QUASI-HAMILTONIAN GEOMETRY OF MEROMORPHIC CONNECTIONS

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Abstract

For each connected complex reductive group G, we find a family of new examples of complex quasi-Hamiltonian G-spaces with G-valued moment maps. These spaces arise naturally as moduli spaces of (suitably framed) meromorphic connections on principal G-bundles over a disk, and they generalise the conjugacy class example of Alekseev, Malkin, and Meinrenken [3] (which appears in the simple pole case). Using the "fusion product" in the theory, this gives a finite-dimensional construction of the natural symplectic structures on the spaces of monodromy/Stokes data of meromorphic connections over arbitrary genus Riemann surfaces, together with a new proof of the symplectic nature of isomonodromic deformations of such connections.

1. Introduction

The quasi-Hamiltonian approach [3] to constructing symplectic moduli spaces of flat connections on G-bundles over surfaces involves "fusing" together some basic pieces and then using a reduction procedure to obtain the symplectic moduli space. Just two types of such basic quasi-Hamiltonian G-spaces are needed to construct all the moduli spaces considered in [3]: conjugacy classes $\mathscr{C} \subset G$ and the *internally fused double* $\mathbf{D} \cong G \times G$. Indeed, given a genus g surface Σ with m boundary components, the quasi-Hamiltonian reduction

$$(\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \mathscr{C}_1 \circledast \cdots \circledast \mathscr{C}_m) /\!\!/ G \tag{1}$$

of the quasi-Hamiltonian fusion of g copies of $\mathbf{D}$ and m conjugacy classes $\mathscr{C}_i$ has a symplectic structure and is isomorphic to the moduli space $\mathrm{Hom}_{\mathscr{C}}(\pi_1(\Sigma), G)/G$ of representations of the fundamental group of Σ with holonomy around the ith boundary component restricted to lie in $\mathscr{C}_i$. Such symplectic moduli spaces have been much studied, and in particular, there are alternative finite-dimensional constructions

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(see [14], [18], [13], [23], [17], [4], [16]). A beautiful feature of the approach of [3] is that the quasi-Hamiltonian moment map condition in the reduction (1) is precisely the monodromy relation in $\mathrm{Hom}_{\mathscr{C}}(\pi_1(\Sigma), G)$.

The aim of this article is to use the quasi-Hamiltonian approach to give a finite-dimensional construction of the natural symplectic structures on more general moduli spaces where there is currently no other finite-dimensional method. This is a continuation of [10], where the Atiyah-Bott infinite-dimensional approach to moduli spaces of flat connections was extended to allow certain singularities in the connections, thereby constructing symplectic structures on moduli spaces of flat singular connections. (Such moduli spaces are naturally isomorphic both to spaces of meromorphic connections with arbitrary order poles over Riemann surfaces and to spaces of monodromy/Stokes data, naturally generalising the space of fundamental group representations noted in the preceding paragraph.)

Due to the quasi-Hamiltonian fusion procedure (which, on the level of surfaces, amounts to gluing two surfaces with one boundary component into two of the holes of a three-holed sphere), we need to understand only moduli spaces of meromorphic connections on G-bundles over a disk having just one pole. This leads to an infinite family of new basic pieces parameterised by the pole order k. These may then be fused together (and with some copies of $\mathbf{D}$) to construct the more general moduli spaces.

The main result of this article is divided into two parts. First, we construct algebraically some new quasi-Hamiltonian G-spaces. Then, we show that these spaces arise from certain Hamiltonian loop group manifolds of framed meromorphic connections over a disk. (These Hamiltonian loop group spaces arise from the extension of the Atiyah-Bott symplectic structure to the case of singular C^∞-connections given in [10].) Of course, historically the two parts were done in the opposite order; they have been reversed here for pedagogical reasons.

In more detail, our quasi-Hamiltonian spaces are as follows. Let $k \geq 2$ be an integer, and suppose that $U_\pm$ are the full unipotent subgroups of a pair of opposite Borels $B_\pm \subset G$ intersecting in a maximal (complex) torus $T = B_+ \cap B_-$ of G. The main space of interest to us is

$$\widetilde{\mathscr{C}} := G \times (U_+ \times U_-)^{k-1} \times \mathfrak{t},$$

where $\mathfrak{t}$ is the Lie algebra of T. This has an action of $G \times T$ defined by

$$(g, t) \cdot (C, S_1, \ldots, S_{2k-2}, \Lambda) = (tCg^{-1}, tS_1t^{-1}, \ldots, tS_{2k-2}t^{-1}, \Lambda) \in \widetilde{\mathscr{C}},$$

where $(g, t) \in G \times T$, $C \in G$, $\Lambda \in \mathfrak{t}$, and $S_{\mathrm{odd/even}} \in U_{+/-}$, respectively. The first main result may then be paraphrased as the following theorem.

THEOREM 1
The space $\widetilde{\mathscr{C}}$ *is a complex quasi-Hamiltonian* $(G \times T)$*-space with moment maps*

$$\mu : \widetilde{\mathscr{C}} \to G, \qquad (C, \mathbf{S}, \Lambda) \mapsto C^{-1} S_{2k-2} \cdots S_2 S_1 e^{2\pi i \Lambda} C$$

and

$$\mu_T : \widetilde{\mathscr{C}} \to T, \qquad (C, \mathbf{S}, \Lambda) \mapsto \exp(-2\pi i \Lambda),$$

where $C \in G$, $\Lambda \in \mathfrak{t}$, *and* $\mathbf{S} = (S_1, S_2, \ldots)$.

Moreover, for each choice of $\Lambda \in \mathfrak{t}$, *the reduction*

$$\mathscr{C} := (\widetilde{\mathscr{C}}|_\Lambda)/T \cong \bigl(G \times (U_+ \times U_-)^{k-1}\bigr)/T$$

is a complex quasi-Hamiltonian G*-space.*

(The explicit formula for the quasi-Hamiltonian two-form ω on $\widetilde{\mathscr{C}}$ appears with the full statement in Theorem 5.)

Geometrically, the unipotent elements S_i here arise as the natural notion of *monodromy data* for irregular meromorphic connections on G-bundles over a curve; namely, they are the Stokes multipliers, which give a precise analytic description of the local moduli of the connection at the pole. (The value of the moment map μ is an expression for the actual monodromy of the corresponding meromorphic connection around the pole, and the integer k is the order of the pole of the connection.) However, we emphasise that Theorem 1 is proved algebraically without referring to meromorphic connections or the Stokes phenomenon.

For example, in the order two pole case $k = 2$, the space $\widetilde{\mathscr{C}}$ may be described as follows. If we define $b_- = e^{-\pi i \Lambda} S_2^{-1}$, and $b_+ = e^{-\pi i \Lambda} S_1 e^{2\pi i \Lambda}$, then

$$\widetilde{\mathscr{C}} \cong G \times G^*, \qquad \mu = C^{-1} b_-^{-1} b_+ C,$$

where G^* is the simply connected Poisson Lie group dual to G with its standard complex Poisson Lie group structure (cf. [9], [11, Appendix B]). Thus, in this case, $\widetilde{\mathscr{C}}$ looks something like a "multiplicative" cotangent bundle of G. In general, the quotient $\widetilde{\mathscr{C}}/G$ has an induced Poisson structure (see [1]), and for $k = 2$, this coincides with the standard Poisson structure on G^*. Moreover, we see that for $k = 2$ the additive analogue $\widetilde{O}$ of $\widetilde{\mathscr{C}}$ is precisely the cotangent bundle T^*G. In this order two pole case, the expression for the quasi-Hamiltonian two-form on $G \times G^*$ is

$$\omega = \frac{1}{2}(\overline{\mathscr{D}}, \overline{\mathscr{E}}) + \frac{1}{2}(\mathscr{D}, \gamma) - \frac{1}{2}(\mathscr{E}, \gamma),$$

where $\overline{\mathscr{D}} = D^*(\overline{\theta})$, $\overline{\mathscr{E}} = E^*(\overline{\theta})$, and $\gamma = C^*(\theta)$, where $\theta, \overline{\theta} \in \Omega^1(G, \mathfrak{g})$ are the left/right-invariant Maurer-Cartan forms, and where the maps $D, E : G \times G^* \to G$ are given by $D = b_- C$, $E = b_+ C$.

To better understand the geometrical origins of these spaces, let us briefly recall the Riemann-Hilbert correspondence in this context (see [7], [11], and the proof of Theorem 6 for more details). Let Δ denote the closed unit disk in $\mathbb{C}$, and let $\mathscr{N}$ be the moduli space of isomorphism classes of four-tuples (P, A, g_0, g_1), where

- $P \to \Delta$ is a holomorphic principal G-bundle;
- A is a meromorphic connection on P with a pole only at the origin (at which it has fixed, generic, irregular type);
- g_0 is a framing of P at zero compatible with the fixed irregular type;
- g_1 is an arbitrary framing of P at $1 \in \partial\Delta$.

This space $\mathscr{N}$ has a natural action of $G \times T$ changing the framings at 1 and 0, respectively.

THEOREM 2 (see [7], [11])
The local irregular Riemann-Hilbert correspondence, taking such connections to their monodromy/Stokes data, yields a $(G \times T)$*-equivariant analytic isomorphism* $\mathscr{N} \cong \widetilde{\mathscr{C}}$.

This gives a description of $\widetilde{\mathscr{C}}$ in terms of meromorphic connections on holomorphic G-bundles. The key to obtaining the quasi-Hamiltonian structure on $\mathscr{N} \cong \widetilde{\mathscr{C}}$ is to give yet another description of these spaces, this time from a complex C^∞ viewpoint (following [10]), where they arise from infinite-dimensional Hamiltonian loop group manifolds.

To see how this helps, recall that Alekseev, Malkin, and Meinrenken [3] prove an equivalence theorem between quasi-Hamiltonian G-spaces (for compact G) and Hamiltonian loop group manifolds (with proper moment maps). Although we do not prove a complex version of this equivalence theorem, we show that the quasi-Hamiltonian spaces $\widetilde{\mathscr{C}}, \mathscr{C}$ of Theorem 1 do correspond (in the sense of [3]) to the complex Hamiltonian loop group manifolds that one obtains naturally from the extended Atiyah-Bott symplectic structures of [10]. Thus, our second main result is Theorem 7 in Section 4, which proves that the explicit quasi-Hamiltonian two-forms on $\widetilde{\mathscr{C}}$ do arise in this way from these generalised Atiyah-Bott-type symplectic structures (and, indeed, were found in this way). In summary, the key results of the present article are to compute explicitly the quasi-Hamiltonian structures corresponding to the generalised Atiyah-Bott structure of [10] and also to give a direct finite-dimensional proof that they are indeed quasi-Hamiltonian.

The main applications of these results are, firstly, to give a finite-dimensional construction of quite general symplectic moduli spaces of meromorphic connections on G-bundles over curves and, secondly, to the theory of isomonodromic deformations:

given a genus g compact Riemann surface Σ with a divisor $D = \sum_{i=1}^m k_i(a_i)$ having each $k_i \geq 1$, the above theorems enable one to construct symplectic moduli spaces of monodromy data for meromorphic connections on Σ of the form

$$(\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \widetilde{\mathscr{C}}_1 \circledast \cdots \circledast \widetilde{\mathscr{C}}_m) /\!\!/ G \tag{2}$$

with g factors of $\mathbf{D}$. (Such symplectic spaces were previously constructed in [10] from an infinite-dimensional viewpoint.) Summing the quasi-Hamiltonian two-forms on each factor in (2) together with the fusion terms gives an explicit expression for the symplectic structure on the manifold (2). Such an explicit expression in terms of monodromy/Stokes data for the symplectic structure on spaces of meromorphic $\mathrm{GL}_n(\mathbb{C})$-connections with one arbitrary order pole over a disk was first obtained by Woodhouse (see [24, equation (14)]), and with any number of poles over arbitrary genus Riemann surfaces, such expression was first obtained by Krichever [19] via an alternative approach that originated in [20].

However, the quasi-Hamiltonian approach here also gives an algebraic proof that it is indeed a symplectic structure, something that interested many people in the case of nonsingular connections (see [18], [13], [23], [17], [4], [16], [3]) because Goldman's finite-dimensional approach [14] appealed to Atiyah and Bott's infinite-dimensional framework to establish the closedness of the symplectic form.

In Section 5, we recall the additive analogues O, $\widetilde{O}$ of the spaces $\mathscr{C}$, $\widetilde{\mathscr{C}}$ for each k. These are symplectic manifolds of matching dimensions ($\dim \mathscr{C} = \dim O$, $\dim \widetilde{O} = \dim \widetilde{\mathscr{C}}$). Indeed, O is just a generic coadjoint orbit of the group

$$G_k := G(\mathbb{C}[z]/z^k)$$

of k-jets of bundle automorphisms, which nicely extends the idea that the conjugacy classes in G are the multiplicative analogues of the coadjoint orbits of G. (Although one should note that for $k \geq 2$, we know of no natural identification of the spaces $\mathscr{C}$ as conjugacy classes of the groups G_k.) The *extended orbits* $\widetilde{O}$ are larger than the coadjoint orbits O by $2 \dim T$ and yield the orbits O upon performing a symplectic quotient by T.

In the genus-zero case, the spaces O, $\widetilde{O}$ enable one to construct global symplectic moduli spaces of meromorphic connections on holomorphically trivial G-bundles explicitly as finite-dimensional symplectic quotients of the form $(\widetilde{O}_1 \times \cdots \times \widetilde{O}_m) /\!\!/ G$, and (at least for $G = \mathrm{GL}_n(\mathbb{C})$ and, presumably, in general) such moduli spaces fill out a dense open subset of the full moduli space of meromorphic connections on topologically trivial bundles. The main result of [10] then leads immediately to the following.

COROLLARY 1
The (global) irregular Riemann-Hilbert map

$$\nu : (\widetilde{O}_1 \times \cdots \times \widetilde{O}_m)/\!\!/ G \hookrightarrow (\widetilde{\mathscr{C}}_1 \circledast \cdots \circledast \widetilde{\mathscr{C}}_m)/\!\!/ G \tag{3}$$

associating monodromy/Stokes data to a meromorphic connection on a trivial G-bundle over $\mathbb{P}^1$ is a symplectic map (provided that the symplectic structure on the right-hand side is divided by $2\pi i$). Moreover, both sides are naturally Hamiltonian $T^{\times m}$-spaces, and ν intertwines these actions and their moment maps.

This map ν depends heavily on the chosen pole positions $a_i \in \mathbb{P}^1$ (and on a choice of irregular type at each pole). However, from the formula for the quasi-Hamiltonian two-form ω, one sees immediately that the symplectic structure on the right-hand side of (3) is independent of these choices. This shows that the isomonodromy connection is a symplectic connection, as was previously shown in [10] from a de Rham point of view. (This fact is the irregular/“wild” analogue of the so-called symplectic nature of the fundamental group of surfaces [14].)

To illustrate Corollary 1, let us briefly describe the case of two order-two poles on $\mathbb{P}^1$. As mentioned above, for $k = 2$ the extended orbits $\widetilde{O}$ are isomorphic to the cotangent bundle T^*G, so that the left-hand side of (3) is also isomorphic to T^*G. On the other hand, we show in Proposition 7 in Section 3 that in this case the right-hand side of (3) turns out to be isomorphic,[†] as a symplectic manifold, to the so-called “symplectic double groupoid” Γ of G and G^*, described by Lu and Weinstein in [21] (which is another, perhaps more global, multiplicative analogue of T^*G). Thus in this case, the monodromy/Stokes map ν is a transcendental, complex symplectic, embedding

$$\nu : T^*G \hookrightarrow \Gamma$$

of the cotangent bundle into the symplectic double groupoid for each choice of the irregular types.

