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Stokes matrices, Poisson Lie groups and Frobenius manifolds

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1. Introduction

The purpose of this paper is to point out and then draw some consequences of the fact that the Poisson Lie group G^* dual to $G = GL_n(\mathbb{C})$ may be identified with a certain moduli space of meromorphic connections over the unit disc having an irregular singularity at the origin. (G^* will be fully described in Sect. 2.)

The key feature of this point of view is that there is a holomorphic map

$$\nu : \mathfrak{g}^* \longrightarrow G^*$$

from the dual of the Lie algebra to the group G^*, for each choice of diagonal matrix A_0 with distinct eigenvalues—the 'irregular type'. This map is essentially the Riemann-Hilbert map or de Rham morphism for such connections (we will call it the 'monodromy map'); it is generically a local analytic isomorphism. The main result is:

Theorem 1. *The monodromy map ν is a Poisson map for each choice of irregular type, where $\mathfrak{g}^*$ has its standard complex Poisson structure and G^* has its canonical complex Poisson Lie group structure, but scaled by a factor of $2\pi i$.*

This was conjectured, and proved in the simplest case, in [6] based on the observation that the space of monodromy/Stokes data of such irregular singular connections 'looks like' the group G^*, and that the symplectic leaves match up.

We will give two applications. First, although ν is neither injective or surjective, upon restricting to the skew-Hermitian matrices $\mathfrak{k}^* \subset \mathfrak{g}^*$ it becomes injective, at least when A_0 is purely imaginary, i.e. diagonal skew-Hermitian (both $\mathfrak{k}^*$ and $\mathfrak{g}^*$ are identified with their duals using the trace here). We also find that the involution $B \mapsto -B^\dagger$ fixing the skew-Hermitian matrices corresponds under ν to an involution fixing the Poisson Lie group K^* dual to the unitary group $K = U(n)$. This leads to:

Theorem 2. *For each purely imaginary irregular type A_0 the monodromy map restricts to a (real) Poisson diffeomorphism $\mathfrak{k}^* \cong K^*$ from the dual of the Lie algebra of K to the dual Poisson Lie group (with its standard Poisson structure, scaled by a factor of π).*

Thus we have a new, direct proof of a theorem of Ginzburg and Weinstein [16], that $\mathfrak{k}^*$ and K^* are (globally) isomorphic as Poisson manifolds. Such diffeomorphisms enable one to convert Kostant's *non-linear* convexity theorem (involving the Iwasawa projection) into Kostant's *linear* convexity theorem (which is due to Schur and Horn in the unitary case, and led to the well-known Atiyah, Guillemin and Sternberg convexity theorem). See [21] and Sect. 6 below. Our approach also gives a new proof of a closely related theorem of Duistermaat [14], as well as a proof of a conjecture of Flaschka and Ratiu [15] concerning convexity theorems for non-Abelian group actions (see Remark 34 below).

Secondly (and this was our original motivation) if we restrict to skew-symmetric (complex) matrices then the corresponding space of Stokes data naturally appears as a moduli space of two-dimensional topological quantum field theories. This is due to B. Dubrovin: in [11] the notion of a *Frobenius manifold* is defined as a geometrical/coordinate-free manifestation of the WDVV equations of Witten-Dijkgraaf-Verlinde-Verlinde governing deformations of 2D topological field theories (see also [12, 13]). One of the main results (Theorem 3.2) of [11] is the identification of the local moduli of semisimple Frobenius manifolds with the entries of a Stokes matrix: an upper triangular matrix $S \in U_+$ with ones on the diagonal. An intriguing aspect of [11] was the explicit formula (F.21 in Appendix F) for a Poisson bracket on this space of matrices in the three dimensional case:

$$S := \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} \{x, y\} &= xy - 2z \\ \{y, z\} &= yz - 2x \\ \{z, x\} &= zx - 2y. \end{aligned} \tag{1}$$

This Poisson structure is invariant under a natural braid group action and has two-dimensional symplectic leaves parameterised by the values of the Markoff polynomial

$$x^2 + y^2 + z^2 - xyz.$$

For example, the quantum cohomology of the complex projective plane $\mathbb{P}^2(\mathbb{C})$ is a 3-dimensional semisimple Frobenius manifold and corresponds to the point $S = \left(\begin{smallmatrix} 1 & 3 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. (The manifold is just the complex cohomology $H^*(\mathbb{P}^2)$ and the Frobenius structure comes from the 'quantum product', deforming the usual cup product.) This is an integer solution of the Markoff equation $x^2 + y^2 + z^2 - xyz = 0$ and quite surprisingly it follows that the solution of the WDVV equations corresponding to the quantum cohomology of $\mathbb{P}^2$ is not an algebraic function, from Markoff's proof (in the nineteenth century) that his equation has an infinite number of integer solutions ([11] Appendix F).

Recently M. Ugaglia [26] has extended Dubrovin's formula to the $n \times n$ case (and found that a constant factor of $-\frac{\pi i}{2}$ is needed in (1)). Our aim here is to obtain these Poisson structures from the standard Poisson structure on G^*:

Theorem 3. *The involution of $\mathfrak{g}^*$ fixing the skew-symmetric matrices corresponds under the monodromy map to an explicit Poisson involution $i_{G^*} : G^* \to G^*$ having fixed point set U_+. The standard ($2\pi i$ scaled) Poisson structure on G^* then induces the Dubrovin-Ugaglia Poisson structure on the fixed point set U_+.*

We note that U_+ is not embedded in G^* as a subgroup. The word 'induces' here means the following: If $S \in U_+ \subset G^*$ then the tangent space $T_S G^*$ decomposes into the ± 1 eigenspaces of the derivative of the involution i_{G^*}. The $+1$ eigenspace is $T_S U_+$ and so there is a projection $\mathrm{pr} : TG^*|_{U_+} \to TU_+$ along the -1 eigenspaces. The 'induced' Poisson bivector on U_+ is simply the projection of the Poisson bivector on G^*.

In symplectic terms Theorem 3 implies that symplectic leaves of U_+ arise as symplectic submanifolds of symplectic leaves of G^*.

There are other ramifications of the identification of the Poisson Lie group G^* as a moduli space of connections that we will postpone. In particular we plan to elucidate in a future publication the Poisson braid group action on G^*, which arises by virtue of it being identified with a moduli space of meromorphic connections: the family of moduli spaces parameterised by the irregular types A_0 has a natural flat Ehresmann connection on it (the isomonodromy connection—which can usefully be thought of as a non-Abelian irregular Gauss-Manin connection [7]). The holonomy of this Ehresmann connection gives a non-linear Poisson braid group action on G^*. This action is intimately related to the braid group action on G^* described explicitly by De Concini-Kac-Procesi [9] in their study of representations of quantum groups at roots of unity.

The organisation of this paper is as follows. The next two sections give background material. Section 2 describes the Poisson Lie groups G^* and K^*, and Sect. 3 describes the monodromy map, associating Stokes matrices to an irregular singular connection. At the end of Sect. 3 we make the basic observation identifying G^* with a space of meromorphic connections. Sections 4 and 5 then give the proofs of Theorems 1 and 2 respectively. Next Sect. 6 gives some more background material on the convexity theorems and explains how Duistermaat's theorem arises naturally. Finally Sect. 7 proves Theorem 3, relating Frobenius manifolds to Poisson Lie groups.

Although we work throughout with $G = GL_n(\mathbb{C})$, the generalisation to arbitrary complex reductive groups appears to be straightforward. This will be addressed elsewhere since the main new issue (the *definition* of Stokes matrices for such groups) is a different type of question to those addressed here.

Acknowledgements. The proof of Theorem 1 is based on the calculation of Poisson structures on certain spaces of Stokes matrices due to N. Woodhouse [28], to whom I am grateful for sending me [28] before publication. I would also like to thank B. Dubrovin for advice and encouragement. A. Weinstein's comment on [16] (Lisbon 1999), that they "didn't know what the map was", was also encouraging.

2. Poisson Lie groups

A Poisson Lie group is a Lie group G with a Poisson structure on it such that the multiplication map $G\times G \to G$ is a Poisson map (where $G\times G$ is given the product Poisson structure). This notion was introduced by Drinfel'd (see [10]); Poisson Lie groups appear as classical limits of quantum groups. In other words, one quantises a Poisson Lie group to obtain a quantum group. A remarkable feature is that Poisson Lie groups come in dual pairs: there is another Poisson Lie group G^* 'dual' to any given Poisson Lie group G. In brief this is because the derivative at the identity of the Poisson bivector on G is a linear map $\mathfrak{g} \to \bigwedge^2 \mathfrak{g}$, and the dual of this map is a Lie bracket on $\mathfrak{g}^*$. The Lie group G^* is defined as a group with this Lie algebra. In turn, the Poisson bivector on G^* is determined by requiring its derivative at the identity to be the dual map of the original Lie bracket on $\mathfrak{g}$; the roles of G and G^* are symmetrical, although the groups G and G^* are often very different.

A list of examples appears (in infinitesimal form) in [10]. Here our main interest is the group G^* dual to $GL_n(\mathbb{C})$ with its standard complex Poisson Lie group structure, so we will proceed immediately to a description of this case, following [1,9,21]. We will see that the Poisson structure on G^* appears as a non-linear analogue of the standard linear Poisson structure on $\mathfrak{g}^*$.

Remark 1. It is relevant to recall that Drinfel'd was motivated by Sklyanin's calculation of the Poisson brackets between matrix entries of a monodromy matrix $M \in G$ and the observation that this Poisson structure has the Poisson Lie group property ([10] Remark 5). The results here are 'dual' to this: a space of Stokes matrices (i.e. the 'monodromy data' of an irregular connection) will be identified, as a Poisson manifold, with G^*.

The Poisson Lie group $\mathbf{G}^*$. Let B_+, B_- be the upper and lower triangular Borel subgroups of $G := GL_n(\mathbb{C})$, let $U_\pm \subset B_\pm$ be the unipotent subgroups and $T = B_+ \cap B_- \subset G$ the subgroup of diagonal matrices. The corresponding Lie algebras will be denoted $\mathfrak{b}_+, \mathfrak{b}_-, \mathfrak{u}_+, \mathfrak{u}_-, \mathfrak{t}$, all subalgebras of the $n \times n$ complex matrices $\mathfrak{g} = \mathrm{Lie}(GL_n(\mathbb{C}))$. The Lie algebra of G^* is defined to be the subalgebra

$$\text{(2)} \qquad \mathrm{Lie}(G^*) := \{(X_-, X_+) \in \mathfrak{b}_- \times \mathfrak{b}_+ \;\big|\; \delta(X_-) + \delta(X_+) = 0\}$$

of the product $\mathfrak{b}_- \times \mathfrak{b}_+$, where $\delta : \mathfrak{g} \to \mathfrak{t}$ takes the diagonal part; $(\delta(X))_{ij} = \delta_{ij} X_{ij}$. This Lie algebra is identified with the (complex) vector space dual of $\mathfrak{g}$ via the pairing:

$$\langle (X_-, X_+), Y\rangle := \operatorname{Tr}((X_+ - X_-)Y) \tag{3}$$

for any $Y \in \mathfrak{g}$. Thus (2) specifies a Lie algebra structure on $\mathfrak{g}^*$ and we define G^* to be the corresponding connected and simply connected complex Lie group. Concretely:

(4)

$$G^* := \{(b_-, b_+, \Lambda) \in B_- \times B_+ \times \mathfrak{t} \bigm| \delta(b_-)\delta(b_+) = 1, \delta(b_+) = \exp(\pi i \Lambda)\}.$$

It is easily seen that this is an n^2 dimensional simply connected (indeed contractible) subgroup of $B_- \times B_+ \times \mathfrak{t}$ (where $\mathfrak{t}$ is a group under $+$) and has the desired Lie algebra. (Conventionally [1,9] one omits the Λ term appearing in (4) and has G^* non-simply connected; the difference—the choice of Λ—is quite trivial, but it is the simply connected group that arises immediately as a moduli space of meromorphic connections.)

