromy data on one branch given by:

$$\theta = (2/7, 2/7, 2/7, 4/7), \quad \sigma = (1/2, 1/3, 1/2). \tag{11}$$

This determines a representation into (a lift to G of) the $(2, 3, 7)$ triangle group (and one may show as in Appendix B its image is not a proper subgroup). Moreover it was proved in [7] that this cannot be obtained by Okamoto transformations from a representation into a finite subgroup of $SL_2(\mathbb{C})$.

Fourthly one may obtain such representations by pulling back certain hypergeometric systems along certain rational maps (cf. Doran [8] and Kitaev [18]). This is closely related Klein's theorem that all second order Fuchsian equations with finite monodromy group are pull-backs of hypergeometric equations with finite monodromy. The basic idea is as follows.

Label two copies of $\mathbb{P}^1$ by u and d (for upstairs and downstairs). Choose four integers $n_0, n_1, n_\infty, N \geq 2$ and suppose we have an algebraic family of branched covers

$$\pi : \mathbb{P}^1_u \to \mathbb{P}^1_d$$

of degree N, parameterised by a curve Π say, such that:

1) π only branches at four points $0, 1, \infty$ and at a variable point $x \in \mathbb{P}^1_d$,

2) All but four of the ramification indices over $0, 1, \infty$ divide n_0, n_1, n_∞ respectively. In other words if $\{e_{i,j}\}$ are the ramification indices over $i = 0, 1, \infty$ then as j varies precisely four of the numbers

$$\frac{e_{0j}}{n_0}, \quad \frac{e_{1j}}{n_1}, \quad \frac{e_{\infty j}}{n_\infty}$$

are not integers. Let t be the cross-ratio of the corresponding four ramification points of $\mathbb{P}^1_u$, in some order, and so we have a coordinate on $\mathbb{P}^1_u$ such that these four points occur at $0, t, 1, \infty$.

3) π has minimal ramification over x, i.e. π ramifies at just one point over x, with ramification index 2.

The idea of [8], [18] is to take a hypergeometric system on $\mathbb{P}^1_d$ with projective monodromy around i equal to an n_i'th root of the identity, for $i = 0, 1, \infty$ and pull it back along π. One then obtains an isomonodromic family on $\mathbb{P}^1_u$ with non-apparent singularities at $0, t, 1, \infty$ and possibly some apparent singularities at the other ramification points. All of the apparent singularities can be removed, for example by applying suitable Schlesinger transformations, yielding an isomonodromic family of systems of the desired form.

In particular the problem of constructing algebraic solutions of P$_{VI}$ now largely becomes a purely algebraic problem about families of covers, although it is only conjectural that all algebraic solutions arise in this way.

However the algebraic construction of such covers seems difficult. First one has the topological problem of finding such covers, then one needs to find the full family of covers explicitly (this amounts to finding a parameterised solution of a large system of algebraic equations, typically with one less equation than the number of variables, so the solution is a curve). See [18] for some interesting examples however (but one should be aware that some of these solutions are equivalent to each other and to known solutions via Okamoto transformations).

Our perspective here is that just the topology of the cover is enough to determine the monodromy of the Fuchsian equation on $\mathbb{P}^1_u$ and we can then apply our previous method [7] to construct the explicit P$_{VI}$ solution. In other words just one topological cover π gives us the desired representation ρ living in a finite $\mathcal{F}_2$ orbit.

To find some interesting topological covers we consider the list appearing in Corollary 4.6 of [8]. Here Doran classified the possible ramification indices of the cover π in the cases where the monodromy group of the hypergeometric system downstairs is an arithmetic triangle group in $\mathrm{SL}_2(\mathbb{R})$. (Contrary to the wording in [8] this does not determine the topology of the cover.) We will content ourselves with looking at the last four entries of Doran's list, which say that the integers N, n_0, n_1, n_∞ and the ramification indices are

$$10, \ 2, 3, 7 \quad [2, \ldots, 2], \ [3, 3, 3, 1], \ [7, 1, 1, 1]$$
$$12, \ 2, 3, 7 \quad [2, \ldots, 2], \ [3, 3, 3, 3], \ [7, 2, 1, 1, 1]$$
$$12, \ 2, 3, 8 \quad [2, \ldots, 2], \ [3, 3, 3, 3], \ [8, 1, 1, 1, 1]$$
$$18, \ 2, 3, 7 \quad [2, \ldots, 2], \ [3, \ldots, 3], \ [7, 7, 1, 1, 1, 1].$$

The basic problem now is to find such covers topologically, in other words to find the possible permutation representations. (The cover of the four punctured sphere $\mathbb{P}^1_d \setminus \{x, 0, 1, \infty\}$ is determined by its monodromy, which amounts to four elements of Sym_N having product equal to the identity and whose conjugacy classes – i.e. cycle types – are as specified by the given ramification indices.)