Finally, we also describe in Section 6 the precise global moduli space of meromorphic connections isomorphic to (2) under the global irregular Riemann-Hilbert correspondence, and we explain how to read off the topological type of the underlying G-bundle from the monodromy/Stokes data and exponents stored in (2).

Notation/Conventions. Throughout this article, G is a connected complex reductive group with maximal torus T and corresponding Lie algebras $\mathfrak{t} \subset \mathfrak{g}$. $B_\pm$ denote a pair of opposite Borel subgroups with $B_+ \cap B_- = T$, and $U_\pm \subset B_\pm$ denote their full unipotent subgroups, with corresponding Lie algebras $\mathfrak{u}_\pm \subset \mathfrak{b}_\pm$.

[†]Precisely, not modulo passing to a covering or some dense open subset, as is common in the field.

Let $(,) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be a symmetric nondegenerate invariant bilinear form. (Note that invariance implies that $(,)$ restricts to zero on $\mathfrak{u}_\pm \otimes \mathfrak{u}_\pm$ and to a pairing on each of $\mathfrak{u}_+ \otimes \mathfrak{u}_-$, $\mathfrak{u}_- \otimes \mathfrak{u}_+$, $\mathfrak{t} \otimes \mathfrak{t}$.)

Let $\theta, \overline{\theta} \in \Omega^1(G, \mathfrak{g})$ denote the tautological left and right invariant $\mathfrak{g}$-valued holomorphic one-forms on G, respectively. (So in any representation $\theta = g^{-1}dg$, $\overline{\theta} = (dg)g^{-1}$.)

Generally, if $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \Omega^1(M, \mathfrak{g})$ are $\mathfrak{g}$-valued holomorphic one-forms on a complex manifold M, then $(\mathscr{A}, \mathscr{B}) \in \Omega^2(M)$ and $[\mathscr{A}, \mathscr{B}] \in \Omega^2(M, \mathfrak{g})$ are defined by wedging the form parts and pairing/bracketing the Lie algebra parts (so, e.g., $(A\alpha, B\beta) = (A, B)\alpha \wedge \beta$ for $A, B \in \mathfrak{g}$, $\alpha, \beta \in \Omega^1(M)$.)

Define $\mathscr{A}\mathscr{B} := (1/2)[\mathscr{A}, \mathscr{B}] \in \Omega^2(M, \mathfrak{g})$ (which works out correctly in any representation of G using matrix multiplication). Then one has $d\theta = -\theta^2$, $d\overline{\theta} = \overline{\theta}^2$.

Define $(\mathscr{A}\mathscr{B}\mathscr{C}) = (\mathscr{A}, \mathscr{B}\mathscr{C}) \in \Omega^3(M)$ (which is totally symmetric in $\mathscr{A}, \mathscr{B}, \mathscr{C}$). The canonical bi-invariant three-form on G is then $\eta := (1/6)(\theta^3)$.

The adjoint action of G on $\mathfrak{g}$ is denoted $gXg^{-1} := \mathrm{Ad}_g X$ for any $X \in \mathfrak{g}$, $g \in G$.

If G acts on M, the fundamental vector field of $X \in \mathfrak{g}$ is minus the tangent to the flow $(v_X)_m = -\frac{d}{dt}(e^{Xt} \cdot m)|_{t=0}$, so that the map $\mathfrak{g} \to \mathrm{Vect}_M; X \to v_X$ is a Lie algebra homomorphism. (This sign convention differs from [3] and agrees with [1]; this leads to sign changes in the quasi-Hamiltonian axioms and the fusion and equivalence theorems.)

2. Quasi-Hamiltonian G-spaces

Definition 2 (cf. [3], [1])
A complex manifold M is a *complex quasi-Hamiltonian G-space* if there is an action of G on M, a G-equivariant map $\mu : M \to G$ (where G acts on itself by conjugation), and a G-invariant holomorphic two-form $\omega \in \Omega^2(M)$ such that

(QH1) the exterior derivative of ω is the pullback of the canonical three-form on G,

$$d\omega = \mu^*(\eta);$$

(QH2) for all $X \in \mathfrak{g}$,

$$\omega(v_X, \cdot\,) = \frac{1}{2}\mu^*(\theta + \overline{\theta}, X) \in \Omega^1(M);$$

(QH3) at each point $m \in M$, the kernel of ω is

$$\ker \omega_m = \left\{ (v_X)_m \;\middle|\; X \in \mathfrak{g} \text{ satisfies } gXg^{-1} = -X, \text{ where } g := \mu(m) \in G \right\}.$$

These axioms are perhaps best motivated in terms of Hamiltonian loop group manifolds (see [3], [22]), as we sketch in Section 4.

A simple but important observation is that if G is abelian (and, in particular, if $G = \{1\}$ is trivial), then these axioms imply that the two-form ω is a complex symplectic form.

Example 3 (Conjugacy classes; see [3])
Let $\mathscr{C} \subset G$ be a conjugacy class with the conjugation action of G and moment map μ given by the inclusion map. Then $\mathscr{C}$ is a quasi-Hamiltonian G-space with two-form ω determined by

$$\omega_g(v_X, v_Y) = \frac{1}{2}\big((X, gYg^{-1}) - (Y, gXg^{-1})\big)$$

for any $X, Y \in \mathfrak{g}$, $g \in \mathscr{C}$. For later use we note that if $g \in \mathscr{C}$ is fixed and we define the surjective map $\pi : G \to \mathscr{C} : C \mapsto C^{-1}gC$, then one may calculate

$$\pi^*(\omega) = \frac{1}{2}(\overline{\theta}, g\overline{\theta}g^{-1}) \in \Omega^2(G). \tag{4}$$

Example 4 (Internally fused double; see [3])
The space $\mathbf{D} = G \times G$ is a quasi-Hamiltonian G-space with G acting by diagonal conjugation ($g(a, b) = (gag^{-1}, gbg^{-1})$), moment map given by the group commutator

$$\mu(a, b) = aba^{-1}b^{-1},$$

and two-form

$$\omega_{\mathbf{D}} = -\frac{1}{2}(a^*\theta, b^*\overline{\theta}) - \frac{1}{2}(a^*\overline{\theta}, b^*\theta) - \frac{1}{2}\big((ab)^*\theta, (a^{-1}b^{-1})^*\overline{\theta}\big).$$

Now, let us recall the quasi-Hamiltonian reduction theorem.

THEOREM 3 (see [3])
Let M be a quasi-Hamiltonian $(G \times H)$-space with moment map $(\mu, \mu_H) : M \to G \times H$, and suppose that the quotient by G of the inverse image $\mu^{-1}(1)$ of the identity under the first moment map is a manifold. Then the restriction of the two-form ω to $\mu^{-1}(1)$ descends to the reduced space

$$M /\!\!/ G := \mu^{-1}(1)/G$$

and makes it into a quasi-Hamiltonian H-space. In particular, if H is abelian (or, in particular, trivial), then $M /\!\!/ G$ is a complex symplectic manifold.

The fusion product, which puts a ring structure on the category quasi-Hamiltonian G-spaces, is defined as follows. (Also, reduction at different values of the moment map may be facilitated by first fusing with a conjugacy class, analogously to the Hamiltonian case.)

THEOREM 4 (see [3])
Let M be a quasi-Hamiltonian $(G \times G \times H)$-space with moment map $\mu = (\mu_1, \mu_2, \mu_3)$. Let $G \times H$ act by the diagonal embedding $(g, h) \to (g, g, h)$. Then M with two-form

$$\widetilde{\omega} = \omega - \frac{1}{2}(\mu_1^*\theta, \mu_2^*\overline{\theta}) \tag{5}$$

and moment map

$$\widetilde{\mu} = (\mu_1 \cdot \mu_2, \mu_3) : M \to G \times H$$

is a quasi-Hamiltonian $(G \times H)$-space.

We refer to the extra term subtracted off in (5) as the "fusion term." If M_i is a quasi-Hamiltonian $(G \times H_i)$-space for $i = 1, 2$, their fusion product

$$M_1 \circledast M_2$$

is defined to be the quasi-Hamiltonian $(G \times H_1 \times H_2)$-space obtained from the quasi-Hamiltonian $(G \times G \times H_1 \times H_2)$-space $M_1 \times M_2$ by fusing the two factors of G.

3. New examples

In this section, we describe the family of quasi-Hamiltonian spaces $\mathscr{C}, \widetilde{\mathscr{C}}$ and prove directly that they are such. However, the motivation for, and geometrical origins of, these spaces only become clear in Section 4, where their infinite-dimensional counterparts are described.

Our main objects of study are the family of complex manifolds

$$\begin{aligned} \widetilde{\mathscr{C}} := \Big\{ (C, \mathbf{d}, \mathbf{e}, \Lambda) \in G \times (B_- \times B_+)^{k-1} \times \mathfrak{t} \;\Big|\; \delta(d_j)^{-1} = e^{\pi i \Lambda/(k-1)} \\ = \delta(e_j) \text{ for all } j \Big\}, \end{aligned} \tag{6}$$

parameterised by an integer $k \geq 2$, where $\mathbf{d} = (d_1, \ldots, d_{k-1})$, $\mathbf{e} = (e_1, \ldots, e_{k-1})$ with $d_{\text{even}}, e_{\text{odd}} \in B_+$ and $d_{\text{odd}}, e_{\text{even}} \in B_-$, and where $\delta : B_\pm \to T$ is the homomorphism with kernel $U_\pm$. This space $\widetilde{\mathscr{C}}$ is isomorphic to $G \times (U_+ \times U_-)^{k-1} \times \mathfrak{t}$, but it is more convenient to use the above definition throughout the remainder of the article. For the record, in terms of the Stokes multipliers S_j mentioned in the introduction, we have

$$d_j = \epsilon^{-j} S_{2k-1-j}^{-1} \epsilon^{j-1}, \qquad e_j = \epsilon^{j+2-2k} S_j \epsilon^{2k-1-j},$$

where $\epsilon := e^{\pi i \Lambda/(k-1)}$. In this description, the action of G on $\widetilde{\mathscr{C}}$ is given by

$$g \cdot (C, \mathbf{d}, \mathbf{e}, \Lambda) = (Cg^{-1}, \mathbf{d}, \mathbf{e}, \Lambda)$$

for $g \in G$, and the action of T is given by

$$t \cdot (C, \mathbf{d}, \mathbf{e}, \Lambda) = (tC, td_1t^{-1}, \dots, td_{k-1}t^{-1}, te_1t^{-1}, \dots, te_{k-1}t^{-1}, \Lambda)$$

for $t \in T$. Independently, these actions are both free, although the combined $(G \times T)$-action is not. Maps $D_i, E_i, \mu : \widetilde{\mathscr{C}} \to G$ are defined as

$$D_i(C, \mathbf{d}, \mathbf{e}, \Lambda) = d_i \cdots d_1 C \quad (i = 0, 1, \dots, k-1),$$

$$E_i(C, \mathbf{d}, \mathbf{e}, \Lambda) = e_i \cdots e_1 C \quad (i = 0, 1, \dots, k-1),$$

$$\mu(C, \mathbf{d}, \mathbf{e}, \Lambda) = C^{-1} d_1^{-1} \cdots d_{k-1}^{-1} e_{k-1} \cdots e_1 C.$$

To lighten the notation, we write $D = D_{k-1}$, $E = E_{k-1}$, so in particular, $\mu = D^{-1}E$. The main result of this section is the following theorem.

THEOREM 5
The manifold $\widetilde{\mathscr{C}}$ is a quasi-Hamiltonian $(G \times T)$-space with the above action, moment map $(\mu, e^{-2\pi i \Lambda}) : \widetilde{\mathscr{C}} \to G \times T$, and two-form

$$\boxed{\omega = \frac{1}{2}(\overline{\mathscr{D}}, \overline{\mathscr{E}}) + \frac{1}{2}\sum_{i=1}^{k-1}(\mathscr{D}_i, \mathscr{D}_{i-1}) - (\mathscr{E}_i, \mathscr{E}_{i-1}),} \tag{7}$$

where $\overline{\mathscr{D}} = D^*(\overline{\theta})$, $\overline{\mathscr{E}} = E^*(\overline{\theta})$, $\mathscr{D}_i = D_i^*(\theta)$, and $\mathscr{E}_i = E_i^*(\theta) \in \Omega^1(\widetilde{\mathscr{C}}, \mathfrak{g})$.

In particular, since T is abelian, this implies that $\widetilde{\mathscr{C}}$ is a quasi-Hamiltonian G-space with moment map μ and the same two-form.

Remark 5
As defined, the spaces $\widetilde{\mathscr{C}}$ are not in the category of *algebraic* quasi-Hamiltonian spaces since the moment maps involve exponentials and so are not algebraic. There are, however, closely related algebraic quasi-Hamiltonian $(G \times T)$-spaces, as follows. Observe that ω is invariant under translations of Λ by the lattice

$$L := \ker\big(\exp(2\pi i\, \cdot\,) : \mathfrak{t} \to T\big).$$

The quotient $\widetilde{\mathscr{C}}/L \cong G \times (U_+ \times U_-)^{k-1} \times T$ is then an *algebraic* quasi-Hamiltonian $(G \times T)$-space. Indeed, all the formulae above make sense directly on the subvariety of $G \times (B_- \times B_+)^{k-1}$ cut out by the equations $\delta(d_i) \cdot \delta(e_j) = 1$, and this subvariety is a finite covering of $\widetilde{\mathscr{C}}/L$ (corresponding to replacing $\exp(\pi i \Lambda/(k-1))$ by $\exp(2\pi i \Lambda)$). We prefer to keep the choice of Λ in our definition of $\widetilde{\mathscr{C}}$ for two reasons:

(1) to obtain (genuine) Hamiltonian T-spaces upon reducing by G and

(2) it enables us (in Section 6) to capture the topological types of the underlying G-bundles under the Riemann-Hilbert correspondence.

Also, we have the following.

COROLLARY 6

Suppose that a value of Λ is fixed. Then the reduction

$$\mathscr{C} := (\widetilde{\mathscr{C}}|_\Lambda)/T$$

is a complex (algebraic) quasi-Hamiltonian G-space.

Proof

$\mathscr{C}$ may also be described as the quasi-Hamiltonian reduction $(\widetilde{\mathscr{C}}/L)/\!\!/T$ of $\widetilde{\mathscr{C}}/L$ at the value $\exp(-2\pi i \Lambda)$ of the moment map for the T action. □

Before proving Theorem 5, let us describe the $k = 2$ case (which in later sections corresponds to meromorphic connections having a pole of order two) and the specialisation to the simple pole case $k = 1$. If $k = 2$ and we denote $b_- = d_1$ and $b_+ = e_1$, then

$$\widetilde{\mathscr{C}} \cong G \times G^*, \qquad \mu = C^{-1} b_-^{-1} b_+ C, \qquad \omega = \frac{1}{2}(\overline{\mathscr{D}}, \overline{\mathscr{E}}) + \frac{1}{2}(\mathscr{D}, \gamma) - \frac{1}{2}(\mathscr{E}, \gamma), \quad (8)$$

where $D = b_- C$, $E = b_+ C$, $\gamma = C^* \theta$, and

$$G^* := \left\{ (b_-, b_+, \Lambda) \in B_- \times B_+ \times \mathfrak{t} \;\middle|\; \delta(b_-)\delta(b_+) = 1, \delta(b_+) = \exp(\pi i \Lambda) \right\}$$

is the Poisson Lie group dual to G (see, e.g., [9], [11, Appendix B]).

Considering two poles of order two on $\mathbb{P}^1$ leads to the following statement, which gives a relationship between symplectic double groupoids and meromorphic connections.

