# On relative trace formulae: the case of Jacquet-Rallis

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**Abstract**

We give an account of recent works on Jacquet-Rallis' approach to the Gan-Gross-Prasad conjecture for unitary groups. We report on the present state of the Jacquet-Rallis relative trace formulae and on some current applications of it. We give also a precise computation of the constant that appears in the statement "Fourier transform and transfer commute up to a constant".

## Contents



# 1 Introduction

## 1.1 The philosophy of the Jacquet-Rallis trace formula

In the emerging relative Langlands program, the main concern is the study of periods of automorphic forms (namely some integrals over subgroups). These objects should have deep relations with special values of $L$-functions and Langlands functorialities (see [SV12]). In these questions, a relative trace formula should play a central role connecting periods to (relative) orbital integrals as advocated by Jacquet (see [Jac97]).

Gan, Gross and Prasad made a series of precise conjectures about the link between the non-vanishing of certain periods and the non-vanishing of special values of $L$-functions (see [GGP12]). Refining the Gan-Gross-Prasad conjecture, Ichino and Ikeda (cf. [II10]) were able to give a conjectural Eulerian factorization of the square modulus of periods of orthogonal groups. Their conjecture extends to the case of unitary groups ([Har14]). In this survey, we will focus on this case. In a seminal paper [JR11], Jacquet-Rallis suggested that the Gan-Gross-Prasad conjecture for unitary groups should follow from three ingredients:

- the classical work of Jacquet-Piatetskii-Shapiro-Shalika on integral representations of $L$-functions of pairs for $GL(n)$ (cf. [JPSS83]);
- two relative trace formulae (one for unitary groups and the other for linear groups) that express periods in terms of relative orbital integrals ;
- a comparison of relating orbital integrals on unitary groups and linear groups.

The idea is roughly that periods and orbital integrals should be dual objects. The comparison of orbital integrals should be dual to a comparison of periods and so it should be possible to transfer part of Jacquet-Piatetskii-Shapiro-Shalika's results to unitary groups. In particular, one has to explore harmonic analysis in this context (density of orbital integrals, transfer of orbital integrals, fundamental lemma etc.). This is an exciting and beautiful program which has seen a lot of progress in last years. An important part of this program has been achieved by Wei Zhang ([Zha14a], [Zha14b]) who used a simple trace formula.

### 1.2 Contents of the paper

It was my intention to review Zhang's work on the subject and also subsequent works ([Xue15], [Beu16], [Zyd16]. [Zyd], [Zyd15] and [CZ]). However, since some parts of the paper are perhaps more detailed than in an usual survey, an explanation is due to the reader. When preparing this paper, I found a (minor) inaccuracy of sign in a local constant that appears in Zhang's statement that "transfer and Fourier transform commute up to a constant" (cf. theorem 3.4.2.1). By a product formula, the signs disappear globally and so this inaccuracy has no global consequences. However, some local statements in the literature certainly depend on this constant. So, I decided to offer the reader the detailed computation of the correct constant. I hope this is of some interest. After the completion of this work, Hang Xue kindly informed me that he also got the correct constant in the archimedean case (see [Xue15]).

Let me now present the structure of the paper. In section 2, we take some time to gather statements and proofs of some basic algebraic results. The section 3 is the technical bulk of the papers. We present Zhang's theorem of the existence of local transfer and the computation of the previously mentioned constant.

In section 4, we present the work of Zydor on the infinitesimal relative trace formula which plays an important technical role. We also explain our recent joint work with Zydor on comparisons of infinitesimal relative trace formulae.

In section 5, we give an overview on Zydor's approach to the relative trace formula. I give some hints about techniques and the current applications of the relative trace formula (including works of Zhang, Hue, Beuzart-Plessis) to the global Gan-Gross-Prasad conjecture.

### 1.3 Acknowledgement

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# 2 Algebraic preliminaries

## 2.1 Linear situation

**2.1.1.** Let $F$ be a field of characteristic 0. Let $n \geqslant 1$ be a an integer and let $V$ be a $F$-vector space of dimension $n$. Let us define

$$\widetilde{\mathfrak{gl}}_F(V) = \mathfrak{gl}_F(V) \oplus V \oplus V^*$$

where $V^*$ is the dual vector space of $V$ and $\mathfrak{gl}_F(V)$ is the space of $F$-endomorphisms of $V$. The group $GL_F(V)$ of $F$-automorphisms of $V$ acts on the left on $\widetilde{\mathfrak{gl}}_F(V)$ in the following way: for any $X = (A, b, c) \in \widetilde{\mathfrak{gl}}_F(V)$ and $g \in GL_F(V)$, one has

$$g \cdot X = (gAg^{-1}, gb, cg^{-1}).$$

**2.1.2.** Let $G = GL_F(V)$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{gl}}_F(V)$. Let

$$\mathcal{A} = \mathcal{A}_V = \widetilde{\mathfrak{g}}//G$$

be the categorical quotient.

In our situation, $\mathcal{A}$ is the affine scheme given by the spectrum

$$\mathcal{A} = \mathrm{Spec}(F[\widetilde{\mathfrak{g}}]^G)$$

of the sub-algebra $F[\widetilde{\mathfrak{g}}]^G$ of invariant functions of the ring $F[\widetilde{\mathfrak{g}}]$ of polynomial functions. Let

$$a : \widetilde{\mathfrak{g}} \to \mathcal{A}. \tag{2.1.2.1}$$

be the canonical morphism dual to the inclusion $F[\widetilde{\mathfrak{g}}]^G \subset F[\widetilde{\mathfrak{g}}]$.

**2.1.3.** One often denotes by $a$ a point of $\mathcal{A}$. Then $\widetilde{\mathfrak{g}}_a$ denotes the fiber of (2.1.2.1) above $a$.

**2.1.4. Regular and semi-simple elements.** — Since we will use these notions in different contexts, it is useful to have a general definition.

**Definition 2.1.4.1.** — Let $G$ be an algebraic group acting on a variety $X$.

- The regular subset $X^{\mathrm{reg}}$ is the open subset of $X$ where the dimension of the centralizer is minimal.
- An element of $X$ is semi-simple if its $G$-orbit is closed.
- We denote by $X^{\mathrm{rss}}$ the set of regular and semi-simple elements of $X$.

We will use these notions and notations in the context of §§ 2.1.1 and 2.1.2. One has the following characterization of semi-simple elements due to Rallis-Schiffmann (cf. [RS08] theorem 6.2).

**Proposition 2.1.4.2.** — *Let $X = (A, b, c)$. Then $X$ is semi-simple if and only if the two conditions are satisfied :*

1. *one has*
$$V = \mathrm{vect}(b, Ab, A^2b, \ldots) \oplus V'$$
*where $V'$ is the orthogonal of* $\mathrm{vect}(c, cA, cA^2, \ldots)$.

*2. the endomorphism of $V'$ induced by $A$ is semi-simple (in the usual sense).*

Let $X = (A, b, c) \in \widetilde{\mathfrak{g}}$. It is easy to choose $X$ such that its centralizer is trivial. In general, its centralizer is always connected (it is an open subset of an affine space and thus it is irreducible). As a consequence, $X$ belongs to the regular locus $\widetilde{\mathfrak{g}}^{\mathrm{reg}}$ if and only if its centralizer is trivial.

Let

$$d_n(X) = \det((cA^{i+j}b)_{0\leqslant i,j\leqslant n-1}).$$

This defines an element in $F[\widetilde{\mathfrak{g}}]^G$.

**Lemma 2.1.4.3**. — *The condition on $X = (A, b, c)$ are equivalent:*

1. *$d_n(X) \neq 0$;*
2. *$(b, Ab, \ldots, A^{n-1}b)$ is a basis of $V$ and $(c, cA, \ldots, cA^{n-1})$ is a basis of $V^*$;*
3. *$X$ is semi-simple and its centralizer is trivial;*
4. *$X$ is semi-simple and regular.*

**Proof.** — The equivalence of 1 and 2 is easy. The equivalence of 2 and 3 follows from proposition 2.1.4.2 . The equivalence of 3 and 4 has already been noticed. □

One then defines the *regular semi-simple* open subsets $\mathcal{A}^{\mathrm{rss}} \subset \mathcal{A}$ and $\widetilde{\mathfrak{g}}^{\mathrm{rss}} \subset \widetilde{\mathfrak{g}}$ by the condition $d_n \neq 0$. Using lemma 2.1.4.3, one sees that they are not empty.

**2.1.5. The ring of invariants.** — For any $X = (A, b, c) \in \widetilde{\mathfrak{g}}$, one defines the elements of $F[\widetilde{\mathfrak{g}}]^G$:

1. $a_i(X) = (-1)^{i-1}\,\mathrm{trace}(\wedge^i A)$ for $1 \leqslant i \leqslant n$;
2. $a_{i+n}(X) = cA^{i-1}b$ for $1 \leqslant i \leqslant n$.

**Lemma 2.1.5.1**. — *Two elements $X$ and $Y$ in $\widetilde{\mathfrak{g}}^{\mathrm{rss}}$ are $G$-conjugate if and only if*

$$a_i(X) = a_i(Y)$$

*for any $1 \leqslant i \leqslant 2n$.*

**Proof.** — Let's write $X = (A, b, c)$ and $Y = (A_1, b_1, c_1)$. Since these elements are regular semi-simple, by lemma 2.1.4.3, there is a unique element $g \in G$ such that $gA^ib = A_1^i b_1$ for $0 \leqslant i \leqslant n-1$. Assume that $a_i(X) = a_i(Y)$ for any $1 \leqslant i \leqslant 2n$. Then $A$ and $A_1$ have the same characteristic polynomial and thus the relation $gA^ib = A_1^ib_1$ is true for any $i \geqslant 0$. Thus one has

$$gAg^{-1}A_1^ib_1 = gAg^{-1}gA^ib = gA^{i+1}b = A_1^{i+1}b_1 = A_1(A_1^ib_1).$$

Once again by lemma 2.1.4.3, one deduces that $gAg^{-1} = A_1$. In the same way, the relation for $0 \leqslant i \leqslant n-1$

$$c_1gA^ib = c_1A_1^ib_1 = a_{i+n+1}(Y) = a_{i+n+1}(X) = cA^ib$$

implies $c_1g = c$. We have proved that $X$ and $Y$ are conjugate by $g$. The converse is obvious. □

Let's fix a basis $e$ of $V$ and the dual basis $e^*$ of $V^*$. For any $(a_i)_{1\leqslant i\leqslant 2n} \in F^{2n}$, the triple

$$X((a_i)_{1\leqslant i\leqslant 2n}) = (A, b, c) \tag{2.1.5.2}$$

defined in the basis $e$ and $e^*$ by the matrices

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ 1 & 0 & \cdots & 0 & a_{n-1} \\ 0 & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} a_{n+1} \\ \vdots \\ a_{2n} \end{pmatrix}$$

satisfies $A_i(X) = a_i$ for $1 \leqslant i \leqslant 2n$.

The map $(a_i)_{1 \leqslant i \leqslant 2n} \mapsto X((a_i)_{1 \leqslant i \leqslant 2n})$ defines an isomorphism from the affine space $\mathbb{A}^{2n}$ to a closed subset of $\widetilde{\mathfrak{g}}$, denoted by $\mathfrak{c}$. It is easy to see that

$$\mathfrak{c} \subset \widetilde{\mathfrak{g}}^{\mathrm{reg}}.$$

**Proposition 2.1.5.2**. — *The restriction to $\mathfrak{c}$ of the canonical morphism $a : \widetilde{\mathfrak{g}} \to \mathcal{A}$ induces an isomorphism from $\mathfrak{c}$ to $\mathcal{A}$. In particular, the algebra $F[\widetilde{\mathfrak{g}}]^G$ is the polynomial algebra in the $2n$ generators $a_i(X)$ for $1 \leqslant i \leqslant n$.*

**Proof.** — The dual morphism $F[\widetilde{\mathfrak{g}}]^G \to F[(t_i)_{1 \leqslant i \leqslant 2n}]$ is given by $P \mapsto P(X((t_i)_{1 \leqslant i \leqslant 2n}))$. Because $a_i(X((t_i)_{1 \leqslant i \leqslant 2n}) = t_i$ this morphism is surjective. Any $P$ in the kernel is 0 because it vanishes on the open dense subset $\widetilde{\mathfrak{g}}^{\mathrm{rss}}$ by lemma 2.1.5.1. Thus it is an isomorphism. □

Let's record a consequence of the proposition.

**Lemma 2.1.5.3**. — *For any $a \in \mathcal{A}(F)$, there exists a triple $X = (A, b, c) \in \widetilde{\mathfrak{g}}(F)$ such that $a(X) = a$.*

**2.1.6. A morphism.** — Let us fix a basis $e$ of $V$. Let $X = (A, b, c) \in \widetilde{\mathfrak{g}}^{\mathrm{rss}}$. Let us denote by $\delta_X$ the matrix of the vectors $(b, Ab, \ldots, A^{n-1}b)$ in the basis $e$. One can view $\delta_X$ as an element of $G$. Note that for $g \in G$ we have

$$\delta_{g \cdot X} = g\delta_X.$$

Thus $X \mapsto \delta_X$ is an $G$-equivariant morphism from $\widetilde{\mathfrak{g}}^{\mathrm{rss}}$ to $G$ (with the left action of $G$ on itself).

## 2.2 Hermitian situation

**2.2.1.** Let $n \geqslant 1$ be an integer. Let $E/F$ be a quadratic extension of fields of characteristic 0. Let $\sigma$ be the generator of the Galois group $\mathrm{Gal}(E/F)$. Let $(V, \Phi)$ be a pair of an $E$-vector space of dimension $n$ and $\Phi$ a non-degenerate $\sigma$-hermitian form on $V$. Let

$$U = U_\Phi = U(V, \Phi)$$

be the unitary group: it is the $F$-algebraic group of automorphisms of $(V, \Phi)$. For any $A \in \mathfrak{gl}_E(V)$, let $A^*$ be its adjoint for the form $\Phi$. Let

$$\widetilde{\mathfrak{u}} = \widetilde{\mathfrak{u}}_\Phi = \widetilde{\mathfrak{u}}(V, \Phi)$$

be the $F$-vector space of pairs $(A, b)$ with $b \in V$ and $A$ a self-adjoint endomorphism of $V$ (i.e. $A = A^*$). When the situation is clear, one omits the pair $(V, \Phi)$ in the notations. We have a left action of $U$ on $\widetilde{\mathfrak{u}}$ defined by

$$g \cdot (A, b) = (gAg^{-1}, gb)$$

for any $g \in U$ and $(A, b) \in \widetilde{\mathfrak{u}}$.

**2.2.2.** There is another useful way to look at the action of $U$ on $\widetilde{\mathfrak{u}}$. With the notations of §2.1.1, one has the action of $GL_E(V)$ on $\widetilde{\mathfrak{gl}}_E(V)$. Any element $X \in \widetilde{\mathfrak{gl}}_E(V)$ can be written $X = (A, b, \Phi(c, \cdot))$ with $b, c \in V$.

We define involutions $\theta$ on $GL_E(V)$ and $\widetilde{\mathfrak{gl}}_E$ respectively by

$$\theta(g) = (g^*)^{-1}$$

and

$$\theta(X) = (A^*, c, \Phi(b, \cdot)).$$

One has the relation

$$\theta(g \cdot X) = \theta(g) \cdot \theta(X).$$

Using the map $(A, b) \in \widetilde{\mathfrak{u}} \mapsto (A, b, \Phi(b, \cdot))$, one gets that $U$ and $\widetilde{\mathfrak{u}}$ are the $\theta$-fixed points of $GL_E(V)$ and $\widetilde{\mathfrak{gl}}_E$. Remember the set $(a_i)_{1 \leqslant i \leqslant 2n}$ of generators of $E[\widetilde{\mathfrak{gl}}_E(V)]^{GL_E(V)}$ (cf. § 2.1.5). One checks that for any $X \in \widetilde{\mathfrak{gl}}_E(V)$, one has

$$\sigma(a_i(\theta(X))) = a_i(X).$$

However, assuming that $V$ has an $F$-structure $V_0$, one has another involution, the Galois involution, on $\widetilde{\mathfrak{gl}}_E(V)$ given by

$$\theta_0(g) = g^\sigma$$

and

$$\theta((A, b, c)) = (A^\sigma, b^\sigma, c^\sigma)$$

for which the fixed-point sets are respectively $GL_F(V_0)$ and $\widetilde{\mathfrak{gl}}_F(V_0)$. One also has $\sigma(a_i(\theta_0(X))) = a_i(X)$.

This implies that the categorical quotients $\widetilde{\mathfrak{u}}//U = \mathrm{Spec}(F[\widetilde{\mathfrak{u}}]^U)$ and $\mathcal{A}_{V_0} = \widetilde{\mathfrak{g}}_F(V_0)//GL_F(V_0)$ which are canonically isomorphic over $E$ are in fact isomorphic over $F$. Using proposition 2.1.5.2, one gets the following proposition.

**Proposition 2.2.2.1.** — *The categorical quotient $\widetilde{\mathfrak{u}}//U$ is isomorphic to the space $\mathcal{A}$ of §2.1.2. Moreover the algebra of invariant functions $F[\widetilde{\mathfrak{u}}]^U$ is the polynomial in the $2n$ generators :*

- $a_i((A, b)) = (-1)^{i-1}\,\mathrm{trace}(\wedge^i A)$ *for* $1 \leqslant i \leqslant n$;
- $a_{i+n}((A, b)) = \Phi(b, A^{i-1}b)$ *for* $1 \leqslant i \leqslant n$.

**2.2.3.** The notions of a regular element or semi-simple are geometric and thus make sense for $\widetilde{\mathfrak{u}}$. Note that $\widetilde{\mathfrak{u}}^{\mathrm{rss}}$ is therefore the inverse image of $\mathcal{A}^{\mathrm{rss}}$.

**2.2.4.** Unlike the linear situation (cf. lemme 2.1.5.3), the canonical map $\widetilde{\mathfrak{u}}(F) \to \mathcal{A}(F)$ is not surjective in general. However, one has the following proposition.

