Regge and Okamoto Symmetries

Philip P. Boalch

École Normale Supérieure et CNRS, 45 rue d'Ulm, 75005 Paris, France.
E-mail: boalch@dma.ens.fr

Abstract: We will relate the surprising Regge symmetry of the Racah-Wigner $6j$ symbols to the surprising Okamoto symmetry of the Painlevé VI differential equation. This then presents the opportunity to give a conceptual derivation of the Regge symmetry, as the representation theoretic analogue of the derivation in [5, 3] of the Okamoto symmetry.

1. Introduction

The $6j$-symbols (or Racah coefficients) are real numbers associated to the choice of six irreducible representations $V_a, \ldots, V_f$ of SU(2). They were first published in work of Racah [16] in 1942, and arise in the addition of three angular momenta, which classically can be viewed as adding three vectors in $\mathbb{R}^3$. Apparently ([15]) "there is hardly any branch of physics involving angular momenta where the use of Racah-coefficients is not needed in order to carry out the simplest computation". (See the two volumes [1, 2] for many more details or the introduction to the tables [20] for a concise summary.) Wigner [23] used a slightly different normalisation so that they have tetrahedral symmetry, and wrote them in the form:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix}. \tag{1}$$

Here a, b, c should be thought of as the lengths of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbb{R}^3$ so the four points $\mathbf{0}$, $\mathbf{a}$, $\mathbf{a} + \mathbf{b}$, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ are the vertices of a (skew) tetrahedron. Then d, e, f should be the lengths of the other three edges of this tetrahedron, i.e. the lengths of $\mathbf{a} + \mathbf{b} + \mathbf{c}$, $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ respectively. (Thus each column of (1) contains the lengths of two opposite edges, and the top row abe is a face. See Fig. 1.) Then the $6j$-symbol is invariant under the possible relabellings of this tetrahedron (preserving the relations so one gets $24 = \#\mathrm{Sym}_4$ possibilities).
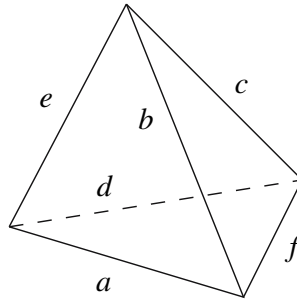
Fig. 1. Labelling of the edges of a skew tetrahedron

Racah established an explicit formula for the $6j$-symbols as a sum, which has since been equated with the value at 1 of certain $_4F_3$ hypergeometric functions (and are closely related to the Wilson orthogonal polynomials [24]).

Using this explicit Racah formula, in 1959 Regge [18] showed the $6j$-symbols also have the following further symmetry, which is more mysterious:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = \begin{Bmatrix} p-a & p-b & e \\ p-c & p-d & f \end{Bmatrix},$$

where $p = (a+b+c+d)/2$. (Combined with the tetrahedral symmetries this generates a symmetry group isomorphic to $\mathrm{Sym}_3 \times \mathrm{Sym}_4$.)

For example classically, recalling that a tetrahedron is determined up to isometry by its edge lengths, one may show (cf. Ponzano–Regge [15] and Roberts [19]) that this Regge action on the set of six edge lengths defines a non-trivial automorphism of the set of (skew) Euclidean tetrahedra, taking a generic tetrahedron to a *non-congruent* tetrahedron.

Earlier Regge [17] found similar extra symmetries of the Clebsch-Gordan $3j$-symbols. The $3j$-symbols are in a sense less canonical, but note that Ponzano–Regge [15] p.7 explain how to obtain the $3j$-symbols as an asymptotic limit of $6j$-symbols and that in this limit the $6j$ Regge symmetry becomes the $3j$ Regge symmetry. They then wrote:

> *The geometrical and physical content of these [Regge] symmetries is still to be understood and they remain a puzzling feature of the theory of angular momenta. Therefore it is a pleasant result to be able to reduce the problem of their interpretation to the Racah [6j] coefficient only.*

The basic aim of this article is to give a conceptual explanation of the Regge symmetry of the Racah $6j$ coefficient. The key idea is to relate the above $6j$-symbols (for the group $SU(2)$) to certain three-dimensional $6j$-symbols (i.e. for the group $SU(3)$). The Regge transformation then arises simply from the natural duality between two dual irreducible representations of $SU(3)$.

The layout of the remainder of this article is as follows. First we will give the definition of the $6j$-symbols, then we will relate the Regge symmetry to a symmetry of a completely different object, this time a non-linear differential equation, the Okamoto symmetry of the Painlevé VI equation. Then we will "quantise" (that is, give the representation theoretical analogue of) the derivation of the Okamoto symmetry given in [5, 3], and so give a conceptual derivation of the Regge symmetry (i.e. without using the Racah formula).

Remark 1. Note that Granovskiǐ and Zhedanov have shown in [8] that the $6j$ symbols are determined by a certain noncommutative algebra and so, as they show, finding the Regge symmetry may be reduced to finding the corresponding symmetry of this algebra.

2. Background

The $6j$-symbols are real numbers associated to the choice of 6 irreducible representations (irreps) of $G := \mathrm{SU}(2)$. We will label irreps V_a by positive integers $a \geq 0$, so that $V_a = \mathrm{Sym}^a(V)$ is the spin $a/2$ representation of dimension $a + 1$, where V is the two-dimensional Hermitian vector space defining G. Given 3 such irreps, with labels a, b, c say, one can form the tensor product

$$V_{abc} := V_a \otimes V_b \otimes V_c$$

which again will be a representation of G and will decompose as a direct sum of irreps. Thus given a fourth representation V_d one may consider the multiplicity space

$$M_{abcd} := \mathrm{Hom}_G(V_d, V_{abc})$$

of G-equivariant maps from V_d into the 3-fold tensor product. Thus M_{abcd} is a Hermitian vector space with dimension equal to the multiplicity of V_d in V_{abc}.