The simplest way to do this is to draw a picture. Suppose we fix $x = -1$ and cut $\mathbb{P}^1_d$ along the interval $I := [-1, \infty]$ from -1 along the positive real axis. Then the preimage of I under π will be a graph in $\mathbb{P}^1_u$ with vertices at each point of $\pi^{-1}(\{x, 0, 1, \infty\})$. The complement of the graph will be the union of exactly N connected components

which are each mapped isomorphically by π onto $\mathbb{P}_d^1 \setminus I$, and in particular the boundary of each component is the same as the boundary of $\mathbb{P}_d^1 \setminus I$. These connected components correspond to the branches of π and the graph specifies how to glue them together. In particular the graph determines the permutation representation of the cover, since it shows us how to lift loops in the base to paths in $\mathbb{P}_u^1$; we just cross the corresponding edges upstairs, and note which connected component we end up in.

Thus we need to draw the graphs in $\mathbb{P}_u^1$. There are four types of vertices, depending on if they lie over $-1, 0, 1, \infty$, which we could draw as circles, squares, blobs and stars (say) respectively. The number of each type of vertices is just the number of points of $\mathbb{P}_u^1$ lying over the corresponding point amongst $-1, 0, 1, \infty$. The corresponding ramification indices give the number of edges coming out of each vertex to each of the neighbouring vertices, and our task is to join these edges together in a consistent manner.

For example for the first row of the above list, there should be 10 branches and, by examining the ramification indices, we see we need to draw a graph on $\mathbb{P}_u^1$ out of the following pieces:

- 8 circles with 1 edge emanating from each, and 1 circle with 2 edges,
- 5 squares with 4 edges,
- 3 blobs with 6 edges and one blob with 2 edges, and
- 1 star with 7 edges and 3 stars with 1 edge.

The graph should divide the sphere into 10 pieces and:

- Each edge from a circle should connect to a square,
- Two edges from each square should connect to a circle and the other two should connect to a blob (and, going around the square, the edges should alternate between going to circles and blobs),
- Similarly half the edges from each blob should connect to squares and, again alternating, the other half should connect to stars,
- Finally each edge from a star should connect to one of the blobs.

We leave the reader to draw such a graph (there are 15 possibilities).[3] Given any such graph we can write down the monodromy of the pulled back Fuchsian system on $\mathbb{P}_u^1$ in terms of that of the hypergeometric system downstairs. Here the projective monodromy downstairs is a $(2, 3, 7)$ triangle group:

$$\Delta_{237} \cong \langle\, a, b, c \mid a^2 = b^3 = c^7 = cba = 1 \,\rangle$$

which can be realised as a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ in various ways (the standard representation into $\mathrm{PSL}_2(\mathbb{R})$ plus its two Galois conjugates, lying in PSU_2).

Puncture $\mathbb{P}_u^1$ at the four exceptional vertices (namely the 3 stars with 1 edge and the blob with 2 edges) and choose generators $l_1, \ldots, l_4$ of the fundamental group of this punctured sphere, with $l_4 \circ \cdots \circ l_1$ contractible. Then we can compute the image

---

[3]To count the possibilities, one may use Theorem 7.2.1 in Serre's book [26] to count the number of such permutation representations, and then divide by conjugation action of the symmetric group, carefully computing the stabiliser. To find all possibilities we draw some and then apply the natural action of the pure three-string braid group to see if we get them all – here all 15 are braid equivalent.

