On the counting of Hitchin bundles

Pierre-Henri Chaudouard

Abstract

We sketch a link between the Lie algebra version of the Arthur-Selberg trace formula and the counting of semistable Hitchin bundles on a projective curve over a finite field.

1 Introduction

This note is a report on papers [4] and [7] which are part of a joint work with Gérard Laumon. Our goal is to get a formula for the counting of semistable Hitchin bundles on a projective curve over a finite field using the Arthur-Selberg trace formula (in fact a variant for Lie algebras). First in section 2 we recall the notion of Hitchin bundles. We also give some geometric motivation to our work. Then, in section 3, we introduce a notion of T-semistability which is inspired by the work of Langlands and Arthur on automorphic forms. We give some basic properties of the counting of T-semistable Hitchin bundles. The main theorem of the section is that, in the most interesting cases, it suffices to compute the constant term of a polynomial function attached to the counting of T-semistable *nilpotent* Hitchin bundles. In section 4, this polynomial function is expressed as a nilpotent adelic integral which is reminiscent of the nilpotent part of Arthur-Selberg trace formula. In section 5, we give an expansion of this polynomial function in terms of polynomial functions attached to nilpotent orbits. Our main philosophy is that the constant term of the latter polynomial functions should admit nice formulae. In the final 6, we give some examples of constant terms we were able to compute. Our hope is that this note will be a useful companion to the papers [4] and [7].

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2 Betti numbers of Hitchin moduli space

2.1. Let k be an algebraically closed field. Let C be an algebraic projective curve over k. We assume that C is smooth and connected. Let g_C be the genus of C.

2.2. Vector bundles. — By vector bundle on C, we mean a locally free $\mathcal{O}_C$-module which is of finite rank. One can attach to any vector bundle $\mathcal{E}$ on C

- its rank $r(\mathcal{E}) \in \mathbb{N}$
- its degree $\deg(\mathcal{E}) \in \mathbb{Z}$
- its slope $\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{r(\mathcal{E})} \in \mathbb{Q}$ whenever $r(\mathcal{E}) > 0$.

2.3. Hitchin bundles. — Let D be a divisor on C. A *Hitchin bundle* (attached to the divisor D) is a pair $(\mathcal{E}, \theta)$ where

- $\mathcal{E}$ is a vector bundle on C ;
- $\theta : \mathcal{E} \to \mathcal{E}(D) = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_C(D)$ is a morphism of $\mathcal{O}_C$-modules.

We will say that θ is a twisted endomorphism of $\mathcal{E}$. When D is a canonical divisor (that is the divisor of a meromorphic differential form) then Hitchin bundles are called Higgs bundles in the literature.

The rank, the degree and the slope of a Hitchin bundle $(\mathcal{E}, \theta)$ are the rank, the degree and the slope of the underlying vector bundle $\mathcal{E}$.

2.4. Characteristic polynomials of Hitchin bundles. — Let $(\mathcal{E}, \theta)$ be a Hitchin bundle attached to the divisor D. We can view the twisted endomorphism θ as a morphism

$$\mathcal{O}_C \to \mathcal{E}^\vee \otimes_{\mathcal{O}_C} \mathcal{E}(D), \tag{2.4.1}$$

where $\mathcal{E}^\vee$ is the dual vector bundle of $\mathcal{E}$. Using the canonical pairing

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_C} \mathcal{E}(D) \to \mathcal{O}(D), \tag{2.4.2}$$

we see that the composition of the two maps (2.4.1) and (2.4.2) gives a morphism

$$\mathcal{O}_C \to \mathcal{O}_C(D).$$

This is the trace of θ denoted by trace(θ). By construction, it belongs to $H^0(C, \mathcal{O}_C(D))$. Using exterior powers, for $1 \leqslant i \leqslant r(\mathcal{E})$, we can form the vector bundle $\wedge^i\mathcal{E}$ of rank $\binom{r(\mathcal{E})}{i}$ and the Hitchin bundle $(\wedge^i\mathcal{E}, \wedge^i\theta)$ where

$$\wedge^i\theta : \wedge^i\mathcal{E} \to \wedge^i\mathcal{E} \otimes_{\mathcal{O}_c} \mathcal{O}_C(iD).$$

So the Hitchin bundle $(\wedge^i\mathcal{E}, \wedge^i\theta)$ is attached to the divisor iD and trace$(\wedge^i\theta)$ belongs to $H^0(C, \mathcal{O}_C(iD))$.

The *characteristic polynomial* of θ (or more precisely $(\mathcal{E}, \theta)$) is then defined by the usual formula

$$\chi_\theta = X^n - a_1(\theta)X^{n-1} + \ldots + (-1)^n a_n(\theta)$$

where

- $n = r(\mathcal{E})$ is the rank of $\mathcal{E}$;
- $a_i(\theta) \in H^0(C, \mathcal{O}_C(iD))$;
- $a_i(\theta) = \text{trace}(\wedge^i\theta)$.

Let us remark that the characteristic polynomial of a Hitchin bundle of rank n attached to a divisor belongs to the *finite dimensional* affine space over k

$$\mathcal{A} \simeq \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(iD)).$$

Using Riemann-Roch formula, one can easily compute its dimension in the following cases :

1. the degree of the divisor D satisfies $\deg(D) > 2g_C - 2$

$$\dim(\mathcal{A}) \quad = \quad n(1 - g_C) + \frac{n(n+1)}{2}\deg(D);$$

2. the divisor D is canonical and $g_C > 1$; in this case one has $\deg(D) = 2g_C - 2$ so by Serre duality one has $\dim(H^1(C, \mathcal{O}(iD))) = 0$ if $i > 1$ and $\dim(H^1(C, \mathcal{O}(D)) = 1$; one gets if we denote by χ the Euler characteristic

$$\begin{aligned} \dim(\mathcal{A}) &= \sum_{i=1}^{n} \chi(C, \mathcal{O}(iD)) + \dim(H^1(C, \mathcal{O}(D)) \\ &= n^2(g_C - 1) + 1; \end{aligned}$$

3. if D is a canonical divisor, the affine space $\mathcal{A}$ is reduced to a point if $g_C = 0$ and it is of dimension n if $g_C = 1$;

4. the degree of the divisor D is < 0. Then $\mathcal{A}$ is reduced to a point.

2.5. Coarse moduli space of Hitchin bundles. — Let us fix a rank $n \in \mathbb{N}^*$ and a degree $e \in \mathbb{Z}$. We want to consider a coarse moduli space which parametrizes the set of isomorphism classes of Hitchin bundles of rank n and degree e. To do this, we have to restrict ourselves to the *semistable* or *stable* Hitchin bundles.