There are two (almost) canonical unitary bases in M_{abcd} ('coupling bases') and the $6j$-symbols arise as matrix entries of the change of basis matrix between these two bases. The coupling bases arise by decomposing V_{abc} in the two possible orders: on one hand we may first decompose $V_{ab} := V_a \otimes V_b$:

$$V_{ab} \cong \bigoplus_e V_e \otimes \mathrm{Hom}_G(V_e, V_{ab})$$

which entails the following direct sum decomposition of M_{abcd}:

$$M_{abcd} = \mathrm{Hom}_G(V_d, V_{ab} \otimes V_c) \cong \bigoplus_e \mathrm{Hom}_G(V_d, V_{ec}) \otimes \mathrm{Hom}_G(V_e, V_{ab}).$$

The key point is that each of the terms in this direct sum is either zero or one-dimensional (since, for SU(2), any irrep appears at most once in the tensor product of two irreps). Thus choosing a real vector of length one in each one dimensional term yields the 1-2 coupling basis $\{v_e\}$ of M_{abcd} as e varies, unique up to the sign of each basis vector. (We set $v_e = 0$ if the space $\mathrm{Hom}_G(V_d, V_{ec}) \otimes \mathrm{Hom}_G(V_e, V_{ab})$ is zero.) Similarly decomposing the 3-fold product in the other order, i.e. first writing

$$V_{bc} \cong \bigoplus_f V_f \otimes \mathrm{Hom}_G(V_f, V_{bc})$$

yields a different basis $\{w_f\}$, the 2-3 coupling basis, adapted to the decomposition

$$M_{abcd} = \bigoplus_f \mathrm{Hom}_G(V_d, V_{af}) \otimes \mathrm{Hom}_G(V_f, V_{bc}).$$

Thus given six irreps with labels a, b, c, d, e, f, and a standard sign-convention, one will get two vectors v_e, w_f in M_{abcd} and thus a number

$$U(a, b, c, d, e, f) = \langle v_e, w_f \rangle$$

by pairing them using the Hermitian form. (As e and f vary these will be the matrix entries of the unitary change of basis matrix alluded to above—in fact by reality it is real orthogonal.)

The $6j$-symbols were defined by Wigner in terms of U by a minor normalisation:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = (-1)^p \frac{U(a, b, c, d, e, f)}{\sqrt{(e+1)(f+1)}}, \quad p := (a+b+c+d)/2.$$

This normalisation is such that the $6j$-symbols admit tetrahedral symmetry, where the coefficients label the six edges of a tetrahedron (containing for example the quadrilateral $abcd$ and faces abe and bcf). Note that if U is non-zero then p will be an integer.

As mentioned in the introduction using the explicit Racah formula for the $6j$-symbols, Regge [18] showed the $6j$-symbols also have the following symmetry:

$$\begin{Bmatrix} a & b & e \\ c & d & f \end{Bmatrix} = \begin{Bmatrix} p-a & p-b & e \\ p-c & p-d & f \end{Bmatrix},$$

where $p = (a+b+c+d)/2$. (Since e, f are fixed, one may just as well view this as a symmetry of the corresponding function U.)

More geometrically one may also view this as a symmetry of the set of tetrahedra in $\mathbb{R}^3$ (cf. [15, 19]). First note that if V_e appears in $V_a \otimes V_b$ then the triangle inequalities

$$|a-b| \le e \le a+b$$

hold; i.e. there exists a Euclidean triangle with sides of lengths a, b, e. (One also has the parity condition that $a+b+e$ be even.) Thus if both vectors v_e and w_f are non-zero then there are four Euclidean triangles with side lengths abe, cde, adf, bcf respectively. Now these four triangles may or may not fit together to form the faces of a Euclidean tetrahedron; the condition that they do is given by requiring the determinant of the 'Cayley-Menger matrix':

$$\begin{pmatrix} 0 & a^2 & e^2 & d^2 & 1 \\ a^2 & 0 & b^2 & f^2 & 1 \\ e^2 & b^2 & 0 & c^2 & 1 \\ d^2 & f^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

to be positive. It is simple to check the Regge transformation preserves the set of all triangle inequalities (although permuting them in a non-trivial way). Moreover a computation will show that the determinant of the Cayley-Menger matrix is preserved too.

Thus one may view the Regge transformation as an automorphism of the set of Euclidean tetrahedra, even if we allow real (not necessarily integral) edge lengths (noting that a tetrahedron with non-zero volume is determined by its edge lengths up to isometry, possibly reversing the orientation).

Our first step in deriving the Regge symmetry is to note a remarkably similar symmetry of a completely different object, this time of a certain nonlinear differential equation.

The Painlevé VI differential equation (henceforth P_{VI}) is the following nonlinear ordinary differential equation for a holomorphic function $y(t)$ with $t \in \mathbb{C} \setminus \{0, 1\}$:

$$
\begin{aligned}
\frac{d^2 y}{dt^2} = {} & \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\
& + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \beta \frac{t}{y^2} + \gamma \frac{(t-1)}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2} \right),
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta$ are four complex constants. P_{VI} is usually thought of as controlling the monodromy preserving (isomonodromic) deformations of rank 2 (traceless) Fuchsian systems with 4 poles on $\mathbb{P}^1$ (whose monodromy is a representation of the free group on 3 generators into $SL_2(\mathbb{C})$). Okamoto [14] proved P_{VI} has a quite nontrivial symmetry:

Theorem 1. *Choose four complex constants* $\theta = (\theta_1, \ldots \theta_4)$ *and set*

$$
\alpha = (\theta_4 - 1)^2 / 2, \qquad \beta = -\theta_1^2 / 2, \qquad \gamma = \theta_3^2 / 2, \qquad \delta = (1 - \theta_2^2)/2.
$$

If y(t) is a solution of P_{VI} with parameters θ then, if defined,

$$
y + \phi/x
$$

solves P_{VI} with parameters

$$
\theta' = (\theta_1 - \phi, \theta_2 - \phi, \theta_3 - \phi, \theta_4 - \phi),
$$

where $\phi = \sum_1^4 \theta_i / 2$ and

$$
2x = \frac{(t-1)\, y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y - t} - \frac{t\, y' + \theta_3}{y - 1},
$$

where y' is the derivative of y with respect to t.