**Proposition 2.2.4.1.** —

1. *For any $a \in \mathcal{A}^{\mathrm{rss}}(F)$ there exists a hermitian form $\Phi$, unique up to isomorphism, such that there exists $X \in \widetilde{\mathfrak{u}}_\Phi(F)$ such that $a(X) = a$.*
2. *If $X, Y$ are two elements of $\widetilde{\mathfrak{u}}_\Phi^{\mathrm{rss}}(F)$ such that $a(X) = a(Y)$ then $X$ and $Y$ are $U(F)$-conjugate.*

**Proof.** — By lemma 2.1.5.3 there exists $X \in \widetilde{\mathfrak{gl}}_F(V_0)^{\mathrm{rss}}$ such that $a(X) = a$. One views $X$ as an element of $\widetilde{\mathfrak{gl}}_E(V)$ such that $\theta_0(X) = X$. Fix an arbitrary non-degenerate hermitian form $\Phi$ on $V$ and let $\theta$ the associated involution (as above). The elements $\theta(X) = \theta\theta_0(X)$ and $X$ have the same invariants and are regular semi-simple. Thus there exists $g \in GL_E(V)$ such that $\theta(X) = g \cdot X$ (see lemma 2.1.5.1). Thus, one has

$$X = (g^{-1}\theta(g)^{-1}) \cdot X$$

and $\theta(g) = g^{-1}$ (cf. lemma 2.1.4.3 assertion 3) i.e. $g^* = g$. In other words, $g$ is self-adjoint for $\Phi$ or $\Phi' = \Phi(g\cdot, \cdot)$ is a hermitian form. It is easy to see that $X \in \widetilde{\mathfrak{u}}_{\Phi'}(F)$.

Replacing $\Phi$ by $\Phi'$, we can assume that $X \in \widetilde{\mathfrak{u}}_\Phi(F)$. Since any hermitian form is of the form $\Phi' = \Phi(g\cdot, \cdot)$ with $g$ self-dual, if $Y \in \widetilde{\mathfrak{u}}_{\Phi'}(F)$ then

$$\theta(Y) = g \cdot Y.$$

If moreover one has $a(Y) = a$ then, as before, there exists $h \in GL_E(V)$ such that $Y = h \cdot X$. From the previous line, we deduce

$$\theta(h) \cdot X = ghX$$

i.e. $g = \theta(h)h^{-1} = (h^*)^{-1}h^{-1}$ and $\Phi' = \Phi(h^{-1}\cdot, h^{-1}\cdot)$ is isomorphic to $\Phi$.

For the second assertion, at least there exists $g \in GL_E(V)$ such that $Y = g \cdot X$. Then

$$Y = \theta(Y) = \theta(g) \cdot \theta(X) = \theta(g) \cdot X = (\theta(g)g^{-1}) \cdot Y.$$

But since $Y$ is regular, one has $g = \theta(g)$ i.e. $g \in U(F)$. □

## 2.3 Classification of hermitian forms

**2.3.1.** The situation is the same as in §2.2.1. We want to recall the classification of hermitian forms on $V$ for different fields. This is of course classical (see [Sch85]) but could nicely rephrased in terms of abelianized Galois cohomology (see [HL04]). We denote by $N$ the norm of $E/F$. To any hermitian form $\Phi$, we attach its discriminant $\mathrm{disc}(\Phi)$ which is an element of $F^\times/N(E^\times)$.

**2.3.2. Archimedean local fields.** — Here the extension $E/F$ is the extension $\mathbb{C}/\mathbb{R}$. The equivalence classes of hermitian forms are parametrized by the signature $(p, q)$ with $p + q = n$. Note that the discriminant is $(-1)^q$ viewed as an element of $\mathbb{R}^\times/\mathbb{R}^\times_+$.

**2.3.3. Non-archimedean local fields.** — Here $F$ is a finite extension of $\mathbb{Q}_p$. The group $F^\times/N(E^\times)$ has only two elements and the equivalence classes of hermitian forms are parametrized by the discriminant. There are only two classes.

**2.3.4. Number fields.** — Here $F$ is a finite extension of $\mathbb{Q}$. Let $\mathcal{V}$ the set of places of $F$. For any $v \in \mathcal{V}$, let $F_v$ be the completion of $F$ at $v$. For any finite $v \in \mathcal{V}$, let $\mathcal{O}_v$ be the ring of integers of $F_v$. Any hermitian form $\Phi$ on $V$ induces by $F_v$-linearity a hermitian form $\Phi_v$ on the $E \otimes_F F_v$-module $V \otimes_F F_v$. If $v$ is split in $E$ i.e. $E \otimes_F F_v$ is not a field, there is obviously only one equivalence class of hermitian form. If $v$ is not split in $E$, we are in one of the situation studied in §§ 2.3.2 and 2.3.3. We say that $\Phi$ and $\Phi'$ are locally equivalent if for any $v \in \mathcal{V}$ the forms $\Phi_v$ and $\Phi'_v$ are equivalent on $V \otimes_F F_v$. Of course it suffices to look at places that are not split in $E$.

**Theorem 2.3.4.1**. — *Two hermitian forms on $V$ are equivalent if and only if they are locally equivalent.*

Let introduce the rings $\mathbb{A}$ and $\mathbb{A}_E$ of adeles of $F$ and $E$ respectively. We have the well-known exact sequence

$$1 \to F^\times/N(E^\times) \to \mathbb{A}^\times/N(\mathbb{A}_E^\times) \to \mathbb{A}^\times/F^\times N(\mathbb{A}_E^\times) \to 1$$

where the group $\mathbb{A}^\times/F^\times N(\mathbb{A}_E^\times)$ has two elements.

Only the injectivity of the second arrow is non-trivial: it says that an element of $F^\times$ is a norm of $E^\times$ if and only if it is locally a norm. Starting form an arbitrary collection $(\Phi'_v)_{v\in\mathcal{V}}$ of hermitian forms on $V \otimes_F F_v$, does there exist a hermitian form $\Phi$ on $V$ such that $\Phi_v$ is equivalent to $\Phi'_v$ for any $v \in \mathcal{V}$ ? Obviously, the two conditions are necessary:

1. the collection $(\mathrm{disc}(\Phi'_v))_{v\in\mathcal{V}}$ defines an element of $\mathbb{A}^\times/N(\mathbb{A}_E^\times)$ ;
2. its image in $\mathbb{A}^\times/F^\times N(\mathbb{A}_E^\times)$ is trivial.

These conditions are in fact also sufficient.

**2.3.5. Quadratic character.** — In the local (resp. global) situation, we denote by $\eta$ the non-trivial quadratic character of $F^\times/N(E^\times)$ (resp. $\mathbb{A}^\times/F^\times N(\mathbb{A}_E^\times)$). In the global situation, for any $v \in \mathcal{V}$, on gets a quadratic character of

$$F_v^\times/N((E \otimes_F F_v)^\times)$$

which is trivial if and only if the quotient is trivial i.e. $v$ is split in $E$.

## 2.4 Fourier transforms

**2.4.1. Pairing.** — Let's consider the case of a reductive group $G$ acting (algebraically and linearly) on a vector space $V$ over a field $F$. Let

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

be a $F$-bilinear symmetric pairing which is non-degenerate and $G$-invariant. Let $V_1 \subset V$ be a $G$-invariant subspace for which the restriction of the pairing is still non-degenerate. Let $V_2$ be the orthogonal complement of $V_1$. Any $X \in V$ can be written $X_1 + X_2$ according to $V = V_1 \oplus V_2$.

We will be interested in the following examples.

**Example 2.4.1.1.** — In the case $V = \widetilde{\mathfrak{g}} = \widetilde{\mathfrak{gl}}(V)$ of §2.1.1, one considers the form given by

$$\langle X, X \rangle = \mathrm{trace}(A^2) + 2cb \tag{2.4.1.1}$$

for any $X = (A, b, c) \in \widetilde{\mathfrak{g}}$. The most interesting subspaces $V_1$ are

1. the subspace $\mathfrak{gl}(V)^0$ generated by elements $(A, 0, 0)$ with $A$ an endomorphism of trace 0;
2. $V \oplus V^*$;
3. the sum $\widetilde{\mathfrak{gl}}(V)^0$ of the two previous subspaces.

**Example 2.4.1.2.** — In the case $\widetilde{\mathfrak{u}}_\Phi$ of §2.2.1, one considers the form given by

$$\langle X, X \rangle = \mathrm{trace}(A^2) + 2\Phi(b, b). \tag{2.4.1.2}$$

for any $X = (A, b) \in \widetilde{\mathfrak{u}}_\Phi$. This situation is parallel to that of example 2.4.1.1 and the corresponding subspaces $V_1$ are

1. the subspace generated by elements $(A, 0, 0)$ with $A$ an self-adjoint endomorphism of trace 0;
2. $V$;
3. the sum of the two previous subspaces.

**2.4.2. Partial Fourier transform.** — Let's assume that $F$ is a number field. Let $\mathcal{S}(V(\mathbb{A}))$ be the Bruhat-Schwartz space of complex functions on $V(\mathbb{A})$.

Let

$$\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$$

be a continuous and non-trivial additive character.

One defines the partial Fourier transform which is an automorphism of $\mathcal{S}(V(\mathbb{A}))$ by

$$\hat{f}_{V_1}(X_1 + X_2) = \int_{V_1(\mathbb{A})} f(Y + X_2)\psi(\langle X_1, Y \rangle)\, dY \tag{2.4.2.3}$$

where the Haar measure $dY$ is autodual.

If $F$ is a local field, $\psi$ denotes a non-trivial additive character on $F$. By a similar formula, one defines a partial Fourier tranform which is an automorphism of the local Schwartz space $\mathcal{S}(V(F))$ if $F$ is archimedean and of the space $C_c^\infty(V(F))$ of smooth (i.e. locally constant) and compactly supported functions on $V(F)$ if $F$ is non-archimedean.

**Remark 2.4.2.1.** — By duality, $G(\mathbb{A})$, resp. $G(F)$ in the local case, acts on spaces of functions on $V(\mathbb{A})$, resp. $V(F)$. The partial Fourier transform commutes with this action.

# 3 Local harmonic analysis: the infinitesimal situation

## 3.1 Orbital integrals: linear case

**3.1.1.** In this section $E/F$ is a quadratic extension of local fields. Let $G = GL_F(V)$ and $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{gl}}_F(V)$ (cf. § 2.1.1). We fix an equivariant morphism

$$\delta : \widetilde{\mathfrak{g}}^{\mathrm{rss}} \to G \tag{3.1.1.1}$$

as in §2.1.6. Using the character $\eta$ (cf. §2.3.5), we set for any $X \in \widetilde{\mathfrak{g}}^{\mathrm{rss}}(F)$

$$\eta(X) = \eta(\det(\delta_X)). \tag{3.1.1.2}$$

We have

$$\eta(g \cdot X) = \eta(\det(g))\eta(X).$$

**3.1.2.** We fix a Haar measure $dg$ on $G(F)$.

**3.1.3. Orbital integrals.** — To any $a \in \mathcal{A}^{\mathrm{rss}}(F)$, there exists $X \in \mathfrak{g}^{\mathrm{rss}}(F)$ such that $a(X) = a$ (cf. lemma 2.1.5.3) and such an element is well-defined up to $G(F)$-action (cf. lemma 2.1.5.1). Thus one can introduce the (regular semi-simple) orbital integral

$$J_a(f) = \int_{G(F)} f(g \cdot X)\eta(g \cdot X)\, dg \tag{3.1.3.3}$$

for any $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$. The $G(F)$-orbit of $X$ is closed so that the function $g \mapsto f(g \cdot X)$ is compactly supported and the integral makes sense.

**Remark 3.1.3.1**. — We could have defined orbital integrals attached to elements in $\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F)$. However, it is more convenient to view orbital integrals as functions on $\mathcal{A}^{\mathrm{rss}}(F)$. There is a price to pay: the definition is somewhat non-canonical because it depends on the choice of the morphism (3.1.1.1) that is on the choice of a basis $e$ of $V$. Let us point out that the basis $e$ also determines a subspace $\mathfrak{c}$ (cf. § 2.1.5) and that the map $X \mapsto \delta_X$ is identically 1 (and thus also $\eta$) on $\mathfrak{c}(F)$. This is analogous to [Kot99].

The distribution $J_a$ is $\eta$-equivariant in the following sense.

**Definition 3.1.3.2**. — A distribution $T$ is $\eta$-invariant if for any $g \in G(F)$ and any $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$

$$T(f^g) = \eta(\det(g))T(f)$$

where $f^g(X) = f(g \cdot X)$.

**Remark 3.1.3.3**. — If $F$ is archimedean then it is not difficult to extend the definition of orbital integrals to the Schwartz space $\mathcal{S}(\widetilde{\mathfrak{g}}(F))$ (cf. [Xue15]).

**3.1.4.** There are two ways to look at orbital integrals: either as distributions on $C_c^\infty(\widetilde{\mathfrak{g}}(F))$ or as functions of $a \in \mathcal{A}^{\mathrm{rss}}(F)$. From the latter point of view we have:

**Proposition 3.1.4.1**. —*([Zha14b] lemma 3.12 and proposition 3.13) Let $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$.*

1. *The function $a \mapsto J_a(f)$ is smooth on $\mathcal{A}^{\mathrm{rss}}(F)$ and its support is relatively compact in $\mathcal{A}(F)$.*
2. *If the support of $f$ is included in $\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F)$ then the function $a \mapsto J_a(f)$ is smooth and compactly supported function on $\mathcal{A}^{\mathrm{rss}}(F)$.*
3. *Conversely any smooth and compactly supported function on $\mathcal{A}^{\mathrm{rss}}(F)$ is equal to $a \mapsto J_a(f)$ for some $f \in C_c^\infty(\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F))$.*

**Remark 3.1.4.2**. — As far as I know, one does not know how to describe in general the space of functions on $\mathcal{A}^{\mathrm{rss}}(F)$ generated by the orbital integrals $J_a(f)$ for various $f \in C_c^\infty(\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F))$ but for $n = 1, 2$ see [JR11] section 6, [Zha12].

**3.1.5. Infinitesimal local relative trace formula.** — Let's consider the pairing (2.4.1.1) and let $\widetilde{\mathfrak{g}}_1$ be one of the subspaces of example 2.4.1.1. Following sec. 2.4, we get a partial Fourier transform. We have an orthogonal decomposition

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_2.$$

Let $X_2 \in \widetilde{\mathfrak{g}}_2(F)$ be a regular semi-simple element.

**Remark 3.1.5.1**. — There is an ambiguity here: the notion of semisimplicity or regularity for an element of $\widetilde{\mathfrak{g}}_2$ is that defined in definition 2.1.4.1 for the action of $G$ on $\widetilde{\mathfrak{g}}_2$. If $\widetilde{\mathfrak{g}}_2 = \mathfrak{gl}(V)^0$ (resp. $F \oplus V \oplus V^*$) then $X_2 = (A, 0, 0)$ (resp. $X_2 = (\lambda, b, c)$) is a regular semi-simple element if and only if the discriminant of the characteristic polynomial of $A$ is not 0 (resp. $cb \neq 0$).

We will focus on the case of a non-archimedean field but the theory for archimedean fields is not more difficult (cf. [Xue15]). For any pair $(f, f')$ of functions with $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and $f' \in C_c^\infty(\widetilde{\mathfrak{g}}_1(F))$ we introduce

$$T_{X_2}(f, f') = \int_{\widetilde{\mathfrak{g}}_1(F)} J_{a(X_1+X_2)}(f)\eta(X_1 + X_2)\bar{f}'(X_1)\, dX_1. \tag{3.1.5.4}$$

where $\bar{f}'$ is the complex-conjugate of $f'$.

**Theorem 3.1.5.2**. —*([Zha14b] theorem 4.6) For any pair $(f, f')$ as above*

1. *The integral defining $T_{X_2}(f, f')$ is convergent.*
2. *We have*
$$T_{X_2}(f, f') = T_{X_2}(\hat{f}_{\widetilde{\mathfrak{g}}_1}, \hat{f}'_{\widetilde{\mathfrak{g}}_1}).$$

**Proof.** — (sketch) One can control the behaviour of orbital integrals near singular points (see [Zha14b] proposition 4.2) and this suffices to prove assertion 1.

The second assertion is a consequence of the Plancherel formula

$$\int_{\widetilde{\mathfrak{g}}_1(F)} f^g(X_1 + X_2)\bar{f}'(X_1)\, dX_1 = \int_{\widetilde{\mathfrak{g}}_1(F)} \hat{f}^g_{\widetilde{\mathfrak{g}}_1}(X_1 + X_2)\overline{\hat{f}'_{\widetilde{\mathfrak{g}}_1}}(X_1)\, dX_1$$

applied to $f^g$ with $g \in G(F)$. One can multiply each side by $\eta(\det(g)) = \eta(g \cdot X)\eta(X)$ with $X = X_1 + X_2$. Then with some care one can integrate each side over $g \in G(F)$ and permute the integrals. □

**3.1.6. A variant.** — In this §3.1.6, we assume that $\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)$ so $\widetilde{\mathfrak{g}}_2 = V \oplus V^*$. We fix $(b, c) \in V \oplus V^*$ such that $cb \neq 0$. We have a decomposition

$$V = V' \oplus Fb$$

where $V'$ is the orthogonal of $c$. According to this decomposition, any $A \in \mathfrak{gl}(V)$ can be written as a matrix

$$A = \begin{pmatrix} A' & b' \\ c' & d' \end{pmatrix}$$

where $A' \in \mathfrak{gl}(V')$, $b' \in V'$, $c' \in (V')^*$ and $d' \in F$. Let $G' \subset G$ be the stabilizer of $(b, c)$: this is a subgroup isomorphic to $GL(V')$. Let $\widetilde{\mathfrak{g}}' = \widetilde{\mathfrak{gl}}(V')$. The map $A \mapsto ((A', b', c'), d)$ gives an isomorphism

$$\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V) \simeq \widetilde{\mathfrak{g}}' \oplus F$$

which is equivariant for the action of $G'$. The map $A \mapsto a(A,b,c) \in \mathcal{A}$ is $G'$-invariant and thus induces a morphism

$$\iota : \mathcal{A}' \to \mathcal{A} \tag{3.1.6.5}$$

where $\mathcal{A}'$ is the categorical quotient $\widetilde{\mathfrak{g}}_1//G' = \widetilde{\mathfrak{g}}'//G' \oplus \mathbb{A}^1$. Up to the affine line $\mathbb{A}^1$, this is the quotient $\widetilde{\mathfrak{g}}'//G'$ for the subspace $V'$. One defines $(\mathcal{A}')^{\mathrm{rss}} = (\widetilde{\mathfrak{g}}'//G')^{\mathrm{rss}} \oplus \mathbb{A}^1$.