Let $(\mathcal{E}, \theta)$ be a Hitchin bundle. Let us say that a subbundle $(0) \subsetneq \mathcal{F} \subsetneq \mathcal{E}$ is θ-invariant if the composition

$$\mathcal{F} \to \mathcal{E} \xrightarrow{\theta} \mathcal{E}(D) \to \mathcal{E}(D)/\mathcal{F}(D)$$

is zero. In this case, $(\mathcal{F}, \theta_{|\mathcal{F}})$ is also a Hitchin bundle.

Definition 2.5.1. — A Hitchin bundle $(\mathcal{E}, \theta)$ is *semistable* (resp. *stable*) if, for any θ-invariant subbundle $(0) \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, we have the slope inequality

$$\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$$

(resp. $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

Remark 2.5.2. — In the following, we will often restrict ourselves to the so-called coprime case: this means that the degree e and the rank n are coprime. One observes that the notions of stability and semistability are the same in this case.

Theorem 2.5.3. —*(Nitsure, [25])*

Assume that either $\deg(D) > 2g_C - 2$ *or D is a canonical divisor.*

There exists a smooth and quasi-projective scheme

$$\mathbf{M}(D, n, e)$$

over k which is a coarse moduli scheme for stable Hitchin bundles $(\mathcal{E}, \theta)$ with $r(\mathcal{E}) = n$ and $\deg(\mathcal{E}) = e$.

Remark 2.5.4. — Nitsure also constructed a moduli scheme for *semistable* Hitchin bundles but one has to replace isomorphism classes by classes under a slightly different equivalence relation. Except the coprime case, it may be non-smooth.

We call the scheme $\mathbf{M}(D, n, e)$ the *Hitchin moduli space*. If there is no ambiguity, we will omit D, n and e and we will simply denote

$$\mathbf{M} = \mathbf{M}(D, n, e).$$

2.6. Properness of the Hitchin morphism. — The Hitchin morphism

$$f : \mathbf{M}(D, n, e) \to \mathbf{A}$$

is given by

$$f(\mathcal{E}, \theta) = \chi_\theta.$$

Theorem 2.6.1. —*(Nitsure, [25])*

Assume that either $\deg(D) > 2g_C - 2$ *or D is a canonical divisor. Assume also that e and n are coprime.*

The Hitchin morphism

$$f : \mathbf{M}(D, n, e) \to \mathbf{A}$$

is proper.

Remark 2.6.2. — If e and n are not coprime, then the Hitchin morphism is not proper. To recover properness, one has to replace $\mathbf{M}(D, n, e)$ by the moduli scheme of semistable Hitchin bundles. But then one may lose smoothness over k (cf. remark 2.5.4). Although generically smooth, the fibers of f may be very singular.

2.7. Betti numbers of the Hitchin moduli space M. — We are in the following situation :

1. D is a canonical divisor;
2. the degree e and the rank n are coprime;
3. the base field k is the complex field $\mathbb{C}$.

A fundamental question is the following one:
Problem : What are the Betti numbers of $\mathbf{M}$?

- They were computed in rank 2 by Hitchin (cf. [17]) and in rank 3 by Gothen (cf. [12]).
- In the beautiful paper [15], Hausel and Rodriguez-Villegas gave a conjectural formula for the mixed Hodge polynomial of some character varieties. These are affine varieties which are known to be diffeomorphic to Hitchin moduli spaces by the so-called non-abelian Hodge theory (cf. in rank 2 the paper [17] of Hitchin, cf. [27] for more general results). Note that there is a recent string theoretic approach to this conjecture by Chuang, Diaconescu, Donagi and Pantev (cf. [9]).
- Since the Poincaré polynomial is just a specialisation of the mixed Hodge polynomial, the Hausel-Rodriguez-Villegas conjecture implies a conjectural formula for the Poincaré polynomial of a character variety and its diffeomorphic Hitchin moduli space.
- The Hausel-Rodriguez-Villegas formula can be refined in a formula which gives the motive of $\mathbf{M}$ (Mozgovoy, [21]).
- García-Prada, Heinloth and Schmitt gave a recursion formula for this motive (cf. [11]). They checked Hausel-Rodriguez-Villegas formula for the Betti numbers of Hitchin moduli space in rank 4 and small genus.
- Schiffmann (cf. [26], cf. also [22] for non-canonical divisors D) has very recently found a closed formula for the Poincaré polynomial of $\mathbf{M}$. In fact, he has obtained a closed formula for the counting of Higgs bundles over finite fields. Although his formula is in the same spirit, it is not clear for the moment if his formula implies Hausel-Rodriguez-Villegas-Mozgovoy conjectural formula.

2.8. Our approach. — Here we still assume the degree e and the rank n are coprime. However the divisor D is either canonical or of degree $> 2g_C - 2$. We follow the approach of Harder-Narasimhan (cf. [14]). To compute the Betti numbers of $\mathbf{M}$ over $\mathbb{C}$, it should suffice to compute the ℓ-adic Betti numbers of

$$\mathbf{M} = \mathbf{M}(D, n, e)$$

in the cases where the base field k is the algebraic closure of a finite field and ℓ is a prime number distinct from the characteristic of k. Let us consider this case. We have a $\mathbb{G}_{m,k}$-action on $\mathbf{M}$ given by

$$t \cdot (\mathcal{E}, \theta) = (\mathcal{E}, t\theta).$$

The action gives a homotopy between the Hitchin moduli space $\mathbf{M}$ and the 0-fiber of the Hitchin morphism f. This fiber $f^{-1}(0)$ is the so-called global nilpotent cone which parametrizes stable Hitchin bundles $(\mathcal{E}, \theta)$ such that θ is everywhere nilpotent. In particular, $\mathbf{M}$ and $f^{-1}(0)$ have the same cohomology groups. By an usual argument which combines Deligne's theorem on weights on ℓ-adic cohomology (cf. [10]) and Poincaré duality for the smooth k-scheme $\mathbf{M}$, we see that the cohomology of $\mathbf{M}$ is *pure*. If the curve C comes from base change from a curve C_0 over $\mathbb{F}_q$ and if the divisor D is also defined over $\mathbb{F}_q$, then $\mathbf{M}$ comes by base change from a scheme $\mathbf{M}_0$ which classifies Hitchin bundles over C_0. In particular, the number $|\mathbf{M}_0(\mathbb{F}_{q^d})|$ of $\mathbb{F}_{q^d}$-points of M_0 is the number of isomorphism classes of Hitchin bundles over $C_0 \times_{\mathbb{F}_q} \mathbb{F}_{q^d}$. Now, we have the Grothendieck-Lefschetz trace formula

$$|\mathbf{M_0}(\mathbb{F}_{q^d})| = \sum_{i=0}^{2\dim(\mathbf{M})} (-1)^i \operatorname{trace}(F^d, H^i_c(\mathbf{M}, \mathbb{Q}_\ell)),$$

where F is the geometric Frobenius.