under π of each l_i in $\mathbb{P}_d^1 \setminus \{0, 1, \infty\}$ and thereby write the monodromy of the pulled back system as words in $a, b, c \in \Delta_{237}$. With one such graph we obtained:

$$L_1 = caca^{-1}c^{-1}, \quad L_2 = c, \quad L_3 = c^{-1}a^{-1}cac, \quad L_4 = c^{-3}bc^3$$

where L_i is the projective monodromy around l_i. By construction $L_4 \ldots L_1 = 1$ in Δ_{237}. Now by choosing an embedding of Δ_{237} in $\mathrm{PSL}_2(\mathbb{C})$ we get $L_i \in \mathrm{PSL}_2(\mathbb{C})$ and we can lift each L_i to a matrix $M_i \in G$, (and possibly negate M_4 to ensure $M_4 \ldots M_1 = 1$). We obtain the representation ρ with data

$$\theta = (2/7, 2/7, 2/7, 1/3), \quad \sigma = (1/3, 1/3, 1/7).$$

This completes our task of producing a representation in a finite $\mathcal{F}_2$ orbit. Now we can apply our previous method to construct the corresponding $\mathrm{P_{VI}}$ solution. Immediately, by computing the $\mathcal{F}_2$ orbit of the conjugacy class of ρ, we find the solution has genus 1 and 18 branches, and that the parameters are not equivalent to those of any known solution. Moreover it turns out that Jimbo's asymptotic formula may be applied to 17 of the 18 branches, and the asymptotics on the remaining branch may be obtained by the lemma in section 7 of [5]. Using this we can get the solution polynomial F explicitly from the Puiseux expansions, and then look for a parameterisation of F. The result is:

2, 3, 7 solution, genus one, 18 branches, $\theta = (2/7, 2/7, 2/7, 1/3)$:

$$y = \frac{1}{2} - \frac{(3\,s^8 - 2\,s^7 - 4\,s^6 - 204\,s^5 - 536\,s^4 - 1738\,s^3 - 5064\,s^2 - 4808\,s - 3199)u}{4\,(s^6 + 196\,s^3 + 189\,s^2 + 756\,s + 154)(s^2 + s + 7)(s + 1)}$$

$$t = \frac{1}{2} - \frac{(s^9 - 84\,s^6 - 378\,s^5 - 1512\,s^4 - 5208\,s^3 - 7236\,s^2 - 8127\,s - 784)u}{432\,s(s + 1)^2(s^2 + s + 7)^2} \quad (12)$$

where

$$u^2 = s\,(s^2 + s + 7).$$

This solution is noteworthy in that currently there is no known relation to a Fuchsian system with finite monodromy group (one might speculate as to the existence of another realisation of $\mathrm{P_{VI}}$ in which this solution corresponds to such a Fuchsian system, but this is unknown).

For the other three entries on the excerpt of Doran's list above, we do not seem to get new solutions, but it is interesting to identify them in any case.

The second entry, a family of degree 12 covers, turns out to give the Klein solution of [7]. The explicit family of covers has been found more recently in [18], p. 27. There are 7 different graphs one could draw, one of which is symmetrical and they are all braid equivalent. Using one of these graphs we obtain, as above, the words:

$$L_1 = c^{-1}a^{-1}cac, \ L_2 = c^3aca^{-1}c^{-3}, \ L_3 = c^2aca^{-1}c^{-2}, \ L_4 = (aca)^{-1}c^2aca.$$

Choosing an appropriate embedding of Δ_{237} and lifting to G we obtain the representation ρ specified in (11). In particular this gives a convenient way to prove that the

projective monodromy group of the family of 2×2 Fuchsian systems determined by the Klein solution is Δ_{237}. We just need to show that the L_i generate all of Δ_{237}, which we will do in Appendix B below.

The third entry indicates a family of degree 12 covers along which one should pull back the $(2, 3, 8)$ triangle group. This time there are 7 graphs one could draw but they are not all braid equivalent, there are two P_3 orbits, distinguished by the fact that the monodromy group of the cover is either Sym_{12} or a group of order 1536. For the degenerate case one finds the P_{VI} solution has just two branches and is $y = t \pm \sqrt{t(t-1)}$ with parameters $\theta = (1, 1, 1, 7)/8$. (This is just the transform of the square root solution $y = \sqrt{t}$ under the Okamoto transformation $(y, t) \mapsto \left(\frac{y-t}{1-t}, \frac{t}{t-1} \right)$). The other case is more interesting; for one graph we obtain:

$$L_1 = aca^{-1}, \quad L_2 = c^{-2}a^{-1}cac^2, \quad L_3 = caca^{-1}c^{-1}, \quad L_4 = a^{-1}caca^{-1}c^{-1}a$$

where now a, b, c generate the $(2, 3, 8)$ triangle group:

$$\Delta_{238} \cong \langle\, a, b, c \mid a^2 = b^3 = c^8 = cba = 1 \,\rangle.$$