The Poisson bivector on G^* may be defined as follows. Consider the map

$$\pi : G^* \to G; \quad (b_-, b_+, \Lambda) \mapsto b_-^{-1}b_+. \tag{5}$$

This is a covering of its image, the 'big cell' $G^0 \subset G$ consisting of matrices that may be factorised into the product of a lower triangular matrix with an upper triangular matrix.

Remark 2. If we define, for each k, a function $\tau_k : G \to \mathbb{C}$ taking the determinant of the top-left $k \times k$ submatrix of $g \in G$ then note that $\tau_k(b_-^{-1}b_+) = \tau_k(e^{2\pi i\Lambda})$ and one can prove that $G^0 = \{g \in G \bigm| \tau_k(g) \neq 0\, \forall k\}$.

The conjugation action of G on itself restricts to an infinitesimal action of $\mathfrak{g}$ on G^0 (since G^0 is open in G) and this lifts canonically along π to an infinitesimal action σ of $\mathfrak{g}$ on G^* (since π is a covering map). By definition $\sigma : \mathfrak{g} \to \operatorname{Vect}(G^*)$ is the Lie algebra homomorphism taking $X \in \mathfrak{g}$ to the corresponding fundamental vector field. This is the (right) infinitesimal *dressing action*. (It is a 'left-action'; the adjective 'right' distinguishes σ from the *left dressing action* which is defined by replacing $b_-^{-1}b_+$ by $b_+b_-^{-1}$ in (5).)

Now, to specify the Poisson bivector $\mathcal{P} \in \Gamma(\bigwedge^2 TG^*)$ it is sufficient to give the associated bundle map $\mathcal{P}^\sharp : T^*G^* \to TG^*$ such that $\mathcal{P}(\alpha, \beta) = \langle \mathcal{P}^\sharp(\alpha), \beta\rangle$. This is defined simply as the composition of left multiplication and the right dressing action:

$$\mathcal{P}^\sharp_p := \sigma_p \circ \varphi; \qquad T^*_pG^* \xrightarrow{\varphi} T^*_eG^* \cong \mathfrak{g} \xrightarrow{\sigma_p} T_pG^*$$

where $p \in G^*$ and $\varphi := l_p^*$ is the dual of the derivative of the map multiplying on the left by p in G^*. That this does indeed define a Poisson Lie group structure on G^* is proved in [21]. (This is really the complexification of [21] and appears in [1,9]—also our sign conventions for $\mathcal{P}^\sharp$ and σ are opposite to [21], however these differences cancel out in the definition of $\mathcal{P}$.) The same bivector is obtained using right multiplications and the left dressing action.

Remark 3. It is worth noting that the standard Poisson structure on $\mathfrak{g}^*$ may be defined analogously in terms of the coadjoint action and the additive group structure of $\mathfrak{g}^*$.

Immediately we can deduce the following well-known fact:

Lemma 4. *The symplectic leaves of G^* are the connected components of the preimages under π of conjugacy classes in G.*

Proof. The tangent space to the symplectic leaf through $p \in G^*$ is the image of $\mathcal{P}^\sharp_p : T^*_pG^* \to T_pG^*$, which by definition is the inverse image under $d\pi$ of the tangent space to the conjugacy class through $\pi(p)$. □

Another fact that was very motivational is as follows. Although the infinitesimal dressing actions above do not integrate to group actions, the restriction to the diagonal subalgebra of both the left and right dressing actions integrate to the following torus action:

$$t \cdot (b_-, b_+, \Lambda) = \left(tb_-t^{-1}, tb_+t^{-1}, \Lambda\right) \tag{6}$$

for any $t \in T$ and $(b_-, b_+, \Lambda) \in G^*$. Moreover this torus action is Hamiltonian:

Lemma 5 (See also [21]). *The map*

$$\mu_T : G^* \longrightarrow \mathfrak{t}^*; \quad (b_-, b_+, \Lambda) \longmapsto (2\pi i)\Lambda$$

is an equivariant moment map for the torus action (6).

Proof. Choose $X \in \mathfrak{t}$ and let $f : G^* \to \mathbb{C}; (b_-, b_+, \Lambda) \mapsto (2\pi i)\mathrm{Tr}(X\Lambda)$ be the X component of μ_T. Observe that the one-form df on G^* is left-invariant and takes the value $X \in T^*_eG^* \cong \mathfrak{g}$ at $e \in G^*$. Thus by definition $\mathcal{P}^\sharp(df) = \sigma(X)$. This says precisely that f is a Hamiltonian for the vector field $\sigma(X)$ generated by X. □

Remark 6. 1) This lemma will also be an immediate consequence of Theorem 1 (since Λ will be essentially the diagonal part of a matrix $B \in \mathfrak{g} \cong \mathfrak{g}^*$ and this is a moment map for the coadjoint action of T on $\mathfrak{g}^*$).

2) It is intriguing to observe that the sum of the first k entries of μ_T is a logarithm of the map $\tau_k \circ \pi : G^* \to \mathbb{C}$, where π is from (5) and τ_k from Remark 2.

Having given the intrinsic formulation of the Poisson bivector on G^*, we now derive some useful formulae. Fix $p = (b_-, b_+, \Lambda) \in G^*$.

Lemma 7. *The right infinitesimal dressing action is given, for any* $X \in \mathfrak{g}$, *by*

$$X \stackrel{\sigma_p}{\longmapsto} (b_- Z_-, b_+ Z_+, \dot{\Lambda}) \in T_p G^*$$

where $(Z_-, Z_+) \in \mathrm{Lie}(G^*)$ *is determined from* X *by the equation*

$$b_+ Z_+ b_+^{-1} - b_- Z_- b_-^{-1} = b_+ X b_+^{-1} - b_- X b_-^{-1}$$

and $\dot{\Lambda} = \delta(Z_+)/(\pi i)$.

Proof. Immediate upon differentiating the map $\pi : G^* \to G$. □

Remark 8. Equivalently, one may readily verify that $Z_\pm$ is given by

$$Z_\pm = X - b_\pm^{-1} \mathrm{Ad}_p^*(X) b_\pm$$

where $\mathrm{Ad}_p^*(X) \in \mathfrak{g}$ is the coadjoint action of G^* on the dual $\mathfrak{g}$ of its Lie algebra.

Corollary 9. *The Poisson bivector on* G^* *is given by*

$$\mathcal{P}_p(\varphi^{-1}(X), \varphi^{-1}(Y)) = \mathrm{Tr}((Z_+ - Z_-)Y)$$

where $\varphi = l_p^* : T_p^* G^* \xrightarrow{\cong} \mathfrak{g}$ *is the isomorphism coming from left multiplication,* $X, Y \in \mathfrak{g}$ *are arbitrary and* $(Z_-, Z_+) \in \mathrm{Lie}(G^*)$ *is determined by* X *as in Lemma 7. In turn, if* $\mathcal{L} \subset G^*$ *is a symplectic leaf and* $p \in \mathcal{L}$ *then the symplectic structure on* $\mathcal{L}$ *is given (in the above notation) by*

$$\omega_{\mathcal{L}}(\sigma_p(X), \sigma_p(Y)) = \mathrm{Tr}((Z_+ - Z_-)Y). \tag{7}$$

Proof. The second statement follows directly from the first which in turn is now immediate: By definition

$$\mathcal{P}_p(\varphi^{-1}(X), \varphi^{-1}(Y)) = \left\langle \mathcal{P}_p^\sharp \varphi^{-1}(X), \varphi^{-1}(Y) \right\rangle = \left\langle \sigma_p(X), \varphi^{-1}(Y) \right\rangle,$$

and since $\varphi = l_p^*$ this is $\langle (l_p^{-1})_* \sigma_p(X), Y \rangle$ which (from Lemma 7 and (3)) is $\langle (Z_-, Z_+), Y \rangle := \mathrm{Tr}((Z_+ - Z_-)Y)$ as required. □

Formula (7) is the G^* analogue of the well-known Kirillov-Kostant formula for the symplectic structure on coadjoint orbits in $\mathfrak{g}^*$.

The unitary case. Let $K = U(n) \subset G$ be the group of $n \times n$ unitary matrices. This is the fixed point set of the involution $g \mapsto g^{-\dagger} = (g^\dagger)^{-1}$ of G. Let $\mathfrak{k} = \mathrm{Lie}(K) \subset \mathfrak{g}$ denote the set of skew-Hermitian matrices.

On the Poisson Lie group G^* we are led (see Lemma 29) to consider the involution

$$(b_-, b_+, \Lambda) \quad \mapsto \quad \left(b_+^{-\dagger}, b_-^{-\dagger}, -\overline{\Lambda}\right). \tag{8}$$

Let K^* be the fixed point set of this involution. Clearly K^* is a subgroup of G^*. Taking the B_+ component projects K^* isomorphically onto the subgroup

$$\{b = b_+ \in B_+ \mid \text{the diagonal entries of } b \text{ are real and positive}\} \tag{9}$$

of B_+. (This is the usual definition of K^*.) Thus on the level of Lie algebras

$$\mathrm{Lie}(K^*) \cong \{Z = Z_+ \in \mathfrak{b}_+ \mid \text{the diagonal entries of } Z \text{ are real}\}$$

and we identify $\mathrm{Lie}(K^*)$ with the (real) vector space dual of $\mathfrak{k}$ by the formula

$$\langle Z, X\rangle = \mathrm{ImTr}(ZX)$$

for any $X \in \mathfrak{k}$. (This is half the imaginary part of the restriction of the bilinear form (3).) The right infinitesimal dressing action of $\mathfrak{g}$ on G^* restricts to an action of $\mathfrak{k}$ on K^*, and moreover this infinitesimal action integrates to a group action; the right dressing action of K on K^*. Two descriptions of this action are as follows.

1) Observe that the map $\pi : G^* \to G$ restricts to a diffeomorphism $\pi|_{K^*} : K^* \to P; b \mapsto b^\dagger b$ onto the set $P \subset G$ of positive definite Hermitian matrices. Then the right dressing action is defined as

$$k \cdot b = \pi|_{K^*}^{-1}(kb^\dagger bk^{-1})$$

for any $k \in K$ and $b \in K^*$.