As a consequence of the purity of the cohomology, one can deduce the Betti numbers of $\mathbf{M}$ from the number of points $|\mathbf{M}_0(\mathbb{F}_{q^d})|$ for any finite extension $\mathbb{F}_{q^d}$ of $\mathbb{F}_q$.

In this way, we are looking for an explicit formula for the number of points $|\mathbf{M}_0(\mathbb{F}_{q^n})|$. Our strategy is to use Langlands-Arthur truncations which appear in the theory of automorphic forms (cf. [18] and [1]). It is indeed possible to understand the number of points $|\mathbf{M}_0(\mathbb{F}_{q^n})|$ as the value of the Arthur-Selberg trace formula on a very simple test function (cf. [4]). For the *elliptic part* of the Hitchin moduli, this has been previously noticed by Ngô in his work (cf. [23] and [24]) on the fundamental lemma ; Laumon and myself have also extended Ngô's observation to the regular semi-simple part of the Hitchin fibration (cf. the first part [8] of our work on the weighted fundamental lemma). So an other goal of our approach (or a byproduct) is to obtain a more explicit form for the nilpotent part of the Arthur-Selberg trace formula. A formula was obtained by Arthur many years ago (cf. [2]) but with essentially unknown global coefficients.

3 Reduction to a counting of nilpotent Hitchin bundles.

3.1. Notations. — Let $\mathbb{F}_q$ be a finite field with q elements. Let C be an algebraic projective curve over $\mathbb{F}_q$. We assume that C is smooth and geometrically connected. Let D be a divisor over C. The other notations are the same as in section 2.

3.2. Groupoid of Hitchin bundles. — For any groupoid $\mathcal{Z}$ (a category in which any morphism is an isomorphism), one defines its mass

$$|\mathcal{Z}| = \sum_z \frac{1}{|\operatorname{Aut}(z)|}$$

where the sum is over the set of isomorphism classes of objects of $\mathcal{Z}$. The group $\operatorname{Aut}(z)$ is the group of automorphisms of z. It may be infinite but in this case we take $\frac{1}{|\operatorname{Aut}(z)|} = 0$. The mass can be finite (it can be of finite support or just convergent) or infinite.

Let

$$\mathcal{M}(D, n, e)$$

be the groupoid of Hitchin bundles $(\mathcal{E}, \theta)$ over C with

- $\theta : \mathcal{E} \to \mathcal{E}(D)$ a twisted endomorphism;
- $\mathcal{E}$ a vector bundle on C of rank n and degree e.

Let

$$\mathcal{M}^{ss}(D, n, e) \subset \mathcal{M}(D, n, e)$$

be the full subgroupoid of semistable Hitchin bundles over C. Up to isomorphism, $\mathcal{M}^{ss}(D, n, e)$ has a finite number of objects (it is the groupoid of $\mathbb{F}_q$-points of the stack of semi-stable Hitchin bundles of degree e which is known to be of finite type) ; so its mass $|\mathcal{M}^{ss}(D, n, e)|$ is finite. Let us emphasize the following observation (here we use the notation $\mathbf{M}_0 = \mathbf{M}_0(D, n, e)$ introduced in §2.8):

Remark 3.2.1. — When $(e, n) = 1$, for any $(\mathcal{E}, \theta) \in \mathcal{M}^{ss}(D, n, e)$, we have

$$\mathrm{Aut}(\mathcal{E}, \theta) \simeq \mathbb{F}_q^{\times}.$$

The set of isomorphism classes of $\mathcal{M}^{ss}(D, n, e)$ can be identified to $\mathbf{M}_0(\mathbb{F}_q)$. We get the simple relation

$$|\mathbf{M}_0(\mathbb{F}_q)| = (q-1)|\mathcal{M}^{ss}(D, n, e)|.$$

We were looking for a formula for $|\mathbf{M}_0(\mathbb{F}_q)|$. In fact, it will be more convenient to work with the groupoids $\mathcal{M}^{ss}(D, n, e)$. By the previous remark, it is equivalent to find a formula for the mass of $\mathcal{M}^{ss}(D, n, e)$.

Remark 3.2.2. — The mass $|\mathcal{M}^{ss}(D, n, e)|$ depends only on $e \mod n$. One has even a stronger result: the category $\mathcal{M}^{ss}(D, n, e)$ and $\mathcal{M}^{ss}(D, n, e + n)$ are equivalent. It suffices to choose a line bundle $\mathcal{L}$ of degree 1 over C. Then $(\mathcal{E}, \theta) \mapsto (\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{L}, \theta \otimes 1)$ gives an equivalence between the categories $\mathcal{M}(D, n, e)$ and $\mathcal{M}(D, n, e + n)$. It is an easy exercise to see that this equivalence preserves stability.

3.3. An example of computation: the case of vector bundles. — Let

$$Bun_{n,e}$$

be the groupoid of vector bundles $\mathcal{E}$ of rank n and degree e over C. One has the beautiful (cf. [28] or [13])

Siegel formula

$$|Bun_{n,e}| = q^{n^2(g_C-1)}\zeta^*(q^{-1})\zeta(q^{-2})\ldots\zeta(q^{-n}) \tag{3.3.1}$$

where

- $\zeta(t) = \exp(\sum_{r=1}^{\infty} |C(\mathbb{F}_{q^r})| t^r/r)$ is the zeta function of C; it is a rational function with denominator $(1-t)(1-qt)$.
- $\zeta^*(t) = (1-qt)\zeta(t)$.

Remark 3.3.1. —

- Here the set of isomorphism classes of $Bun_{n,e}$ is infinite. Nonetheless the mass is convergent.
- The answer does not depend on the degree e.

From the Siegel formula, it is possible to get a closed formula for the mass of the groupoid of semistable vector bundles (one has to solve the Harder-Narasimhan recursion cf. for example [19]; for a more direct argument see [4] §2.7).