The space $\widetilde{\mathfrak{g}}_1$ is equipped with the quadratic form $A \mapsto \mathrm{trace}(A^2)$. By transport, one gets a $G'$-invariant non-degenerate quadratic form on $\widetilde{\mathfrak{g}}' \oplus F$ given by

$$(A', b', c') \mapsto \mathrm{trace}((A')^2) + 2c'b' + (d')^2.$$

We have a morphism

$$\delta' : (\widetilde{\mathfrak{g}}')^{\mathrm{rss}} \to G'$$

associated to a basis of $V'$. This defines $\eta((A', b', c'))$ for (any) $(A', b', c') \in (\widetilde{\mathfrak{g}}')^{\mathrm{rss}}$. We have $(A, b, c) \in \widetilde{\mathfrak{g}}^{\mathrm{rss}}$ if and only if $(A', b', c') \in (\widetilde{\mathfrak{g}}')^{\mathrm{rss}}$. Moreover $\eta(A', b', c')$ and $\eta(A, b, c)$ are equal up to a constant independant of $A$. For $X = (A', b', c', d)$, let $\eta(X) = \eta(A', b', c')$.

Let $dg$ be a Haar measure on $G'(F)$. For any $f' \in C_c^\infty(\widetilde{\mathfrak{g}}'(F))$, we have an orbital integral

$$J_a(f') = \int_{G'(F)} f(g \cdot X)\eta(g \cdot X)\, dg$$

for any $a \in (\mathcal{A}')^{\mathrm{rss}}(F)$ and $X \in \mathfrak{gl}_F(V)$ of invariant $a$. Let $da$ be the measure on $\mathcal{A}(F)$ such that

$$\int_{\widetilde{\mathfrak{g}}_1(V)} f'(X)\, dX = \int_{\mathcal{A}'(F)} \int_{G'(F)} f'(g \cdot X)\, dg\, da.$$

We will use the following variant of (3.1.5.4) for $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and $f' \in C_c^\infty(\widetilde{\mathfrak{g}}'(F))$

$$T_{(b,c)}(f, f') = \int_{\mathcal{A}'(F)} J_{\iota(a)}(f)\overline{J_a(f')}\, da.$$

As a consequence of theorem 3.1.5.2, we have

$$T_{(b,c)}(f, f') = T_{(b,c)}(\hat{f}_{\widetilde{\mathfrak{g}}_1}, \hat{f}'_{\widetilde{\mathfrak{g}}_1}). \tag{3.1.6.6}$$

**3.1.7.** Let's introduce the notion of $\eta$-stable distributions.

**Definition 3.1.7.1**. — A distribution $T$ is $\eta$-stable if it is in the weak closure of the space generated by the orbital integrals $J_a$ for $a \in \mathcal{A}^{\mathrm{rss}}(F)$.

In other words, $T$ is $\eta$-stable if for any $f$ such that $J_a(f) = 0$ for any $a \in \mathcal{A}^{\mathrm{rss}}$, one has

$$T(f) = 0.$$

**Remark 3.1.7.2**. — It is clear that a $\eta$-stable distribution is also $\eta$-invariant. The converse might be true (for the case $n \leqslant 2$ see [Zha12]).

**3.1.8. A property of $\eta$-stable distribution.** – One has:

**Proposition 3.1.8.1**. — *If $T$ is $\eta$-stable then the partial Fourier transform $\hat{T}_{\widetilde{\mathfrak{g}}_1}$ is also $\eta$-stable.*

It is an immediate consequence of the following:

**Corollary 3.1.8.2**. —*(of theorem 3.1.5.2) Suppose that $J_a(f) = 0$ for any $a \in \mathcal{A}^{\mathrm{rss}}$. Then the same property holds for $\hat{f}_{\widetilde{\mathfrak{g}}_1}$.*

**Proof.** — The hypothesis implies that $T_{X_2}(f,f')=0$ for any $f'\in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and any regular semi-simple $X_2\in\widetilde{\mathfrak{g}}_2(F)$. By theorem 3.1.5.2, one also has

$$T_{X_2}(\hat{f}_{\widetilde{\mathfrak{g}}_1},\hat{f}'_{\widetilde{\mathfrak{g}}_1})=0.$$

But since $f'$ is arbitrary, this implies that $J_{a(X_1+X_2)}(\hat{f}_{\widetilde{\mathfrak{g}}_1})=0$ for any $X_1\in\widetilde{\mathfrak{g}}_2(F)$ such that $X_1+X_2\in\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F)$. The set of $a(X_1+X_2)$ when such $X_1$ and $X_2$ vary is dense in $\mathcal{A}^{\mathrm{rss}}(F)$. The result then follows from proposition 3.1.4.1 assertion 1. □

## 3.2 Orbital integrals: hermitian case

**3.2.1.** In this section $E/F$ is a quadratic extension of local fields. We use notations of sections 2.2 and 2.3.

**3.2.2.** We fix a Haar measure $dg$ on $U(F)$. We fix also a set $\mathcal{H}$ of representatives of equivalence classes of (non-degenerate) hermitian forms on the $E$-vector space $V$. Thus if $F$ is non-archimedean the set $\mathcal{H}$ has two elements.

**3.2.3. Local orbital integrals.** — Let $\Phi\in\mathcal{H}$ and $a\in\mathcal{A}^{\mathrm{rss}}(F)$. If the fiber $\widetilde{\mathfrak{u}}_{\Phi,a}(F)$ is not empty then it is a unique $U_\Phi(F)$-orbit. As in the linear case, one defines the (regular semi-simple) orbital integral

$$I_a^\Phi(f)=\begin{cases}0 & \text{if } \widetilde{\mathfrak{u}}_{\Phi,a}(F)=\emptyset,\\ \displaystyle\int_{U(F)} f(g\cdot X)\,dg, & \text{for any } X\in\widetilde{\mathfrak{u}}_{\Phi,a}(F) \quad \text{otherwise.}\end{cases} \tag{3.2.3.1}$$

for any $f\in C_c^\infty(\widetilde{\mathfrak{u}}_\Phi(F))$. It is an invariant distribution.

**3.2.4.** There are a lot of properties of the orbital integrals $I^\Phi$ that are shared with their linear analogues: we will not state all of them. Let's just mention:

- analog of proposition 3.1.4.1;
- analog of the local trace formula (theorem 3.1.5.2);
- the notion of stable distribution and the fact that the (partial) Fourier transform of a stable distribution is also stable (see proposition 3.1.8.1).

**3.2.5.** We would like to spend more time of the variant of the local trace formula (see §3.1.6) in the hermitian case. In this case, $\widetilde{\mathfrak{u}}_1$ is the space of self-adjoint endomorphisms of $V$. We start from an element $b\in V$ such that $\kappa\neq 0$ where

$$\kappa=\Phi(b,b).$$

We then have an orthogonal decomposition

$$V=V'\oplus Fb.$$

According to this decomposition, any $A\in\widetilde{\mathfrak{u}}_1$ can be written as a matrix (with the basis $b$ of $Fb$)

$$A=\begin{pmatrix}A' & b'\\ c' & d'\end{pmatrix}$$

where $A'\in\mathfrak{gl}_E(V')$ is self-adjoint for $\Phi$, $b'\in V'$, $d'\in F$ and

$$c'=\kappa^{-1}\Phi(Ab,\cdot)\in(V')^*.$$

One has

$$\begin{aligned}\operatorname{trace}(A^2) &= \operatorname{trace}((A')^2) + 2\kappa^{-1}\Phi(Ab, b') + (d')^2 \\ &= \operatorname{trace}((A')^2) + 2\kappa^{-1}\Phi(b', b') + (d')^2 \\ &= \operatorname{trace}((A')^2) + 2\Phi'(b', b') + (d')^2\end{aligned}$$

where we introduce the hermitian form $\Phi' = \kappa^{-1}\Phi$ on $V'$.

Recall that $U$ is the unitary group of $(V, \Phi)$. Let $U'$ be the stabilizer of $b$ in $U$: it is also identified with the unitary group of $(V', \Phi')$. The map

$$A \mapsto (A', b)$$

induces a $U'$-equivariant isomorphism from $\widetilde{\mathfrak{u}}_1$ to the space $\widetilde{\mathfrak{u}}' \oplus F$ where $\widetilde{\mathfrak{u}}'$ is a shortcut for $\widetilde{\mathfrak{u}}(V', \Phi')$. The categorical quotient of $\widetilde{\mathfrak{u}}' \oplus F$ by $U'$ is identified with the space $\mathcal{A}'$ introduced in §3.1.6. Moreover, the map $A \mapsto a(A, b)$ induces also a map $\iota : \mathcal{A}' \to \mathcal{A}$.

**Remark 3.2.5.1**. — We have also introduce a map $\iota$ in (3.1.6.5) that depends on a pair $(b, c) \in V$ (with the notations of §3.1.6). If $cb = \kappa$ this is the *same* map.

For a choice of Haar measure on $U'(F)$, we have for the measure $da$ on $\mathcal{A}(F)$ of §3.1.6

$$\int_{\widetilde{\mathfrak{u}}_1(F)} f'(X)\, dX = \int_{\mathcal{A}'(F)} \int_{U'(F)} f'(g \cdot X)\, dg\, da.$$

For any $f' \in C_c^\infty(\widetilde{\mathfrak{u}}'(F))$, we have an orbital integral

$$I_a^{\Phi'}(f') = \int_{U'(F)} f(g \cdot X)\, dg$$

for any $a \in (\mathcal{A}')^{\mathrm{rss}}(F)$ and $X \in \widetilde{\mathfrak{u}}_1(F)$ of invariant $a$. We introduce for $f \in C_c^\infty(\widetilde{\mathfrak{u}}(F))$ and $f' \in C_c^\infty(\widetilde{\mathfrak{u}}'(F))$

$$T_b(f, f') = \int_{\mathcal{A}'(F)} I_{\iota(a)}^{\Phi}(f)\overline{I_a^{\Phi'}(f')}\, da.$$

We have

$$T_b(f, f') = T_b(\hat{f}_{\widetilde{\mathfrak{u}}_1}, \hat{f}'_{\widetilde{\mathfrak{u}}_1}). \tag{3.2.5.2}$$

**3.2.6.** We define

$$C_c^\infty(\mathcal{H}) = \oplus_{\Phi\in\mathcal{H}} C_c^\infty(\widetilde{\mathfrak{u}}_\Phi(F)).$$

We slightly extend the definition of orbital integrals: for any $f = (f_\Phi)_{\Phi\in\mathcal{H}} \in C_c^\infty(\mathcal{H})$ and any $a \in \mathcal{A}^{\mathrm{rss}}(F)$ let

$$I_a(f) = \sum_{\Phi\in\mathcal{H}} I_a^\Phi(f_\Phi).$$

## 3.3 Comparison of Fourier transforms: the case of dimension 1

**3.3.1.** We want to begin with the situations of sections 2.1 and 2.2 with $n = 1$. In this case, the group acts trivially on the "endomorphism part" of $\widetilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{u}}$. So we will simply remove this factor. In §3.3.2 the character $\eta$ is either the quadratic character attached to the exension $E/F$ or the trivial character. However from §3.3.4 on, the character $\eta$ is the quadratic character.

In the following $F$ is a non-archimedean field. For the case of an archimedean field, see §3.3.7.

**3.3.2. Linear case.** — In this case, $F^\times$ acts on $F \oplus F$ by

$$t \cdot (b, c) = (tb, t^{-1}c).$$

The quotient is $F$ and the canonical invariant map is given by $(b, c) \mapsto cb$. One defines an orbital integral by

$$J_a(f) = \int_{F^\times} f(tb, t^{-1}c)\eta(tb)\, dt^\times \tag{3.3.2.1}$$

where $(b, c) \in F^2$ is any pair such that $cb = a$. The integral depends of course on the choice of the multiplicative Haar measure $dt^\times$.

**Remark 3.3.2.1.** — The integral is somewhat non-canonical: we could have replaced $\eta(tb)$ by $\eta(\delta(tb, t^{-1}c))$ where $\delta : F^2 - \{xy = 0\} \to F$ is any $F^\times$-equivariant (algebraic) morphism, e.g. $(b, c) \mapsto c^{-1}$.

**3.3.3. Fourier transform.** — Let $\psi$ be a non-trivial additive character of $F$. Following section 2.4 with the invariant quadratic form $(b, c) \mapsto 2cb$, one defines a Fourier transform on $C_c^\infty(F^2)$ by

$$\hat{f}(b', c') = \int_{F^2} f(b, c)\psi(c'b + cb')\, dbdc$$

where $db = dc$ is the autodual measure on $F$ for the character $\psi$. Explicitly, one has

$$db = \frac{|c_\psi|^{-1/2}}{\mathrm{vol}(\mathcal{O}_F, dx)} dx \tag{3.3.3.2}$$

where $dx$ is any Haar measure on $F$, $|\cdot|$ is the normalized absolute value and $c_\psi \mathcal{O}_F$ is the maximal fractionnal ideal on which $\psi$ is trivial.

**Proposition 3.3.3.1.** — *There exists a unique locally constant function on*

$$\hat{j}_\psi : F^\times \times F^\times \to \mathbb{C}$$

*such that for all $a \in F^\times$*

$$J_a(\hat{f}) = \int_F \hat{j}_\psi(a, a') J_{a'}(f) da'$$

*where $da'$ is a Haar measure on $F$.*

**Remarks 3.3.3.2.** — The function $\hat{j}_\psi$ depends on the choices of $da'$ (although the notation seems to suggest the contrary) and $\psi$. However, it does not depend on the choice of the morphism $\delta$ (see remark 3.3.2.1) nor on the choice of the multiplicative Haar measure $dt^\times$ on $F^\times$.

**Proof.** — The uniqueness follows easily from the fact that any smooth and compactly supported function on $F^\times$ is an orbital integral (see proposition 3.1.4.1 assertion 3).

The existence can be deduced from a local trace formula and a finiteness property à la Howe (cf. [Zha14b] corollary 4.7). However, we prefer to give here a direct proof. We fix a Haar measure $da$ on $F$. Let $dt^\times$ be the Haar measure on $F^\times$ such as we have the following integration formula

$$\int_{F^2} f(x, y)\, dxdy = \int_F \int_{F^\times} f(ta, t^{-1})\, dt^\times da$$

where in the LHS we use the autodual measure associated to $\psi$. For any $(a_1, a_2) \in (F^\times)^2$ there exists $m$ such that for all $k \in \mathbb{Z}$ such that if $|k| \geqslant m$ then

$$\int_{\varpi^k \mathcal{O}^\times} \psi(a_1 t^{-1} + a_2 t)\eta(t)\, dt^\times = 0$$

where $\varpi$ is an uniformizer and $\mathcal{O}$ the ring of integers. Hence we can define

$$\begin{aligned} \hat{j}_\psi(a_1,a_2) &= \int^*_{F^\times} \psi(a_1t^{-1}+a_2t)\eta(t)\,dt^\times \\ &= \lim_{m\to+\infty}\sum_{k=-m}^{m}\int_{\varpi^k\mathcal{O}^\times}\psi(a_1t^{-1}+a_2t)\eta(t)\,dt^\times. \end{aligned} \tag{3.3.3.3}$$

It is easy to see that $\hat{j}_\psi$ is locally constant on $(F^\times)^2$. The change of variables $t\mapsto t^{-1}$ shows that

$$\hat{j}_\psi(a_1,a_2)=\hat{j}_\psi(a_2,a_1). \tag{3.3.3.4}$$

Moreover the change of variable $t\mapsto \alpha t$ shows that

$$\hat{j}_\psi(\alpha a_1,a_2)=\eta(\alpha)\hat{j}_\psi(a_1,\alpha a_2). \tag{3.3.3.5}$$

We have for a large enough $m\in\mathbb{Z}$ (which depends on $f$ and $a$)

$$\begin{aligned} J_a(\hat{f}) &= \int_{F^\times}\hat{f}(ta,t^{-1})\eta(ta)\,dt \\ &= \sum_{k=-m}^{m}\int_{\varpi^k\mathcal{O}^\times}\hat{f}(ta,t^{-1})\eta(ta)\,dt \\ &= \sum_{k=-m}^{m}\int_{F^2}f(x,y)\int_{\varpi^k\mathcal{O}^\times}\psi(tay+t^{-1}x)\eta(ta)\,dt \\ &= \int_{F^2}f(x,y)\eta(a)\hat{j}_\psi(x,ay)\,dxdy \\ &= \int_{F^2}f(x,y)\eta(x)\hat{j}_\psi(a,xy)\,dxdy \\ &= \int_F J_{a'}(f)\hat{j}_\psi(a,a')\,da'. \end{aligned}$$

The integral is absolutely convergent since $\hat{j}_\psi(a,a')$ is locally $L^1$ (see lemme 4.9 of [Zha14b]) and $J_{a'}(f)$ is bounded and vanishes outside a bounded set (see [JR11] lemma 1 of section 6). □

**Lemma 3.3.3.3**. — *Let $\alpha\in F^\times$ and $\psi_\alpha=\psi(\alpha\cdot)$. Then for all $(a,a')\in(F^\times)^2$ one has*

1. $$\hat{j}_{\psi_\alpha}(a,a')=\eta(\alpha)|\alpha|\hat{j}_\psi(\alpha^2a,a').$$
2. $$\hat{j}_\psi(\alpha a,a')=\eta(\alpha)\hat{j}_\psi(a,\alpha a').$$
3. $$\hat{j}_\psi(a,a')=0$$ *unless $aa'\in N_{E/F}(E^\times)$.*

**Proof.** — The assertion 2 is proved in (3.3.3.5) above. The assertion 3 is a consequence of assertion 2 and (3.3.3.4) above.