PROPOSITION 7

Let $\widetilde{\mathscr{C}}_1$ and $\widetilde{\mathscr{C}}_2$ be two copies of $\widetilde{\mathscr{C}}$ with $k = 2$. Then the quasi-Hamiltonian reduction of the fusion of $\widetilde{\mathscr{C}}_1$ and $\widetilde{\mathscr{C}}_2$ is isomorphic as a symplectic manifold to the symplectic *double groupoid* Γ *of* G *and* G^* *appearing in [21]:*

$$(\widetilde{\mathscr{C}}_1 \circledast \widetilde{\mathscr{C}}_2)/\!\!/ G \cong \Gamma. \tag{9}$$

Proof
First, recall that $\widetilde{\mathscr{C}}_i \cong G \times G^*$ as manifolds. We assume that the Borels chosen at the first pole are opposite to those chosen at the second (which we may since isomonodromy gives symplectic isomorphisms with the spaces arising from any other choice of Borels intersecting in T). Thus, $\widetilde{\mathscr{C}}_1 = \{(C_1, b_-, b_+, \Lambda_1) \mid \delta(b_\pm) = e^{\pm\pi i\Lambda_1}\}$ and $\widetilde{\mathscr{C}}_2 = \{(C_2, c_+, c_-, \Lambda_2) \mid \delta(c_\pm) = e^{\mp\pi i\Lambda_2}\}$ with $b_\pm, c_\pm \in B_\pm$. The moment map on $\widetilde{\mathscr{C}}_1 \circledast \widetilde{\mathscr{C}}_2$ is $\mu = C_1^{-1}b_-^{-1}b_+C_1C_2^{-1}c_+^{-1}c_-C_2$. If we write $h := C_2C_1^{-1}$, the condition $\mu = 1$ becomes $hb_-^{-1}b_+h^{-1}c_+^{-1}c_- = 1$, and if we define $g := c_-hb_-^{-1}$, then this condition is clearly equivalent to $c_+h = gb_+$. Thus (omitting the Λ-terms in order to simplify notation), we have defined a surjective map

$$\mu^{-1}(1) \to \Gamma := \Big\{(g, b_-, b_+, h, c_+, c_-) \,\Big|\, c_\pm h = gb_\pm\Big\} \subset (G \times G^*)^2$$

whose fibres are precisely the G-orbits. This is the definition of the manifold Γ given in [21]. The symplectic structures may be shown to agree as follows.

The map $\Gamma \to G \times G; (g, b_-, b_+, h, c_+, c_-) \mapsto (gb_-, gb_+)$ expresses Γ as the covering of a dense subset of $G \times G$. This subset is the big symplectic leaf of the Heisenberg double Poisson structure on $G \times G$, and the symplectic structure on Γ is defined to be the pullback of the symplectic structure on this leaf. An explicit formula for this pullback (i.e., for the symplectic structure on Γ) is given in [2, Theorem 3]. On the other hand, we have an explicit formula for the symplectic structure on $(\widetilde{\mathscr{C}}_1 \circledast \widetilde{\mathscr{C}}_2)/\!\!/ G$ (involving seven terms, the fusion term plus three terms (8) for each factor). A straightforward calculation shows that these explicit formulae on each side agree. □

In the simple pole case $k = 1$, we *define* $\widetilde{\mathscr{C}} = G \times \mathfrak{t}_1$, where $\mathfrak{t}_1 \subset \mathfrak{t}$ is the complement of the affine root hyperplanes: $\mathfrak{t}_1 := \{\Lambda \in \mathfrak{t} \mid \alpha(\Lambda) \notin \mathbb{Z}$ for all roots $\alpha\}$. The correct specialisation of (7) to this case is

$$\omega = \frac{1}{2}(\overline{\mathscr{D}}, \overline{\mathscr{E}}) + \frac{1}{2}(\mathscr{D}, \gamma) - \frac{1}{2}(\mathscr{E}, \gamma) = 2\pi i(\overline{\gamma}, d\Lambda) + \frac{1}{2}(\overline{\gamma}, e^{2\pi i\Lambda}\overline{\gamma}e^{-2\pi i\Lambda}),$$

where $D = e^{-\pi i\Lambda}C$, $E = e^{\pi i\Lambda}C$, and $\overline{\gamma} = C^*\overline{\theta}$. This is the restriction of the two-form (7) to the submanifold with $d_i, e_i \in T, \Lambda \in \mathfrak{t}_1$ (for any k) and makes $\widetilde{\mathscr{C}}$ into a quasi-Hamiltonian $(G \times T)$-space with moment map $(D^{-1}E, e^{-2\pi i\Lambda})$. (It also arises as a cross-section of the double in [3].) Given a fixed choice of Λ, the restriction of ω to $G \times \{\Lambda\}$ clearly agrees with (4), and so we deduce that the reduction $\mathscr{C} := (\widetilde{\mathscr{C}}|_\Lambda)/T \cong G/T$ is isomorphic as a quasi-Hamiltonian G-space to the conjugacy class through $e^{2\pi i\Lambda}$.

Proof of Theorem 5
To establish (QH1), that $\mu^*(\theta^3) = 6\,d\omega$, we observe that the expression (7) defines a two-form on $G \times (B_+ \times B_-)^{k-1}$, and working algebraically, we view (QH1) as a statement about the differential algebra generated by the symbols C, d_i, e_j, using only the restriction that $(d_i^*\theta^3) = (e_j^*\theta^3) = 0$ (which follows from the fact that d_i, e_j live in Borel subgroups). By restriction, the result for $\widetilde{\mathscr{C}}$ then follows. This viewpoint enables us to use induction on k. From the definition, one finds $\mu^*(\theta^3) = ((\overline{\mathscr{E}} - \overline{\mathscr{D}})^3)$, which expands to give

$$\mu^*(\theta^3) = 3(\overline{\mathscr{D}\mathscr{D}\mathscr{E}}) - 3(\overline{\mathscr{D}\mathscr{E}\mathscr{E}}) + (\overline{\mathscr{E}}^3) - (\overline{\mathscr{D}}^3).$$

On the other hand,

$$2d\omega = (\overline{\mathscr{D}\mathscr{D}\mathscr{E}}) - (\overline{\mathscr{D}\mathscr{E}\mathscr{E}}) + F_{k-1},$$

where

$$F_{k-1} := \sum_{i=1}^{k-1} (\mathscr{D}_i\mathscr{D}_{i-1}\mathscr{D}_{i-1}) - (\mathscr{D}_i\mathscr{D}_i\mathscr{D}_{i-1}) - (\mathscr{E}_i\mathscr{E}_{i-1}\mathscr{E}_{i-1}) + (\mathscr{E}_i\mathscr{E}_i\mathscr{E}_{i-1}),$$

so that what we must prove is $(\overline{\mathscr{E}}^3) - (\overline{\mathscr{D}}^3) = 3F_{k-1}$ or, equivalently (assuming $(\overline{\mathscr{E}}^3_{k-2}) - (\overline{\mathscr{D}}^3_{k-2}) = 3F_{k-2}$ inductively), that $(\overline{\mathscr{E}}^3) - (\overline{\mathscr{D}}^3)$ equals

$$(\overline{\mathscr{E}}^3_{k-2}) - (\overline{\mathscr{D}}^3_{k-2}) + 3\big((\mathscr{D}\mathscr{D}_{k-2}\mathscr{D}_{k-2}) - (\mathscr{D}\mathscr{D}\mathscr{D}_{k-2}) - (\mathscr{E}\mathscr{E}_{k-2}\mathscr{E}_{k-2}) + (\mathscr{E}\mathscr{E}\mathscr{E}_{k-2})\big). \tag{10}$$

To establish this, write $E = b_+E_{k-2}$, $D = b_-D_{k-2}$, where $b_+ := e_{k-1}$, $b_- := d_{k-1}$. (Note that we do not necessarily have $b_\pm \in B_\pm$, only that they are in opposite Borels.) Thus,

$$\mathscr{E} = E^{-1}_{k-2}\theta_+E_{k-2} + \mathscr{E}_{k-2}, \qquad \mathscr{D} = D^{-1}_{k-2}\theta_-D_{k-2} + \mathscr{D}_{k-2}, \tag{11}$$

$$\overline{\mathscr{E}} = \overline{\theta}_+ + b_+\overline{\mathscr{E}}_{k-2}b_+^{-1}, \qquad \overline{\mathscr{D}} = \overline{\theta}_- + b_-\overline{\mathscr{D}}_{k-2}b_-^{-1},$$

where $\theta_\pm = b^*_\pm(\theta)$, $\overline{\theta}_\pm = b^*_\pm(\overline{\theta})$, and so

$$\begin{aligned}(\overline{\mathscr{E}}^3) - (\overline{\mathscr{D}}^3) &= \big((\theta_+ + \overline{\mathscr{E}}_{k-2})^3\big) - \big((\theta_- + \overline{\mathscr{D}}_{k-2})^3\big)\\ &= (\overline{\mathscr{E}}^3_{k-2}) - (\overline{\mathscr{D}}^3_{k-2}) + 3\big((\theta_+\theta_+\overline{\mathscr{E}}_{k-2}) + (\theta_+\overline{\mathscr{E}}_{k-2}\overline{\mathscr{E}}_{k-2})\\ &\qquad - (\theta_-\theta_-\overline{\mathscr{D}}_{k-2}) - (\theta_-\overline{\mathscr{D}}_{k-2}\overline{\mathscr{D}}_{k-2})\big),\end{aligned} \tag{12}$$

using the fact that $(\theta^3_\pm) = 0$. Thus, we must show that the coefficients of 3 in (10) and (12) are the same; this, however, follows easily by substituting the expressions (11) for $\mathscr{E}, \mathscr{D}$ into (10) and expanding. Finally, the $k = 2$ case may be proved directly, justifying the induction; namely, we must show that $(\overline{\mathscr{E}}^3) - (\overline{\mathscr{D}}^3) = 3F_1$, and this comes about simply by expanding both sides in terms of $b_\pm$ and C. (The $k = 1$ case is similar.)

Next, we check (QH2) for the G-action. Choose $X \in \mathfrak{g}$ and an arbitrary holomorphic vector field Y on $\widetilde{\mathscr{C}}$. We denote derivatives along v_X by primes and along Y by dots, so for example, $\dot{\mathscr{D}}_i = \langle Y, \mathscr{D}_i\rangle \in \mathfrak{g}$ and $\mathscr{E}'_j = \langle v_X, \mathscr{E}_j\rangle \in \mathfrak{g}$ (and in any representation of G, we have $\dot{\mathscr{D}}_i = D_i^{-1}\dot{D}_i$, and so forth). By definition of the action $\mathscr{D}'_i = \mathscr{E}'_i = X$ for all i, and $\overline{\mathscr{D}}' = DXD^{-1}, \overline{\mathscr{E}}' = EXE^{-1}$. Thus,

$$2\omega(v_X, Y) = (DXD^{-1}, \dot{\overline{\mathscr{E}}}) - (EXE^{-1}, \dot{\overline{\mathscr{D}}}) + \sum_{i=1}^{k-1}(X, \dot{\mathscr{D}}_{i-1} - \dot{\mathscr{D}}_i - \dot{\mathscr{E}}_{i-1} + \dot{\mathscr{E}}_i),$$

which simplifies to $(X, D^{-1}\dot{\overline{\mathscr{E}}}D - E^{-1}\dot{\overline{\mathscr{D}}}E - \dot{\mathscr{D}} + \dot{\mathscr{E}})$. On the other hand, since $\mu = D^{-1}E$,

$$\langle(\mu^*\theta + \mu^*\overline{\theta}, X), Y\rangle = (\mu^{-1}\dot{\mu} + \dot{\mu}\mu^{-1}, X) = (\dot{\mathscr{E}} - \dot{\mathscr{D}} + D^{-1}\dot{\overline{\mathscr{E}}}D - E^{-1}\dot{\overline{\mathscr{D}}}E, X),$$

so we have established (QH2) for the G-action.

For the T-action, if $X \in \mathfrak{t}$ then the derivatives along the corresponding fundamental vector field v_X (for the T action) are: $\dot{\overline{\mathscr{D}}}_i = \dot{\overline{\mathscr{E}}}_i = -X$, $\dot{\mathscr{D}}_i = -D_i^{-1}XD_i$, $\dot{\mathscr{E}}_i = -E_i^{-1}XE_i$. Thus, for any vector field Y on $\widetilde{\mathscr{C}}$,

$$2\omega(v_X, Y) = \Big(X, -\overline{\mathscr{E}}' + \overline{\mathscr{D}}' + \sum_{i=1}^{k-1} -D_i\mathscr{D}'_{i-1}D_i^{-1} + E_i\mathscr{E}'_{i-1}E_i^{-1}$$
$$+ D_{i-1}\mathscr{D}'_iD_{i-1}^{-1} - E_{i-1}\mathscr{E}'_iE_{i-1}^{-1}\Big),$$

where the primes denote the derivatives along Y. Now $D_i = d_iD_{i-1}$, so that $D_i\mathscr{D}'_{i-1}D_i^{-1} = \overline{\mathscr{D}}'_i - \overline{\delta}'_i$ and $D_{i-1}\mathscr{D}'_iD_{i-1}^{-1} = \overline{\mathscr{D}}'_{i-1} + \delta'_i$ (and similarly for the $\mathscr{E}_i$'s), where $\delta_i := d_i^*\theta$, and so forth. Substituting thus shows

$$2\omega(v_X, Y) = \Big(X, \sum_{i=1}^{k-1}\delta'_i + \overline{\delta}'_i - \varepsilon'_i - \overline{\varepsilon}'_i\Big).$$

Since $X \in \mathfrak{t}$, we may take the $\mathfrak{t}$ component of the right-hand side yielding

$$\omega(v_X, Y) = -(2\pi i)(X, \Lambda') = -(2\pi i)\langle(d\Lambda, X), Y\rangle,$$

which is what appears on the right-hand side of (QH2) if the moment map is $e^{-2\pi i\Lambda}$.

The proof of the minimal degeneracy condition (QH3) is rather complicated, so it has been put in the appendix. □

4. Derivation

In this section, we explain how the quasi-Hamiltonian spaces $\mathscr{C}, \widetilde{\mathscr{C}}$ were found. In brief, the extension of the Atiyah-Bott symplectic structure to the meromorphic case in [10] leads to new (infinite-dimensional) Hamiltonian loop group manifolds, and $\mathscr{C}, \widetilde{\mathscr{C}}$ are the corresponding quasi-Hamiltonian spaces.