3.4. The case of Hitchin bundles. — First we have the following theorem (which can be proved as in the case of vector bundles)

Theorem 3.4.1. —*(Harder-Narasimhan filtration.)*
For any Hitchin bundle $(\mathcal{E}, \theta)$, there is a unique *filtration of θ-invariant subbundles*

$$(0) = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_r = \mathcal{E}$$

such that

- *the quotients $(\mathcal{F}_{i+1}/\mathcal{F}_i, \theta)$ are semistable Hitchin bundles;*

- *the slopes of the quotients are decreasing*

$$\mu(\mathcal{F}_1/\mathcal{F}_0) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \dots$$

From the theorem, we deduce the Harder-Narasimhan recursion formula: formally, the difference

$$|\mathcal{M}(D,n,e)| - |\mathcal{M}^{ss}(D,n,e)|$$

can be expressed in terms of $|\mathcal{M}^{ss}(D,n',e')|$ for $n' < n$. However there is a serious difficulty : the mass

$$\begin{aligned} |\mathcal{M}(D,n,e)| &= \sum_{(\mathcal{E},\theta)} \frac{1}{|\operatorname{Aut}(\mathcal{E},\theta)|} \\ &= \sum_{\mathcal{E}\in Bun_{n,e}} \frac{|\operatorname{Hom}(\mathcal{E},\mathcal{E}(D))|}{|\operatorname{Aut}(\mathcal{E})|} \end{aligned}$$

is infinite if $n > 1$. The second equality comes from the obvious equality

$$|\operatorname{Hom}(\mathcal{E},\mathcal{E}(D))| = \sum_{\theta} \frac{|\operatorname{Aut}(\mathcal{E})|}{|\operatorname{Aut}(\mathcal{E},\theta)|}$$

where the sum of is over the set of orbits of $|\operatorname{Aut}(\mathcal{E})|$ on $\operatorname{Hom}(\mathcal{E},\mathcal{E}(D))$. In fact the mass of $\mathcal{M}(D,n,e)$ diverges in a very strong sense: the ratios

$$\frac{|\operatorname{Hom}(\mathcal{E},\mathcal{E}(D))|}{|\operatorname{Aut}(\mathcal{E})|}$$

are essentially constant on very unstable Harder-Narasimhan strata of $Bun_{n,e}$ of given ranks.

3.5. The notion of T-semistability. — Since $\mathcal{M}(D,n,e)$ is of infinite mass, we will approximate it by a large full subcategory of finite mass. The construction is directly inspired by Arthur's truncation (cf. [1]). Let $T \in \mathbb{Z}^n$ a stability parameter. We assume that $T \geqslant 0$ in the sense that $T_1 \geqslant T_2 \geqslant \ldots \geqslant T_n$. Let us introduce the following definition

Definition 3.5.1. — A Hitchin bundle $(\mathcal{E},\theta)$ is T-semistable if for any θ-invariant subbundle $(0) \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, we have the slope inequality

$$\mu(\mathcal{F}) - \frac{T_1 + \ldots + T_{r(\mathcal{F})}}{r(\mathcal{F})} \leqslant \mu(\mathcal{E}) - \frac{T_1 + \ldots + T_{r(\mathcal{E})}}{r(\mathcal{E})}.$$

Remarks 3.5.2. —

- If $T = 0$, the T-semistability is nothing else but the usual semistability.
- One has also an obvious notion of T-semistability for vector bundles where the slope inequality must hold for *any* subbundle $(0) \subsetneq \mathcal{F} \subsetneq \mathcal{E}$.
- So we can consider two notions of T-semistability on a Hitchin bundle $(\mathcal{E},\theta)$: one for the pair $(\mathcal{E},\theta)$ and the other for the underlying vector bundle $\mathcal{E}$. For a comparison of the two notions see lemma 3.8.2 below.

The T-semistability defines a full subgroupoid

$$\mathcal{M}^{\leqslant T}(D,n,e) \subset \mathcal{M}(D,n,e).$$

Since $\mathcal{M}^{\leqslant T}(D,n,e)$ has only a finite number of objects up to isomorphism (the stack of T-semistable Hitchin bundles of given rank and degree is of finite type), its mass is indeed finite

$$|\mathcal{M}^{\leqslant T}(D,n,e)| < \infty.$$

Now the problem is the following:

- can we compute $|\mathcal{M}^{\leqslant T}(D,n,e)|$ for large T and apply Harder-Narasimhan recursion to get $|\mathcal{M}^{ss}(D,n,e)|$?

For the second part of the question, the answer is yes and it is nicely formulated in term of the qualitative behavior of $|\mathcal{M}^{\leqslant T}(D,n,e)|$ in T.

3.6. A quasi-polynomial counting. — Let us introduce the following definition.

Definition 3.6.1. — A function $\mathbb{Z}^n \to \mathbb{C}$ is a *quasi-polynomial* if is a finite sum of products $\psi \cdot P$ where $\psi : \mathbb{Z}^n \to \mathbb{C}^\times$ is a character of finite order and P is a polynomial.

Let $\alpha \in \mathbb{R}$. We write $T \geqslant \alpha$ if $T_i - T_{i+1} \geqslant \alpha$ for all $1 \leqslant i \leqslant n-1$.

Theorem 3.6.2. —*(cf. [4])*

1. *For any divisor D, there exists a unique quasi-polynomial $T \mapsto J_D^{T,e}$ such that*

$$|\mathcal{M}^{\leqslant T}(D,n,e)| = J_D^{T,e}$$

for any $T \in \mathbb{Z}^n$ such that $T \geqslant \max(0, \deg(D), 2g-2-D)$.

2. *If $\deg(D) \geqslant 2g_C - 2$, one can replace the above condition on T by the weaker condition*

$$T \geqslant 0.$$

In particular, one has

$$|\mathcal{M}^{ss}(D,n,e)| = J_D^{0,e}.$$

Proof. — The second assertion is corollary 4.5.6 of [4]. The first assertion is not explicitly stated in [4]. But it is a straightforward combination of formula (4.1.3), proposition 5.1.1 and theorem 5.2.1 of [4]. Let us sketch the proof in the case where $\deg(D) \geqslant 2g_C - 2$. One could also obtain a rather direct proof of assertion 1 (and a better lower bound on T) by a refinement of the arguments given below.

