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Symplectic Manifolds and Isomonodromic Deformations

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We study moduli spaces of meromorphic connections (with arbitrary order poles) over Riemann surfaces together with the corresponding spaces of monodromy data (involving Stokes matrices). Natural symplectic structures are found and described both explicitly and from an infinite dimensional viewpoint (generalising the Atiyah–Bott approach). This enables us to give an intrinsic symplectic description of the isomonodromic deformation equations of Jimbo, Miwa and Ueno, thereby putting the existing results for the six Painlevé equations and Schlesinger's equations into a uniform framework.

Contents.

1. INTRODUCTION

Moduli spaces of representations of fundamental groups of Riemann surfaces have been intensively studied in recent years and have an incredibly rich structure: For example, a theorem of Narasimhan and Seshadri [56] identifies the space of irreducible unitary representations of the fundamental group of a compact Riemann surface with the moduli space of stable holomorphic vector bundles on the surface. In particular, this description puts a Kähler structure on the space of fundamental group representations—it has a symplectic structure together with a compatible

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complex structure. A remarkable fact is that although the complex structure on the space of representations will depend on the complex structure of the surface, the *symplectic* structure only depends on the topology, a fact often referred to as "the symplectic nature of the fundamental group" [22].

The geometry is richer still if we consider the moduli space of *complex* fundamental group representations: Due to results of Hitchin, Donaldson and Corlette, the Kähler structure above now becomes a *hyper-Kähler* structure and the symplectic structure becomes a *complex symplectic* structure, which is still topological. One of the main aims of this paper is to generalise this complex symplectic structure. (Hyper-Kähler structures will not be considered here.)

First recall that, over a Riemann surface, there is a one to one correspondence between complex fundamental group representations and *holomorphic* connections (obtained by taking a holomorphic connection to its monodromy/holonomy representation). Then replace the word "holomorphic" by "meromorphic"—we will study the symplectic geometry of moduli spaces of meromorphic connections.

In fact, as in the holomorphic case, these moduli spaces may also be realised in a more topological way, using a generalised notion of monodromy data. By restricting a meromorphic connection to the complement of its polar divisor and taking the corresponding monodromy representation, a map is obtained from the moduli space of meromorphic connections to the moduli space of representations of the fundamental group of the punctured Riemann surface. For connections with only simple poles this map is generically a covering map and so we are essentially in the well-known case of representations of fundamental groups of punctured Riemann surfaces. However in general there are local moduli at the poles—it is not sufficient to restrict to the complement of the polar divisor and take the monodromy representation as above.

Fortunately this extra data—the local moduli of meromorphic connections—has been studied in the theory of differential equations for many years and has a monodromy-type description in terms of "Stokes matrices", which encode (as we will explain) the change in asymptotics of solutions on sectors at the poles. The Stokes matrices and the fundamental group representation fit together in a natural way and the main question we ask is simply: "What is the symplectic geometry of these moduli spaces of generalised monodromy data?"

Recently, Martinet and Ramis [50] have constructed a huge group associated to any Riemann surface, the "wild fundamental group", whose set of finite dimensional representations naturally corresponds to the set of meromorphic connections on the surface. Although we will not directly use this perspective, the question above can then be provocatively rephrased as asking: "What is the symplectic nature of the *wild* fundamental group?"

The motivation behind these questions is to understand intrinsically the symplectic geometry of the full family of isomonodromic deformation equations of Jimbo, Miwa and Ueno [40]. The initial impetus was the theorem of B. Dubrovin [19] identifying the local moduli space of semisimple Frobenius manifolds with a space of Stokes matrices. (In brief, this means certain Stokes matrices parameterise certain two-dimensional topological quantum field theories.) The original aim was to find a more intrinsic approach to the intriguing (braid group invariant) Poisson structure written down by Dubrovin on this space of Stokes matrices in the rank three case (see [19] appendix F and also the recent paper [67] of M. Ugaglia for the higher rank formula). The key step in the proof of Dubrovin's theorem is that (in the semisimple case) the WDVV equations (of Witten–Dijkgraaf–Verlinde–Verlinde) are *equivalent* to the equations for isomonodromic deformations of certain meromorphic connections on the Riemann sphere with just two poles, of orders one and two respectively (so the space of solutions corresponds to the moduli of such connections—the Stokes matrices).

More generally Jimbo, Miwa and Ueno [40] have written down a vast family of nonlinear differential equations, governing isomonodromic deformations of meromorphic connections over $\mathbb{P}^1$ having arbitrarily many poles of arbitrary order (on arbitrary rank bundles). These are of independent interest and can be thought of as a *universal* family of nonlinear equations: They are the largest family of nonlinear differential equations known to have the "Painlevé property" (that, except on fixed critical varieties, solutions will only have poles as singularities). Special cases include the six Painlevé equations (which arise as the isomonodromic deformation equations for connections on rank two bundles over $\mathbb{P}^1$, with total pole multiplicity four) and Schlesinger's equations (the simple pole case—see below).

In brief, the six Painlevé equations were found almost a hundred years ago, as a means to construct new transcendental functions (namely their solutions—the Painlevé transcendents); R. Fuchs discovered then that the sixth Painlevé equation arises as an isomonodromic deformation equation. The subject then lay more or less dormant until the late 1970's when (spectacularly) Wu, McCoy, Tracey and Barouch [73, 52] found that the correlation functions of certain quantum field theories satisfied Painlevé equations. Subsequently Jimbo, Miwa, Môri and Sato [60, 39] showed that this was a special case of a more general phenomenon and developed the theory of "holonomic quantum fields" which led to [40]. See for example [36, 69] for more background material.

One expects isomonodromic deformations to lurk underneath most integrable partial differential equations since the heuristic "Painlevé integrability test" says that a nonlinear PDE will be "integrable" if it admits some reduction to an ODE with the Painlevé property (for example the KdV equation has a reduction to the first Painlevé equation and all six Painlevé equations appear as reductions of the anti-self-dual Yang–Mills equations; see [1, 51]). They certainly appear in a diverse range of non-linear problems in geometry and theoretical physics, such as Frobenius manifolds [19] or in the construction of Einstein metrics [65, 30].

On the other hand general solutions of isomonodromy equations cannot be given explicitly in terms of known special functions; as mentioned above general solutions are *new* transcendental functions (see [68]). This is the reason we turn to geometry to understand more about these equations. Recent work on isomonodromic deformations seems to have focused mainly on particular examples, in particular exploring the rich geometry of the six Painlevé equations and searching for the few, very special, explicit solutions that they do admit. The question we address here is simply "What is the symplectic geometry of the full family of isomonodromic deformation equations of Jimbo, Miwa and Ueno?"

Geometrically the isomonodromy equations constitute a flat (Ehresmann) connection on a fibre bundle over a space of deformation parameters, the fibres being certain moduli spaces of meromorphic connections over $\mathbb{P}^1$. Thus the idea is to find natural symplectic structures on such moduli spaces and then prove they are preserved by the isomonodromy equations. In certain special cases, such as the Schlesinger or six Painlevé equations the symplectic geometry is well-known (see [28, 29, 57). The main results of this paper are analogous to those of Hitchin [29] and Iwasaki [35] who explained intrinsically why Schlesinger's equations and certain rank two higher genus isomonodromy equations (respectively) admit a time-dependent Hamiltonian description. However for the general isomonodromy equations considered here a Hamiltonian description is still not known: this work indicates strongly that such description should exist. (See also Remark 7.1.)

For example, understanding the symplectic geometry of the isomonodromic deformation equations enables us to ask questions about their quantisation. This has been addressed in certain cases by Reshetikhin [59] and Harnad [27] and leads to Knizhnik–Zamolodchikov type equations.

A key step (Theorem 6.1) is to establish that the transcendental map taking a meromorphic connection to its (generalised) monodromy data, is a symplectic map. This is the "inverse monodromy theory" version of the well-known result in inverse scattering theory, that the map from the set of initial potentials to scattering data is a symplectic map (see [20] Part 1, Chapter III).

Although apparently not mentioned in the literature, a useful perspective (explained in Section 7) has been to interpret the paper [40] of Jimbo, Miwa and Ueno, as stating that the Gauss–Manin connection in non-Abelian cohomology (in the sense of Simpson [64]) generalises to the case of *meromorphic* connections. This offers a fantastic guide for future generalisation.

The Prototype: Simple Poles

The simplest way to explain the results of this paper is to first describe the intrinsic symplectic geometry of Schlesinger's equations, following Hitchin [29].

Choose matrices $A_1, \dots, A_m \in \mathrm{End}(\mathbb{C}^n)$, distinct numbers $a_1, \dots, a_m \in \mathbb{C}$ and consider the following meromorphic connection on the trivial rank n holomorphic vector bundle over the Riemann sphere:

$$\nabla := d - \left(A_1 \frac{dz}{z-a_1} + \cdots + A_m \frac{dz}{z-a_m} \right). \tag{1}$$

This has a simple pole at each a_i and will have no further pole at ∞ if and only if $A_1 + \cdots + A_m = 0$, which we will assume to be the case. Thus, on removing a small open disc D_i from around each a_i and restricting ∇ to the m-holed sphere $S := \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_m)$, we obtain a (nonsingular) holomorphic connection. In particular it is flat and so, taking its monodromy, a representation of the fundamental group of the m-holed sphere is obtained. This procedure defines a holomorphic map, which we will call the *monodromy map*, from the set of such connection coefficients to the set of complex fundamental group representations

$$\left\{ (A_1, \dots, A_m) \,\middle|\, \sum A_i = 0 \right\} \xrightarrow{\nu_{\mathbf{a}}} \left\{ (M_1, \dots, M_m) \,\middle|\, M_1 \cdots M_m = 1 \right\}, \tag{2}$$

where appropriate loops generating the fundamental group of S have been chosen and the matrix $M_i \in G := GL_n(\mathbb{C})$ is the automorphism obtained by parallel translating a basis of solutions around the ith loop.

This map is the key to the whole theory and is generically a local analytic isomorphism. It is tempting to think of $\nu_{\mathbf{a}}$ as a generalisation of the exponential function, but note the dependence on the pole positions $\mathbf{a}$ is rather complicated since the monodromy map *solves* Painlevé type equations (see below).

We can however study the geometry of the monodromy map, particularly the *symplectic* geometry. First, to remove the base-point dependence, quotient both sides of (2) by the diagonal conjugation action of G. Second, restrict the matrices A_i to be in fixed adjoint orbits. (These may be identified with coadjoint orbits using the trace, and so have natural complex symplectic structures.) Thus we pick m generic (co)adjoint orbits $O_1, \dots, O_m$ and require $A_i \in O_i$. Also define $\mathscr{C}_i \subset G$ to be the conjugacy class containing

$\exp(2\pi\sqrt{-1}\,A_i)$ for any $A_i \in O_i$. Fixing $A_i \in O_i$ implies $M_i \in \mathscr{C}_i$. The key fact now is that the sum $\sum A_i$ is a moment map for the diagonal conjugation action of G on $O_1 \times \cdots \times O_m$ and so (2) becomes

$$O_1 \times \cdots \times O_m /\!/ G \xrightarrow{\nu_{\mathbf{a}}} \mathrm{Hom}_{\mathscr{C}}(\pi_1(S), G)/G, \tag{3}$$

where the subscript $\mathscr{C}$ means we restrict to representations having local monodromy around a_i in the conjugacy class $\mathscr{C}_i$. The symplectic geometry of this set of representations has been much studied recently. The primary symplectic description is due to Atiyah and Bott [6, 5] and involves interpreting it as an infinite dimensional symplectic quotient, starting with all C^∞ connections on the manifold-with-boundary S (see also [7]). Alternatively, a purely finite dimensional description of the symplectic structure is given by the cup product in parabolic group cohomology [13] and finding a finite dimensional proof of the closedness of this symplectic form has occupied many people. (See [41, 21, 4, 25, 3].)

By construction the left-hand side of (3) is a finite dimensional symplectic quotient and one of the key results of [29] was that, for any choice of pole positions $\mathbf{a}$, the monodromy map $\nu_{\mathbf{a}}$ is *symplectic*; it pulls back the Atiyah–Bott symplectic structure on the right to the symplectic structure on the left, coming from the coadjoint orbits. This fact is the key to understanding intrinsically why Schlesinger's equations are symplectic, as we will now explain.

Observe that if we vary the positions of the poles slightly then the spaces on the left and the right of (3) do not change. However the monodromy map $\nu_{\mathbf{a}}$ does vary. Schlesinger [61] wondered how the matrices A_i should vary with respect to the pole positions $a_1, \ldots, a_m$ such that the monodromy representation $\nu_{\mathbf{a}}(A_1, \ldots, A_m)$ stays fixed, and thereby discovered the beautiful family of nonlinear differential equations which now bear his name:

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if} \quad i \neq j, \quad \text{and} \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}.$$

These are the equations for *isomonodromic deformations* of the logarithmic connections ∇ on $\mathbb{P}^1$ that we began with in (1). Hitchin's observation now is that the local self-diffeomorphisms of the symplectic manifold $O_1 \times \cdots \times O_m /\!/ G$ induced by integrating Schlesinger's equations, are clearly symplectic diffeomorphisms, because they are of the form $\nu_{\mathbf{a}'}^{-1} \circ \nu_{\mathbf{a}}$ for two sets of pole positions $\mathbf{a}$ and $\mathbf{a}'$ and the monodromy map is a local *symplectic* isomorphism for all $\mathbf{a}$.

This is the picture we will generalise to the case of higher order poles, after rephrasing it in terms of symplectic fibrations. The main missing ingredient is the Atiyah–Bott construction of a symplectic structure on the generalised monodromy data; when the discs are removed any local moduli at the poles is lost. We will work throughout over $\mathbb{P}^1$ since the weight of this paper is to see what to do locally at a pole of order at least two, and because our main interest is the Jimbo–Miwa–Ueno isomonodromy equations, which are for connections over $\mathbb{P}^1$. However, apart from in Section 2 and for the explicit form of the isomonodromy equations, global coordinates on $\mathbb{P}^1$ are not used and so most of this work generalises immediately to arbitrary genus compact Riemann surfaces, possibly with boundary.

The organisation of this paper is as follows. The next three sections each give a different approach to meromorphic connections. In Section 2 we generalise the left-hand side of (3) and prove the results we will need later regarding the symplectic geometry of these spaces. Section 3 describes the generalised monodromy data of a meromorphic connection on a Riemann surface, both the local data (the Stokes matrices) and the global data fitting together the local data at each pole. This generalises the spaces of fundamental group representations above—the notion of fixing the conjugacy class of local monodromy is replaced by fixing the "formal equivalence class". In Section 4 we introduce an appropriate notion of C^∞ singular connections and prove the basic results one might guess from the non-singular case, relating flat singular connections to spaces of monodromy data and to meromorphic connections on degree zero holomorphic bundles. The notion of fixing the formal type of a meromorphic connection corresponds nicely to the notion of fixing the "C^∞ Laurent expansion" of a flat C^∞ singular connection. Section 5 then shows that the Atiyah–Bott symplectic structure generalises naturally to these spaces of C^∞ singular connections, and that as in the non-singular case, the curvature, when defined appropriately, is a moment map for the gauge group action. Thus the spaces of generalised monodromy data also appear as infinite dimensional symplectic quotients. (One should note that the "naive" extension of the Atiyah–Bott symplectic structure to C^∞ singular connections does not work since the two-forms that arise are too singular to be integrated.) Section 6 summarises the preceding sections in a commutative diagram and then proves the key result, that the monodromy map pulls back the Atiyah–Bott type symplectic structure on the generalised monodromy data to the explicit symplectic structure of Section 2, on the spaces of meromorphic connections. Section 7 explains geometrically what the Jimbo–Miwa–Ueno isomonodromy equations are (we will write them explicitly in the appendix), and then proves the main result, Theorem 7.1, that the isomonodromic deformation equations of Jimbo–Miwa–Ueno are equivalent to a flat *symplectic* connection on a *symplectic* fibre bundle having the moduli spaces of Section 2 as fibre. Note that in the general case there are more deformation parameters: we may vary the "irregular types" of the connections at the poles, as well as the pole positions. This produces, in particular, many nonlinear symplectic braid group representations on the spaces of monodromy data. Finally, we end by sketching a relationship between Stokes matrices and Poisson Lie groups.

2. MEROMORPHIC CONNECTIONS ON TRIVIAL BUNDLES

Let $D = k_1(a_1) + \cdots + k_m(a_m) > 0$ be an effective divisor on $\mathbb{P}^1$ (so that $a_1, \ldots, a_m \in \mathbb{P}^1$ are distinct points and $k_1, \ldots, k_m > 0$ are positive integers) and let $V \to \mathbb{P}^1$ be a rank n holomorphic vector bundle.

DEFINITION 2.1. A *meromorphic connection* ∇ on V with poles on D is a map $\nabla: V \to V \otimes K(D)$ from the sheaf of holomorphic sections of V to the sheaf of sections of $V \otimes K(D)$, satisfying the Leibniz rule: $\nabla(fv) = (df) \otimes v + f\nabla v$, where v is a local section of V, f is a local holomorphic function and K is the sheaf of holomorphic one-forms on $\mathbb{P}^1$.

If we choose a local coordinate z on $\mathbb{P}^1$ vanishing at a_i then in terms of a local trivialisation of V, ∇ has the form $\nabla = d - A$, where

$$A = A_{k_i} \frac{dz}{z^{k_i}} + \cdots + A_1 \frac{dz}{z} + A_0\, dz + \cdots \tag{4}$$

is a matrix of meromorphic one-forms and $A_j \in \mathrm{End}(\mathbb{C}^n)$, $j \leqslant k_i$.

DEFINITION 2.2. A meromorphic connection ∇ will be said to be *generic* if at each a_i the leading coefficient A_{k_i} is diagonalisable with distinct eigenvalues (if $k_i \geqslant 2$), or diagonalisable with distinct eigenvalues mod $\mathbb{Z}$ (if $k_i = 1$).

This condition is independent of the trivialisation and coordinate choice. We will restrict to such generic connections since they are simplest yet sufficient for our purpose (to describe the symplectic nature of isomonodromic deformations).

DEFINITION 2.3. A *compatible framing at* a_i of a vector bundle V with generic connection ∇ is an isomorphism $g_0: V_{a_i} \to \mathbb{C}^n$ between the fibre V_{a_i} and $\mathbb{C}^n$ such that the leading coefficient of ∇ is *diagonal* in any local trivialisation of V extending g_0.

Given a trivialisation of V in a neighbourhood of a_i so that $\nabla = d - A$ as above, then a compatible framing is represented by a constant matrix (also denoted g_0) that diagonalises the leading coefficient: $g_0 \in G$ such that $g_0 A_{k_i} g_0^{-1}$ is diagonal.

At each point a_i choose a germ $d-{}^i\!A^0$ of a diagonal generic meromorphic connection on the trivial rank n vector bundle. (We use the terminology that *a* trivial bundle is just trivialisable, but *the* trivial bundle has a chosen trivialisation. Also pre-superscripts ${}^i\!A$, whenever used, will signify local information near a_i.) Thus ${}^i\!A^0$ is a matrix of germs of meromorphic one-forms, which we require (without loss of generality) to be diagonal. If z_i is a local coordinate vanishing at a_i, write

$$ {}^i\!A^0 = d({}^iQ) + {}^i\!\Lambda^0 \frac{dz_i}{z_i}, \tag{5} $$

where ${}^i\!\Lambda^0$ is a constant diagonal matrix and ${}^iQ$ is a diagonal matrix of meromorphic functions.

DEFINITION 2.4. A connection (V, ∇) with compatible framing g_0 at a_i has *irregular type* ${}^i\!A^0$ if g_0 extends to a formal trivialisation of V at a_i, in which ∇ differs from $d-{}^i\!A^0$ by a matrix of one-forms with just simple poles.

Equivalently this means, if $\nabla = d - A$ in some local trivialisation, we require $gAg^{-1} + (dg)\,g^{-1} = d({}^iQ) + {}^i\!\Lambda\, dz_i/z_i$ for some diagonal matrix ${}^i\!\Lambda$ not necessarily equal to ${}^i\!\Lambda^0$ and some formal bundle automorphism $g \in G[\![z_i]\!] = GL_n(\mathbb{C}[\![z_i]\!])$ with $g(a_i) = g_0$. The diagonal matrix ${}^i\!\Lambda$ appearing here will be referred to as the *exponent of formal monodromy* of (V, ∇, g_0).

Let $\mathbf{A}$ denote the choice of the effective divisor D and all the germs ${}^i\!A^0$. The spaces which generalise those on the left-hand side of (3) are defined as follows.

DEFINITION 2.5. The moduli space $\mathcal{M}^*(\mathbf{A})$ is the set of isomorphism classes of pairs (V, ∇) where V is a *trivial* rank n holomorphic vector bundle over $\mathbb{P}^1$ and ∇ is a meromorphic connection on V which is formally equivalent to $d-{}^i\!A^0$ at a_i for each i and has no other poles.

Following [40], we also define slightly larger moduli spaces:

DEFINITION 2.6. The *extended moduli space* $\widetilde{\mathcal{M}}^*(\mathbf{A})$ is the set of isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of a generic connection ∇ (with poles on D) on a *trivial* holomorphic vector bundle V over $\mathbb{P}^1$ with compatible framings $\mathbf{g} = ({}^1g_0, \ldots, {}^mg_0)$ such that $(V, \nabla, \mathbf{g})$ has irregular type ${}^i\!A^0$ at each a_i.

The term "extended moduli space" is taken from the paper [37] of L. Jeffrey, since these spaces play a similar role (but are not the same).

Remark 2.1. For use in later sections we also define spaces $\mathcal{M}(\mathbf{A})$ and $\widetilde{\mathcal{M}}(\mathbf{A})$ simply by replacing the word "trivial" by "degree zero" in Definitions 2.5 and 2.6 respectively.

Since $\mathcal{M}^*(\mathbf{A})$ and $\widetilde{\mathcal{M}}^*(\mathbf{A})$ are moduli spaces of connections on trivial bundles we can obtain explicit descriptions of them. First define G_k to be the group of $(k-1)$-jets of bundle automorphisms

$$G_k := GL_n(\mathbb{C}[\zeta]/\zeta^k),$$

where ζ is an indeterminate. Then the main result of this section is:

PROPOSITION 2.1.

- *$\mathcal{M}^*(\mathbf{A})$ is isomorphic to a complex symplectic quotient*

$$\mathcal{M}^*(\mathbf{A}) \cong O_1 \times \cdots \times O_m /\!/ G \tag{6}$$

where $G := GL_n(\mathbb{C})$ and $O_i \subset \mathfrak{g}^*_{k_i}$ is a coadjoint orbit of G_{k_i}.

- *Similarly there are complex symplectic manifolds (extended orbits) $\widetilde{O}_i$ with $\dim(\widetilde{O}_i) = \dim(O_i) + 2n$ and (free) Hamiltonian G actions, such that*

$$\widetilde{\mathcal{M}}^*(\mathbf{A}) \cong \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!/ G. \tag{7}$$

- *In this way $\widetilde{\mathcal{M}}^*(\mathbf{A})$ inherits (intrinsically) the structure of a complex symplectic manifold, the torus actions changing the choices of compatible framings are Hamiltonian (with moment maps given by the values of the ${}^i\Lambda$'s) and $\mathcal{M}^*(\mathbf{A})$ arises as a symplectic quotient by these m torus actions.*

Because of the third statement here (and that $\mathcal{M}^*(\mathbf{A})$ may not be Hausdorff) we will mainly work with the extended moduli spaces. They will be the phase spaces of the isomonodromy equations. Before proving Proposition 2.1 we first collect together all the results we will need regarding the extended orbits $\widetilde{O}_i$.

Extended Orbits

Fix a positive integer $k \geqslant 2$. Let B_k be the subgroup of G_k of elements having constant term 1. This is a unipotent normal subgroup and in fact G_k is the semi-direct product $G \ltimes B_k$ (where $G := GL_n(\mathbb{C})$ acts on B_k by conjugation). Correspondingly the Lie algebra of G_k decomposes as a vector space direct sum and dualising we have:

$$\mathfrak{g}^*_k = \mathfrak{b}^*_k \oplus \mathfrak{g}^*. \tag{8}$$

Concretely if we have a matrix of meromorphic one-forms A as in (4) with $k = k_i$ then the principal part of A can be identified as an element of $\mathfrak{g}_k^*$ simply by replacing the coordinate z by the indeterminate ζ:

$$A_k \frac{d\zeta}{\zeta^k} + \cdots + A_1 \frac{d\zeta}{\zeta} \in \mathfrak{g}_k^*. \tag{9}$$

Abusing notation, this element of $\mathfrak{g}_k^*$ will also be denoted by A. Such A's are identified as elements of $\mathfrak{g}_k^*$ via the pairing $\langle A, X \rangle := \mathrm{Res}_0(\mathrm{Tr}(A(\zeta)\cdot X)) = \sum_{i=1}^k \mathrm{Tr}(A_i X_{i-1})$ where $X = X_0 + X_1\zeta + \cdots + X_{k-1}\zeta^{k-1} \in \mathfrak{g}_k$. Then from (8), $\mathfrak{b}_k^*$ is identified with the set of A in (9) having zero residue and $\mathfrak{g}^*$ with those having only a residue term (zero irregular part). Let $\pi_{\mathrm{res}}\colon \mathfrak{g}_k^* \to \mathfrak{g}^*$ and $\pi_{\mathrm{irr}}\colon \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ denote the corresponding projections.

Now choose a diagonal element $A^0 = A_k^0\, d\zeta/\zeta^k + \cdots + A_2^0\, d\zeta/\zeta^2$ of $\mathfrak{b}_k^*$ whose leading coefficient A_k^0 has distinct eigenvalues. For example if $k = k_i$, such A^0 arises from the irregular part $d({}^iQ)$ of ${}^iA^0$ in (5). Let $O_B \subset \mathfrak{b}_k^*$ denote the B_k coadjoint orbit containing A^0.

DEFINITION 2.7. The *extended orbit* $\tilde{O} \subset G \times \mathfrak{g}_k^*$ associated to O_B is

$$\tilde{O} := \{(g_0, A) \in G \times \mathfrak{g}_k^* \mid \pi_{\mathrm{irr}}(g_0 A g_0^{-1}) \in O_B\},$$

where $\pi_{\mathrm{irr}}\colon \mathfrak{g}_k^* \to \mathfrak{b}_k^*$ is the natural projection removing the residue.

If $(g_0, A) \in \tilde{O}$ then eventually A will correspond to the principal part of a generic meromorphic connection and g_0 to a compatible framing.

LEMMA 2.2. *The extended orbit $\tilde{O}$ is canonically isomorphic to the symplectic quotient of the product $T^*G_k \times O_B$ by B_k, where both the cotangent bundle T^*G_k and the coadjoint orbit O_B have their natural symplectic structures.*

Proof. B_k acts by the coadjoint action on O_B and by the standard (free) left action on T^*G_k (induced from left multiplication of the groups). A moment map is given by $\mu\colon T^*G_k \times O_B \to \mathfrak{b}_k^*; (g, A, B) \mapsto -\pi_{\mathrm{irr}}(\mathrm{Ad}_g^*(A)) + B$ where $B \in O_B$ and $(g, A) \in G_k \times \mathfrak{g}_k^* \cong T^*G_k$ via the left trivialisation. Thus

$$\mu^{-1}(0) = \{(g, A, B) \mid \pi_{\mathrm{irr}}(gAg^{-1}) = B\}. \tag{10}$$

It is straightforward to check that the map

$$\chi\colon \mu^{-1}(0) \to \tilde{O}; \qquad (g, A, B) \mapsto (g(0), A) \tag{11}$$

is well-defined, surjective and has precisely the B_k orbits as fibres. ∎

This gives $\widetilde{O}$ the structure of a complex symplectic manifold. Next we examine the torus action on $\widetilde{O}$ corresponding to changing the choice of compatible framing. If $(g_0, A) \in \widetilde{O}$ then by hypothesis there is some $g \in G_k$ such that $gAg^{-1} = A^0 + \Lambda\, d\zeta/\zeta$ for some matrix Λ. It is easy now to modify g such that Λ is in fact diagonal. (Conjugating by $1 + X\zeta^{k-1}$ for an appropriate matrix X will remove any off-diagonal part of Λ.) It follows that there is a well-defined map

$$\mu_T\colon \widetilde{O} \to \mathfrak{t}^*; \qquad (g_0, A) \mapsto -\Lambda \frac{d\zeta}{\zeta},$$

where, as above, if $R = \Lambda\, d\zeta/\zeta \in \mathfrak{t}^*$ and $\Lambda' \in \mathfrak{t}$ then $\langle R, \Lambda' \rangle = \mathrm{Tr}(\Lambda\Lambda')$.

LEMMA 2.3. (1) *The map μ_T is a moment map for the free action of $T \cong (\mathbb{C}^*)^n$ on $\widetilde{O}$ defined by $t(g_0, A) = (tg_0, A)$ where $t \in T$.*

(2) *The symplectic quotient at the value $-R$ of μ_T is the G_k coadjoint orbit through the element $A^0 + R$ of $\mathfrak{g}_k^*$.*

(3) *Any tangents v_1, v_2 to $\widetilde{O} \subset G \times \mathfrak{g}_k^*$ at (g_0, A) are of the form*

$$v_i = (X_i(0), [A, X_i] + g_0^{-1} \dot{R}_i\, g_0) \in \mathfrak{g} \times \mathfrak{g}_k^*$$

for some $X_1, X_2 \in \mathfrak{g}_k$ and $\dot{R}_1, \dot{R}_2 \in \mathfrak{t}^*$ (where $\mathfrak{g} \cong T_g G$ via left multiplication), and the symplectic structure on $\widetilde{O}$ is then given explicitly by the formula

$$\omega_{\widetilde{O}}(v_1, v_2) = \langle \dot{R}_1, \widetilde{X}_2 \rangle - \langle \dot{R}_2, \widetilde{X}_1 \rangle + \langle A, [X_1, X_2] \rangle, \tag{12}$$

where $\widetilde{X}_i := g_0 X_i g_0^{-1} \in \mathfrak{g}_k$ for $i = 1, 2$.