In more detail, recall that the equivalence theorem [3, Theorem 8.3] gives a correspondence between Hamiltonian LK-manifolds (with proper moment maps) and quasi-Hamiltonian K-spaces, where K is a compact (connected) Lie group and $LK = C^\infty(S^1, K)$ is the corresponding loop group. The main examples of such Hamiltonian LK-spaces are moduli spaces of framed flat connections on principal K-bundles over compact two-manifolds Σ with precisely one boundary component: given Σ and K, one defines a space of connections

$$\mathscr{A} := \left\{\alpha \in \Omega^1_{C^\infty}(\Sigma, \mathfrak{k})\right\}$$

on the trivial C^∞-principal K-bundle over Σ (where $\mathfrak{k} = \mathrm{Lie}(K)$) and a gauge group

$$\mathscr{K} := C^\infty(\Sigma, K).$$

This has normal subgroup $\mathscr{K}_\partial := \{g \in \mathscr{K} \mid g|_{\partial\Sigma} = 1\}$ consisting of bundle automorphisms equal to the identity on the boundary circle. The quotient $\mathscr{K}/\mathscr{K}_\partial$ is thus isomorphic to the loop group LK (at least if K is simply connected). Atiyah and Bott [5] define the following symplectic structure on $\mathscr{A}$:

$$\omega_{\mathscr{A}}(\phi, \psi) = \int_\Sigma (\phi, \psi),$$

where (,) denotes a chosen pairing on $\mathfrak{k}$. Then, taking the curvature of the connections in $\mathscr{A}$ gives a moment map for the action of $\mathscr{K}_\partial$ (see [6]), and so the symplectic quotient at the zero value of the moment map is the moduli space of flat connections with a framing along the boundary circle

$$\widehat{\mathscr{N}} := \mathscr{A}_{\mathrm{flat}}/\mathscr{K}_\partial.$$

This infinite-dimensional symplectic manifold is a Hamiltonian LK-space in the sense of [3] (and such spaces constitute the main class of examples). The action of LK is simply the residual action of $\mathscr{K}$, and the moment map is the restriction of the connections to the boundary circle

$$\widehat{\mu} : \widehat{\mathscr{N}} \to \mathscr{A}_{S^1}, \qquad \alpha \mapsto \alpha|_{\partial\Sigma}.$$

(Strictly speaking, $\widehat{\mathcal{N}}$ is a Hamiltonian $\widehat{LK}$-space, where $\widehat{LK}$ is the centrally extended loop group and the central circle acts trivially on $\widehat{\mathcal{N}}$; the space $\mathcal{A}_{S^1}$ of connections on the trivial K-bundle over the circle is naturally identified with the level-one hyperplane in the dual of the Lie algebra of $\widehat{LK}$. However, this complication is incorporated into the definition of Hamiltonian LK-spaces in [3] and [22].)

Now, choose a point $p \in \partial\Sigma$ of the boundary circle of Σ. The equivalence theorem of [3] implies that the quotient $\mathcal{N} := \widehat{\mathcal{N}}/\Omega K$ of $\widehat{\mathcal{N}}$ by the based loop group $\Omega K = \{g \in LK \mid g(p) = 1\}$ is a (finite-dimensional) quasi-Hamiltonian K-space. In other words, moduli spaces of flat connections on Σ with a framing at one point on the boundary are naturally quasi-Hamiltonian K-spaces.

The two-form and moment map on $\mathcal{N}$ are constructed as follows. One has a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{N}} & \xrightarrow{\widehat{\mu}} & \mathcal{A}_{S^1} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle h} \\ \mathcal{N} & \xrightarrow{\mu} & K \end{array} \tag{13}$$

where π is the ΩK quotient and the maps μ and h take the holonomy of the connections around the boundary circle (in a positive sense, starting at p with initial condition $1 \in K$). The quasi-Hamiltonian two-form $\omega_{\mathcal{N}}$ on $\mathcal{N}$ is defined by†

$$-\pi^*(\omega_{\mathcal{N}}) = \omega_{\widehat{\mathcal{N}}} - \widehat{\mu}^*(\varpi),$$

where $\omega_{\widehat{\mathcal{N}}}$ is the symplectic form on $\widehat{\mathcal{N}}$ and ϖ is the following two-form on $\mathcal{A}_{S^1}$. For each point $z \in S^1$, define a map $h_z : \mathcal{A}_{S^1} \to K$ taking a connection α to its holonomy along the positive arc from p to z, with initial condition $1 \in K$. Thus, $h_z^*\overline{\theta}$ is a z-dependent $\mathfrak{k}$-valued one-form on $\mathcal{A}_{S^1}$, and ϖ is defined to be

$$\varpi = \frac{1}{2}\int_{S^1} (h_z^*\overline{\theta}, dh_z^*\overline{\theta}), \tag{14}$$

where d is the exterior derivative on S^1. It is worth noting that this procedure of subtracting off $\widehat{\mu}^*(\varpi)$ simply amounts to forgetting part of an integral in the computation below.

Remark 8

Under this map from surfaces with just one boundary component to quasi-Hamiltonian K-spaces, the quasi-Hamiltonian fusion operation corresponds to gluing two surfaces, each with one boundary component, into two of the holes of a three-holed sphere, so that the resulting surface again has one boundary component (see [3], [22]). Also,

†The signs differ from [3] because (1) we give the boundary circle the induced orientation; and (2) an overall sign change has been made anyway.

quasi-Hamiltonian reduction corresponds to fixing the conjugacy class of monodromy around the boundary component and forgetting the framing, thereby giving the usual symplectic moduli space of flat connections. The upshot is that once we allow fusion, all the symplectic manifolds that arise as moduli spaces of flat connections on surfaces may be constructed from just two types of quasi-Hamiltonian K-spaces: conjugacy classes (one for each boundary component), and the internally fused double ($\cong K \times K$), which corresponds to the one-holed torus.

Now, we apply the above philosophy to the extension of the Atiyah-Bott symplectic structure to singular connections (C^∞-connections with poles) given in [10]. First, we point out that the above story may be complexified; if Σ has just one boundary component and G is a connected complex reductive group (e.g., the complexification of K), then the moduli space of flat connections on G-bundles over Σ with framings at one point on the boundary are complex quasi-Hamiltonian G-spaces. In turn, if Σ has a complex structure, such moduli spaces may be identified with the moduli space of holomorphic connections on holomorphic G-bundles over Σ (together with a framing at one point on the boundary). (Both spaces are isomorphic to the manifold $\mathrm{Hom}(\pi_1(\Sigma, p), G)$ of fundamental group representations.)

In a similar way, the moduli spaces of flat C^∞ singular connections, which we define below, correspond both to moduli spaces of meromorphic connections on holomorphic G-bundles (cf. [10, Proposition 4.5]) and to spaces of monodromy/Stokes data (cf. [10, Proposition 4.8]).

Due to fusion, it is sufficient to consider only C^∞ singular connections on a disk having just one pole. Fix an integer $k \geq 1$ (the pole order) and an *irregular type*

$$\widetilde{A}^0 := A_0 \frac{dz}{z^k} + \cdots + A_{k-2} \frac{dz}{z^2} \in \Omega^1[D](\Delta, \mathfrak{g}),$$

where $A_i \in \mathfrak{t}$, $A_0 \in \mathfrak{t}_{\mathrm{reg}}$, z is a coordinate on the closed unit disk Δ, and $D := k(0)$ is a divisor on Δ supported at the origin. If $k = 1$, we set $\widetilde{A}^0 = 0$. The spaces of C^∞ singular connections which we are interested in have their full infinite jets of derivatives fixed, except for the residue term

$$\widetilde{\mathscr{A}} := \left\{ \alpha \in \Omega^1_{C^\infty}[D](\Delta, \mathfrak{g}) \;\middle|\; L_0(\alpha) = \widetilde{A}^0 + \Lambda \frac{dz}{z} \text{ for some } \Lambda \in \mathfrak{t}_k \right\},$$

where L_0 takes the full C^∞ Laurent expansion of α at the origin and $\mathfrak{t}_k = \mathfrak{t}$ if $k \geq 2$, but $\mathfrak{t}_1$ is the affine regular Cartan: $\mathfrak{t}_1 = \{\Lambda \in \mathfrak{t} \mid \beta(\Lambda) \notin \mathbb{Z} \text{ for all roots } \beta\}$. Let

$$\mathscr{G}_T := \left\{ g \in C^\infty(\Delta, G) \;\middle|\; L_0(g) \in T \subset G[\![z, \bar{z}]\!] \right\}$$

be the group of bundle automorphisms having Taylor expansion zero at the origin except for the constant term, which should be in T. Clearly, the tangent space to $\widetilde{\mathscr{A}}$ at a connection α is

$$T_\alpha \widetilde{\mathscr{A}} = \left\{ \phi \in \Omega^1_{C^\infty}[D](\Delta, \mathfrak{g}) \;\middle|\; L_0(\phi) \in \mathfrak{t}\frac{dz}{z} \right\}.$$

Thus, as in [10], we may still use the Atiyah-Bott formula in this singular situation and define a symplectic structure on $\widetilde{\mathscr{A}}$ as

$$\omega_{\widetilde{\mathscr{A}}}(\phi, \psi) = \int_\Delta (\phi, \psi).$$

LEMMA 9
The gauge action of the subgroup $\mathscr{G}_{1,\partial} := \{g \in \mathscr{G}_T \mid g|_{\partial\Delta} = 1, g(0) = 1\}$ *on* $\widetilde{\mathscr{A}}$ *is Hamiltonian with moment map given by the curvature.*

Proof
See [10, Proposition 5.4]. □

The symplectic quotient of $\widetilde{\mathscr{A}}$ at the zero value of the moment map is thus

$$\widehat{\mathscr{N}} := \widetilde{\mathscr{A}}_{\text{flat}} / \mathscr{G}_{1,\partial},$$

which has a residual action of $\mathscr{G}_T / \mathscr{G}_{1,\partial} \subset T \times LG$. The T-action is Hamiltonian with moment map

$$\alpha \mapsto -2\pi i \Lambda = -(2\pi i)\,\text{Res}_0 L_0(\alpha)$$

as in [10, Proposition 5.5], and (as above) the LG-action makes $\widehat{\mathscr{N}}$ into a Hamiltonian LG-space (in the sense of [3]) with moment map

$$\widehat{\mu} : \widehat{\mathscr{N}} \to \mathscr{A}_{S^1}, \qquad \alpha \mapsto \alpha|_{\partial\Delta}.$$

(Strictly speaking, if G is not simply connected, we may only have an action of the identity component of LG, but that is sufficient for our purposes.) Now, fix the point $p = -1 \in \partial\Delta$. Thus (momentarily forgetting the T-action), the quotient $\mathscr{N} := \widehat{\mathscr{N}} / \Omega G$ by the based loop group should be a quasi-Hamiltonian G-space. First, we use the irregular Riemann-Hilbert correspondence to identify $\mathscr{N}$ as a complex manifold. Let

$$\mathscr{G}_{1,p} := \left\{ g \in \mathscr{G}_T \;\middle|\; g(p) = 1 = g(0) \right\}$$

so that

$$\mathscr{N} = \widehat{\mathscr{N}} / \Omega G = \widetilde{\mathscr{A}}_{\text{flat}} / \mathscr{G}_{1,p}$$

which has a residual action of $\mathscr{G}_T / \mathscr{G}_{1,p} \cong G \times T$.

THEOREM 6 (see [10], [11])
The quotient $\widetilde{\mathscr{A}}_{\mathrm{flat}}/\mathscr{G}_{1,p}$ *is isomorphic to* $\widetilde{\mathscr{C}}$ *as a* $(G \times T)$*-space.*

Proof
As in [10, Proposition 4.5, Corollary 4.6], this quotient may be shown to be canonically isomorphic to the set of isomorphism classes of 4-tuples (P, A, g_0, g_p), where $P \to \Delta$ is a holomorphic principal G-bundle, A is a meromorphic connection on P with irregular type $\widetilde{A}^0$ and compatible framing g_0 at the origin, and g_p is an arbitrary framing of P at p. This was how $\mathscr{N}$ was described in the introduction, and it is achieved simply by noticing that the $(0, 1)$-parts of the C^∞ singular connections are actually nonsingular and so define holomorphic structures on the underlying G-bundle. Then, by the irregular Riemann-Hilbert correspondence of [11, Section 2] (extending [7]), the moduli space of such triples (P, A, g_0) is analytically isomorphic to the space $(U_+ \times U_-)^{k-1} \times \mathfrak{t}$ of Stokes multipliers and exponents of formal monodromy (Λ's). The inclusion of the framing g_p in the moduli problem simply adds a factor of G, so the result follows. The formula for the G-action is immediate, and for the T-action, see [10, Corollary 3.5]. □

Remark 10
The monodromy map $\widetilde{\nu} : \widetilde{\mathscr{A}}_{\mathrm{flat}} \twoheadrightarrow \widetilde{\mathscr{C}}$, whose fibres are precisely the $\mathscr{G}_{1,p}$ orbits, is described directly in the proof of Theorem 7. Also, the five pages of [11, Section 2] summarise all that one really needs to know about G-valued Stokes multipliers, at least in the generic case that we are considering here.

Now, if $\omega_{\widehat{\mathscr{N}}}$ is the symplectic structure on $\widehat{\mathscr{N}}$ and ϖ is the complex analogue of the two-form (14) on $\mathscr{A}_{S^1}$ (defined exactly the same way), then, by the general theory described above, we expect the two-form $-\omega_{\widehat{\mathscr{N}}} + \widehat{\mu}^*(\varpi)$ on $\widehat{\mathscr{N}}$ to be the pullback of some quasi-Hamiltonian two-form on $\widetilde{\mathscr{C}}$ along the map $\pi : \widehat{\mathscr{N}} \to \mathscr{N} \cong \widetilde{\mathscr{C}}$. Indeed, we have the following theorem, which is our second main result.

THEOREM 7
Let ω *be the two-form on* $\widetilde{\mathscr{C}}$ *defined in (7). Then we have*

$$-\pi^*(\omega) = \omega_{\widehat{\mathscr{N}}} - \widehat{\mu}^*(\varpi).$$

Proof
Since $\widehat{\mathscr{N}}$ is the symplectic quotient of $\widetilde{\mathscr{A}}$, this is equivalent to proving $\iota^*\omega_{\widetilde{\mathscr{A}}} - \mathrm{pr}^*\widehat{\mu}^*\varpi = -\widetilde{\nu}^*\omega$, where $\iota : \widetilde{\mathscr{A}}_{\mathrm{flat}} \to \widetilde{\mathscr{A}}$ is the inclusion, $\mathrm{pr} : \widetilde{\mathscr{A}}_{\mathrm{flat}} \to \widehat{\mathscr{N}}$ is the projection, and $\widetilde{\nu} : \widetilde{\mathscr{A}}_{\mathrm{flat}} \to \widetilde{\mathscr{C}}$. To this end, suppose that we have a two-parameter family $\alpha(s, t) \in \widetilde{\mathscr{A}}_{\mathrm{flat}}$ of flat singular connections depending holomorphically on

s, t. We evaluate the two-form $\iota^*\omega_{\widetilde{\mathscr{A}}} - \mathrm{pr}^*\widehat{\mu}^*\varpi$ on the pair $\alpha', \dot\alpha \in \Omega^1_{C^\infty}[D](\Delta, \mathfrak{g})$ of tangent vectors to $\widetilde{\mathscr{A}}_{\text{flat}}$ at $\alpha = \alpha(0,0)$, where $\alpha' = \frac{d}{ds}\alpha\big|_{s=t=0}$, $\dot\alpha = \frac{d}{dt}\alpha\big|_{s=t=0}$. If $X = \widetilde{\nu}_*(\alpha')$, $Y = \widetilde{\nu}_*(\dot\alpha) \in T_{\widetilde{\nu}(\alpha)}\widetilde{\mathscr{C}}$, then we should obtain $-\omega(X, Y)$, where by definition

$$2\omega(X,Y) = (\overline{\mathscr{D}}', \dot{\overline{\mathscr{E}}}) - (\dot{\overline{\mathscr{D}}}, \overline{\mathscr{E}}') + \sum_{j=1}^{k-1} (\mathscr{D}'_j, \dot{\mathscr{D}}_{j-1}) - (\dot{\mathscr{D}}_j, \mathscr{D}'_{j-1}) \\ - (\mathscr{E}'_j, \dot{\mathscr{E}}_{j-1}) + (\mathscr{E}'_j, \dot{\mathscr{E}}_{j-1})$$

with $\mathscr{D}'_j = \langle \mathscr{D}_j, X\rangle$, and so forth.