Let's prove assertion 1. Let $dx$ be the autodual measure for $\psi$. Then by (3.3.3.2), the autodual measure for $\psi_\alpha$ is $|\alpha|^{1/2}dx$. Let $\tilde{f}$ be the Fourier transform of $f$ written relatively to the character $\psi_\alpha$. Thus we have

$$\tilde{f}(b,c)=|\alpha|\hat{f}(\alpha b,\alpha c).$$

As a consequence, one has
$$J_a(\tilde{f}) = |\alpha|\eta(\alpha)J_{\alpha^2 a}(\hat{f}).$$
The equality 1 is then straightforward.

□

**3.3.4. Hermitian case.** — In this case $E^1$, the subgroup of elements of norm 1 in $E^\times$, acts on $E$ equipped with a hermitian form
$$\Phi(x,y) = \kappa x^\sigma y$$
for some $\kappa \in F^\times$. The canonical invariant map $E \to F$ is given by $x \mapsto \Phi(x,x)$. For any non-trivial additive character $\psi$ of $F$, let $\psi_E = \psi \circ \mathrm{trace}_{E/F}$. One defines a Fourier transform on $C_c^\infty(E)$ by

$$\hat{f}(y) = \int_E f(x)\psi_E(\Phi(y,x))\,dx \tag{3.3.4.6}$$

where $dx$ is the autodual measure on $E$. By choosing a Haar measure $dt^\times$ on $E^1$, one defines an orbital integral for any $a \in F^\times$ by
$$I_a^\Phi(f) = \begin{cases} \displaystyle\int_{F^\times} f(tb)\,dt^\times \text{ if there is } \in E \text{ such that } \Phi(b,b) = a \\ 0 \text{ otherwise.} \end{cases}$$

**Proposition 3.3.4.1.** — *There exists a unique locally constant function*
$$\hat{i}_\psi^\Phi : F^\times \times F^\times \to \mathbb{C}$$
*such that for all $a \in F^\times$*
$$I_a^\Phi(\hat{f}) = \int_F \hat{i}_\psi^\Phi(a,a')I_{a'}^\Phi(f)da'$$
*where $da'$ is a Haar measure on $F$ and*
$$\hat{i}_\psi^\Phi(a,a') = 0$$
*if $a'$ is not of the form $\Phi(b,b)$ for some $b \in E$.*

**Remarks 3.3.4.2.** —

- The function $\hat{i}_\psi$ depends on the choices of $da'$ but not on the choice of the Haar measure on $E^1$.
- Clearly $\hat{i}_\psi^\Phi(a,\cdot) = 0$ if $a$ is not of the form $\Phi(b,b)$ for some $b \in E$.
- The lemma 3.3.4.3 below implies that the function $\hat{i}_\psi^\Phi$ depends only on the isomorphism class of $\Phi$ (that is on the class of $\kappa$ in $F^\times/N_{E/F}(E^\times)$).

**Proof.** — The proof is the same (it is even simpler) as the proof of proposition 3.3.3.1. We fix a Haar measure $da$ on $F$. Let $dt^\times$ be the Haar measure on $E^1$ such as we have the following integration formula
$$\int_E f(x)\,dxdy = \int_{E^\times/E^1}\int_{E^1} f(ta)\,dt^\times da.$$
Here we use the norm map to dentify $E^\times/E^1$ with an open subset of $F$. The measure on $E$ is the autodual one (in the sense of the Fourier transform (3.3.4.6)). Then it is easy to show that

$$\hat{i}_\psi(\Phi(x_1,x_1),\Phi(x_2,x_2)) = \int_{E^1} \psi(\kappa t x_1 x_2))\,dt^\times \tag{3.3.4.7}$$

for any $(x_1,x_2) \in (E^\times)^2$.

□

**Lemma 3.3.4.3.** — *Let $\alpha \in F^\times$ and $\psi_\alpha = \psi(\alpha\cdot)$. Let $\beta \in E^\times$ and $\Phi' = \beta\beta^\sigma\Phi$. Then for all $(a,a') \in (F^\times)^2$*

1.
$$\hat{i}^{\Phi}_{\psi_\alpha}(a,a') = |\alpha|\hat{i}^{\Phi}_{\psi}(\alpha^2 a, a').$$

2.
$$\hat{i}^{\Phi}_{\psi}(N(\beta)a,a') = \hat{i}^{\Phi}_{\psi}(a, N(\beta)a').$$

3.
$$\hat{i}^{\Phi'}_{\psi}(a,a') = \hat{i}^{\Phi}_{\psi}(a,a').$$

**Proof.** — Let $dx$ be the autodual measure on $E$ associated to $\Phi$ and $\psi_\alpha$. One checks that that the autodual measure for $\psi_\alpha$ and $\Phi$ is
$$|\alpha|_F dx.$$
Let $\tilde{f}$ be the Fourier transform of $f$ written relatively to the character $\psi_\alpha$ and $\Phi$. Thus we have
$$\tilde{f}(b) = |\alpha|_F \hat{f}(\alpha b)$$
and
$$I^{\Phi}_a(\tilde{f}) = |\alpha| I^{\Phi}_{\alpha^2 a}(\hat{f}).$$
Then assertion 1 is straightforward.

The assertion 2 can be proved directly or it is a consequence of (3.3.4.7).

Let $\tilde{f}$ be the Fourier transform of $f$ written relatively to the character $\psi$ and the form $\Phi'$. We have
$$\tilde{f}(b) = |N\beta|_F \hat{f}(N(\beta)b)$$
Thus
$$\begin{aligned} I^{\Phi'}_a(\tilde{f}) &= |N\beta|_F I^{\Phi}_{N\beta a}(\hat{f}) \\ &= |N\beta|_F \int_F \hat{i}^{\Phi}_{\psi}(N\beta a a, a') I^{\Phi}_{a'}(f) da' \\ &= |N\beta|_F \int_F \hat{i}^{\Phi}_{\psi}(N\beta a a, a') I^{\Phi'}_{N\beta a'}(f) da' \\ &= \int_F \hat{i}^{\Phi}_{\psi}(N\beta a, N\beta^{-1} a') I^{\Phi'}_{a'}(f) da' \end{aligned}$$
hence assertion 3 follows from assertion 2. □

**3.3.5. Tate's $\gamma$-factors.** — From now on, $\eta$ is the quadratic character attached to the extension $E/F$. For any additive character $\psi$ on $F$, any multiplicative character $\chi$ and any $s \in \mathbb{C}$, one defines the $\gamma$ and $\varepsilon$ factors $\gamma(\chi|\cdot|^s,\psi)$ and $\varepsilon(\chi|\cdot|^s,\psi)$. These are the factors denoted by $\gamma(\chi|\cdot|^s,\psi,dx)$ and $\gamma(\chi|\cdot|^s,\psi,dx)$ in [RV99] theorem 7.2 for the autodual measure $dx$. Recall that the two factors differ by a ratio of $L$-functions (*ibid.*) .

One checks that for $\alpha \in F^\times$

(3.3.5.8) $$\gamma(\chi|\cdot|^s,\psi_\alpha) = \chi(\alpha)|\alpha|^{s-1/2}\gamma(\chi|\cdot|^s,\psi).$$

For the field $E$ and the characters $\psi_E = \psi \circ \mathrm{trace}_{E/F}$, $\chi_E = \chi \circ N_{E/F}$ and the absolute value $|\cdot|_E = |N_{E/F}(\cdot)|$, one also has $\gamma$ and $\varepsilon$ factors. One checks first that one has

(3.3.5.9) $$\frac{\gamma(\chi|\cdot|^s,\psi)\gamma(\eta\chi|\cdot|^s,\psi)}{\gamma(\chi_E|\cdot|^s_E,\psi_E)} = \frac{\varepsilon(\chi|\cdot|^s,\psi)\varepsilon(\eta\chi|\cdot|^s,\psi)}{\varepsilon(\chi_E|\cdot|^s_E,\psi_E)}.$$

However the RHS depends neither on $\chi$ nor on $s$ (see e.g. theorem 29.4 and corollary 30.4 of [BH06]). So for $\chi = 1$ and $s = 1/2$, we get that (3.3.5.9) is equal to $\varepsilon(\eta|\cdot|^{1/2},\psi)$ which we denote simply by
$$\varepsilon(\eta, 1/2, \psi).$$

Because of the functional equation for $\varepsilon$ factors (see e.g. corollary 2 of § 23.4 of [BH06]), one has

$$\varepsilon(\eta, 1/2, \psi)^2 = \eta(-1) \tag{3.3.5.10}$$

and $\varepsilon(\eta, 1/2, \psi)$ is a fourth root of unity.

**3.3.6. Comparison of Fourier transforms.** — In the following theorem the functions $\hat{j}_\psi$ and $\hat{i}_\psi^\Phi$ are implicitly defined relatively to the *same* measure on $F$.

**Theorem 3.3.6.1**. — *For any non-trivial additive character $\psi$ of $F$ and any $(a, a') \in (F^\times)^2$ one has*

$$\hat{j}_\psi(a, a') = \varepsilon(\eta, 1/2, \psi) \sum_\Phi \eta(\mathrm{disc}(\Phi))\hat{i}_\psi^\Phi(a, a')$$

*where the sum is over the set (of cardinality 2) of isomorphism classes of non-degenerate hermitian forms $\Phi$ on $E$.*

**Proof.** — Let $(a, a') \in (F^\times)^2$. Clearly, every term vanishes if $aa' \notin N(E^\times)$ (see lemma 3.3.3.3 and remarks 3.3.4.2). Therefore we may assume that $aa' \in N(E^\times)$. But there is exactly (up to isomorphism) one hermitian form such that both $a$ and $a'$ belongs to $\{\Phi(b, b) | b \in E\}$. For any $\Phi'$ not isomorphic to $\Phi$, we have

$$\hat{i}_\psi^{\Phi'}(a, a') = 0$$

So we have to show that

$$\hat{j}_\psi(a, a') = \varepsilon(\eta, 1/2, \psi)\eta(\mathrm{disc}(\Phi))\hat{i}_\psi^\Phi(a, a') \tag{3.3.6.11}$$

Since any smooth and compactly supported function on $F^\times$ (resp. on $N(E^\times)$) is an orbital integral (see proposition 3.1.4.1), (3.3.6.11) is a consequence of lemma 3.3.6.2 below. □

**Lemma 3.3.6.2**. — *Let $\kappa = \mathrm{disc}(\Phi)$. Let $f \in C_c^\infty(F^2)$ and $f' \in C_c^\infty(E)$ such that for any $b \in E$ and $a = \Phi(b, b)$*

$$J_a(f) = I_a^\Phi(f'). \tag{3.3.6.12}$$

*Then, for any $b \in E$ and $a = \Phi(b, b)$, one has*

$$J_a(\hat{f}) = \varepsilon(\eta, 1/2, \psi)\eta(\kappa)I_a^\Phi(\hat{f}') \tag{3.3.6.13}$$

*where the Fourier transform are defined relatively to $\psi$ and $\Phi$ as above.*

**Proof.** — Using Fourier transform on the group $F^\times$, it suffices to prove that, for any unitary character $\chi$ of $F^\times$ and any $s \in \mathbb{C}$ of large real part $\Re(s)$, one has

$$\int_{F^\times} J_a(\hat{f})\chi(a)|a|^s(\eta(\kappa) + \eta(a))da^\times = \varepsilon(\eta, 1/2, \psi)\eta(\kappa) \int_{F^\times} I_a^\Phi(\hat{f}')\chi(a)|a|^s(\eta(\kappa) + \eta(a))da^\times. \tag{3.3.6.14}$$

In the following, the measure $da^\times$ will be $da/|a|$ where $da$ is the $\psi$-autodual measure.

We are free to choose the Haar measures that define orbital integrals. For the orbital integral $J_a$ we also take $dx/|x|$ on $F^\times$ where $dx$ is the $\psi$-autodual measure. By a change of variables the LHS of (3.3.6.14) becomes

$$\int_{(F^\times)^2} \hat{f}(t, t^{-1}a)\chi(a)|a|^s(\eta(t\kappa) + \eta(ta))da^\times dt^\times = \int_{(F^\times)^2} \hat{f}(t, a)\chi(ta)|ta|^s(\eta(t\kappa) + \eta(a))da^\times dt^\times.$$

Then one can use Tate's local functional equation for zeta integrals (see e.g. [RV99] theorem 7.2) to get [1]

$$\gamma(\chi^{-1}|\cdot|^{1-s}, \psi)\gamma(\eta\chi^{-1}|\cdot|^{1-s}, \psi) \int_{(F^\times)^2} f(a, t)\chi(ta)^{-1}|ta|^{1-s}(\eta(\kappa t) + \eta(a))da^\times dt^\times$$

[1]Since it may be a source of confusion, we emphasize that it is $f(a, t)$ in the expression below and *not* $f(t, a)$.

$$= \gamma(\chi^{-1}|\cdot|^{1-s},\psi)\gamma(\eta\chi^{-1}|\cdot|^{1-s},\psi)\int_{F^\times} J_a(f)\chi(a)^{-1}|a|^{1-s}(\eta(a\kappa)+1)da^\times.$$

$$= \eta(\kappa)\gamma(\chi^{-1}|\cdot|^{1-s},\psi)\gamma(\eta\chi^{-1}|\cdot|^{1-s},\psi)\int_{F^\times} J_a(f)\chi(a)^{-1}|a|^{1-s}(\eta(\kappa)+\eta(a))da^\times. \tag{3.3.6.15}$$

The expression above is convergent for negative $\Re(s)$ but in general it has to be understood in the sense of analytic continuation.

We can do a similar computation for the integral of the RHS of (3.3.6.14). We first choose the Haar measure $dx$ on $E$ which is autodual for the character $\psi_E(\kappa\cdot)$. We choose the measure $dx^\times = dx/|x|$ on $E^\times$ and the measure on $E^1$ is such that the quotient measure on $E^1\backslash E^\times$ is identified with the measure on $\kappa N(E^\times)$ induced by the previous measure on $F^\times$. By Tate's local functional equation and (3.3.5.8), the integral of the RHS of (3.3.6.14) is equal to

$$\begin{aligned}
2\eta(\kappa)\int_{E^1\backslash E^\times}\int_{E^1}\hat{f}'(tb)dt^\times\chi(\kappa N(b))|\kappa N(b)|^s db^\times &= 2\eta(\kappa)\int_{E^\times}\hat{f}'(b)\chi(\kappa N(b))|\kappa N(b)|^s db^\times \\
&= 2\eta(\kappa)\chi(\kappa)|\kappa|^s\gamma(\chi_E^{-1}|\cdot|_E^{1-s},\psi_E(\kappa\cdot))\int_{E^\times} f'(b)\chi_E(b)^{-1}|b|_E^{1-s}db^\times \\
= \chi(\kappa)^2|\kappa|^{2s-1}\gamma(\chi_E^{-1}|\cdot|_E^{1-s},\psi_E(\kappa\cdot))&\int_{F^\times} I_a^\Phi(f')\chi(a)^{-1}|a|^{1-s}(\eta(\kappa)+\eta(a))da^\times \\
&= \gamma(\chi_E^{-1}|\cdot|_E^{1-s},\psi_E))\int_{F^\times} I_a^\Phi(f')\chi(a)^{-1}|a|^{1-s}(\eta(\kappa)+\eta(a))da^\times
\end{aligned}$$

Now it is easy to conclude with the considerations of §3.3.5.

□

**3.3.7. Archimedean cases.** — The only archimedean quadratic extension $E/F$ is the case $\mathbb{C}/\mathbb{R}$. But one needs to consider also the cases $F=\mathbb{R}$ or $F=\mathbb{C}$ for the trivial character $\eta$.

The propositions 3.3.3.1, 3.3.4.1 and theorem 3.3.6.1 have then obvious analogs. The analog of 3.3.3.3 is an improper integral (see [Xue15] section 7).

## 3.4 Fourier transform and matching functions

**3.4.1.** The notations are borrowed from sections 3.1 and 3.2. Let $\mathrm{Orb}(\widetilde{\mathfrak{g}})$ and $\mathrm{Orb}(\mathcal{H})$ be the space of functions $\mathcal{A}^{\mathrm{rss}}(F)\to\mathbb{C}$ generated respectively by $a\mapsto J_a(f)$ for $f\in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and $a\mapsto I_a(f)$ for $f\in C_c^\infty(\mathcal{H})$.

**Conjecture 3.4.1.1**. — *We have*

$$\mathrm{Orb}(\widetilde{\mathfrak{g}})=\mathrm{Orb}(\mathcal{H}).$$

**Remark 3.4.1.2**. — Equivalently, the conjecture predicts that for any $f\in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ there exists $f'\in C_c^\infty(\mathcal{H})$ (and vice-versa) such that $(f,f')$ is a pair of matching functions in the sense that

$$J_a(f)=I_a(f')$$

for any $\mathcal{A}^{\mathrm{rss}}(F)$.

**Theorem 3.4.1.3**. —*([Zha14b] theorem 2.6) If $F$ is non-archimedean, the conjecture 3.4.1.1 holds.*

**Remark 3.4.1.4**. — In the archimedean case, we have only a partial but useful result due to H. Xue (cf. [Xue15]).

In the following we would like to explain some key steps in the proof of the theorem 3.4.1.3. So, from now on, $F$ is non-archimedean. The contents of theorem 3.4.1.3 are an equality of two spaces of functions. We will explain only the inclusion $\mathrm{Orb}(\widetilde{\mathfrak{g}}) \subset \mathrm{Orb}(\mathcal{H})$. The same argument proves the reverse inclusion.