Now we can choose an embedding of Δ_{238} into $\mathrm{PSL}_2(\mathbb{C})$ and a lift to G (negating M_4 if necessary) such that we obtain the representation ρ with data

$$\theta = (3/8, 3/8, 3/8, 5/8), \quad \sigma = (1/2, 2/3, 1/3).$$

This is precisely that obtained above by applying an Okamoto transformation to the 16 branch octahedral solution (and gives a convenient way to prove, in Appendix B, that the projective monodromy group is Δ_{238}).

Finally there are 9 graphs corresponding to the last row of Doran's list, all braid equivalent. Even though the graphs are the most complicated in this case (and there is a quite attractive one with 4-fold symmetry), this case leads again to the 2-branch P_{VI} solution $y = t \pm \sqrt{t(t-1)}$ with the parameters $\theta = (1, 1, 1, 6)/7$ (and $\sigma = (1/2, 1/2, 5/7)$ on one branch).

In conclusion we should mention that we do not know any other finite $\mathcal{F}_2$ orbits of triples of elements of $\mathrm{SL}_2(\mathbb{C})$ (e.g. up to isomorphism as 'sets with $\mathcal{F}_2$-action'); so far they all either come from a finite subgroup or one of the two $(2, 3, 7)$ cases (the Klein solution or the genus one solution above).

Appendix A

Here are the explicit formula from [16] for the residue matrices A_i, of the isomonodromic family of Fuchsian systems corresponding to a P_{VI} solution $y(t)$ with parameters $\theta_1, \ldots, \theta_4$. The matrix entries are rational functions of $y, t, y' = \frac{dy}{dt}, \{\theta_i\}$. (Our coordinate x is denoted $\widetilde{z}$ in [16] and is related simply to p which is the usual dual variable to $y = q$ in the Hamiltonian formulation of P_{VI}. Also one should add

diag$(\theta_i, \theta_i)/2$ to our A_i to obtain that of [16].)

$$A_i := \begin{pmatrix} z_i + \theta_i/2 & -u_i z_i \\ (z_i + \theta_i)/u_i & -z_i - \theta_i/2 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

where

$$z_1 = y \frac{E - k_2^2(t+1)}{t\theta_4}, \quad z_2 = (y-t)\frac{E + t\theta_4(y-1)xk_2^2 - tk_1k_2}{t(t-1)\theta_4}$$

$$z_3 = -(y-1)\frac{E + \theta_4(y-t)x - k_2^2 t - k_1k_2}{(t-1)\theta_4},$$

$$x = p - \frac{\theta_1}{y} - \frac{\theta_2}{y-t} - \frac{\theta_3}{y-1}, \quad 2p = \frac{\theta_1 + (t-1)\,y'}{y} + \frac{\theta_2 - 1 + y'}{y-t} + \frac{\theta_3 - t\,y'}{y-1}$$

$$u_1 = \frac{y}{tz_1}, \quad u_2 = \frac{y-t}{t(t-1)z_2}, \quad u_3 = -\frac{y-1}{(t-1)z_3}$$

$$E = y(y-1)(y-t)x^2 + \big(\theta_3(y-t) + t\theta_2(y-1) - 2k_2(y-1)(y-t)\big)x + k_2^2 y - k_2(\theta_3 + t\theta_2)$$

$$k_1 = (\theta_4 - \theta_1 - \theta_2 - \theta_3)/2, \quad k_2 = (-\theta_4 - \theta_1 - \theta_2 - \theta_3)/2.$$

Appendix B

Proposition 6. *The 2×2 Fuchsian systems corresponding to the Klein solution and to the 18 branch genus 1 solution of Section 5 have projective monodromy group isomorphic to Δ_{237}, and that corresponding to the transformation of the 16 branch octahedral appearing in Section 5 has projective monodromy group isomorphic to Δ_{238}.*

Proof. Since in Section 5 the projective monodromy groups were expressed as words in the generators of the respective triangle groups, it is sufficient to check in each case that these words are in fact generators. To do this we will repeatedly use the fact that in the group

$$\Delta = \langle\, a, b, c \mid a^2 = b^3 = c^n = cba = 1 \,\rangle$$

one has $c^b = bc^{-1}$ and $c^{b^{-1}} = c^{-1}b$, where in general we write x^y for $y^{-1}xy$. (These are easily verified, for example the first is true since

$$b^{-1}cbc = b^{-1}(cb)(cb)b^{-1} = b^{-2} = b,$$

using the fact that $a = a^{-1} = cb$.) In particular we immediately deduce $\Delta = \langle c, c^b \rangle = \langle c, c^{b^{-1}} \rangle$.