Proof. There is a surjective "winding" map $w\colon G_k \times \mathfrak{t}^* \to \widetilde{O}$ defined by $(g, R) \mapsto (g(0), g^{-1}(A^0 + R)\, g)$. It fits into the commutative diagram

$$\begin{array}{ccccc} G_k \times \mathfrak{t}^* & \overset{\iota}{\hookrightarrow} & \mu^{-1}(0) \subset T^*G_k \times O_B & \xrightarrow{\mathrm{pr}} & T^*G_k \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \chi} & & \\ \widetilde{O} & = & \widetilde{O} & & \end{array} \tag{13}$$

where χ is from (11), $\iota(g, R) := (g, g^{-1}(A^0 + R)\, g, A^0)$ and pr is the projection. Since the O_B component of ι is constant the pullback of the symplectic structure on T^*G_k along $\mathrm{pr} \circ \iota$ is the pullback of the symplectic structure on $\widetilde{O}$ along w. Let T act on T^*G_k by the standard left action $t(g, A) = (tg, A)$, on O_B by conjugation ($t(B) = tBt^{-1}$) and on $G_k \times \mathfrak{t}^*$ by left multiplication: $t(g, R) = (tg, R)$. Observe that all the maps in (13) are then T-equivariant and that a moment map on T^*G_k is given by $\nu\colon T^*G_k \to \mathfrak{t}^*; (g, A) \mapsto -\delta(\pi_{\mathrm{res}}(gAg^{-1}))$ since the map $\delta \circ \pi_{\mathrm{res}}$ (taking the diagonal part of the residue term of an element of $\mathfrak{g}_k^*$) is the dual of the derivative of the inclusion $T \hookrightarrow G_k$. Statement 1) now follows from the fact that the pullback of ν along $\mathrm{pr} \circ \iota$ is the pullback of μ_T along w (both maps pullback to the projection $G_k \times \mathfrak{t}^* \to \mathfrak{t}^*$). The third statement is proved by directly calculating the pullback of the symplectic structure on T^*G_k along $\mathrm{pr} \circ \iota$. (Note $(X_i, \dot{R}_i)$ is just a lift of v_i to $G_k \times \mathfrak{t}^*$.) The second statement follows directly from (12). ∎

Thus, by projecting to $\mathfrak{g}_k^*$, we see $\widetilde{O}$ is a principal T bundle over an n-parameter family of G_k coadjoint orbits. An alternative description will also be useful:

LEMMA 2.4 (Decoupling). *The following map is a symplectic isomorphism*

$$\widetilde{O} \cong T^*G \times O_B; \quad (g_0, A) \mapsto (g_0, \pi_{\mathrm{res}}(A), \pi_{\mathrm{irr}}(g_0 A g_0^{-1})),$$

where $T^*G \cong G \times \mathfrak{g}^*$ via the left trivialisation.

Proof. It is an isomorphism as the map $(g_0, S, B) \mapsto (g_0, g_0^{-1} B g_0 + S) \in \widetilde{O}$ (where $(g_0, S, B) \in T^*G \times O_B$) is an inverse. Under this identification, a section s of the projection χ in (13) is given by: $s: T^*G \times O_B \to T^*G_k \times O_B$; $(g_0, S, B) \mapsto (g_0, g_0^{-1} B g_0 + S, B)$ where left multiplication is used to trivialise the cotangent bundles. A straightforward calculation shows s is symplectic. ∎

This will be important because O_B admits *global* Darboux coordinates.

COROLLARY 2.5. *The free G action $h(g_0, A) := (g_0 h^{-1}, hAh^{-1})$ on $\widetilde{O}$ is Hamiltonian with moment map $\mu_G \colon \widetilde{O} \to \mathfrak{g}^*$; $(g_0, A) \mapsto \pi_{\mathrm{res}}(A)$.*

Proof. After decoupling $\widetilde{O}$, G acts only on the T^*G factor and it does so by the standard action coming from right multiplications, which has moment map μ_G. ∎

Finally in the simple pole case ($k = 1$) not yet considered we define

$$\widetilde{O} := \{(g_0, A) \in G \times \mathfrak{g}^* \mid g_0 A g_0^{-1} \in \mathfrak{t}'\} \subset G \times \mathfrak{g}^*$$

where $\mathfrak{t}' \subset \mathfrak{t}^*$ is the subset containing diagonal matrices whose eigenvalues are distinct mod $\mathbb{Z}$. If we identify $G \times \mathfrak{g}^*$ with T^*G then $\widetilde{O}$ is in fact a symplectic submanifold (see [24, Theorem 26.7]). The formula (12) holds unchanged and the free G and T actions are still Hamiltonian with the same moment maps as above (the diagonalisation of A used to define μ_T is simply $g_0 A g_0^{-1}$). Note that the winding map $w \colon G \times \mathfrak{t}' \to \widetilde{O}$; $(g_0, R) \mapsto (g_0, g_0^{-1} R g_0)$ is now an isomorphism.

Proof (of Proposition 2.1). Choose a coordinate z to identify $\mathbb{P}^1$ with $\mathbb{C}\cup\infty$ such that each a_i is finite. Define $z_i := z - a_i$. The chosen meromorphic connection germs $d - {}^i\!A^0$ determine G_{k_i} coadjoint orbits O_i and extended orbits $\widetilde{O}_i$ as above: Define O_i to be the coadjoint orbit through the point of $\mathfrak{g}^*_{k_i}$ determined (using the coordinate choice z_i) by the principal part of ${}^i\!A^0$ in (5). Similarly the irregular part of ${}^i\!A^0$ determines a point of $\mathfrak{b}^*_{k_i}$ and $\widetilde{O}_i$ is the extended orbit associated to the B_{k_i} coadjoint orbit through this point.

Now suppose ∇ is a meromorphic connection on a holomorphically trivial bundle V over $\mathbb{P}^1$ with poles on the divisor D. Upon trivialising V we find $\nabla = d - A$ for a matrix A of meromorphic one-forms of the form

$$A = \sum_{i=1}^{m} \left({}^i\!A_{k_i} \frac{dz}{(z-a_i)^{k_i}} + \cdots + {}^i\!A_1 \frac{dz}{(z-a_i)} \right), \tag{14}$$

where the ${}^i\!A_j$ are $n \times n$ matrices. The principal part of A at a_i determines an element ${}^i\!A \in \mathfrak{g}^*_{k_i}$ as above (replacing $z - a_i$ by ζ in the ith term of the sum (14)).

The crucial fact now is that ∇ is formally equivalent to $d - {}^i\!A^0$ at a_i if and only if ${}^i\!A$ is in O_i. The "only if" part is clear since the gauge action restricts to the coadjoint action on the principal parts of A. The converse is not true in general (even if formal *meromorphic* transformations are allowed: see [8]), but it does hold in the generic case we are considering here, and is well-known (see [11]). Also, using the description of the extended orbits as principal T bundles, it follows that if ∇ is generic and has compatible framings $\mathbf{g} = ({}^1g_0, \ldots, {}^mg_0)$ then $(V, \nabla, \mathbf{g})$ has irregular type ${}^i\!A^0$ at a_i if and only if $({}^ig_0, {}^i\!A)$ is in $\widetilde{O}_i$.

Thus any meromorphic connection on the trivial bundle with the correct formal type determines and is determined by a point of the product $O_1 \times \cdots \times O_m$. Observe however that a general point of $O_1 \times \cdots \times O_m$ will give a connection with an additional pole at $z = \infty$ unless we impose the constraint

$$ {}^1\!A_1 + \cdots + {}^m\!A_1 = 0. \tag{15}$$

Also observe that the choice of global trivialisation of V corresponds to the action of G on $O_1 \times \cdots \times O_m$ by diagonal conjugation. The first statement in Proposition 2.1 follows simply by observing that the left-hand side of (15) is a moment map for this G action on $O_1 \times \cdots \times O_m$ (since the G action on each O_i factor is the restriction of the coadjoint action to $G \subset G_{k_i}$).

The proof of the second statement is completely analogous. (The G action on $\widetilde{O}_i$ is given in Corollary 2.5.)

Lemma 2.4 makes it transparent that $\widetilde{\mathcal{M}}^*(\mathbf{A})$ is a smooth complex manifold: the symplectic quotient by G just removes a factor of T^*G from the product of extended orbits. It is straightforward to check the complex symplectic structure so defined on $\widetilde{\mathcal{M}}^*(\mathbf{A})$ is independent of the coordinate choices used above. (In fact arbitrary *local* coordinates z_i may be used.)

Finally the statements concerning the torus actions are immediate from Lemma 2.3, since the G and T action on each extended orbit commute. ∎

Remark 2.2. Open subsets of the symplectic quotients $O_1 \times \cdots \times O_m /\!/ G$ in (6) have been previously studied: They are algebraically completely integrable Hamiltonian systems [2, 12]. See also [18] Sections 4.3 and 5.3. The perspective there is to regard these as spaces of meromorphic Higgs fields, rather than as spaces of meromorphic connections.

3. GENERALISED MONODROMY

This section describes the monodromy data of a generic meromorphic connection, involving both a fundamental group representation and Stokes matrices, largely following [8, 11, 40, 43, 50]. The presentation here is quite nonstandard however and care has been taken to keep track of all the choices made and thereby see what is intrinsically defined.

Fix the data $\mathbf{A}$ of a divisor $D = \sum k_i(a_i)$ on $\mathbb{P}^1$ and connection germs $d - {}^iA^0$ at each a_i as in Section 2. Let $(V, \nabla, \mathbf{g})$ be a compatibly framed meromorphic connection on a holomorphic vector bundle $V \to \mathbb{P}^1$ with irregular type $\mathbf{A}$.

In brief, the monodromy data arises as follows. The germ $d - {}^iA^0$ canonically determines some directions at a_i ("anti-Stokes directions") for each i and (using local coordinate choices) we can consider the sectors at each a_i bounded by these directions (and having some small fixed radius). Then the key fact is that the framings $\mathbf{g}$ (and a choice of branch of logarithm at each pole) determine, in a canonical way, a choice of basis of solutions of the connection ∇ on each such sector at each pole. Now along any path in the punctured sphere $\mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ between two such sectors we can extend the two corresponding bases of solutions and obtain a constant $n \times n$ matrix relating these two bases. The monodromy data of $(V, \nabla, \mathbf{g})$ is simply the set of all such constant matrices, plus the exponents of formal monodromy.

Before filling in the details of this procedure we will give a concrete definition of the monodromy manifolds that store this monodromy data and so give a clear idea of where we are going. All of the monodromy manifolds are of the following form. Suppose $N_1, \dots, N_m$ are manifolds, we have maps $\rho_i \colon N_i \to G'$ to some group G' for each i and that there is an action of $G = GL_n(\mathbb{C})$ on G' (via group automorphisms) and on each N_i such that ρ_i is G-equivariant. Define a map $\boldsymbol{\rho}$ to be the (reverse ordered) product of the ρ_i's:

$$\boldsymbol{\rho}: N_1 \times \cdots \times N_m \to G'; \qquad (n_1, \ldots, n_m) \mapsto \rho_m(n_m) \cdots \rho_2(n_2)\,\rho_1(n_1).$$

Since G acts on G' by automorphisms, $\boldsymbol{\rho}$ is G-equivariant and $\boldsymbol{\rho}^{-1}(1)$ is a G-invariant subset of the product $N_1 \times \cdots \times N_m$. We will write the quotient as:

$$N_1 \times \cdots \times N_m /\!/ G := \boldsymbol{\rho}^{-1}(1)/G. \tag{16}$$

This is viewed simply as a convenient way to write down the various sets of monodromy data that arise.[1] There is no conflict of notation since the symplectic quotients of Section 2 arise in this way by taking $N_i = O_i$ (or $\widetilde{O}_i$), $G' = (\mathfrak{g}^*, +)$ and the ρ_i's as the moment maps for the G actions. All of the examples in this section however will have $G' := G$ acting on itself by conjugation.

Recall in the simple pole case that we fixed generic coadjoint orbits $O_1, \ldots, O_m$ of G to define a symplectic space of connections on trivial bundles over $\mathbb{P}^1$. By choosing appropriate generators of the fundamental group of the punctured sphere we see that the corresponding space of monodromy data is of the form

$$\mathrm{Hom}_{\mathscr{C}}(\pi_1(\mathbb{P}^1 \backslash \{a_1, \ldots, a_m\}), G)/G \cong \mathscr{C}_1 \times \cdots \times \mathscr{C}_m /\!/ G, \tag{17}$$

where G acts on each conjugacy class $\mathscr{C}_i$ by conjugation and each map $\rho_i: \mathscr{C}_i \to G$ is just the inclusion.

Considering higher order poles in Section 2 amounted to replacing the coadjoint orbits of G above by coadjoint orbits of G_{k_i} (still denoted O_i) or by extended orbits $\widetilde{O}_i$. By analogy, in this section we now replace each conjugacy class in the simple pole case by a larger manifold $\mathscr{C}_i$ (the multiplicative version of O_i), or by $\widetilde{\mathscr{C}}_i$ (the multiplicative version of $\widetilde{O}_i$). The basic definition is somewhat surprising:

DEFINITION 3.1.

- Let $U_{+/-}$ be the upper/lower triangular unipotent subgroups of G, then

$$\widetilde{\mathscr{C}}_i := G \times (U_+ \times U_-)^{k_i - 1} \times \mathfrak{t},$$

[1] The relationship with [3] will be discussed elsewhere.

where $\mathfrak{t}$ is the set of diagonal $n \times n$ matrices and k_i is the pole order at a_i. (If $k_i = 1$ replace $\mathfrak{t}$ by $\mathfrak{t}'$ here; the elements with distinct eigenvalues mod $\mathbb{Z}$.) A point of $\widetilde{\mathscr{C}}_i$ will be denoted $(C_i, {}^i\mathbf{S}, {}^i\Lambda')$ where ${}^i\mathbf{S} = ({}^iS_1, \ldots, {}^iS_{2k_i-2}) \in U_+ \times U_- \times U_+ \times \cdots \times U_-$.

- The formula $t(C_i, {}^i\mathbf{S}, {}^i\Lambda') = (t \cdot C_i, (t \cdot {}^iS_1 \cdot t^{-1}, \ldots, t \cdot {}^iS_{2k_i-2} \cdot t^{-1}), {}^i\Lambda')$ defines a free action of the torus T on $\widetilde{\mathscr{C}}_i$ and, given some fixed choice of ${}^i\Lambda'$, we define $\mathscr{C}_i$ to be the subset of the quotient having $\mathfrak{t}$ component fixed to this value: $\mathscr{C}_i := (\widetilde{\mathscr{C}}_i / T)|_{{}^i\Lambda'}$.
- The map $\rho_i : \widetilde{\mathscr{C}}_i \to G$ is defined by the formula

$$\rho_i(C_i, {}^i\mathbf{S}, {}^i\Lambda') = C_i^{-1} \cdot {}^iS_{2k_i-2} \cdots {}^iS_2 \cdot {}^iS_1 \cdot \exp\left((2\pi\sqrt{-1}\,){}^i\Lambda'\right) \cdot C_i.$$

(This is T invariant so descends to define $\rho_i \colon \mathscr{C}_i \to G$.)

- Finally G acts on $\widetilde{\mathscr{C}}_i$ (and on $\mathscr{C}_i$) via $g(C_i, {}^i\mathbf{S}, {}^i\Lambda') = (C_i g^{-1}, {}^i\mathbf{S}, {}^i\Lambda')$ (so that ρ_i is clearly G-equivariant, where G acts on itself by conjugation).

The triangular matrices ${}^iS_j$ (with 1's on their diagonals) appearing here are the Stokes matrices. Note that in every case the dimension of $\widetilde{\mathscr{C}}_i$ is the same as the dimension of the extended orbit $\widetilde{O}_i$ (and similarly $\dim(\mathscr{C}_i) = \dim(O_i)$). Also note that if the pole is simple ($k_i = 1$) then $\widetilde{\mathscr{C}}_i = G \times \mathfrak{t}'$ and that $\mathscr{C}_i$ can naturally be identified with the conjugacy class through $\exp(2\pi\sqrt{-1} \cdot {}^i\Lambda') \in G$.

Our aim in the rest of this section is to define an (abstract) space of monodromy data $M(\mathbf{A})$ and an intrinsic holomorphic map ν from the moduli space $\mathscr{M}^*(\mathbf{A})$ of Section 2 to $M(\mathbf{A})$, obtained by taking monodromy data. We will call ν the monodromy map, although the names Riemann–Hilbert map or de Rham morphism are also appropriate. Recall in Proposition 2.1 that after making some choices (of local coordinates in that case) a concrete description of the moduli space $\mathscr{M}^*(\mathbf{A})$ was obtained. Analogously here, after making some choices (of some "tentacles"; something like a choice of generators of the fundamental group—see Definition 3.9), we will see that the quotient $\mathscr{C}_1 \times \cdots \times \mathscr{C}_m /\!/ G$ is a concrete realisation of the space of monodromy data. Thus we will have the diagram:

$$\begin{array}{ccc} O_1 \times \cdots \times O_m /\!/ G & & \mathscr{C}_1 \times \cdots \times \mathscr{C}_m /\!/ G \\ \uparrow \cong & & \uparrow \cong \\ \mathscr{M}^*(\mathbf{A}) & \xrightarrow{\quad \nu \quad} & M(\mathbf{A}). \end{array} \tag{18}$$

As in Section 2 we will work mainly with the extended version (putting tildes on all the spaces in the above diagram) since the spaces are then manifolds (and again the non-extended version may be deduced by considering torus actions).

LEMMA 3.1. *The extended monodromy manifold* $\widetilde{M}(\mathbf{A}) \cong \widetilde{\mathscr{C}}_1 \times \cdots \times \widetilde{\mathscr{C}}_m /\!/ G$ *is indeed a complex manifold and has the same dimension as* $\widetilde{\mathscr{M}}^*(\mathbf{A})$.

Proof. Remove the G action by fixing $C_1 = 1$, so $\widetilde{M}(\mathbf{A})$ is identified with the subvariety $\rho_m \cdots \rho_1 = 1$ of the product $\widetilde{\mathscr{C}}'_1 \times \widetilde{\mathscr{C}}_2 \times \cdots \times \widetilde{\mathscr{C}}_m$ where $\widetilde{\mathscr{C}}'_1$ is the subset of $\widetilde{\mathscr{C}}_1$ having $C_1 = 1$. The result now follows from the implicit function theorem since the map $\rho_m \cdots \rho_1 : \widetilde{\mathscr{C}}'_1 \times \widetilde{\mathscr{C}}_2 \times \cdots \times \widetilde{\mathscr{C}}_m \to G$ is surjective on tangent vectors (except in the trivial case $m = 1, k_1 = 1$). In particular $\dim \widetilde{M}(\mathbf{A}) = \sum \dim(\widetilde{\mathscr{C}}_i) - 2n^2$ and, from Proposition 2.7, this is $\dim \widetilde{\mathscr{M}}^*(\mathbf{A})$. ∎

Local Moduli: Stokes Matrices

First we will set up the necessary auxiliary data. Let $d - A^0$ be a diagonal generic meromorphic connection on the trivial rank n vector bundle over the unit disc $\mathbb{D} \subset \mathbb{C}$ with a pole of order $k \geqslant 2$ at 0 and no others. Let z be a coordinate on $\mathbb{D}$ vanishing at 0. Thus (as in Section 2) $A^0 = dQ + \Lambda^0 \frac{dz}{z}$ where Λ^0 is a constant diagonal matrix and Q is a diagonal matrix of meromorphic functions. Q is determined by A^0 and z by requiring that it has constant term zero in its Laurent expansion with respect to z. Write $Q = \mathrm{diag}(q_1, \ldots, q_n)$ and define $q_{ij}(z)$ to be the leading term of $q_i - q_j$. Thus if $q_i - q_j = a/z^{k-1} + b/z^{k-2} + \cdots$ then $q_{ij} = a/z^{k-1}$.

Let the circle S^1 parameterise rays (directed lines) emanating from $0 \in \mathbb{C}$; intrinsically one can think of this circle as being the boundary circle of the real oriented blow up of $\mathbb{C}$ at the origin. If $d_1, d_2 \in S^1$ then $\mathrm{Sect}(d_1, d_2)$ will denote the (open) sector swept out by rays rotating in a positive sense from d_1 to d_2. The radius of these sectors will be taken sufficiently small when required later.

DEFINITION 3.2. The *anti-Stokes directions* $\mathbb{A} \subset S^1$ are the directions $d \in S^1$ such that for some $i \neq j$: $q_{ij}(z) \in \mathbb{R}_{<0}$ for z on the ray specified by d.

These are the directions along which $e^{q_i - q_j}$ decays most rapidly as z approaches 0 and it follows that $\mathbb{A}$ is independent of the coordinate choice z. (For uniform notation later, define $\mathbb{A}$ to contain a single, arbitrary direction if $k = 1$; this will be used only as a local branch cut.)

DEFINITION 3.3. Let $d \in S^1$ be an anti-Stokes direction.

- The *roots* of d are the ordered pairs (ij) "supporting" d:

$$\mathrm{Roots}(d) := \{(ij) \mid q_{ij}(z) \in \mathbb{R}_{<0} \text{ along } d\}.$$

- The *multiplicity* Mult(d) of d is the number of roots supporting d.
- The *group of Stokes factors* associated to d is the group

$$\mathbb{S}\mathrm{to}_d(A^0) := \{K \in G \mid (K)_{ij} = \delta_{ij} \text{ unless } (ij) \text{ is a root of } d\}.$$

It is straightforward to check that $\mathbb{S}\mathrm{to}_d(A^0)$ is a unipotent subgroup of $G = GL_n(\mathbb{C})$ of dimension equal to the multiplicity of d. Beware that the terms "Stokes factors" and "Stokes matrices" are used in a number of different senses in the literature. (Our terminology is closest to Balser, Jurkat and Lutz [11]. However our approach is perhaps closer to that of Martinet and Ramis [50] but what we call Stokes factors, they call Stokes matrices, and they do not use the things we call Stokes matrices.)

The anti-Stokes directions $\mathbb{A}$ have $\pi/(k-1)$ rotational symmetry: if $q_{ij}(z) \in \mathbb{R}_{<0}$ then $q_{ji}(z \exp(\frac{\pi\sqrt{-1}}{k-1})) \in \mathbb{R}_{<0}$. Thus the number $r := \#\mathbb{A}$ of anti-Stokes directions is divisible by $2k-2$ and we define $l := r/(2k-2)$. We will refer to an l-tuple $\mathbf{d} = (d_1, \ldots, d_l) \subset \mathbb{A}$ of *consecutive* anti-Stokes directions as a "half-period". When weighted by their multiplicities, the number of anti-Stokes directions in any half-period is $n(n-1)/2 = \mathrm{Mult}(d_1) + \cdots + \mathrm{Mult}(d_l)$. Also a half-period $\mathbf{d}$ determines a total ordering of the set $\{q_1, \ldots, q_n\}$ defined by:

$$q_i \underset{\mathbf{d}}{<} q_j \qquad \Leftrightarrow \qquad (ij) \text{ is a root of some } d \in \mathbf{d}. \tag{19}$$

A simple check shows (ij) is a root of some $d \in \mathbf{d}$ precisely if $e^{q_i}/e^{q_j} \to 0$ as $z \to 0$ along the ray $\theta(\mathbf{d}) \in S^1$ bisecting $\mathrm{Sect}(d_1, d_l)$ (so that (19) is the natural dominance ordering along $\theta(\mathbf{d})$). In turn there is a permutation matrix $P \in G$ associated to $\mathbf{d}$ given by $(P)_{ij} = \delta_{\pi(i)\, j}$ where π is the permutation of $\{1, \ldots, n\}$ corresponding to (19): $q_i \underset{\mathbf{d}}{<} q_j \Leftrightarrow \pi(i) < \pi(j)$. A key result is then:

Lemma 3.2. *Let $\mathbf{d} = (d_1, \ldots, d_l) \subset \mathbb{A}$ be a half-period (where d_{i+1} is the next anti-Stokes direction after d_i in a positive sense).*

(1) *The product of the corresponding groups of Stokes factors is isomorphic as a variety, via the product map, to the subgroup of G conjugate to U_+ via P:*

$$\prod_{d \in \mathbf{d}} \mathbb{S}to_d(A^0) \cong PU_+P^{-1}; \qquad (K_1, \ldots, K_l) \mapsto K_l \cdots K_2 K_1 \in G.$$

(2) *Label the rest of* $\mathbb{A}$ *uniquely as* $d_{l+1}, \ldots, d_r$ *(in order) then the following map from the product of all the groups of Stokes factors, is an isomorphism of varieties*

$$\prod_{d\in\mathbb{A}} \mathbb{S}\mathrm{to}_d(A^0) \cong (U_+ \times U_-)^{k-1}; \qquad (K_1, \ldots, K_r) \mapsto (S_1, \ldots, S_{2k-2})$$

where $S_i := P^{-1}K_{il} \cdots K_{(i-1)l+1}P \in U_{+/-}$ *if* i *is odd/even.*

Proof. (1) holds since the groups of Stokes factors are a set of "direct spanning" subgroups of PU_+P^{-1}; see Borel [15, Section 14]. It is also proved directly in Lemma 2 of [11]. (2) follows from (1) simply by observing that the orderings associated to neighbouring half-periods are opposite. ∎

Now we move on to the local moduli of meromorphic connections. Let $\mathrm{Syst}(A^0)$ denote the set of germs at $0 \in \mathbb{C}$ of meromorphic connections on the trivial rank n vector bundle, that are formally equivalent to $d - A^0$. Concretely we have

$$\mathrm{Syst}(A^0) = \{d - A \mid A = \hat{F}[A^0] \text{ for some } \hat{F} \in G[\![z]\!]\},$$

where A is a matrix of germs of meromorphic one-forms, $G[\![z]\!] = GL_n(\mathbb{C}[\![z]\!])$ and $\hat{F}[A^0] = (d\hat{F})\hat{F}^{-1} + \hat{F}A^0\hat{F}^{-1}$. The group $G[\![z]\!]$ does not act on $\mathrm{Syst}(A^0)$ since in general $\hat{F}[A^0]$ will not have convergent entries. The subgroup $G\{z\} := GL_n(\mathbb{C}\{z\})$ of germs of holomorphic bundle automorphisms does act however and we wish to study the quotient $\mathrm{Syst}(A^0)/G\{z\}$ which is by definition the set of isomorphism classes of germs of meromorphic connections formally equivalent to A^0. Note that any *generic* connection is formally equivalent to some such A^0.

In the Abelian case ($n = 1$) and in the simple pole case ($k = 1$) $\mathrm{Syst}(A^0)/G\{z\}$ is just a point; the notions of formal and holomorphic equivalence coincide. In the non-Abelian, irregular case ($n, k \geqslant 2$) however, $\mathrm{Syst}(A^0)/G\{z\}$ is non-trivial and we will explain how to describe it explicitly in terms of Stokes matrices.

It is useful to consider spaces slightly larger than $\mathrm{Syst}(A^0)$:

DEFINITION 3.4.

- Let $\widehat{\mathrm{Syst}}_{\mathrm{cf}}(A^0)$ be the set of compatibly framed connection germs with both irregular and formal type A^0.
- Let $\widehat{\mathrm{Syst}}_{\mathrm{mp}}(A^0) := \{(A, \hat{F}) \mid A \in \mathrm{Syst}(A^0), \hat{F} \in G[\![z]\!], A = \hat{F}[A^0]\}$, be the set of *marked pairs.*

Thus $\widehat{\mathrm{Syst}}_{\mathrm{cf}}(A^0)$ is the set of pairs (A, g_0) with $A \in \mathrm{Syst}(A^0)$ and $g_0 \in G$, such that $g_0[A]$ and A^0 have the same leading term. Clearly the projection to the first factor is a surjection $\widehat{\mathrm{Syst}}_{\mathrm{cf}}(A^0) \twoheadrightarrow \mathrm{Syst}(A^0)$ and the fibres are the orbits of the torus action $t(A, g_0) = (A, t \cdot g_0)$ (where $t \in T \cong (\mathbb{C}^*)^n$).

LEMMA 3.3. *There is a canonical isomorphism* $\widehat{\mathrm{Syst}}_{\mathrm{mp}}(A^0) \cong \widehat{\mathrm{Syst}}_{\mathrm{cf}}(A^0)$: *For each compatibly framed connection germ* $(A, g_0) \in \widehat{\mathrm{Syst}}_{\mathrm{cf}}(A^0)$ *there is a* unique *formal isomorphism* $\hat{F} \in G[\![z]\!]$ *with* $A = \hat{F}[A^0]$ *and* $\hat{F}(0) = g_0^{-1}$.

Proof. It is sufficient to prove that if $\hat{F}[A^0] = A^0$ then $\hat{F} \in T$, since this implies the map $(A, \hat{F}) \mapsto (A, g_0)$ with $g_0 := \hat{F}(0)^{-1}$, is bijective. Now if $\hat{F}[A^0] = A^0$ then the (ij) matrix entry f of $\hat{F}$ is a power series solution to $df = (d(q_i - q_j) + (\lambda_i - \lambda_j)\, dz/z)\, f$, where $\lambda_i = (\Lambda^0)_{ii}$. It follows (using Definition 2.2 if $k = 1$) that $f = 0$ unless $i = j$ when it is a constant. ∎

See for example [8] for an algorithm to determine $\hat{F}$ from g_0. Below we will use "$\widehat{\mathrm{Syst}}(A^0)$" to denote either of these two sets. Heuristically the action $g(A, \hat{F}) = (g[A], g \circ \hat{F})$ of $G\{z\}$ on the marked pairs is free and so one expects the quotient

$$\mathscr{H}(A^0) := \widehat{\mathrm{Syst}}(A^0)/G\{z\}$$

to be in some sense nice (as is indeed the case). Moreover the actions of T and $G\{z\}$ on $\widehat{\mathrm{Syst}}(A^0)$ commute so $\mathrm{Syst}(A^0)/G\{z\} \cong \mathscr{H}(A^0)/T$.