**3.4.2. Fourier transform and matching functions.** — Let $\widetilde{\mathfrak{g}}_1$ be one of the space of example 2.4.1.1. The following proposition is the main key step in the proof of theorem 3.4.1.3. Let $\psi$ be a non-trivial character of $F$. For any $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$, let $\hat{f}_1$ be the partial Fourier transform associated to $\psi$ and $\widetilde{\mathfrak{g}}_1$. For any $\Phi \in \mathcal{H}$, there is a companion subspace $\widetilde{\mathfrak{u}}_{\Phi,1}$ of $\widetilde{\mathfrak{u}}_\Phi$ attached to $\widetilde{\mathfrak{g}}_1$ (cf. example 2.4.1.2). We define a constant $\gamma_\psi(\Phi, \widetilde{\mathfrak{g}}_1)$ (a fourth root of unity) attached to $\psi$ and any $\Phi \in \mathcal{H}$ by

$$\gamma_\psi(\Phi, \widetilde{\mathfrak{g}}_1) = \begin{cases} \eta(\mathrm{disc}(\Phi))\varepsilon(\eta, 1/2, \psi)^n \text{ if } \widetilde{\mathfrak{g}}_1 = V \oplus V^* \\ \eta(\mathrm{disc}(\Phi))^{n-1}\varepsilon(\eta, 1/2, \psi)^{n(n-1)/2} \text{ if } \widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)^0 \\ \eta(\mathrm{disc}(\Phi))^{n}\varepsilon(\eta, 1/2, \psi)^{n(n+1)/2} \text{ if } \widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{gl}}(V)^0 \end{cases}$$

For any $f' = (f'_\Phi)_{\Phi \in \mathcal{H}} \in C_c^\infty(\mathcal{H})$, let

$$\hat{f}'_1 = (\gamma_\psi(\Phi, \widetilde{\mathfrak{g}}_1)\hat{f}'_{\Phi, \widetilde{\mathfrak{u}}_{\Phi,1}})_{\Phi \in \mathcal{H}}$$

where $\hat{f}'_{\Phi, \widetilde{\mathfrak{u}}_{\Phi,1}}$ is the partial Fourier transform associated to $\psi$ and $\widetilde{\mathfrak{u}}_{\Phi,1}$.

**Theorem 3.4.2.1.** —*([Zha14b] theorem 4.17) Let $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and $f' = (f'_\Phi)_{\Phi \in \mathcal{H}} \in C_c^\infty(\mathcal{H})$ be a pair of matching functions. Then the pair $(\hat{f}_1, \hat{f}'_1)$ is also a pair of matching functions.*

**Remark 3.4.2.2.** — We emphasize that the constants $\gamma_\psi(\Phi, \widetilde{\mathfrak{g}}_1)$ are somewhat different from the constants that could be deduced from the computations of [Zha14b].

**Remark 3.4.2.3.** — The statement of the theorem is also true in the archimedean situation. In this case, the constants $\gamma_\psi(\Phi, \widetilde{\mathfrak{g}}_1)$ have been obtained in [Xue15] section 9.

**3.4.3. Proof of theorem 3.4.2.1: case of $\widetilde{\mathfrak{g}}_1 = V \oplus V^*$.** — Let $a_0 \in \mathcal{A}^{\mathrm{rss}}(F)$ and let $(A, e, e^*) \in \widetilde{\mathfrak{g}}_{a_0}(F)$. We assume moreover that the endomorphism $A$ is regular semi-simple. Let $\chi_A = \prod_{i \in I} \chi_i$ be the decomposition of the characteristic polynomial of $A$ into two by two distinct $F$-irreducible factors. Then the algebra $F_A = F[X]/(\chi_A)$ is isomorphic to $\prod_{i \in I} F_i$ where $F_i$ is the field $F[X]/(\chi_i)$. The choice of $A$ and $e$ makes $V$ a free $F_A$-module of rank 1. In particular, one can write $e = \sum_{i \in I} e_i$ according to the decomposition

$$V = \oplus_{i \in I} V_i$$

where $V_i$ is the free $F_i$-module of rank 1 generated by $e_i$. Dually, by the choice of $A$ and $e^*$, one has that $V^*$ is also a free $F_A$-module of rank 1. And we have a decomposition $e^* = \sum_{i \in I} e_i^*$ w.r.t. $V^* = \oplus_{i \in I} V_i^*$. We will use the identifications $V_i \simeq F_i$, $V_i^* \simeq F_i$ etc. given by the choice of generators. Moreover $A$ is idenfied with $(\alpha_i)_{i \in I} \in \prod_{i \in I} F_i$.

The centralizer $T$ of $A$ in $G$ is then identified with $\prod_{i \in I} \mathrm{Res}_{F_i/F}\, \mathbb{G}_m$. Let $\mathcal{A}_T$ be the categorical quotient of $(V \oplus V^*)//T$. The canonical map $V \oplus V^* \to \mathcal{A}_T$ at the level of $F$-points is identified with the map

$$\prod_{i \in I} (F_i \times F_i) \to \mathcal{A}_T(F) = \prod_{i \in I} F_i$$

given factor by factor by $(x_i, x_i) \mapsto x_i y_i$. The map $(b, c) \in V \oplus V^* \mapsto a(A, b, c) \in \mathcal{A}$ is $T$-invariant and this it factorizes through a unique morphism

$$\iota : \mathcal{A}_T \to \mathcal{A}.$$

Note that $\iota^{-1}(\mathcal{A}^{\mathrm{rss}}) = \mathcal{A}_T^{\mathrm{rss}}$ and $\mathcal{A}_T^{\mathrm{rss}}(F) = \prod_{i\in I} F_i^\times$ (see [CZ] lemma 3.4.1.1). The canonical pairing $V \times V^* \to F$ induces a pairing $\prod_{i\in I}(F_i \times F_i) \to F$ given by

$$(x_i, y_i)_{i\in I} \mapsto \sum_{i\in I} \mathrm{trace}_{F_i/F}(\lambda_i x_i y_i) \tag{3.4.3.1}$$

for uniquely defined $\lambda_i \in F_i$. The transport of the autodual measure on $V \oplus V^*$ is the measure on $\prod_{i\in I}(F_i \times F_i)$ induced by the autodual measures $dx_i$ on $F_i$ for the character

$$\psi_{F_i,\lambda_i} = \psi_{F_i}(\lambda_i \cdot) = (\psi \circ \mathrm{trace}_{F_i/F})(\lambda_i \cdot)$$

We then define a function $\hat{j}_\psi$ on $\mathcal{A}_T^{\mathrm{rss}}(F)\times\mathcal{A}_T^{\mathrm{rss}}$ in the following way: let $a = (a_i)_{i\in I}$ and $a' = (a'_i)_{i\in I}$ two elements of $\mathcal{A}_T^{\mathrm{rss}}(F)$, then

$$\hat{j}_\psi(a, a') = \prod_{i\in I} \hat{j}_{\psi_{F_i,\lambda_i}}(a_i, a'_i)$$

where $\hat{j}_{\psi_{F_i,\lambda_i}}$ is the function defined in §3.3.2 for the field $F_i$, the characters $\eta \circ N_{F_i/F}$ and $\psi_{F_i,\lambda_i}$ and an arbitrary Haar measure on $F_i$. Via $\mathcal{A}_T(F) \simeq \prod_{i\in I} F_i$, the product of such Haar measures gives a measure $da$ on $\mathcal{A}_T(F)$.

**Lemma 3.4.3.1**. — *For any $f \in C_c^\infty(\widetilde{\mathfrak{g}}(F))$ and any $a \in \mathcal{A}_T^{\mathrm{rss}}(F)$ we have*

$$J_{\iota(a)}(\hat{f}_1) = \int_{\mathcal{A}_T(F)} J_{\iota(a')}(f)\hat{j}_\psi(a, a')da'.$$

**Proof.** — One can assume that $f = h_1 \otimes h_2$ where $h_i \in C_c^\infty(\widetilde{\mathfrak{g}}_i(F))$. We have for any $g \in G(F)$

$$(\hat{f}_{\mathfrak{g}_1})^g = \widehat{(h_1^g)} \otimes h_2^g$$

where in the LHS we consider the Fourier transform on $\widetilde{\mathfrak{g}}_1(F)$. Using the identification $T(F) \simeq \prod_{i\in F_i}$ the character $\eta$ becomes $\prod_{i\in I} \eta \circ N_{F_i/F}$. Using the idenfication $V \simeq \prod_{i\in I} F_i$, we write $b = (b_i)_{i\in I}$ and we define

$$\eta_I(b) = \prod_{i\in I} \eta \circ N_{F_i/F}(b_i).$$

Thus we have for $X = (A, b, c)$ of invariant $\iota(a)$ and for $\frac{dg}{dt}$ the quotient measure

$$J_{\iota(a)}(\hat{f}_{\mathfrak{g}_1}) \quad = \quad \int_{G(F)/T(F)} h_2(gAg^{-1})\eta(g \cdot X)\eta_I(b) \int_{T(F)} \widehat{h_1^g}(tb, ct^{-1})\eta_I(tb)\, dt\, \frac{dg}{dt}.$$

By proposition 3.3.3.1, we have

$$\int_{T(F)} \widehat{h_1^g}(tb, ct^{-1})\eta_I(tb)\, dt = \int_{\mathcal{A}_T(F)} \hat{j}_\psi(a, a')(\int_{T(F)} h_1^g(tb', c't^{-1})\eta_I(tb')\, dt)da'$$

where $(b', c')$ is such that the invariant of $X_{b',c'} = (A, b', c')$ is $\iota(a')$. Then we get

$$J_{\iota(a)}(\hat{f}_{\mathfrak{g}_1}) \quad = \quad \int_{\mathcal{A}_T(F)} \hat{j}_\psi(a, a') \int_{G(F)} f(g \cdot X_{b',c'})\eta(g \cdot X)\eta_I(b)\eta_I(b')\, dg.$$

To conclude, it is enough to notice that

$$\eta(X_{b',c'}) = \eta(X)\eta_I(b)\eta_I(b').$$

In fact, one can even show that

$$\det(\delta_{X_{b',c'}})N_{F_i/F}(b'_i)^{-1}$$

does not depend on $(b', c')$ (where $b' = (b'_i)_{i\in I}$ ; see e.g. the proof of lemma 3.4.1.1 of [CZ]). □

Let $V_E = V \otimes_F E$. We have the decomposition

$$V_E = \oplus_{i\in I} V_i \otimes_F E$$

So it is identified with $\prod_{i\in I} E_i$ where $E_i = F_i \otimes_F E$. We still denote by $\sigma$ the automorphism $1 \otimes \sigma$ of $E_i$ where $\sigma \neq 1 \in \mathrm{Gal}(E/F)$. We view the element $(\alpha_i)_{i\in I}$ as an endomorphism of $V_E$ (by multiplication) and 1 as a vector of $V_E$.

The element $a_0 \in \mathcal{A}^{\mathrm{rss}}(F)$ determines a hermitian form $\Phi$ on $V_E$ such that the pair $((\alpha_i)_{i\in I}, 1)$ belongs to $\widetilde{\mathfrak{u}}(V_E, \Phi)(F)$ and its invariant is $a_0$. One checks easily that this hermitian form can be taken of the form

$$\Phi((x_i)_{i\in I}, (x_i)_{i\in I}) = \sum_{i\in I} \mathrm{trace}_{E_i/E}(\lambda_i x_i x_i^\sigma)$$

where $\lambda_i \in F_i^\times$ is uniquely defined by (3.4.3.1). Let $T'$ be the centralizer of $(\alpha_i)_{i\in I}$ in the unitary group $U(V_E, F)$. Then $T'(F)$ is identified with the group of element of norm 1 in $\prod_{i\in I} E_i^\times$. The categorical quotient $V_E//T'$ is identified with the space $\mathcal{A}_T$. The canonical map $V_E \to \mathcal{A}_T(F)$ is given by $(x_i)_{i\in I} \mapsto (N_{E_i/F_i}(x_i))_{i\in I}$.

Let $I' \subset I$ be the subset of $i$ such that $E_i$ is a field. We have $i \notin I'$ if and only if $F_i$ contains $E$ and thus the degree $[F_i : F]$ is even. If $i \notin I'$, the algebra $E_i$ is a product of two copies of $F_i$ and we are in the situation of §3.3.2 (with the trivial character) whereas if $i \in I'$ is a field we are in the situation of §3.3.4. In both cases, we have a function $\hat{i}_{\Psi_{F_i,\lambda_i}}$ on $F_i^\times \times F_i^\times$ for the character $\psi_{F_i,\lambda_i} = \psi \circ \mathrm{trace}_{F_i/F}(\lambda_i \cdot)$, the hermitian form $x_i x_i^\sigma$ and the measure on $F_i$ that is used to define $\hat{j}_{\psi_{F_i,\lambda_i}}$. We define for $a = (a_i)_{i\in I}$ and $a' = (a'_i)_{i\in I} \in \mathcal{A}_T^{rs}(F)$

$$\hat{i}_\psi(a, a') = \prod_{i\in I} \hat{i}_{\psi_{F_i,\lambda_i}}(a_i, a'_i).$$

**Lemma 3.4.3.2**. — *For any $a \in \mathcal{A}_T^{\mathrm{rss}}(F)$ such that $\iota(a) = a_0$ and any $a' \in \mathcal{A}_T^{\mathrm{rss}}(F)$, we have*

$$\hat{j}_\psi(a, a') = \varepsilon(\eta, 1/2, \psi)^n \eta(\mathrm{disc}(\Phi)) \hat{i}_\psi(a, a').$$

**Proof.** — By theorem 3.3.6.1 for $i \in I'$ (for $i \notin I'$ the comparison is almost tautological), we have for $a \in \mathcal{A}_T^{\mathrm{rss}}(F)$ such that $\iota(a) = a_0$ and any $a \in \mathcal{A}_T^{\mathrm{rss}}(F)$:

$$\hat{j}_\psi(a, a') = \big(\prod_{i\in I'} \varepsilon(\eta \circ N_{F_i/F}, 1/2, \psi_{F_i,\lambda_i})(\eta \circ N_{F_i/F})(\mathrm{disc}(x_i x_i^\sigma))\big) \hat{i}_\psi(a, a').$$

The problem is to prove that the coefficient above is the expected coefficient. Note that

$$(\eta \circ N_{F_i/F})(\mathrm{disc}(x_i x_i^\sigma)) = 1.$$

Using relation (3.3.5.8), one has for $i \in I'$

$$\varepsilon(\eta \circ N_{F_i/F}, 1/2, \psi_{F_i,\lambda_i}) = \eta(N_{F_i/F}(\lambda_i)) \varepsilon(\eta \circ N_{F_i/F}, 1/2, \psi_{F_i}).$$

Let $\Phi_i$ be the restriction of $\Phi$ to $V_i \otimes_F E$. Using a $F$-basis $(x_k)_{1\leqslant k\leqslant d_i}$ of $F_i$ with $d_i = [F_i : F]$, one sees that (the equality is in the group $F_i^\times / N_{E_i/F_i}(E_i^\times)$)

$$\begin{aligned} \mathrm{disc}(\Phi_i) &= \det(\mathrm{trace}_{E_i/E}(\lambda_i x_k x_{k'}^\sigma))_{1\leqslant k,k'\leqslant d_i} \\ &= N_{F_i/F}(\lambda_i) \det(\mathrm{trace}_{F_i/F}(x_k x_{k'}^\sigma))_{1\leqslant k,k'\leqslant d_i}. \end{aligned}$$

But one also has (cf. [Zha14b] theorem 4.13 *et infra*)

$$\varepsilon(\eta \circ N_{F_i/F}, 1/2, \psi_{F_i}) = \varepsilon(\eta, 1/2, \psi)^{d_i} \eta(\det(\mathrm{trace}_{F_i/F}(x_k x_{k'}^\sigma))_{1\leqslant k,k'\leqslant d_i})$$

Thus one gets

$$\prod_{i\in I'} \varepsilon(\eta\circ N_{F_i/F}, 1/2, \psi_{F_i,\lambda_i}) \quad = \quad \prod_{i\in I'} \varepsilon(\eta, 1/2, \psi)^{d_i}\eta(\mathrm{disc}(\Phi_i)).$$

The lemma follows because

$$\eta(\mathrm{disc}(\Phi)) = \prod_{i\in I} \eta(\mathrm{disc}(\Phi_i))$$

and the discriminant of $\Phi_i$ for $i \notin I'$ is $(-1)^{d_i/2}$. Thus for $i \notin I'$, one has (cf. (3.3.5.10))

$$\eta(\mathrm{disc}(\Phi_i)) = \eta(-1)^{d_i/2} = \varepsilon(\eta, 1/2, \psi)^{d_i}.$$

□

As a consequence of propositions 3.3.3.1 and 3.3.4.1, we have:

**Lemma 3.4.3.3**. — *For any $f' \in C_c^\infty(\widetilde{\mathfrak{u}}(V_E, \Phi))$ and any $a \in \mathcal{A}_T^{\mathrm{rss}}(F)$ we have*

$$I^\Phi_{\iota(a)}(\hat{f}'_{V_E}) = \int_{\mathcal{A}_T(F)} I^\Phi_{\iota(a')}(f)\hat{i}_\psi(a,a')da'.$$

Using lemma 3.4.3.1, lemma 3.4.3.3 and lemma 3.4.3.2, we get

$$J_{a_0}(\hat{f}_1) = \varepsilon(\eta, 1/2, \psi)^n \,\mathrm{disc}(\Phi) I^\Phi_{a_0}(\hat{f}'_{\Phi,\widetilde{\mathfrak{u}}_{\Phi},1})$$

for the specific $a_0$. But such $a_0$'s are dense in $\mathcal{A}^{\mathrm{rss}}(F)$ and the result extends to any $a_0 \in \mathcal{A}^{\mathrm{rss}}(F)$ (see proposition 3.1.4.1 assertion 1 and its analog for $\widetilde{\mathfrak{u}}_{\Phi,1}$). This proves theorem 3.4.2.1 in the case $\widetilde{\mathfrak{g}}_1 = V \oplus V^*$.