(Observe the striking similarity with the Regge transformation.)

In the next section we will describe exactly how the Okamoto and Regge symmetries are related (in effect showing precisely how the complicated Okamoto action on the pair (y, x) relates to the trivial Regge action on the pair (e, f)).

Remark 2. Since the parameters appearing in P_{VI} are now quadratic functions of the θ's, if y solves P_{VI} with parameters θ then y will also solve P_{VI} for any parameters obtained from θ by negating any combination of $\theta_1, \theta_2, \theta_3$ and possibly replacing θ_4 by $2 - \theta_4$. Together with the Okamoto transformation these four 'trivial' transformations generate a group isomorphic to the affine Weyl group of type D_4 (see [14]). Further one may add in transformations corresponding to the Sym_4 symmetry group of the affine D_4 Dynkin diagram and obtain a symmetry group isomorphic to the affine Weyl group of type F_4. The confusing fact to note is that one still does not obtain symmetries corresponding to all the tetrahedral $6j$ symmetries, basically because the P_{VI} flows vary y, x and fix the θ's.

3. Regge and Okamoto

We will relate the Regge action on Euclidean tetrahedra to the Okamoto action.

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$ be three vectors, so that $\mathbf{0}, \mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$ are the vertices of a tetrahedron. Denote the other three edge vectors of the tetrahedron by $\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6$ so that:

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4 = 0,$$

$$\mathbf{a}_5 = \mathbf{a}_1 + \mathbf{a}_2, \qquad \mathbf{a}_6 = \mathbf{a}_2 + \mathbf{a}_3.$$

Denote the lengths of these six vectors $\mathbf{a}_1, \ldots, \mathbf{a}_6$ by a, b, c, d, e, f respectively.

Now let $\mathcal{H}$ be the set of traceless 2×2 Hermitian matrices. Thus $\mathcal{H}$ is a real three-dimensional vector space and we give it a Euclidean inner product by defining

$$\langle A_1, A_2 \rangle := 2\mathrm{Tr}(A_1 A_2).$$

Thus we can view the tetrahedron as living in $\mathcal{H}$ by choosing an isometry $\varphi : \mathbb{R}^3 \cong \mathcal{H}$ such as

$$\varphi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x & y + iz \\ y - iz & -x \end{pmatrix}.$$

Then we set

$$A_j = \varphi(\mathbf{a}_j) \in \mathcal{H}$$

for $j = 1, \ldots, 6$. Thus the Regge symmetry becomes an action on the set of these 6 Hermitian matrices A_j (clearly determined by its action on the first three matrices).

Now in the standard isomonodromy interpretation [11] of the Painlevé VI equation the Okamoto symmetry becomes a (birational) action on the set of Fuchsian systems of the form

$$\frac{d}{dz} - A, \qquad A := \frac{A_1}{z} + \frac{A_2}{z - t} + \frac{A_3}{z - 1}, \qquad A_i \in \mathfrak{sl}_2(\mathbb{C}), \tag{2}$$

where the coefficients A_1, A_2, A_3 are 2×2 traceless complex matrices and $t \in \mathbb{C} \setminus \{0, 1\}$ fixed. As above we will set $A_4 = -(A_1 + A_2 + A_3)$ which is now the residue of $A dz$ at $z = \infty$. This interpretation comes about by using explicit local coordinates on the space of such systems; up to overall conjugation by $\mathrm{SL}_2(\mathbb{C})$, the set of such Fuchsian systems is of complex dimension 6 and local coordinates (near a generic system) are given by

$$\theta_1, \theta_2, \theta_3, \theta_4, x, y,$$

where θ_i is such that A_i has eigenvalues $\pm \theta_i / 2$, and where x, y are two explicit algebraic functions of A defined for example in [5] p.199, following [11]. (These are the same x, y appearing in Theorem 1.)

Of course one would prefer to view the birational transformation as the intrinsic object, and its explicit coordinate expression as secondary. In particular one might hope for a simpler expression than that given in terms of x, y in Theorem 1. One way to do this, which will be useful here, was observed in [5] Lemma 34. To describe this we should first modify slightly the matrices A_1, A_2, A_3: Let

$$\widehat{A}_i = A_i + \theta_i / 2 \qquad i = 1, 2, 3$$

so that $\widehat{A}_i$ has eigenvalues $0, \theta_i$ (i.e. it has rank one and trace θ_i).

Lemma 3. ([5] Lemma 34) *Each of the five expressions*

$$\text{Tr}(\widehat{A}_1\widehat{A}_2), \quad \text{Tr}(\widehat{A}_2\widehat{A}_3), \quad \text{Tr}(\widehat{A}_1\widehat{A}_3),$$

$$\text{Tr}(\widehat{A}_1\widehat{A}_2\widehat{A}_3), \quad \text{Tr}(\widehat{A}_3\widehat{A}_2\widehat{A}_1)$$

is preserved by the birational Okamoto transformation of Theorem 1.

This may be proved by a direct coordinate computation; the geometric origin of it is given in [5] (see especially Lemma 34, Remark 30) and may be thought of as the (complexification of the) classical analogue of the ideas we will use in the next section.