2) Alternatively recall the Iwasawa decomposition of G. This says (rephrasing slightly) that the product map $K \times K^* \to G; (k, b) \mapsto kb$ is a diffeomorphism. In particular there is a map $\rho : G \to K^*; g = kb \mapsto b$ taking the K^* component of g. It easy to see then that the right dressing action is also given by

$$k \cdot b = \rho(bk^{-1}).$$

The standard (real) Poisson Lie group structure on K^* can be defined as for G^* in terms of left multiplication and the right dressing action ([21] Remark 4.12). In particular the symplectic leaves are the orbits of the dressing action which, by 1), are isomorphic to spaces of Hermitian matrices with fixed positive eigenvalues. One should note that the symplectic structures on the leaves are *not* K invariant; rather the dressing actions are Poisson—i.e. such that the action map $K \times K^* \to K^*$ is a Poisson map, where K has its standard non-trivial Poisson Lie group structure ([21] Remark 4.14). The basic formulae are as follows (and are established exactly as in Corollary 9).

Lemma 10. *Let b be a point of K^*, $\mathcal{L} \subset K^*$ the symplectic leaf containing b and choose $X, Y \in \mathfrak{k}$. Then, at b the symplectic form on $\mathcal{L}$ and Poisson bivector on K^* are given by*

$$\omega_{\mathcal{L}}(\sigma_b(X), \sigma_b(Y)) = \operatorname{ImTr}(ZY) = \mathcal{P}_{K^*}(\varphi^{-1}(X), \varphi^{-1}(Y))$$

where $\sigma_b : \mathfrak{k} \to T_b\mathcal{L}$ is the right dressing action, $\varphi : T_b^* K^* \xrightarrow{\cong} T_e^* K^* = \mathfrak{k}$ is induced from left multiplication by b and $Z := X - b^{-1}\mathrm{Ad}_b^*(X)b \in \mathrm{Lie}(K^*)$.

3. The monodromy map

Now we will jump and describe some spaces of meromorphic connections. Choose a diagonal $n \times n$ matrix A_0 with distinct eigenvalues. Given a matrix $B \in \mathfrak{g}$ we will consider the meromorphic connection

$$\nabla = d - A; \qquad A = \left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz \tag{10}$$

on the trivial rank n holomorphic vector bundle over the Riemann sphere. Thus ∇ has an order two pole at 0 (irregular singularity) and (if $B \neq 0$) a first order pole at ∞ (logarithmic singularity). We will call A_0 the 'irregular type' of ∇ and once fixed, the only variable is B, which we identify with the element $\mathrm{Tr}(B\,\cdot\,)$ of $\mathfrak{g}^*$.

In this section we will define a moduli space of meromorphic connections $\mathcal{M}(A_0)$ over the closed unit disc $\Delta \subset \mathbb{P}^1$ having principal parts at 0 of the form (10). Restricting the connections in (10) to the unit disc will give a map $\mathfrak{g}^* \to \mathcal{M}(A_0)$. Then $\mathcal{M}(A_0)$ will be identified transcendentally, via the irregular Riemann-Hilbert correspondence, with a space of monodromy data $M(A_0)$, containing a pair of Stokes matrices and the so-called 'exponents of formal monodromy'. As a manifold there will be a simple identification $M(A_0) \cong G^*$ between $M(A_0)$ and the Poisson Lie group G^* defined above. Thus for each A_0 (plus a certain discrete choice—of initial sector and branch of $\log(z)$) the composition

$$\mathfrak{g}^* \to \mathcal{M}(A_0) \xrightarrow{\mathrm{RH}} M(A_0) \cong G^*$$

defines a holomorphic map $\nu : \mathfrak{g}^* \to G^*$; the *monodromy map*. Our aim in this section is to fill in the details of this description.

Suppose ∇ is any meromorphic connection on a rank n vector bundle V over the unit disc Δ with an order two pole at 0 and no others. Upon choosing a trivialisation of V we find

$$\nabla = d - A; \qquad A = \left(\frac{A_0'}{z^2} + \frac{B}{z}\right) dz + \Theta \tag{11}$$

for some matrices $A_0', B \in \mathfrak{g}$ and a matrix Θ of holomorphic one-forms on Δ.

A framing of V at 0 is an isomorphism $g_0 : V_0 \cong \mathbb{C}^n$ between the fibre of V at 0 and $\mathbb{C}^n$. We will say a connection with framing (∇, V, g_0) has *irregular type* A_0 if we have $A_0' = A_0$ in any trivialisation $V \cong \Delta \times \mathbb{C}^n$ extending the framing g_0.

Definition 11. *The moduli space $\mathcal{M}(A_0)$ is the set of isomorphism classes of triples (∇, V, g_0) consisting of a meromorphic connection ∇ on a rank n vector bundle $V \to \Delta$ with just one pole, of second order at 0, together with a framing g_0 at 0 in which ∇ has irregular type A_0.*

Concretely, if $\mathrm{Syst}_\Delta(A_0)$ denotes the infinite dimensional space of connections (11) on the trivial bundle over Δ having $A_0' = A_0$, then (by choosing arbitrary trivialisations of the bundles V extending their framings g_0) we obtain an isomorphism

$$\mathcal{M}(A_0) \cong \mathrm{Syst}_\Delta(A_0)/\mathcal{G}_\Delta$$

where the gauge group $\mathcal{G}_\Delta$ is the group of holomorphic maps $g : \Delta \to GL_n(\mathbb{C})$ with $g(0) = 1$. We will denote the gauge action by square brackets:

$$g[\nabla] = d - g[A]; \qquad g[A] = gAg^{-1} + (dg)g^{-1}.$$

The remarkable fact is that we can give an explicit description of $\mathcal{M}(A_0)$ as a complex manifold in terms of the natural monodromy data for irregular connections: the Stokes matrices and exponents of formal monodromy.

Remark 12. *Generically* a connection (11) (with $A_0' = A_0$) is gauge equivalent to a connection of the form (10); indeed (10) is often called the 'Birkhoff normal form'. However not every connection can be reduced to this form and even if possible, the form is not unique: the monodromy map is neither injective or surjective (see [18] for a detailed analysis in the $n = 2$ case).

Stokes matrices. Here we mainly follow Balser, Jurkat and Lutz [5] and Martinet and Ramis [24]. The presentation and notation is close to [7].

Let $Q := -A_0/z$, so that $dQ = A_0 dz/z^2$ and write $Q(z) = \mathrm{diag}(q_1, \ldots, q_n)$.

Definition 13. *1) The* anti-Stokes directions *at* 0 *associated to* A_0 *are the directions along which $e^{q_i - q_j}$ decays most rapidly as $z \to 0$ for some $i \neq j$. (Equivalently they are the directions between pairs of eigenvalues of A_0, when plotted in the z-plane.) The number of distinct anti-Stokes directions (clearly even) will be denoted $2l$.*

2) The monodromy manifold $M(A_0)$ *is*

$$M(A_0) := U_+ \times U_- \times \mathfrak{t},$$

where, for $(S_+, S_-, \Lambda) \in M(A_0)$, the matrices (S_+, S_-) will be called Stokes matrices *and Λ is the* permuted exponent of formal monodromy.

The aim now is to define a surjective map

$$\widetilde{\nu} : \mathrm{Syst}_\Delta(A_0) \longrightarrow M(A_0)$$

having precisely the $\mathcal{G}_\Delta$ orbits as fibres, thereby inducing an isomorphism $\mathcal{M}(A_0) \cong M(A_0)$; the (irregular) Riemann-Hilbert isomorphism. The map $\widetilde{\nu}$ will be holomorphic with respect to any finite number of coefficients of ∇ ([5] Remark 1.8).

The auxiliary choices needed in order to define $\widetilde{\nu}$ are: 1) A choice of initial sector Sect_0 at 0 bounded by two adjacent anti-Stokes directions and 2) A choice of branch of $\log(z)$ on Sect_0.

Given such a choice of initial sector we will label the anti-Stokes directions $d_1, d_2, \dots, d_{2l}$ going in a positive sense and starting on the positive edge of Sect_0. We will write $\mathrm{Sect}_i = \mathrm{Sect}(d_i, d_{i+1})$ for the open sector swept out by rays moving from d_i to d_{i+1} in a positive sense. (Indices are taken modulo $2l$—so $\mathrm{Sect}_0 = \mathrm{Sect}(d_{2l}, d_1)$.)

Suppose $\nabla = d - A \in \mathrm{Syst}_\Delta(A_0)$. The first step in defining $\widetilde{\nu}(\nabla)$ is to find a formal transformation simplifying ∇. Some straightforward algebra yields:

Lemma 14 (See [5]). *There is a unique formal gauge transformation diagonalising ∇ and removing the holomorphic terms. In other words there is a unique $\widehat{F} \in G[\![z]\!] = GL_n(\mathbb{C}[\![z]\!])$ with $\widehat{F}(0) = 1$ such that*

$$\widehat{F}\left[\frac{A_0}{z^2}dz + \frac{\delta(B)}{z}dz\right] = \frac{A_0}{z^2}dz + \frac{B}{z}dz + \Theta$$

as formal series, where $\delta(B)$ is the diagonal part of B.

Thus ∇ is *formally* isomorphic to the simple diagonal connection $\nabla^0 := d - \left(\frac{A_0}{z^2} + \frac{\delta(B)}{z}\right) dz$ (the 'formal normal form of ∇'). Clearly the matrix $z^{\delta(B)}e^Q$ is a local fundamental solution for ∇^0 (i.e. its columns are a basis of solutions). Thus in turn $\widehat{F}z^{\delta(B)}e^Q$ is a *formal* fundamental solution for ∇.

The radius of convergence of the series $\widehat{F}$ will in general be zero however so we do not immediately obtain analytic solutions of ∇. The way to proceed is via the following result, which is the outcome of work of many people (see in the references below).

Theorem 4. *1) On each sector* Sect_i *there is a canonical way to choose an invertible $n \times n$ matrix of holomorphic functions $\Sigma_i(\widehat{F})$ such that $\Sigma_i(\widehat{F})[\nabla^0] = \nabla$.*

2) The matrix of functions $\Sigma_i(\widehat{F})$ can be analytically continued to the ith 'supersector' $\widehat{\mathrm{Sect}}_i := \widehat{\mathrm{Sect}}\,(d_i - \pi/2, d_{i+1} + \pi/2)$ and then $\Sigma_i(\widehat{F})$ is asymptotic to $\widehat{F}$ at 0 within $\widehat{\mathrm{Sect}}_i$.

3) If $g \in G\{z\}$ is a germ of a convergent gauge transformation and $t \in T$ then

$$\Sigma_i(g \circ \widehat{F} \circ t^{-1}) = g \circ \Sigma_i(\widehat{F}) \circ t^{-1}.$$

The point is that on Sect_i there are generally many holomorphic isomorphisms between ∇^0 and ∇ which are asymptotic to $\widehat{F}$ and one is being chosen in a canonical way; it is in fact characterised by property 2). There are various ways to construct $\Sigma_i(\widehat{F})$, although the details will not be needed here. In particular the series $\widehat{F}$ is '1-summable' along each ray in Sect_i, with sum $\Sigma_i(\widehat{F})$—see [4,23,24]; the singular directions of the summation operator are (contained in) the set of anti-Stokes directions. Other approaches appear in [5,20]. The directions which bound the supersectors, where the asymptoticity may be lost, are often referred to as *Stokes directions*. See for example [22,27] regarding asymptotic expansions on sectors.