**3.4.4. Proof of theorem 3.4.2.1: general case.** — We want to prove the remaining cases of theorem 3.4.2.1 by recursion on $n$. The case $\widetilde{\mathfrak{g}}_1 = V \oplus V^*$ has been proved in the previous §. The conjunction of the cases $\widetilde{\mathfrak{g}}_1 = V \oplus V^*$ and $\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)^0$ gives the case $\widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{gl}}(V)^0$. It is also obvious that the cases $\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)^0$ and $\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)$ (resp. $\widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{gl}}(V)^0$ and $\widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{gl}}(V)$) are equivalent for the *same* constant. The case $n = 1$ is also obvious.

So in the following, we assume that theorem 3.4.2.1 holds for any space $V'$ of dimension $< \dim(V)$. From now on $\widetilde{\mathfrak{g}}_1 = \mathfrak{gl}(V)$.

Let $a_0 \in \mathcal{A}^{\mathrm{rss}}(F)$. There exists $(A_0, b_0, c_0) \in \widetilde{\mathfrak{g}}(F)$ such that $a(A_0, b_0, c_0) = a_0$. Moreover there exists a unique element $\Phi \in \mathcal{H}$ such that there exists $(A'_0, b'_0) \in \widetilde{\mathfrak{u}}_\Phi(F)$ such that $a(A'_0, b'_0) = a_0$. Let

$$\kappa = \Phi(b'_0, b'_0) = c_0 b_0.$$

Let $f$ and $f'$ be as in the statement of theorem 3.4.2.1. To simplify the notation, let $\hat{f}'_{\Phi,1} = \hat{f}'_{\Phi,\widetilde{\mathfrak{u}}_{\Phi},1}$. One wants to show the equality

$$J_{a_0}(\hat{f}_1) = \eta(\mathrm{disc}(\Phi))^{n-1}\varepsilon(\eta, 1/2, \psi)^{n(n-1)/2} I^\Phi_{a_0}(\hat{f}'_{\Phi,1}). \tag{3.4.4.2}$$

We introduced in §3.1.6 a $F$-vector space $V'$, an action of $G' = GL(V')$ and $\widetilde{\mathfrak{g}}' = \widetilde{\mathfrak{gl}}(V')$ such that $\widetilde{\mathfrak{g}}_1 = \widetilde{\mathfrak{g}}' \oplus F$. The quotient of $\widetilde{\mathfrak{g}}_1$ by $G'$ is $\mathcal{A}'$ and moreover we have a map $\iota : \mathcal{A}' \to \mathcal{A}$. For any $g \in C_c^\infty(\widetilde{\mathfrak{g}}_1(F))$ we have (see (3.1.6.6))

$$\int_{\mathcal{A}'(F)} J_{\iota(a)}(f)\overline{J_a(g)}\, da = \int_{\mathcal{A}'(F)} J_{\iota(a)}(\hat{f}_1)\overline{J_a(\hat{g})}\, da. \tag{3.4.4.3}$$

On the other hand, following §3.1.6, we get an hermitian space $(V', \Phi')$ and a space $\widetilde{\mathfrak{u}}'$ on which the unitary group $U' = U(V', \Phi')$ acts. We have $\widetilde{\mathfrak{u}}_1 = \widetilde{\mathfrak{u}}' \oplus F$. The quotient of $\widetilde{\mathfrak{u}}_1$ by $U'$ is the same space $\mathcal{A}'$ as before. For any $g \in C_c^\infty(\widetilde{\mathfrak{u}}_1(F))$ we have (see (3.2.5.2))

$$(3.4.4.4) \qquad \int_{\mathcal{A}'(F)} I^{\Phi}_{\iota(a)}(f'_{\Phi})\overline{I^{\Phi'}_a(g')}\, da = \int_{\mathcal{A}'(F)} I^{\Phi}_{\iota(a)}(\hat{f}'_{\Phi,1})\overline{I^{\Phi'}_a(\hat{g}')}\, da.$$

We are free to choose $\hat{g}$ and $\hat{g}'$. We assume that both are supported on the regular semi-simple locus. Moreover, we may also assume that for any $a \in (\mathcal{A}')^{\mathrm{rss}}(F)$

$$J_a(\hat{g}) = \eta(\mathrm{disc}(\Phi'))^{n-1}\varepsilon(\eta, 1/2, \psi)^{n(n-1)/2} I^{\Phi'}_a(\hat{g}').$$

This is possible thanks to proposition 3.1.4.1 and its variant for unitary groups. But by theorem 3.4.2.1 for $\widetilde{\mathfrak{g}}'$ which we know by recursion we also have for any $a \in (\mathcal{A}')^{\mathrm{rss}}(F)$

$$J_a(g) = I^{\Phi'}_a(g').$$

Hence for such $(g, g')$ the LHS of (3.4.4.3) and (3.4.4.4) are equal. As a consequence the LHS are also equal. Using the fact that the complex conjugate of $\varepsilon(\eta, 1/2, \psi)$ is its inverse and that $\mathrm{disc}(\Phi) = \mathrm{disc}(\Phi')$, we get:

$$\int_{\mathcal{A}'(F)} \left(J_{\iota(a)}(\hat{f}_1) - \eta(\mathrm{disc}(\Phi))^{n-1}\varepsilon(\eta, 1/2, \psi)^{n(n-1)/2} I^{\Phi}_{\iota(a)}(f'_{\Phi})\right)\overline{J_a(\hat{g})}\, da.$$

Now, we have enough freedom on the choice of $g$ to conclude that (3.4.4.2) holds on a dense subset of $(\mathcal{A}')^{\mathrm{rss}}(F)$ and hence on $(\mathcal{A}')^{\mathrm{rss}}(F)$.

**3.4.5. Aizenbud's result.** — Let's denote by $\mathcal{N}$ the central fiber of $\widetilde{\mathfrak{g}} \to \mathcal{A}$ (i.e. the fiber that contains 0). Let's call it the nilpotent cone of $\widetilde{\mathfrak{g}}$.

**Theorem 3.4.5.1**. —*([Aiz13] theorem 6.2.1) There is no non-zero distribution $T$ on $\widetilde{\mathfrak{g}}(F)$ such that:*

- *$T$ is $\eta$-invariant ;*
- *the support of $T$ is included in $\mathcal{N}(F)$ ;*
- *the support of any partial Fourier transform of $T$ (in the short list of example 2.4.1.1) is included in $\mathcal{N}(F)$.*

By duality, one gets the interesting corollary.

**Corollary 3.4.5.2**. — *Any function $f \in C^\infty_c(\widetilde{\mathfrak{g}}(F))$ can be written as*

$$(3.4.5.5) \qquad f = f^0 + f^1 + \hat{f}^2_{V\oplus V^*} + \hat{f}^3_{\mathfrak{gl}^0(V)} + \hat{f}^4_{\widetilde{\mathfrak{gl}}^0(V)}$$

*where*

1. *$D(f^0) = 0$ for any $\eta$-invariant distribution $D$;*
2. *the functions $f^i$ for $1 \leqslant i \leqslant 4$ belong to $C^\infty_c(\widetilde{\mathfrak{g}}(F) \setminus \mathcal{N}(F))$.*

**Remark 3.4.5.3**. — The theorem 3.4.5.1 holds even for archimedean fields and Schwartz functions. However in this case one cannot deduce corollary 3.4.5.2 from theorem 3.4.5.1. As far as I know, one does not know how to prove corollary 3.4.5.2 for archimedean fields. What one gets is a weaker result: namely any function can be approximated by a sum as in (3.4.5.5) (see [Xue15]).

**3.4.6. Proof of $\mathrm{Orb}(\widetilde{\mathfrak{g}}) \subset \mathrm{Orb}(\mathcal{H})$.** — By using the standard technique of "descent" around semi-simple elements (see [Zha14b] section 3.2) and a recursion on the dimension, one shows that if $f \in C^\infty_c(\widetilde{\mathfrak{g}}(F) \setminus \mathcal{N}(F))$ then $a \mapsto J_a(f) \in \mathrm{Orb}(\mathcal{H})$. Now the inclusion $\mathrm{Orb}(\widetilde{\mathfrak{g}}) \subset \mathrm{Orb}(\mathcal{H})$ is a consequence of corollary 3.4.5.2 and theorem 3.4.2.1.

# 4 A global relative trace formula: infinitesimal situation

## 4.1 Linear case

**4.1.1.** From now on $F$ is a number field with ring of adèles $\mathbb{A}$. Let $E/F$ be a quadratic extension and $\eta$ be the quadratic character of $\mathbb{A}^\times$ attached to $E/F$. The other notations are the same as before (see in particular sec. 2.1).

**4.1.2. Fourier transform.** — We will fix a non-trivial continuous additive character

$$\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times.$$

Recall that one defines partial Fourier transforms (see § 2.4.2) attached to any of the three subspaces defined in example 2.4.1.1.

**4.1.3. A compact subset.** — Adding a line $Fe_0$ to the linear space $V$, one can embed $G$ in the group $\widetilde{G} = GL(V \oplus Fe_0)$ as the subgroup preserving $V$ and acting trivially on $e_0$. To any sufficiently positive parameter $T \in \mathbb{R}^{n+1}$ (i.e. $T = (T_1, \ldots, T_{n+1})$ with $T_i - T_{i+1}$ large enough for $1 \leqslant i \leqslant n$) and other auxiliary data, one defines a Siegel set $\mathfrak{S}^T \subset \widetilde{G}(\mathbb{A})$ and let $F^T$ be the characteristic function of $\widetilde{G}(F)\mathfrak{S}^T$ (see [Art78] section 6). When restricted to

$$[G] = G(F)\backslash G(\mathbb{A})$$

the function $F^T$ is the characteristic function of a compact subset of $[G]$. Let's fix a Haar measure on $G(\mathbb{A})$ and let $dg$ be the quotient measure on $[G]$ (by the counting measure on $G(F)$).

Thus one can define for $f \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}))$

$$J^T(f) = \int_{[G]} F(g,T) \sum_{X \in \widetilde{\mathfrak{g}}(F)} f(g^{-1} \cdot X)\eta(\det(g))\, dg$$

and for any $a \in \mathcal{A}(F)$

$$J_a^T(f) = \int_{[G]} F(g,T) \sum_{X \in \widetilde{\mathfrak{g}}_a(F)} f(g^{-1} \cdot X)\eta(\det(g))\, dg.$$

We have $J^T(f) = \sum_{a \in \mathcal{A}(F)} J_a^T(f)$. Using any partial Fourier transform $f \mapsto \hat{f}_{\widetilde{\mathfrak{g}}_1}$ and the Poisson summation formula for the sum $\sum_{X \in \widetilde{\mathfrak{g}}(F)} f(g^{-1} \cdot X)$ one gets

$$\sum_{a \in \mathcal{A}(F)} J_a^T(f) = \sum_{a \in \mathcal{A}(F)} J_a^T(\hat{f}_{\widetilde{\mathfrak{g}}_1}).$$

This formula is a precursor for the trace formula we are looking for. However to get a useful formula, we must get rid of the parameter $T$ (and the other auxiliary data). This is achieved in the following theorem.

**Theorem 4.1.3.1.** —*(Zydor, cf. [Zyd])*

1. *Let $a \in \mathcal{A}(F)$. There exists a unique exponential-polynomial $P_a(f,T)$ in $T$ such that one has for any $\varepsilon > 0$*
$$J_a^T(f) = P_a(f,T) + O(\exp(-\varepsilon \| T \|))$$
*for sufficiently positive $T$. Let*
$$J_a(f)$$
*be the constant term of $P_a(f,T)$.*

2. *The distribution $J_a$ is $\eta$-invariant (in the sense of definition 3.1.3.2) and does not depend on auxiliary choices but the Haar measure on $G(\mathbb{A})$.*

3. *The support of the distribution is included in* $\widetilde{\mathfrak{g}}_a(\mathbb{A})$.

4. *For any partial Fourier transform* $f \mapsto \hat{f}_{\widetilde{\mathfrak{g}}_1}$, *we have* the infinitesimal relative trace formula

$$\sum_{a\in\mathcal{A}(F)} J_a(f) = \sum_{a\in\mathcal{A}(F)} J_a(\hat{f}_{\widetilde{\mathfrak{g}}_1}).$$

**Remark 4.1.3.2**. — In Zydor's work, there is also a concrete expression for $P_a(f, T)$ based on an integral of a "modified" kernel. This construction is clearly inspired by Arthur's approach to the usual trace formula (see [Art05], for an infinitesimal analog see [Cha02]). However, unlike the usual trace formula, the geometric terms are directly ($\eta$-)invariant. This is a quite remarkable and pleasant: in the usual trace formula the procedure to make the trace formula invariant is rather delicate.

**4.1.4.** Let $a \in \mathcal{A}^{\mathrm{rss}}(F)$. Then there exists $X \in \widetilde{\mathfrak{g}}^{\mathrm{rss}}(F)$, well-defined up to $G(F)$-conjugacy, such that $a(X) = a$ (see lemmas 2.1.5.1 and 2.1.5.3). Then one can express $J_a(f)$ as a global regular semi-simple orbital integral

$$J_a(f) = \int_{G(\mathbb{A})} f(g^{-1}\cdot X)\eta(\det(g))\,dg. \tag{4.1.4.1}$$

When $f$ is a pure tensor $\otimes_{v\in\mathcal{V}} f_v$, it is possible to write this contribution as a product of local regular semi-simple orbital integrals. To be specific, one needs to choose a Haar measure on $G(F_v)$ such that for almost all $v$ the measure of $G(\mathcal{O}_v)$ is 1 and the product of local Haar measures gives the measure on $G(\mathbb{A})$. One also needs to choose an $F$-basis of $V$ that gives the morphism $\delta$ as in §2.1.6. Using this morphism and the decomposition $\eta = \otimes_{v\in\mathcal{V}}\eta_v$, one gets a function $\eta_v$ on $\widetilde{\mathfrak{g}}^{\mathrm{rss}}(F_v)$ (see (3.1.1.2)). Let $a_v$ be the image of $a$ by the canonical map $\mathcal{A}(F) \to \mathcal{A}(F_v)$. We denote by $J_{a_v}(f_v)$ the local regular semi-simple orbital defined in (3.1.3.3). Note that if $v$ is split in $E$ then the character $\eta_v$ is trivial. However everything we state in section 3.1 is still true, *mutatis mutandis*, when one replaces $\eta$ by the trivial character. Then we have

$$J_a(f) = \prod_{v\in\mathcal{V}} J_{a_v}(f_v) \tag{4.1.4.2}$$

where almost all factors are in fact equal to 1.

In general, one does not know how to express the global contribution in terms of local contributions. However for some specific but non-regular semi-simple $a$ see [Zyd].

**4.1.5.** Let $v \in \mathcal{V}$ be a place of $F$. To simplify notation, we assume that $v$ is finite. But the result we will state is also true for archimedean places and Schwartz spaces.

Let $f^v \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}^v))$ where $\mathbb{A}^v$ is the ring of adèles of $F$ "outside $v$" i.e. $\mathbb{A} = F_v \times \mathbb{A}^v$. From the global contribution $J_a$, we get a local distribution

$$f_v \in C_c^\infty(\widetilde{\mathfrak{g}}(F_v)) \mapsto J_a(f_v \otimes f^v).$$

We have the following theorem.

**Theorem 4.1.5.1**. — *(see [CZ] theorem 6.2.2.1) Let* $a \in \mathcal{A}(F)$. *The distribution*

$$f_v \mapsto J_a(f_v \otimes f^v)$$

*is* $\eta_v$*-stable in the sense of definition 3.1.7.1.*

**Remark 4.1.5.2**. — For $a \in \mathcal{A}^{\mathrm{rss}}$, the theorem is a straightforward consequence of formula (4.1.4.1) and splitting formula, e.g. (4.1.4.2).

**4.1.6. Some words on the proof of theorem 4.1.5.1.** — Let's explain some steps in the proof of theorem 4.1.5.1. What we have to show is the following: let $f_v \in C_c^\infty(\widetilde{\mathfrak{g}}(F_v))$ such that all local regular semisimple orbital integrals vanish. Then $J_a(f_v \otimes f^v) = 0$ for any $a \in \mathcal{A}(F_v)$ and any $f^v \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}^v))$. In remark 4.1.5.2, we have already noticed that this holds for $a \in \mathcal{A}^{\mathrm{rss}}(F)$. By recursion and descent to auxiliary situation, one can assume the results for all $a$ but the central one denoted by 0 (the $a$ associated to $0 \in \widetilde{\mathfrak{g}}$). Moreover by proposition 3.1.8.1, we have also

$$J_a(\hat{f}_{v,\widetilde{\mathfrak{g}}_1} \otimes f^v) = 0$$

for any non-central $a$, any $f^v \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}^v))$ and for any partial Fourier transform $f_v \mapsto \hat{f}_{v,\widetilde{\mathfrak{g}}_1}$. As a consequence, applying the trace formula of 4.1.3.1 assertion 4, we get

$$J_0(f_v \otimes f^v) = J_0(\hat{f}_{v,\widetilde{\mathfrak{g}}_1} \otimes \hat{f}^v_{\widetilde{\mathfrak{g}}_1}) \tag{4.1.6.3}$$

for any $f^v \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}^v))$ and any partial Fourier transform. Now let $u \neq v$ be an auxiliary finite place and let $g \in \mathcal{S}(\widetilde{\mathfrak{g}}(\mathbb{A}^{u,v}))$. We can introduce the distribution on $C_c^\infty(\widetilde{\mathfrak{g}}(F_u))$

$$D(f_u) = J_0(f_v \otimes f_u \otimes g).$$

The distribution is $\eta_v$-invariant, with support included in the nilpotent cone (cf. theorem 4.1.3.1 assertion 3). Moreover its partial Fourier transform defined by

$$\hat{D}_{\widetilde{\mathfrak{g}}_1}(f_u) = D(\hat{f}_{u,\mathfrak{g}_1})$$

satisfies (cf. (4.1.6.3))

$$\hat{D}_{\widetilde{\mathfrak{g}}_1}(f_u) = J_0(\hat{f}_{v,\widetilde{\mathfrak{g}}_1} \otimes f^\vee_{u,\widetilde{\mathfrak{g}}_1} \otimes \hat{g}_{\widetilde{\mathfrak{g}}_1}))$$

where $f^\vee_{u,\widetilde{\mathfrak{g}}_1}$ means the partial Fourier transform applied twice. As a consequence, the support of $\hat{D}_{\widetilde{\mathfrak{g}}_1}$ is also included in the nilpotent cone. By theorem 3.4.5.1, one has $D = 0$ and this gives the desired result.