Now it is straightforward to prove (see [9]) that generically the first two of these expressions (viewed as functions on the set of Fuchsian systems) together with the four θ's make up a system of local coordinates. Let us write $\lambda_{12} = \text{Tr}(\widehat{A}_1\widehat{A}_2)$ and $\lambda_{23} = \text{Tr}(\widehat{A}_2\widehat{A}_3)$. Thus, *in these coordinates* the Okamoto transformation acts simply as

$$(\theta, \lambda_{12}, \lambda_{23}) \mapsto (\theta_1 - \phi, \theta_2 - \phi, \theta_3 - \phi, \theta_4 - \phi, \lambda_{12}, \lambda_{23})$$

with $\phi = \sum_1^4 \theta_i/2$, which looks even more like the Regge transformation, and easily yields the main result of this section:

Theorem 2. *Suppose A_1, A_2, A_3 are 2×2 traceless Hermitian matrices with eigenvalues $\pm\theta_i/2$ with $\theta_i > 0$ and let a, b, c, d, e, f be the edge lengths of the corresponding tetrahedron in $\mathbb{R}^3$ (under the isometry φ) i.e. the lengths of the vectors corresponding to the six matrices*

$$A_1, A_2, A_3, \quad A_1 + A_2 + A_3, \quad A_1 + A_2, \quad A_2 + A_3$$

respectively. Then the Okamoto transformation of (A_1, A_2, A_3) corresponds to the Regge transformation of the tetrahedron with edge lengths a, b, c, d, e, f.

Proof. For the first four edge lengths this is easy since $(a, b, c, d) = (\theta_1, \theta_2, \theta_3, \theta_4)$ (note that the triangle inequalities imply $\phi - \theta_i \geq 0$). We need to also show that the Okamoto transformation preserves the edge lengths e and f. But this is now a simple computation: First note

$$e^2 = 2\text{Tr}(A_1 + A_2)^2 = \theta_1^2 + \theta_2^2 + 4\text{Tr}A_1A_2.$$

Then observe $4\text{Tr}A_1A_2 = 4\text{Tr}(\widehat{A}_1 - \theta_1/2)(\widehat{A}_2 - \theta_2/2) = 4\text{Tr}\widehat{A}_1\widehat{A}_2 - 2\theta_1\theta_2$ so that

$$e^2 = (\theta_1 - \theta_2)^2 + 4\text{Tr}\widehat{A}_1\widehat{A}_2.$$

The first term on the right here is preserved, as is the second term by the lemma above, and so e is preserved since it is positive. Similarly for f. $\square$

Remark 4. Returning briefly to the complex (not-necessarily Hermitian) picture, the above argument implies that the Okamoto transformation is also characterised as preserving $\text{Tr}A_5^2$ and $\text{Tr}A_6^2$, where $A_5 = A_1 + A_2$ and $A_6 = A_2 + A_3$ (and similarly it preserves $\text{Tr}(A_1 + A_3)^2$).

Remark 5. (Spherical tetrahedra, cf. [21].) Consider three elements M_1, M_2, $M_3 \in \mathrm{SU}(2)$ of the 3-sphere $S^3 \cong \mathrm{SU}(2)$, and the spherical tetrahedron with vertices I, M_1, $M_1 M_2$, $M_1 M_2 M_3$. This has edge lengths l_i, where $\mathrm{Tr} M_i = 2 \cos(l_i)$ ($i = 1, \dots, 6$), where $M_4 = (M_1 M_2 M_3)^{-1}$, $M_5 = M_1 M_2$, $M_6 = M_2 M_3$. One may define a Regge symmetry of the set of such tetrahedra, by acting on the edge lengths exactly as before. This action may be complexified in the obvious way (allow $M_i \in \mathrm{SL}_2(\mathbb{C})$ and $l_i \in \mathbb{C}$). On the other hand the Okamoto action on the Fuchsian systems (2) induces an action on their monodromy data (i.e. essentially on the space of $\mathrm{SL}_2(\mathbb{C})$ representations of the fundamental group of the four-punctured sphere). The fact to be noted is that this action coincides with the above spherical Regge action (taking M_i to be the monodromy around the ith puncture for $i = 1, 2, 3, 4$); in other words the Okamoto action fixes the functions $\mathrm{Tr}(M_1 M_2)$ and $\mathrm{Tr}(M_2 M_3)$ of the monodromy data—this was the main result of [10], proved differently in Corollary 35 of [5].

4. Conceptual Regge Symmetry

We will give a conceptual derivation of the fact that the Regge symmetry preserves the $6j$-symbols (i.e. without using the Racah formula).

There are two basic steps:

- Identify the SU(2) $6j$ symbols with certain SU(3) $6j$-symbols,
- Use a natural symmetry of these SU(3) $6j$-symbols.

These are representation theoretic analogues of the derivation given in [5, 3] of the Okamoto symmetry. (The article [22] helped us to understand this—see also [4] where we first realised that [22] describes a representation theoretic analogue of some things the present author had been thinking about.)

First we will set up notation for representations of $H := \mathrm{SU}(3)$. Let W be the three-dimensional Hermitian vector space defining H. For any integer $a \geq 0$ write

$$W_a := \mathrm{Sym}^a(W)$$

for the ath symmetric power of W; an irreducible representation of H. Similarly for integers $a \geq b \geq 0$ write $W_{(a,b)}$ for the irrep of H corresponding to the Young diagram with 3 rows of lengths a, b, 0 resp. (cf. [6]). (Thus in particular $W_a = W_{(a,0)}$.)

One would like to define the SU(3) $6j$-symbols using the same framework as described above for SU(2). This is difficult though since SU(3) is *not* multiplicity-free and in general one will not obtain a decomposition of the multiplicity spaces into one-dimensional pieces, but into pieces of higher dimension. (There are numerous articles discussing this multiplicity problem, and methods to circumvent it.) However things are simpler if we take the three initial representations to be symmetric representations (i.e. of the form W_a). Then the Pieri rules imply one will again get the desired one-dimensional decomposition and we may proceed as before (and this is the only case we will need here).