Thus we can immediately construct many fundamental solutions of ∇:

Definition 15. *The* canonical fundamental solution *of* ∇ *on* Sect_i *is*

$$\Phi_i := \Sigma_i(\widehat{F})z^{\delta(B)}e^Q$$

where (by convention) the branch of $\log(z)$ *chosen on* Sect_0 *is extended to the other sectors in a* negative *sense.*

The Stokes matrices are essentially the transition matrices between the canonical fundamental solution Φ_0 on Sect_0 and Φ_l on the opposite sector Sect_l, when they are continued along the two possible paths in the punctured disk joining these sectors. (In fact these two Stokes matrices encode *all* the possible transition matrices between any of the canonical bases of solutions, although this may not be clear from the definition below—see [5,7].)

Some work is required to get these Stokes matrices to be in U_+, U_- however, and the standard method is as follows:

Definition 16. *1) The* permutation matrix $P \in G$ *associated to the choice of* Sect_0 *is defined by* $(P)_{ij} = \delta_{\pi(i)j}$ *where* π *is the permutation of* $\{1,\dots,n\}$ *corresponding to the dominance ordering of* $\{e^{q_1},\dots,e^{q_n}\}$ *along the direction* θ *bisecting the sector* $\mathrm{Sect}(d_1,d_l)$*:*

$$\pi(i) < \pi(j) \qquad \Longleftrightarrow \qquad e^{q_i}/e^{q_j} \to 0 \text{ as } z \to 0 \text{ along } \theta.$$

2) The Stokes matrices (S_+, S_-) *of* ∇ *are the unique matrices such that:*
- *If* Φ_l *is continued in a positive sense to* Sect_0 *then* $\Phi_l = \Phi_0 \cdot PS_-P^{-1}$*, and*
- *If* Φ_0 *is continued in a positive sense to* Sect_l *then* $\Phi_0 = \Phi_l \cdot PS_+P^{-1}M_0$*, where* $M_0 := \exp(2\pi\sqrt{-1}\delta(B)) \in T$ *is the* formal monodromy *of* ∇*; it is the actual monodromy of the formal normal form* ∇^0.

3) The exponent of formal monodromy *of* ∇ *is* $\delta(B)$ *and the* permuted exponent of formal monodromy *is* $\Lambda := P^{-1}\delta(B)P \in \mathfrak{t}$.

The crucial fact, motivating the definition of P, is:

Lemma 17. $S_+ \in U_+$ *and* $S_- \in U_-$.

Proof. Observe θ and $-\theta$ are the bisecting directions of the two components of the intersection $\widehat{\mathrm{Sect}}_0 \cap \widehat{\mathrm{Sect}}_l$ of the supersectors. (Recall from Theorem 4, for each i, $\Sigma_i(\widehat{F})$ is asymptotic to $\widehat{F}$ at 0 when continued within $\widehat{\mathrm{Sect}}_i$.) Thus $z^{\delta(B)}e^{Q}(PS_-P^{-1})e^{-Q}z^{-\delta(B)} = \Sigma_0(\widehat{F})^{-1}\Sigma_l(\widehat{F})$ is asymptotic to 1 within the component of $\widehat{\mathrm{Sect}}_0 \cap \widehat{\mathrm{Sect}}_l$ containing $-\theta$. The exponentials dominate so we must have $(PS_-P^{-1})_{ij} = \delta_{ij}$ unless $e^{q_i-q_j} \to 0$ as $z \to 0$ along $-\theta$. This says, equivalently, that $S_- \in U_-$. The argument for S_+ is the same once the change in choice of $\log(z)$ is accounted for. □

Thus we have now defined the desired map $\widetilde{\nu} : \mathrm{Syst}_\Delta(A_0) \to M(A_0)$ taking the Stokes matrices and (permuted) exponents of formal monodromy. Part 3) of Theorem 4 implies that the $\mathcal{G}_\Delta$ orbits are contained in the fibres of $\widetilde{\nu}$ so that $\widetilde{\nu}$ induces a well-defined map $\mathcal{M}(A_0) \to M(A_0)$.

Theorem 5 (See [5,3]). *The induced map $\mathcal{M}(A_0) \to M(A_0)$ is bijective.*

Proof. For injectivity, suppose two connections $\nabla_1, \nabla_2 \in \mathrm{Syst}_\Delta(A_0)$ have the same Stokes matrices and exponent of formal monodromy $\delta(B)$. Let $\widehat{F}_1, \widehat{F}_2$ be the associated formal isomorphisms (with the same normal form ∇^0) and let $\phi_i = \Sigma_i(\widehat{F}_2) \circ \Sigma_i(\widehat{F}_1)^{-1}$ for $i = 0$ and $i = l$. ϕ_i is a holomorphic solution of the connection $\mathrm{Hom}(\nabla_1, \nabla_2)$ asymptotic to $\widehat{F}_2 \circ \widehat{F}_1^{-1}$ at 0 in $\widehat{\mathrm{Sect}}_i$. Since the Stokes matrices are equal, $\phi_0 = \phi_l$ on both components of the intersection $\widehat{\mathrm{Sect}}_0 \cap \widehat{\mathrm{Sect}}_l$ and so they fit together to define an isomorphism ϕ between ∇_1 and ∇_2 on the punctured disc. By Riemann's removable singularity theorem it follows that ϕ extends across 0 (and has *Taylor* expansion $\widehat{F}_2 \circ \widehat{F}_1^{-1}$) and so is the desired element of $\mathcal{G}_\Delta$. Surjectivity follows from a result of Sibuya (see [5] Sect. 6, Theorem V), together with the (straightforward to prove) fact that any meromorphic connection germ is gauge equivalent to the germ of a meromorphic connection on the unit disc. □

Next we observe how the Stokes matrices encode the local monodromy conjugacy class, and how they behave under the torus action changing the framing at 0:

Lemma 18. *If $[(\nabla, V, g_0)] \in \mathcal{M}(A_0)$ has monodromy data $(S_+, S_-, \Lambda) \in M(A_0)$ then*

1) The monodromy (in the usual sense) of ∇ around a simple positive loop in the punctured disc, is conjugate to

$$S_-S_+e^{2\pi i\Lambda} \in G.$$

2) For any $t \in T$, the framed connection $(\nabla, V, t \circ g_0)$ has monodromy data $(sS_+s^{-1}, sS_-s^{-1}, \Lambda)$ where $s := P^{-1}tP \in T$.

Proof. 1) When continued in a positive sense, Φ_0 becomes $\Phi_l PS_+P^{-1}M_0$ on Sect_l, which will become $\Phi_0 PS_-S_+P^{-1}M_0 = \Phi_0 PS_-S_+e^{2\pi i\Lambda}P^{-1}$ on continuing around, back to Sect_0.

2) Observe that changing g_0 to $t\circ g_0$ corresponds to changing $\widehat{F}$ to $t\widehat{F}t^{-1}$ and so, by 3) of Theorem 4, the canonical solution Φ_i changes to $t\Phi_i t^{-1}$, whence the result is clear. □

The monodromy map. Combining the maps above we thus obtain a map $\mathfrak{g}^* \to M(A_0)$, taking a matrix B to the monodromy data at 0 of the connection $d-(A_0/z^2+B/z)dz$. The final step is to identify the monodromy manifold $M(A_0)$ with the Poisson Lie group G^*. This is motivated by the following simple observation. Let $\mathcal{O}\subset\mathfrak{g}$ be an adjoint orbit which is generic in the sense that no pair of distinct eigenvalues of an element of $\mathcal{O}$ differ by an integer. Define $\mathfrak{C}\subset G$ to be the conjugacy class $e^{2\pi i\mathcal{O}}$.

Lemma 19. *If $B\in\mathcal{O}$ then $S_-S_+e^{2\pi i\Lambda}\in\mathfrak{C}$, where $(S_+,S_-,\Lambda)\in M(A_0)$ is the monodromy data at* 0 *of the connection $\nabla=d-(A_0/z^2+B/z)dz$.*

Proof. By Lemma 18 the local monodromy of ∇ around zero is conjugate to $S_-S_+e^{2\pi i\Lambda}$. However ∇ has only one other pole in $\mathbb{P}^1$: a first order pole at ∞ (logarithmic/regular singularity). The connection ∇ has residue B at infinity and this implies it has local monodromy conjugate to $e^{-2\pi iB}$ (see e.g. [27] or Sect. 4 below). Clearly a simple positive loop around ∞ is a simple *negative* loop around 0. □

Now recall from Lemma 4 that the symplectic leaves of the Poisson Lie group G^* are obtained by fixing the conjugacy class of $b_-^{-1}b_+$. Thus we are led to the following:

Definition 20. *The isomorphism $M(A_0)\cong G^*$ is defined to be $(S_+,S_-,\Lambda)\mapsto(b_-,b_+,\Lambda)$ where $b_-=e^{-\pi i\Lambda}S_-^{-1}$ and $b_+=e^{-\pi i\Lambda}S_+e^{2\pi i\Lambda}$, so that $b_-^{-1}b_+=S_-S_+\exp(2\pi i\Lambda)$.*

Thus we have completed the last step in the definition of the monodromy map ν as the composition

$$\mathfrak{g}^*\to\mathcal{M}(A_0)\xrightarrow{\cong}M(A_0)\cong G^*.$$

(This will be streamlined in Sect. 4.) In summary the above considerations lead us to:

Proposition 21. *For each choice of irregular type A_0, initial sector and branch of* $\log(z)$*, the monodromy map is a holomorphic map $\nu:\mathfrak{g}^*\to G^*$ such that:*

1) Any generic symplectic leaf of $\mathfrak{g}^*$ maps to a symplectic leaf of G^*,

2) If A_0 and the initial sector are chosen such that the permutation matrix $P=1$, then ν is T-equivariant, where T acts on $\mathfrak{g}^*$ by the coadjoint action and on G^* by the dressing action, as described in (6).

Remark 22. In [7] the author generalised the well-known Atiyah-Bott construction of symplectic structures on moduli spaces of holomorphic connections on compact Riemann surfaces, to the case of meromorphic connections with arbitrary order poles. (Holomorphic connections correspond to complex representations of the fundamental group of the surface—one needs to incorporate Stokes data in the general meromorphic case.) If the surface has boundary a Poisson structure is obtained on the moduli space, the symplectic leaves of which are specified by fixing monodromy conjugacy classes on each boundary component (exactly as in the holomorphic case). On specialising to the closed unit disc, this gives another a priori definition of the Poisson structure on $\mathcal{M}(A_0)$. One can show (as in [7]) that the monodromy map is Poisson with respect to this Poisson structure. Hence (by Theorem 1 and the above identification $\mathcal{M}(A_0) \cong G^*$) we obtain a gauge-theoretic construction of the standard Poisson Lie group structure on G^*.

4. The monodromy map is Poisson

The main step in the proof of Theorem 1 is to see that the monodromy map restricts to a symplectic map between generic symplectic leaves, i.e. that it relates the Kirillov-Kostant symplectic structure and the analogue (7) on the leaves in G^*.