Let $1 \leqslant r \leqslant n$. Let $(e_i)_{1\leqslant i\leqslant r} \in \mathbb{Z}^r$ and $(n_i)_{1\leqslant i\leqslant r} \in (\mathbb{N}^*)^r$ such that the three following conditions hold

$$e_1 + \ldots + e_r = e \tag{3.6.1}$$

$$n_1 + \ldots + n_r = n \tag{3.6.2}$$

$$\frac{e_1}{n_1} > \frac{e_2}{n_2} > \ldots > \frac{e_r}{n_r}. \tag{3.6.3}$$

Let $\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$ be the groupoid of Hitchin bundles $(\mathcal{E},\theta)$ whose Harder-Narasimhan filtration $\mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_r = \mathcal{E}$ satisfies

$$r(\mathcal{F}_{i+1}/\mathcal{F}_i) = n_i$$

and

$$\deg(\mathcal{F}_{i+1}/\mathcal{F}_i) = e_i.$$

When $r=1$, one has $n_1 = n$ and $e_1 = e$ and $\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$ is nothing else but $\mathcal{M}^{ss}(D,n,e)$ the groupoid of semistable Hitchin bundles. By uniqueness of the Harder-Narasimhan filtration, $\mathcal{M}(D,n,e)$ is a *disjoint union* of $\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$ when $1 \leqslant r \leqslant n$ and $(e_i)_{1\leqslant i\leqslant r} \in \mathbb{Z}^r$ and $(n_i)_{1\leqslant i\leqslant r} \in (\mathbb{N}^*)^r$ satisfy the three conditions (3.6.1), (3.6.2) and (3.6.3).

Let us fix $1 \leqslant r \leqslant n$ and $(n_i)_{1\leqslant i\leqslant r} \in (\mathbb{N}^*)^r$ such that (3.6.2) holds. Let

$$\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r}) = \bigcup \mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$$

where the union is over the degrees which satisfy the conditions (3.6.1) and (3.6.3). Let us introduce

$$\mathcal{M}^{\leqslant T}(D,(n_i)_{1\leqslant i\leqslant r}) = \mathcal{M}^{\leqslant T}(D,e,n)\cap\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r}).$$

If $r=1$ this is simply $\mathcal{M}^{ss}(D,n,e)$ whose mass does not depend on T. Assume $r>1$. We just have to check that the mass of $\mathcal{M}^{\leqslant T}(D,(n_i)_{1\leqslant i\leqslant r})$ coincides with a quasi-polynomial in T whose value at $T=0$ is 0.

Let $(\mathcal{E},\theta)$ with Harder-Narasimhan filtrations $\mathcal{F}_0\subsetneq\mathcal{F}_1\subsetneq\ldots\subsetneq\mathcal{F}_r=\mathcal{E}$ with rank $r(\mathcal{F}_{i+1}/\mathcal{F}_i)=n_i$. The map

$$(\mathcal{E},\theta)\mapsto((\mathcal{F}_1/\mathcal{F}_0,\theta),\ldots,(\mathcal{F}_r/\mathcal{F}_{r-1},\theta)).$$

gives rise to a functor

$$\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})\to\prod_{i=1}^{r}\mathcal{M}^{ss}(D,n_i,e_i) \tag{3.6.4}$$

for any $(e_i)_{1\leqslant i\leqslant r}\in\mathbb{Z}^r$.

This functor enables us to analyse the mass of

$$\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$$

in terms of the mass of $\mathcal{M}^{ss}(D,n_i,e_i)$ and the mass of the fibers of (3.6.4). Under our assumption $\deg(D)\geqslant 2g_C-2$, the mass of the fibers of (3.6.4) does not depend on $(e_i)_{1\leqslant i\leqslant r}$ (this is a computation of extensions, cf. proof of proposition 3.9.2 of [4]): one gets (*ibid.*):

$$|\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})| = q^{(n^2-\sum_{i=1}^r n_i^2)\deg(D)}\prod_{i=1}^{r}|\mathcal{M}^{ss}(D,n_i,e_i)|$$

Then the mass of $\mathcal{M}^{ss}(D,n_i,e_i)$ depends only on $e_i \mod n_i$ (cf. remark 3.2.2).

Now the crucial observation (cf. proposition 4.4.2 of [4]) is that either

$$\mathcal{M}^{\leqslant T}(D,n,e)\cap\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})=\emptyset \tag{3.6.5}$$

or

$$\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})\subset\mathcal{M}^{\leqslant T}(D,n,e). \tag{3.6.6}$$

The counting of the degrees $(e_i)_{1\leqslant i\leqslant r}$ such that the condition (3.6.6) holds is a quasi-polynomial function in T (this is the content of §4.5 of [4]). Thus $\mathcal{M}^{\leqslant T}(D,(n_i)_{1\leqslant i\leqslant r})$ is a finite disjoint union of a finite number of strata

$$\mathcal{M}(D,(n_i)_{1\leqslant i\leqslant r},(e_i)_{1\leqslant i\leqslant r})$$

and its mass is indeed a quasi-polynomial function in T. Its value at $T=0$ is clearly 0. □

3.7. A refinement of the theorem. — Let

$$\mathcal{A}=\mathcal{A}_{D,n}=\bigoplus_{i=1}^{n}H^0(C,\mathcal{O}(iD))$$

be the (finite) set of characteristic polynomials (cf. §2.4). For any $\chi\in\mathcal{A}$, let

$$\mathcal{M}_\chi^{\leqslant T}(D,n,e)\subset\mathcal{M}(D,n,e)$$

be the full groupoid of T-semistable Hitchin bundles $(\mathcal{E},\theta)$ with $\chi_\theta=\chi$.

Theorem 3.7.1. —*(cf. theorem 6.1.1 of [4])*

1. *For any divisor D, there exists a unique quasi-polynomial $T\mapsto J_{D,\chi}^{T,e}$ such that*

$$|\mathcal{M}_\chi^{\leqslant T}(D,n,e)|=J_{D,\chi}^{T,e}$$

for any $T\in\mathbb{Z}^n$ such that $T\geqslant\max(0,\deg(D),2g-2-D)$.

2. If $\deg(D) \geqslant 2g_C - 2$*, one can replace the above condition on* T *by the weaker condition*

$$T \geqslant 0.$$

In particular, we have

$$|\mathcal{M}^{ss}_{\chi}(D,n,e)| = J^{0,e}_{D,\chi}.$$

Proof. — The proof is the same as for theorem 3.6.2. The main point is that the functor (3.6.4) is compatible with characteristic polynomials.

□

Remark 3.7.2. — Since we have a disjoint union

$$\mathcal{M}^{\leqslant T}(D,n,e) = \bigcup_{\chi \in \mathcal{A}_{D,n}} \mathcal{M}^{\leqslant T}_{\chi}(D,n,e),$$

we get

$$J^{T,e}_{D} = \sum_{\chi} J^{T,e}_{D,\chi}.$$

3.8. Symmetry $D \leftrightarrow K - D$.

— Let K be a canonical divisor. Let us recall that $\deg(K) = 2g_C - 2$.