Thus we choose 3 symmetric irreps, with labels a, b, c say, and form the tensor product

$$W_{abc} := W_a \otimes W_b \otimes W_c.$$

Now, given an arbitrary representation W_λ with $\lambda = (p, q)$ one obtains a multiplicity space as before

$$N_{abc\lambda} := \mathrm{Hom}_H(W_\lambda, W_{abc}).$$

Similarly to before the two expansions

$$W_a \otimes W_b \cong \bigoplus_{(r,s)} W_{(r,s)} \otimes \mathrm{Hom}_H(W_{(r,s)}, W_a \otimes W_b),$$

$$W_b \otimes W_c \cong \bigoplus_{(t,u)} W_{(t,u)} \otimes \mathrm{Hom}_H(W_{(t,u)}, W_b \otimes W_c)$$

yield two decompositions of the multiplicity space:

$$N_{abc\lambda} \cong \bigoplus_{(r,s)} \mathrm{Hom}_H(W_\lambda, W_{(r,s)} \otimes W_c) \otimes \mathrm{Hom}_H(W_{(r,s)}, W_a \otimes W_b),$$

$$N_{abc\lambda} \cong \bigoplus_{(t,u)} \mathrm{Hom}_H(W_\lambda, W_a \otimes W_{(t,u)}) \otimes \mathrm{Hom}_H(W_{(t,u)}, W_b \otimes W_c).$$

Now the Pieri rules (see [6]) imply that the tensor product of a symmetric representation with any irrep will be multiplicity free (i.e. each irrep that appears in the tensor product will appear exactly once). Thus both of these decompositions of $N_{abc\lambda}$ will be into one-dimensional pieces and by choosing real basis vectors of length one v_{rs}, w_{tu} in the corresponding pieces we can define matrix entries as before:

$$U^{(3)}(a, b, c, \lambda, (r, s), (t, u)) = \langle v_{rs}, w_{tu} \rangle.$$

(Again a sign-convention is needed to fix the signs of the basis vectors, but we will not worry about this here—the correct choices will be such that the following result is true.)

The basic fact that we will use is that these 3-dimensional $6j$ coefficients are equal to 2-dimensional $6j$ coefficients:

Proposition 6. *Choose six integers $a, b, c, d, e, f \geq 0$. Then [up to sign]*

$$U(a, b, c, d, e, f) = U^{(3)}(a, b, c, (p, q), (r, s), (t, u)),$$

where

$$p = \frac{a+b+c+d}{2}, \quad r = \frac{a+b+e}{2}, \quad t = \frac{b+c+f}{2},$$

$$q = p - d, \quad s = r - e, \quad u = t - f.$$

This will be proved below (it is probably quite well-known). First we will describe the desired three-dimensional symmetry and deduce the Regge symmetry:

Proposition 7. *The three-dimensional $6j$-symbol is symmetric as follows:*

$$U^{(3)}(a, b, c, (p, q), (r, s), (t, u))$$
$$= U^{(3)}(p - a, p - b, p - c, (p, p - q), (p - s, p - r), (p - u, p - t)).$$

Note that this does indeed project back to give the Regge symmetry[1]. Thus our task is reduced to justifying the above two propositions. At first sight it may appear that little progress has been made, replacing the Regge symmetry by the symmetry of Proposition 7. But as we shall see, this symmetry arises simply by pairing two dual representations of SU(3). [It is not however simply a matter of dualising all the representations in sight, since the dual of W_a is not a symmetric representation, for $a > 0$.]

Proof (of Proposition 6). The first step is to identify the corresponding weight spaces M_{abcd} and $N_{abc\lambda}$. This can be done easily using "Howe duality for GL_k-GL_n", as follows (cf. [25, 22])[2]. Choose two positive integers k, n and let V, W be complex vector spaces of dimensions k, n respectively. Then their tensor product $V \otimes W$ is a representation of $GL(V) \times GL(W)$ and so its dth symmetric power $\text{Sym}^d(V \otimes W)$ is also a $GL(V) \times GL(W)$-module. This decomposes as a direct sum of irreducible $GL(V) \times GL(W)$-modules in the following way (see [6] Exercise 6.11; apparently this goes back to Cauchy):

$$\text{Sym}^d(V \otimes W) \cong \bigoplus_{\substack{\#\lambda \leq k,n \\ |\lambda|=d}} V_\lambda \otimes W_\lambda,$$

where the sum is over all Young diagrams λ with d boxes and having no more than k or n rows, and V_λ (resp. W_λ) is the irreducible $GL(V)$-module (resp. $GL(W)$-module) corresponding to λ. Choosing bases of V and W allows us to be more explicit. In particular it identifies $GL(V) \cong GL_k(\mathbb{C})$ and so picks out a maximal torus (the diagonal subgroup), as well as a Borel subgroup (the upper triangular subgroup) and so allows us to speak of weights and highest weight vectors of $GL(V)$ modules (similarly for $GL(W)$). Also we can now view $V \otimes W$ as the space of linear functions ψ on the set $M_{k\times n}$ of $k \times n$ matrices $X = (x_{ij})$ with the following action of $GL_k(\mathbb{C}) \times GL_n(\mathbb{C})$:

$$(g_k, g_n)(\psi)(X) = \psi(g_k^T X g_n).$$

(To avoid confusion below when $k = n = 3$, we will refer to this $GL_k(\mathbb{C})$-action, respectively $GL_n(\mathbb{C})$-action, as the action on the left, resp. right.) Then $\text{Sym}^d(V \otimes W) = \text{Sym}^d(M_{k\times n}^*)$ is the set of such functions which are homogeneous polynomials of degree d, with the same action. Thus as $GL_k(\mathbb{C}) \times GL_n(\mathbb{C})$-modules,

$$\text{Sym}^d(M_{k\times n}^*) \cong \bigoplus_{\substack{\#\lambda \leq k,n \\ |\lambda|=d}} V_\lambda^{(k)} \otimes V_\lambda^{(n)}, \tag{3}$$

where $V_\lambda^{(r)}$ is the $GL_r(\mathbb{C})$ irrep. with Young diagram λ.