Let Δ_r denote the slit annulus obtained by cutting Δ along the ray from zero to $p = -1$ and removing the open disk of radius r centred on the origin. Denote by $\overline{\Delta}_r$ the closure of Δ_r in the universal cover of the punctured disk $\Delta \setminus \{0\}$. Thus, $\overline{\Delta}_r$ has two straight edges l_+, l_- lying over the interval $[-1, -r]$ and has interior isomorphic to the interior of $\Delta_r \subset \Delta$. In particular, $\overline{\Delta}_r$ is contractible. We identify the lower lip l_- with the interval $[-1, -r] \subset \Delta$, so that one arrives at the upper lip l_+ by turning a full turn in a positive sense from l_-. For each s, t, let

$$\chi(s,t) : \overline{\Delta}_r \to G$$

be the fundamental solution of the connection $\alpha(s,t)$ taking the value $1 \in G$ at $p \in l_-$. (In other words, $\chi(s,t)$ is the map solving the differential equation $\alpha(s,t) = \chi^*(\overline{\theta})$.) Then, for each $z \in \overline{\Delta}_r$, let $\chi'(z) := \frac{d}{dt}\chi(s,t,z)\big|_{s=t=0} \in T_{\chi(z)}G$, and so

$$\chi^{-1}\chi' := l_{\chi^{-1}}\chi'$$

is a $\mathfrak{g}$-valued function on $\overline{\Delta}_r$, where $l_{\chi^{-1}(z)} : T_{\chi(z)}G \to \mathfrak{g}$ denotes the derivative of left multiplication by $\chi^{-1}(z)$ in the group G. Now, define a one-form φ on $\overline{\Delta}_r$ by

$$\varphi := \frac{1}{2}(\varphi_1 - \varphi_2), \qquad \varphi_1 := \big(\chi^{-1}\chi', d(\chi^{-1}\dot\chi)\big),\ \varphi_2 := \big(d(\chi^{-1}\chi'), \chi^{-1}\dot\chi\big),$$

where d is the exterior derivative on $\overline{\Delta}_r$. Thus, $d\varphi = (\alpha', \dot\alpha)$ as two-forms on $\overline{\Delta}_r$ (since, e.g., $\alpha' = \chi d(\chi^{-1}\chi')\chi^{-1}$). In turn, since $(\alpha', \dot\alpha)$ is a smooth two-form on Δ, we have

$$\omega_{\widetilde{\mathscr{A}}}(\alpha', \dot\alpha) = \int_\Delta (\alpha', \dot\alpha) = \lim_{r\to 0}\int_{\overline{\Delta}_r} d\varphi = \lim_{r\to 0}\int_{\partial\overline{\Delta}_r}\varphi.$$

This integral is evaluated along each arc of the boundary of $\overline{\Delta}_r$, neglecting any terms that vanish in the limit. A similar calculation appears in [24].

First, around the outer boundary of $\overline{\Delta}_r$ (the circle of radius one), we recognise that the integral of φ is precisely $(\mathrm{pr}^*\widehat{\mu}^*\varpi)(\alpha', \dot\alpha)$ (since, on this circle, χ restricts to the map h_z used to define ϖ), which is the term to be subtracted off.

For the other arcs, we first need to describe directly the map $\widetilde{\nu} : \widetilde{\mathscr{A}}_{\text{flat}} \to \widetilde{\mathscr{C}}$ associating monodromy data $(C, \mathbf{d}, \mathbf{e})$ to a flat singular connection α. The key point is that any $\alpha \in \widetilde{\mathscr{A}}_{\text{flat}}$ has canonical fundamental solutions

$$\Phi_i : \mathrm{Sect}_i \to G, \qquad (d\Phi_i)\Phi_i^{-1} = \alpha$$

on certain distinguished sectors Sect_i defined as follows (see [10, Lemma 4.7], [11, Section 2]). The leading coefficient $A_0 \in \mathfrak{t}_{\text{reg}}$ of the chosen irregular type $\widetilde{A}^0$ determines the *anti-Stokes directions* $\mathbb{A}$ at $0 \in \Delta$ defined as

$$z \in \Delta \setminus \{0\} \text{ lies on an anti-Stokes direction } \iff \frac{\beta(A_0)}{z^{k-1}} \in \mathbb{R}$$
$$\text{for some root } \beta \in \mathscr{R}.$$

This determines a finite set $\mathbb{A}$ of directions which is clearly invariant under rotation by $\pi/(k-1)$, and so the number $l := \#\mathbb{A}/(2k-2)$ is an integer. The sectors Sect_i are just the sectors bounded by consecutive anti-Stokes directions. Without loss of generality, we assume that the positive real axis $\mathbb{R}_+$ is not an anti-Stokes direction, and we label these sectors in a positive sense and so that $\mathbb{R}_+ \subset \mathrm{Sect}_0$. In turn, the anti-Stokes directions $a_i \in \mathbb{A}$ are labelled (modulo $\#\mathbb{A}$) so that $\mathrm{Sect}_i = \mathrm{Sect}(a_i, a_{i+1})$. By [11, Lemma 2.4], we know that the set of roots

$$\mathscr{R}_+ := \left\{ \beta \in \mathscr{R} \;\middle|\; \frac{\beta(A_0)}{z^{k-1}} \in \mathbb{R}_+ \text{ for } z \text{ on one of the directions } a_1, \dots, a_l \right\}$$

"supporting" one of the first l anti-Stokes directions is a set of positive roots, and we define B_+ to be the corresponding Borel subgroup containing T. Now, to define Φ_i, we recall that the Laurent expansion of α is

$$L_0(\alpha) = dQ + \Lambda \frac{dz}{z} =: A^0$$

for some $\Lambda \in \mathfrak{t}$, where $Q := \sum_{j=1}^{k-1} \frac{z^{j-k}}{j-k} A^0_{j-1}$ (so $dQ = \widetilde{A}^0$). In particular, the $(0,1)$-part of α is nonsingular across the origin, and so we may solve the $\overline{\partial}$-problem

$$(\overline{\partial} g)g^{-1} = \alpha^{0,1}$$

for a smooth map $g : U \to G$ defined in some neighbourhood $U \subset \Delta$ of the origin. Given such g, one observes (see [10, Lemma 4.3]) that the Taylor expansion $\widehat{F} = L_0(g^{-1})$ is in $G[\![z]\!]$ (has no $\overline{z}$ terms), and that $A := \widehat{F}[A^0] = g^{-1}[\alpha]$ is the germ of a (convergent) meromorphic connection. In turn, this implies (see [11, Theorem 2.5]) that there is a unique holomorphic map

$$\Sigma_i(\widehat{F}) : \mathrm{Sect}_i \to G$$

on each sector such that $\Sigma_i(\widehat{F})[A^0] = A$ and that the analytic continuation of $\Sigma_i(\widehat{F})$ to the *supersector*

$$\widehat{\mathrm{Sect}}_i := \mathrm{Sect}\Big(a_i - \frac{\pi}{2k-2}, a_{i+1} + \frac{\pi}{2k-2}\Big)$$

is asymptotic to $\widehat{F}$ at zero in $\widehat{\mathrm{Sect}}_i$. Now, we are led to the following definition because $z^\Lambda e^Q$ is a fundamental solution of the connection A^0, $\Sigma_i(\widehat{F})$ is an isomorphism between A^0 and A, and g is an isomorphism between A and α. (Here z^Λ is defined on Sect_0 using the branch of $\log(z)$ which is real on $\mathbb{R}_+$, and by convention, we extend this to the other sectors in a *negative* sense.)

Definition 11
The *canonical fundamental solution* of $\alpha \in \widetilde{\mathscr{A}}_{\mathrm{flat}}$ on Sect_i is the map

$$\Phi_i := g\Sigma_i(L_0 g^{-1})z^\Lambda e^Q : \mathrm{Sect}_i \to G$$

for *any* solution g of $(\overline{\partial} g)g^{-1} = \alpha^{0,1}$.

The Stokes multipliers S_i of α can now be defined (as in [11, Definition 2.6]) as the constant elements of G relating the fundamental solutions Φ_{il} and $\Phi_{(i+1)l}$. However, to define directly the elements d_i, e_i, we first define new fundamental solutions Ψ_i, Θ_i as

$$\Psi_i := \Phi_{il}\epsilon^{2k-2-i} : \mathrm{Sect}_{il} \to G \quad (i = 1, \ldots, 2k-2), \qquad \Theta_i = \Psi_{2k-2-i},$$

where $\epsilon := e^{\pi i\Lambda/(k-1)}$. The indices of Ψ_i, Θ_i are taken modulo $2k-2$, so $\Psi_0 = \Theta_0 = \Phi_0$ on Sect_0. For $i = 0, \ldots, k-2$, the sector on which Ψ_i or Θ_i is defined intersects the slit annulus Δ_r in a contractible set, and so we may extend Ψ_i, Θ_i uniquely (as fundamental solutions of α) to maps from $\overline{\Delta}_r$ to G. Now, the intersection of $\mathrm{Sect}_{(k-1)l}$ (the sector containing $\mathbb{R}_-$) and Δ_r has two components, and we extend Ψ_{k-1} from the upper component of this intersection onto $\overline{\Delta}_r$, and we extend Θ_{k-1} from the lower component. Thus, we have $2k$ generally distinct fundamental solutions of α on $\overline{\Delta}_r$:

$$\chi, \Phi_0 = \Psi_0 = \Theta_0, \Psi_1, \ldots, \Psi_{k-1}, \Theta_1, \ldots, \Theta_{k-1}.$$

The monodromy data C, d_i, e_i is defined to be the set of (z-independent) group elements relating them, as follows:

$$\Phi_0 C = \chi, \qquad \Psi_i e_i = \Psi_{i-1}, \qquad \Theta_i d_i = \Theta_{i-1} \quad (i = 1, \ldots, k-1).$$

If d_i, e_i are defined in this way, it follows from [11, Lemma 2.7] that $d_{\mathrm{even}}, e_{\mathrm{odd}} \in B_+$, $d_{\mathrm{odd}}, e_{\mathrm{even}} \in B_-$, and $\delta(d_j)^{-1} = \epsilon = \delta(e_j)$, so we have indeed associated a point of $\widetilde{\mathscr{C}}$

to α. Note also that the maps $D_i, E_i : \widetilde{\mathscr{C}} \to G$ arise as

$$\Psi_i E_i = \chi, \qquad \Theta_i D_i = \chi \quad (i = 0, \dots, k-1).$$

It follows that χ has holonomy $D^{-1}E$ since $\chi|_{l_+} = \Psi_{k-1}|_{l_+}E = \Theta_{k-1}|_{l_-}E = \chi|_{l_-}D^{-1}E$, and so this is the quasi-Hamiltonian moment map.

Now, we return to the boundary integral. Choose a point q_i of distance r from the origin and in the intersection $\widehat{\mathrm{Sect}}_{il} \cap \widehat{\mathrm{Sect}}_{(i-1)l}$ of two of the supersectors, for $i = 1, \dots, k-1$. Thus, we know that both $\Phi_{il}(q_i)$ and $\Phi_{(i-1)l}(q_i)$ are asymptotic to $z^\Lambda e^Q$ at zero as $r \to 0$, and in turn we know the asymptotics of $\Psi_i(q_i)$ and $\Psi_{i-1}(q_i)$ at zero. Similarly, choose $p_i \in \widehat{\mathrm{Sect}}_{-il} \cap \widehat{\mathrm{Sect}}_{-(i-1)l}$ of modulus r so that we know the asymptotics of both $\Theta_i(p_i)$ and $\Theta_{i-1}(p_i)$ at zero as $r \to 0$. Let $p_k = -r \in l_-$, and let q_k be the point of the upper lip l_+ lying over $-r$. Thus, we may divide the inner boundary circle of $\overline{\Delta}_r$ into $2k-1$ arcs by breaking it at the points p_i, q_i. Now, since $\chi = \Psi_i E_i$ and E_i is z-independent, we find

$$\varphi_1 = \big(\Psi_i^{-1}\Psi_i', d(\Psi_i^{-1}\dot{\Psi}_i)\big) + d(\overline{\mathscr{E}}_i', \Psi_i^{-1}\dot{\Psi}_i), \tag{15}$$

where $\overline{\mathscr{E}}_i' = \langle E_i^*\overline{\theta}, X\rangle$, and similarly for φ_2 (swapping the dot and the prime).

LEMMA 12
The first term in (15) may be neglected in the integral from q_{i+1} to q_i.

Proof
The first term of (15) and the corresponding term of φ_2 contribute

$$\frac{1}{2}\int_{q_{i+1}}^{q_i} \big(\Psi_i^{-1}\Psi_i', d(\Psi_i^{-1}\dot{\Psi}_i)\big) - \big(\Psi_i^{-1}\dot{\Psi}_i, d(\Psi_i^{-1}\Psi_i')\big) \tag{16}$$

to the integral of φ. However, $\Psi_i \simeq z^\Lambda e^Q \epsilon^{2k-2-i}$ at zero uniformly in $\widehat{\mathrm{Sect}}_{il}$ (which contains the integration path). Substituting in this approximation gives that the integrand in (16) is zero. This implies that in the limit $r \to 0$, the integral (16) really is zero. □

Thus modulo negligible terms

$$\int_{q_{i+1}}^{q_i} \varphi_1 = (\overline{\mathscr{E}}_i', \Psi_i^{-1}\dot{\Psi}_i)|_{q_{i+1}}^{q_i}.$$

If we sum this integral for $i = 1, \dots, k-1$, then the contribution at q_i is

$$(\overline{\mathscr{E}}_i', \Psi_i^{-1}\dot{\Psi}_i)(q_i) - (\overline{\mathscr{E}}_{i-1}', \Psi_{i-1}^{-1}\dot{\Psi}_{i-1})(q_i),$$

provided that $i \neq 1, k$. Now, using $\Psi_{i-1} = \Psi_i e_i$ to remove Ψ_{i-1}, this becomes

$$(\overline{\mathscr{E}}_i' - e_i \overline{\mathscr{E}}_{i-1}' e_i^{-1}, \Psi_i^{-1} \dot{\Psi}_i) - (\overline{\mathscr{E}}_{i-1}', \dot{\varepsilon}_i),$$

where $\dot{\varepsilon}_i = \langle e_i^* \theta, Y \rangle$. In turn, using $E_i = e_i E_{i-1}$, this becomes

$$(\varepsilon_i', \Psi_i^{-1} \dot{\Psi}_i) - (\mathscr{E}_{i-1}', \dot{\mathscr{E}}_i). \tag{17}$$

If we also repeat the above for φ_2, we get the same but with the dots and primes swapped. Now, since $\Psi_i \simeq z^\Lambda e^Q \epsilon^{2k-2-i}$ and the T-component of e_i is ϵ, we deduce $(\varepsilon_i', \Psi_i^{-1} \dot{\Psi}_i) - (\dot{\varepsilon}_i, \Psi_i^{-1} \Psi_i') \to 0$ as $r \to 0$. Thus, the contribution at q_i $(i \neq 1, k)$ to the integral of φ from q_k to q_1 is

$$-\frac{1}{2}\big((\mathscr{E}_{i-1}', \dot{\mathscr{E}}_i) - (\dot{\mathscr{E}}_{i-1}, \mathscr{E}_i')\big) = \frac{1}{2}(\mathscr{E}_i, \mathscr{E}_{i-1})(X, Y),$$

which is a term appearing in $-\omega(X, Y)$. Writing $p_0 := q_1$ and performing the same manipulations for the Θ_i's, integrating φ from p_0 to p_k yields a contribution of

$$\frac{1}{2}\big((\mathscr{D}_{i-1}', \dot{\mathscr{D}}_i) - (\dot{\mathscr{D}}_{i-1}, \mathscr{D}_i')\big) = -\frac{1}{2}(\mathscr{D}_i, \mathscr{D}_{i-1})(X, Y)$$

at p_i, provided that $i \neq 0, k$. The two leftover contributions at $q_1 = p_0$ combine to give the term $(1/2)(\mathscr{E}_1, \mathscr{E}_0)(X, Y)$. (Thus, all terms of $-\omega$ except $-(1/2)(\overline{\mathscr{D}}, \overline{\mathscr{E}})(X, Y)$ have been obtained so far.) The leftover contributions at q_k and p_k are

$$\frac{1}{2}\big((\overline{\mathscr{D}}', \Theta_{k-1}^{-1} \dot{\Theta}_{k-1}) - (\dot{\overline{\mathscr{D}}}, \Theta_{k-1}^{-1} \Theta_{k-1}')\big)(p_k) - \frac{1}{2}\big((\overline{\mathscr{E}}', \Psi_{k-1}^{-1} \dot{\Psi}_{k-1}) - (\dot{\overline{\mathscr{E}}}, \Psi_{k-1}^{-1} \Psi_{k-1}')\big)(q_k).$$