Theorem 3.8.1. — *(Corollary 5.2.3 of [4]) For any parameter* $T \in \mathbb{Z}^n$ *and any degree* $e \in \mathbb{Z}$*, one has*

$$J^{T,e}_{D} = q^{n^2(1-g_C+\deg(D))} J^{T,e}_{K-D}.$$

Proof. — It is obvious that if a vector bundle $\mathcal{E}$ is T-semistable then any Hitchin bundle $(\mathcal{E},\theta)$ is also T-semistable. The converse is also true under some condition on T:

Lemma 3.8.2. — *(cf. lemma 4.2.2 of [4]) For any* $T \geqslant \max(0,\deg(D))$*, the Hitchin bundle* $(\mathcal{E},\theta) \in \mathcal{M}(D,n,e)$ *is* T*-semistable if and only if the underlying vector bundle* $\mathcal{E}$ *is* T*-semistable.*

Using the above lemma and the notation $Bun^{\leqslant T}_{n,e}$ for the groupoid of T-semistable vector bundles of degree e and rank n, we have, for any T such that $T \geqslant \max(0,\deg(D))$,

$$|\mathcal{M}^{\leqslant T}(D,n,e)| = \sum_{\mathcal{E} \in Bun^{\leqslant T}_{n,e}} \frac{|\operatorname{Hom}(\mathcal{E},\mathcal{E}(D))|}{|\operatorname{Aut}(\mathcal{E})|}.$$

The we can use the Riemann-Roch theorem and Serre duality to get

$$\begin{aligned} |\mathcal{M}^{\leqslant T}(D,n,e)| &= q^{n^2(1-g_C+\deg(D))} \sum_{\mathcal{E} \in Bun^{\leqslant T}_{n,e}} \frac{|\operatorname{Ext}^1(\mathcal{E},\mathcal{E}(D))|}{|\operatorname{Aut}(\mathcal{E})|} \\ &= q^{n^2(1-g_C+\deg(D))} \sum_{\mathcal{E} \in Bun^{\leqslant T}_{n,e}} \frac{|\operatorname{Hom}(\mathcal{E},\mathcal{E}(K-D))|}{|\operatorname{Aut}(\mathcal{E})|}. \end{aligned}$$

by Riemann-Roch formula and Serre duality. Applying again lemma 3.8.2, one concludes that for any T such that $T \geqslant \max(0,\deg(D),\deg(K-D))$, we have

$$|\mathcal{M}^{\leqslant T}(D,n,e)| = q^{n^2(1-g_C+\deg(D))} |\mathcal{M}^{\leqslant T}(K-D,n,e)|.$$

By theorem 3.6.2, one gets the equality

$$J_D^{T,e} = q^{n^2(1-g_C+\deg(D))} J_{K-D}^{T,e}$$

for any $T \geqslant \max(0, \deg(D), \deg(K-D))$. But since both sides are quasi-polynomials in T, the equality is true for any T. □

3.9. A reduction to a counting of T-semistable nilpotent Hitchin bundles. — Let us denote

$$\mathcal{N}(D,n,e) := \mathcal{M}_{\chi=X^n}(D,n,e)$$

where the right-hand side is the groupoid of Hitchin bundles $(\mathcal{E},\theta)$ with characteristic polynomial $\chi_\theta = X^n$ (in other words: θ is nilpotent). Let us introduce the quasi-polynomial

$$J_{D,nilp}^{T,e} := J_{D,\chi=X^n}^{T,e}$$

and its value at $T=0$

$$J_{D,nilp}^{e} := J_{D,\chi=X^n}^{T=0,e}.$$

By theorem 3.7.1, this quasi-polynomial gives (at least for large T) the mass of the subcategory

$$\mathcal{N}^{\leqslant T}(D,n,e) \subset \mathcal{N}(D,n,e)$$

of T-semistable nilpotent Hitchin bundles.

Theorem 3.9.1. —*(cf. corollary 6.1.2 of [4]) Assume* $\deg(D) \geqslant 2g_C - 2$. *Let K be a canonical divisor.*

1. *If* $\deg(D) > 2g_C - 2$

$$|\mathcal{M}^{ss}(D,n,e)| = q^{n^2(1-g_C+\deg(D))} \cdot J_{K-D,nilp}^{e}$$

2. *If $D = K$ and the degree e is* prime *to the rank n*

$$|\mathcal{M}^{ss}(K,n,e)| = q^{n^2(1-g_C+\deg(K))+1} \cdot J_{D=0,nilp}^{e}$$

Proof. — Using theorem 3.6.2 assertion 2 and theorem 3.8.1, one gets for any divisor of degree $\deg(D) \geqslant 2g_C - 2$

$$\begin{aligned} |\mathcal{M}^{ss}(D,n,e)| &= J_D^{T=0,e} \\ &= q^{n^2(1-g_C+\deg(D))} J_{K-D}^{T=0,e} \end{aligned}$$

If moreover $\deg(D) > 2g_C - 2$, one has

$$\mathcal{A}_{K-D,n} = \bigoplus_{i=1}^{n} H^0(C, \mathcal{O}(i(K-D))) = 0$$

so

$$J_{K-D}^{T,e} = J_{K-D,nilp}^{T,e}.$$

Thus, we get the conclusion

$$|\mathcal{M}^{ss}(D,n,e)| = q^{n^2(1-g_C+\deg(D))} J_{K-D,nilp}^{T=0,e}.$$

If $D = 0$, the characteristic polynomials $\chi \in \mathcal{A}_{D,n}$ have coefficients in $\mathbb{F}_q$. If e and n are coprime one can show that (cf. theorem 6.2.1 of [4])

$$J_{D=0,\chi}^{T,e} = \begin{cases} J_{D=0,nilp}^{T,e} \text{ if } \chi = (X-a)^n \text{ with } a \in \mathbb{F}_q \\ 0 \text{ otherwise.} \end{cases}$$