Thus choose a generic matrix $J \in \mathfrak{g}$ (which at this stage only needs to have the property that no pair of distinct eigenvalues differ by an integer) and let $\mathcal{O}$ be the adjoint orbit of J (which we identify with a coadjoint orbit using the trace and so $\mathcal{O}$ inherits a complex symplectic structure). Let $\mathcal{C} = \exp(2\pi i \mathcal{O})$ be the corresponding conjugacy class and let $\mathcal{L} = \pi^{-1}(\mathcal{C} \cap G^0) \subset G^*$ be the symplectic leaf of G^* over $\mathcal{C}$ (more precisely each connected component of $\mathcal{L}$ is a symplectic leaf). Take the symplectic form on $\mathcal{L}$ to be that given by formula (7) but divided by $2\pi i$.

Choose an irregular type A_0 and initial sector and branch of $\log(z)$. (For notational simplicity we will assume that these are chosen such that the corresponding permutation matrix P is the identity—the extension to the general case is simple.)

We will now associate the following list of data to a matrix $g \in G$:

- Matrices: $B := gJg^{-1} \in \mathcal{O}$ and $\Lambda := \delta(B) \in \mathfrak{t}$,
- Meromorphic connections on the trivial rank n vector bundle over $\mathbb{P}^1$:

$$\nabla = d - A, \quad \nabla^0 = d - A^0, \quad \nabla^\infty = d - A^\infty$$

where

$$A = \left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz, \quad A^0 = \left(\frac{A_0}{z^2} + \frac{\Lambda}{z}\right) dz, \quad A^\infty = \frac{Jdz}{z}.$$

- Formal series:

$\widehat{F} \in G[[z]]$ such that $\widehat{F}[A^0] = A$ and $\widehat{F}(0) = 1$ (see Lemma 14), and similarly:

$\widehat{H} \in G[[z^{-1}]]$ such that $\widehat{H}[A^\infty] = A$ and $\widehat{H}(\infty) = g$.

• Fundamental solutions of ∇:

$$\Phi := \Phi_0 \text{ on } \mathrm{Sect}_0, \qquad \Psi := \Phi_l e^{\pi i \Lambda} \text{ on } \mathrm{Sect}_l \text{ (see Definition 15) and}$$

$$\chi := Hz^J \text{ on a neighbourhood of } \infty \text{ slit along } d_1.$$

Here the first anti-Stokes ray d_1 is extended to ∞ and the chosen branch of $\log(z)$ on Sect_0 is extended to $\mathbb{P}^1 \setminus d_1$. The series $\widehat{H}$ is a formal isomorphism at $z = \infty$ between ∇^∞ and ∇ and so is a series solution of $\mathrm{Hom}(\nabla^\infty, \nabla)$; a connection with a simple pole at ∞. This implies $\widehat{H}$ is actually convergent and defines a holomorphic map $H : \mathbb{P}^1 \setminus \{0\} \to G$. (See e.g. [27] for the existence, uniqueness and convergence of $\widehat{H}$.) Finally we obtain:

• Monodromy data $b_-, b_+, C \in G$ relating these fundamental solutions, as indicated schematically in Fig. 1. (For example the arrow $\chi \xrightarrow{C} \Phi$ means that if χ is extended along the arrow then $\chi = \Phi \cdot C$ in the domain of definition of Φ.)

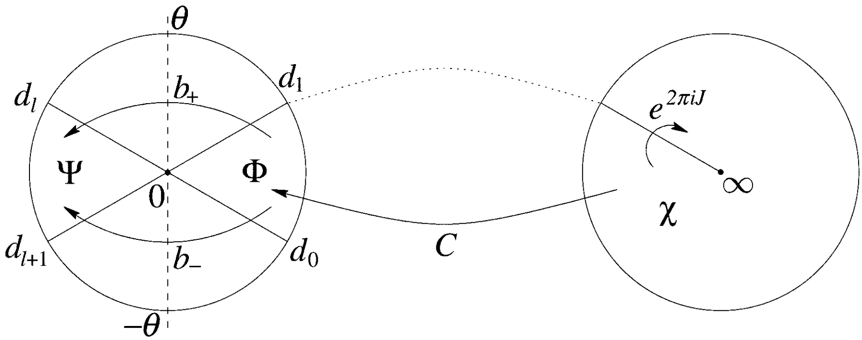

Fig. 1 Configuration in $\mathbb{P}^1$

The fact that a simple positive loop around 0 is also a simple negative loop around ∞ translates into the important monodromy relation:

$$b_-^{-1} b_+ = C e^{2\pi i J} C^{-1}. \tag{12}$$

Note that $b_\pm$ only depend on B and not on all of g and that by definition $\nu(B) = (b_-, b_+, \Lambda) \in \mathcal{L} \subset G^*$.

Proposition 23. *The restricted monodromy map $\nu : \mathcal{O} \to \mathcal{L}$ is a symplectic map.*

Proof. We will now vary the initial matrix g in the procedure above. Note that the fundamental solutions (and therefore all the monodromy data) will vary holomorphically with g ([5] Remark 1.8). Choose $X_0, Y_0 \in \mathfrak{g}$ arbitrarily and suppose we have a two parameter holomorphic family $g(s,t) \in G$ with $\dot{g}g^{-1} = X_0$ and $g'g^{-1} = Y_0$ at $s = t = 0$ (for example $g = e^{X_0 t + Y_0 s} g_0$).

Generally we will write $\dot{M} = \frac{\partial M}{\partial t}$ and $M' = \frac{\partial M}{\partial s}$ and will exclusively be interested in the point $s = t = 0$; this will be tacitly assumed in all the expressions below.

By definition $\dot{B} = [X_0, B]$ and $B' = [Y_0, B]$ and the Kirillov-Kostant symplectic structure on $\mathcal{O}$ evaluated on these tangents is:

$$\omega_{\mathcal{O}}([X_0, B], [Y_0, B]) = \mathrm{Tr}([X_0, Y_0]B). \tag{13}$$

On the other side, on the leaf $\mathcal{L} \subset G^*$ we have tangents

$$\nu_*(\dot{B}) = (\dot{b}_-, \dot{b}_+, \dot{\Lambda}) = (b_- Z_-, b_+ Z_+, \dot{\Lambda})$$

where $(Z_-, Z_+) := (b_-^{-1}\dot{b}_-, b_+^{-1}\dot{b}_+) \in \mathrm{Lie}(G^*)$ (and similarly for $\nu_*(B')$). The monodromy relation (12) implies that if we define $X := -\dot{C}C^{-1} \in \mathfrak{g}$ then the value of the fundamental vector field of X under the right dressing action is $\nu_*(\dot{B})$, i.e. $\sigma_p(X) = \nu_*(\dot{B})$ where $p := (b_-, b_+, \Lambda) \in G^*$. Similarly $\sigma_p(Y) = \nu_*(B')$ with $Y := -C'C^{-1}$. Thus formula (7) (after rescaling) says

$$\omega_{\mathcal{L}}(\nu_*(\dot{B}), \nu_*(B')) = \frac{1}{2\pi i}\mathrm{Tr}\left(\left(b_-^{-1}\dot{b}_- - b_+^{-1}\dot{b}_+\right)C'C^{-1}\right). \tag{14}$$

Our task is to show that (13) and (14) are equal. This will be accomplished via the following intermediate expression:

Lemma 24.

$$\frac{1}{2\pi i}\oint_{\partial\Delta} \mathrm{Tr}\left(\dot{H}H^{-1}\nabla\left(H'H^{-1}\right)\right) = \mathrm{Tr}\left([X_0, Y_0]B\right)$$

where $\Delta \subset \mathbb{P}^1$ is the unit disc $\{z : |z| \leq 1\}$ with its natural orientation and ∇ acts in the adjoint representation: $\nabla\left(H'H^{-1}\right) = d\left(H'H^{-1}\right) - [A, H'H^{-1}]$.

Proof. Recall H is holomorphic on the opposite hemisphere $\Delta^+ = \mathbb{P}^1 \setminus \{z : |z| < 1\}$ and that $H(w) = g + O(w)$ where $w = z^{-1}$. A direct calculation then gives that, on Δ^+:

$$\begin{aligned}\mathrm{Tr}\left(\dot{H}H^{-1}\nabla\left(H'H^{-1}\right)\right) &= \mathrm{Tr}\left(\dot{H}H^{-1}\left[B, H'H^{-1}\right]\right)\frac{dw}{w} + O(1)dw \\ &= \mathrm{Tr}\left(X_0[B, Y_0]\right)\frac{dw}{w} + O(1)dw.\end{aligned}$$

The lemma now follows immediately from the residue theorem. □

Remark 25. In other words this says that the map $\mathcal{O} \to \widehat{\mathcal{O}}$; $B \mapsto \nabla|_{\partial\Delta}$ is symplectic, where $\widehat{\mathcal{O}}$ is the set of connections on the trivial bundle over the circle $\partial\Delta$ that have monodromy in the conjugacy class $\mathcal{C}$. $\widehat{\mathcal{O}}$ can be naturally identified with a coadjoint orbit of the central extension of the loop group of

G and so inherits the Kirillov-Kostant symplectic structure, which is (upto scale):

$$\omega_\alpha(\nabla_\alpha\phi, \nabla_\alpha\psi) = \frac{1}{2\pi i}\oint_{\partial\Delta} \mathrm{Tr}(\phi\nabla_\alpha\psi).$$

In our situation $\alpha = A|_{\partial\Delta} = d\chi\chi^{-1}$ and, since J is fixed, $\dot\chi\chi^{-1} = \dot H H^{-1}$. Then it follows that $\dot\alpha = \nabla(\dot H H^{-1})$ and $\alpha' = \nabla(H'H^{-1})$.

The strategy now is to re-evaluate the integral in Lemma 24 in terms of the monodromy data. First note that the integrand is a holomorphic one-form on $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$, since both H and ∇ are holomorphic there. Thus (by Cauchy's theorem) the value of the integral is independent of the radius of the circle we integrate around: we will calculate the limit as the radius tends to zero, capitalising on the fact that we know the asymptotics at 0 (in appropriate sectors) of Φ and Ψ.

Divide the circle $\partial\Delta_r$ (bounding the disc of radius r centred at 0) into two arcs a_r^0, a_r^l by breaking it at the points p_r, q_r of intersection with the directions θ and $-\theta$ respectively. (Recall θ was defined to bisect $\mathrm{Sect}(d_1, d_l)$.) a_r^0 is an arc in a positive sense from q_r to p_r and is wholly contained in the supersector $\widehat{\mathrm{Sect}}_0$ (on which we know $\Phi \sim \widehat{F} z^\Lambda e^Q$) and a_r^l is a positive arc from p_r to q_r contained in $\widehat{\mathrm{Sect}}_l$ (on which we know $\Psi \sim \widehat{F} z^\Lambda e^Q e^{\pi i\Lambda}$).

Define φ to be the holomorphic one-form

$$\varphi := -\mathrm{Tr}\left(\nabla\big(\dot H H^{-1}\big) H' H^{-1}\right)$$

on $\mathbb{P}^1 \setminus \{0, \infty\}$, so (by Stokes theorem and Leibniz) $\frac{1}{2\pi i}\oint_{\partial\Delta}\varphi$ appears in Lemma 24.