Now each case is an easy exercise. For the Klein case we need to show that $\langle L_i, \ i = 1, 2, 3 \rangle = \Delta$ where $n = 7$ and

$$L_1 = c^{ac}, \quad L_2 = c^{ac^{-3}}, \quad L_3 = c^{ac^{-2}}.$$

Up to conjugacy in Δ, we have $\langle L_2, L_3 \rangle = \langle p, c \rangle$ where $p = c^{ac^{-1}a}$. However, using $a = cb$ we see $p = c^{cb^2} = c^{b^2} = c^{b^{-1}}$ so we are done.

For the other $(2, 3, 7)$ case corresponding to the genus one solution we have

$$L_1 = c^{ac^{-1}}, \quad L_2 = c, \quad L_3 = c^{ac}.$$

Thus $\langle L_2, L_3 \rangle = \langle c, c^a \rangle = \langle c, c^b \rangle$ since $a = cb$.

For the $(2, 3, 8)$ case we have $L_1 = c^a$, $L_2 = c^{ac^2}$, $L_3 = c^{ac^{-1}}$. Up to conjugacy $\langle L_1, L_3 \rangle = \langle c, c^{ac^{-1}a} \rangle$ and as in the Klein case above $c^{ac^{-1}a} = bcb^{-1}$. $\square$

Appendix C

At the request of the editors and of A. Kitaev, we will add some remarks to aid the reader interested in reproducing the results of this article. The main results are of two types: 1) classification of P_{VI} solutions coming from the binary tetrahedral and octahedral groups and 2) construction of explicit P_{VI} solutions using Jimbo's asymptotic formula. For both 1) and 2) the details are parallel to those described in [5] and [7] resp., with the precise references as in the body of this article. For 1) there are 3 steps:

• Prove that the relation of Okamoto equivalence is sandwiched between the relations of geometric and parametric equivalence, i.e. in symbols one has GE $\Rightarrow$ OE $\Rightarrow$ PE. The second arrow is immediate by definition and the first is proved in Lemma 9 of [5].

• Compute the parameter equivalence classes in the set of parameters coming from triples of generators of either the tetrahedral or octahedral group. This is as in section 3 of [5]. One first writes down the set of possible parameters θ. This is a finite subset of $\mathbb{Q}^4 \subset \mathbb{R}^4$. Then one uses a simple algorithm to move each of these points into a chosen affine F_4 alcove, using the standard action of the affine Weyl group $W_a(F_4)$ on $\mathbb{R}^4$ (this is entirely standard and the details are written in [5], Proposition 6). Then we count the number of distinct alcove points obtained. By definition this is the number of "parameter equivalence classes".

• Compute the geometric equivalence classes in the set of linear representations ρ coming from either the binary tetrahedral or octahedral group. This amounts to computing the orbits of an explicit action of a group on a finite set (of size 520, 3360 resp.) and is carefully described in section 4 of [5].

Some confidence that there is no computational error comes from the fact that the geometric and parametric equivalence classes turn out to coincide in both the cases considered here. Also Hall's formulae [11] (computing the number of generating triples) gives confidence that all the generating triples have been computed correctly – since we get the right number of them. (In principle one can go through all triples of elements of the finite group $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ and throw out those that do not generate Γ. In the two cases at hand this is feasible, but some simple tricks are useful in the icosahedral case.)

Now we will move on to 2), constructing the solutions. The main steps of the procedure used are as in [7] (see especially p. 193). However with experience various tricks have been developed to speed up the computation, so we will also detail some of these below (they are inessential if one has a fast enough computer, as presumably future readers will have). The underlying strategy is analogous to that used in [10] although we do not in fact use any of their results. (It was particularly troublesome to get the correct form of Jimbo's formula in [7], which is the main ingredient and was not used in [10].)