Let us just give an example. Assume $n \geqslant 2$. Let $\chi \in \mathbb{F}_q[X]$ be an irreducible polynomial. Let $(\mathcal{E}, \theta)$ be a Hitchin bundle with characteristic polynomial $\chi_\theta = \chi$. By looking at the generic fiber of $\mathcal{E}$, one easily shows that there is no non-trivial subbundle of $\mathcal{E}$ which is θ-invariant. So any such $(\mathcal{E}, \theta)$ is semistable and, as such, T-semistable. In this case, one has

$$J^{T,e}_{D=0,\chi} = |\mathcal{M}_\chi(D = 0, n, e)|.$$

After base change to $\mathbb{F}_{q^n}$, the vector bundle $\mathcal{E}$ is a sum of line bundles of same degrees so n must divide $\deg(\mathcal{E})$. Thus if we assume that n and e are coprime, we get

$$J^{T,e}_{D=0,\chi} = |\mathcal{M}_\chi(D = 0, n, e)| = 0.$$

□

Example 3.9.2. — Consider the case $C = \mathbb{P}^1_{\mathbb{F}_q}$, $n = 2$ and $e = 1$. Here a canonical divisor K is of degree -2. Consider a divisor D of degree 2. By Grothendieck classification, one knows that a vector bundle on $\mathbb{P}^1_{\mathbb{F}_q}$ of rank 2 and degree 1 is isomorphic to $\mathcal{O}(a) \oplus \mathcal{O}(1-a)$ with $a \in \mathbb{N}^*$. From this one can compute (see also [4] eq.(4.3.2) of §4.3)

$$|\mathcal{M}^{ss}(D, 2, 1)| = q^8 \frac{q+1}{q-1}.$$

Let $T = (t, -t)$ a truncation parameter with $t \in \mathbb{N}$. Using the methods and the computations of [4] §4.3, it is easy to get the following formula

$$\begin{aligned} |\mathcal{M}^{\leqslant T}(K - D, 2, 1)| &= \frac{1}{q^2(q-1)^2} + \sum_{a=2}^{t} \frac{q^{2a-4}}{q^{2a}(q-1)^2} \\ &= \frac{1}{q^2(q-1)^2} + (t-1)\frac{1}{q^4(q-1)^2}. \end{aligned}$$

We see that the formula is a polynomial in t. Thus the constant term is

$$J^1_{K-D} = J^1_{K-D,nilp}$$

(this last equality is due to the fact that $\deg(K - D) = -4 < 0$). In this way, one gets

$$J^1_{K-D,nilp} = \frac{q+1}{q^4(q-1)}. \tag{3.9.1}$$

So we have checked that

$$|\mathcal{M}^{\leqslant T}(K - D, 2, 1)| = q^{12} J^1_{K-D,nilp}$$

as predicted by the theorem 3.9.1.

Remark 3.9.3. — We emphasize that in general we have

$$J^e_{K-D,nilp} \neq |\mathcal{N}^{ss}(K - D, n, e)|.$$

Let us consider the situation of example 3.9.2. Let $(\mathcal{E}, \theta) \in \mathcal{M}^{ss}(K - D, 2, 1)$. We are in the coprime case: so $(\mathcal{E}, \theta)$ is stable. If $\theta \neq 0$, one must have strict inequalities among slopes of $\mathcal{E}$ and the (co)-image of θ :

$$\mu(\mathcal{E}) < \mu(coIm(\theta)) \leqslant \mu(Im(\theta)) < \mu(\mathcal{E}(K - D) = \mu(\mathcal{E}) - 4.$$

This is not possible so $\theta = 0$. Thus $\mathcal{E}$ is a *stable vector bundle* of rank 2 and degree 1.

However, any vector bundle of rank 2 and degree 1 is isomorphic to $\mathcal{O}(a) \oplus \mathcal{O}(1-a)$ (with $a \in \mathbb{N}^*$) and thus cannot be stable. So

$$|\mathcal{N}^{ss}(K - D, 2, 1)| = 0,$$

and it cannot be equal to (3.9.1). Note that the same example gives that, in general,

$$J^e_{K-D} \neq |\mathcal{M}^{ss}(K - D, n, e)|.$$

4 The language of adèles

4.1. In this section, we will give adelic integral expressions for the quasi-polynomials $J_{D,\chi}^{T,e}$ defined in section 3.

4.2. Dictionary between adèles and bundles. — We keep the notations of section 3. Let F be the field of rational functions on C and $\mathbb{A}$ be the ring of adèles of F. Let $\mathcal{O} \subset \mathbb{A}$ be its maximal compact subring. The inclusion $F \subset \mathbb{A}$ is discrete and cocompact. We have the degree morphism

$$\deg : F^\times \backslash \mathbb{A}^\times / \mathcal{O}^\times \to \mathbb{Z}$$

Let $G = GL(n)$ and $G(\mathbb{A})^e = \{g \in G(\mathbb{A}) \mid \deg(\det(g)) = -e\}$. The group $G(F)$ acts on the quotient $G(\mathbb{A})^e/G(\mathcal{O})$. Let $[G(F)\backslash G(\mathbb{A})^e/G(\mathcal{O})]$ denote the quotient groupoid. Weil has constructed an equivalence of category $g \mapsto \mathcal{E}_g$ (roughly speaking, one views adelic elements as glueing data between vector bundles on the generic fiber and vector bundles on formal neighborhoods of closed points, cf. also the descent lemma in [3] and [16])

$$[G(F)\backslash G(\mathbb{A})^e/G(\mathcal{O})] \simeq Bun_{n,e}$$

for which

$$\mathrm{Aut}(\mathcal{E}_g) \simeq G(F) \cap gG(\mathcal{O})g^{-1}.$$

We will give a group theoretic counterpart of the slope inequalities. Let $T_0 \subset G$ be the standard maximal subtorus of diagonal matrices. Let $B \subset G$ be the standard Borel subgroup of upper-triangular matrices. The parabolic sugroups of G are the stabilizers of flags. For any standard parabolic subgroup P (abbreviated by p.s.g., "standard" means that P contains B) one introduces

- the group $X^*(P)$ of algebraic characters of P;
- the lattice and the real vector space $\mathfrak{a}_P = \mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{Z}) \subset a_P = \mathrm{Hom}_{\mathbb{Z}}(X^*(P), \mathbb{R})$.
- ${}^+a_P \subset a_P$ the *obtuse* cone defined by the fundamental P-dominant weights and $\hat{\tau}_P$ its characteristic function.
- the map $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ defined by for $\chi \in X^*(P)$, $p \in P(\mathbb{A})$, $k \in G(\mathcal{O})$

$$\langle \chi, H_P(pk) \rangle = -\deg(\chi(p)).$$

The last map is well-defined by the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})G(\mathcal{O})$.

We can quickly summarize the correspondence between adelic (or group-theoretic) constructions and vector bundles:

- An element $g \in G(\mathbb{A})/G(\mathcal{O})$ corresponds to a pair of a bundle $\mathcal{E}_g$ on C and a trivialization of the generic fiber of $\mathcal{E}_g$;
- A p.s.g. P of G and $\delta \in P(F)\backslash G(F)$ corresponds to a flag $\mathcal{F}_\bullet$ of subbundles of $\mathcal{E}_g$;
- the vector $H_P(\delta g)$ is equivalent to the collection of degrees $(\deg(\mathcal{F}_1/\mathcal{F}_0), \deg(\mathcal{F}_2/\mathcal{F}_1), \ldots)$;
- the condition $\hat{\tau}_P(H_P(\delta g)) = 1$ is equivalent to the slope inequalities $\mu(\mathcal{F}_i) > \mu(\mathcal{E}_g)$ $\forall i$.