Now, as a GL_k module $\text{Sym}^\bullet(M_{k\times n}^*)$ can be viewed as the tensor product of the functions on each of the columns of X, i.e. $\text{Sym}^\bullet(M_{k\times n}^*) \cong \bigotimes_1^n \text{Sym}^\bullet \mathbb{C}^k$. Moreover if we choose n positive integers $\mu = (\mu_1, \ldots, \mu_n)$ then we can consider the subspace:

$$S^\mu \mathbb{C}^k := \text{Sym}^{\mu_1} \mathbb{C}^k \otimes \cdots \otimes \text{Sym}^{\mu_n} \mathbb{C}^k \subset \text{Sym}^\bullet(M_{k\times n}^*)$$

---

[1] Inverting the equations appearing in Proposition 6 yields $d = p - q, e = r - s, f = t - u$ so that Proposition 6 implies $U^{(3)}(p - a, \ p - b, \ p - c, \ (p, \ p - q), \ (p - s, \ p - r), \ (p - u, \ p - t)) = U(p - a, \ p - b, \ p - c, \ q, \ r - s, \ t - u)$ and so, since $q = p - d, \ r - s = e, \ t - u = f$ we do obtain the Regge symmetry $U(a, \ b, \ c, \ d, \ e, \ f) = U(p - a, \ p - b, \ p - c, \ p - d, \ e, \ f)$ as desired.

[2] To proceed explicitly (using similar ideas) see [13, 7].

of functions which are homogeneous of degree μ_i in the ith column. This is equivalent to saying they are the vectors of weight μ for the GL_n action on $\text{Sym}^\bullet(M^*_{k \times n})$.

Now what we are really interested in are the GL_k multiplicity spaces of the form:

$$M^\mu_\lambda := \text{Hom}_{\text{GL}_k}(V^{(k)}_\lambda, S^\mu \mathbb{C}^k)$$

whose dimension is the multiplicity of $V^{(k)}_\lambda$ in this n-fold product of symmetric representations. This multiplicity space may be realised explicitly as the subspace of $S^\mu \mathbb{C}^k$ of vectors of highest weight λ for the GL_k action (since each copy of $V^{(k)}_\lambda$ in $S^\mu \mathbb{C}^k$ has a unique highest weight vector, up to scale). On the other hand by the decomposition (3) the subspace of $\text{Sym}^\bullet(M^*_{k \times n})$ of vectors of highest weight λ for the GL_k action is a single copy of $V^{(n)}_\lambda$. Intersecting this with the subspace $S^\mu \mathbb{C}^k$ (i.e. the vectors with $\text{GL}_n(\mathbb{C})$ weight μ) yields the basic result we need (cf. [22] Lemma 3.4):

Lemma 8. *The above discussion gives an isomorphism* $V^{(n)}_\lambda[\mu] \cong M^\mu_\lambda$ *between the weight μ subspace of the* $\text{GL}_n(\mathbb{C})$ *representation* $V^{(n)}_\lambda$ *and the* $\text{GL}_k(\mathbb{C})$ *multiplicity space* M^μ_λ.

In the $6j$-symbol situation, we are interested in three-fold tensor products of symmetric representations, so $n = 3$, and our Young diagrams always have at most two non-zero rows, *so we may take any $k \geq 2$. For $k = 2$, Lemma 8 implies*

$$M_{abcd} \cong V^{(3)}_\lambda[\mu],$$

where $\mu = (a, b, c)$ and where λ has at most 2 rows and is such that $V^{(2)}_\lambda \cong V_d$ as $\text{SU}(2)$ representations and the number of boxes in λ equals $a+b+c$. This implies $\lambda = (p, q, 0)$, where $p + q = a + b + c$ and $p - q = d$, i.e. $p = (a+b+c+d)/2$ and $q = p - d$ as in the statement of Proposition 6. In turn, for $k = 3$, Lemma 8 implies

$$V^{(3)}_\lambda[\mu] \cong N_{abc\lambda},$$

and so combining the two gives the desired isomorphism of multiplicity spaces. (Explicitly if we view the 2×3 matrices as the first two rows of the 3×3 matrices, then these multiplicity spaces are actually equal as spaces of polynomial functions on 3×3 matrices since in both cases they do not depend on the variables in the third row, as λ has at most two non-zero rows.)

Remark 9. For the reader familiar with Gelfand-Tsetlin tableaux, we should mention that the weight space $V^{(3)}_\lambda[\mu]$ (and thus the common multiplicity space) admits a basis parameterised by tableaux of the form

$$\begin{pmatrix} p & & q & & 0 \\ & \alpha & & \beta & \\ & & \gamma & & \end{pmatrix},$$

where α, β, γ are integers satisfying interlacing inequalities: $p \geq \alpha \geq q \geq \beta \geq 0$, $\alpha \geq \gamma \geq \beta$ and should be such that the tableau has 'weight' $\mu = (a, b, c)$—the weight of a tableau is the differences of the row sums, i.e. we require $\gamma = a$, $\alpha + \beta - \gamma = b$, $p+q-(\alpha+\beta) = c$. This gives a simple way to compute the dimension of the multiplicity spaces (i.e. count the tableaux) although we will not need to use this Gelfand-Tsetlin basis (in general it does not coincide with any of the coupling bases).