The basic steps are as follows:

1) We start with a linear representation ρ living in a finite mapping class group orbit. The conjugacy class of ρ is encoded in the seven-tuple

$$\theta_1, \ \theta_2, \ \theta_3, \ \theta_4, \ \sigma_{12}, \ \sigma_{23}, \ \sigma_{13}.$$

Specifying these seven numbers is equivalent to specifying the numbers $m_i = 2\cos(\pi\theta_i)$, $m_{ij} = 2\cos(\pi\sigma_{ij})$ provided we agree $\theta_i, \sigma_{ij} \in [0, 1]$. We compute the orbit of this 7-tuple under the pure mapping class group $\cong \mathcal{F}_2$. The formula for this action is given in [5], section 4 (cf. also (7) above). This gives a list, of length N say, of 7-tuples, one for each branch of the corresponding P_{VI} solution. The values of the θ's will not vary on different branches so the branches are parameterised by the values of the sigmas. Let their values on the kth branch be denoted $\sigma_{ij}^k, k = 1, \ldots, N$.

2) Plug each 7-tuple into Jimbo's asymptotic formula (in the form in [7], Theorem 4). This gives N leading terms $y_k = a_k t^{b_k} + \cdots$ for $k = 1, \ldots, N$ of the Puiseux expansion at 0 of the P_{VI} solution $y(t)$ on the N branches. One will have $b_k = 1 - \sigma_{12}^k$ but a_k is given by a very complicated, but explicit, formula. (Jimbo's formula is not always applicable – cf. the discussion of 'good' solutions in [5], but often there is an equivalent solution for which Jimbo's formula can be applied on every branch, or there is a degeneration of Jimbo's formula (as in [10] or [5], Lemma 19) which will compute the remaining leading terms.)

3) Compute lots of terms in the Puiseux expansions of the solutions $y(t)$ on each branch. These will be expansions in $s = t^{1/d_k}$ where d_k is the denominator of b_k (when written in lowest terms). Geometrically d_k is the number of branches of y that meet the given branch over $t = 0$, i.e. it is the cycle length of the given branch in the permutation representation of the solution curve as a cover of the t-line. The expansions are computed recursively by substituting back into the P_{VI} equation; at each step this leads to a linear equation for the coefficient of the next term in the expansion.

4) Use these expansions to determine the coefficients of the solution polynomial $F(y, t)$. (This determines y as an algebraic function of t by the condition $F(y, t) = 0$.) Since F is a polynomial (of degree N in y) this is clearly possible since we have arbitrarily many terms of each Puiseux expansion; $F(y_k(s), s^{d_k}) = 0$ for all s and for each branch y_k of the solution. (Thus in principle just one expansion is needed, not the expansion for all branches.) Given $F(y, t)$ one may check symbolically that it specifies a solution to P_{VI}, using implicit differentiation.

5) Compute a parameterisation of the resulting curve $F(y, t) = 0$. (As mentioned in the acknowledgments the author is grateful to Mark van Hoeij for help with this last step.) In general this will be simpler to write down than the polynomial F.

Now we will list some of the tricks we have found useful in carrying out the above steps.

1) One needs to convert the numbers a_k given by Jimbo's formula into algebraic numbers. In the examples so far this can be done by raising a_k (and/or its real/imaginary parts) to the d_k-th power until a rational number is obtained (which can be ascertained by looking at continued fraction expansions).

2) Reduce the number of Puiseux expansions to compute: the d_k branches which meet the given branch over zero will have Galois conjugate expansions. These can be obtained from one another by multiplying s by a d_k-th root of unity. Also, when choosing which of these d_k expansions to actually compute it is good to choose the real one, if possible. (Also sometimes some expansions are complex conjugate to others so further optimisations are possible.)

3) Reduce the degree of the field extension used to compute the expansion: In computing the Puiseux expansions one is often working over a finite extension of $\mathbb{Q}$, such as $\mathbb{Q}(6^{1/7})$. Often the degree of this extension can be reduced by taking the expansion in a variable $h = c \times s$ for a suitable constant c, rather than in $s = t^{1/d_k}$. This trick was very useful for computing the larger solutions (with ≥ 15 branches say).