The fundamental technical result we need to quote in order to describe $\mathscr{H}(A^0)$ is the following theorem. First we set up a labelling convention, that will behave well when we vary A^0 in later sections. Choose a point p in one of the r sectors at 0 bounded by anti-Stokes rays. Label the first anti-Stokes ray when turning in a positive sense from p as d_1 and label the subsequent rays $d_2, \ldots, d_r$ in turn. Write $\mathrm{Sect}_i := \mathrm{Sect}(d_i, d_{i+1})$; the "$i$th sector" (indices are taken modulo r). Note $p \in \mathrm{Sect}_r = \mathrm{Sect}_0$; the "last sector". Also define the "ith supersector" to be $\widehat{\mathrm{Sect}}_i := \mathrm{Sect}(d_i - \frac{\pi}{2k-2}, d_{i+1} + \frac{\pi}{2k-2})$. This is a sector containing the ith sector symmetrically (the same direction bisects both) and has opening greater than $\pi/(k-1)$. (The rays bounding these supersectors are usually referred to as "Stokes rays".)

THEOREM 3.1. *Suppose* $\hat{F} \in G[\![z]\!]$ *is a formal transformation such that* $A := \hat{F}[A^0]$ *has convergent entries. Set the radius of the sectors* Sect_i, $\widehat{\mathrm{Sect}}_i$ *to be less than the radius of convergence of* A. *Then the following hold:*

(1) *On each sector* Sect_i *there is a canonical way to choose an invertible* $n \times n$ *matrix of holomorphic functions* $\Sigma_i(\hat{F})$ *such that* $\Sigma_i(\hat{F})[A^0] = A$.

(2) $\Sigma_i(\hat{F})$ *can be analytically continued to the supersector* $\widehat{\mathrm{Sect}}_i$ *and then* $\Sigma_i(\hat{F})$ *is asymptotic to* $\hat{F}$ *at* 0 *within* $\widehat{\mathrm{Sect}}_i$.

(3) *If* $g \in G\{z\}$ *and* $t \in T$ *then* $\Sigma_i(g \circ \hat{F} \circ t^{-1}) = g \circ \Sigma_i(\hat{F}) \circ t^{-1}$.

The point is that on a narrow sector there are generally many holomorphic isomorphisms between A^0 and A which are asymptotic to $\hat{F}$ and one is being chosen in a canonical way; $\Sigma_i(\hat{F})$ is in fact uniquely characterised by property (2). There are various ways to construct $\Sigma_i(\hat{F})$, although the details will not be needed here. In particular the series $\hat{F}$ is "$(k-1)$-summable" on Sect_i, with sum $\Sigma_i(\hat{F})$—see [10, 48, 50]. Other approaches appear in [11, 43]. See also [47, 72] regarding asymptotic expansions on sectors.

Functions on the quotient $\mathscr{H}(A^0)$ are now obtained as follows. Let $(A, g_0) \in \widehat{\text{Syst}}(A^0)$ be a compatibly framed connection germ and let $\hat{F} \in G[\![z]\!]$ be the associated formal isomorphism from Lemma 3.3. The sums of $\hat{F}$ on the two sectors adjacent to some anti-Stokes ray $d_i \in \mathbb{A}$ may be analytically continued across d_i and they will generally be different on the overlap. Thus for each anti-Stokes ray d_i there is a matrix of holomorphic functions $\kappa_i := \Sigma_i(\hat{F})^{-1} \circ \Sigma_{i-1}(\hat{F})$ asymptotic to 1 on a sectorial neighbourhood of d_i. Moreover clearly $\kappa_i[A^0] = A^0$; it is an automorphism of A^0. A concrete description of κ_i is obtained by choosing a basis of solutions of A^0, which is made via a choice of branch of $\log(z)$.

Thus choose a branch of $\log(z)$ along d_1 and extend it in a positive sense across $\text{Sect}_1, d_2, \text{Sect}_2, d_3, \ldots, \text{Sect}_r = \text{Sect}_0$ in turn. In particular we get a lift $\tilde{p}$ of the point $p \in \text{Sect}_0$ to the universal cover of the punctured disc $\mathbb{D} \setminus \{0\}$ and we will say that these $\log(z)$ choices are *associated to* $\tilde{p}$.

DEFINITION 3.5. Fix data $(A^0, z, \tilde{p})$ as above. The *Stokes factors* of a compatibly framed connection $(A, g_0) \in \widehat{\text{Syst}}(A^0)$ are

$$K_i := e^{-Q} z^{-A^0} \cdot \kappa_i \cdot z^{A^0} e^{Q}, \qquad i = 1, \ldots, r = \#\mathbb{A}$$

using the choice of $\log(z)$ along d_i, where $\kappa_i := \Sigma_i(\hat{F})^{-1} \circ \Sigma_{i-1}(\hat{F})$.

Since $z^{A^0} e^Q$ is a fundamental solution of A^0 (i.e. its columns are a basis of solutions) we have $d(K_i) = 0$; the Stokes factors are constant invertible matrices. By part (3) of Theorem 3.1, K_i only depends on the $G\{z\}$ orbit of (A, g_0) and so matrix entries of K_i are functions on $\mathscr{H}(A^0)$. A useful equivalent definition is:

DEFINITION 3.6. Fix data $(A^0, z, \tilde{p})$ and choose $(A, g_0) \in \widehat{\text{Syst}}(A^0)$.

- The *canonical fundamental solution* of A on the ith sector is $\Phi_i := \Sigma_i(\hat{F})\, z^{A^0} e^Q$ where z^{A^0} uses the choice (determined by $\tilde{p}$) of $\log(z)$ on $\widehat{\text{Sect}}_i$. (Note $\Phi_{i+r} = \Phi_i$.)
- If Φ_i is continued across the anti-Stokes ray d_{i+1} then on Sect_{i+1} we have: $K_{i+1} := \Phi_{i+1}^{-1} \circ \Phi_i$ for all i except $K_1 := \Phi_{i+1}^{-1} \circ \Phi_i \circ M_0^{-1}$ for $i = r$, where $M_0 := e^{2\pi\sqrt{-1}\cdot A^0}$ is the so-called "formal monodromy".

Taking care to use the right $\log(z)$ choices it is straightforward to prove the equivalence of these two definitions of the Stokes factors. The basic fact then is:

LEMMA 3.4. *The Stokes factor K_i is in the group $\mathbb{S}\mathrm{to}_{d_i}(A^0)$.*

Proof. From Theorem 3.1, $\Sigma_j(\hat{F})$ is asymptotic to $\hat{F}$ at 0 when continued within the supersector $\widehat{\mathrm{Sect}}_j$, for each j. Thus (if $i \neq 1$) $z^{A^0}e^Q K_i e^{-Q} z^{-A^0} = \Sigma_i(\hat{F})^{-1}\Sigma_{i-1}(\hat{F})$ is asymptotic to 1 within the intersection $\widehat{\mathrm{Sect}}_i \cap \widehat{\mathrm{Sect}}_{i-1}$. As K_i is constant we must therefore have $(K_i)_{ab} = \delta_{ab}$ unless $e^{q_a - q_b} \to 0$ as $z \to 0$ along any ray in $\widehat{\mathrm{Sect}}_i \cap \widehat{\mathrm{Sect}}_{i-1}$. It is straightforward to check this is equivalent to (ab) being a root of d_i. (The $i = 1$ case is similar.) ∎

Thus as in Lemma 3.2 we can define the Stokes matrices of $(A, g_0) \in \widehat{\mathrm{Syst}}(A^0)$

$$S_i := P^{-1}K_{il} \cdots K_{(i-1)l+1}P \in U_{+/-}$$

if i is odd/even, where $i = 1, ..., 2k-2$ and P is the permutation matrix associated to the half-period $(d_1, ..., d_l)$. To go directly from the canonical solutions to the Stokes matrices, simply observe that if Φ_{il} is continued in a positive sense across all the anti-Stokes rays $d_{il+1}, ..., d_{(i+1)l}$ and onto $\mathrm{Sect}_{(i+1)l}$ we have: $\Phi_{il} = \Phi_{(i+1)l}PS_{i+1}P^{-1}$ for $i = 1, ..., 2k-3$, and $\Phi_{il} = \Phi_l PS_1 P^{-1} M_0$ for $i = 2k-2 = r/l$ where $M_0 = e^{2\pi\sqrt{-1}A^0}$. The main fact we need is then:

THEOREM 3.2 (Balser, Jurkat, Lutz [11]). *Fix the data $(A^0, z, \tilde{p})$ as above. Then the "local monodromy map" taking the Stokes matrices induces a* bijection

$$\mathscr{H}(A^0) \xrightarrow{\cong} (U_+ \times U_-)^{k-1}; \qquad [(A, g_0)] \longmapsto (S_1, ..., S_{2k-2}).$$

In particular $\mathscr{H}(A^0)$ is isomorphic to the vector space $\mathbb{C}^{(k-1)n(n-1)}$.

Sketch. For injectivity, suppose two compatibly framed systems in $\widehat{\mathrm{Syst}}(A^0)$ have the same Stokes matrices. Let $\hat{F}_1, \hat{F}_2$ be their associated formal isomorphisms (from Lemma 3.3). Since the Stokes matrices (and therefore the Stokes factors and the automorphisms κ_j) are equal, the holomorphic matrix $\Sigma_i(\hat{F}_2) \circ \Sigma_i(\hat{F}_1)^{-1}$ has no monodromy around 0 and does not depend on i. Thus on any sector it has asymptotic expansion $\hat{F}_2 \circ \hat{F}_1^{-1}$ and so (by Riemann's removable singularity theorem) we deduce the power series $\hat{F}_2 \circ \hat{F}_1^{-1}$ is actually convergent with the function $\Sigma_i(\hat{F}_2) \circ \Sigma_i(\hat{F}_1)^{-1}$ as sum. This gives an isomorphism between the systems we began with: they represent the same point in $\mathscr{H}(A^0)$. Surjectivity follows from a result of Sibuya: See [11, Section 6]. ▮

Remark 3.1. The set $\mathscr{H}(A^0)$ is also described (by the Malgrange–Sibuya isomorphism) as the first cohomology of a sheaf of non-Abelian unipotent groups over the circle S^1, explaining our notation. However we will not use this viewpoint: the sums $\Sigma_i(\hat{F})$ lead to canonical choices of representatives of the cohomology classes that occur. See [9, 43, 50] and the survey [70].

Finally two (by now easy) facts that we will need are:

COROLLARY 3.5.

- *The torus action on $\mathscr{H}(A^0)$ changing the compatible framing corresponds to the conjugation action $t(\mathbf{S}) = (tS_1t^{-1}, \dots, tS_{2k-2}t^{-1})$ on the Stokes matrices, and so there is a bijection $\mathrm{Syst}(A^0)/G\{z\} \cong (U_+ \times U_-)^{k-1}/T$ between the set of isomorphism classes of germs of meromorphic connections formally equivalent to A^0 and the set of T-orbits of Stokes matrices.*
- *If Φ_0 is continued once around 0 in a positive sense, then on return to Sect_0 it will become*

$$\Phi_0 \cdot PS_{2k-2} \cdots S_2 S_1 P^{-1} M_0,$$

where $M_0 = e^{2\pi\sqrt{-1}\cdot A^0}$ is the formal monodromy.

Proof. The first part is immediate from Theorem 3.1 statement (3). For the second part, from Definition 3.6 we see Φ_0 becomes $\Phi_i \cdot K_i \cdots K_2 K_1 M_0$ when continued to Sect_i. Then observe $K_r \cdots K_1 = PS_{2k-2} \cdots S_1 P^{-1}$. ▮

Global Monodromy

Recall we have fixed the data $\mathbf{A}$ of a divisor $D = \sum k_i(a_i)$ on $\mathbb{P}^1$ and connection germs $d - {}^iA^0$ at each a_i. Now also choose m disjoint open discs D_i on $\mathbb{P}^1$ with $a_i \in D_i$ and, for each i, a coordinate z_i on D_i vanishing at a_i. Thus the local picture above is repeated on each such disc. Abstractly the monodromy manifolds will be defined as spaces of representations of the following groupoid $\tilde{\Gamma}$.

Choose a base-point $p_0 \in \mathbb{P}^1 \setminus \{a_1, \dots, a_m\}$ and a point b_ξ in each of the sectors bounded by anti-Stokes directions at each pole a_i, where ξ ranges over some finite set indexing these sectors. Let $\tilde{B}_i$ denote the (discrete) subset of points of the universal cover of the punctured disc $D_i \setminus \{a_i\}$, which are above one of the b_ξ's. Let $\tilde{B} := \{p_0\} \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_m$. If $\tilde{p} \in \tilde{B}$ we will write p for the point of $\mathbb{P}^1$ underlying $\tilde{p}$ (namely p_0 or one of the b_ξ's).

DEFINITION 3.7.

(1) The set of objects of the groupoid $\widetilde{\Gamma}$ is the set $\widetilde{B}$.

(2) If $\widetilde{p}_1, \widetilde{p}_2 \in \widetilde{B}$, the set of morphisms of $\widetilde{\Gamma}$ from $\widetilde{p}_1$ to $\widetilde{p}_2$ is the set of homotopy classes of paths γ: $[0, 1] \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ from p_1 to p_2.

This is clearly groupoid with multiplication (of composable morphisms) defined by path composition.

Now let $(V, \nabla, \mathbf{g})$ be a compatibly framed meromorphic connection with irregular type ${}^iA^0$ at a_i for each i. (Thus, if V is trivial, $(V, \nabla, \mathbf{g})$ represents a point of the extended moduli space $\widetilde{\mathcal{M}}^*(\mathbf{A})$.) For each choice of basis of the fibre V_{p_0} of V at p_0 such $(V, \nabla, \mathbf{g})$ naturally determines a representation of the groupoid $\widetilde{\Gamma}$ in the group $G = GL_n(\mathbb{C})$, as follows.

Suppose $[\gamma_{\widetilde{p}_2\widetilde{p}_1}]$ is a morphism in $\widetilde{\Gamma}$, represented by a path $\gamma_{\widetilde{p}_2\widetilde{p}_1}$ in the punctured sphere from p_1 to p_2. Then from Definition 3.10 (with $\widetilde{p} = \widetilde{p}_i$) we obtain a canonical choice of basis Φ_i: $\mathbb{C}^n \to V$ of ∇-horizontal sections of V in a neighbourhood of p_i for $i = 1, 2$. (First use any local trivialisation of V, and then observe the basis obtained is independent of this choice. In the case $\widetilde{p}_i = p_0$, use the choice of basis of V_{p_0} to determine Φ_i.) Now both bases extend uniquely (as solutions of ∇) along the track $\gamma_{\widetilde{p}_2\widetilde{p}_1}([0, 1])$ of the path $\gamma_{\widetilde{p}_2\widetilde{p}_1}$. Since they are both ∇-horizontal bases we have $\Phi_1 = \Phi_2 \cdot C$ on the track of $\gamma_{\widetilde{p}_2\widetilde{p}_1}$, for some constant invertible matrix $C \in G$. The representation ρ of $\widetilde{\Gamma}$ is defined by setting

$$\rho(\gamma_{\widetilde{p}_2\widetilde{p}_1}) := C = {\Phi_2}^{-1}\Phi_1. \tag{20}$$

Clearly C only depends on the homotopy class of the path in $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ and it is easy to check this is indeed a representation. (For example ρ maps contractible loops to 1 and has composition property $\rho(\gamma_{\widetilde{p}_3\widetilde{p}_2} \cdot \gamma_{\widetilde{p}_2\widetilde{p}_1}) = \rho(\gamma_{\widetilde{p}_3\widetilde{p}_2}) \cdot \rho(\gamma_{\widetilde{p}_2\widetilde{p}_1})$.)

Thus ρ encodes all possible "connection matrices" between sectors at different poles as well as all the Stokes factors and Stokes matrices at each pole. To characterise the representations of $\widetilde{\Gamma}$ that arise in this way we observe:

LEMMA 3.6. *The representation ρ has the following two properties:*

(SR1) *For any i, if $\widetilde{p}_1 \in \widetilde{B}_i$ and $\widetilde{p}_2$ is the next element of $\widetilde{B}_i$ after $\widetilde{p}_1$ in a positive sense and $\gamma_{\widetilde{p}_2\widetilde{p}_1}$ is a small arc in D_i from p_1 to p_2 then $\rho(\gamma_{\widetilde{p}_2\widetilde{p}_1}) \in \mathbb{S}\mathrm{to}_d({}^iA^0)$, where d is the unique anti-Stokes ray that $\gamma_{\widetilde{p}_2\widetilde{p}_1}$ crosses.*

(SR2) *For each i there is a diagonal matrix ${}^i\Lambda$ (which has distinct eigenvalues mod $\mathbb{Z}$ if $k_i = 1$) such that for any $\tilde{p}_1 \in \widetilde{B}_i$, $\tilde{p}_2 \in \widetilde{B}$ and morphism $\gamma_{\tilde{p}_2\tilde{p}_1}$,*

$$\rho(\gamma_{\tilde{p}_2(\tilde{p}_1+2\pi)}) = \rho(\gamma_{\tilde{p}_2\tilde{p}_1}) \cdot \exp(2\pi\sqrt{-1} \cdot {}^i\Lambda),$$

where $\gamma_{\tilde{p}_2(\tilde{p}_1+2\pi)} = \gamma_{\tilde{p}_2\tilde{p}_1}$ as paths, but $(\tilde{p}_1 + 2\pi)$ is the next point of $\widetilde{B}_i$ after $\tilde{p}_1$ (in a positive sense) which is also above p_1.

Proof. The first part is immediate from Definition 3.5 and Lemma 3.4 whilst the second is clear from the definition of the canonical solutions, with ${}^i\Lambda$ the exponent of formal monodromy of $(V, \nabla, \mathbf{g})$ at a_i. ∎

DEFINITION 3.8.

- A *Stokes representation* ρ is a representation of the groupoid $\widetilde{\Gamma}$ into G together with a choice of m diagonal matrices ${}^i\Lambda$ such that (SR1) and (SR2) hold. The set of Stokes representations will be denoted $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G)$.
- The matrices ${}^i\Lambda$ associated to a Stokes representation ρ will be called the *exponents of formal monodromy* of ρ and the number $\deg(\rho) := \sum_i \mathrm{Tr}({}^i\Lambda)$ is the *degree* of ρ.
- Two Stokes representations are *isomorphic* if they are in the same orbit of the following G action on $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G)$: if $\tilde{p}_1, \tilde{p}_2 \in \widetilde{B} \setminus \{p_0\}$, $g \in G$ define

$$(g\cdot\rho)(\gamma_{p_0p_0}) = g\rho(\gamma_{p_0p_0})\,g^{-1}, \qquad (g\cdot\rho)(\gamma_{p_0\tilde{p}_1}) = g\rho(\gamma_{p_0\tilde{p}_1}),$$
$$(g\cdot\rho)(\gamma_{\tilde{p}_2p_0}) = \rho(\gamma_{\tilde{p}_2p_0})\,g^{-1}, \qquad (g\cdot\rho)(\gamma_{\tilde{p}_2\tilde{p}_1}) = \rho(\gamma_{\tilde{p}_2\tilde{p}_1}).$$

- The *extended monodromy manifold* $\widetilde{M}(\mathbf{A}) := \mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G)/G$ is the set of isomorphism classes of Stokes representations.

Observe that this G action on $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G)$ corresponds to the choice of basis of the fibre V_{p_0} made above, and so a compatibly framed meromorphic connection $(V, \nabla, \mathbf{g})$ canonically determines a point of $\widetilde{M}(\mathbf{A})$. (Also $\widetilde{M}(\mathbf{A})$ does not depend on the choices of the base-points p_0, b_ξ that were used to define the groupoid $\widetilde{\Gamma}$.)

PROPOSITION 3.7 (see [40]). *Two compatibly framed meromorphic connections are isomorphic if and only if they have isomorphic Stokes representations.*

Proof. Suppose $(V^1, \nabla^1, \mathbf{g}^1)$, $(V^2, \nabla^2, \mathbf{g}^2)$ are isomorphic and both have irregular type ${}^iA^0$ at each a_i. Thus there is a vector bundle isomorphism

$\varphi: V^1 \to V^2$ which relates the connections and the framings. It is easy to check now that, for each i, φ also relates the canonical bases $\Phi^{1,2}(z_i)$: $\mathbb{C}^n \to V^{1,2}$ of solutions on each sector at a_i, associated to any point $\tilde{p} \in \tilde{B}_i$. This implies the Stokes representations are isomorphic. Conversely if the Stokes representations are isomorphic the local isomorphisms $\Phi^2 \circ (\Phi^1)^{-1}$: $V^1 \to V^2$ extend to $\mathbb{P}^1$ to give the desired isomorphism φ, as in the proof of Theorem 3.2. ▮

Thus on restricting attention to connections on trivial bundles we get a well-defined injective map $\tilde{\nu}: \widetilde{\mathcal{M}}^*(\mathbf{A}) \to \widetilde{M}(\mathbf{A})$ from the extended moduli space $\widetilde{\mathcal{M}}^*(\mathbf{A})$ of Section 2. This is the (extended) monodromy map and is the key ingredient in the whole isomonodromy story. It is a map between complex manifolds of the same dimension (see Lemma 3.1 and Proposition 3.8) and moreover results of Sibuya and Hsieh [63, 32, 62] imply it is *holomorphic*. (They prove each canonical fundamental solution varies holomorphically with parameters and therefore so does all the monodromy data—see also [40, Proposition 3.2].) It follows immediately that $\tilde{\nu}$ is surjective on tangent vectors and biholomorphic onto its image (since any injective holomorphic map between equi-dimensional complex manifolds has these properties—see for example [58, Theorem 2.14]). We will see in Section 7 that the image of $\tilde{\nu}$ is the complement of a divisor in the degree zero component of $\widetilde{M}(\mathbf{A})$.

Now we wish to describe the monodromy manifold $\widetilde{M}(\mathbf{A})$ more explicitly and this requires the following choices:

Definition 3.9. A choice of *tentacles* $\mathcal{T}$ is a choice of:

(1) A point p_i in some sector at a_i between two anti-Stokes rays $(i = 1, \ldots, m)$.

(2) A lift $\tilde{p}_i$ of each p_i to the universal cover of the punctured disc $D_i \setminus \{a_i\}$.

(3) A base-point $p_0 \in \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$.

(4) A path $\gamma_i: [0, 1] \to \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ in the punctured sphere, from p_0 to p_i for $i = 1, \ldots, m$, such that the loop

$$(\gamma_m^{-1} \cdot \beta_m \cdot \gamma_m) \cdot \cdots \cdot (\gamma_2^{-1} \cdot \beta_2 \cdot \gamma_2) \cdot (\gamma_1^{-1} \cdot \beta_1 \cdot \gamma_1) \tag{21}$$

based at p_0 is contractible in $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$, where β_i is any loop in $D_i \setminus \{a_i\}$ based at p_i encircling a_i once in a positive sense.

Proposition 3.8. *For each choice of tentacles $\mathcal{T}$ there is an explicit algebraic isomorphism $\tilde{\varphi}_{\mathcal{T}}: \widetilde{M}(\mathbf{A}) \to \widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m /\!/ G$ from the extended* *monodromy manifold to the "explicit monodromy manifold" of Definition* 3.1 *and* (16).

Proof. The choice $\mathscr{T}$ determines an isomorphism $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G) \xrightarrow{\cong} \boldsymbol{\rho}^{-1}(1) \subset \widetilde{\mathscr{C}}_1 \times \cdots \times \widetilde{\mathscr{C}}_m$ as follows. Recall, using the convention used before, that the chosen point $\widetilde{p}_i \in \mathscr{T}$ determines a labelling of, and a $\log(z_i)$ choice on, each sector and anti-Stokes ray at a_i. Let ${}^i\widetilde{b}_j$ be the element of $\widetilde{B}_i$ lying in the corresponding lift of the jth sector ${}^i\mathrm{Sect}_j$ at a_i to the universal cover of the punctured disc $D_i \setminus \{a_i\}$. Without loss of generality we assume that ${}^i\widetilde{b}_0 = \widetilde{p}_i$ and that the base-point p_0 of $\widetilde{\Gamma}$ and $\mathscr{T}$ is the same. Also the labelling determines a permutation matrix P_i associated to each a_i (see Lemma 3.2). (If $k_i = 1$ set $P_i = 1$.) Let $\gamma_{\widetilde{p}_i p_0}$ be the morphism of $\widetilde{\Gamma}$ from p_0 to $\widetilde{p}_i$ corresponding to the path γ_i and define $C_i := P_i^{-1}\rho(\gamma_{\widetilde{p}_i p_0}) \in G$ for $i = 1, \ldots, m$. Next let ${}^i\sigma_j$ be the morphism from ${}^i\widetilde{b}_{(j-1)\cdot l}$ to ${}^i\widetilde{b}_{j\cdot l}$ with underlying path a simple arc in $D_i \setminus \{a_i\}$ from ${}^i b_{(j-1)\cdot l}$ to ${}^i b_{j\cdot l}$ in a positive sense (where $l = l_i = r_i/(2k_i - 2)$ and $r_i = \#{}^i\mathbb{A}$). Then define the Stokes matrices (as explained before Theorem 3.2) by the formulae: ${}^iS_j := P_i^{-1}\rho({}^i\sigma_j)\, P_i$ for $j = 2, \ldots, 2k_i - 2$ and ${}^iS_1 := P_i^{-1}\rho({}^i\sigma_0) \cdot {}^iM_0^{-1} \cdot P_i$. Finally set ${}^i\Lambda' = P_i^{-1}\,{}^i\Lambda P_i$ where ${}^i\Lambda$ is the ith exponent of formal monodromy of ρ (Definition 3.8). Thus a Stokes representation ρ determines a point $(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}')$ of the product $\widetilde{\mathscr{C}}_1 \times \cdots \times \widetilde{\mathscr{C}}_m$, where $\mathbf{C} = (C_1, C_2, \ldots, C_m)$, $\mathbf{S} = ({}^1\mathbf{S}, \ldots, {}^m\mathbf{S})$, ${}^i\mathbf{S} := ({}^iS_1, \ldots, {}^iS_{2k_i-2})$ and $\boldsymbol{\Lambda}' = ({}^1\Lambda', \ldots, {}^m\Lambda')$. Now observe that the value $\rho(\gamma_i^{-1} \cdot \beta_i \cdot \gamma_i)$ of the representation ρ on the loop $\gamma_i^{-1} \cdot \beta_i \cdot \gamma_i$ based at p_0 is equal to the value $\rho_i(C_i, {}^i\mathbf{S}, {}^i\Lambda')$ of the map $\rho_i : \widetilde{\mathscr{C}} \to G$, and so the contractibility of the loop (21) implies the monodromy data $(\mathbf{C}, \mathbf{S}, \boldsymbol{\Lambda}')$ satisfies the constraint $\rho_m \cdots \rho_1 = 1$. This defines the map $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G) \to \boldsymbol{\rho}^{-1}(1)$ and it is straightforward to see it is an isomorphism (using Lemma 3.2 and knowledge of the fundamental group of the punctured sphere). This map is G-equivariant and so descends to give $\widetilde{\varphi}_{\mathscr{T}}$. ∎

Now we turn to the non-extended version. First, taking the exponents of formal monodromy $\boldsymbol{\Lambda} = ({}^1\Lambda, \ldots, {}^m\Lambda)$ of any Stokes representation ρ induces a map

$$\mu_{T^m} : \widetilde{M}(\mathbf{A}) \longrightarrow \mathfrak{t}^m; \qquad \rho \longmapsto \boldsymbol{\Lambda}.$$

Also for each pole a_i there is a torus action on $\mathrm{Hom}_{\mathbb{S}}(\widetilde{\Gamma}, G)$ defined by the formulae

$$\begin{aligned} (t\cdot\rho)(\gamma_{\widetilde{p}_2\widetilde{p}_1}) &= t\rho(\gamma_{\widetilde{p}_2\widetilde{p}_1})\, t^{-1} & \quad (t\cdot\rho)(\gamma_{\widetilde{q}_2\widetilde{p}_1}) &= \rho(\gamma_{\widetilde{q}_2\widetilde{p}_1})\, t^{-1} \\ (t\cdot\rho)(\gamma_{\widetilde{q}_2\widetilde{q}_1}) &= \rho(\gamma_{\widetilde{q}_2\widetilde{q}_1}) & \quad (t\cdot\rho)(\gamma_{\widetilde{p}_2\widetilde{q}_1}) &= t\rho(\gamma_{\widetilde{p}_2\widetilde{q}_1}) \end{aligned} \tag{22}$$

for any $\widetilde{p}_1, \widetilde{p}_2 \in \widetilde{B}_i$ and $\widetilde{q}_1, \widetilde{q}_2 \in \widetilde{B} \setminus \widetilde{B}_i$, where $t \in T$.

DEFINITION 3.10.

- The (non-extended) space of monodromy data $M(\mathbf{A})$ is the set of T^m orbits in $\widetilde{M}(\mathbf{A})$ which have exponents of formal monodromy equal to $\mathbf{\Lambda}^0 = ({}^1\Lambda^0, \ldots, {}^m\Lambda^0)$, where ${}^i\Lambda^0 = \mathrm{Res}_{a_i}({}^iA^0)$:

$$M(\mathbf{A}) := \mu_{T^m}^{-1}(\mathbf{\Lambda}^0)/T^m.$$

- The *monodromy map* is the map ν: $\mathcal{M}^*(\mathbf{A}) \to M(\mathbf{A})$ induced from the extended monodromy map.