To complete the proof of Proposition 6 we need to see the corresponding coupling bases match up under the above isomorphism of multiplicity spaces. (One may use the Bargmann-Segal-Fock Hermitian form on $\mathrm{Sym}^\bullet(M^*_{k\times n})$ and so see the Hermitian forms coincide.) One way to do this is to first observe that the 1-2 coupling spaces are the eigenspaces of the $\mathrm{GL}_k(\mathbb{C})$ quadratic Casimir operator $\mathcal{C}_k$ acting on the first two tensor factors of $S^\mu\mathbb{C}^k = \mathrm{Sym}^a\mathbb{C}^k \otimes \mathrm{Sym}^b\mathbb{C}^k \otimes \mathrm{Sym}^b\mathbb{C}^k$, where $\mu = (a, b, c)$ and $k = 2$ or 3. This holds since in general $\mathcal{C}_k$ acts (see e.g. [25] p.161) by multiplication by the scalar

$$\sum m_i^2 + \sum_{i<j} m_i - m_j \tag{4}$$

on the GL_k irrep. with Young diagram $(m_1, \ldots, m_k)$. Denote this operation by $\mathcal{C}_k^{12}$ and note it preserves the common multiplicity space

$$M_{abcd} = N_{abc\lambda} \subset S^\mu\mathbb{C}^2 \subset S^\mu\mathbb{C}^3$$

for $k = 2, 3$. Then we just observe, using (4), that on the multiplicity space the Casimirs differ by a scalar: $\mathcal{C}_3^{12} = \mathcal{C}_2^{12} + (a + b)$ and so have the same eigenspaces (and the eigenspace labels match up as stated, that is: $r - s = e, r + s = a + b$). (Similarly for the 2-3 coupling.) □

Next we will describe the three-dimensional symmetry which lifts the Regge symmetry: If $V_\lambda^{(3)}$ is the irrep of $\mathrm{GL}_3(\mathbb{C})$ with Young diagram $\lambda = (p, q, 0)$ then there is a pairing

$$V_\lambda^{(3)} \otimes (V_\lambda^{(3)})^\vee \to \mathbb{C},$$

where $(V_\lambda^{(3)})^\vee$ is the dual representation. Tensoring this by the pth power D^p of the determinant representation yields a pairing $V_\lambda^{(3)} \otimes V_{\lambda'}^{(3)} \to D^p$, where $\lambda' = (p, p - q, 0)$. This pairs the $\mu = (a, b, c)$ weight space of $V_\lambda^{(3)}$ with the μ' weight space of $V_{\lambda'}^{(3)}$, where $\mu' = (p - a, p - b, p - c)$ and so yields a perfect pairing:

$$V_\lambda^{(3)}[\mu] \otimes V_{\lambda'}^{(3)}[\mu'] \to \mathbb{C}. \tag{5}$$

Now to prove Proposition 7 one just needs to check that the corresponding coupling bases are dual with respect to this pairing.

Remark 10. On the level of Gelfand-Tsetlin tableaux the above duality corresponds to negating each tableau element then adding p to each element and finally flipping the tableau about its vertical axis. (Observe that the tableau's weight has transformed as stated.)

Proof (of Proposition 7). We will first describe the above pairing in a different way which will be more convenient. Write $G = \mathrm{GL}_3(\mathbb{C})$ and let $S^a = \mathrm{Sym}^a\mathbb{C}^3$. By the Pieri rules there is a unique G-equivariant map

$$S^a \otimes S^{p-a} \to S^p$$

since S^p appears precisely once in this tensor product[3]. (Similarly replacing a by b or c.) Putting these together there is a G-equivariant map

$$S^a S^b S^c \otimes S^{p-a} S^{p-b} S^{p-c} \to S^p S^p S^p$$

pairing the corresponding factors (and omitting to write several $\otimes$ symbols). Again by the Pieri rules there is a unique (projection) map $S^p S^p S^p \to D^p$ to the pth power of the determinant representation. Composing with the above map we get a (degenerate) bilinear form:

$$\nu : S^a S^b S^c \otimes S^{p-a} S^{p-b} S^{p-c} \to D^p.$$

In terms of pairs of polynomials on $M_{3\times 3}$ (viewing $S^\alpha S^\beta S^\gamma$ as polynomials homogeneous of degrees α, β, γ in the columns $1, 2, 3$ resp. as above) this bilinear form ν amounts to multiplication followed by orthogonal projection onto D^p (which is just the line spanned by the pth power of the polynomial det : $M_{3\times 3} \to \mathbb{C}$).

Now we wish to relate ν to the natural pairing $V_\lambda^{(3)}[\mu] \otimes V_{\lambda'}^{(3)}[\mu'] \to \mathbb{C}$ of (5). For this we view $V_\lambda^{(3)}$ as a space of polynomials on the 3×3 matrices as in Lemma 8 (as the polynomials with highest weight λ for the GL_3-action on the left), but we view $V_{\lambda'}^{(3)}$ differently as the space of polynomials with *lowest* weight $(0, p-q, p)$ for the GL_3-action on the left. Then multiplication of functions followed by orthogonal projection onto D^p gives a pairing $V_\lambda^{(3)} \otimes V_{\lambda'}^{(3)} \to D^p$ which one may check is nonzero directly (observing that a highest weight vector of $V_\lambda^{(3)}$ pairs non-degenerately with a lowest weight vector of $V_{\lambda'}^{(3)}$). Thus by Schur's lemma this pairing coincides with the natural one up to scale. Restricting to the μ and μ' weight spaces respectively and using Lemma 8 thus yields a non-degenerate pairing

$$N_{\mu\lambda} \otimes N_{\mu'\lambda'} \to \mathbb{C} \tag{6}$$

which is a restriction of ν. (To identify $N_{\mu'\lambda'} \cong V_{\lambda'}^{(3)}[\mu']$ we use the analogue of Lemma 8 with "highest weight" replaced by "lowest weight" throughout the proof.) Now we need to show that, with respect to the pairing (6), the 1-2 coupling bases on each side are dual (and similarly for the 2-3 bases). For this it is sufficient to prove the non-corresponding coupling basis vectors are orthogonal; the non-degeneracy of the pairing then forces the corresponding coupling basis vectors to pair up. We will show this for the 1-2 coupling (the other coupling being analogous).