4) To obtain the coefficients of the polynomial F from the Puiseux expansion we use the trick suggested in [10]: Write F in the form

$$F = q(t)y^N + p_{N-1}(t)y^{N-1} + \cdots + p_1(t)y + p_0(t)$$

for polynomials p_i, q in t and define rational functions $r_i(t) := p_i/q$ for $i = 0, \ldots, N-1$. If $y_1, \ldots, y_N$ denote the (locally defined) solutions on the branches then for each t we have that $y_1(t), \ldots, y_N(t)$ are the roots of $F(t, y) = 0$ and it follows that

$$y^N + r_{N-1}(t)y^{N-1} + \cdots + r_1(t)y + r_0(t) = (y - y_1(t))(y - y_2(t)) \ldots (y - y_N(t)).$$

Thus, expanding the product on the right, the rational functions r_i are obtained as symmetric polynomials in the y_i:

$$r_0 = (-1)^N y_1 \ldots y_N, \ldots, \quad r_{N-1} = -(y_1 + \cdots + y_N).$$

Since the r_i are global rational functions, the Puiseux expansions of the y_i give the Laurent expansions at 0 of the r_i. Clearly only a finite number of terms of each Laurent expansion are required to determine each r_i, and indeed it is simple to convert these truncated Laurent expansions into global rational functions. (This is easily done by Padé approximation, e.g. as implemented in the Maple command "convert(, ratpoly)".)

5) Much time may be saved by carefully choosing the representative for the solution in the first place (i.e. try to choose an equivalent solution for which the polynomial F is simpler). Heuristically this can be estimated by seeing how complicated the algebraic

numbers a_k are (or by seeing how complicated the coefficients of the polynomial $q(t)$ are; this is usually easily obtained from $(y_1 + \cdots + y_N)$, i.e. before having to compute complicated symmetric functions).

6) Use Okamoto symmetries wherever possible: e.g. if (we can arrange that) the solution has the symmetry $(y, t) \mapsto (y/t, 1/t)$, swapping θ_2 and θ_3 then the coefficients of each p_i, q should be symmetrical, thereby essentially halving the number of coefficients that need to be computed. (Also for the 24-branch icosahedral solution in [5] it was too cumbersome to compute the longest symmetric functions, corresponding to the 'middle' polynomials p_i, but, by using another Okamoto symmetry, the outstanding coefficients could be determined in terms of those we were able to compute.)

7) Finally there are various optimisations that can be made (especially in computing the symmetric functions of the Puiseux expansions) if we expect F to have integer coefficients (which is the case for all examples so far).

Acknowledgments. The author is very grateful to Mark van Hoeij for help computing the more difficult parameterisations of the curves $F = 0$, and to both C. Doran and A. Kitaev for explaining various aspects of their work.

After this work was complete A. Kitaev informed the author that he had found an explicit family of covers corresponding to the genus one $(2, 3, 7)$ solution of Section 5 and had obtained a similar solution. Happily the solution here and that of Kitaev are not related by Okamoto transformations, but arise by choosing different embeddings of Δ_{237} into $\mathrm{PSL}_2(\mathbb{C})$. In fact there are three inequivalent choices, corresponding to the three conjugacy classes of order 7 elements in $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$. (This is analogous to the sibling solutions of [5] which arose from the two classes of order 5 elements.) The third inequivalent P_{VI} solution is:

$$y = \frac{1}{2} - \frac{(s^{10} + 5\,s^9 + 24\,s^8 + 20\,s^7 - 266\,s^6 - 2874\,s^5 - 14812\,s^4 - 40316\,s^3 - 85359\,s^2 - 100067\,s - 67396)u}{16(s+1)(s^2 + s + 7)(5\,s^6 + 63\,s^5 + 252\,s^4 + 854\,s^3 + 1449\,s^2 + 1827\,s + 2030)}$$

with t, u, s exactly as in (12) and $\theta = (4/7, 4/7, 4/7, 1/3)$.

References

[1] F. V. Andreev and A. V. Kitaev, Transformations $RS_4^2(3)$ of the ranks ≤ 4 and algebraic solutions of the sixth Painlevé equation, *Comm. Math. Phys.* 228 (1) (2002), 151–176.

[2] D. V. Anosov and A. A. Bolibruch, *The Riemann-Hilbert problem*, Aspects Math. E 22, Friedr. Vieweg & Sohn, Braunschweig 1994.