## 4.2 Hermitian case

**4.2.1.** The situation is the same as in section 4.1. We fix a non-degenerate hermitian form $\Phi$ on $V \otimes_F E$. From this we deduce as in section 2.2 a unitary group $U$ acting on a space $\widetilde{\mathfrak{u}}$. We fix a Haar measure on $U(\mathbb{A})$ and this gives an invariant measure on the quotient $[U] = U(F)\backslash U(\mathbb{A})$. As in the linear case for any $a \in \mathcal{A}(F)$ and $f \in \mathcal{S}(\widetilde{\mathfrak{u}}(\mathbb{A}))$, we can consider the truncated integral

$$I_a^T(f) = \int_{[U]} F(g,T) \sum_{X \in \widetilde{\mathfrak{u}}_A(F)} f(g^{-1} \cdot X)\, dg.$$

Here the function $F(\cdot, T)$ is compactly supported but it is simpler than the one used in the linear case: this is the characteristic function of $U(F)\mathfrak{S}^T$ where $\mathfrak{S}^T$ is a Siegel set of $U(\mathbb{A})$ depending on a parameter $T$ and other auxiliary data.

We have almost word-for-word the analog of theorem 4.1.3.1.

**Theorem 4.2.1.1.** —*(Zydor, cf. [Zyd16])*

1. *Let $a \in \mathcal{A}(F)$. There exists a unique exponential-polynomial $P_a(f,T)$ in $T$ such that one has for any $\varepsilon > 0$*

$$I_a^T(f) = P_a(f,T) + O(\exp(-\varepsilon \| T \|))$$

*for sufficiently positive $T$. Let*

$$I_a(f)$$

*be the constant term of $P_a(f,T)$.*

2. *The distribution $I_a$ is invariant (under the natural action of $U(\mathbb{A})$) and does not depend on auxiliary choices but the Haar measure on $U(\mathbb{A})$.*

3. *The support of the distribution is included in $\widetilde{\mathfrak{u}}_a(\mathbb{A})$.*

4. *For any partial Fourier transform $f \mapsto \hat{f}_{\widetilde{\mathfrak{u}}_1}$, we have* the infinitesimal relative trace formula

$$\sum_{a\in\mathcal{A}(F)} I_a(f) = \sum_{a\in\mathcal{A}(F)} I_a(\hat{f}_{\widetilde{\mathfrak{g}}_1}).$$

The main difference is that $\widetilde{\mathfrak{u}}_a(F)$ may happen to be empty and in this case we have $I_a(f) = 0$. For any $a \in \mathcal{A}^{\mathrm{rss}}(F)$ such that $\widetilde{\mathfrak{u}}_a(F) \neq \emptyset$, the fiber $\widetilde{\mathfrak{u}}_a(F)$ is a $U(F)$-orbit of an element say $X \in \widetilde{\mathfrak{u}}_a(F)$. Then we have

$$I_a(f) = \int_{U(\mathbb{A})} f(g^{-1}\cdot X)\,dg. \tag{4.2.1.1}$$

If moreover $f$ is a pure tensor then this expression is a product over all places $v \in \mathcal{V}$ of local orbital integrals defined in §3.2.3.

**4.2.2.** Without any surprise, we also have the following theorem whose proof is parallel to the proof of theoreme 4.1.5.1.

**Theorem 4.2.2.1**. — *(see [CZ] theorem 6.2.2.1) Let $a \in \mathcal{A}(F)$. The distribution*

$$f_v \mapsto I_a(f_v \otimes f^v)$$

*is stable.*

## 4.3 Comparison of relative trace formulae

**4.3.1.** The notations are the same as before. We would like to compare the relative trace formulae in the linear case and in the hermitian case. However, instead of considering one specific hermitian form on $V_E = V \otimes_F E$, one has to consider all of them. Let $\mathcal{H}$ be the set of non-degenerate hermitian on $V_E$ up to equivalence (it is identified with a set of representatives). For $v \in \mathcal{V}$, one has the local analog $\mathcal{H}_v$ of this set. For any $\Phi \in \mathcal{H}$, we have thus distributions $I_a^{\Phi}$.

**4.3.2.** To state the main result we will assume for simplicity that all archimedean places of $F$ split in $E$. In general, we have to deal with the lack of knowledge of conjecture 3.4.1.1. Of course we need also to know the conjecture 3.4.1.1 when the place $v$ split in $E$. Since in this case one can identifiy the hermitian and linear cases, the conjecture is not difficult to prove (even for an archimedean place; see [Zha14b] proposition 2.5).

**4.3.3.** Let $S$ be a finite set of places of $\mathcal{V}$ containing the archimedean places. Let

$$\mathrm{Orb}_S(\widetilde{\mathfrak{g}}) = \otimes_{v\in S}\,\mathrm{Orb}_v(\widetilde{\mathfrak{g}})$$

where $\mathrm{Orb}_v(\widetilde{\mathfrak{g}})$ is the space of functions on $\mathcal{A}(F_v)$ given by local orbital integrals (see §3.4.1). Recall that in §4.1.4 we fix a basis to define a section $\delta$ that is used to split global orbital integrals. Using this basis, one equips $\widetilde{\mathfrak{g}}$ with a structure over the ring of integers of $F$. As a consequence, we get for any $v \in V$ an $\mathcal{O}_v$-module $\widetilde{\mathfrak{g}}(\mathcal{O}_v)$. Let $\mathbf{1}^S$ be the characteristic function of $\prod_{v\notin S}\widetilde{\mathfrak{g}}(\mathcal{O}_v)$. The content of theorem 4.1.5.1 is that the map

$$(f_v)_{v\in S} \mapsto J_a(\otimes_{v\in S} f_v \otimes \mathbf{1}^S)$$

factors through a linear map

$$\mathcal{J}_{a,S} : \mathrm{Orb}_S(\widetilde{\mathfrak{g}}) \to \mathbb{C}.$$

We have indeed the map $(f_v)_{v\in S} \mapsto \otimes_{v\in S} J_{a_v}(f_v) \in \mathrm{Orb}_S(\widetilde{\mathfrak{g}})$.

We can introduce

$$\mathrm{Orb}(\widetilde{\mathfrak{g}}) = \varinjlim_{S} \mathrm{Orb}_S(\widetilde{\mathfrak{g}})$$

where the transition map $\mathrm{Orb}_S(\widetilde{\mathfrak{g}}) \to \mathrm{Orb}_{S'}(\widetilde{\mathfrak{g}})$ is given by the tensor product with orbital integrals of the characteristic function of $\widetilde{\mathfrak{g}}(\mathcal{O}_v)$, $v \in S' \setminus S$. In this case, the linear forms $\mathcal{J}_{a,S}$ give rise to a linear form

$$\mathcal{J}_a : \mathrm{Orb}(\widetilde{\mathfrak{g}}) \to \mathbb{C}.$$

**4.3.4.** Let $\Phi \in \mathcal{H}$. Assume $S$ contains all archimedean places and all places that are ramified in $E$.

Thanks to the conjecture 3.4.1.1 (which is known in our case) the local orbital integrals in the hermitian case (cf. § 3.2.3) give also a map

$$\prod_{v\in S} C_c^\infty(\widetilde{\mathfrak{u}}_\Phi(F_v)) \to \mathrm{Orb}_S(\widetilde{\mathfrak{g}}).$$

Let $\mathcal{O}^S$ be the ring of integers outside $S$. If $S$ is large enough, we may assume that for $v \notin S$ the $\mathcal{O}_v$-lattice generated by the fixed basis of $V$ is autodual. This gives an $\mathcal{O}^S$-structure to $\widetilde{\mathfrak{u}}_\Phi$. Let $\mathbf{1}^S$ be the characteristic function of $\widetilde{\mathfrak{u}}_\Phi(\mathcal{O}^S)$. Then the map

$$(f_v)_{v\in S} \in \prod_{v\in S} C_c^\infty(\widetilde{\mathfrak{u}}_\Phi(F_v)) \to I_a^\Phi(\otimes_{v\in S} f_v \otimes \mathbf{1}^S)$$

factors through a unique linear map

$$\mathcal{I}_{a,S}^\Phi : \mathrm{Orb}_S(\widetilde{\mathfrak{g}}) \to \mathbb{C}$$

such that $\mathcal{I}_a^\Phi(I_S) = 0$ if one of the component of $I_S \in \mathrm{Orb}_S(\widetilde{\mathfrak{g}})$ is trivial on the image of $\widetilde{\mathfrak{u}}_\Phi^{\mathrm{rss}}(F_v)$.

One has to appeal to the following theorem.

**Theorem 4.3.4.1**. — *(Yun-Gordon, cf. [Yun11]) There is finite set $S'$ of places of $F$, which contains all archimedean places and which depends only on $E/F$ and the integer $n$, such that the following holds: for any $v \notin S$ such that $\Phi_v$ has an autodual lattice, one has*

$$J_{a_v}(\mathbf{1}_{\widetilde{\mathfrak{g}}(\mathcal{O}_v)}) = I_{a_v}^{\Phi_v}(\mathbf{1}_{\widetilde{\mathfrak{u}}_{\Phi_v}(\mathcal{O}_v)}) \tag{4.3.4.1}$$

*for any $a_v \in \mathcal{A}^{\mathrm{rss}}(F_v)$.*

**Remark 4.3.4.2**. — This statement is not at all easy: it is the "fundamental lemma" in our situation! Yun first proved the result in the equal characteristic case by geometric methods à la Ngô. Then Gordon used "transfer methods" of Cluckers-Hales-Loeser to deduce the result in the unequal characteristic case (see the appendix of [Yun11]).

Thanks to theorem 4.3.4.1, one gets from linear forms $\mathcal{I}_{a,S}^\Phi$ a linear form

$$\mathcal{I}_a^\Phi : \mathrm{Orb}(\widetilde{\mathfrak{g}}) \to \mathbb{C}$$

**4.3.5.** Finally one can introduce

$$\mathcal{I}_a = \sum_{\Phi\in\mathcal{H}} \mathcal{I}_a^\Phi.$$

This is well-defined because of theorem 4.3.4.1. Indeed, an easy part of it shows that the LHS of (4.3.4.1) vanishes for $a_v$ such that the unique form $\Phi_0 \in \mathcal{H}_v$ such that $\widetilde{\mathfrak{u}}_{\Phi_0,a_v}(F_v) \neq \emptyset$ (cf. proposition 2.2.4.1) has no auto-dual lattice (recall that $\Phi_v$ in (4.3.4.1) does have such a lattice).

Let $I \in \mathrm{Orb}_S(\widetilde{\mathfrak{g}})$. This implies that $\mathcal{I}_a^{\Phi}(I) = 0$ unless $\Phi_v$ has an autodual lattice. But there is only one such form. So $\Phi$ is determined outside $S$. So there is only a finite number of possibilities for $\Phi$. For example, if $a \in \mathcal{A}^{\mathrm{rss}}(F)$, there is only one possible $\Phi$.

**Theorem 4.3.5.1**. — *For any $a \in \mathcal{A}(F)$, we have*

$$\mathcal{J}_a = \mathcal{I}_a.$$

For $a \in \mathcal{A}^{\mathrm{rss}}(F)$, the theorem is a direct consequence of the fact that distributions $J_a$ and $I_a^{\Phi}$ are global orbital integrals (and as such can be expressed as products of local ones). However in general the proof needs a huge recursion and an argument similar to the one used in the proof of theorem 4.1.5.1 (see the sketch of proof in § 4.1.6).

# 5 The Jacquet-Rallis relative trace formulae

## 5.1 Linear case

**5.1.1.** We are still working with a quadratic extension $E/F$ of number fields and an $F$-vector space $V$ of dimension $n$. However in this section we will change the other notations. Let $W = V \oplus Fe_0$. Let $V_E = V \otimes_F E$ and $W_E = W \otimes_F E$. Let $G = GL_E(V_E) \times GL_E(W_E)$ viewed as an $F$-group. The group $G$ has two interesting sugroups. Namely

- $H_1 = GL_E(V_E)$ viewed as an $F$-group and embedded diagonally in $G$;
- $H_2 = GL_F(V) \times GL_F(W)$.

The subgroup $H_1$ acts by left translation and the subgroup $H_2$ acts by right translation. We will denote by $\mathcal{A}$ the categorical quotient $H_1\backslash\backslash G//H_2$. We will denote $\det_V$ and $\det_W$ the two characters of $H_2$ respectively induced by the determinant on the first and second factors.

**5.1.2.** There is an other useful way to look at this action. We can identify the quotient $H_1\backslash G/GL_F(W)$ with the symmetric space $X = GL_E(W)/GL_F(W)$ in a $GL_F(V)$-equivariant way. But now, thanks to the Cayley map, $X$ with the $GL_F(V)$-action is birationally equivalent to the space $\widetilde{\mathfrak{gl}}(V) \oplus F$ (cf. sec. 2.1) with its $GL_F(V)$-action (that is trivial on the factor $F$).

**5.1.3.** Let $f \in C_c^\infty(G(\mathbb{A}))$. The right convolution action $R$ of the (convolution) algebra $C_c^\infty(G(\mathbb{A}))$ on the space $L^2([G])$ gives rise to an operator $R(f)$ whose kernel is

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)$$

where $x, y \in G(\mathbb{A})$. Roughly speaking, in the usual trace formula, one would like to compute the trace of $R(f)$ and one is led to compute the integral of $K_f$ over the diagonal subgroup of $G(\mathbb{A})^2$. However in Jacquet's relative trace formula philosophy, one would like to get other invariants of this action and one is led to consider different subgroups of $G(\mathbb{A})^2$. In the Jacquet-Rallis trace formula (see the seminal paper [JR11]) we would like to interpret the (in general divergent) integral

$$\int_{[H_1]} \int_{[H_2]} k(h_1, h_2)\eta(h_2)\, dh_1 dh_2 \tag{5.1.3.1}$$

where $\eta(h_2)$ is a shorthand for $\eta(\det_V(h_2))^{n+1}\eta(\det_W(h_2))^n$ and $\eta$ is the quadratic character associated to $E/F$. The integral over $[H_1]$ is called a Rankin-Selberg period and it appears in the integral representation of $L$-functions of pairs due to Jacquet-Piatetskii-Shapiro-Shalika. The integral over $[H_2]$ is the Flicker-Rallis period and it is expected to kill the part of the automorphic spectrum of $G$ that does not come from a unitary group via standard base change.

**5.1.4.** In [Zyd15], Zydor introduces a truncated variant of the automorphic kernel inspired by his work on the infinitesimal situation (cf. [Zyd16] and [Zyd]). He thus replaces (5.1.3.1) by a convergent integral which however depends on a truncation parameter. As in the infinitesimal situation, Zydor shows that the expression has a simple behaviour in the parameter (it is a polynomial exponential). The main interesting term is the contant term. To get a "trace formula", Zydor uses two expansions of the truncated kernels: one in terms of geometric invariants (namely $a \in \mathcal{A}(F)$) and the other based on Langlands decomposition of $L^2([G])$ parametrized by the set $\mathfrak{X}$ of cuspidal data $\chi$ (namely pairs $(M, \pi)$ (up to equivalence) of a Levi subgroup $M$ of $G$ and a cuspidal automorphic representation $\pi$ of $M(\mathbb{A})$ ; for more details see [Art05] sections 7 and 10, and also [MW94], [LW13] and the original work of Langlands [Lan76]). We can now state Zydor's theorem.

**Theorem 5.1.4.1**. —*(Zydor, cf. [Zyd15]) Let $f \in C_c^\infty(G(\mathbb{A}))$. There is an absolutely convergent equality of $H_1(\mathbb{A})$-invariant and $(H_2(\mathbb{A}), \eta)$-invariant distributions :*

$$\sum_{a \in \mathcal{A}(F)} J_a(f) = \sum_{\chi \in \mathfrak{X}} J_\chi(f).$$

*These distributions are canonical and depend only on the choice of Haar measures on $H_1(\mathbb{A})$ and $H_2(\mathbb{A})$.*

In the following we will give precise expressions for these distributions for specific data $a$ or $\chi$.

**5.1.5.** First let's consider $a \in \mathcal{A}^{\mathrm{rss}}(F)$. The fiber $G_a$ of the canonical map $G \to \mathcal{A}$ above $a$ has $F$-rational points and moreover $G_a(F)$ is the $H_1(F) \times H_2(F)$-orbit of an element $\gamma$. Then $J_a(f)$ is the global relative orbital integral:

$$J_a(f) = \int_{H_1(\mathbb{A})} \int_{H_2(\mathbb{A})} f(h_1^{-1} \gamma h_2) \eta(h_2) \, dh_1 dh_2.$$

To split the orbital integral into local integrals that depend only on the image $a_v$ of $a$ in $\mathcal{A}(F_v)$ for $v \in \mathcal{V}$, it is convenient to use a specific function (see [Zha14b] section 2.4)

$$\Omega = \otimes_{v \in \mathcal{V}} \Omega_v : G(\mathbb{A}) \to \mathbb{C}^\times$$

trivial on $G(F)$ and that is $H_1(\mathbb{A})$-invariant and $(H_2(\mathbb{A}), \eta)$-equivariant. Then for a pure tensor $f = \otimes_{v \in \mathcal{V}} f_v$, one has

$$J_a(f) = \prod_{v \in \mathcal{V}} J_{a_v}(f_v)$$

where one defines the local orbital integral

$$J_{a_v}(f_v) = \int_{H_1(F_v)} \int_{H_2(F_v)} f(h_1^{-1} \delta h_2) \Omega(h_1^{-1} \delta h_2) \, dh_1 dh_2.$$

In general, it is possible to relate the distributions $J_a$ and their local variants to their infinitesimal counterparts studied in sections 3.1 and 4.1.