The monodromy map is well-defined since part (3) of Theorem 3.1 implies that the extended monodromy map is T^m-equivariant and also since it is clear that $\mu_{T^m} \circ \tilde{\nu}$ is the moment map for the T^m action on the extended moduli space $\widetilde{\mathcal{M}}^*(\mathbf{A})$ (defined in Proposition 2.1).

COROLLARY 3.9. *For each choice of tentacles $\mathcal{T}$ there is an explicit algebraic isomorphism $\varphi_{\mathcal{T}}$: $M(\mathbf{A}) \to \mathscr{C}_1 \times \cdots \times \mathscr{C}_m /\!/ G$ from the monodromy space to the explicit set of monodromy data from Definition* 3.1 *(with $\mathscr{C}_i$ depending on $\mathcal{T}$).*

Proof. The choice of tentacles determines a permutation matrix P_i for $i = 1, \ldots, m$. Then define ${}^i\Lambda' := P_i^{-1} \cdot {}^i\Lambda^0 \cdot P_i$ and use this value to define $\mathscr{C}_i$. The rest now follows from Proposition 3.8 since $\tilde{\varphi}_{\mathcal{T}}$ is T^m-equivariant, where $(t_1, \ldots, t_m)$ acts on $\widetilde{\mathscr{C}}_i$ via $(P_i^{-1} t_i P_i) \in T$ and the T-action of Definition 3.1. ∎

Remark 3.2 (Degree). If ρ is a Stokes representation having exponents of formal monodromy $\mathbf{\Lambda}$ then the degree $\deg(\rho) = \sum_i \mathrm{Tr}({}^i\Lambda)$ of ρ is an integer. One way to see this is to choose some tentacles so ρ determines (via Proposition 3.8) a point $(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}')$ of $\widetilde{\mathscr{C}}_1 \times \cdots \times \widetilde{\mathscr{C}}_m$ satisfying the constraint $\rho_m \cdots \rho_1 = 1$. By taking the determinant of this constraint we see that $\sum_i \mathrm{Tr}({}^i\Lambda') = \sum_i \mathrm{Tr}({}^i\Lambda) \in \mathbb{Z}$. (It is also clear that the fixed-degree components $\widetilde{M}_d(\mathbf{A})$ of the extended monodromy manifolds are pairwise isomorphic.) On the other hand suppose $(V, \nabla, \mathbf{g})$ is a compatibly framed meromorphic connection on a holomorphic vector bundle $V \to \mathbb{P}^1$ with irregular type $\mathbf{A}$ and exponents of formal monodromy $\mathbf{\Lambda}$. Then by considering the induced connection on the determinant line bundle $\Lambda^n V$ of V one finds that $\sum_i \mathrm{Tr}({}^i\Lambda)$ is equal to the degree of the vector bundle V. The only point we need to make here is that the germs ${}^iA^0$ must be chosen such that $\sum_i \mathrm{Tr}({}^i\Lambda^0) = 0$, if the moduli spaces $\mathcal{M}^*(\mathbf{A})$ are to be non-empty, and so we will tacitly assume this throughout.

To end this section we describe the dependence on the local coordinate choices z_i that were made right at the start. Let $\mathbf{A}$ be a choice of divisor D

and connection germs $d-{}^i\!A^0$ as above. This determines all the spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$, $\widetilde{M}(\mathbf{A})$, $\mathcal{M}^*(\mathbf{A})$ and $M(\mathbf{A})$.

PROPOSITION 3.10.

(1) *The extended monodromy map* $\tilde{\nu}: \widetilde{\mathcal{M}}^*(\mathbf{A}) \to \widetilde{M}(\mathbf{A})$ *depends (only) on the choice of a* k_i*-jet of a coordinate* z_i *at each* a_i.

(2) *This coordinate dependence is only within the* T^m *orbits: The monodromy map* $\nu: \mathcal{M}^*(\mathbf{A}) \to M(\mathbf{A})$ *is completely intrinsic.*

Proof. The key point is to see how a fundamental solution $\Phi = \Sigma_j(\widehat{F})\, z^{\Lambda} e^{Q}$ changes when the coordinate z is changed. Here $A^0 = dQ + \Lambda\, dz/z$ is fixed and Q is determined by (A^0, z) by requiring it to have zero constant term in its Laurent expansion with respect to z. Suppose $z' = ze^f$ is a new coordinate choice, for some local holomorphic function f. One finds $Q' = Q - \Lambda f + \Lambda f(0) - \mathrm{Res}_0(Q\, df)$ (as meromorphic functions near $z=0$), since then $\mathrm{Res}_0(Q'\, dz'/z') = 0$. In turn $\Phi' = \Phi \cdot t^{-1}$ where $t = \exp(\mathrm{Res}_0(Q\, df) - \Lambda f(0)) \in T$. (The function $\Sigma_j(\widehat{F})$ is intrinsic.) Then observe: 1) If $f = O(z^k)$ then $t = 1$, since Q has a pole of order $k-1$, and 2) This action of $t \in T$ corresponds to the torus action we have defined. ∎

Remark 3.3. One should also note that all the spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$, $\widetilde{M}(\mathbf{A})$, $\mathcal{M}^*(\mathbf{A})$ and $M(\mathbf{A})$ only depend on the principal part of each germ ${}^i\!A^0$. For the monodromy manifolds this is immediate and for the moduli spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$, $\mathcal{M}^*(\mathbf{A})$ it is because all ${}^i\!A^0$ with the same principal part are formally equivalent via a transformation with constant term 1 (as explained in Section 2).

4. C^∞ APPROACH TO MEROMORPHIC CONNECTIONS

This section gives a third viewpoint on meromorphic connections: a C^∞ approach. Although we work exclusively with "generic" connections over $\mathbb{P}^1$ (as we wish to study isomonodromic deformations of such connections) we remark that this C^∞ approach works over arbitrary compact Riemann surfaces (maybe with boundary) and the generic hypothesis is also superfluous (see Remark 4.2).

Singular Connections: C^∞ Connections with Poles

Let $D = k_1(a_1) + \cdots + k_m(a_m)$ be an effective divisor on $\mathbb{P}^1$ as usual and choose m disjoint discs $D_i \subset \mathbb{P}^1$ with $a_i \in D_i$ and a coordinate z_i on D_i

vanishing at a_i. Define the sheaf of "smooth functions with poles on D" to be the sheaf of C^∞ sections of the holomorphic line bundle associated to the divisor D

$$C^\infty[D] := \mathcal{O}[D] \otimes_{\mathcal{O}} C^\infty$$

where $\mathcal{O}$ is the sheaf of holomorphic functions and C^∞ the infinitely differentiable complex functions. Any local section of $C^\infty[D]$ near a_i is of the form $f/z_i^{k_i}$ for a C^∞ function f. Similarly define sheaves $\Omega^r[D]$ of C^∞ r-forms with poles on D (so in particular $\Omega^0[D] = C^\infty[D]$). A basic feature is that "C^∞-Laurent expansions" can be taken at each a_i. This gives a map

$$L_i \colon \Omega^*[D](\mathbb{P}^1) \to \mathbb{C}[\![z_i, \bar{z}_i]\!]\, z_i^{-k_i} \otimes \bigwedge^* \mathbb{C}^2, \tag{23}$$

where $\mathbb{C}^2 = \mathbb{C}dz_i \oplus \mathbb{C}d\bar{z}_i$. For example if f is a C^∞ function defined in a neighbourhood of a_i then $L_i(f/z_i^{k_i}) = L_i(f)/z_i^{k_i}$ where $L_i(f)$ is the Taylor expansion of f at a_i.

The Laurent map L_i has nice morphism properties, for example $L_i(\omega_1 \wedge \omega_2) = L_i(\omega_1) \wedge L_i(\omega_2)$ and L_i commutes with the exterior derivative d, where d is defined on the right-hand side of (23) in the obvious way ($d(z_i^{-1}) = -dz_i/z_i^2$).

We will repeatedly make use of the fact that the kernel of L_i consists of nonsingular forms, that is: if $L_i(\omega) = 0$ then ω is nonsingular at a_i. This apparently innocuous statement is surprisingly tricky to prove directly, but since it is crucial for us we remark it follows from the following:

Lemma 4.1 (Division). *Let $\mathbb{D} \subset \mathbb{C}$ be a disk containing the origin. Suppose $f \in C^\infty(\mathbb{D})$ and that the Taylor expansion of f at 0 is in the ideal in $\mathbb{C}[\![z, \bar{z}]\!]$ generated by z. Then $f/z \in C^\infty(\mathbb{D})$.*

Proof. This is a special case of a much more general result of Malgrange [44]. The particular instance here is discussed by Martinet [49, p. 115]. ∎

Another fact we will use is that the C^∞ Laurent expansion map L_i in (23) is *surjective* for each i. This is due to a classical result of E. Borel which we quote here in the relative case that will be needed later:

Theorem 4.2 (E. Borel). *Suppose U is a differentiable manifold, I is a compact neighbourhood of the origin in $\mathbb{R}$ and $\hat{f} \in \mathbb{C}[\![x, y]\!] \otimes C^\infty(U)$ (where x, y are real coordinates on $\mathbb{C} \cong \mathbb{R}^2$). Then there exists a smooth function $f \in C^\infty(U \times I \times I)$ such that the Taylor expansion of f at $x = y = 0$ is $\hat{f}$.*

Proof. This is easily deduced, via partitions of unity, by using two applications of the version of Borel's theorem proved on p. 16 of Hörmander's book [31]. ∎

Now let $V \to \mathbb{P}^1$ be a rank n, C^∞ complex vector bundle.

DEFINITION 4.1. A C^∞ *singular connection* ∇ *on* V *with poles on* D is a map $\nabla: V \longrightarrow V \otimes \Omega^1[D]$ from the sheaf of (C^∞) sections of V to the sheaf of sections of $V \otimes \Omega^1[D]$, satisfying the Leibniz rule: $\nabla(fv) = (df) \otimes v + f\nabla v$ where v is a local section of V and f is a local C^∞ function.

Concretely in terms of the local coordinate z_i on $\mathbb{P}^1$ vanishing at a_i and a local trivialisation of V, ∇ has the form: $\nabla = d - {}^iA/z_i^{k_i}$ where ${}^iA$ is an $n \times n$ matrix of C^∞ one-forms. In this paper, to study the Jimbo–Miwa–Ueno isomonodromy equations, we need only to consider the case when V is the trivial rank n, C^∞ vector bundle over $\mathbb{P}^1$. (Recall any degree zero vector bundle over $\mathbb{P}^1$ is C^∞ trivial.)

DEFINITION 4.2.

- Let $\mathscr{A}[D]$ denote the set of C^∞ singular connections with poles on D on the trivial C^∞ rank n vector bundle: $\mathscr{A}[D] := \{d - \alpha \mid \alpha \in \mathrm{End}_n(\Omega^1[D](\mathbb{P}^1))\}$ where $\Omega^1[D]$ is the sheaf of C^∞ one-forms with poles on D.
- The *gauge group* of C^∞ bundle automorphisms is $\mathscr{G} := GL_n(C^\infty(\mathbb{P}^1))$.
- The *curvature* of a singular connection $d - \alpha \in \mathscr{A}[D]$ is the matrix of singular two-forms $\mathscr{F}(\alpha) := (d-\alpha)^2 = -d\alpha + \alpha^2 \in \mathrm{End}_n(\Omega^2[2D](\mathbb{P}^1))$.
- The *flat* connections are those with zero curvature and the subset of flat singular connections will be denoted $\mathscr{A}_{\mathrm{fl}}[D]$.

Remark 4.1. Occasionally one comes across notions of curvature of singular connections involving distributional derivatives. For example a meromorphic connection on a Riemann surface is sometimes said to have a δ-function singularity in its curvature at the pole, to account for the monodromy around the pole. The definition above of curvature does *not* involve distributional derivatives, and so, for us, *any* meromorphic connection over a Riemann surface is flat.

The group $\mathscr{G}$ of bundle automorphisms clearly acts on the singular connections $\mathscr{A}[D]$ and explicitly this is given by the formula $g[\alpha] = g\alpha g^{-1} + (dg)g^{-1}$. This restricts to an action on $\mathscr{A}_{\mathrm{fl}}[D]$ since $\mathscr{F}(g[\alpha]) = g(\mathscr{F}(\alpha))g^{-1}$ for $g \in \mathscr{G}$.

Now choose a generic diagonal connection germ $d - {}^iA^0$ at a_i for each i and let **A** denote this m-tuple of germs and the divisor D, as usual. Since $d - \alpha \in \mathscr{A}[D]$ is on the trivial vector bundle, and $d - {}^iA^0$ is a germ of a connection on the trivial bundle, we can compare the Laurent expansion of α at a_i with ${}^i\!A^0$. In particular the following definition makes sense:

DEFINITION 4.3.

• Let $\mathscr{A}(\mathbf{A})$ be the set of singular connections with fixed Laurent expansions:

$$\mathscr{A}(\mathbf{A}) := \{d-\alpha \in \mathscr{A}[D] \mid L_i(\alpha) = {}^i\!A^0 \text{ for each } i\}.$$

• Let $\widetilde{\mathscr{A}}(\mathbf{A})$ be the following *extended* set of singular connections with fixed Laurent expansions:

$$\widetilde{\mathscr{A}}(\mathbf{A}) := \left\{ d-\alpha \in \mathscr{A}[D] \;\middle|\; L_i(\alpha) = {}^i\!A^0 + ({}^i\!\Lambda - {}^i\!\Lambda^0)\frac{dz_i}{z_i} \text{ for some } {}^i\!\Lambda \in \mathfrak{t}_i \right\},$$

where ${}^i\!\Lambda^0 = \mathrm{Res}_0({}^i\!A^0)$, $\mathfrak{t}_i = \mathfrak{t}$ if $k_i \geqslant 2$ and $\mathfrak{t}_i = \mathfrak{t}'$ if $k_i = 1$.

• Let $\mathscr{G}_T$ and $\mathscr{G}_1$ denote the subgroups of $\mathscr{G}$ of elements having Taylor expansion equal to a constant diagonal matrix or the identity, respectively, at each a_i.

The basic motivation for this definition is Corollary 4.4 below. Note that $\mathscr{A}(\mathbf{A})$ is an affine space and that if $d-\alpha \in \mathscr{A}(\mathbf{A})$ then (from the division lemma above) the (0, 1) part of α is *nonsingular* over all of $\mathbb{P}^1$.

Smooth Local Picture

Now we will give a C^∞ description of the sets $\mathscr{H}(A^0)$ and the local analytic classes $\mathrm{Syst}(A^0)/G\{z\}$ defined in Section 3.

We begin with a straightforward observation. Let z be a complex coordinate on the unit disc $\mathbb{D} \subset \mathbb{C}$. From Borel's theorem we have an exact sequence of groups

$$1 \longrightarrow {}^0\mathscr{G}_1 \longrightarrow {}^0\mathscr{G} \xrightarrow{L_0} GL_n(\mathbb{C}[\![z, \bar{z}]\!]) \longrightarrow 1,$$

where ${}^0\mathscr{G}$ is the group of germs at 0 of gauge transformations $g \in \mathscr{G}$ and ${}^0\mathscr{G}_1 := \ker(L_0)$ is the subgroup of germs with Taylor expansion 1.

Fix a generic diagonal connection germ $d - A^0$ with an order k pole at $z = 0$. By projecting a marked pair $(A, \widehat{F})$ onto its second factor we obtain an injection $\widehat{\mathrm{Syst}}(A^0) \hookrightarrow G[\![z]\!]$ (see Lemma 3.3) and so in turn may regard $\widehat{\mathrm{Syst}}(A^0)$ as a subset of $GL_n(\mathbb{C}[\![z, \bar{z}]\!])$. Define $\widehat{\mathscr{S}}(A^0) := L_0^{-1}(\widehat{\mathrm{Syst}}(A^0))$ to be the lift of this subset to ${}^0\mathscr{G}$. Also lift the stabiliser torus $T \cong (\mathbb{C}^*)^n$, that is define ${}^0\mathscr{G}_T := L_0^{-1}(T)$.

LEMMA 4.2. *Taking Taylor series at* 0 *induces isomorphisms*

$$G\{z\}\backslash\widehat{\mathcal{S}}(A^0)/{}^0\mathcal{G}_1 \cong \mathcal{H}(A^0) \tag{24}$$

and (by considering the residual action of ${}^0\mathcal{G}_T/{}^0\mathcal{G}_1 \cong T$*):*

$$G\{z\}\backslash\widehat{\mathcal{S}}(A^0)/{}^0\mathcal{G}_T \cong \mathrm{Syst}(A^0)/G\{z\}. \tag{25}$$

Proof. First observe L_0 induces isomorphisms $\widehat{\mathcal{S}}(A^0)/{}^0\mathcal{G}_1 \cong \widehat{\mathrm{Syst}}(A^0)$ and ${}^0\mathcal{G}_T/{}^0\mathcal{G}_1 \cong T$. Then recall $\mathcal{H}(A^0) := G\{z\}\backslash\widehat{\mathrm{Syst}}(A^0)$. ∎

Having lifted things up into a smooth context a new interpretation of the smooth quotients above will be given. In particular it is desirable to remove the groups $G\{z\}$ occurring on the left-hand sides in (24) and (25).

Let ${}^0_0\mathcal{A}[k] = {}^0_0\mathcal{A}[k(0)]$ denote the set of germs at 0 of C^∞ singular connections on the trivial bundle, with poles of order at most k.

Now given $g \in \widehat{\mathcal{S}}(A^0)$ we can apply the formal transformation $L_0(g)$ to A^0 to obtain a meromorphic connection $A := L_0(g)[A^0]$. Now apply the C^∞ gauge transformation g^{-1} to A to define a singular connection $\sigma(g) := g^{-1}[A] = g^{-1}[L_0(g)[A^0]]$. This defines a map $\sigma\colon \widehat{\mathcal{S}}(A^0) \to {}^0_0\mathcal{A}[k]$. Observe that

- $\sigma(g)$ has Laurent expansion A^0 (from the morphism properties of L_0),
- If $h \in G\{z\}$ is holomorphic then $\sigma(hg) = \sigma(g)$, as $h[A] = L_0(h)[A]$, and
- $\sigma(g)$ is a *flat* singular connection, since it is C^∞ gauge equivalent to the meromorphic connection A.

Thus σ gives a map into the flat connection germs with Laurent expansion A^0, i.e. into ${}^0_0\mathcal{A}_{\mathrm{fl}}(A^0)$. In fact it is surjective and its fibres are precisely the $G\{z\}$ orbits:

PROPOSITION 4.3. *The map* σ *defined above induces an isomorphism*

$$G\{z\}\backslash\widehat{\mathcal{S}}(A^0) \xrightarrow{\cong} {}^0_0\mathcal{A}_{\mathrm{fl}}(A^0)$$

onto the set of flat singular connection germs with Laurent expansion A^0.

Proof. We have seen the induced map is well defined and now show it is bijective. For surjectivity, suppose $d-\alpha \in {}^0_0\mathcal{A}_{\mathrm{fl}}(A^0)$ is a flat singular connection with Laurent expansion A^0. Thus the $d\bar{z}$ component $\alpha^{0,1}$ of α has zero Laurent expansion at 0 and so in particular is nonsingular. It follows (see [6, p. 555] or [9, p. 67]) that there exists $g \in {}^0\mathcal{G}$ with $(\bar{\partial}g)\,g^{-1} = \alpha^{0,1}$ and so $A := g^{-1}[\alpha]$ is still flat and has no (0, 1) part. By writing $A = \gamma\, dz/z^k$ for smooth γ observe that flatness implies $\bar{\partial}\gamma = 0$ and so A is meromorphic.

We claim now that A is formally equivalent to A^0, and that $L_0(g)$ is a formal isomorphism between them. First, $L_0(g)$ has no terms containing $\bar{z}$ because $L_0(\bar{\partial}g) = L_0(\alpha^{0,1}g) = 0$ since $L_0(\alpha^{0,1}) = 0$. Second, just observe

$$L_0(g^{-1})[A^0] = L_0(g^{-1})[L_0(\alpha)] = L_0(g^{-1}[\alpha]) = L_0(A) = A$$

and so the claim follows. In particular $g^{-1} \in \widehat{\mathcal{S}}(A^0)$ and by construction $\sigma(g^{-1}) = \alpha$ and so σ is onto. Finally if $g_1[A] = g_2[B]$ with A, B meromorphic then $h[A] = B$ with $h := g_2^{-1}g_1$. Looking at (0, 1) parts gives $(\bar{\partial}h)\,h^{-1} = 0$ and so h is holomorphic. This proves injectivity. ∎

Combining this with Lemma 4.2 immediately yields the main local result:

Corollary 4.4. *There are canonical isomorphisms*

$$ {}^0_{}\mathcal{A}_{\text{fl}}(A^0)/{}^0\mathcal{G}_1 \cong \mathcal{H}(A^0) \qquad \textit{and} \qquad {}^0_{}\mathcal{A}_{\text{fl}}(A^0)/{}^0\mathcal{G}_T \cong \operatorname{Syst}(A^0)/G\{z\}$$

between the ${}^0\mathcal{G}_1$ orbits of flat singular connection germs with Laurent expansion A^0 and the set of analytic equivalence classes of compatibly framed systems with formal type A^0, and between the ${}^0\mathcal{G}_T$ orbits of flat singular connection germs with Laurent expansion A^0 and the set of analytic equivalence classes of connection germs formally equivalent to A^0.

Proof. This follows directly by substituting ${}^0\mathcal{A}_{\text{fl}}(A^0)$ for $G\{z\}\backslash\widehat{\mathcal{S}}(A^0)$ in Lemma 4.2. In summary: to go from a flat singular connection $d-\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ to $\mathcal{H}(A^0)$ just solve $(\bar{\partial}g)\,g^{-1} = \alpha^{0,1}$ and take the $G\{z\}$ orbit of $L_0(g^{-1}) \in G[\![z]\!]$ to give an element of $\mathcal{H}(A^0)$ (see the proof of Proposition 4.3). Conversely, given $\widehat{F} \in G[\![z]\!]$ such that $A := \widehat{F}[A^0]$ is convergent, use E. Borel's theorem to find $g \in {}^0\mathcal{G}$ such that $L_0(g) = \widehat{F}^{-1}$. Then set $\alpha = g[A]$ to give $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$. ∎

Thus the analytic equivalence classes may be encoded in an entirely C^∞ way. These bijections can be thought of as relating the two distinguished types of elements (the meromorphic connections and the connections with fixed Laurent expansion) within the ${}^0\mathcal{G}$ orbits in ${}^0\mathcal{A}_{\text{fl}}[k]$. That is, they relate the conditions $\alpha \in \operatorname{Syst}(A^0)$ and $\alpha \in {}^0\mathcal{A}_{\text{fl}}(A^0)$ on $\alpha \in {}^0\mathcal{A}_{\text{fl}}[k]$ by moving within α's ${}^0\mathcal{G}$ orbit.

Remark 4.2. Corollary 4.4 easily extends to the general (non-generic) case, with the same proof. The precise statement is as follows (but won't be needed elsewhere in this paper). Let $d-A$ be *any* meromorphic connection germ and let ${}^0\mathcal{G}_{\text{Stab}(A)}$ be the subgroup of ${}^0\mathcal{G}$ consisting of elements g whose Taylor expansion stabilises A (i.e. $L_0(g)[A] = A$). Then the set of analytic isomorphism classes of meromorphic connection germs formally equivalent to

A is canonically isomorphic to the set of ${}^0\mathcal{G}_{\mathrm{Stab}(A)}$-orbits of flat singular connection germs with Laurent expansion A: $\mathrm{Syst}(A)/G\{z\} \cong {}^0\!\mathcal{A}_{\mathrm{fl}}(A)/{}^0\mathcal{G}_{\mathrm{Stab}(A)}$. Similarly $\mathcal{H}(A) := \widehat{\mathrm{Syst}}_{\mathrm{mp}}(A)/G\{z\}$ is canonically isomorphic to ${}^0\!\mathcal{A}_{\mathrm{fl}}(A)/{}^0\mathcal{G}_1$ (but in general this cannot be interpreted in terms of compatibly framed systems, only in terms of marked pairs).

Globalisation

Recall we have fixed the data $\mathbf{A}$ (of a divisor $D = \sum k_i(a_i)$ on $\mathbb{P}^1$ and connection germs $d - {}^iA^0$) and defined $\mathcal{A}(\mathbf{A})$ to be the set of singular connections on the trivial rank n vector bundle on $\mathbb{P}^1$ having Laurent expansion ${}^iA^0$ at a_i for each i. Following the results of the last section we are led to consider such connections which are *flat*. The main result is:

PROPOSITION 4.5. *There is a canonical bijection between the set of* $\mathcal{G}_T$ *orbits of flat* C^∞ *singular connections with fixed Laurent expansions* $\mathbf{A}$ *and the set of isomorphism classes of meromorphic connections with formal type* $\mathbf{A}$ *on* degree zero *holomorphic bundles over* $\mathbb{P}^1$:

$$\mathcal{M}(\mathbf{A}) \cong \mathcal{A}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_T.$$

Proof. Suppose (V, ∇) represents an isomorphism class in $\mathcal{M}(\mathbf{A})$. The meromorphic connection ∇ is in particular a C^∞ singular connection, according to Definition 4.1. Since V is degree zero it is C^∞ trivial so, by choosing a trivialisation, (V, ∇) determines a singular connection $d - \alpha$ on the trivial bundle over $\mathbb{P}^1$.

From the local picture just described, since ∇ is formally equivalent to ${}^iA^0$ at a_i, we can choose $g \in \mathcal{G}$ such that $g[\alpha]$ has Laurent expansion ${}^iA^0$ at a_i for all i. This gives an element $g[\alpha]$ of $\mathcal{A}_{\mathrm{fl}}(\mathbf{A})$ and we take the $\mathcal{G}_T$ orbit through it to define the required map. We need to check this $\mathcal{G}_T$ orbit only depends on the isomorphism class of (V, ∇) and that the map is bijective.

Suppose we have two such pairs (V, ∇) and (V', ∇') and we choose C^∞ trivialisations of V and V' so that ∇, ∇' give singular connections $d - \alpha_1$, $d - \alpha_2$ respectively. Now a standard $\bar{\partial}$-operator argument implies $(V, \nabla) \cong (V', \nabla')$ if and only if α_1 and α_2 are in the same $\mathcal{G}$ orbit. Thus an isomorphism class $[(V, \nabla)]$ of meromorphic connections determines (and is determined by) a $\mathcal{G}$ orbit of singular connections on the trivial bundle. This $\mathcal{G}$ orbit has a subset of singular connections having Laurent expansion ${}^iA^0$ at a_i for each i. This subset is a $\mathcal{G}_T$ orbit of singular connections (since T is the stabiliser of ${}^iA^0$) and is the element of $\mathcal{A}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_T$ corresponding to $[(V, \nabla)]$. ∎

COROLLARY 4.6. *The set* $\widetilde{\mathscr{M}}(\mathbf{A})$ *of isomorphism classes of triples* $(V, \nabla, \mathbf{g})$ *consisting of a generic meromorphic connection* ∇ *(with poles on* D*) on a* degree zero *holomorphic vector bundle* V *over* $\mathbb{P}^1$ *with compatible framings* $\mathbf{g}$ *such that* $(V, \nabla, \mathbf{g})$ *has irregular type* $\mathbf{A}$ *is canonically isomorphic to the set of* $\mathscr{G}_1$ *orbits of flat connections in* $\widetilde{\mathscr{A}}(\mathbf{A})$*:*

$$\widetilde{\mathscr{M}}(\mathbf{A}) \cong \widetilde{\mathscr{A}}_{\text{fl}}(\mathbf{A})/\mathscr{G}_1.$$

Proof. As in Corollary 4.4, replacing $\mathscr{G}_T$ by $\mathscr{G}_1$ in Proposition 4.5 corresponds to incorporating a compatible framing as required for $\widetilde{\mathscr{M}}(\mathbf{A})$. The desired isomorphism is then obtained by simply repeating the proof of Proposition 4.5 for each possible set of choices of exponents of formal monodromy $\mathbf{\Lambda}$. ∎

Monodromy of Flat Singular Connections

Having related the C^∞ approach to meromorphic connections we now relate it to the monodromy approach of Section 3. The key step is to define the generalised monodromy data of flat C^∞ singular connections with fixed Laurent expansions, but this is easy since they too have canonical solutions on sectors:

LEMMA 4.7. *Suppose* $\alpha \in {}^0\mathscr{A}_{\text{fl}}(A^0)$. *For each choice of* $\log(z)$ *there is a canonical choice* Φ_i *of fundamental solution of* α *on* Sect_i*, given by*

$$\Phi_i := g\Sigma_i(L_0(g^{-1}))\, z^{\Lambda^0} e^Q$$

for any $g \in {}^0\mathscr{G}$ *solving* $(\bar{\partial} g)\, g^{-1} = \alpha^{0,1}$.