[3] F. Baldassarri and B. Dwork, On second order linear differential equations with algebraic solutions, *Amer. J. Math.* 101 (1) (1979), 42–76.

[4] F. Beukers and A. van der Waall, Lamé equations with algebraic solutions, *J. Differential Equations* 197 (1) (2004), 1–25.

[5] P. P. Boalch, The fifty-two icosahedral solutions to Painlevé VI, *J. Reine Angew. Math.*, to appear; arXiv:math.AG/0406281, v.7.

[6] ——, Higher genus icosahedral Painlevé curves, arXiv:math.AG/0506407.

[7] ——, From Klein to Painlevé via Fourier, Laplace and Jimbo, *Proc. London Math. Soc.* 90 (3) (2005), 167–208.

[8] C. F. Doran, Algebraic and geometric isomonodromic deformations, *J. Differential Geom.* 59 (1) (2001), 33–85.

[9] B. Dubrovin, Geometry of 2D topological field theories, in *Integrable Systems and Quantum Groups* (M. Francaviglia and S. Greco, eds.), Lecture Notes in Math. 1620, Springer-Verlag, Berlin 1995, 120–348.

[10] B. Dubrovin and M. Mazzocco, Monodromy of certain Painlevé-VI transcendents and reflection groups, *Invent. Math.* 141 (1) (2000), 55–147.

[11] P. Hall, The Eulerian functions of a group, *Quart. J. Math. Oxford Ser.* 7 (1936), 134–151.

[12] N. J. Hitchin, Poncelet polygons and the Painlevé equations, in *Geometry and analysis* (Bombay, 1992), Tata Inst. Fund. Res., Bombay 1995, 151–185.

[13] ——, A lecture on the octahedron, *Bull. London Math. Soc.* 35 (2003), 577–600.

[14] M. Inaba, K. Iwasaki, and M.-H. Saito, Bäcklund transformations of the sixth Painlevé equation in terms of Riemann-Hilbert correspondence, *Internat. Math. Res. Notices* 2004 (1) (2004), 1–30.

[15] M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations, *Publ. Res. Inst. Math. Sci.* 18 (3) (1982), 1137–1161.

[16] M. Jimbo and T. Miwa, Monodromy preserving deformations of linear differential equations with rational coefficients II, *Physica 2D* 4 (1981), 407–448.

[17] N. M. Katz, Algebraic solutions of differential equations (*p*-curvature and the Hodge filtration), *Invent. Math.* 18 (1972), 1–118.

[18] A. V. Kitaev, Dessins d'enfants, their deformations and algebraic the sixth Painlevé and Gauss hypergeometric functions, nlin.SI/0309078, v.3.

[19] ——, Quadratic transformations for the sixth Painlevé equation, *Lett. Math. Phys.* 21 (2) (1991), 105–111.

[20] ——, Special functions of isomonodromy type, rational transformations of the spectral parameter, and algebraic solutions of the sixth Painlevé equation, *Algebra i Analiz* 14 (3) (2002), 121–139.

[21] Felix Klein, *Lectures on the icosahedron and the solution of equations of the fifth degree*, Dover Publications Inc., New York, N.Y., 1956.

[22] B. Malgrange, Sur les deformations isomonodromiques. I. Singularites regulieres, in *Mathématique et physique*, Séminaire E.N.S., Paris, 1979/1982 (L. Boutet de Monvel, A. Douady, and J.-L. Verdier, eds.), Progr. Math. 37, Birkhäuser, Boston, Mass., 1983, 401–426.

[23] M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum fields I-V, *Publ. Res. Inst. Math. Sci.* 14 (1978), 223–267; 15 (1979), 201–278; 15 (1979), 577–629; 15 (1979), 871–972; 16 (1980), 531–584.

[24] L. Schlesinger, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten, *J. für Math.* 141 (1912), 96–145.

[25] H. A. Schwarz, Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elements darstellt, *J. Reine Angew. Math.* 75 (1873), 292–335.

[26] J.-P. Serre, *Topics in Galois theory*, Res. Notes in Math. 1, Jones and Bartlett Publishers, Boston, Mass., 1992.

[27] M. van der Put and F. Ulmer, Differential equations and finite groups, *J. Algebra* 226 (2) (2000), 920–966.

[28] H. Watanabe, Birational canonical transformations and classical solutions of the sixth Painlevé equation, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 27 (3–4) (1998), 379–425.