Write $W_{(x,y)}$ for the irrep. of G with Young diagram $(x, y, 0)$. By the Pieri rules there is a unique map $S^p S^p \to W_{(p,p)}$ and so we have a G-equivariant map,

$$S^a S^b S^{p-a} S^{p-b} \to S^p S^p \to W_{(p,p)}. \tag{7}$$

This enables us to factor ν as follows:

$$S^a S^b S^c S^{p-a} S^{p-b} S^{p-c} \to W_{(p,p)} S^c S^{p-c} \to W_{(p,p)} S^p \to D^p.$$

To see how the coupling subspaces pair up, first expand both $S^a S^b$ and $S^{p-a} S^{p-b}$ into sums of irreps, so $S^a S^b S^{p-a} S^{p-b}$ becomes a sum of tensor products of the form

---

[3] More precisely there is a unique subspace of $S^a S^{p-a}$ which is G-equivariantly isomorphic to S^p, and thus a (unique) orthogonal projection onto this subspace. To lighten the notation we will call this subspace S^p here (and similarly below).

$W_{(r,s)} \otimes W_{(x,y)}$, where $W_{(r,s)} \subset S^a S^b$ etc., and (7) maps these to $W_{(p,p)}$. However using the Littlewood–Richardson rule it is easy to see that there is a nonzero map $W_{(r,s)} \otimes W_{(x,y)} \to W_{(p,p)}$ if and only if $x = p - s$, $y = p - r$ (and if so it is unique up to scale). This gives the stated correspondence between the 1-2 coupling bases (in the fifth slot of $U^{(3)}$).   □

Remark 11. Presumably the argument above extends to the Regge symmetry of the q-deformation of the $6j$-coefficients [12].

References

1. Biedenharn, L.C., Louck, J.D.: Angular momentum in quantum physics. In: *Encyclopedia of Mathematics and its Applications*, Vol. **8**, Reading, MA:Addison-Wesley, 1981
2. Biedenharn, L.C., Louck, J.D.: The Racah-Wigner algebra in quantum theory. In: *Encyclopedia of Mathematics and its Applications*, Vol. **9**, Reading, MA:Addison-Wesley, 1981
3. Boalch, P.P.: *Six results on Painlevé VI.* SMF, Séminaires et congrès, Vol. **14**, to appear, available at http://arxiv.org/abs/math/0503043, 2005
4. Boalch, P.P.: G-bundles, isomonodromy and quantum Weyl groups. Int. Math. Res. Not. no. 22, 1129–1166, (2002)
5. Boalch, P.P.: From Klein to Painlevé via Fourier, Laplace and Jimbo. Proc. London Math. Soc. **90**(3), 167–208 (2005)
6. Fulton W., Harris J.: *Representation theory.* GTM, Vol. **129**, Berlin Heidelberg-New York:Springer, 1991
7. Gliske, S., Klink, W.H., Ton-That, T.: Algorithms for computing generalized U(N) Racah coefficients. Acta Appl. Math. **88**(2), 229–249 (2005)
8. Granovskiĭ, Ya.A., Zhedanov, A.S.: Nature of the symmetry group of the $6j$-symbol. Sov. Phys. JETP **67**(10), 1982–1985 (1988)
9. Hitchin, N.J.: Geometrical aspects of Schlesinger's equation. J. Geom. Phys. **23**(3–4), 287–300 (1997)
10. Inaba, M., Iwasaki, K., Saito, M.-H.: Bäcklund transformations of the sixth Painlevé equation in terms of Riemann-Hilbert correspondence. I. M. R. N. **2004**(1), 1–30 (2004)
11. Jimbo, M., Miwa, T.: Monodromy preserving deformations of linear differential equations with rational coefficients II. Physica **2D**, 407–448 (1981)
12. Kirillov, A.N., Yu, N.: Reshetikhin Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links. In: Adv. Ser. Math. Phys., Vol. **7**, V. Kac, ed., reprinted in: Adv. Ser. Math. Phys. Vol. **11**, T. Kohno, ed., Singapore:World Scientific, 1989, pp. 285–339, 202–256 resp.
13. Klink, W.H., Ton-That, T.: Calculation of Clebsch-Gordan and Racah coefficients using symbolic manipulation programs. J. Comput. Phys. **80**(2), 453–471 (1989)
14. Okamoto, K.: Studies on the Painlevé equations. I. Sixth Painlevé equation P_{VI}. Ann. Mat. Pura Appl. (4) **146**, 337–381 (1987)
15. Ponzano, G., Regge, T.: Semiclassical limit of Racah coefficients. In: *Spectroscopic and group theoretical methods in physics*, Bloch, F. ed., NewYork: John Wiley and Sons, 1968, pp. 1–58
16. Racah, G.: Theory of complex spectra II. Phys. Rev. **62**, 438–462 (1942)
17. Regge, T.: Symmetry properties of Clebsch–Gordan coefficients. Il Nuovo Cimento **X**, 544–545 (1958)
18. Regge, T.: Symmetry properties of Racah's coefficients. Il Nuovo Cimento **XI**, 116–117 (1959)
19. Roberts, J.: Classical $6j$-symbols and the tetrahedron. Geom. Topol. **3**, 21–66 (1999) (electronic)
20. Rotenberg, M., Bivins, R., Metropolis, N., Wooten, J.K.: *The 3-j and 6-j symbols*. London: Crosby Lockwood and Son Ltd, 1959
21. Taylor, Y., Woodward, C.: $6j$ symbols for $U_q(sl_2)$ and non-Euclidean tetrahedra. Selecta Math. **11**, 539–571 (2005)
22. Toledano Laredo, V.: A Kohno-Drinfeld theorem for quantum Weyl groups. Duke Math. J. **112**(3), 421–451 (2002)
23. Wigner, E.P.: *On the matrices which reduce the Kronecker products of representations of S.R. groups.* (1940). Published in: Quantum theory of angular momentum L.C. Biedenharn, ed., New York: Acad. Press, 1965
24. Wilson, J.A.: Some hypergeometric orthgonal polynomials. SIAM J. Math. Anal. **11**(4), 690–701 (1980)
25. Želobenko, D.P.: *Compact Lie groups and their representations.* A.M.S. Trans. Math. Monog. Vol. **40**, Providence, RI: Amer. Math. Soc, 1973