Proof. From the proof of Proposition 4.3, such g is unique up to right multiplication by $h \in G\{z\}$ and $A := L_0(g^{-1})[A^0] = g^{-1}[\alpha]$ is a convergent meromorphic connection germ. Theorem 3.1 then provides an analytic isomorphism $\Sigma_i(L_0(g^{-1}))$ between A^0 and A on Sect_i. It follows that $g\Sigma_i(L_0(g^{-1}))$ is an isomorphism between A^0 and α which is independent of the choice of g. Composing this with the fundamental solution $z^{\Lambda^0} e^Q$ of A^0 gives the result. ∎

Thus, exactly as in Section 3, a singular connection $d - \alpha \in \widetilde{\mathscr{A}}(\mathbf{A})$ determines a Stokes representation ρ (upto isomorphism). This gives a map (which will be referred to as the C^∞ monodromy map):

$$\tilde{\nu}: \widetilde{\mathscr{A}}_{\text{fl}}(\mathbf{A}) \longrightarrow \widetilde{M}(\mathbf{A}); \qquad \alpha \longmapsto [\rho].$$

Since the connections in $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ are on a degree zero bundle, the image of $\tilde{\nu}$ is in the degree zero component $\widetilde{M}_0(\mathbf{A})$ of the extended monodromy manifold (see Remark 3.2). The main result is then:

PROPOSITION 4.8. *The C^∞ monodromy map $\tilde{\nu}\colon \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A}) \to \widetilde{M}_0(\mathbf{A})$ is surjective and has precisely the $\mathscr{G}_1$ orbits in $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ as fibres, so that*

$$\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathscr{G}_1 \cong \widetilde{M}_0(\mathbf{A}).$$

Moreover $\tilde{\nu}$ intertwines the $\mathscr{G}_T$ action on $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ with the $\mathscr{G}_T$ action on $\widetilde{M}_0(\mathbf{A})$ defined via the evaluation map $\mathscr{G}_T \to T^m$ and the torus actions of (22).

Before proving this we deduce what the monodromy data corresponds to in the meromorphic world:

COROLLARY 4.9. *Taking monodromy data induces bijections*

$$\widetilde{\mathscr{M}}(\mathbf{A}) \cong \widetilde{M}_0(\mathbf{A}) \qquad \text{and} \qquad \mathscr{M}(\mathbf{A}) \cong M(\mathbf{A})$$

between the spaces of meromorphic connections on degree zero bundles and the corresponding spaces of monodromy data. In particular $\widetilde{\mathscr{M}}(\mathbf{A})$ inherits the structure of a complex manifold from $\widetilde{M}_0(\mathbf{A})$.

Proof. The first bijection follows directly from Propositions 4.5 and 4.8. The second bijection follows from the first by fixing the exponents of formal monodromy and quotienting by the T^m action (using the intertwining property of $\tilde{\nu}$). ∎

Proof (of Proposition 4.8). Choose some tentacles $\mathscr{T}$ and a thickening $\bar{\gamma}_i\colon [0,1]\times[0,1] \to \mathbb{P}^1\setminus\{a_1,\dots,a_m\}$ of each path γ_i (i.e. a ribbon following γ_i whose track $|\bar{\gamma}_i|$ is a closed tubular neighbourhood of the track of γ_i). Let D_0 be a disc in $\mathbb{P}^1$ containing p_0 and disjoint from each disc $D_1,\dots,D_m$. Let $|\mathscr{T}| := \bar{D}_0 \cup \bigcup_{i=1}^m (\bar{D}_i \cup |\bar{\gamma}_i|) \subset \mathbb{P}^1$ be the union of the $m+1$ discs $\bar{D}_i$ and the m ribbons $|\bar{\gamma}_i|$. We will suppose (as is clearly possible) that $\mathscr{T}$, $|\bar{\gamma}_i|$, D_0 have been chosen such that: 1) if $i \neq j$ then $|\bar{\gamma}_i|$ only intersects $|\bar{\gamma}_j|$ inside D_0 and 2) that $|\mathscr{T}|$ is homeomorphic to a (closed) disc.

For surjectivity, let ρ be any degree zero Stokes representation. From Proposition 3.8, specifying ρ is equivalent to specifying the matrices $(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}') = \widetilde{\varphi}_{\mathscr{T}}(\rho)$. (Also as in Proposition 3.8, P_i will denote the permutation matrix associated to a_i via the tentacle choice.) Since the Stokes matrices classify germs of singular connections up to C^∞ gauge transformations with Taylor expansion 1, germs $\alpha_i \in {}^i\widetilde{\mathscr{A}}_{\mathrm{fl}}({}^iA^0)$ may be obtained having any given Stokes matrices and residue for each $i = 1,\dots,m$ (combine Theorem 3.2 with Corollary 4.4). It is straightforward to extend

α_i arbitrarily to $\bar{D}_i$. Next the α_i's are patched together along the ribbons $|\bar{\gamma}_i|$. Let ${}^i\Phi_0$ be the canonical solution of α_i on ${}^i\mathrm{Sect}_0$ from Lemma 4.7. Since $G = GL_n(\mathbb{C})$ is path connected it is possible to choose a smooth map $\chi_i : |\bar{\gamma}_i| \to G$ such that $\chi_i = 1$ on $\bar{D}_0 \cap |\bar{\gamma}_i|$ and $\chi_i = {}^i\Phi_0 P_i C_i$ on ${}^i\mathrm{Sect}_0 \cap |\bar{\gamma}_i|$ for $i = 1, \ldots, m$. Define α over $|\mathcal{T}|$ as follows: $\alpha|_{\bar{D}_0} = 0$ and for $i = 1, \ldots, m$ $\alpha|_{\bar{D}_i} = \alpha_i$ and $\alpha|_{|\bar{\gamma}_i|} = (d\chi_i)\,\chi_i^{-1}$. It is easy to check these definitions agree on the overlaps and that when the basis ${}^i\Phi_0$ is extended over $|\bar{\gamma}_i|$ as a solution of α then $\rho(\gamma_{\bar{p}_i p_0}) = {}^i\Phi_0^{-1} \cdot \Phi$ on $|\bar{\gamma}_i|$, where Φ is the basis which equals 1 on D_0.

Now we must extend α to the rest of $\mathbb{P}^1$. First the product relation $\rho_m \cdots \rho_1 = 1$ ensures that α has no monodromy around the boundary circle $\partial|\mathcal{T}| \cong S^1$, so that any local fundamental solution Ψ extends to give a map $\Psi\colon \partial|\mathcal{T}| \to G$. Then the hypothesis that $\deg(\rho) = 0$ ensures that this loop Ψ in G is contractible. To see this, firstly recall that the determinant map $\det\colon G \to \mathbb{C}^*$ expresses G as a fibre bundle over $\mathbb{C}^*$, with fibres diffeomorphic to $SL_n(\mathbb{C})$, and that $SL_n(\mathbb{C})$ is simply connected. Then, from the homotopy long exact sequence for fibrations, it follows that det induces an isomorphism of fundamental groups: $\pi_1(G) \cong \pi_1(\mathbb{C}^*) \cong \mathbb{Z}$. Thus we need to see that the loop $\psi := \det(\Psi)\colon \partial|\mathcal{T}| \to \mathbb{C}^*$ in the punctured complex plane does not wind around zero. But the winding number of ψ is

$$\frac{1}{2\pi i}\oint_{\partial|\mathcal{T}|} \frac{d\psi}{\psi} = \frac{1}{2\pi i}\oint_{\partial|\mathcal{T}|} \mathrm{Tr}(\alpha)$$

and the C^∞ version of Cauchy's integral theorem (see Lemma 6.3) implies this is equal to $\sum \mathrm{Tr}({}^i\Lambda) = \deg(\rho)$ (using the flatness of α to deduce $d\,\mathrm{Tr}(\alpha) = 0$).

Thus the loop Ψ in G may be extended to a smooth map from the complement of $|\mathcal{T}|$ in $\mathbb{P}^1$ to G. We then define $\alpha = (d\Psi)\,\Psi^{-1}$ on this complement and thereby obtain $\alpha \in \widetilde{\mathcal{A}}_{\mathrm{fl}}(\mathbf{A})$ having the desired monodromy data. Hence the C^∞ monodromy map is indeed surjective.

Next observe (from Theorem 3.1 and Lemma 4.7) that if $h \in \mathcal{G}_T$ and $\alpha' = h[\alpha]$ then the canonical solutions of α and α' are related by: ${}^i\Phi'_j = h \cdot {}^i\Phi_j \cdot t_i^{-1}$ where $t_i = h(a_i) \in T$. The intertwining property and the fact that the $\mathcal{G}_1$ orbits are contained in the fibres of $\tilde{\nu}$ are then immediate from the definition of the Stokes representation in terms of the canonical solutions

The proof that the fibres are precisely the $\mathcal{G}_1$ orbits is much like the proof of Proposition 3.7: Suppose $\alpha, \alpha' \in \widetilde{\mathcal{A}}_{\mathrm{fl}}(\mathbf{A})$ have the same monodromy data. Let $\varphi := {}^1\Phi'_0({}^1\Phi_0)^{-1}$ be the induced isomorphism between α and α' on ${}^1\mathrm{Sect}_0$. Then φ is single-valued when extended to $\mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$ as a solution of the induced connection $\mathrm{Hom}(\alpha, \alpha')$ on the trivial bundle $\mathrm{End}(\mathbb{C}^n)$. (When φ is extended around any loop γ based at p_1 it has no monodromy since, when extended around this loop, ${}^1\Phi_0'$ and ${}^1\Phi_0$ are both multiplied on the right by the same constant matrix.) Also, since the monodromy data encodes the transitions between the various canonical fundamental solutions it follows that $\varphi = {}^i\Phi_j'({}^i\Phi_j)^{-1}$ for any i, j. Now observe (from Theorem 3.1 and Lemma 4.7) that ${}^i\Phi_j'({}^i\Phi_j)^{-1}$ is asymptotic to 1 at a_i on some sectorial neighbourhood of ${}^i\mathrm{Sect}_j$ ($j = 1, \ldots, r_i$, $i = 1, \ldots, m$). It follows that φ extends smoothly to $\mathbb{P}^1$ and has Taylor expansion 1 at each a_i. By construction $\alpha' = \varphi[\alpha]$ so α and α' are in the same $\mathcal{G}_1$ orbit. ▮

5. SYMPLECTIC STRUCTURE AND MOMENT MAP

In this section we observe that the well known Atiyah–Bott symplectic structure on nonsingular connections naturally generalises to the singular case we have been studying. Moreover, as in the nonsingular case we find that the curvature is a moment map for the action of the gauge group. Thus the moduli spaces of *flat* singular connections with fixed expansions arise as infinite dimensional symplectic quotients.

The main technical difficulty here is that standard Sobolev/Banach space methods cannot be used since we want to fix infinite-jets of derivatives at the singular points $a_i \in \mathbb{P}^1$. Instead the infinite dimensional spaces here are naturally Fréchet manifolds. We will not use any deep properties of Fréchet spaces but do need a topology and differential structure. (The explicitness of our situation means we can get by without using an implicit function theorem—the monodromy description gives $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1$ the structure of a complex manifold and local slices for this $\mathcal{G}_1$ action will be constructed directly.) The reference used for Fréchet spaces is Treves [66] and for Fréchet manifolds or Lie groups see Hamilton [26] and Milnor [54]; we will give direct references to these works rather than full details here.

The Atiyah–Bott Symplectic Structure on $\widetilde{\mathscr{A}}(\mathbf{A})$

Let E denote the trivial rank n complex vector bundle over $\mathbb{P}^1$ and consider the complex vector space $\Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$ of $n \times n$ matrices of global C^∞ singular one-forms on $\mathbb{P}^1$ with poles on D (see Section 4). This is the space of sections of a C^∞ vector bundle and so can be given a Fréchet topology in a standard way [26, p. 68]. Now define W to be the vector subspace

$$W := \left\{ \phi \in \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E)) \;\middle|\; L_i(\phi) \in \mathfrak{t}\frac{dz_i}{z_i} \text{ for } i = 1, \ldots, m \right\}$$

of $\Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$ of elements having Laurent expansion zero at each i, except for a possibly nonzero, diagonal residue term. This is a closed subspace[2] and so inherits a Fréchet topology.

LEMMA 5.1. *The extended space $\widetilde{\mathscr{A}}(\mathbf{A})$ of singular connections is a complex Fréchet manifold and if $\alpha \in \widetilde{\mathscr{A}}(\mathbf{A})$ then the tangent space to $\widetilde{\mathscr{A}}(\mathbf{A})$ at α is canonically isomorphic to the complex Fréchet space W defined above: $T_\alpha \widetilde{\mathscr{A}}(\mathbf{A}) \cong W$.*

Proof. If all $k_i \geqslant 2$ then $\widetilde{\mathscr{A}}(\mathbf{A})$ is an affine space modelled on W: If $\alpha_0 \in \widetilde{\mathscr{A}}(\mathbf{A})$ then $\widetilde{\mathscr{A}}(\mathbf{A}) = \{\alpha_0 + \phi \mid \phi \in W\}$. In general (some $k_i = 1$), $\widetilde{\mathscr{A}}(\mathbf{A})$ is identified in this way with an open subset of W (recall the residues must be regular mod $\mathbb{Z}$): if $\alpha_0 \in \widetilde{\mathscr{A}}(\mathbf{A})$ then the map $\{\alpha_0 + \phi \mid \phi \in W\} \to \mathfrak{t}^m$; $\alpha \mapsto (\mathrm{Res}_i L_i(\alpha))_{i=1}^m$ taking the residues is continuous and $\widetilde{\mathscr{A}}(\mathbf{A})$ is the inverse image of an open subset of $\mathfrak{t}^m$. Thus $\widetilde{\mathscr{A}}(\mathbf{A})$ is identified with an open subset of W; it is thus a Fréchet manifold (with just one coordinate chart) and the tangent spaces are canonically identified with W as in the finite dimensional case (see discussion [54, p. 1030]). ▮

Thus following Atiyah–Bott [6] we can define a two-form

$$\omega_\alpha(\phi, \psi) := \frac{1}{2\pi i} \int_{\mathbb{P}^1} \mathrm{Tr}(\phi \wedge \psi) \tag{26}$$

on $\widetilde{\mathscr{A}}(\mathbf{A})$, where $\alpha \in \widetilde{\mathscr{A}}(\mathbf{A})$ and $\phi, \psi \in T_\alpha \widetilde{\mathscr{A}}(\mathbf{A})$. This integral is well defined since the two-form $\mathrm{Tr}(\phi \wedge \psi)$ on $\mathbb{P}^1$ is nonsingular; its Laurent expansion at a_i is a $(2, 0)$ form and so zero. Thus ω_α is a skew-symmetric complex bilinear form on the tangent space $T_\alpha \widetilde{\mathscr{A}}(\mathbf{A})$. It is nondegenerate in the sense that if $\omega_\alpha(\phi, \psi) = 0$ for all ψ then $\phi = 0$ (if $\phi \neq 0$ then ϕ is nonzero at some point $p \neq a_1, ..., a_m$ and it is easy then to construct ψ vanishing outside a neighbourhood of p and such that $\omega_\alpha(\phi, \psi) \neq 0$). Also ω_α is continuous as a map $W \times W \to \mathbb{C}$, since it is continuous in each factor, and (for Fréchet spaces) such "separately continuous" bilinear maps are continuous ([66, p. 354]). Finally the right-hand side of (26) is independent of α, so ω is a constant two-form on $\widetilde{\mathscr{A}}(\mathbf{A})$ and in particular it is closed. Owing to these properties we will say ω is a complex *symplectic* form on $\widetilde{\mathscr{A}}(\mathbf{A})$. (See for example Kobayashi [42] for a discussion of the more well-known theory of symplectic *Banach* manifolds.)

[2] Since the C^∞ Laurent expansion maps L_i are *continuous* (if we put the topology of simple convergence of coefficients on the formal power series ring which is the image of the Laurent expansion map L_i); see [66 p. 390], where this fact is used to prove E. Borel's theorem on the surjectivity of L_i.

Group Actions

First, the full gauge group $\mathcal{G} := GL_n(C^\infty(\mathbb{P}^1))$ is a Fréchet Lie group; that is, it is a Fréchet manifold such that the group operations $g, h \mapsto g \cdot h$ and $g \mapsto g^{-1}$ are C^∞ maps (see [54, Example 1.3]). $\mathcal{G}$ is locally modelled on the Fréchet space $\text{Lie}(\mathcal{G}) := C^\infty(\mathbb{P}^1, \mathfrak{gl}_n(\mathbb{C}))$ and has a complex analytic structure coming from the exponential map exp: $\text{Lie}(\mathcal{G}) \to \mathcal{G}$; $x \mapsto \exp(x)$ which is defined pointwise in terms of the exponential map for G. This implies $\text{Lie}(\mathcal{G})$ is a canonical coordinate chart for $\mathcal{G}$ in a neighbourhood of the identity since exp has a local inverse $g \mapsto \log(g)$ (also defined pointwise). In particular $\text{Lie}(\mathcal{G})$ is so identified with the tangent space to $\mathcal{G}$ at the identity; the Lie algebra of $\mathcal{G}$.

The group we are really interested in here is $\mathcal{G}_1$, the subgroup of $\mathcal{G}$ consisting of elements $g \in \mathcal{G}$ having Taylor expansion 1 at each $a_i \in \mathbb{P}^1$. As above, the Taylor expansion maps are continuous and so $\mathcal{G}_1$ (the intersection of their kernels) is a closed subgroup of $\mathcal{G}$. It follows that $\mathcal{G}_1$ is a complex Fréchet Lie group with Lie algebra

$$\text{Lie}(\mathcal{G}_1) := \{x \in \text{Lie}(\mathcal{G}) \mid L_i(x) = 0 \text{ for } i = 1, \ldots, m\},$$

where L_i is the Taylor expansion map at a_i. (The same statements also hold for $\mathcal{G}_T$ except now $\text{Lie}(\mathcal{G}_T) := \{x \in \text{Lie}(\mathcal{G}) \mid L_i(x) \in \mathfrak{t} \text{ for } i = 1, \ldots, m\}$.)

LEMMA 5.2. *The groups $\mathcal{G}_1$ and $\mathcal{G}_T$ act holomorphically on $\widetilde{\mathcal{A}}(\mathbf{A})$ by gauge transformations and the fundamental vector field of $x \in \text{Lie}(\mathcal{G}_T)$ takes the value $-d_\alpha x \in T_\alpha \widetilde{\mathcal{A}}(\mathbf{A})$ at $\alpha \in \widetilde{\mathcal{A}}(\mathbf{A})$, where d_α is the singular connection on* $\text{End}(E)$ *induced from α.*

Proof. First the action map $\mathcal{G}_T \times \widetilde{\mathcal{A}}(\mathbf{A}) \to \widetilde{\mathcal{A}}(\mathbf{A}); (g, \alpha) \mapsto g\alpha g^{-1} + (dg)\, g^{-1}$ can be factored into simpler maps each of which is holomorphic (see [26]). By convention the fundamental vector field is minus the tangent field to the flow generated by x, which may be calculated using the exponential map for $\mathcal{G}_T$. ∎

The Curvature is a Moment Map

It is clear that the action of $\mathcal{G}_T$ on $\widetilde{\mathcal{A}}(\mathbf{A})$ preserves the symplectic form ω: If $g \in \mathcal{G}_T$ and $\alpha \in \widetilde{\mathcal{A}}(\mathbf{A})$ then the derivative of the action of g is simply conjugation:

$$(g[\,\cdot\,])_* : T_\alpha \widetilde{\mathcal{A}}(\mathbf{A}) \to T_{g[\alpha]} \widetilde{\mathcal{A}}(\mathbf{A}); \qquad \phi \mapsto g\phi g^{-1},$$

and so ω is preserved as $\text{Tr}(\phi \wedge \psi) = \text{Tr}(g\phi g^{-1} \wedge g\psi g^{-1})$ for any $\phi, \psi \in T_\alpha \widetilde{\mathcal{A}}(\mathbf{A})$.

More interestingly, this action is Hamiltonian. If we firstly look at the smaller group $\mathcal{G}_1$, then, as observed by Atiyah and Bott in the nonsingular case, the curvature is a moment map. To start with observe:

LEMMA 5.3. *The curvature map* $\mathcal{F}: \widetilde{\mathcal{A}}(\mathbf{A}) \longrightarrow \Omega^2(\mathbb{P}^1, \mathrm{End}(E))$ *is an infinitely differentiable (even holomorphic) map to the Fréchet space of* End(E) *valued* nonsingular *two-forms on* $\mathbb{P}^1$. *The derivative of* $\mathcal{F}$ *at* $\alpha \in \widetilde{\mathcal{A}}(\mathbf{A})$ *is*

$$(d\mathcal{F})_\alpha : T_\alpha \widetilde{\mathcal{A}}(\mathbf{A}) \to \Omega^2(\mathbb{P}^1, \mathrm{End}(E)); \qquad \phi \longmapsto -d_\alpha \phi,$$

where $\phi \in T_\alpha \widetilde{\mathcal{A}}(\mathbf{A}) = W$ *and* $d_\alpha : \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E)) \to \Omega^2[2D](\mathbb{P}^1, \mathrm{End}(E))$ *is the operator naturally induced from the singular connection* α.

Proof. Recall the curvature is given explicitly by $\mathcal{F}(\alpha) = -d\alpha + \alpha \wedge \alpha$ and observe (by looking at Laurent expansions and using the division lemma) that this is a matrix of *nonsingular* two-forms. That $\mathcal{F}$ is C^∞ with the stated derivative follows from basic facts about calculus on Fréchet spaces (see [26] Part I). ∎

Next there is a natural inclusion from $\Omega^2(\mathbb{P}^1, \mathrm{End}(E))$ to the dual of the Lie algebra of $\mathcal{G}_1$ given by taking the trace and integrating

$$\imath : \Omega^2(\mathbb{P}^1, \mathrm{End}(E)) \to \mathrm{Lie}(\mathcal{G}_1)^*; \qquad \mathcal{F}(\alpha) \mapsto \left(x \mapsto \frac{1}{2\pi i} \int_{\mathbb{P}^1} \mathrm{Tr}(\mathcal{F}(\alpha)\, x) \right)$$

where $x \in \mathrm{Lie}(\mathcal{G}_1)$ is a matrix of functions on $\mathbb{P}^1$. Using this inclusion we will regard $\mathcal{F}$ as a map into the dual of the Lie algebra of the group. We then have

PROPOSITION 5.4. *The curvature* $\mathcal{F}: \widetilde{\mathcal{A}}(\mathbf{A}) \longrightarrow \mathrm{Lie}(\mathcal{G}_1)^*$ *is an equivariant moment map for the* $\mathcal{G}_1$ *action on the extended space* $\widetilde{\mathcal{A}}(\mathbf{A})$ *of singular connections.*

Proof. Everything has been set up so that the arguments from the nonsingular case still work, as we will now show. Given $x \in \mathrm{Lie}(\mathcal{G}_1)$, define a (Hamiltonian) function H_x on $\widetilde{\mathcal{A}}(\mathbf{A})$ to be the x component of $\mathcal{F}$,

$$H_x := \langle \mathcal{F}, x \rangle : \widetilde{\mathcal{A}}(\mathbf{A}) \to \mathbb{C}; \qquad H_x(\alpha) = \frac{1}{2\pi i} \int_{\mathbb{P}^1} \mathrm{Tr}(\mathcal{F}(\alpha)\, x),$$

where the angled brackets denote the natural pairing between $\mathrm{Lie}(\mathcal{G}_1)$ and its dual. We need to show that the fundamental vector field of x is the Hamiltonian vector field of H_x, i.e. that $(dH_x)_\alpha = \omega_\alpha(\cdot, -d_\alpha x)$ as elements of $T^*_\alpha \widetilde{\mathcal{A}}(\mathbf{A})$. Now if $\phi \in T_\alpha \widetilde{\mathcal{A}}(\mathbf{A})$ then

$$(dH_x)_\alpha(\phi) = -\frac{1}{2\pi i} \int_{\mathbb{P}^1} \mathrm{Tr}((d_\alpha \phi)\, x) \tag{27}$$

from Lemma 5.3 and the chain rule. Now observe that $\mathrm{Tr}(\phi x)$ is a *nonsingular* one-form on $\mathbb{P}^1$ (as $L_i(x)=0$ for all i). Therefore Stokes' theorem implies $d\,\mathrm{Tr}(\phi x)=\mathrm{Tr}((d_\alpha\phi)\,x)-\mathrm{Tr}(\phi\wedge d_\alpha x)$ integrates to zero over $\mathbb{P}^1$. Hence (27) becomes

$$(dH_x)_\alpha(\phi)=-\frac{1}{2\pi i}\int_{\mathbb{P}^1}\mathrm{Tr}(\phi\wedge d_\alpha x)=\omega_\alpha(\phi,-d_\alpha x)$$

proving that the curvature is indeed a moment map.

The equivariance follows directly from the definition of the coadjoint action of $\mathscr{G}_1$: If $x\in\mathrm{Lie}(\mathscr{G}_1)$ then $\langle\mathrm{Ad}^*_g(\mathscr{F}(\alpha)),x\rangle:=\langle\mathscr{F}(\alpha),\mathrm{Ad}_{g^{-1}}(x)\rangle=\langle\mathscr{F}(g[\alpha]),x\rangle$ using the fact that $\mathrm{Tr}(\mathscr{F}(\alpha)\,g^{-1}xg)=\mathrm{Tr}(\mathscr{F}(g[\alpha])\,x)$. ∎

Thus the subset of flat connections is the preimage of zero under the moment map: $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})=\mathscr{F}^{-1}(0)$. Therefore, at least in a formal sense, the moduli space is a symplectic quotient: $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathscr{G}_1=\mathscr{F}^{-1}(0)/\mathscr{G}_1$. (Recall $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathscr{G}_1$ was identified in Section 4 with the space $\widetilde{M}_0(\mathbf{A})$ of monodromy data, analogously to the non-singular case.) In the next section we will show that this prescription does define a genuine symplectic structure on at least the dense open subset of $\widetilde{M}_0(\mathbf{A})$ which is the image of the extended monodromy map $\tilde{\nu}$.

Torus Actions

To end this section we consider the action of the larger group $\mathscr{G}_T$ on the extended space of singular connections $\widetilde{\mathscr{A}}(\mathbf{A})$. This action is also Hamiltonian:

PROPOSITION 5.5. *Let $\mu:\widetilde{\mathscr{A}}(\mathbf{A})\to\mathrm{Lie}(\mathscr{G}_T)^*$ be the map given by taking the curvature together with the residue at each a_i: If $x\in\mathrm{Lie}(\mathscr{G}_T)$ and $\alpha\in\widetilde{\mathscr{A}}(\mathbf{A})$*

$$\langle\mu(\alpha),x\rangle:=\frac{1}{(2\pi\sqrt{-1})}\int_{\mathbb{P}^1}\mathrm{Tr}(\mathscr{F}(\alpha)\,x)-\sum_{i=1}^{m}\mathrm{Res}_i L_i(\mathrm{Tr}(\alpha x)).$$

Then μ is an equivariant moment map for the $\mathscr{G}_T$ action on $\widetilde{\mathscr{A}}(\mathbf{A})$.

Proof. For any $x\in\mathrm{Lie}(\mathscr{G}_T)$ define the function $H_x:\widetilde{\mathscr{A}}(\mathbf{A})\to\mathbb{C}$ to be the x component of μ: $H_x(\alpha):=\langle\mu(\alpha),x\rangle$. If $\phi\in T_\alpha\widetilde{\mathscr{A}}(\mathbf{A})$ then

$$(dH_x)_\alpha(\phi)=-\frac{1}{(2\pi\sqrt{-1})}\int_{\mathbb{P}^1}\mathrm{Tr}((d_\alpha\phi)\,x)-\sum_i\mathrm{Res}_i L_i(\mathrm{Tr}(\phi x)).\qquad(28)$$

Our task is to show $\omega_\alpha(\phi,-d_\alpha x)=(dH_x)_\alpha(\phi)$. We do this by using the C^∞ Cauchy integral theorem (see Lemma 6.3). Recall ϕ is a matrix of C^∞ one-forms on $\mathbb{P}^1$ with (at worst) first order poles in its $(1,0)$ part at each a_i.

Also $x \in \mathrm{Lie}(\mathcal{G}_T)$ is a matrix of functions on $\mathbb{P}^1$ and has Taylor expansion equal to a constant diagonal matrix at each a_i. Thus for each i we can choose a C^∞ function $f_i\colon \mathbb{P}^1 \to \mathbb{C}$ which vanishes outside D_i, such that $\mathrm{Tr}(\phi x) = \theta + f_1 dz_1/z_1 + \cdots f_m\, dz_m/z_m$ for some *nonsingular* one-form θ on $\mathbb{P}^1$. Thus $d\,\mathrm{Tr}(\phi x) = d\theta - \sum_i \frac{\partial f_i}{\partial \bar{z}_i}\frac{dz_i \wedge d\bar{z}_i}{z_i}$ and so by Stokes' theorem and Cauchy's integral theorem:

$$\int_{\mathbb{P}^1} d\mathrm{Tr}(\phi x) = \sum_i \int_{\bar{D}_i} \frac{\partial f_i}{\partial \bar{z}_i}\frac{dz_i \wedge d\bar{z}_i}{z_i} = -(2\pi\sqrt{-1}\,)\sum_i f_i(a_i).$$

(Note $f_i(a_i) = \mathrm{Res}_i\, L_i(\mathrm{Tr}(\phi x))$.) Then the equality $\omega_\alpha(\phi, -d_\alpha x) = (dH_x)_\alpha(\phi)$ follows from the fact that $d\mathrm{Tr}(\phi x) = \mathrm{Tr}(d_\alpha(\phi x)) = \mathrm{Tr}((d_\alpha \phi)\, x) - \mathrm{Tr}(\phi \wedge d_\alpha x)$. The equivariance follows exactly as before since $\mathcal{G}_T/\mathcal{G}_1 \cong T^m$ is Abelian. ∎

Instead we could do the reduction in stages, and consider the T^m action on $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1$. This matches up with the Hamiltonian T^m actions considered in Section 2, since the residues above are the exponents of formal monodromy ${}^i\Lambda$.