Now, consider the two straight edges $l_\pm$ of $\overline{\Delta}_r$. Recall that $\Theta_{k-1}|_{l_-} = \Psi_{k-1}|_{l_+}$, so that from (15),

$$\int_{l_+ + l_-} \varphi_1 = \int_{p_k}^{p} d(\overline{\mathscr{D}}' - \overline{\mathscr{E}}', \Theta_{k-1}^{-1} \dot{\Theta}_{k-1}) = (\overline{\mathscr{D}}' - \overline{\mathscr{E}}', \Theta_{k-1}^{-1} \dot{\Theta}_{k-1})|_{p_k}^{p},$$

and similarly for φ_2. Observe that the contribution at p_k to the integral of φ along $l_\pm$ cancels precisely with the leftover terms at p_k, q_k displayed above. Finally, since $\Theta_{k-1}(p) = \chi(p) D^{-1} = D^{-1}$, the contribution at p is

$$-\frac{1}{2}\big((\overline{\mathscr{D}}' - \overline{\mathscr{E}}', \dot{\overline{\mathscr{D}}}) - (\dot{\overline{\mathscr{D}}} - \dot{\overline{\mathscr{E}}}, \overline{\mathscr{D}}')\big) = -\frac{1}{2}(\overline{\mathscr{D}}, \overline{\mathscr{E}})(X, Y). \qquad \square$$

5. Additive analogues

Here we recall (from [10, Section 2]) the complex symplectic manifolds $O, \widetilde{O}$, which are the additive analogues of the quasi-Hamiltonian spaces $\mathscr{C}, \widetilde{\mathscr{C}}$.

Fix an integer $k \geq 2$. Let $G_k := G(\mathbb{C}[z]/z^k)$ be the group of k-jets of bundle automorphisms, and let $\mathfrak{g}_k = \mathrm{Lie}(G_k)$ be its Lie algebra, which contains elements of the form $X = X_0 + X_1 z + \cdots + X_{k-1} z^{k-1}$ with $X_i \in \mathfrak{g}$. Let B_k be the subgroup of G_k of elements having constant term 1. The group G_k is the semidirect product $G \ltimes B_k$ (where G acts on B_k by conjugation). Correspondingly, the Lie algebra of G_k decomposes as a vector space direct sum, and dualising we have: $\mathfrak{g}_k^* = \mathfrak{b}_k^* \oplus \mathfrak{g}^*$. Elements of $\mathfrak{g}_k^*$ are written as

$$A = A_0 \frac{dz}{z^k} + \cdots + A_{k-1} \frac{dz}{z} \tag{18}$$

via the pairing with $\mathfrak{g}_k$ given by $\langle A, X \rangle := \mathrm{Res}_0(A, X) = \sum_{i+j=k-1} (A_i, X_j)$. In this way, $\mathfrak{b}_k^*$ is identified with the set of A having zero residue and $\mathfrak{g}^*$ with those having only a residue term (zero irregular part). Let $\pi_{\mathrm{res}} : \mathfrak{g}_k^* \to \mathfrak{g}^*$, and let $\pi_{\mathrm{irr}} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose an element (an "irregular type")

$$\widetilde{A}^0 = A_0^0 \frac{dz}{z^k} + \cdots + A_{k-2}^0 \frac{dz}{z^2}$$

of $\mathfrak{b}_k^*$ with $A_i^0 \in \mathfrak{t}$ and with regular leading coefficient $A_0^0 \in \mathfrak{t}_{\mathrm{reg}}$. Let $O_B \subset \mathfrak{b}_k^*$ denote the B_k-coadjoint orbit containing $\widetilde{A}^0$.

Definition 13
The *extended orbit* $\widetilde{O} \subset G \times \mathfrak{g}_k^*$ associated to O_B is

$$\widetilde{O} := \left\{ (g_0, A) \in G \times \mathfrak{g}_k^* \;\middle|\; \pi_{\mathrm{irr}}(g_0 A g_0^{-1}) \in O_B \right\},$$

where $\pi_{\mathrm{irr}} : \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue.

If $(g_0, A) \in \widetilde{O}$, then A corresponds to the principal part of a generic meromorphic connection and g_0 to a compatible framing.

In the simple pole case $k = 1$, we define

$$\widetilde{O} := \left\{ (g_0, A) \in G \times \mathfrak{g}^* \;\middle|\; g_0 A g_0^{-1} \in \mathfrak{t}_1 \right\} \subset G \times \mathfrak{g}^*,$$

where $\mathfrak{t}_1 \subset \mathfrak{t}^* \cong \mathfrak{t}$ is the complement of the affine root hyperplanes. If we identify $G \times \mathfrak{g}^*$ with T^*G, then $\widetilde{O}$ is, in fact, a symplectic submanifold (see [15, Theorem 26.7]).

The basic properties of these extended orbits may be summarised as follows. Given $(g_0, A) \in \widetilde{O}$, then by hypothesis, there is some $g \in G_k$ such that $gAg^{-1} = \widetilde{A}^0 + R\,dz/z$ for some $R \in \mathfrak{g}$, and we define a map $\Lambda = \delta(R) : \widetilde{O} \to \mathfrak{t} \cong \mathfrak{t}^*$ by taking the $\mathfrak{t}$ component of R (which is independent of g).

PROPOSITION 14 (see [10, Section 2])

(1) *The extended orbit* $\widetilde{O}$ *is isomorphic to the symplectic quotient* $(T^*G_k \times O_B)/\!\!/ B_k$.

(2) *(Decoupling). The map* $\widetilde{O} \to (T^*G) \times O_B$; $(g_0, A) \mapsto (g_0, \pi_{\text{res}}(A), \pi_{\text{irr}}(g_0 A g_0^{-1}))$ *is a symplectic isomorphism, where* $T^*G \cong G \times \mathfrak{g}^*$ *via the left trivialisation.*

(3) *The map* $-\Lambda$ *is a moment map for the free action of* T *on* $\widetilde{O}$ *defined by* $t(g_0, A) = (tg_0, A)$, *where* $t \in T$.

(4) *The symplectic quotient of* $\widetilde{O}$ *by* T *at the value* $-\Lambda$ *of the moment map is the* G_k*-coadjoint orbit* $O \subset \mathfrak{g}_k^*$ *through the element* $\widetilde{A}^0 + \Lambda\, dz/z$ *of* $\mathfrak{g}_k^*$.

(5) *The free* G*-action* $h(g_0, A) := (g_0 h^{-1}, hAh^{-1})$ *on* $\widetilde{O}$ *is Hamiltonian with moment map* $\mu_G : \widetilde{O} \to \mathfrak{g}^*$; $(g_0, A) \mapsto \pi_{\text{res}}(A)$.

In particular, $\widetilde{O}$ is a Hamiltonian $(G \times T)$-manifold with T reductions equal to G_k-coadjoint orbits O; these properties are viewed as natural analogues of those of $\widetilde{\mathscr{C}}$ (and they do indeed match up under the Riemann-Hilbert correspondence). Note that the coadjoint orbit O_B is a point if $k = 2$, so that part (2) says $\widetilde{O} \cong T^*G$, the additive analogue of the fact that $\widetilde{\mathscr{C}} \cong G \times G^*$ for $k = 2$.

6. Global moduli spaces

In this section, we describe the global moduli spaces that arise upon fusing together some of the basic quasi-Hamiltonian spaces described so far. This section is less self-contained than the rest of the article, and some familiarity with [10] and [11] is assumed. (Note that [10] was written with the present article in mind; the spaces $\widetilde{\mathscr{C}}, \mathscr{C}$ do already appear there, but without the quasi-Hamiltonian two-forms.) The main purpose of this section is to explain how the results of [10] extend to the current context, so we focus on the necessary modifications to the proofs there rather than repeating everything.

The two main new features to incorporate are

(I) higher genus curves; and

(II) arbitrary structure groups G.

Increasing the genus causes no problems, and we only really notice it in that there are some commutators added to the main monodromy relation. Changing the structure group is also now quite straightforward since the local theory was studied in [11] (see [11, pages 1132–1137] for a summary). The main steps left are to define the "topological type" $d \in \pi_1(G)$ of a set of global Stokes/monodromy data and to see that this agrees with the topological type of the underlying G-bundle. We do this here.

The general setup is as follows. Fix integers $g \geq 0, m \geq 1$, a genus g compact Riemann surface Σ, and m distinct points $\{a_1, \ldots, a_m\} \subset \Sigma$ with multiplicities $k_1, \ldots, k_m \geq 1$, respectively. We suppose that there is at least one irregular singularity (some $k_i \geq 2$), and if there is just one pole and the curve is rational $(m = 1, g = 0)$, then the pole is of order at least three ($k_1 \geq 3$). Let G be a connected complex reductive group with maximal torus T (with corresponding Lie algebras $\mathfrak{t} \subset \mathfrak{g}$), and fix a topological type $d \in \pi_1(G)$ of G-bundle on Σ. (Recall that any holomorphic G-bundle on Σ may be described by a holomorphic clutching map $h : \mathbb{A} \to G$, from a small annulus $\mathbb{A} \subset \Sigma$, and the topological type of the bundle is given by the homotopy class of the image $d = h_*(\gamma) \in \pi_1(G)$ of a positive loop γ in $\mathbb{A}$ generating $\pi_1(\mathbb{A})$.)

At each marked point a_i, we also choose an irregular type: the principal part of a (residue-less) $\mathfrak{t}$-valued meromorphic one-form with regular leading coefficient. That is,

$$ {}^i\widetilde{A^0} := {}^iA_0 \frac{dz}{z^{k_i}} + \cdots + {}^iA_{k_i-2} \frac{dz}{z^2}, $$

where ${}^iA_j \in \mathfrak{t}$, ${}^iA_0 \in \mathfrak{t}_{\mathrm{reg}}$, and z is a coordinate defined on a neighbourhood of a_i vanishing at a_i. For consistent notation, if $k_i = 1$ we set ${}^i\widetilde{A^0} = 0$.

Let $\mathbf{A}$ denote the continuous subset of the above data, which may vary if we fix just $g, m, k_1, \ldots, k_m, G, T$, and d; namely, $\mathbf{A}$ denotes the choice of the points a_i, the complex structure of Σ, and the irregular types ${}^i\widetilde{A^0}$.

Given this data, the moduli spaces we are interested in are as follows.

Definition 15

- $\widetilde{\mathscr{M}}(\mathbf{A})$ is the (extended) moduli space of isomorphism classes of triples $(P, A, \mathbf{s})$ consisting of a holomorphic principal G-bundle $P \to \Sigma$ and a compatibly framed meromorphic connection $(A, \mathbf{s})$ with irregular type ${}^i\widetilde{A^0}$ at a_i and no other poles.
- $\widetilde{\mathscr{M}}_d(\mathbf{A}) \subset \widetilde{\mathscr{M}}(\mathbf{A})$ is the subset where the bundles have topological type d.

Here $\mathbf{s} = (s_1, \ldots, s_m)$ is an m-tuple of framings of the G-bundle P, where $s_i \in P_{a_i}$. Given $\mathbf{s}$, we may choose a local section s of P near a_i with $s(a_i) = s_i$ and so identify A with the local $\mathfrak{g}$-valued meromorphic one-form $A^s := -s^*(A)$. The statement that $(A, \mathbf{s})$ is compatibly framed with irregular type ${}^i\widetilde{A^0}$ at a_i means simply that we can choose such s so that near a_i,

$$ A^s = {}^i\widetilde{A^0} + {}^i\Lambda \frac{dz}{z} + \text{holomorphic terms} $$

for some (uniquely determined) ${}^i\Lambda \in \mathfrak{t}$, the *exponent* of A at a_i. If the pole is simple ($k_i = 1$), then the framing should be such that ${}^i\Lambda \in \mathfrak{t}$, and we further insist that ${}^i\Lambda$ should be affine-regular.

A basic result that we extend from [10] is as follows. (The proofs of the following three results are deferred to the end of this section.)

THEOREM 8
The moduli space $\widetilde{\mathcal{M}}_d(\mathbf{A})$ may be described as a complex symplectic quotient, starting with an Atiyah-Bott-type symplectic structure on an (infinite-dimensional) space of C^∞ singular connections.

This defines a complex symplectic structure on $\widetilde{\mathcal{M}}(\mathbf{A})$, and it turns out (see [10]) that the action of $T^{\times m}$ on $\widetilde{\mathcal{M}}(\mathbf{A})$ (defined by changing the compatible framings) is Hamiltonian with moment map given by (minus) the exponents $(-{}^1\Lambda, \dots, -{}^m\Lambda)$.

Now, we identify $\widetilde{\mathcal{M}}(\mathbf{A})$ with a finite-dimensional quasi-Hamiltonian quotient and so give a finite-dimensional construction of $\widetilde{\mathcal{M}}(\mathbf{A})$ as a complex symplectic manifold, which was the principal motivation of this article.

Let $\widetilde{\mathcal{C}}_i$ be the basic quasi-Hamiltonian space from Theorem 1 with $k = k_i$, and consider the space

$$\widetilde{M}(\mathbf{A}) := (\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m) /\!\!/ G$$

with g copies of $\mathbf{D}$. By construction, this is a complex symplectic manifold. The remaining action of $T^{\times m}$ coming from the T-action on each $\widetilde{\mathcal{C}}_i$ makes it into a quasi-Hamiltonian $T^{\times m}$-space with moment map $(\exp(-2\pi i \cdot {}^1\Lambda), \dots, \exp(-2\pi i \cdot {}^m\Lambda))$. Thus (since ${}^i\Lambda$ is a well-defined function), $\widetilde{M}(\mathbf{A})$ is also a Hamiltonian T^m-space with moment map $(-{}^1\Lambda, \dots, -{}^m\Lambda)$. A basic observation yielding the topological types of the corresponding principal bundles is the following theorem.

LEMMA 16
There is a surjective map (which is defined precisely below)

$$\mathrm{top} : \widetilde{M}(\mathbf{A}) \to \pi_1(G)$$

with pairwise isomorphic fibres.

Then the main result is now essentially a corollary of the previous results and of the results of [3] concerning fusion, as in the following.

THEOREM 9
The irregular Riemann-Hilbert correspondence, associating monodromy/Stokes data to meromorphic connections, gives an analytic isomorphism

$$\widetilde{\mathcal{M}}(\mathbf{A}) \xrightarrow{\cong} \widetilde{M}(\mathbf{A}) = (\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \widetilde{\mathcal{C}}_1 \circledast \cdots \circledast \widetilde{\mathcal{C}}_m) /\!\!/ G$$

of Hamiltonian $T^{\times m}$-spaces, mapping $\widetilde{\mathcal{M}}_d(\mathbf{A})$ isomorphically onto $\mathrm{top}^{-1}(d) \subset \widetilde{M}(\mathbf{A})$.