Lemma 26. *On the supersector* $\widehat{\mathrm{Sect}}_0$ *we have*

$$\varphi = -df_0 + \varepsilon_0,$$
$$f_0 = \mathrm{Tr}\left(F_0^{-1}\dot F_0 \Lambda' \log_0 z + \tfrac{1}{2}\dot\Lambda\Lambda'(\log_0 z)^2 + \Phi^{-1}\dot\Phi C' C^{-1}\right)$$

where we have continued $F_0 := \Sigma_0(\widehat{F})$ *and* $\log_0(z) = \log(z)$ *from* Sect_0 *to* $\widehat{\mathrm{Sect}}_0$, *and* ε_0 *is a one-form such that* $\int_{a_r^0}\varepsilon_0 \to 0$ *as* $r \to 0$. *Similarly on* $\widehat{\mathrm{Sect}}_l$ *we have*

$$\varphi = -df_l + \varepsilon_l,$$
$$f_l = \mathrm{Tr}\Big(F_l^{-1}\dot F_l\Lambda'(\log_l z + \pi i) + \tfrac{1}{2}\dot\Lambda\Lambda'(\log_l z + \pi i)^2 + \Psi^{-1}\dot\Psi(b_-C)'(b_-C)^{-1}\Big)$$

where we have continued $F_l := \Sigma_l(\widehat{F})$ *and* $\log_l(z) = \log(z)$ *from* Sect_l *to* $\widehat{\mathrm{Sect}}_l$, *and* $\int_{a_r^l}\varepsilon_l \to 0$ *as* $r \to 0$.

Proof. First we recall (see e.g. [27]) that if ϵ is a holomorphic function on $\widehat{\mathrm{Sect}}_0$ with asymptotic expansion at 0 a power series $\epsilon \sim \sum_0^\infty a_n z^n$, then $\int_{a_r^0}\epsilon dz \to 0$ as $r \to 0$.

Now $H'H^{-1}=\chi'\chi^{-1}$ and $\chi=\Phi\cdot C$ (when χ is extended along C's arrow) and so

$$-\varphi=\mathrm{Tr}\left(\nabla\left(\dot\Phi\Phi^{-1}\right)\Phi'\Phi^{-1}\right)+\mathrm{Tr}\left(\nabla\left(\dot\Phi\Phi^{-1}\right)\Phi C'C^{-1}\Phi^{-1}\right).$$

The second term is $d\mathrm{Tr}(\Phi^{-1}\dot\Phi C'C^{-1})$ by Leibniz and then a direct calculation substituting the definition $\Phi:=F_0z^\Lambda e^Q$ into the first term, yields

$$-\varphi=df_0+\mathrm{Tr}\bigg(d\left(F^{-1}\dot F\right)F^{-1}F'+\left[F^{-1}F',F^{-1}\dot F\right]A^0 \\ +F^{-1}\left(F'\dot\Lambda-\dot F\Lambda'\right)\frac{dz}{z}\bigg)$$

where $F=F_0$. Since $F_0\sim\widehat F$ in $\widehat{\mathrm{Sect}}_0$ it follows that the long expression here is indeed negligible. This proves the first statement and the second is analogous. □

Thus $\oint_{\partial\Delta_r}\varphi=(-f_0+f_l)\big|_{q_r}^{p_r}+\epsilon_r$ where $\epsilon_r\to0$ as $r\to0$. If we write $v_r=\log_0(p_r)$ then $\log_0(q_r)=\log_l(q_r)=v_r-\pi i$ and $\log_l(p_r)=v_r-2\pi i$ and we find that

$$\oint_{\partial\Delta_r}\varphi=\mathrm{Tr}\left(\Phi^{-1}\dot\Phi C'C^{-1}-\Psi^{-1}\dot\Psi(b_-C)'(b_-C)^{-1}\right)\Big|_{p_r}^{q_r} \\ -\pi i(2v_r-\pi i)\mathrm{Tr}(\dot\Lambda\Lambda')+\epsilon_r. \tag{15}$$

Lemma 27. *We have*

$$\mathrm{Tr}\left(\Phi^{-1}\dot\Phi C'C^{-1}-\Psi^{-1}\dot\Psi(b_-C)'(b_-C)^{-1}\right)(q_r) \\ =\mathrm{Tr}\left(b_-^{-1}\dot b_-C'C^{-1}\right)+\pi iv_r\mathrm{Tr}\dot\Lambda\Lambda'+\epsilon_r$$

and

$$\mathrm{Tr}\left(\Phi^{-1}\dot\Phi C'C^{-1}-\Psi^{-1}\dot\Psi(b_-C)'(b_-C)^{-1}\right)(p_r) \\ =\mathrm{Tr}\left(b_+^{-1}\dot b_+C'C^{-1}\right)-\pi i(v_r-\pi i)\mathrm{Tr}\dot\Lambda\Lambda'+\epsilon_r$$

where each $\epsilon_r\to0$ as $r\to0$.

Proof. Along $-\theta$ we have $\Phi=\Psi\cdot b_-$. Using this to remove Φ from the left-hand side of the first formula and expanding $(b_-C)'$ yields:

$$\mathrm{Tr}\left(b_-^{-1}\dot b_-C'C^{-1}\right)-\mathrm{Tr}\left(\Psi^{-1}\dot\Psi b_-'b_-^{-1}\right).$$

To deal with the second term here recall that the diagonal part of b_- is $e^{-\pi i\Lambda}$ so $b_-'b_-^{-1}=-\pi i\Lambda'+n_-$ for some constant strictly lower triangular matrix n_-. Now $\Psi=F_le^{\Lambda(\pi i+\log_l z)}e^Q$ by definition and so

$$\mathrm{Tr}\left(\Psi^{-1}\dot\Psi b_-'b_-^{-1}\right)=-\pi i\mathrm{Tr}\left(F_l^{-1}\dot F_l\Lambda'\right) \\ +\mathrm{Tr}\left(F_l^{-1}\dot F_lz^\Lambda e^{\pi i\Lambda}e^Qn_-e^{-Q}e^{-\pi i\Lambda}z^{-\Lambda}\right)-\pi iv_r\mathrm{Tr}\left(\dot\Lambda\Lambda'\right).$$

The first two terms on the right here tend to zero as $z = q_r \to 0$ along $-\theta$ (see Lemma 17 for the second term) and so we have established the first formula. The second formula arises similarly (using the fact that $\Phi = \Psi \cdot b_+$ along θ) once we note that the monodromy relation (12) implies $(b_-C)'(b_-C)^{-1} = (b_+C)'(b_+C)^{-1}$. $\square$

Substituting these into (15) we happily find that the $\mathrm{Tr}(\dot{\Lambda}\Lambda')$ terms cancel, so that

$$\oint_{\partial\Delta} \varphi = \lim_{r\to 0} \oint_{\partial\Delta_r} \varphi = \mathrm{Tr}\left((b_-^{-1}\dot{b}_- - b_+^{-1}\dot{b}_+)C'C^{-1} \right)$$

thereby completing the proof of Proposition 23. $\square$

Remark 28. The method of Lemma 31 below can be used to also show that the restricted monodromy map $\nu : \mathcal{O} \to \mathcal{L}$ is *injective*.

Proof of Theorem 1. Let $U \subset \mathfrak{g}^*$ be the subset of all matrices having distinct eigenvalues mod $\mathbb{Z}$. Thus U is a regular Poisson submanifold of $\mathfrak{g}^*$ and each of its symplectic leaves is a coadjoint orbit of the type appearing in Proposition 23. It follows then (using local Darboux-Weinstein coordinates for example) that $\nu|_U : U \to G^*$ is a Poisson map (where G^* has its canonical Poisson structure, as defined in Sect. 2, but multiplied by $2\pi i$). Now choose arbitrary holomorphic functions f, g on G^* and consider the holomorphic function

$$\nu^*\{f, g\}_{G^*} - \{\nu^* f, \nu^* g\}_{\mathfrak{g}^*}$$

on $\mathfrak{g}^*$. We have shown this function vanishes on the dense subset $U \subset \mathfrak{g}^*$ and so it vanishes everywhere. $\square$

5. Ginzburg-Weinstein isomorphisms

In this section we will consider the restriction of the monodromy map to the skew-Hermitian matrices and prove Theorem 2.

We will fix the irregular type A_0 to be purely imaginary, so that there are only two anti-Stokes directions; the two halves of the imaginary axis. We will take Sect_0 to be the sector containing the positive real axis $\mathbb{R}_+$ and use the branch of $\log(z)$ which is real on $\mathbb{R}_+$. Thus, by convention, on Sect_1 (the opposite sector) $\log(z)$ has imaginary part $-\pi$ on the negative real axis.

In the previous section we explained how to associate monodromy data $(b_-, b_+, \Lambda, C) \in G^* \times G$ to a matrix $g \in G$, given a choice of matrix J which has no distinct eigenvalues differing by an integer. In other words we have defined a map

$$\widehat{\nu} : G \times \mathfrak{g}'' \to G^* \times G; \quad \widehat{\nu}(g, J) := (b_-, b_+, \Lambda, C).$$

where $\mathfrak{g}'' = \{ J \in \mathfrak{g} \bigm| \text{ if } p \neq q \text{ are eigenvalues of } J \text{ then } p - q \notin \mathbb{Z} \}$. Note that the set of skew-Hermitian matrices sits inside $\mathfrak{g}''$ and that $\mathfrak{g}''$ is *open* in $\mathfrak{g}$.

Lemma 29. *The extended monodromy map $\widehat{\nu}$ is equivariant as follows:*

$$\widehat{\nu}\big(g^{-\dagger}, -J^{\dagger}\big) = \big(b_+^{-\dagger}, b_-^{-\dagger}, -\overline{\Lambda}, C^{-\dagger}\big)$$

where $(b_-, b_+, \Lambda, C) := \widehat{\nu}(g, J)$. In particular $\nu(-B^{\dagger}) = (b_+^{-\dagger}, b_-^{-\dagger}, -\overline{\Lambda})$ where $B = gJg^{-1}$, so that if B is skew-Hermitian then $\nu(B) \in K^* \subset G^*$.