6. THE MONODROMY MAP IS SYMPLECTIC

Most of the story so far can be summarised in the commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathscr{M}}(\mathbf{A}) & \xrightarrow{\ \cong\ } & \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1 \\ \cup & & \downarrow \cong \\ \widetilde{O}_1 \times \cdots \times \widetilde{O}_m /\!/ G \cong \widetilde{\mathscr{M}}^*(\mathbf{A}) & \xrightarrow{\ \tilde{\nu}\ } & \widetilde{M}_0(\mathbf{A}). \end{array} \tag{29}$$

The extended moduli space $\widetilde{\mathscr{M}}^*(\mathbf{A})$ was defined in Section 2 to be the set of isomorphism classes of compatibly framed meromorphic connections on trivial rank n vector bundles with irregular type $\mathbf{A}$. It was given an intrinsic complex symplectic structure explicitly in terms of (finite dimensional) coadjoint orbits and cotangent bundles. The extended monodromy manifold $\widetilde{M}(\mathbf{A})$ was defined as the set of isomorphism classes of Stokes representations and looks like a multiplicative version of $\widetilde{\mathscr{M}}^*(\mathbf{A})$ (when both are described explicitly). $\widetilde{M}_0(\mathbf{A})$ is the degree zero component of $\widetilde{M}(\mathbf{A})$ and was identified with the set of $\mathcal{G}_1$ orbits in the extended space $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ of flat C^∞ singular connections. Moreover the curvature was shown to be a moment map for the action of the gauge group $\mathcal{G}_1$ on the symplectic Fréchet manifold $\widetilde{\mathscr{A}}(\mathbf{A})$, so that (formally) $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})/\mathcal{G}_1$ is a complex symplectic quotient. $\widetilde{\mathscr{M}}(\mathbf{A})$ has the same definition as $\widetilde{\mathscr{M}}^*(\mathbf{A})$ except with the word "trivial" replaced by "degree zero". The act of taking monodromy data defines both the right-hand isomorphism in the diagram and the monodromy map $\tilde{\nu}$, which is a biholomorphic map onto its image (a dense open submanifold of $\widetilde{M}_0(\mathbf{A})$).

Basically the bottom line appears in the work [40] of Jimbo, Miwa and Ueno but the symplectic structures and the rest of the diagram do not. The torus $T^m \cong (\mathbb{C}^*)^{nm}$ acts on each space in (29) and these actions are intertwined by all the maps. The non-extended picture arises by taking the symplectic quotient (fixing the exponents of formal monodromy and quotienting by T^m). We then obtain another commutative diagram as above but with all the tildes removed and $\mathcal{G}_1$ replaced by $\mathcal{G}_T$.

In this section we show that the symplectic structure on $\widetilde{\mathscr{A}}(\mathbf{A})$ does induce a symplectic structure on (at least) the dense open submanifold of $\widetilde{M}_0(\mathbf{A})$ that is the image of the monodromy map $\tilde{\nu}$, and that this symplectic form pulls back along $\tilde{\nu}$ to the explicit symplectic form on $\widetilde{\mathscr{M}}^*(\mathbf{A})$. In other words we will prove:

THEOREM 6.1. *The monodromy map $\tilde{\nu}$ is symplectic.*

Analogous results have been proved in the simple pole case independently by Hitchin [29] and by Iwasaki [34, 35]. (Note that Iwasaki considers only $PSL_2(\mathbb{C})$ Fuchsian equations, but he does so over (fixed) arbitrary genus Riemann surfaces.)

Factorising the Monodromy Map

Recall from Proposition 4.5 how the isomorphism at the top of the above diagram arose: a meromorphic connection gives rise to a $\mathcal{G}$ orbit of C^∞ singular connections and we consider the subset with fixed Laurent expansion at each a_i to define the map. In other words we can choose $g \in \mathcal{G}$ to "straighten" a meromorphic connection to have fixed C^∞ Laurent expansions and thereby specify an element of $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$. Here we show that this straightening procedure can be carried out for a family of connections all at the same time, and so the monodromy map factorises through $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$.

As usual we fix the data $\mathbf{A}$ consisting of an effective divisor $D = \sum k_i(a_i)$ and diagonal generic connections germs $d - {}^iA^0$. Also choose a coordinate z to identify $\mathbb{P}^1$ with $\mathbb{C} \cup \infty$ such that each a_i is finite and let $D_1, \ldots, D_m \subset \mathbb{P}^1$ be disjoint open disks with $a_i \in D_i$, so that $z_i := z - a_i$ is a coordinate on D_i.

PROPOSITION 6.1. *Let $U \subset \widetilde{\mathscr{M}}^*(\mathbf{A})$ be an open subset. Then there exists a universal family $d_{\mathbb{P}^1} - A$ of meromorphic connections on the trivial bundle over $\mathbb{P}^1$ (with compatible framings $\mathbf{g} = ({}^1g_0, \ldots, {}^mg_0)$) parameterised by $u \in U$ and a family of smooth bundle automorphisms $g \in GL_n(C^\infty(U \times \mathbb{P}^1))$ such that for each $u \in U$ and each $i = 1, 2, \ldots, m$:*

• *$g(u, a_i) \in GL_n(\mathbb{C})$ is the compatible framing ${}^ig_0(u)$ at a_i specified by $u \in U$,*

• *The singular connection $\alpha(u) := g(u)[A(u)]$ on $\mathbb{P}^1$ has Laurent expansion ${}^iA^0 + {}^iR(u)$ at $a_i \in \mathbb{P}^1$, where ${}^iR(u) = ({}^i\Lambda - {}^i\Lambda^0)\, dz/z_i$, ${}^i\Lambda^0 =$* $\mathrm{Res}_i({}^iA^0)$ *and ${}^i\Lambda$ is the exponent of formal monodromy of $(d - A(u), \mathbf{g})$ at a_i,*

• *If $z \in \mathbb{P}^1 \setminus (D_1 \cup \cdots \cup D_m)$ then $g(u, z) = 1$.*

Proof. The construction of the universal family is immediate from the proof of Proposition 2.1: Using the coordinate choices, $\widetilde{\mathcal{M}}^*(\mathbf{A})$ can be identified with the submanifold of $\mu_G^{-1}(0) \subset \widetilde{O}_1 \times \cdots \times \widetilde{O}_m$ which has ${}^1g_0 = 1$. This subset was identified as a set of matrices of meromorphic one-forms $A(u)$ on $\mathbb{P}^1$, together with compatible framings $\mathbf{g}$. (Although we do not need this fact, it is easy to check that the family $(d - A, \mathbf{g})$ of compatibly framed connections on the trivial bundle has the appropriate universal property for $\widetilde{\mathcal{M}}^*(\mathbf{A})$; it is a *fine* moduli space.)

Now consider the Laurent expansion $L_i(A) \in \mathrm{End}_n(\mathbb{C}\{z_i\} \otimes \mathcal{O}(U))\, dz/z_i^{k_i}$ of A at $a_i \in \mathbb{P}^1$, where the coefficients are now holomorphic functions on U. (If $u \in U$ then $L_i(A)(u) = L_i(A(u))$ as elements of $\mathrm{End}_n(\mathbb{C}\{z_i\})\, dz/z_i^{k_i}$.) Recall from Lemma 3.3 that the compatible framings determine formal isomorphisms: In the relative case here this means that, for each i, there is a unique invertible matrix ${}^i\hat{g} \in GL_n(\mathbb{C}[\![z_i]\!] \otimes \mathcal{O}(U))$ of formal power series with coefficients in $\mathcal{O}(U)$ which agrees with the compatible framing at a_i and for each $u \in U$ satisfies

$$ {}^i\hat{g}(u)[A(u)] = {}^iA^0 + {}^iR(u) \in \mathrm{End}_n(\mathbb{C}[\![z_i]\!])\, \frac{dz}{z_i^{k_i}} \tag{30} $$

with ${}^iR(u)$ as in the statement of the proposition. (The algorithm to construct such ${}^i\hat{g}$'s is as before; it works with coefficients in $\mathcal{O}(U)$.)

The crucial step is to now use E. Borel's result that the Taylor expansion map is surjective (Theorem 4.1 above). Applying this to each matrix entry of each ${}^i\hat{g}$ in turn for $i = 1, \ldots, m$ gives matrices of functions ${}^ig \in \mathrm{End}_n(C^\infty(U \times \bar{D}_i))$ such that for each $u \in U$ the Taylor expansion of ${}^ig$ at a_i is ${}^i\hat{g}(u)$. Since $\det {}^ig(u, a_i) = \det {}^ig_0(u)$ is nonzero for all $u \in U$, there is a neighbourhood of $U \times \{a_i\} \subset U \times \mathbb{P}^1$ throughout which $\det({}^ig)$ is nonzero. It follows (as $GL_n(\mathbb{C})$ is connected) that there is a smooth bundle automorphism $g \in GL_n(C^\infty(U \times \mathbb{P}^1))$ that equals ${}^ig$ in some neighbourhood of $U \times \{a_i\} \subset U \times \mathbb{P}^1$ for each i and equals 1 outside $U \times (D_1 \cup \cdots \cup D_m)$. In particular g has the desired Taylor expansions at each a_i so that $\alpha = g[A]$ has the desired C^∞ Laurent expansions by construction. ∎

COROLLARY 6.2. *The monodromy map $\tilde{\nu}$ factorises through $\widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$: It is possible to choose a map $\hat{\nu}$ from the extended moduli space $\widetilde{\mathscr{M}}^*(\mathbf{A})$ to the extended space of flat singular connections determined by* **A** *such that the diagram*

$$\begin{array}{ccccc} \widetilde{\mathscr{M}}^*(\mathbf{A}) & \xrightarrow{\hat{\nu}} & \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A}) & \xhookrightarrow{i} & \widetilde{\mathscr{A}}(\mathbf{A}) \\ \Vert & & \downarrow{\scriptstyle \mathscr{G}_1} & & \\ \widetilde{\mathscr{M}}^*(\mathbf{A}) & \xrightarrow{\tilde{\nu}} & \widetilde{M}_0(\mathbf{A}) & & \end{array}$$

commutes and the composition $i \circ \hat{\nu}$ into the Fréchet manifold $\widetilde{\mathscr{A}}(\mathbf{A})$ is holomorphic.

Proof. Construct g as in Proposition 6.1 with $U = \widetilde{\mathscr{M}}^*(\mathbf{A})$ and then define $\hat{\nu}(u) = g(u)[A(u)]$ for all $u \in \widetilde{\mathscr{M}}^*(\mathbf{A})$. All that remains is to see that the composition $i \circ \hat{\nu}$ is holomorphic. Recall (from Lemma 5.1) that by choosing a basepoint $\widetilde{\mathscr{A}}(\mathbf{A})$ is identified with a Fréchet submanifold of the Fréchet space $\Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$ of matrices of C^∞ one-forms with poles on the divisor D. Thus we must prove that the map $\widetilde{\mathscr{M}}^*(\mathbf{A}) \to \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$; $u \mapsto \alpha(u) := g(u)[A(u)]$ is holomorphic. Now if $u_0 \in \widetilde{\mathscr{M}}^*(\mathbf{A})$ and $W_0 \in T_{u_0}\widetilde{\mathscr{M}}^*(\mathbf{A})$ is a tangent vector at u_0, then we will denote the partial derivative of α along W_0 by $W_0(\alpha) \in \Omega^1[D](\mathbb{P}^1, \mathrm{End}(E))$. Here we think of α as a section of the C^∞ vector bundle $\pi^*(\mathrm{End}_n(\Omega^1[D]))$ over $\mathbb{P}^1 \times U$ (where π: $\mathbb{P}^1 \times U \to \mathbb{P}^1$ is the obvious projection). This vector bundle is trivial in the U directions so the partial derivative makes sense. (Concretely, local sections are of the form $\sum h_i\theta_i$ for C^∞ functions h_i on U and sections θ_i of $\mathrm{End}_n(\Omega^1[D])$. Then W_0 differentiates just the h_i's: $W_0(\sum h_i\theta_i) = \sum W_0(h_i)\,\theta_i$.) It then follows from basic facts about calculus on Fréchet spaces that the map $i \circ \hat{\nu}$ is holomorphic and has derivative $W_0(\alpha)$ along W_0 at u_0. (This can be deduced from Examples 3.1.6 and 3.1.7 in [26].) ∎

Main Proof

Proof (of Theorem 6.1). Choose g as in Corollary 6.2 above and let $\hat{\nu}$: $\widetilde{\mathscr{M}}^*(\mathbf{A}) \to \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ be the corresponding lift of the monodromy map. It is sufficient for us to prove that the composite map $i \circ \hat{\nu} : \widetilde{\mathscr{M}}^*(\mathbf{A}) \to \widetilde{\mathscr{A}}(\mathbf{A})$ is symplectic. This is because the symplectic form on $\widetilde{M}_0(\mathbf{A})$ is defined locally as $(i \circ s)^*\,\omega_{\widetilde{\mathscr{A}}(\mathbf{A})}$ for any local slice $s : \widetilde{M}_0(\mathbf{A}) \to \widetilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A})$ of the $\mathscr{G}_1$ action. But, over the subset $\tilde{\nu}(\widetilde{\mathscr{M}}^*(\mathbf{A}))$, such a slice is given by $\hat{\nu} \circ \tilde{\nu}^{-1}$. Thus $\tilde{\nu}^*\omega_{\widetilde{M}_0(\mathbf{A})} = (i \circ \hat{\nu})^*\,\omega_{\widetilde{\mathscr{A}}(\mathbf{A})}$ and, if $i \circ \hat{\nu}$ is symplectic, this is $\omega_{\widetilde{\mathscr{M}}^*(\mathbf{A})}$.

Now choose a point $u_0 \in \widetilde{\mathscr{M}}^*(\mathbf{A})$ and two tangent vectors $W_1, W_2 \in T_{u_0}\widetilde{\mathscr{M}}^*(\mathbf{A})$. Define two matrices of singular one-forms on $\mathbb{P}^1$, $\phi_j := W_j(\alpha) \in \mathrm{End}_n(\Omega^1[D](\mathbb{P}^1))$ $(j = 1, 2)$, to be the corresponding partial derivatives of $\alpha(u) := g(u)[A(u)]$. As in the proof of Corollary 6.2, ϕ_j is the derivative $(i \circ \hat{v})_* (W_j)$ of the map $i \circ \hat{v}$ along W_j. Therefore what we must prove is:

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} \operatorname{Tr}(\phi_1 \wedge \phi_2) = \omega_{\widetilde{\mathcal{M}}^*(\mathbf{A})}(W_1, W_2). \tag{31}$$

The first step is to obtain a formula for the right-hand side in terms of g. This comes from Lemma 2.3 since, by construction, the first k_i terms of the Taylor expansion of g at a_i give a section of the ith "winding map" w. For $j = 1, 2$ define ${}^i\dot{\Lambda}_j = W_j({}^i\Lambda) \in \mathfrak{t}$ where ${}^i\Lambda$ is the ith exponent of formal monodromy (which is regarded as a $\mathfrak{t}$-valued function on $\widetilde{\mathcal{M}}^*(\mathbf{A})$). Let ${}^i\dot{R}_j := {}^i\dot{\Lambda}_j dz/z_i$ and denote the derivatives of g as $\dot{g}_j := W_j(g) \in \operatorname{End}_n(C^\infty(\mathbb{P}^1))$. Then according to Lemma 2.3, if we define ${}^iX_j \in \mathfrak{g}_{k_i}$ to be the first k_i terms in the Taylor expansion of $g(u_0)^{-1}\,\dot{g}_j$ at a_i then

$$\omega_{\widetilde{\mathcal{M}}^*(\mathbf{A})}(W_1, W_2) = \sum_{i=1}^{m} (\langle {}^i\dot{R}_1, {}^i\widetilde{X}_2\rangle - \langle {}^i\dot{R}_2, {}^i\widetilde{X}_1\rangle + \langle {}^iA(u_0), [{}^iX_1, {}^iX_2]\rangle), \tag{32}$$

where ${}^iA$ is the Laurent expansion of A at a_i and ${}^i\widetilde{X}_j = {}^ig_0(u_0)\cdot {}^iX_j \cdot {}^ig_0(u_0)^{-1} \in \mathfrak{g}_{k_i}$ for $j = 1, 2$ and $i = 1, \ldots, m$.

Now we will calculate the left-hand side of (31). First observe that the two-form $\operatorname{Tr}(\phi_1 \wedge \phi_2)$ on $\mathbb{P}^1$ is non-singular. Indeed the C^∞ Laurent expansion of ϕ_j at a_i is ${}^i\dot{R}_j$, and so the expansion of $\operatorname{Tr}(\phi_1 \wedge \phi_2)$ is a $(2, 0)$ form and so zero. Then the division lemma implies $\operatorname{Tr}(\phi_1 \wedge \phi_2)$ is non-singular.

Next, by differentiating the expression $g(u)[A(u)]$ for α along W_j we find

$$\phi_j = g(u_0)\cdot \widetilde{\phi}_j \cdot g(u_0)^{-1}, \qquad \widetilde{\phi}_j := \dot{A}_j + d_{A(u_0)}(g(u_0)^{-1}\,\dot{g}_j) \tag{33}$$

for $j = 1, 2$, where $\dot{A}_j := W_j(A(u))$ and $g(u) = g(u, \cdot) \in \mathcal{G}$. (Note that this formula is the basic reason why the "straightening" procedure makes the Atiyah–Bott formula (26) non-trivial in this situation.) In particular we have $\operatorname{Tr}(\phi_1 \wedge \phi_2) = \operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2)$. Observe this two-form is zero outside of the disks D_i, since $\dot{g}_j$ is zero there and each $\dot{A}_j$ has type $(1, 0)$. It follows that the integral splits up into integrals over the closed disks:

$$\int_{\mathbb{P}^1} \operatorname{Tr}(\phi_1 \wedge \phi_2) = \sum_{i=1}^{m} \int_{\bar{D}_i} \operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2). \tag{34}$$

We break each term in this sum into two pieces, using the definition (33) of $\widetilde{\phi}_2$: $\operatorname{Tr}(\widetilde{\phi}_1 \wedge \widetilde{\phi}_2) = \operatorname{Tr}(\widetilde{\phi}_1 \wedge \dot{A}_2) + \operatorname{Tr}(\widetilde{\phi}_1 \wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_2))$. Therefore by comparing with the expression (32), the theorem now follows immediately from:

Claim.

(1) $\frac{1}{(2\pi\sqrt{-1})}\int_{\bar{D}_i}\operatorname{Tr}(\widetilde{\phi}_1\wedge\dot{A}_2)=\langle{}^iA(u_0),[{}^iX_1,{}^iX_2]\rangle-\langle{}^i\dot{R}_2,{}^i\widetilde{X}_1\rangle,$

(2) $\frac{1}{(2\pi\sqrt{-1})}\int_{\bar{D}_i}\operatorname{Tr}(\widetilde{\phi}_1\wedge d_{A(u_0)}(g(u_0)^{-1}\dot{g}_2))=\langle{}^i\dot{R}_1,{}^i\widetilde{X}_2\rangle.$

The basic tool we will use to evaluate these integrals is:

LEMMA 6.3 (Modified C^∞ Cauchy Integral Theorem). *Let k be a non-negative integer, $a\in\mathbb{C}$ a complex number and D_a a disk in $\mathbb{C}$ containing the point a. Suppose $f\in C^\infty(\bar{D}_a)$ and $(\frac{\partial f}{\partial\bar{z}})/(z-a)^k\in C^\infty(\bar{D}_a)$. Then $(\frac{\partial f}{\partial\bar{z}})\,dz\wedge d\bar{z}/(z-a)^{k+1}$ is absolutely integrable over $\bar{D}_a$ and*

$$\frac{(2\pi i)}{k!}\frac{\partial^k f}{\partial z^k}(a)=\oint_{\partial\bar{D}_a}\frac{f(z)\,dz}{(z-a)^{k+1}}+\int_{\bar{D}_a}\frac{\partial f}{\partial\bar{z}}\frac{dz\wedge d\bar{z}}{(z-a)^{k+1}},$$

where the line integral is taken in an anti-clockwise direction.

Proof. The $k=0$ case is the usual C^∞ Cauchy integral theorem; see [23, p. 2]. Differentiating with respect to a then gives the above result: we may reorder the integration and differentiation due to the absolute integrability. ∎

Part (1) of the claim arises as follows. Since $\dot{A}_2$ is a matrix of meromorphic one-forms we have $\operatorname{Tr}(\widetilde{\phi}_1\wedge\dot{A}_2)=\operatorname{Tr}(\widetilde{\phi}_1^{(0,1)}\wedge\dot{A}_2)$, and from (33):

$$\widetilde{\phi}_1^{(0,1)}=\bar{\partial}(g(u_0)^{-1}\dot{g}_1)=\frac{\partial(g(u_0)^{-1}\dot{g}_1)}{\partial\bar{z}}\,d\bar{z}.$$

Also $\dot{A}_2$ has a pole of order at most k_i at a_i and so we can define a smooth function on $\bar{D}_i$, $f\in C^\infty(\bar{D}_i)$, by the prescription $f\,dz=(z-a_i)^{k_i}\cdot\operatorname{Tr}(g(u_0)^{-1}\dot{g}_1\dot{A}_2)$ on $\bar{D}_i$. By taking the exterior derivative of both sides of this and dividing through by $(z-a_i)^{k_i}$ we deduce

$$\operatorname{Tr}(\widetilde{\phi}_1\wedge\dot{A}_2)=-\frac{\partial f}{\partial\bar{z}}\frac{dz\wedge d\bar{z}}{(z-a_i)^{k_i}}\qquad\text{on}\quad\bar{D}_i,$$

where the minus sign occurs since we have reversed the order of dz and $d\bar{z}$. Observe that the Taylor expansion of $f\,dz$ at a_i has no terms containing $\bar{z}_i$. Thus $\partial f/\partial\bar{z}$ has zero Taylor expansion at a_i and in particular using the division lemma we see f satisfies the conditions in Lemma 6.3. Also f is zero on the boundary $\partial\bar{D}_i$ since $\dot{g}_1$ is zero there. Therefore Cauchy's integral theorem gives

$$\frac{1}{(2\pi\sqrt{-1}\,)}\int_{\bar{D}_i}\mathrm{Tr}(\widetilde{\phi}_1\wedge\dot{A}_2)=-\frac{1}{k!}\frac{\partial^k f}{\partial z^k}(a_i)\qquad\text{with}\quad k=k_i-1. \tag{35}$$

This value is just $-\mathrm{Res}_i(f\,dz/(z-a_i)^{k_i})=-\mathrm{Res}_i(\mathrm{Tr}(g(u_0)^{-1}\,\dot{g}_1\dot{A}_2))$, where "residue" just means taking the coefficient of dz/z_i in the (C^∞) Laurent expansion. This last expression only involves the principal part of $\dot{A}_2$ at a_i and the first k_i terms of the Taylor expansion of $g(u_0)^{-1}\,\dot{g}_1$. By definition these first k_i terms are given by ${}^iX_1$. Also, by construction, the principal part of A at a_i is the same as the principal part of $g(u)^{-1}\,({}^i\!A^0+({}^i\!A-{}^i\!A^0)\,dz/z_i)\,g(u)$. It follows directly that

$$\mathrm{PP}_i(\dot{A}_2)=\mathrm{PP}_iW_2(A(u))=[{}^i\!A(u_0),\,{}^iX_2]+{}^ig_0(u_0)^{-1}\cdot{}^i\dot{R}_2\cdot{}^ig_0(u_0).$$

Statement (1) of the claim is now immediate, upon substituting this and ${}^iX_1$ into the expression $-\mathrm{Res}_i(\mathrm{Tr}(g(u_0)^{-1}\dot{g}_1\dot{A}_2))$ for the integral (35).

Now for part (2) of the claim. First observe that $d_{A(u_0)}\widetilde{\phi}_1=0$ as a matrix of two-forms on $\mathbb{P}^1$. This is equivalent to $d_{\alpha(u_0)}\phi_1=0$ (since $\widetilde{\phi}_1=g(u_0)^{-1}\,\phi_1\,g(u_0)$ and $\alpha(u)=g(u)[A(u)]$), which follows immediately by differentiating the equation $d(\alpha(u))=\alpha(u)\wedge\alpha(u)$ for the flatness of α along W_1.

Therefore, by Leibniz $\mathrm{Tr}(\widetilde{\phi}_1\wedge d_{A(u_0)}(g(u_0)^{-1}\,\dot{g}_2))=-d\,\mathrm{Tr}(\widetilde{\phi}_1\,g(u_0)^{-1}\,\dot{g}_2)$. Now, the Laurent expansion of ϕ_1 at a_i is just ${}^i\dot{R}_1$, so that $\phi_1={}^i\psi_1+{}^i\dot{R}_1$ on $\bar{D}_i$ for some matrix of non-singular one-forms ${}^i\psi_1$. Thus the integrand in (2) is

$$-d\,\mathrm{Tr}(g(u_0)^{-1}\cdot{}^i\psi_1\cdot\dot{g}_2)-d\,\mathrm{Tr}(g(u_0)^{-1}\cdot{}^i\dot{R}_1\cdot\dot{g}_2).$$

The first term integrates to zero over the disk by Stokes' theorem (the boundary term is zero as $\dot{g}_2$ vanishes on $\partial\bar{D}_i$). Now from the definition ${}^i\dot{R}_1={}^i\dot{A}_1(dz/(z-a_i))$ we find that the second term is $\frac{\partial f}{\partial\bar{z}}\,dz\wedge d\bar{z}/(z-a_i)$ where $f:=\mathrm{Tr}(g(u_0)^{-1}\,{}^i\!\dot{A}_1\dot{g}_2)$. This smooth function f vanishes on $\partial\bar{D}_i$ and so part (2) of the claim follows using Cauchy's integral theorem since $f(a_i)=\langle{}^iR_1,\,{}^i\widetilde{X}_2\rangle$. This completes the proof of Theorem 6.1. ∎

7. ISOMONODROMIC DEFORMATIONS AND SYMPLECTIC FIBRATIONS

Now we will consider smoothly varying the data **A** that was previously held fixed (consisting of the pole positions and the choices of generic connection germs $d - {}^i\!A^0$ at the poles)—all that is now fixed throughout is the rank n of the bundles, the number m of distinct poles and the multiplicities $k_1, \ldots, k_m$ of the poles. This leads naturally to the notion of "isomonodromic deformations" of meromorphic connections. Our aim is to explain, and then prove, the following:

THEOREM 7.1. *The Jimbo–Miwa–Ueno isomonodromic deformation equations are equivalent to a flat symplectic Ehresmann connection on a symplectic fibre bundle, having the moduli spaces* $\widetilde{\mathcal{M}}^*(\mathbf{A})$ *as fibre.*

The Betti Approach to Isomonodromy

A choice of data $\mathbf{A}$ determines all the spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$, $\widetilde{M}(\mathbf{A})$, $\mathcal{M}^*(\mathbf{A})$ and $M(\mathbf{A})$. Note however that the extended spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$ and $\widetilde{M}(\mathbf{A})$ only depend on the principal part of each diagonal matrix $d({}^iQ)$ of meromorphic one-forms, where ${}^i\!A^0 = d({}^iQ) + {}^i\Lambda^0\, dz_i/z_i$. (cf. Remark 3.3.) Thus if $a \in \mathbb{P}^1$ it is useful to define the set $X_k(a)$ of "order k irregular types at a", to be the set of such principal parts. Upon choosing a local coordinate z vanishing at a we have an isomorphism

$$X_k(a) \cong (\mathbb{C}^n \setminus \text{diagonals}) \times (\mathbb{C}^n)^{k-2} \tag{36}$$

obtained by taking the coefficients of dz/z^j of the Laurent expansion in z of A^0, for $j = k, k-1, \ldots, 2$. (If $k = 1$ define $X_k(a) :=$ (point).)

For the rest of this section we will change notation slightly, and let $\mathbf{A}$ denote data $(a_1, a_2, \ldots, a_m, {}^1\!A^0, \ldots, {}^m\!A^0)$ where ${}^i\!A^0 \in X_{k_i}(a_i)$ and the a_i are pairwise distinct points of $\mathbb{P}^1$. Thus such $\mathbf{A}$ determines the extended spaces $\widetilde{\mathcal{M}}^*(\mathbf{A})$ and $\widetilde{M}(\mathbf{A})$ (although we need to further specify exponents of formal monodromy ${}^i\Lambda^0$ to define $\mathcal{M}^*(\mathbf{A})$ and $M(\mathbf{A})$).

There are three manifolds of deformation parameters we will consider:

DEFINITION 7.1.

- The *basic manifold of deformation parameters* X is simply the set of such $\mathbf{A}$.
- The *extended manifold of deformation parameters* $\widetilde{X}$ is the set of such $\mathbf{A}$ together with the choice of a k_i-jet of a coordinate z_i at each a_i.
- If z is a fixed coordinate identifying $\mathbb{P}^1$ with $\mathbb{C} \cup \infty$, the *Jimbo–Miwa–Ueno manifold of deformation parameters* X_{JMU} is the set of all such $\mathbf{A}$ having $a_1 = \infty$.

It is easy to see these are complex manifolds, with $\dim(X) = \dim(X_{\text{JMU}}) + 1 = \dim(\widetilde{X}) - \sum k_i = m - mn + n\sum k_i$. There is an obvious embedding $X_{\text{JMU}} \hookrightarrow X$ and a projection $\widetilde{X} \twoheadrightarrow X$ forgetting the jets of local coordinates. Moreover using the chosen coordinate z there is an embedding $X_{\mathrm{JMU}} \hookrightarrow \widetilde{X}$ obtained by using the jets of the coordinates $z_i := z - a_i$ for $i = 2, \ldots, m$ and $z_1 := 1/z$. X_{JMU} can be described very explicitly: via (36) these coordinates identify it with

$$(\mathbb{C}^{m-1} \setminus \text{diagonals}) \times (\mathbb{C}^n \setminus \text{diagonals})^{m-l} \times (\mathbb{C}^n)^{l+\sum(k_i-2)},$$

where $l = \#\{i \mid k_i = 1\}$ is the number of simple poles. However our aim here is more to understand the intrinsic geometry of isomonodromic deformations, than seek explicitness, and so we will mainly use X and $\widetilde{X}$.