**5.1.6.** Then, let's consider a cuspidal datum $\chi = (G, \pi)$ where $\pi$ is a cuspidal automorphic representation of $G(\mathbb{A})$. On the space of $\pi$, we can define two linear forms:

- the Rankin-Selberg period given by

$$\mathcal{P}_{H_1} : \phi \mapsto \int_{[H_1]} \phi(h_1) \, dh_1$$

- the Flicker-Rallis period given by

$$\mathcal{P}_{H_2} : \phi \mapsto \int_{[H_2]^1} \phi(h_2) \eta(h_2) \, dh_2$$

where $[H_2]^1$ is the subset of $[H_2]$ determined by the condition $|\det_V|_{\mathbb{A}} = 1$ and $|\det_W|_{\mathbb{A}} = 1$.

Then the distribution $J_\chi(f)$ is equal to the relative character $J_\pi(f)$ defined by

$$J_\pi(f) = \sum_{\phi\in\mathcal{B}_\pi} \mathcal{P}_{H_1}(\pi(f)\phi)\overline{\mathcal{P}_{H_2}(\phi)}$$

where the sum is taken over an orthonormal basis for the Petersson inner product.

**5.1.7. Decomposition of relative characters.** — Let's still consider the cuspidal datum $(G,\pi)$. We fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$. From this, we construct a generic character of the unipotent radical of the usual Borel subgroup of $G(\mathbb{A})$ that is trivial on the unipotent radical $N_1$ of the usual Borel subgroup of $H_1(\mathbb{A})$. We have an abstract decomposition $\pi = \otimes_{v\in\mathcal{V}_E}\pi_v$ and $\pi_v$ is automatically a generic representation namely it can be identified with the representation of $G(F_v)$ on a space $\mathcal{W}(\pi_v)$ of Whittaker functions for the local component of the character we have fixed. This space comes with a natural inner product (see [Zha14a] section 3.1). We then define two local periods for $W \in \mathcal{W}(\pi_v)$ :

- the local Rankin-Selberg period

$$\mathcal{P}_{H_{1,v}}(W)$$

  which is the value at $s = 0$ of the analytic continuation of

$$\frac{1}{L(s+1/2,\pi_v)}\int_{N_1(F_v)\backslash H_1(F_v)} W(h)|\det(h)|^s\,dh$$

  where $L(s+1/2,\pi_v)$ is a Rankin-Selberg local $L$-factor (more precisely a product of two if $v$ splits in $E$).

- the local Flicker-Rallis period (see [Zha14a] section 3.2)

$$\mathcal{P}_{H_{2,v}}(W).$$

Then as in the global case, we define for $f \in C_c^\infty(G(F_v))$ the local relative character

$$J_{\pi_v}(f) = \sum_W \mathcal{P}_{H_{1,v}}(\pi(f)W)\overline{\mathcal{P}_{H_{2,v}}(W)}.$$

where the sum is over an orthonormal basis of $\mathcal{W}(\pi_v)$.

As a consequence of the work of Jacquet-Piatetskii-Shapiro-Shalika (cf. [JPSS83]) and the work of Flicker on Asai L-functions (cf. [Fli88]) we have (using Tamagawa measures and a decomposition of them)

**Proposition 5.1.7.1.** —*([Zha14a] proposition 3.6) Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A})$ and let $f = \otimes_{v\in\mathcal{V}} f_v$. Then*

$$J_\pi(f) = \frac{L(1/2,\pi)}{L(1,\pi,As^-)}\prod_{v\in\mathcal{V}} J_{\pi_v}(f_v).$$

Here, if $\pi = \pi_V \times \pi_W$ one defines

$$L(s,\pi,As^-) := L(s,\Pi_V,As^{(-1)^n})L(s,\Pi_W,As^{(-1)^{n+1}}) \tag{5.1.7.2}$$

where in the RHS we use Asai L-functions (see [GGP12] §7).

## 5.2 Unitary case and refined Gan-Gross-Prasad conjecture

**5.2.1.** We continue with the same notations as in the previous section. For any hermitian form $\Phi \in \mathcal{H}$ (see §4.3.1), let $\tilde{\Phi}$ on $W_E$ be the hermitian form on $W_E$ such that $V_E$ is orthogonal to $e_0$ and it induces $\Phi$ on $V_E$ and satisfies $\tilde{\Phi}(e_0, e_0) = 1$. Let $G_\Phi$ and $H_\Phi$ be the unitary groups respectively of $(V_E, \Phi)$ and $(V_E \oplus W_E, \Phi \oplus \tilde{\Phi})$. One can embed diagonally $H_\Phi$ in $G_\Phi$. The group $H_\Phi \times H_\Phi$ then acts on $G_\Phi$ by left translation of the first factor and right translation of the second factor.

The categorical quotient $H_\Phi \backslash\backslash G_\Phi // H_\Phi$ can be canonically identified to $\mathcal{A}$ (this identification is tautological after base change to $E$).

**5.2.2.** The quotient $H_\Phi \backslash\backslash G_\Phi$ equipped with the action of $H_\Phi$ is identified with the unitary group $U(W_E, \tilde{\Phi})$ equipped with the action by conjugation of $H_\Phi$. Then thanks to the Cayley map, one can identify birationally this situation to the action of $H_\Phi$ on the space $\widetilde{\mathfrak{u}}(V_E, \Phi) \oplus F$ (see sec. 2.2).

**5.2.3.** Let $f \in C_c^\infty(G_\Phi(\mathbb{A}))$ and $K_f$ be the automorphic kernel. The Jacquet-Rallis trace formula for unitary groups consists in interpreting the integral (which is in general divergent)

$$\int_{[H_\Phi]} \int_{[H_\Phi]} k(h_1, h_2)\, dh_1 dh_2.$$

Here the integral $[H_\Phi]$ is a Gan-Gross-Prasad period that appears in Gan-Gross-Prasad conjectures and their refined versions (see [GGP12], [II10] and [Har14])

**5.2.4.** As in the linear case, Zydor obtains the following theorem.

**Theorem 5.2.4.1**. *—(Zydor, cf. [Zyd15]) Let $f \in C_c^\infty(G_\Phi(\mathbb{A}))$. There is an absolutely convergent equality of $H_\Phi(\mathbb{A}) \times H_\Phi(\mathbb{A})$-invariant distributions :*

$$\sum_{a \in \mathcal{A}(F)} I_a^\Phi(f) = \sum_{\chi \in \mathfrak{X}_\Phi} I_\chi^\Phi(f)$$

*where $\mathfrak{X}_\Phi$ is the set of cuspidal data of $G_\Phi$. These distributions are canonical and depend only on the choice of the Haar measure on $H_\Phi(\mathbb{A})$.*

**5.2.5.** Let $a \in \mathcal{A}^{\mathrm{rss}}(F)$. If the fiber $G_{\Phi,a}(F)$ is empty then $I_a^\Phi(f) = 0$. Otherwise it is the $H_\Phi(F) \times H_\Phi(F)$-orbit of an element $\gamma \in G_\Phi(F)$. Then $I_a(f)$ is the global relative orbital integral:

$$I_a^\Phi(f) = \int_{H_\Phi(\mathbb{A})} \int_{H_\Phi(\mathbb{A})} f(h_1^{-1} \gamma h_2)\, dh_1 dh_2.$$

If $f = \otimes_{v \in \mathcal{V}} f_v$ then

$$I_a^\Phi(f) = \prod_{v \in \mathcal{V}} I_{a_v}^\Phi(f_v)$$

where one defines the local orbital integral

$$I_{a_v}^\Phi(f_v) = \int_{H_\Phi(F_v)} \int_{H_\Phi(F_v)} f(h_1^{-1} \delta h_2)\, dh_1 dh_2.$$

**5.2.6.** Let $\chi = (G, \pi) \in \mathfrak{X}_\Phi$ where $\pi$ is a cuspidal automorphic representation of $G(\mathbb{A})$. Let

$$\mathcal{P}_\Phi : \phi \mapsto \int_{[H_\Phi]} \phi(h)\, dh$$

be the Gan-Gross-Prasad period. The distribution $I_\chi(f)$ is given by the relative character

$$I_\chi(f) = I_\pi(f) = \sum_{\phi \in \mathcal{B}_\pi} \mathcal{P}_\Phi(\pi(f)\phi) \overline{\mathcal{P}_\Phi(\phi)}$$

where the sum is taken over an orthonormal basis for the Petersson inner product.

**5.2.7. Refined Gan-Gross-Prasad conjecture.** — To simplify matters, we assume that $\pi$ is a cuspidal representation of $G_\Phi(\mathbb{A})$ that is everywhere tempered. We set

$$\mathcal{L}_v(s,\pi) = \prod_{i=1}^{n+1} L_v(i,\eta^i) \cdot \frac{L_v(s, BC(\pi))}{L_v(s+\frac{1}{2}, BC(\pi), As^-)}$$

where $BC(\pi)$ is the base change to $G$ (see [Mok15],[KMSW14] which extend the work of Arthur [Art13]), $L_v(s, BC(\pi))$ is the local Rankin-Selberg factor, and $L_v(s, BC(\pi), As^-)$ is defined by the local analog of (5.1.7.2). Sometimes, this factor is called an adjoint $L$-function. In the same way, we define a global factor (which for $s$ of large real part is the Eulerian product of local ones)

$$\mathcal{L}(s,\pi).$$

We assume that the Haar measures $dg$ and $dh$ on $G_\Phi(\mathbb{A})$ and $H_\Phi(\mathbb{A})$ are the Tamagawa measures and we fix local decompositions $dg = \prod_v dg_v$ and $dh = \prod_v dh_v$. We assume that almost everywhere the volume of a hyperspecial subgroup is 1.

We define for $f \in C_c^\infty(G_\Phi(F_v))$ the local relative character by

$$J_{\pi_v}(f) = \frac{1}{\mathcal{L}_v(\frac{1}{2},\pi_v)} \int_{H_\Phi(F_v)} \mathrm{trace}(\pi_v(h_v)\pi_v(f))\, dh_v.$$

The local relative character may vanish identically but this happens if and only if $\pi_v$ is not $H_\Phi(F_v)$-distinguished in the sense that $\mathrm{Hom}_{H_\Phi(F_v)}(\pi_v, \mathbb{C}) = 0$ (see [Beu15] theorem 8.2.1). We say that $\pi$ is $H_\Phi(\mathbb{A})$-distinguished if $\pi_v$ is $H_\Phi(F_v)$-distinguished at all places $v \in \mathcal{V}$. If $J_\pi \neq 0$, then $\pi$ must be $H_\Phi(\mathbb{A})$-distinguished. If $\pi_v$ is $H_\Phi(F_v)$-distinguished we have the multiplicity 1 result (cf. [AGRS10] and [SZ12]).

$$\dim(\mathrm{Hom}_{H_\Phi(F_v)}(\pi_v, \mathbb{C})) = 1.$$

Thus we get:

**Proposition 5.2.7.1**. — *Let assume that $\pi$ is $H_\Phi(\mathbb{A})$-distinguished. There exists a unique constant $c(\pi)$ such that*

$$J_\pi(f) = c(\pi) \prod_{v\in V} J_{\pi_v}(f_v)$$

*for all $f = \otimes_{v\in\mathcal{V}} f_v \in C_c^\infty(G_\Phi(F_v))$.*

We have the following conjecture due to Ichino-Ikeda [II10] and N. Harris [Har14].

**Conjecture 5.2.7.2**. — *We have*

$$c(\pi) = \frac{1}{|S_\pi|}\mathcal{L}(\frac{1}{2},\pi).$$

*where $S_\pi$ is an elementary 2-group associated to $\pi$.*

**Theorem 5.2.7.3**. —*([Zha14a], [BP12]) The conjecture 5.2.7.2 holds if all archimedean places of $F$ split in $E$ and if there is a finite place $v$ such that $BC(\pi_v)$ is supercuspidal.*

**Remark 5.2.7.4**. — R. Beuzart-Plessis has announced the following results ([Beu]). Without the hypothesis on archimedean places the conjecture is true at least up to a sign. Moreover it is possible to formulate the conjecture without assuming $\pi$ is everywhere tempered, the hypothesis that the Arthur parameter is generic suffices. Then the theorem is still true (with the hypothesis at the finite place $v$).

Now we would like to give some ideas on the proof of the theorem 5.2.7.3. Of course one cannot help but notice the strong similarity of the conjecture 5.2.7.2 with the proposition 5.1.7.1. The strategy is indeed to deduce the theorem 5.2.7.3 from the proposition 5.1.7.1.

## 5.3 Comparison of trace formulae

**5.3.1.** We continue with previous notations. In particular $\pi$ is a cuspidal representation of $G_\Phi(\mathbb{A})$ that is $H_\Phi(\mathbb{A})$-distinguished and that satisfies the hypothesis of § 5.2.7. We assume also the hypothesis of theorem 5.2.7.3.

All the results in sections 3 and 4 have obvious analogs in the situation considered since section 5. In our situation, we have, after [Zha14b], a fundamental lemma and the comparison of local orbital integrals (recall that we assume that all archimedean places of $F$ are split in $E$ to avoid the problem of transfer at real and non-split places). We have also an analog of theorem 4.3.5.1 (see [CZ]). All the theorems are deduced from their infinitesimal analogs through a Cayley map. Hence for test functions that give the same orbital integrals on $\prod_v \mathcal{A}^{\mathrm{rss}}(F_v)$ we have full comparisons of geometric sides of relative trace formulae. We get automatically an equality of spectral sides. At the moment, due to a lack of a *fine* spectral expansion of the trace formula, we really understand only the cuspidal parts of the spectral sides. This is where the hypothesis that $BC(\pi_v)$ is supercuspidal enters. This implies that $\pi_v$ is also supercuspidal (see [Beu16] lemma 2.3.1). Moreover $BC(\pi)$ must belong to the cuspidal spectrum.

**5.3.2.** At this point one wants to use coefficients of $\pi_v$ and $BC(\pi_v)$ as local components at $v$ . But we do not know *a priori* that such coefficients give the same orbital integrals. So we begin with the simpler case where the place $v$ splits in $E$ so that the groups $G$ and $G_\Phi$ can be identified at this place. Using the identification for (almost) all places that split in $E$ and an automorphic density theorem à la Cebotarev due to Ramakrishnan (cf. [Ram15]), Zhang proves the following statement. In fact, it is an improved version of it that used results of [CZ] (see [Beu16]). Note also that this theorem uses deep results of [Mok15] and [KMSW14] and also solution of the local Gan-Gross-Prasad conjecture (cf. [Beu15]).

**Theorem 5.3.2.1**. —*(improved version of [Zha14a]) Let $f = \otimes_{w\in\mathcal{V}} f_w \in C_c^\infty(G(\mathbb{A}))$ and $f^\Phi = \otimes f^\Phi_{w\in\mathcal{V}} \in C_c^\infty(G^\Phi(\mathbb{A}))$ such that $f_v \simeq f_v^\Phi$ is a coefficient of $\pi_v$ and that at all places the functions $f_w$ and $f_w^\Phi$ match. Then we have*

$$J_{BC(\pi)}(f) = 4 I_\pi(f^\Phi).$$

Here and in the following we say that $f_w$ and $f_w^\Phi$ match if their regular semisimple orbital integrals are equal on the image of $G_\Phi^{\mathrm{rss}}(F_w)$ in $\mathcal{A}^{\mathrm{rss}}(F_w)$.

We have the following corollary:

**Corollary 5.3.2.2**. — *For $w \neq v$, there exists $\kappa(\pi_w) \in \mathbb{C}$ such that*

$$J_{BC(\pi_w)}(f_w) = \kappa(\pi_w) I_{\pi_w}(f_w^\Phi)$$

*if $f_w$ and $f_w^\Phi$ match.*

From this corollary, by a globalisation and density result, one can show :

**Corollary 5.3.2.3**. —*([Beu16] proposition 4.2.1) For any place $w$ and any tempered $H_\Phi(F_w)$-distinguished representation $\sigma$ of $G_\Phi(F_w)$, there exists $\kappa(\sigma) \in \mathbb{C}$ such that*

$$J_{BC(\sigma)}(f_w) = \kappa(\sigma) I_\sigma(f_w^\Phi) \tag{5.3.2.1}$$

*if $f_w$ and $f_w^\Phi$ match.*

In theorem 5.3.2.1, the integer 4 has to be interpreted as $|S_\pi|$. So the next point is to compute $\kappa(\sigma)$ and to show that we have:

$$\prod_{v\in\mathcal{V}} \kappa(\pi_v) = \prod_{i=1}^{n+1} L(i, \eta^i).$$

This is achieved by using:

- some kind of "local character expansion" of relative characters on $G$ in terms of Fourier transform of a regular nilpotent orbital integral in the infinitesimal situation;
- the results of section 3;
- the following local variant of the relative trace formula (see [Zha14b] part 2, [Beu16] section 4).

**Theorem 5.3.2.4.** —*([Beu16] section 4.3) Let $w$ be a non-split of $F$. For any $f_1, f_2 \in C_c^\infty(G_\Phi(F_w))$, we have*

$$\int_{\mathcal{A}(F_w)} I_a(f_1)I_a(f_2)\,da = \int_{Temp_{H_\Phi}(G_\Phi(F_w))} J_\pi(f_1)J_{\pi^\vee}(f_2)\mathcal{L}(1/2,\pi)\mathcal{L}(1/2,\pi^\vee)d\mu(\pi) \tag{5.3.2.2}$$

*where $da$ is a suitable measure, $d\mu$ is the Harish-Chandra-Plancherel measure and $Temp_{H_\Phi}(G_\Phi(F_w))$ is the set of tempered $H_\Phi(F_w)$-distinguished representations of $G_\Phi(F_w)$.*

**5.3.3.** Finally to get rid of the hypothesis that $v$ splits in $E$, one uses the following result (based on the relative trace formula (5.3.2.2) which is a kind of converse to the corollary 5.3.2.3.

**Proposition 5.3.3.1.** —*([Beu16] corollary 4.5.1) If* (5.3.2.1) *holds for any tempered $H_\Phi(F_w)$-distinguished representation $\sigma$ of $G_\Phi(F_w)$ then $f_w$ and $f_w^\Phi$ match.*

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