In particular, if we perform the symplectic reduction by T^m at a generic fixed value of the exponents $({}^1\Lambda, \dots, {}^m\Lambda)$, then we obtain a symplectic map

$$\mathscr{M}(\mathbf{A}) \xrightarrow{\cong} M(\mathbf{A}) = (\mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \mathscr{C}_1 \circledast \cdots \circledast \mathscr{C}_m) /\!\!/ G,$$

where $\mathscr{M}(\mathbf{A})$ is the moduli space obtained by forgetting the compatible framings and fixing the exponents ${}^i\Lambda$, and $\mathscr{C}_i$ is the basic quasi-Hamiltonian space from Theorem 1 with $k = k_i$, $\Lambda = {}^i\Lambda$. For $G = \mathrm{GL}_n(\mathbb{C})$, these moduli spaces $\mathscr{M}(\mathbf{A})$ have been shown to support complete hyper-Kähler metrics (see [8]). (In general—nongeneric exponents—the quotient is not well behaved, and stability conditions are needed; see [8].) Note that the full topological type is not necessarily fixed in $M(\mathbf{A})$, although the degree (denoted "deg" below) is.

Remark 17
We conjecture that the spaces $\widetilde{\mathscr{M}}_d(\mathbf{A}) \cong \widetilde{M}_d(\mathbf{A})$ are connected. For example, if one were able to prove that the moment map

$$\mu : \mathbf{D} \circledast \cdots \circledast \mathbf{D} \circledast \widetilde{\mathscr{C}}_1 \circledast \cdots \circledast \widetilde{\mathscr{C}}_m \to G$$

was a fibration (the restrictions we made imply that it is surjective and submersive), then we would be able to prove that $\widetilde{M}_d(\mathbf{A})$ was connected directly using the homotopy long exact sequence. This is not pursued further here since it is not directly needed.

In the genus-zero, topologically trivial case ($g = 0$, $d = \mathrm{id}$), we may go further and describe a big open subset of $\widetilde{\mathscr{M}}(\mathbf{A})$ directly using the additive analogues of Section 5: the argument of [10, Proposition 2.1] explains how the symplectic manifold

$$(\widetilde{O}_1 \times \cdots \times \widetilde{O}_m) /\!\!/ G,$$

where $\widetilde{O}_i$ is the extended orbit corresponding to ${}^i\widetilde{A^0}$, is isomorphic to the open submanifold of $\widetilde{\mathscr{M}}(\mathbf{A})$, where the G-bundles are holomorphically trivial. (This is achieved simply by choosing a global holomorphic trivialisation and writing out what one gets.) Moreover, [10, Theorem 6.1] implies that it is a symplectic submanifold, provided that we first scale the symplectic structure by $2\pi i$. Theorem 9 specialises to Corollary 1 of the introduction, that the Riemann-Hilbert map restricts to a (transcendental) symplectic map

$$(\widetilde{O}_1 \times \cdots \times \widetilde{O}_m) /\!\!/ G \hookrightarrow (\widetilde{\mathscr{C}}_1 \circledast \cdots \circledast \widetilde{\mathscr{C}}_m) /\!\!/ G$$

between these two symplectic manifolds.

Proof of Lemma 16
First, we recall some basic facts about connected complex reductive groups (see, e.g., [5, page 561] for the parallel facts for compact groups). Let $H = Z(G)_e$ be the identity

component of the centre of G, $S = [G, G]$ the commutator subgroup (which is also the maximal connected semisimple subgroup), and let $D = H \cap S$, which is a finite abelian group. Let $\widetilde{S}$ be the universal cover of S, let $\overline{S} = S/D$, and let $\overline{H} = H/D$. Then $G = H \times_D S$, and quotienting further by D, we have a short exact sequence

$$1 \to D \to G \to \overline{H} \times \overline{S} \to 1, \tag{19}$$

and so the homotopy long exact sequence for fibrations gives the exact sequence

$$1 \to \pi_1(G) \to \pi_1(\overline{H}) \times \pi_1(\overline{S}) \to D \to 1. \tag{20}$$

Thus, we may identify $d \in \pi_1(G)$ with a pair $(a, b) \in \pi_1(\overline{H}) \times \pi_1(\overline{S})$ having opposite images in D; if we denote the map $G \to \overline{H} \times \overline{S}$ by $(\pi_{\overline{H}}, \pi_{\overline{S}})$, then $a = (\pi_{\overline{H}})_*(d)$, $b = (\pi_{\overline{S}})_*(d)$.

Also, the group $\widetilde{G} := H \times \widetilde{S}$ (having $\pi_1(\widetilde{G}) = \pi_1(H)$) is a finite covering of G, so if we let $\widetilde{D}$ be the kernel, then there is an exact sequence

$$1 \to \widetilde{D} \to \widetilde{G} \to G \to 1 \tag{21}$$

whose homotopy long exact sequence yields

$$1 \to \pi_1(H) \to \pi_1(G) \to \widetilde{D} \to 1. \tag{22}$$

Now we define $\mathrm{top}(p) \in \pi_1(G)$ associated to some monodromy/Stokes data (and exponents)

$$p = (\mathbf{a}, \mathbf{b}, \mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}) \in \mu^{-1}(1) \subset \mathbf{D}^g \circledast \widetilde{\mathscr{C}}_1 \circledast \cdots \circledast \widetilde{\mathscr{C}}_m$$

with $(a_i, b_i) \in \mathbf{D}$, $(C_j, \mathbf{S}_j, \Lambda_j) \in \widetilde{\mathscr{C}}_j$, $i = 1, \ldots, g$, $j = 1, \ldots, m$. Due to (20), it is sufficient to define the components of $\mathrm{top}(p)$ in $\pi_1(\overline{H})$ and $\pi_1(\overline{S})$, which we will call $\deg(p)$ and $\mathrm{tor}(p)$, respectively:

$$\mathrm{top}(p) = \big(\deg(p), \mathrm{tor}(p)\big) \in \pi_1(\overline{H}) \times \pi_1(\overline{S}).$$

For $\deg(p)$, simply apply $\pi_{\overline{H}}$ to the monodromy relation $\mu(p) = 1$; this reduces to

$$\pi_{\overline{H}}\bigg(\exp\Big(2\pi i \sum \Lambda_j\Big)\bigg) = 1$$

since $\overline{H}$ is abelian. (Each a_i, b_i, C_j cancels exactly, and each Stokes matrix is unipotent, so it maps to the identity.) Thus, $\pi_{\mathfrak{h}}\big(\sum \Lambda_j\big) \in \ker(\exp_{\overline{H}}(2\pi i \cdot) : \mathfrak{h} \to \overline{H})$, where $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$ is the derivative of $\pi_{\overline{H}}$ at the identity and $\mathfrak{h}$ is the Lie algebra of

$\overline{H}$. However, $\exp_{\overline{H}}(2\pi i\cdot) : \mathfrak{h} \to \overline{H}$ is the universal covering, so its kernel is $\pi_1(\overline{H})$, and we define

$$\deg(p) := \pi_{\mathfrak{h}}\Big(\sum \Lambda_j\Big) \in \pi_1(\overline{H}).$$

For $\operatorname{tor}(p)$, just choose an arbitrary lift of each a_i, b_i, C_j to $\widetilde{G}$ and so get a set of monodromy data $\widetilde{p}$ for the group $\widetilde{G}$ lying over p. (Each Stokes matrix has a canonical lift to $\widetilde{G}$, and each $\Lambda_j \in \mathfrak{t}$ is unchanged.) If $\widetilde{\mu}$ is the corresponding moment map for $\widetilde{G}$, then $\pi_G(\widetilde{\mu}(\widetilde{p})) = \mu(p) = 1 \in G$, so that $\widetilde{\mu}(\widetilde{p}) \in \widetilde{D} \subset Z(\widetilde{G})$. Hence, projecting onto the semisimple components, we see that $\pi_{\widetilde{S}}(\widetilde{\mu}(\widetilde{p}))$ is in the centre of $\widetilde{S}$ and lies over the identity of $\overline{S}$; that is, it represents an element of $\pi_1(\overline{S})$ since $\widetilde{S}$ is the universal cover. Thus, we define

$$\operatorname{tor}(p) = \pi_{\widetilde{S}}\big(\widetilde{\mu}(\widetilde{p})\big) \in \pi_1(\overline{S}).$$

Different choices of lifts of a_i, b_i, C_j give the same element $\operatorname{tor}(p)$ since the lifts differ by elements of $Z(\widetilde{G})$ which cancel with their inverses on applying $\widetilde{\mu}$.

It is straightforward to see that $(\deg(p), \operatorname{tor}(p))$ do constitute an element of $\pi_1(G)$; that is, their projections to D are opposite. Namely, one observes that $\pi_D(\deg(p)) = \exp_H(\deg(p))$ (viewing D as $\ker(H \to \overline{H}) \subset G$) and that this equals $\pi_H(\widetilde{\mu}(\widetilde{p}))$ (where $(\pi_H, \pi_{\widetilde{S}}) : \widetilde{G} \xrightarrow{=} H \times \widetilde{S}$). Then, note that $\pi_D(\operatorname{tor}(p)) = \pi_G\big(\pi_{\widetilde{S}}(\widetilde{\mu}(\widetilde{p}))\big)$, where $\pi_G : \widetilde{S} \to S \subset G$. Hence, the projection to G of the tautology $\widetilde{\mu}(\widetilde{p}) = \pi_H(\widetilde{\mu}(\widetilde{p})) \cdot \pi_{\widetilde{S}}(\widetilde{\mu}(\widetilde{p}))$ is the desired statement that

$$\pi_D\big(\deg(p)\big) \cdot \pi_D\big(\operatorname{tor}(p)\big) = 1 \in D \subset G.$$

This defines the map top (since it is G-invariant, it descends to $\widetilde{M}(\mathbf{A})$), and it is easy to check that it is continuous. (This is clear for deg and, for tor, is simply because $\widetilde{G} \to G$ is a covering, so the lifts may be chosen smoothly.) Thus, top is constant on each connected component of $\widetilde{M}(\mathbf{A})$. One may easily show that the fibres are pairwise isomorphic by translating one of the exponents Λ_j by a suitable element of $\mathfrak{t}$. □

Remark 18

Note that for $G = \mathrm{GL}_n(\mathbb{C})$, $\overline{H} = \mathbb{C}^*$, the map $\pi_{\overline{H}}$ is just the determinant (so that $\pi_{\mathfrak{h}}$ is the trace) and that $S = \widetilde{S} = \mathrm{SL}_n(\mathbb{C})$, so top $=$ deg (under the isomorphism $\det_* : \pi_1(G) \cong \pi_1(\overline{H})$), and so the definition here does extend that of [10] for $\mathrm{GL}_n(\mathbb{C})$.

Proofs of Theorems 8 and 9

The proof of Theorem 8 is very similar to the genus-zero degree-zero vector bundle case appearing in [10, Sections 4, 5], the main difference being that one now starts with a C^∞ principal G-bundle with arbitrary topological type. (The novelty of the approach of [10], beyond [5], was only local—at the poles—and this generalises immediately, using local coordinates and trivialisations. Since the connections have poles, we are always able to reduce at the zero value of the moment map, yielding the flat connections.) □

The analytic isomorphism in Theorem 9 is very similar to [10, Proposition 4.8, Corollary 4.9]; the main difference is that the "tentacles" (see [10, page 163]) should be modified in the obvious way to include the g handles of Σ. That the topological type of the underlying G-bundle arises from the monodromy data, as in Lemma 16, may be seen as follows.

Suppose that we have a meromorphic connection A of the desired type on a G-bundle P on Σ. Choose an annulus $\mathbb{A} \subset \Sigma$ away from all the poles of A, and trivialise P via some clutching map $h : \mathbb{A} \to G$. (Thus, P is obtained by gluing the trivial bundles $\Delta \times G$ and $\Sigma_+ \times G$ via h, where Σ_+ is the part of Σ outside of the inner boundary circle of $\mathbb{A}$ and Δ is the disk in Σ inside the outer boundary circle of $\mathbb{A}$.) Consider a local fundamental solution $\Phi_0 : \Delta \to G$ of A defined on Δ. Written in the other trivialisation, this extends to a fundamental solution $\Phi : U \to G$ of A defined throughout some neighbourhood U of $\mathbb{A}$ in Σ_+. By definition, the topological type of P is given by the image of a positive generating loop γ under the map

$$\Phi_* : \pi_1(\mathbb{A}) \to \pi_1(G).$$

We need to compute this in terms of (the monodromy/exponents of) connection A.

First, recall that after (20) it is sufficient to compute the images $\deg := \pi_{\overline{H}}(\Phi_*(\gamma))$ in $\pi_1(\overline{H})$ and $\mathrm{tor} := \pi_{\overline{S}}(\Phi_*(\gamma))$ in $\pi_1(\overline{S})$. Now, in the trivialisation $\Sigma_+ \times G$ of P, A is given by a meromorphic one-form α on Σ_+ with values in $\mathfrak{g}$. Then, recall from the proof of Lemma 16 that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{s}$ with $\mathfrak{h}$ abelian and $\mathfrak{s} := \mathrm{Lie}(\overline{S})$ semisimple. Now, for deg, we observe that $\pi_{\overline{H}}(\Phi)$ is a fundamental solution of the connection $\pi_{\mathfrak{h}}(\alpha)$ (thought of as a meromorphic connection on the trivial principal $\overline{H}$-bundle). But $\overline{H}$ is abelian, so deg can be computed in terms of the residues of $\pi_{\mathfrak{h}}(\alpha)$ (as in [10, page 175] for $\mathrm{GL}_n(\mathbb{C})$). The residue at the jth pole is $\pi_{\mathfrak{h}}(\Lambda_j)$, and so the residue formula implies $\deg = \pi_{\mathfrak{h}}\left(\sum \Lambda_j\right)$, as expected.

For tor, we view α as a meromorphic connection on $\Sigma_+ \times \widetilde{G}$, where $\widetilde{G} = H \times \widetilde{S}$ with $\widetilde{S}$ the universal cover of S. Thus, Φ lifts to a multivalued fundamental solutions $\widetilde{\Phi}$ of α with some monodromy, say, $M \in \widetilde{G}$ around γ. By construction, $\pi_{\widetilde{S}}(M)$ maps to 1 under the isogeny $\widetilde{S} \to \overline{S}$ and represents $\mathrm{tor} \in \pi_1(\overline{S})$. On the other hand, computing the Stokes/monodromy data of α expresses M in terms of a lift of the monodromy data to $\widetilde{G}$, as in the definition of $\mathrm{tor}(p)$ in Lemma 16.

Finally, we note that the coincidence of the symplectic structures in Theorem 9 is now a formal consequence of the definitions of the symplectic structures on both sides (namely, it follows from the fact that fusing quasi-Hamiltonian spaces corresponds to fusing the corresponding Hamiltonian loop group manifolds). □

Appendix. Kernel calculation

We establish the minimal degeneracy condition (QH3) for the two-form ω on $\widetilde{\mathscr{C}}$.

Proof (of (QH3))

LEMMA 19

The two-form 2ω on $\widetilde{\mathscr{C}}$ is also given by the formula

$$\big(\overline{\gamma}, (11)\overline{\gamma}(11)^{-1}\big) + \sum_{i=1}^{k-1}\big(\overline{\gamma}, (1i)\varepsilon_i(1i)^{-1} + \{i1\}^{-1}\varepsilon_i\{i1\} - (1i)^{-1}\delta_i(1i) - [i1]^{-1}\delta_i[i1]\big) + \sum_{1\le i,j\le k-1}\big(\delta_i, (ij)\varepsilon_j(ij)^{-1}\big) + \sum_{1\le j<i\le k-1}(\delta_i, [ij]\delta_j[ij]^{-1}) - (\varepsilon_i, \{ij\}\varepsilon_j\{ij\}^{-1}),$$

where $\delta_i = d_i^*(\theta)$, $\varepsilon_i = e_i^*(\theta)$, $\overline{\gamma} = C^*(\overline{\theta})$, $(ij) := d_i^{-1}d_{i+1}^{-1}\cdots d_{k-1}^{-1}e_{k-1}\cdots e_{j+1}e_j$, $[ij] := d_{i-1}\cdots d_j$, and $\{ij\} := e_{i-1}\cdots e_j$.