Proof. Let $i : \mathbb{P}^1 \to \mathbb{P}^1$ denote complex conjugation; $z \mapsto \overline{z}$. If $F : U \to G$ is a smooth map, where $U = i(U) \subset \mathbb{P}^1$ is an open subset invariant under i, then define $F^{\iota} := i^*(F^{-\dagger}) : U \to G$. Similarly if $\nabla = d - A$ is a connection on the trivial rank n vector bundle over U, define $\nabla^{\iota} = d - A^{\iota}$ where $A^{\iota} := -i^*(A^{\dagger})$. Since both i and $\dagger$ are anti-holomorphic, ι takes holomorphic maps/connections to holomorphic maps/connections. (Similarly for formal power series, meromorphic connections etc.) We will repeatedly use the (easily verified) fact that $(F[A])^{\iota} = F^{\iota}[A^{\iota}]$, where the square brackets denote the gauge action. Note that if $A = \left(\frac{A_0}{z^2} + \frac{B}{z}\right) dz$ then $A^{\iota} = \left(\frac{A_0}{z^2} - \frac{B^{\dagger}}{z}\right) dz$. It follows then that the list of data associated in Sect. 4 to $(g^{-\dagger}, -J^{\dagger})$ is (in terms of the corresponding data associated to (g, J)):

$$(-B^{\dagger}, -\overline{\Lambda}, \nabla^{\iota}, (\nabla^0)^{\iota}, (\nabla^{\infty})^{\iota}, \widehat{F}^{\iota}, \widehat{H}^{\iota}, \Phi^{\iota}, \Psi^{\iota}, \chi^{\iota}).$$

The only subtlety here involves the fundamental solution $\Psi := \Sigma_1(\widehat{F}) z^{\Lambda} e^{Q} e^{\pi i \Lambda}$ on Sect_1. By definition the new Ψ is $\Sigma_1(\widehat{F}^{\iota}) z^{-\overline{\Lambda}} e^{Q} e^{-\pi i \overline{\Lambda}}$. To see this is Ψ^{ι} we just need to observe that on Sect_1 we have $(z^{\Lambda})^{\iota} = z^{-\overline{\Lambda}} e^{-2\pi i \overline{\Lambda}}$. The lemma now follows immediately. For example, since $\Phi = \Psi b_+$ on the positive imaginary axis, we have $\Phi^{\iota} = \Psi^{\iota} b_+^{\iota}$ on the negative imaginary axis, and b_+ is constant so $b_+^{\iota} = b_+^{-\dagger}$. □

Remark 30. The involution on the monodromy data is much less attractive when written in terms of the Stokes matrices; one is thus led to believe that G^* is a more natural receptacle.

Next we examine the injectivity of $\nu|_{\mathfrak{k}^*}$.

Lemma 31. *1) For each $J \in \mathfrak{g}''$ the map $\widehat{\nu}_J : G \to G^* \times G;\ g \mapsto \widehat{\nu}(g, J)$, is injective.*
2) If $h \in G$ then $\widehat{\nu}(gh^{-1}, hJh^{-1}) = (b_-, b_+, \Lambda, Ch^{-1})$ where $(b_-, b_+, \Lambda, C) := \widehat{\nu}(g, J)$.
3) $\nu|_{\mathfrak{k}^*} : \mathfrak{k}^* \to K^*$ is injective and its derivative is bijective.

Proof. Part 1) is similar to Theorem 5: Suppose $\widehat{\nu}_J(g_1) = \widehat{\nu}_J(g_2)$. We will use subscripts 1, 2 to denote the corresponding auxiliary data. Thus Φ_1, Φ_2 denote the corresponding fundamental solutions on Sect_0. Consider the holomorphic matrix $X := \Phi_1 \Phi_2^{-1}$. X has asymptotic expansion $\widehat{F}_1 \widehat{F}_2^{-1}$ on the supersector $\widehat{\mathrm{Sect}}_0$. On continuation to Sect_1, we find $X := \Psi_1 \Psi_2^{-1}$, since $(b_+)_1 = (b_+)_2$. Similarly X is unchanged on return to Sect_0, and on continuation to ∞ it becomes $\chi_1\chi_2^{-1}$. Thus X has no monodromy around 0 and has the same asymptotic expansion $\widehat{F}_1\widehat{F}_2^{-1}$ on $\widehat{\mathrm{Sect}}_1$. Riemann's removable singularity theorem then implies X is holomorphic across 0 and across ∞ (with Taylor expansions $\widehat{F}_1\widehat{F}_2^{-1}$ and $\widehat{H}_1\widehat{H}_2^{-1}$ respectively). Thus X is a matrix of holomorphic functions on $\mathbb{P}^1$ and so is constant. Its value at 0 is $\widehat{F}_1(0)\widehat{F}_2^{-1}(0) = 1$ and its value at ∞ is $(\widehat{H}_1\widehat{H}_2^{-1})\big|_{z=\infty} = g_1g_2^{-1}$.

Part 2) is straightforward. For 3) we argue as follows. We have a commutative diagram:

$$
(16)\qquad
\begin{array}{ccccccc}
G\times\mathfrak{g}'' & \longrightarrow & G^*\times G\times\mathfrak{g}''; & \qquad & (g,J) & \mapsto & (b_-,b_+,\Lambda,C,J)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\mathfrak{g}'' & \longrightarrow & G^*\times\mathfrak{g}''; & & gJg^{-1} & \mapsto & (b_-,b_+,\Lambda,CJC^{-1})
\end{array}
$$

where $(b_-,b_+,\Lambda,C) := \widehat{\nu}(g,J)$. The top map is injective by 1). Also 2) implies that the top map takes fibres of the left map into fibres of the right map (so the bottom map is well-defined) and moreover distinct fibres go to distinct fibres (so the bottom map is injective). Now if $B = gJg^{-1}$ is skew-Hermitian then so is $R := CJC^{-1}$ by Lemma 29. The monodromy relation (12) says $b_-^{-1}b_+ = e^{2\pi iR}$, so we see R is determined by (b_-,b_+); the unique Hermitian logarithm of $b_-^{-1}b_+$ is $2\pi iR$. Thus restricting the bottom map to $\mathfrak{k}^*\subset\mathfrak{g}''$ and 'forgetting R' on the right-hand side yields an injective map $\mathfrak{k}^*\to G^*$. This is of course the restriction of the monodromy map to $\mathfrak{k}^*$.

Finally we must show that $\nu|_{\mathfrak{k}^*}$ is bijective on tangent vectors. First observe the above argument extends to show that there is an open subset $U\subset\mathfrak{g}''$ which contains $\mathfrak{k}^*$ and on which the monodromy map is injective. (The unique choice of logarithm on the Hermitian matrices extends uniquely to matrices sufficiently close to being Hermitian.) Thus $\nu|_U : U\to G^*$ is an injective holomorphic map between equi-dimensional complex manifolds. This implies it is biholomorphic onto its image (see e.g. [25] Theorem 2.14). It follows immediately that $d\nu|_{\mathfrak{k}^*}$ is bijective. $\square$

All that is left is to consider surjectivity and the Poisson structures:

Proof of Theorem 2. Choose any point $b\in K^*$ and let $\mathcal{L}\subset K^*$ be its symplectic leaf (dressing orbit of K). The fact that we can diagonalise the Hermitian matrix $b^\dagger b$ implies we can choose a diagonal element of $\mathcal{L}$, which we will write as $e^{\pi iJ}$ (with J diagonal and purely imaginary). It is straightforward to see that $\nu|_{\mathfrak{k}^*}(J) = e^{\pi iJ}\in K^*$.

Now let $\mathcal{O}\subset\mathfrak{k}^*$ be the K-coadjoint orbit of J. By the monodromy relation (12) and Lemma 29 we deduce $\nu(\mathcal{O})\subset\mathcal{L}$. Thus $\nu|_{\mathcal{O}}$ is a smooth map between two equi-dimensional compact connected manifolds. As such it has a well-defined degree which, since it is injective, is ±1. This implies $\nu(\mathcal{O}) = \mathcal{L}$ since non-surjective maps have degree zero (cf. [8]). Thus the monodromy map ν maps $\mathfrak{k}^*$ *onto* K^*, as b was arbitrary.

Now we will examine the symplectic structures of $\mathcal{O}$ and $\mathcal{L}$. Let $B\in\mathfrak{k}^*$ be the skew-Hermitian matrix with $\nu(B) = b$ and choose $g\in K$ such that

$B = gJg^{-1}$. Thus given arbitrary $X_0, Y_0 \in \mathfrak{k}$, Proposition 23 says that

$$\mathrm{Tr}([X_0, Y_0]B) = \frac{1}{2\pi i}\mathrm{Tr}((Z_+ - Z_-)Y) = \frac{1}{\pi}\mathrm{Im}\mathrm{Tr}(ZY)$$

where $Z = Z_+$ and the rest of the notation is as in Proposition 23. In other words (recalling Lemma 10) $\nu|_{\mathcal{O}} : \mathcal{O} \to \mathcal{L}$ is symplectic (if we divide the symplectic form on $\mathcal{L}$ by π). Arguing as for Theorem 1 it follows that $\nu|_{\mathfrak{k}^*}$ is Poisson (provided we multiply the Poisson structure on K^* by π). Finally since we have also proved the derivative of $\nu|_{\mathfrak{k}^*} : \mathfrak{k}^* \to K^*$ is bijective we deduce this map is indeed a Poisson diffeomorphism. □

Remark 32. The behaviour of the coadjoint action of the maximal diagonal torus T_K of K under the restricted monodromy map $\nu|_{\mathfrak{k}^*} : \mathfrak{k}^* \to K^*$ follows from Lemma 18 and Proposition 21: If the irregular type A_0 is such that the permutation matrix $P = 1$ then T_K acts on K^* by the dressing action (6). In general one needs to permute this action as indicated in Lemma 18.

6. Kostant's non-linear convexity theorem and the theorem of Duistermaat

Let $\mathfrak{p}$ be the set of $n \times n$ Hermitian matrices and $P = \exp(\mathfrak{p})$ the set of positive definite Hermitian matrices. Multiplying by $\sqrt{-1}$ identifies $\mathfrak{p}$ with the skew-Hermitian matrices $\mathfrak{k}$. In turn $\mathfrak{k} \cong \mathfrak{k}^*$ via the trace, so $\mathfrak{p}$ inherits the standard Poisson structure from $\mathfrak{k}^*$. The symplectic leaves $\mathcal{O} \subset \mathfrak{p}$ are the $\mathrm{Ad}(K)$ orbits, consisting of matrices with the same n-tuple of eigenvalues. The map taking the diagonal part

$$\delta : \mathfrak{p} \longrightarrow \mathbb{R}^n$$

is a moment map for the adjoint action of the (maximal) diagonal torus T_K of K.

Schur and Horn proved classically that the set of diagonal entries appearing in a fixed orbit $\mathcal{O}$ is a convex polytope; $\delta(\mathcal{O})$ is the convex hull of the Sym_n orbit of the n-tuple of eigenvalues of $\mathcal{O}$. Kostant [19] extended this to arbitrary semisimple groups. Subsequently Atiyah, Guillemin and Sternberg [2, 17] put these results into the very general context of convexity of the images of moment maps for Hamiltonian torus actions on compact symplectic manifolds.

Now for the non-linear version: Let $\mathcal{C} = \exp(\mathcal{O}) \subset P$ be a set of positive definite Hermitian matrices with fixed eigenvalues. The *Iwasawa projection* $\widehat{\delta} : G \to \mathbb{R}^n$ is the map

$$g = kan \longmapsto \log(a),$$

where kan is the Iwasawa (Gram-Schmidt) decomposition of $g \in G = GL_n(\mathbb{C})$ into the product of a unitary matrix k and diagonal positive real matrix a and a unipotent upper-triangular complex matrix n. Clearly $\mathcal{C} \subset G$.

Kostant's non-linear convexity theorem [19] says that on $\mathcal{C}$ the image of the (non-linear) Iwasawa projection is the same as the convex polytope appearing above: $\widehat{\delta}(\mathcal{C}) = \delta(\mathcal{O})$.