Now we move on to the construction of bundles over these parameter spaces.

DEFINITION 7.2. The *bundle of extended moduli spaces* $\widetilde{\mathcal{M}}^*$ is the set of isomorphism classes of data $(V, \nabla, \mathbf{g}, \mathbf{a})$ consisting of a *generic* meromorphic connection ∇ (with compatible framings $\mathbf{g}$) on a *trivial* rank n holomorphic vector bundle V over a fixed copy of $\mathbb{P}^1$ such that ∇ has m poles which are labelled $a_1, \ldots, a_m$ and the order of the pole at a_i is k_i.

It is clear from the discussion in Section 2 that a generic compatibly framed connection determines an irregular type at each pole and it follows that there is a natural projection $\widetilde{\mathcal{M}}^* \twoheadrightarrow X$ onto the manifold X of deformation parameters, taking the pole positions and the irregular types. The fibre of this projection over a point $\mathbf{A} \in X$ is the extended moduli space $\widetilde{\mathcal{M}}^*(\mathbf{A})$. The results of Section 2 now yield the following, which will amount to half of Theorem 7.1:

PROPOSITION 7.1. *The bundle $\widetilde{\mathcal{M}}^*$ of extended moduli spaces is a complex manifold and the projection above expresses it as a locally trivial* symplectic *fibre bundle over X. In particular $\widetilde{\mathcal{M}}^*$ has an intrinsic complex Poisson structure, its foliation by symplectic leaves is fibrating and the space of leaves is X.*

Proof. The only non-trivial part left is to see that $\widetilde{\mathcal{M}}^*$ is locally trivial as a bundle of symplectic manifolds. The decoupling lemma from Section 2 is useful here. Choose m disjoint open disks $D_i \subset \mathbb{P}^1$ and choose a coordinate z on $\mathbb{P}^1$ which is non-singular on all the D_i's. Restrict to the open subset X' of X having $a_i \in D_i$ for each i. Let $z_i := z - a_i$. Now, from Proposition 2.7, over X' any fibre $\widetilde{\mathcal{M}}^*(\mathbf{A})$ can be identified (using the coordinates z_i) with a symplectic submanifold of $\widetilde{O}_1 \times \cdots \times \widetilde{O}_m$ (e.g. as the subset of $\mu_G^{-1}(0)$ which has ${}^1g_0 = 1$). In turn, using Lemma 2.14, $\widetilde{\mathcal{M}}^*(\mathbf{A})$ is identified (if all $k_i \geqslant 2$) with the symplectic manifold

$$(T^*G)^{m-1} \times {}^1O_B \times \cdots \times {}^mO_B,$$

where ${}^iO_B$ is the B_{k_i}-coadjoint orbit through the element of $\mathfrak{b}^*_{k_i}$ determined by ${}^iA^0$ (on expanding ${}^iA^0$ with respect to z_i and replacing z_i by ζ). Thus the dependence of $\widetilde{\mathcal{M}}^*(\mathbf{A})$ on $\mathbf{A}$ is clear: as $\mathbf{A}$ varies, the orbit ${}^iO_B$ moves around in $\mathfrak{b}^*_{k_i}$. The key fact now is that B_{k_i} is a nilpotent Lie group: coadjoint orbits of nilpotent Lie groups are diffeomorphic to vector spaces and admit *global* Darboux coordinates. Indeed M. Vergne [72] shows how to find $\dim({}^iO_B)$ functions on $\mathfrak{b}^*_{k_i}$ which restrict to global Darboux coordinates on any ${}^iO_B$ that arises as ${}^iA^0$ varies (${}^iO_B$ is always a generic orbit). Such coordinates immediately give a symplectic trivialisation of $\widetilde{\mathcal{M}}^*$ over X'. (If $k_i = 1$ for some i then $\widetilde{O}_i$ is a fixed symplectic submanifold of T^*G; there is no ${}^iO_B$ factor to worry about.) ∎

Similarly there is a fibre bundle $\widetilde{M}$ over X whose fibres are the extended monodromy manifolds $\widetilde{M}(\mathbf{A})$. The key feature of the bundle $\widetilde{M}$ is that it has a canonical complete flat Ehresmann connection on it—in other words there is a canonical isomorphism between nearby fibres. In essence this connection arises by "keeping the monodromy data constant" so we will call it the *isomonodromy connection*. There is a subtlety however because it is the Stokes matrices which are held constant locally, rather than the Stokes factors: For example any anti-Stokes direction with multiplicity greater than one can break up into distinct anti-Stokes directions under arbitrarily small deformations of the data $\mathbf{A}$, and the dimensions of the groups of Stokes factors jump accordingly (so the notion of keeping the Stokes factors constant makes no sense directly, in general). A precise description of the isomonodromy connection is as follows.

Suppose $\mathbf{A} \in X$ is a choice of pole positions $(a_1, \ldots, a_m)$ and irregular types. Choose disjoint open discs $D_i \subset \mathbb{P}^1$ with $a_i \in \mathbb{P}^1$, together with a coordinate on each disc (so directions at a_i can be drawn as lines on D_i). If we choose a set of tentacles $\mathcal{T}$ (see Definition 3.9) then there is, from Proposition 3.8, an isomorphism $\widetilde{\varphi}_{\mathcal{T}}: \widetilde{M}(\mathbf{A}) \to \widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m /\!/ G$ to the explicit monodromy manifold (which is completely independent of $\mathbf{A}$). The point is that, by continuity, there is a small open neighbourhood $U_{\mathbf{A}}$ of $\mathbf{A}$ in X such that if $\mathbf{A}'$ moves around in $U_{\mathbf{A}}$, then none of the anti-Stokes directions at a_i' cross over the base-point $p_i \in D_i$ chosen as part of the tentacles. Thus using the maps $\widetilde{\varphi}_{\mathcal{T}}$ (with $\mathcal{T}$ fixed and $\mathbf{A}'$ varying) we get a local trivialisation of $\widetilde{M}$ over $U_{\mathbf{A}}$. Repeating this process gives an open cover of X with a choice of trivialisation of $\widetilde{M}$ over each patch. This describes the bundle $\widetilde{M}$ explicitly with clutching functions of the form $\widetilde{\varphi}_{\mathcal{T}_1} \circ \widetilde{\varphi}^{-1}_{\mathcal{T}_2}$. Now the fact that these clutching functions are constant with respect to the parameters $\mathbf{A} \in X$ means that we have a well defined flat connection on $\widetilde{M}$ (the local horizontal sections of which have constant explicit monodromy data $(\mathbf{C}, \mathbf{S}, \mathbf{\Lambda}') \in \widetilde{\mathcal{C}}_1 \times \cdots \times \widetilde{\mathcal{C}}_m /\!/ G$). This is the isomonodromy connection.

Now we want to define the relative version of the extended monodromy map. However recall from Proposition 3.10, that this requires a choices of coordinate jets. Thus we first pull both bundles $\widetilde{\mathscr{M}}^*$ and $\widetilde{M}$ back to the extended manifold $\widetilde{X}$ of deformation parameters along the projection $\widetilde{X} \twoheadrightarrow X$. (These bundles over $\widetilde{X}$ will also be denoted $\widetilde{\mathscr{M}}^*$ and $\widetilde{M}$ but this should not lead to confusion.) Then, using the jets of coordinates encoded in $\widetilde{X}$, the fibrewise monodromy maps fit together to define a holomorphic bundle map, $\tilde{\nu}: \widetilde{\mathscr{M}}^* \to \widetilde{M}$ between the bundles over $\widetilde{X}$. (As before this is holomorphic since the canonical solutions depend holomorphically on parameters.)

Definition 7.3. The *isomonodromy connection on* $\widetilde{\mathscr{M}}^*$ is the pull-back of the isomonodromy connection on $\widetilde{M}$ along $\tilde{\nu}$.

See Fig. 1. The point is that $\tilde{\nu}$ is a highly nonlinear map with respect to the explicit descriptions of the bundles $\widetilde{\mathscr{M}}^*$ and $\widetilde{M}$; whilst being trivial on $\widetilde{M}$, the isomonodromy connection defines interesting nonlinear differential equations on $\widetilde{\mathscr{M}}^*$, such as the Painlevé or Schlesinger equations (indicated by a wavy line in the figure).

Equivalently one may view $\tilde{\nu}$ as a kind of nonlinear Fourier–Laplace transform (the "monodromy transform"), converting hard nonlinear equations on the left-hand side into trivial equations on the right. The image of $\tilde{\nu}$ is a subset of the degree zero component $\widetilde{M}_0$ and as before, for dimensional reasons (since it is injective and holomorphic) $\tilde{\nu}$ is biholomorphic onto its image. Moreover Miwa [55] has proved that the inverse $\tilde{\nu}^{-1}: \widetilde{M}_0 \to \widetilde{\mathscr{M}}^*$ is *meromorphic*, so that local horizontal sections of the isomonodromy connection on $\widetilde{\mathscr{M}}^*$ will develop at worst poles when extended around $\widetilde{X}$: this is the Painlevé property of the equations. In particular this implies the image of $\tilde{\nu}$ is the complement of a divisor in $\widetilde{M}_0$.

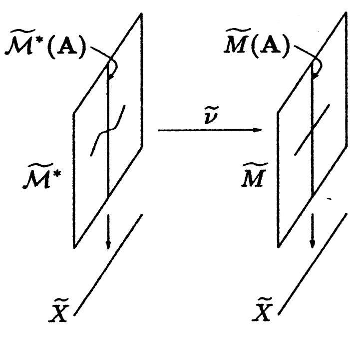

FIG. 1. Isomonodromic Deformations.

Note that the isomonodromy connections are equivariant under the $PSL_2(\mathbb{C})$ action on the bundles $\widetilde{\mathcal{M}}^*$, $\widetilde{M}$, induced from automorphisms of $\mathbb{P}^1$.

To be precise, the isomonodromy equations of Jimbo, Miwa and Ueno are the equations for horizontal sections of the restriction of the isomonodromy connection on $\widetilde{\mathcal{M}}^*$ to $X_{\text{JMU}} \hookrightarrow \widetilde{X}$, as we will explain in the appendix. The key idea required to actually write down equations for such horizontal sections is the following recharacterisation of the isomonodromy connection, which will also be very useful in the proof of Theorem 7.1.

Remark 7.1. Observe that in the order two pole case $k_i = 2$, the B_{k_i} coadjoint orbit ${}^i\mathcal{O}_B$ in Proposition 7.1 is just a point. Thus if there are no poles of order three or more, the bundle $\widetilde{\mathcal{M}}^*$ has a *canonical* symplectic trivialisation (which is not directly related to the isomonodromy connection) and so the isomonodromy equations will be naturally identified with time-dependent flows on a fixed symplectic manifold. In general however, choices are needed in the use of Vergne's theorem in Proposition 7.1, so we do not know a natural way to make such an identification. In particular, one must find/choose such a symplectic trivialisation before the notion of time-dependent Hamiltonians for isomonodromy even makes sense. This is a question we hope to return to in the future. (One suspects such a trivialisation arises naturally by requiring Hamiltonians to come from the logarithmic derivative of the Jimbo–Miwa–Ueno τ function.)

De Rham Approach to Isomonodromy

Suppose $\pi : Y \to X$ is some fibration over X, with manifolds Y_t as fibres. Replacing each Y_t by its cohomology $\mathrm{H}^\bullet(Y_t, \mathbb{C})$ yields a vector bundle $\mathrm{H}^\bullet_{\mathrm{Rel}}(Y, \mathbb{C}) \to X$. This vector bundle has a natural flat connection on it: the Gauss–Manin connection. One way to see this is from the homotopy invariance of cohomology: if $\Delta \subset X$ is an open ball then $Y|_\Delta$ is homotopy equivalent to any fibre $Y_t \subset Y|_\Delta$ so there is a canonical isomorphism $\mathrm{H}^\bullet(Y_t, \mathbb{C}) \cong \mathrm{H}^\bullet(Y_s, \mathbb{C})$ for any $s, t \in \Delta$. Alternatively there is a de Rham approach as follows. Given a closed differential form θ_t on a fibre Y_t, choose any closed form θ on $Y|_\Delta$ extending θ_t, and let θ_s be the restriction of θ to Y_s. The cohomology class of θ in $\mathrm{H}^\bullet(Y|_\Delta, \mathbb{C})$ is uniquely determined by the cohomology class of θ_t or of θ_s: this process defines the isomorphism $\mathrm{H}^\bullet(Y_t, \mathbb{C}) \cong \mathrm{H}^\bullet(Y_s, \mathbb{C})$ over Δ.

At least for H^1, this generalises to non-Abelian cohomology, replacing $\mathbb{C}$ by $G = GL_n(\mathbb{C})$. Topologically $\mathrm{H}^1(Y_t, G) = \mathrm{Hom}(\pi_1(Y_t), G)/G$ is the set of conjugacy classes of fundamental group representations. These fit together into a (non-linear) fibre bundle $\mathrm{H}^1_{\mathrm{Rel}}(Y, G) \to X$, which again clearly has a natural flat (Ehresmann) connection on it, due to the homotopy invariance of the fundamental group: the Gauss–Manin connection in non-Abelian cohomology. Simpson [64] refers to this as the Betti approach and studies the corresponding de Rham version. In the non-Abelian case, one-forms are replaced by connections on vector bundles, closedness is replaced by flatness, and the notion of differing by an exact form is replaced by gauge equivalence. Thus the de Rham version of $\mathrm{H}^1(Y_t, G)$ is the set of isomorphism classes of flat connections on rank n vector bundles over Y_t, and the isomorphism $\mathrm{H}^1(Y_t, G) \cong \mathrm{H}^1(Y_s, G)$ arises by extending a flat connection over a fibre Y_t to a flat connection over the family $Y|_\Delta$ and then restricting to Y_s.

The main realisation now is that one can very usefully view the isomonodromy connection described above as the analogue in the meromorphic case of this non-Abelian Gauss–Manin connection. This emphasises the basic geometrical nature of isomonodromy and suggests many generalisations (we are, after all, working over $\mathbb{P}^1$ with $G = GL_n(\mathbb{C})$). Note however, the necessity of having explicit descriptions of the moduli spaces in order to have explicit equations: the distinction between $\widetilde{\mathcal{M}}^*(\mathbf{A})$ and $\widetilde{\mathcal{M}}(\mathbf{A})$ is important.

Thus, in the de Rham approach, horizontal sections of the isomonodromy connection on $\widetilde{\mathcal{M}}^*$ over some ball $\Delta \subset \widetilde{X}$ are related to flat meromorphic connections on vector bundles over $\mathbb{P}^1 \times \Delta$. This alternative approach was one of the main results of [40] (although not expressed in these terms). More precisely, in the extended case, the following holds:

THEOREM 7.2 (see [40]). *Let $\Delta \subset \widetilde{X}$ be an open ball. Then there is a canonical one to one correspondence between horizontal sections of the isomonodromy connection on $\widetilde{\mathcal{M}}^*$ over Δ and isomorphism classes of triples $(V, \nabla, \mathbf{g})$ consisting of flat meromorphic connections ∇ on vector bundles V over $\mathbb{P}^1 \times \Delta$ with* good *compatible framings $\mathbf{g}$, such that for any $t \in \Delta$ the restriction of $(V, \nabla, \mathbf{g})$ to the projective line $\mathbb{P}^1 \times \{t\}$ represents an element in the fibre $\widetilde{\mathcal{M}}^*_t$.*

Sketch. See the appendix for more details, and in particular for the definition of "good" compatible framings. To go from such triples $(V, \nabla, \mathbf{g})$ to sections of $\widetilde{\mathcal{M}}^*$ over Δ, simply restrict to the $\mathbb{P}^1$ fibres. Lemma A.2 shows why the flatness of ∇ implies the isomonodromicity of this family of connections over $\mathbb{P}^1$. Conversely, suppose we have a horizontal section of the isomonodromy connection on $\widetilde{\mathcal{M}}^*$ over Δ, or equivalently a compatibly framed isomonodromic family ∇_t of meromorphic connections over $\mathbb{P}^1$, parameterised by $t \in \Delta$. Then for each fixed t we have a canonical basis of horizontal solutions of ∇_t on each sector at each pole on $\mathbb{P}^1 \times \{t\}$. The key idea is that as t varies, these bases (where defined) vary holomorphically with t and ∇ is defined by declaring all of these bases to be horizontal sections of it. The isomonodromicity of the original family implies this ∇ is well-defined and flat. Moreover one can deduce that ∇ is meromorphic and, by summing its principal parts, write down an algebraic expression for ∇ in terms of the original horizontal section. This leads directly to the explicit deformation equations. ▮

The non-extended version can easily be deduced from the above result, by forgetting the framings, and is closer in spirit to the non-singular (Gauss–Manin) case. First choose an m-tuple Λ of diagonal $n \times n$ matrices. Then define bundles $\mathcal{M}^*(\Lambda)$ and $M(\Lambda)$ over the space X of deformation parameters, by restricting the bundles $\widetilde{\mathcal{M}}^* \to X$ and $\widetilde{M} \to X$ to the subsets which have exponents of formal monodromy Λ and quotienting by the action of $T^m \cong (\mathbb{C}^*)^{nm}$. The isomonodromy connection on $\widetilde{M}$ descends to induce a canonical isomorphism between nearby fibres of $M(\Lambda) \to X$, and in turn we obtain a well-defined notion of local horizontal sections of the isomonodromy connection on $\mathcal{M}^*(\Lambda) \to X$. Immediately we obtain the following (see Malgrange [45, 46] for some global statements along these lines):

COROLLARY 7.2. *Horizontal sections of the isomonodromy connection on $\mathcal{M}^*(\Lambda)$ over $\Delta \subset X$ correspond canonically to isomorphism classes of pairs (V, ∇) consisting of flat meromorphic connections ∇ on vector bundles V over $\mathbb{P}^1 \times \Delta$, such that for any $t \in \Delta$ the restriction of (V, ∇) to the projective line $\mathbb{P}^1 \times \{t\}$ represents an element in the fibre $\mathcal{M}^*(\Lambda)_t$.*

Isomonodromic Deformations Are Symplectic

Now we will establish the second part of Theorem 7.1, thereby revealing the symplectic nature of the full family of Jimbo–Miwa–Ueno isomonodromic deformation equations:

THEOREM 7.3. *The isomonodromy connection on the bundle $\widetilde{\mathcal{M}}^* \to \widetilde{X}$ of extended moduli spaces, is a symplectic connection. In other words, the local analytic diffeomorphisms induced by the isomonodromy connection between the fibres of $\widetilde{\mathcal{M}}^*$ are symplectic diffeomorphisms.*

Proof. We will show that arbitrary, small, isomonodromic deformations induce symplectomorphisms. Let u_0 be any point of $\widetilde{\mathcal{M}}^*$ and let x_0 be the image of u_0 in $\widetilde{X}$. Let γ be any holomorphic map from the open unit disk $\mathbb{D} \subset \mathbb{C}$ into $\widetilde{X}$ such that $\gamma(0) = x_0$. For $t \in \mathbb{D}$, let $\widetilde{\mathcal{M}}^*_t$ denote the (symplectic) extended moduli space which is the fibre of $\widetilde{\mathcal{M}}^*$ over $\gamma(t)$. The standard vector field $\partial/\partial t$ on $\mathbb{D}$ gives a vector field on $\gamma(\mathbb{D}) \subset \widetilde{X}$ which we lift to a vector field V on $\widetilde{\mathcal{M}}^*|_{\gamma(\mathbb{D})}$, transverse to the fibres $\widetilde{\mathcal{M}}^*_t$, using the isomonodromy connection. This lifted vector field may be integrated throughout a neighbourhood of u_0 in $\widetilde{\mathcal{M}}^*|_{\gamma(\mathbb{D})}$. Concretely, this means that there is a contractible neighbourhood U of u_0 in $\widetilde{\mathscr{M}}^*_0$, a neighbourhood $\Delta \subset \mathbb{D}$ of 0 in $\mathbb{C}$ and a holomorphic map $F\colon U \times \Delta \to \widetilde{\mathscr{M}}^*|_{\gamma(\Delta)}$ such that for all $u \in U$ and $t \in \Delta$:

$$F(u, t) \in \widetilde{\mathscr{M}}^*_t, \qquad F(u, 0) = u \in \widetilde{\mathscr{M}}^*_0 \qquad \text{and} \qquad \frac{\partial F}{\partial t}(u, t) = V_{F(u, t)}.$$

In particular for each $t \in \Delta$ we have a symplectic form $\omega_t := (F|_t)^*(\omega_{\widetilde{\mathscr{M}}^*_t})$ on U, where $\omega_{\widetilde{\mathscr{M}}^*_t}$ is the symplectic form defined on the extended moduli space $\widetilde{\mathscr{M}}^*_t$ in Section 2 and $F|_t = F(\cdot, t) : U \to \widetilde{\mathscr{M}}^*_t$. Now, given any two tangent vectors W_1, W_2 to U at u_0, it is sufficient for us to show that the function $\omega_t(W_1, W_2)$ of t is *constant* in some neighbourhood of $0 \in \Delta$.

First, as in Proposition 6.1 it is easy to construct a local universal family over the image of F in $\widetilde{\mathscr{M}}^*$. Pulling back along F yields a family of meromorphic connections on the trivial bundle over $\mathbb{P}^1$ parameterised by $U \times \Delta$. For each fixed $u \in U$ we get an isomonodromic family parameterised by Δ, that is, a "vertical" meromorphic connection on the trivial bundle over $\Delta \times \mathbb{P}^1$ (where $\mathbb{P}^1$ is the vertical direction), such that each connection on $\mathbb{P}^1$ has the "same" monodromy data. The result of Jimbo, Miwa and Ueno (Theorem 7.2 above) then tells us how to extend this vertical connection to a full *flat* connection over $\Delta \times \mathbb{P}^1$. From the algebraic formula (A.4) for this extension it is clear that this process behaves well as we vary $u \in U$: for each $u \in U$ we obtain a flat meromorphic connection, which we will denote ∇_u, on the trivial bundle over $\Delta \times \mathbb{P}^1$, that depends holomorphically on u. The poles of ∇_u will be denoted by $a_1(t), \ldots, a_m(t)$ and the polar divisor in $\Delta \times \mathbb{P}^1$ of ∇_u by $\widetilde{D} = \sum k_i \Delta_i$ (these are all independent of $u \in U$). Shrinking Δ if necessary, choose disjoint open discs D_i in $\mathbb{P}^1$ such that $a_i(t) \in D_i$ for all $t \in \Delta$. For each i let $z_i\colon D_i \times \Delta \to \mathbb{C}$ be a function which, for each fixed $t \in \Delta$ is a coordinate on D_i, vanishing at $a_i(t)$ and having the k_i-jet at $a_i(t)$ as specified by the point of the base $\widetilde{X}$ below $\gamma(t)$.

The next step is to push everything over to the C^∞ picture where the symplectic forms are expressed simply as integrals. To do this we choose a smooth bundle automorphism: $g \in GL_n(C^\infty(U \times \Delta \times \mathbb{P}^1))$ which "straightens" the whole family of connections ∇_u at the same time, as in Section 6. The map F into $\widetilde{\mathscr{M}}^*$ specifies a family of good compatible framings ${}^ig_0\colon U \times \Delta_i \to GL_n(\mathbb{C})$ of ∇_u along Δ_i for each i and all $u \in U$. Use the coordinate z_i to define uniquely a family ${}^iA^0 := d_{\mathbb{P}^1}({}^iQ)$ of diagonal matrices of meromorphic one-forms on D_i, parameterised by $U \times \Delta$. (Recall only the principal part of ${}^iQ$ is specified by $\widetilde{X}$: declare the other terms are zero in its Laurent expansion with respect to z_i.) As in Proposition 6.1 the framings extend uniquely to formal isomorphisms ${}^i\hat{g} \in GL_n(\mathbb{C}[\![z_i]\!] \otimes \mathscr{O}(U \times \Delta_i))$ to (uniquely determined) diagonal connections $d_{\mathbb{P}^1} - d_{\mathbb{P}^1}({}^iQ) - {}^i\Lambda(u)\, d_{\mathbb{P}^1} z_i / z_i$.

By definition, that the framings are good, means ${}^i\hat{g}$ satisfies a stronger condition: it transforms the Laurent expansion of ∇_u along Δ_i into a standard full connection associated to the normal forms for each u,

$$ {}^i\hat{g}[L_i(\nabla_u)] = d - d({}^iQ(t)) - {}^i\Lambda(u)\,\frac{d(z_i)}{z_i}, \tag{37} $$

where d denotes the exterior derivative on the product $\Delta \times \mathbb{P}^1$, rather than just $\mathbb{P}^1$. The automorphism g is now constructed using Borel's theorem, as in Proposition 6.1 to have Taylor expansion at $a_i(t)$ equal to ${}^i\hat{g}$ for all $t \in \Delta$ and for all $u \in U$.

Thus we can use g to straighten the whole family ∇_u at the same time. Define two families of C^∞ singular connections. First a family $\tilde{\nabla}_u := g[\nabla_u]$ on $\Delta \times \mathbb{P}^1$ parameterised by U, and second $d_\alpha = d_{\mathbb{P}^1} - \alpha := \tilde{\nabla}_u|_{\mathbb{P}^1}$ on $\mathbb{P}^1$ parameterised by $U \times \Delta$. By construction the C^∞ Laurent expansion of $\tilde{\nabla}_u$ at a_i is given by (37). It follows, for all $u \in U$ and $t \in \Delta$, that d_α is an element of the extended space $\tilde{\mathscr{A}}_{\mathrm{fl}}(\mathbf{A}_t) \subset \tilde{\mathscr{A}}(\mathbf{A}_t)$ of flat singular connections associated to $\mathbf{A}_t := \{{}^i\!A^0\}$.

Now differentiate $\tilde{\nabla}_u$ and d_α with respect to u along both W_1 and W_2 at $u = u_0$. Define these derivatives to be $\Psi_j := W_j(\tilde{\nabla}_u)$ and $\psi_j := W_j(d_\alpha) = \Psi_j|_{\mathbb{P}^1}$ respectively, for $j = 1, 2$. Each Ψ_j is a matrix of singular one-forms on $\Delta \times \mathbb{P}^1$ and each ψ_j is a matrix of singular one-forms on $\mathbb{P}^1$ parameterised by Δ. Clearly $\mathrm{Tr}(\psi_1 \wedge \psi_2) = \mathrm{Tr}(\Psi_1 \wedge \Psi_2)|_{\mathbb{P}^1}$. Also since the Laurent expansion of $\tilde{\nabla}_u$ is given by (37) at each a_i we can deduce what the Laurent expansions of Ψ_1 and Ψ_2 are: $L_i(\Psi_j) = W_j({}^i\Lambda(u))\, d_{\Delta\times\mathbb{P}^1}(z_i)/z_i$ for $j = 1, 2$ and $i = 1, \ldots, m$. It follows that $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ is a *nonsingular* two-form on $\Delta \times \mathbb{P}^1$ since $L_i(\Psi_1 \wedge \Psi_2) = L_i(\Psi_1) \wedge L_i(\Psi_2) = 0$ for each i.

Now observe that for each $u \in U$ the flatness of ∇_u implies the flatness of $\tilde{\nabla}_u$. By differentiating the equation $\tilde{\nabla}_u \circ \tilde{\nabla}_u = 0$ with respect to u along W_1 and W_2 we find $\tilde{\nabla}_{u_0}\Psi_1 = 0$ and $\tilde{\nabla}_{u_0}\Psi_2 = 0$. In particular, by Leibniz, the two-form $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ on $\Delta \times \mathbb{P}^1$ is *closed*.

Thus if we do the fibre integral over $\mathbb{P}^1$ we obtain a zero-form on Δ (i.e. a function of t):

$$ \int_{\mathbb{P}^1} \mathrm{Tr}(\Psi_1 \wedge \Psi_2) = \int_{\mathbb{P}^1} \mathrm{Tr}(\psi_1 \wedge \psi_2). $$

This is a *closed* 0-form (i.e. a *constant* function) since integration over the fibre commutes with exterior differentiation. See for example Bott and Tu [16] Proposition 6.14.1 (it is important here that $\mathrm{Tr}(\Psi_1 \wedge \Psi_2)$ is nonsingular).

Finally we appeal to Theorem 6.1 to see that for all $t \in \Delta$

$$\frac{1}{2\pi i}\int_{\mathbb{P}^1} \mathrm{Tr}(\psi_1 \wedge \psi_2) = \omega_t(W_1, W_2)$$

and so the symplectic form is indeed independent of t. ∎

Closing Remarks

One upshot of Theorem 7.1 is that the symplectic structure on each monodromy manifold is independent of the choice of deformation parameters; the isomonodromy connection on $\widetilde{M}$ is symplectic. This is the generalisation of the "symplectic nature of the fundamental group". As in the non-singular case, one then wonders if there is an intrinsic finite-dimensional/algebraic approach to this symplectic structure (generalising the cup product in group cohomology). This should be possible by combining the C^∞ approach here with the ideas of Alekseev, Malkin and Meinrenken [3].