Proof

This is a straightforward direct calculation expanding each term in (7). □

Now, suppose we choose a pair of tangent vectors X, Y to $\widetilde{\mathscr{C}}$ at some point p, such that X is in the kernel of ω_p and Y is arbitrary. We use dots/primes to denote derivatives along Y/X, respectively, so, for example, $\dot{\delta}_i = \langle Y, \delta_i\rangle \in \mathfrak{g}$ and $\varepsilon'_j = \langle X, \varepsilon_j\rangle \in \mathfrak{g}$ (and in any representation of G, we have $\dot{\delta}_i = d_i^{-1}\dot{d}_i$, and so forth). Our aim is to prove that $\delta'_i = \varepsilon'_i = 0$ for all i (so X is tangent to the G-action) and that $\mathrm{Ad}_{\mu(p)}(\gamma') = -\gamma'$, which is the required degeneracy condition. The equation expressing the fact that X is in the kernel of ω_p is equivalent to

$$2\omega(Y, X) = (\dot{\gamma}, \Gamma) + \sum_{i=1}^{k-1}(\dot{\delta}_i, \Delta_i) + (\dot{\varepsilon}_i, \xi_i) = 0 \tag{23}$$

for all Y, where $\Gamma, \Delta_i, \xi_i \in \mathfrak{g}$ are the corresponding coefficients involving just X-derivatives; explicitly from Lemma 19, we have

$$\Delta_i = (i1)\overline{\gamma}'(i1)^{-1} + [i1]\overline{\gamma}'[i1]^{-1} + \sum_{j=1}^{k-1}(ij)\varepsilon'_j(ij)^{-1} + \sum_{j<i}[ij]\delta'_j[ij]^{-1} - \sum_{j>i}[ji]^{-1}\delta'_j[ji],$$

$$\xi_i = -(1i)^{-1}\overline{\gamma}'(1i) - \{i1\}\overline{\gamma}'\{i1\}^{-1} - \sum_{j=1}^{k-1}(ji)^{-1}\delta'_j(ji) - \sum_{j<i}\{ij\}\varepsilon'_j\{ij\}^{-1} + \sum_{j>i}\{ji\}^{-1}\varepsilon'_j\{ji\},$$

$$\Gamma = (11)\overline{\gamma}'(11)^{-1} - (11)^{-1}\overline{\gamma}'(11)$$
$$+\sum_{j=1}^{k-1}(1j)\varepsilon'_j(1j)^{-1} + \{j1\}^{-1}\varepsilon'_j\{j1\} - (j1)^{-1}\delta'_j(j1) - [j1]^{-1}\delta'_j[j1].$$

Since Y is arbitrary, (23) implies that $\Gamma = 0$ and (since $(\, , \,)$ pairs opposite Borels) that the piece of Δ_i in the unipotent subalgebra opposite the Borel containing $\dot{\delta}_i$ is zero (and, similarly, that the piece of ξ_i in the unipotent subalgebra opposite the Borel containing $\dot{\varepsilon}_i$ is zero). The only other information about X in (23) concerns the $\mathfrak{t}$ components as follows. Since Y is tangent to $\widetilde{\mathscr{C}}$, we have $\delta(\dot{\varepsilon}_j) = -\delta(\dot{\delta}_j) = \pi i \dot{\Lambda} \in \mathfrak{t}$ for all j, where $\delta : \mathfrak{g} \to \mathfrak{t}$ is the projection along the root spaces. Thus, as $\dot{\Lambda}$ is arbitrary, (23) implies

$$\sum_{i=1}^{k-1}\delta(\Delta_i) = \sum_{i=1}^{k-1}\delta(\xi_i), \tag{24}$$

where $\delta : \mathfrak{g} \to \mathfrak{t}$. Now we will proceed to deduce the required result. From the formula for Δ_i, it follows that $d_i\Delta_i d_i^{-1} - \Delta_{i+1} = -\overline{\delta}'_i - \delta'_{i+1}$. Thus, if we define $T_i := d_i\Delta_i d_i^{-1} + \delta'_{i+1} = \Delta_{i+1} - \overline{\delta}'_i$ (for $i = 1, \dots, k-2$), then the restrictions on the unipotent pieces of Δ_i, Δ_{i+1} imply that $T_i = \delta'_{i+1} - \overline{\delta}'_i + H_i$ for some $H_i \in \mathfrak{t}$, and so in turn,

$$\Delta_i = -\delta'_i + d_i^{-1}H_i d_i, \qquad \Delta_{i+1} = \delta'_{i+1} + H_i.$$

Thus $\Delta_i = \delta'_i + H_{i-1} = -\delta'_i + d_i^{-1}H_i d_i$, so that $2\delta'_i = d_i^{-1}H_i d_i - H_{i-1}$ for $i = 2, \dots, k-2$. Taking the $\mathfrak{t}$ component of this implies that $H_{i-1} = H + H_i$, where $H := (2\pi i)\Lambda'$ (so $H = -2\delta(\delta'_j) = 2\delta(\varepsilon'_j)$ for all j). If we *define* $H_{k-1} := \delta(\Delta_{k-1}) - H/2$, then $\delta(\Delta_i) = H/2 + H_i$ for all i, and since $H_i = (k-1-i)H + H_{k-1}$, this implies

$$\sum_{i=1}^{k-1}\delta(\Delta_i) = (k-1)H_{k-1} + (k-1)^2H/2.$$

A similar exercise in terms of the ε_i and ξ_i yields analogous formulae with some sign changes: $Y_i := e_i\xi_i e_i^{-1} - \varepsilon'_{i+1} = \xi_{i+1} + \overline{\varepsilon}'_i$ (for $i = 1, \dots, k-2$), so that $Y_i = \overline{\varepsilon}'_i - \varepsilon'_{i+1} + K_i$ for some $K_i \in \mathfrak{t}$, and in turn, $2\varepsilon'_i = -e_i^{-1}K_i e_i + K_{i-1}$ for $i = 2, \dots, k-2$. Similarly, this implies that $K_{i-1} = H + K_i$, and then $\delta(\xi_i) = H/2 + K_i$ for all i, so that $\sum_{i=1}^{k-1}\delta(\xi_i) = (k-1)K_{k-1} + (k-1)^2H/2$. Thus, equation (24) is equivalent to $H_{k-1} = K_{k-1}$.

Now we reconsider the equations $\Delta_i + \delta'_i = d_i^{-1}H_i d_i$ and $\xi_i - \varepsilon'_i = e_i^{-1}K_i e_i$. Using these and the initial formulae for Δ_i, ξ_i, one finds that $[i1]^{-1}(\Delta_i + \delta'_i)[i1] + (1i)$

$(\xi_i - \varepsilon_i')(1i)^{-1}$ is equal to both sides of

$$2\sum_{j>i}\big((1j)\varepsilon_j'(1j)^{-1} - [j1]^{-1}\delta_j'[j1]\big) = [i1]^{-1}(d_i^{-1}H_i d_i)[i1] + (1i)(e_i^{-1}K_i e_i)(1i)^{-1}. \tag{25}$$

Conjugating by $(11)^{-1}$, this is equivalent to

$$\begin{aligned} &2\sum_{j>i}\big(\{j1\}^{-1}\varepsilon_j'\{j1\} - (j1)^{-1}\delta_j'(j1)\big) \\ &\quad = (i+11)^{-1}H_i(i+11) + \{i+11\}^{-1}K_i\{i+11\}. \end{aligned} \tag{26}$$

Putting $i = k-2$ in equation (25) (so the sum has just one term), we find

$$2\overline{\varepsilon}_{k-1}' - 2\overline{\delta}_{k-1}' = d_{k-1}H_{k-2}d_{k-1}^{-1} + e_{k-1}K_{k-2}e_{k-1}^{-1}.$$

First, the $\mathfrak{t}$ component of this says that $2H = H_{k-2} + K_{k-2}$, but $H_{k-2} = H + H_{k-1} = H + K_{k-1} = K_{k-2}$, and thus we deduce $H_{k-1} = K_{k-1} = 0$ (so that now, $H_i = K_i = (k-1-i)H$ for all i). Second, rewriting gives

$$2\overline{\varepsilon}_{k-1}' - e_{k-1}K_{k-2}e_{k-1}^{-1} = 2\overline{\delta}_{k-1}' + d_{k-1}H_{k-2}d_{k-1}^{-1},$$

the two sides of which live in opposite Borel subalgebras and have zero $\mathfrak{t}$ component, and so they are both zero; that is, $\varepsilon_{k-1}' = H/2 = -\delta_{k-1}'$.

Similarly, considering the difference $[i1]^{-1}(\Delta_i + \delta_i')[i1] - (1i)(\xi_i - \varepsilon_i')(1i)^{-1}$ instead, and setting $i = 1$, one obtains

$$\begin{aligned} &2(11)\varepsilon_1'(11)^{-1} + 2(11)\overline{\gamma}'(11)^{-1} \\ &\quad = -2\delta_1' - 2\overline{\gamma}' + (k-2)\big(d_1^{-1}Hd_1 - (12)H(12)^{-1}\big). \end{aligned} \tag{27}$$

Conjugating by $(11)^{-1}$, this is equivalent to

$$\begin{aligned} &2(11)^{-1}\delta_1'(11) + 2(11)^{-1}\overline{\gamma}'(11) \\ &\quad = -2\varepsilon_1' - 2\overline{\gamma}' + (k-2)\big((21)^{-1}H(21) - e_1^{-1}He_1\big). \end{aligned} \tag{28}$$

Finally, we return to the equation $\Gamma = 0$. Observe that every term of 2Γ appears on the left-hand side of one of the equations (25), (26), (27), or (28) (where we set $i = 1$ in (25), (26)), except for the terms $2\varepsilon_1' - 2\delta_1'$. Upon substituting the right-hand sides of (25)–(28) into 2Γ, most terms cancel, and we are left with

$$2\Gamma = 4\varepsilon_1' - 4\delta_1'$$

and so $\varepsilon_1' = \delta_1'$. First, taking the $\mathfrak{t}$ component of this implies that $H = 0$ (and so $\delta_i' = 0 = \varepsilon_i'$ for $i > 1$); second, ε_1' and δ_1' are in opposite Borels with zero $\mathfrak{t}$ component and so must both be zero. Now, returning to equation (27), we see that

$$(11)\overline{\gamma}'(11)^{-1} = -\overline{\gamma}',$$

which says precisely that $\mathrm{Ad}_{\mu(p)}\gamma' = -\gamma'$ since $\mu(p) = C^{-1}(11)C$ in the notation we are using. □

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References

[1] A. Y. ALEKSEEV, Y. KOSMANN-SCHWARZBACH, and E. MEINRENKEN, *Quasi-Poisson manifolds*, Canad. J. Math. **54** (2002), 3 – 29. MR 1880957 371, 375

[2] A. Y. ALEKSEEV and A. Z. MALKIN, *Symplectic structures associated to Lie-Poisson groups*, Comm. Math. Phys. **162** (1994), 147 – 173. MR 1272770 380

[3] A. Y. ALEKSEEV, A. Z. MALKIN, and E. MEINRENKEN, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), 445 – 495. MR 1638045 369, 370, 372, 373, 375, 376, 377, 380, 383, 384, 386, 396

[4] J. E. ANDERSEN, J. MATTES, and N. RESHETIKHIN, *The Poisson structure on the moduli space of flat connections and chord diagrams*, Topology **35** (1996), 1069 – 1083. MR 1404925 370, 373

[5] M. F. ATIYAH and R. BOTT, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523 – 615. MR 0702806 383, 397, 399

[6] M. AUDIN, "Lectures on gauge theory and integrable systems" in *Gauge Theory and Symplectic Geometry (Montréal, 1995)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **488**, Kluwer, Dordrecht, Netherlands, 1997, 1 – 48. MR 1461568 383

[7] W. BALSER, W. B. JURKAT, and D. A. LUTZ, *Birkhoff invariants and Stokes' multipliers for meromorphic linear differential equations*, J. Math. Anal. Appl. **71** (1979), 48 – 94. MR 0545861 372, 387

[8] O. BIQUARD and P. P. BOALCH, *Wild non-abelian Hodge theory on curves*, Compos. Math. **140** (2004), 179 – 204. MR 2004129 397

[9] P. P. BOALCH, *Stokes matrices, Poisson Lie groups and Frobenius manifolds*, Invent. Math. **146** (2001), 479 – 506. MR 1869848 371, 379

[10] ———, *Symplectic manifolds and isomonodromic deformations*, Adv. Math. **163** (2001), 137–205. MR 1864833 370, 372, 373, 374, 383, 385, 386, 387, 389, 392, 394, 395, 396, 397, 399, 400

[11] ———, *G-bundles, isomonodromy, and quantum Weyl groups*, Int. Math. Res. Not. **2002**, no. 22, 1129–1166. MR 1904670 371, 372, 379, 387, 389, 390, 394

[12] ———, *Quasi-Hamiltonian geometry of meromorphic connections*, preprint, arXiv:math/0203161v2 [math.DG] 404

[13] V. V. FOCK and A. A. ROSLY, *Flat connections and polyubles* (in Russian), Teoret. Mat. Fiz. **95**, no. 2 (1993), 228–238; English translation in Theoret. and Math. Phys. **95** (1993), 526–534. MR 1243250 370, 373

[14] W. M. GOLDMAN, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), 200–225. MR 0762512 370, 373, 374

[15] V. GUILLEMIN and S. STERNBERG, *Symplectic Techniques in Physics*, Cambridge Univ. Press, Cambridge, 1984. MR 0770935 393

[16] K. GURUPRASAD, J. HUEBSCHMANN, L. JEFFREY, and A. WEINSTEIN, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. **89** (1997), 377–412. MR 1460627 370, 373

[17] L. C. JEFFREY, *Group cohomology construction of the cohomology of moduli spaces of flat connections on* 2*-manifolds*, Duke Math. J. **77** (1995), 407–429. MR 1321064 370, 373

[18] Y. KARSHON, *An algebraic proof for the symplectic structure of moduli space*, Proc. Amer. Math. Soc. **116** (1992), 591–605. MR 1112494 370, 373

[19] I. M. KRICHEVER, *Isomonodromy equations on algebraic curves, canonical transformations and Whitham equations*, Mosc. Math. J. **2** (2002), 717–752. MR 1986088 373

[20] I. M. KRICHEVER and D. H. PHONG, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories*, J. Differential Geom. **45** (1997), 349–389. MR 1449977 373

[21] J.-H. LU and A. WEINSTEIN, *Groupoïdes symplectiques doubles des groupes de Lie-Poisson*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 951–954. MR 1054741 374, 380

[22] E. MEINRENKEN and C. WOODWARD, *Cobordism for Hamiltonian loop group actions and flat connections on the punctured two-sphere*, Math. Z. **231** (1999), 133–168. MR 1696761 375, 384

[23] A. WEINSTEIN, “The symplectic structure on moduli space” in *The Floer Memorial Volume*, Progr. Math. **133**, Birkhäuser, Basel, 1995, 627–635. MR 1362845 370, 373

[24] N. M. J. WOODHOUSE, *The symplectic and twistor geometry of the general isomonodromic deformation problem*, J. Geom. Phys. **39** (2001), 97–128. MR 1844829 373, 388

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