What one would like to have is a map $\eta : \mathfrak{p} \to P$ taking each orbit $\mathcal{O}$ to $\mathcal{C} = \exp(\mathcal{O})$ and converting δ into $\widehat{\delta}$—i.e. such that following diagram commutes:

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\delta} & \mathbb{R}^n \\ \downarrow{\scriptstyle \eta} & & \| \\ P & \xrightarrow{\widehat{\delta}} & \mathbb{R}^n. \end{array} \tag{17}$$

Clearly taking $\eta(X) = e^X$ maps the orbits correctly, but then the diagram does not commute. However one may 'twist' the exponential map appropriately:

Theorem 6 (Duistermaat [14]).
There is a real analytic map $\psi : \mathfrak{p} \to K$ such that for each $X \in \mathfrak{p}$:

1) $\widehat{\delta}\big(\psi(X)^{-1} \cdot \exp(X) \cdot \psi(X)\big) = \delta(X)$, and
2) The map $\phi_X : k \mapsto k \cdot \psi(k^{-1}Xk)$ is a diffeomorphism from K onto K.

Duistermaat's motivation was to reparameterise certain integrals over K, converting terms involving $\widehat{\delta}$ into terms involving the linear map δ. The proof of the existence of such maps ψ in [14] is for connected real semisimple groups G (with finite centre) and involves an indirect homotopy argument. Our work in the previous sections immediately gives a new proof (in the case $G = GL_n(\mathbb{C})$); one may take ψ to be the inverse of the connection matrix C:

Proof of Theorem 6. Given $X \in \mathfrak{p}$, let $J := X/(\pi i)$. Then we have monodromy data $(b_-, b_+, \Lambda, C) := \widehat{\nu}(1, J)$ as defined in Sect. 5, (taking $g = 1$). By Lemma 29, since J is skew-Hermitian, C is unitary and $b_- = b_+^{-\dagger}$. The monodromy relation (12) implies

$$b^\dagger b = Ce^{2X}C^{-1} = h^\dagger h \tag{18}$$

where $b := b_+$ and h is the Hermitian matrix Ce^XC^{-1}. Now let $h = kan$ be the Iwasawa decomposition of h. Clearly $h^\dagger h = (an)^\dagger an$ and so, from (18), we deduce $b = an$. Thus

$$\widehat{\delta}(Ce^XC^{-1}) = \log(a) = \log(\delta(b)) = \pi i \Lambda,$$

and by definition $\Lambda = \delta(J)$. Hence if we define $\psi(X) = C^{-1}$ we have established 1).

The real analyticity of ψ is clear: it is the restriction of a holomorphic map. Property 2) is also straightforward: from Lemma 31 we know the map $\widehat{\nu}_J : K \to K^* \times K$ is injective. Projecting further onto the K factor yields an injective map $\mathrm{pr}_K \circ \widehat{\nu}_J : K \to K$ (since the monodromy relation determines the K^* component from the K component). This map is onto for degree reasons and a diffeomorphism since it is the restriction of a biholomorphic map. Finally from 2) of Lemma 31, observe that ϕ_X is just the composition of $\mathrm{pr}_K \circ \widehat{\nu}_J$ with the inversion map $K \to K$. □

Remark 33. Restricting to $G = SL_n(\mathbb{C})$ does indeed give a special case of Duistermaat's result since, as a real Lie group, $SL_n(\mathbb{C})$ is semisimple and has finite centre.

Let us briefly continue the story to motivate Theorem 2. After Duistermaat, the next step was taken by Lu and Ratiu [21] who gave P a Poisson structure by identifying it with the Poisson Lie group K^*: The Cartan decomposition $G = KP$ combined with the Iwasawa decomposition $G = KAN$ identifies P with AN, and in turn $K^* \cong AN$. Then the symplectic leaves are the orbits $\mathcal{C} \subset P$ and $\widehat{\delta}$ is a moment map for the dressing action of the maximal torus T_K of K: Kostant's non-linear convexity theorem may now be deduced from the Atiyah, Guillemin and Sternberg convexity theorem.

It was conjectured in [21] that there is in fact a T_K-equivariant Poisson diffeomorphism $\mathfrak{k}^* \cong K^*$. (So Kostant's non-linear convexity theorem is reduced to the linear case.) This was proved explicitly for $K = SU(2)$ by P. Xu and then in general by Ginzburg-Weinstein [16], building on Duistermaat's indirect homotopy argument mentioned above. Theorem 2 here points out that such diffeomorphisms arise naturally as monodromy maps for irregular singular connections on the unit disc.

Remark 34. Since writing the first version of this paper the author has learnt of a conjecture of Flaschka and Ratiu [15] p. 50, that "*there is a Ginzburg-Weinstein isomorphism that fixes the positive Weyl chamber.*" However for our Ginzburg-Weinstein isomorphisms $\nu|_{\mathfrak{k}^*}$ this follows immediately from the simple fact that $\nu|_{\mathfrak{k}^*}(J) = e^{\pi i J}$ for any diagonal skew-Hermitian matrix J (provided the permutation matrix P is the identity). Flaschka and Ratiu point out that such Ginzburg-Weinstein isomorphisms enable one to immediately deduce convexity theorems for Poisson actions of Poisson Lie groups on symplectic manifolds, from the "classical" non-Abelian convexity theorem of Guillemin-Sternberg and Kirwan for Hamiltonian actions. Note that the hope of [15], that the property of fixing a positive Weyl chamber would pick out a 'distinguished' Ginzburg-Weinstein isomorphism, does not hold: the dependence of the monodromy map on the irregular type is highly non-trivial.

7. Frobenius manifolds and Poisson Lie groups

Now we will consider the space U_+ of Stokes matrices arising in the theory of Frobenius manifolds. Our aim is to prove Theorem 3 which stated that the standard Poisson structure on G^* induces the Dubrovin-Ugaglia Poisson structure on U_+.

Proof of Theorem 3. The space U_+ appears by restricting the monodromy map to the skew-symmetric (complex) matrices, as can be seen from the following:

Lemma 35. *The monodromy map* ν *intertwines the following involutions*

$$i_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*; \qquad\qquad i_{G^*} : G^* \to G^*;$$
$$B \mapsto -B^T \qquad\qquad (b_-, b_+, \Lambda) \mapsto \left(b_+^T, b_-^T, -\Lambda\right)$$

of $\mathfrak{g}^*$ *and* G^**. In other words:* $\nu \circ i_{\mathfrak{g}^*} = i_{G^*} \circ \nu$.

Proof. This is similar to Lemma 29; just modify the involutions appearing there to be $i(z) = -z$, $F^\iota := i^*(F^{-T})$, $A^\iota := -i^*(A^T)$, and then the proof goes the same: The original fundamental solutions Φ, Ψ become Ψ^ι, Φ^ι. (Using the fact that, under ι, the function z^Λ on Sect_0 becomes $(z^\Lambda)^\iota = z^{-\Lambda} e^{-\pi i \Lambda}$ on Sect_l.) The lemma is now immediate: for example the relation $\Psi b_+ = \Phi$ along the direction θ implies $\Phi^\iota b_+^T = \Psi^\iota$ along $-\theta$. □

Thus the map $U_+ \hookrightarrow G^*;\ S \mapsto (S^T, S, 0)$ identifies U_+ with the fixed point set of the involution i_{G^*} of G^*. (Note that U_+ is not embedded as a subgroup, and that one would not expect it to be, since the irregular type A_0 is manifestly not skew-symmetric.) Let $\mathfrak{h} \subset \mathfrak{g}$ denote the set of skew-symmetric matrices and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using the trace: $B \leftrightarrow \mathrm{Tr}(B\,\cdot\,)$. Then Lemma 35 implies the monodromy map restricts to a map

$$\nu|_{\mathfrak{h}^*} : \mathfrak{h}^* \to U_+.$$

This is generically a local analytic isomorphism and the Dubrovin-Ugaglia Poisson structure on U_+ is characterised by the fact that $\nu|_{\mathfrak{h}^*}$ is a Poisson map for any value of the irregular type A_0. (The diagonal entries of A_0 are the 'canonical coordinates' in the language of Frobenius manifolds.) Thus we have a commutative diagram:

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "\nu"] & G^* \\ \mathfrak{h}^* \arrow[u] \arrow[r, "\nu|_{\mathfrak{h}^*}"] & U_+ \arrow[u] \end{tikzcd}$$

where the vertical maps are the inclusions. By Theorem 1 the top map is Poisson and by definition the bottom map is Poisson (where $\mathfrak{g}^*, \mathfrak{h}^*$ have their standard complex Poisson structures, U_+ has the Dubrovin-Ugaglia structure and G^* has its standard Poisson structure, but scaled by $2\pi i$).

Now to complete the proof of the theorem we just need to make the simple observation that the Poisson structure on $\mathfrak{h}^*$ is 'induced' from that on $\mathfrak{g}^*$ via the involution $i_{\mathfrak{g}^*}$ (in the sense described after the statement of the theorem). □

For example in the 4×4 case Ugaglia's remarkable explicit description [26] of the Dubrovin-Ugaglia Poisson structure on U_+ is $\{\cdot\,,\,\cdot\}_{\mathrm{DU}} = \frac{\pi i}{2}\{\cdot\,,\cdot\}$ where $\{\cdot\,,\,\cdot\}$ is determined by the following formulae for its values on coordinate functions:

$$S := \begin{pmatrix} 1 & u & v & w \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} \{u, z\} &= 0 \\ \{v, y\} &= 2uz - 2xw \\ \{w, x\} &= 0 \end{aligned}$$

$$\begin{array}{lll} \{u, v\} = 2x - uv & \{u, w\} = 2y - uw & \{x, u\} = 2v - xu \\ \{y, u\} = 2w - yu & \{v, w\} = 2z - vw & \{v, x\} = 2u - vx \\ \{z, v\} = 2w - zv & \{w, y\} = 2u - wy & \{w, z\} = 2v - wz \\ \{x, y\} = 2z - xy & \{z, x\} = 2y - zx & \{y, z\} = 2x - yz. \end{array}$$

Remark 36. Although our proof is transcendental in nature, the relationship between the Poisson structures on G^* and U_+ is entirely algebraic. Indeed, as a plausibility check for Theorem 1 of this paper, in [6] a computer algebra program (Mathematica) was used to derive Ugaglia's formula above, from the Poisson structure on G^*.

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Notes added in proof:

1) A. Alekseev has informed me that the Ginzburg-Weinstein isomorphisms he constructed (A.Yu. Alekseev, *On Poisson actions of compact Lie groups on symplectic manifolds*, J. Differential Geom. **45** (1997), 241–256) also have the property required to prove the conjecture of Flaschka and Ratiu referred to in Remark 34 above.

2) Questions concerning 'Poisson-ness of monodromy maps' seem to have been first considered (in very special cases, using different methods) in the paper: H. Flaschka and A.C. Newell, *The inverse monodromy transform is a canonical transformation*, Nonlinear problems: present and future (Los Alamos, NM, 1981) (A. Bishop et al., ed.), North-Holland Math. Studies 61, 1982, pp. 65–89.