Alternatively (or perhaps equivalently) a direct connection between Stokes matrices and Poisson Lie groups was observed in [14], which we will briefly sketch here since it is quite intriguing. Consider the case of connections on $\mathbb{P}^1$ with just two poles, of orders one and two respectively. The choice of an irregular type at the order two pole determines the moduli space $\widetilde{\mathcal{M}}^*(\mathbf{A})$ and the monodromy manifold $\widetilde{M}(\mathbf{A})$. If we forget the framing at the order one pole, we obtain $\widetilde{\mathcal{M}}^*(\mathbf{A})/T$ which is isomorphic as a Poisson manifold to (a covering of a dense open subset of) $\mathfrak{g}^*$. Also $\widetilde{M}(\mathbf{A})/T$ is isomorphic to a covering of a dense open subset of $U_+ \times U_- \times \mathfrak{t}$. The monodromy map extends to a map $\nu : \mathfrak{g}^* \to U_+ \times U_- \times \mathfrak{t}$, taking the Stokes matrices and the exponent of formal monodromy at 0 of the connection $d-(U dz/z^2 + V dz/z)$, where $V \in \mathfrak{g} \cong \mathfrak{g}^*$ and U is a fixed diagonal matrix with distinct eigenvalues. The basic observation now is that $U_+ \times U_- \times \mathfrak{t}$ may be identified with the simply connected Poisson Lie group G^* dual to $GL_n(\mathbb{C})$. We then claim that, under such identification, $\nu : \mathfrak{g}^* \to G^*$ is a Poisson map, where $\mathfrak{g}^*$ and G^* both have their standard Poisson structures.[3] In particular, taking V to be skew-symmetric, this claim yields a new approach to the Poisson bracket on Dubrovin's local moduli space of semisimple Frobenius manifolds.

APPENDIX

We will give more details regarding Theorem 7.2, relating flat connections to horizontal sections of the isomonodromy connection. This differs

[3] This has now been proved, cf. P. P. Boalch, math.DG/0011062.

from [40] in that the coordinate dependence is isolated here. At the end we will write down the deformation equations.

First some generalities on the local structure of meromorphic connections in higher dimensions. The local model is of a meromorphic connection $\nabla = d - \widetilde{A}$ on the trivial rank n vector bundle over a product $\mathbb{D}\times\varDelta$ of the unit disc $\mathbb{D}\subset\mathbb{C}$ and some contractible space of parameters $\varDelta$. We suppose, for each $t\in\varDelta$ that the restriction $\nabla_t := \nabla|_{\mathbb{D}\times\{t\}}$ to the corresponding disc has only one pole (of order k) at some point $a(t)\in\mathbb{D}$ and is formally equivalent to a generic diagonal connection $d_{\mathbb{D}} - A^0(t)$ depending holomorphically on t. Assume the divisor $\varDelta_0 := \{(a(t),t)\}\subset\mathbb{D}\times\varDelta$ is smooth. Let $z_0\colon \mathbb{D}\times\varDelta\to\mathbb{C}$ be any holomorphic function vanishing on $\varDelta_0$ which restricts to a coordinate on $\mathbb{D}\times\{t\}$ for each $t\in\varDelta$ (only the k-jet of the Taylor expansion of z_0 along $\varDelta_0$ will be significant below). Write $A^0 = d_{\mathbb{D}}Q + \Lambda^0(t)\, d_{\mathbb{D}}z_0/z_0$, as usual and define the "standard full connection" to be $d-\widetilde{A}^0$ where $\widetilde{A}^0 := dQ + \Lambda^0(t)\, d(z_0)/z_0$ and d denotes the full exterior derivative on $\mathbb{D}\times\varDelta$.

If we choose a compatible framing g_0 of ∇ along $\varDelta_0$ then, as in Proposition 6.1, there is a unique family of formal isomorphisms $\hat{g}\in GL_n(\mathbb{C}[\![z_0]\!]\otimes\mathcal{O}(\varDelta_0))$ satisfying $\hat{g}|_{\varDelta_0} = g_0$ and $\hat{g}_t[\nabla_t] = d_{\mathbb{D}} - A^0$ for each fixed t (after possibly permuting the entries of A^0). The basic structural result is then:

Lemma A.1 (see [46]). *If ∇ is* flat *then Λ^0 is constant and there is a diagonal matrix valued holomorphic function $F\in \mathrm{End}_n(\mathcal{O}(\varDelta_0))$ (which is unique upto the addition of a constant diagonal matrix) such that*

$$\hat{g}[\nabla]_{\mathbb{D}\times\varDelta} = d - (\widetilde{A}^0 + \pi^*(d_{\varDelta_0}F)),$$

where $\pi\colon \mathbb{D}\times\varDelta\to\varDelta_0$ is the projection along the $\mathbb{D}$ direction.

Proof. Let $d_{\varDelta_0} - B$ be the $\varDelta_0$ component of the Laurent expansion of $\hat{g}[\nabla]_{\mathbb{D}\times\varDelta}$ so that $\hat{g}[\nabla]_{\mathbb{D}\times\varDelta} = d_{\mathbb{D}\times\varDelta} - (A^0 + B)$. This is flat because ∇ is. The $(\mathbb{D}$-$\varDelta_0)$ part of the equation for this flatness is:

$$d_{\mathbb{D}}B + d_{\varDelta_0}A^0 = A^0\wedge B + B\wedge A^0. \tag{A.1}$$

Since A^0 is diagonal this equation splits into two independent pieces, the diagonal part and the off-diagonal part. First we deduce that the off-diagonal part B^{od} of B is zero: Suppose $B^{\mathrm{od}}\neq 0$ and let M/z_0^r be its leading term, $M\in\mathrm{End}_n^{\mathrm{od}}(\Omega^1_{\mathrm{hol}}(\varDelta_0))$. Equation (A.1) implies $d_{\mathbb{D}}B^{\mathrm{od}} = A^0\wedge B^{\mathrm{od}} + B^{\mathrm{od}}\wedge A^0$. Counting the pole orders we deduce $B^{\mathrm{od}}=0$ unless $k=1$. If $k=1$, say $A^0 = A_1^0 dz_0/z_0$, then considering coefficients of dz_0/z_0^{r+1} we see $(-r)\,M = [A_1^0, M]$ which implies $M=0$ (and therefore $B^{\mathrm{od}}=0$) since A^0 is generic; the difference between any two eigenvalues of A_1^0 is never the integer $-r$. Thus B is diagonal, and so (A.1) now reads $d_{\mathbb{D}}B+d_{\Delta_0}A^0=0$. This implies $d_{\Delta_0}A^0(t)=0$ since $d_{\mathbb{D}}B$ will have no residue term, and so $\widetilde{A}^0$ is flat. Thus $d_{\mathbb{D}}B=-d_{\Delta_0}\widetilde{A}^0=d_{\mathbb{D}}\widetilde{A}^0$. Hence $B=\widetilde{A}^0_{\Delta_0}+\phi(t)$ for some diagonal matrix of one-forms $\phi\in \mathrm{End}_n(\Omega^1_{\mathrm{hol}}(\Delta_0))$ where $\widetilde{A}^0_{\Delta_0}$ is the Δ_0 component of $\widetilde{A}^0$. Finally the (Δ_0-Δ_0) part of the equation for the flatness of $d-A^0-B$ implies $d_{\Delta_0}B=0$. It follows that $d_{\Delta_0}(\phi(t))=0$ and so, since Δ_0 is contractible, $\phi=d_{\Delta_0}F$ for some diagonal $F\in\mathrm{End}_n(\mathcal{O}(\Delta_0))$ ∎

This leads us to make the following:

Definition A.1. If ∇ is flat then a compatible framing g_0 of ∇ along Δ_0 is *good* if $\hat{g}[\nabla]_{\mathbb{D}\times\Delta}=d-\widetilde{A}^0$ where $\hat{g}$ is the formal series associated to g_0.

Thus an arbitrary compatible framing g_0 can be made good by replacing it by $e^{-F}g_0$ where F is from Lemma A.1. It is worth saying the same thing slightly differently. In the convention we are using, the columns of the inverse g_0^{-1} of the compatible framing are a basis of sections of $V|_{\Delta_0}$, where V is the bundle that ∇ is on. Thus, since good compatible framings are determined upto a constant, there is a flat holomorphic connection ∇_0 on $V|_{\Delta_0}$ whose horizontal sections are the columns of g_0^{-1} for any good compatible framing g_0. A direct definition is:

Definition A.2. If g_0 is any compatible framing of ∇ along Δ_0 then the *induced connection along* Δ_0 is $\nabla_0=(\nabla+\hat{g}^{-1}\cdot\widetilde{A}^0\cdot\hat{g})|_{\Delta_0}$, where $\hat{g}$ is the formal series associated to g_0.

It is easy to check this definition is independent of the choice of compatible framing and, if g_0 is good, then the columns of g_0^{-1} are horizontal. Moreover this definition makes sense for non-flat ∇, but then ∇_0 may not be flat. One may also check that ∇_0 only depends on ∇ and the choice of k-jets of coordinates z_0. (Also in the logarithmic case $k=1$, ∇_0 coincides with the usual (canonical) notion of induced connection $\nabla|_{\Delta_0}$, provided z_0 satisfies $(dz_0/z_0)|_{\Delta_0}=0$.) Thus one can alternatively define good framings to be the compatible framings g_0 such that the columns of g_0^{-1} are horizontal for ∇_0. The reason for restricting how the framings vary along Δ_0 is the following:

Lemma A.2 (see [40] Theorem 3.3). *Let ∇ be a full flat connection as above and let g_0 be a good compatible framing with corresponding formal series $\hat{g}$. Fix any point $t_0\in\Delta$, choose a labelling of the sectors between the anti-Stokes directions at $a(t_0)\in\mathbb{D}\times\{t_0\}$, and choose $\log(z_0)$ branches on $\mathbb{D}\times\{t_0\}$. Let Δ' be a neighbourhood of $t_0\in\Delta$ such that the last sector at $a(t_0)$ deforms into a unique sector at $a(t)$ for all $t\in\Delta'$ (the last sector at $a(t)$).*

Then the canonical fundamental solution $\Phi_0 := \Sigma_0(\hat{g}^{-1})\, z_0^{A^0_0} e^{Q}$ *of* $\nabla|_{\mathrm{Vert}}$ *on the last sector at* $a(t) \in \mathbb{D}\times\{t\}$ *varies holomorphically with* $t \in \Delta'$ *and* $\Phi_0(z,t)$ *is a local fundamental solution of the original full connection* ∇. *(Similarly on the other sectors: just relabel.)*

Proof. Write $\nabla = d - \widetilde{A}$ and let Ω be the Δ component of $\widetilde{A}$ so that $\widetilde{A} = A + \Omega$. The aim is to show that $d_\Delta \Phi_0 = \Omega\Phi_0$. From the definition of $\hat{g}$ we have $A + \Omega = \hat{g}^{-1}[\widetilde{A}^0]_{\mathbb{D}\times\Delta}$ and this has Δ component $\Omega = \hat{g}^{-1}\cdot \widetilde{A}^0_\Delta \cdot \hat{g} - \hat{g}^{-1} d_\Delta \hat{g}$. Now the key observation is that the equation $d_\Delta A = -d_{\mathbb{D}}\Omega + A\wedge\Omega + \Omega\wedge A$ (from the flatness of ∇) implies that the matrix of one-forms $d_\Delta\Phi_0 - \Omega\Phi_0$ satisfies the equation $d_{\mathbb{D}}(d_\Delta\Phi_0 - \Omega\Phi_0) = A(d_\Delta\Phi_0 - \Omega\Phi_0)$ (also using the fact that $d_{\mathbb{D}}\Phi_0 = A\Phi_0$). Then if we define a matrix $K := \Phi_0^{-1}(d_\Delta\Phi_0 - \Omega\Phi_0)$ of one-forms it follows that $d_{\mathbb{D}}K = 0$ so that K is constant in the $\mathbb{D}$ direction. Then using the fact that the asymptotic expansion of Φ_0 in the last sector at $a(t)$ is $\hat{g}^{-1} z_0^{A^0_0} e^{Q}$, it follows that K has zero asymptotic expansion there. (This uses the fact that the asymptotic expansions are uniform in t to see that d_Δ commutes with the operation of taking the asymptotic expansion.) It follows immediately that $K = 0$ because K is constant in the $\mathbb{D}$ direction, and so $d_\Delta\Phi_0 = \Omega\Phi_0$. ∎

This is the main result needed to prove Theorem 7.2 as sketched. All that remains is to write down the deformation equations of Jimbo, Miwa and Ueno. Restrict the parameter space to $X_{\mathrm{JMU}} \hookrightarrow \widetilde{X}$. The bundle $\widetilde{\mathcal{M}}^*$ over X_{JMU} can be decribed explicitly: using Proposition 2.1 (and removing the G action by fixing ${}^1g_0 = 1$) it is identified as a subbundle of the trivial bundle over X_{JMU} with fibre

$$(GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_1}) \times \cdots \times (GL_n(\mathbb{C}) \times \mathfrak{g}^*_{k_m}).$$

When described in this way the bundle $\widetilde{\mathcal{M}}^* \to X_{\mathrm{JMU}}$ is identified as the "manifold of singularity data" of [40]. Now suppose we have a horizontal section of the isomonodromy connection on $\widetilde{\mathcal{M}}^*$ over some ball $\Delta \hookrightarrow X_{\mathrm{JMU}}$. From Section 2 this determines a family of meromorphic connections $d_{\mathbb{P}^1} - A$ on the trivial bundle over $\mathbb{P}^1$ and compatible framings ${}^ig_0$ (the principal parts of A lie in the $\mathfrak{g}^*_{k_i}$'s using the coordinate choices). As above we also get (algebraically) formal isomorphisms ${}^i\hat{g}$, connection germs $d_{\mathbb{P}^1} - {}^iA^0$ and "full" connection germs $d_{\mathbb{P}^1\times\Delta} - {}^i\widetilde{A}^0$, where d is the full exterior derivative on $\mathbb{P}^1\times\Delta$. (The holomorphic terms in the expansion of ${}^iA^0$ with respect to z_i are defined to be zero.)

From the sketch of the proof of Theorem 7.2, $d_{\mathbb{P}^1} - A$ is the vertical component of a full connection $\nabla = d - \widetilde{A}$, where $\widetilde{A} = (d\Phi)\,\Phi^{-1}$ for any local canonical fundamental solution $\Phi(z,t) := {}^i\Sigma_j({}^i\hat{g}^{-1})\, z_i^{{}^i\!\Lambda} e^{{}^iQ}$ on (say) the jth sector at the ith pole. These local definitions agree as the family $d_{\mathbb{P}^1} - A$ is isomonodromic. Let Ω denote the Δ component of $\widetilde{A}$, so $\widetilde{A} = A + \Omega$.

From the definition we know the asymptotics of $\Phi(z,t)$ (uniformly) on the jth (super)sector at the ith pole and so we can deduce the asymptotics of Ω:

$$Æ_i(\Omega) = ({}^i\hat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\hat{g}) - {}^i\hat{g}^{-1} \cdot d_\Delta({}^i\hat{g}). \tag{A.2}$$

A priori this only holds on some sector at the ith pole, but choosing a different Φ, we get the same expansion on every sector. It follows that Ω is meromorphic, with *Laurent* expansion (A.2). First, it follows immediately from this expression that the compatible framing ${}^ig_0$ is a good compatible framing of ∇. Secondly it is clear that the ith principal part of Ω is the principal part of ${}^i\hat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\hat{g}$ and so is determined algebraically. Also we need a formula for the induced connections ∇_i on the polar divisors of ∇. Upon pulling ∇_i down to the base Δ, from Definition A.2, one finds that ∇_i becomes $d_\Delta - \Theta_i$ where

$$\Theta_i = {}^ig_0^{-1}(d_\Delta a_i)^i\, g_1 + \mathrm{Const}_{z_i}({}^i\hat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\hat{g}) - \mathrm{Const}_{z_i}(\Omega) \tag{A.3}$$

with ${}^i\hat{g} = {}^ig_0 + {}^ig_1 \cdot z_i + O(z_i^2)$ and where Const_{z_i} takes the constant term in the Laurent expansion with respect to z_i. Since we are working in the trivialisation determined by the first framing (${}^1g_0 = 1$), we have $\Theta_1 = 0$ and so the expression (A.3) determines the constant term in the expansion of Ω at $a_1 = \infty$. Thus Ω is completely determined by this constant and the principal parts:

$$\Omega = \mathrm{Const}_{z_1}({}^1\hat{g}^{-1} \cdot {}^1\widetilde{A}^0_\Delta \cdot {}^1\hat{g}) + \sum_{i=1}^{m} \mathrm{PP}_{z_i}({}^i\hat{g}^{-1} \cdot {}^i\widetilde{A}^0_\Delta \cdot {}^i\hat{g}). \tag{A.4}$$

Now the flatness of the full connection ∇ over $\Delta \times \mathbb{P}^1$ implies two equations. Firstly $d_\Delta\Omega = \Omega \wedge \Omega$, which says that Ω is a family of flat connections on Δ depending rationally on the "spectral parameter" z; a situation that often arises in soliton theory. Secondly

$$d_\Delta A = -d_{\mathbb{P}^1}\Omega + A \wedge \Omega + \Omega \wedge A. \tag{A.5}$$

Also the "goodness" of the compatible framings ${}^ig_0$ implies that

$$d_\Delta({}^ig_0) = -({}^ig_0)\,\Theta_i. \tag{A.6}$$

Note that the formulae (A.3) and (A.4) for Ω and Θ_i make sense for an arbitrary section of the bundle $\widetilde{\mathcal{M}}^*$ so that the equations (A.5) and (A.6) amount to a coupled system of nonlinear *algebraic* differential equations for horizontal sections $s = (\mathbf{g}, {}^1A, \ldots, {}^mA)$ of the isomonodromy connection on $\widetilde{\mathscr{M}}^*$ over X_{JMU}: They are the Jimbo–Miwa–Ueno isomonodromic deformation equations [40].

A number of examples are given in [38, 40] and in particular the cases of the Schlesinger equations and the six Painlevé equations are explained.

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REFERENCES

1. M. J. Ablowitz and P. A. Clarkson, "Solitons, Nonlinear Evolution Equations and Inverse Scattering," Cambridge Univ. Press, Cambridge, UK, 1991.
2. M. R. Adams, J. Harnad, and J. Hurtubise, Isospectral Hamiltonian flows in finite and infinite dimensions, II. Integration of the flows, *Comm. Math. Phys.* **134** (1990), 555–585.
3. A. Alekseev, A. Malkin, and E. Meinrenken, Lie group valued moment maps, *J. Differential Geom.* **48** (1998), 445–495, dg-ga/9707021.
4. J. E. Andersen, J. Mattes, and N. Reshetikhin, The Poisson structure on the moduli space of flat connections and chord diagrams, *Topology* **35** (1996), 1069–1083.
5. M. F. Atiyah, "The Geometry and Physics of Knots," Cambridge Univ. Press, Cambridge, UK, 1990.
6. M. F. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, *Phil. Trans. Roy. Soc. London* **308** (1982), 523–615.
7. M. Audin, Lectures on integrable systems and gauge theory, *in* "Gauge Theory and Sympletic Geoemtry" (J. Hurtubise and F. Lalonde, Eds.), NATO ASI Series C: Math & Phys., Kluwer, Dordrecht/Boston/London, 1995.
8. D. G. Babbitt and V. S. Varadarajan, Formal reduction theory of meromorphic differential equations: a group theoretic view, *Pacific J. Math.* **109** (1983), 1–80.
9. D. G. Babbitt and V. S. Varadarajan, Local moduli for meromorphic differential equations, *Astérisque* **169–170** (1989), 1–217.
10. W. Balser, B. J. L. Braaksma, J. P. Ramis, and Y. Sibuya, Multisummability of formal power series solutions of linear ordinary differential equations, *Asymptotic Anal.* **2** (1991), 27–45.
11. W. Balser, W. B. Jurkat, and D. A. Lutz, Birkhoff in variants and Stokes' multipliers for meromorphic linear differential equations, *J. Math. Anal. Appl.* **71** (1979), 48–94.
12. A. Beauville, Jacobiennes des courbes spectrales et systèmes Hamiltoniens complètement intégrables, *Acta Math.* **164** (1990), 211–235.

13. I. Biswas and K. Guruprasad, Principal bundles on open surfaces and in variant functions on Lie groups, *Internat. J. Math.* **4** (1993), 535–544.
14. P. P. Boalch, “Symplectic Geometry and Isomonodromic Deformations,” D.Phil. thesis, Oxford University, 1999.
15. A. Borel, “Linear Algebraic Groups,” Springer-Verlag, New York, second ed., 1991.
16. R. Bott and L. W. Tu, “Differential Forms in Algebraic Topology,” Graduate Texts in Mathematics, Vol. 82, Springer-Verlag, Berlin/New York, second ed., 1995.
17. L. Boutet de Monvel, A. Douady, and J.-L. Verdier (Eds.),“Séminaire E.N.S. Mathématique et Physique,” Progress in Math., Vol. 37, Birkhäuser, Boston, 1983.
18. R. Donagi and E. Markman, Spectral covers, algebraically integrable Hamiltonian systems and moduli of bundles, *in* “Integrable Systems and Quantum Groups” (M. Francaviglia and S. Greco, Eds.), Lecture Notes in Mathematics, Vol. 1620, pp. 1–119, Springer-Verlag, Berlin, 1995.
19. B. Dubrovin, Geometry of 2D topological field theories, *in* “Integrable Systems and Quantum Groups” (M. Francaviglia and S. Greco, Eds.), Lecture Notes in Mathematics, Vol. 1620, pp. 120–348, Springer-Verlag, Berlin, 1995.
20. L. D. Faddeev and L. A. Takhtajan, “Hamiltonian Methods in the Theory of Solitons,” Springer-Verlag, Berlin, 1987.
21. V. V. Fock and A. A. Rosly, Flat connections and Polyubles, *Teoret. Mat. Fiz.* **95** (1993), 228–238.
22. W. M. Goldman, The symplectic nature of fundamental groups of surfaces, *Adv. in Math.* **54** (1984) 200–225.
23. P. Griffiths and J. Harris, “Principles of Algebraic Geometry,” Wiley Interscience, New York, 1994.
24. V. Guillemin and S. Sternberg, “Symplectic Techniques in Physics,” Cambridge Univ. Press, Cambridge, UK, 1984.
25. K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, Group systems, groupoids and moduli spaces of parabolic bundles, *Duke Math. J.* **89** (1997), 377–412, dg-ga/9510006.
26. R. S. Hamilton, The inverse function theorem of Nash and Moser, *Bull. Amer. Math. Soc. (New Ser.)* **7** (1982), 65–222.
27. J. Harnad, Quantum isomonodromic deformations and the Knizhnik–Zamolodchikov equations, *in* “Symmetries and Integrability of Difference Equations Estérel, PQ, 1994,” pp. 155–161, Amer. Math. Soc., Providence, RI, 1996.
28. J. Harnad and M. A. Wisse, Loop algebra moment maps and Hamiltonian models for the Painlevé transcendants, *Fields Inst. Commun.* **7** (1993), 155–169, hep-th/9305027.
29. N. J. Hitchin, Frobenius manifolds (notes by D. Calderbank), *in* “Gauge Theory and Sympletic Geometry” (J. Hurtubise and F. Lalonde, Eds.), NATO ASI Series C: Maths and Phys., Vol. 488, Kluwer, Dordrecht/Boston/London, 1995.
30. N. J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, *J. Differential Geom.* **42** (1995), 30–112.
31. L. Hörmander, “The Analysis of Linear Partial Differential Operators. I,” Springer-Verlag, Berlin, 1983.
32. P. Hsieh and Y. Sibuya, Note on regular perturbations of linear ordinary differential equations at irregular singular points, *Funkcial. Ekvac.* **8** (1966), 99–108.
33. J. Hurtubise and F. Lalonde (Eds.), “Gauge Theory and Symplectic Geometry,” NATO ASI Series C: Maths Phys., Vol. 488, Kluwer, Dordrecht/Boston/London, 1995.
34. K. Iwasaki, Moduli and deformation for Fuchsian projective connections on a Riemann surface, *J . Fac. Sci. Univ. Tokyo, Sect. 1A, Math.* **38** (1991), 431–531.
35. K. Iwasaki, Fuchsian moduli on a Riemann surface - its Poisson structure and Poincaré Lefschetz duality, *Pacific J. Math.* **155** (1992), 319–340.

36. K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida, "From Gauss to Painlevé: A Modern Theory of Special Functions," Aspects of Mathematics, Vieweg, Braunschweig, 1991.
37. L. Jeffrey, Extended moduli spaces of flat connections on Riemann surfaces, *Math. Ann.* **298** (1994), 667–692.
38. M. Jimbo and T. Miwa, Monodromy preserving deformations of linear differential equations with rational coefficients II, *Physica 2D* (1981), 407–448.
39. M. Jimbo, T. Miwa, Y. Mori, and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendant, *Physica 1D* (1980), 80–158.
40. M. Jimbo, T. Miwa, and Kimio Ueno, Monodromy preserving deformations of linear differential equations with rational coefficients I, *Physica 2D* (1981), 306–352.
41. Y. Karshon, An algebraic proof for the symplectic structure of moduli space, *Proc. Amer. Math. Soc.* **116** (1992), 591–605.
42. S. Kobayashi, "Differential Geometry of Complex Vector Bundles," Kanô Memorial Lectures, Vol. 5, Iwanami Shoten/Princeton University Press, Tokyo/Princeton, 1987.
43. M. Loday-Richaud, Stokes phenomenon, multisummability and differential Galois groups, *Ann. Inst. Fourier* **44** (1994), 849–906.
44. B. Malgrange, "Ideals of Differentiable Functions," TATA Institute of fundamental Research, Bombay/Oxford University Press, Oxford, 1966.
45. B. Malgrange, Sur les déformations isomonodromiques, I, singularités regulières, *in* "Seminaire E.N.S. Mathématique et Physique" (L. Boutet de Monvel, A. Douady, and J.-L. Verdier, Eds.), Progress in Math., Vol. 37, pp. 401–426, Birkhäuser, Boston, 1983.
46. B. Malgrange, Sur les déformations isomonodromiques, II, singularités irregulières, *in* "Seminaire E.N.S. Mathématique et Physique" (L. Boutet de Monvel, A. Douady, and J.-L. Verdier, Eds.), Progress in Math., Vol. 37, pp. 427–438, Birkhäuser, Boston, 1983.
47. B. Malgrange, Sommation des séries divergentes, *Exposition. Math.* **13** (1995), 163–222.
48. B. Malgrange and J.-P. Ramis, Fonctions multisommables, *Ann. Inst. Fourier (Grenoble)* **42** (1992), 353–368.
49. J. Martinet, "Singularities of Smooth Functions and Maps," Cambridge Univ. Press, Cambridge, UK, 1982.
50. J. Martinet and J. P. Ramis, Elementary acceleration and multisummability, *Ann. inst. H. Poincaré, Phys. Théorique* **54** (1991), 331–401.
51. L. J. Mason and N. M. J. Woodhouse, "Integrability, Self-Duality and Twistor Theory," Oxford Univ. Press, Oxford, 1996.
52. B. M. McCoy, C. A. Tracy, and T. T. Wu, Painlevé functions of the third kind, *J. Math. Phys.* **18** (1977), 1058–1092.
53. M. Francaviglia and S. Greco, Eds., "Integrable Systems and Quantum Groups," Lecture Notes in Mathematics, Vol. 1620, Springer, Berlin, 1995.
54. J. Milnor, Remarks on infinite-dimensional Lie groups, *in* "Relativity, Groups and Topology, II Les Houches, 1983," pp. 1007–1057, North-Holland, Amsterdam, 1984.
55. T. Miwa, Painlevé property of monodromy preserving deformation equations and the analyticity of τ functions, *Publ. RIMS, Kyoto Univ.* **17** (1981), 703–721.
56. M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–567.
57. K. Okamoto, On the τ function of the Painlevé equations, *Physica 2D* (1981), 525–535.
58. R. M. Range, "Holomorphic Functions and Integral Representations in Several Complex Variables," Graduate Texts in Mathematics, Vol. 108, Springer-Verlag, Berlin, 1986.
59. N. Reshetikhin, The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem, *Lett. Math. Phys.* **26** (1992), 167–177.
60. M. Sato, T. Miwa, and M. Jimbo, Holonomic quantum fields I-V, *Publ. RIMS Kyoto Univ.* **14** (1978), 223–267; **15** (1979), 201–278, 577–629, 871–972; **16** (1980) 531–584.

61. L. Schlesinger, Über eine Klasse von Differentialsystemen beliebiger Ordnung mit feten kritischen Punkten, *J. Math.* **141** (1912), 96–145.
62. Y. Sibuya, Simplification of a system of linear ordinary differential equations about a singular point, *Funkcial. Ekvac.* **4** (1962), 29–56.
63. Y. Sibuya, Perturbation of linear ordinary differential equations at irregular singular points, *Funkcial. Ekvac.* **11** (1968), 235–246.
64. C. T . Simpson, Moduli of representations of the fundamental group of a smooth projective variety, I, II, *Publ. Math. I.H.E.S.* **79** (1994), 47–129; **80** (1994), 5–79.
65. K. P. Tod, Self-dual Einstein metrics from the Painlevé VI equation, *Phys. Lett. A* **190** (1994), 221–224.
66. F. Treves, "Topological Vector Spaces, Distributions and Kernels," Pure and Applied Mathematics Monographs, Vol. 25, Acadamic Press, New York, London, 1967.
67. M. Ugaglia, On a Poisson structure on the space of Stokes matrices, *Internat. Math. Res. Notices* **9** (1999), 473–493, math.AG/9902045.
68. H. Umemura, Irreducibility of Painlevé transcendental functions, *Sugaku Expositions* **2** (1989), 231–252.
69. H. Umemura, Painlevé equations and classical functions, *Sugaku Expositions* **11** (1998), 77–100.
70. V. S. Varadarajan, Linear meromorphic differential equations: a modern point of view, *Bull. AMS* **33** (1996), 1–42.
71. M. Vergne, La structure de Poisson sur l'algèbre symétrique d'une algèbre de Lie nilpotente, *Bull. Soc. Math. France* **100** (1972), 301–335.
72. W. Wasow, "Asymptotic Expansions for Ordinary Differential Equations," Wiley Interscience, New York, 1976.
73. T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region, *Phys. Rev. B* **13** (1976), 316–374.