Let $\mathfrak{g} = Lie(G)$ the Lie algebra of G. Let $\mathfrak{p} = Lie(P)$ the Lie algebra of any standard parabolic subgroup P of G. We can attach an idèle $\varpi^{-D} = (\varpi_c^{-mult_c(D)})_{c\in|C|} \in \mathbb{A}^\times$ to any divisor D : it depends on the choice of local uniformizers ϖ_c. We have denoted $mult_c(D)$ the multiplicity of D at the closed point c of C. Let $\mathbf{1}_D$ be the characteristic function of $\varpi^{-D}\mathfrak{g}(\mathcal{O})$. It does not depend on auxiliary choices. Let $g \in G(\mathbb{A})/G(\mathcal{O})$. Any $\gamma \in \mathfrak{g}(F)$ with the integrality condition

$$g^{-1}\gamma g \in \varpi^{-D}\mathfrak{g}(\mathcal{O})$$

corresponds to an endomorphism $\theta \in \mathrm{Hom}(\mathcal{E}_g, \mathcal{E}_g(D))$. In particular, we have

$$|\,\mathrm{Hom}(\mathcal{E}_g, \mathcal{E}_g(D))| = \sum_{\gamma \in \mathfrak{g}(F)} \mathbf{1}_D(g^{-1}\gamma g).$$

4.3. A variant of Arthur's truncated kernel. — Let $g \in G(\mathbb{A})$ and let $\chi \in F[X]$ be a characteristic polynomial of some element in $\mathfrak{g}(F)$. For any standard parabolic subgroup P of G with standard Levi decomposition $P = MN$, one defines

$$k_{D,\chi}^P(g) := \sum_{\gamma \in \mathfrak{m}(F), \chi_\gamma = \chi} \int_{\mathfrak{n}(\mathbb{A})} \mathbf{1}_D(g^{-1}(\gamma + U)g)\, dU$$

where $\mathfrak{m} = Lie(M)$ and $\mathfrak{n} = Lie(N)$. The Haar measure dU is normalized by $\mathrm{vol}(\mathfrak{n}(\mathbb{A})/\mathfrak{n}(F)) = 1$. The sum over γ is in fact of finite support because $\mathbf{1}_D$ is compactly supported. The integral over U is also finite. So $k_{D,\chi}^P(g)$ is well-defined.

Remark 4.3.1. — At first approximation, one may think that the expression counts the twisted endomorphisms of $\mathcal{E}_g$ (the vector bundle corresponding to g) which admit χ as characteristic polynomial and which stabilize a flag of subbundles corresponding to P. This is not literally true. More correctly, the expression is intermediate between a counting and a Euler-Poincaré characteristic. For more precise statements we refer to [4] sections 5.1 and 6.1.

The following object is a Lie algebra analog of Arthur's truncated kernel (cf. [1], it has already appeared in the case of number fields in [5]):

$$k_{D,\chi}^T(g) = \sum_{P \text{ p.s.g.}} \varepsilon_P^G \sum_{\delta \in P(F)\backslash G(F)} \hat{\tau}_P(H_P(\delta g) - T_P) k_{D,\chi}^P(\delta g) \tag{4.3.1}$$

where T_P is the image of $T \in \mathfrak{a}_B$ under the canonical map $\mathfrak{a}_B \to \mathfrak{a}_P$ and

$$\varepsilon_P^G := (-1)^{\dim(a_P) - \dim(a_G)}.$$

One can show that the sum over δ is in fact of finite support in (4.3.1). Again, $k_{D,\chi}^T(g)$ is well-defined.

For large T, the expression $k_{D,\chi}^T(g)$ counts the twisted endomorphisms θ of $\mathcal{E}_g$ such that $(\mathcal{E}_g, \theta)$ is T-semistable and the characteristic polynomial of θ is χ. For example, if χ is irreducible, the semistability condition is vacuous (there is no θ-stable proper subbundle) and one simply gets

$$\begin{aligned} k_{D,\chi}^T(g) &= \sum_{\gamma \in \mathfrak{g}(F), \chi_\gamma = \chi} \mathbf{1}_D(g^{-1}\gamma g) \\ &= |\{\theta \in \mathrm{Hom}(\mathcal{E}_g, \mathcal{E}_g(D)) \mid \chi_\theta = \chi\}|. \end{aligned}$$

The map $g \mapsto k_{D,\chi}^T(g)$ is compactly supported on $G(F)\backslash G(\mathbb{A})^e$ (cf. theorem 6.1.1 of [4]).

Let dg be the quotient of the Haar measure on $G(\mathbb{A})$ normalized by $\mathrm{vol}(G(\mathcal{O})) = 1$ by the counting measure on $G(F)$. Then we have the simple formula

$$|Bun_{n,e}| = \int_{G(F)\backslash G(\mathbb{A})^e} dg.$$

The main point in introducing the function $k_{D,\chi}^T$ is the next proposition (cf. theorem 6.1.1 of [4]). In the following, we use the natural identification

$$\mathbb{Z}^n \simeq \mathfrak{a}_B$$

deduced from the characters $(t_1, \ldots, t_n) \mapsto t_i$.

Proposition 4.3.2. — *For any $T \in \mathbb{Z}^n$ and any characteristic polynomial $\chi \in \mathcal{A}_{D,n}$, we have*

$$J^{T,e}_{D,\chi} = \int_{G(F)\backslash G(\mathbb{A})^e} k^T_{D,\chi}(g)\, dg.$$

For example, let χ be irreducible and let $\gamma \in \mathfrak{g}(F)$ such that $\chi_\gamma = \chi$. Let $G_\gamma \subset G$ the centralizer of γ. Then one has

$$\begin{aligned} J^{T,e}_{D,\chi} &= \int_{G(F)\backslash G(\mathbb{A})^e} \sum_{\delta \in \mathfrak{g}(F), \chi_\delta = \chi} \mathbf{1}_D(g^{-1}\delta g) \\ &= \int_{G_\gamma(F)\backslash G(\mathbb{A})^e} \mathbf{1}_D(g^{-1}\gamma g)\, dg \end{aligned}$$

which is an adelic orbital integral. In general, they are very difficult to compute (they appear in the famous fundamental lemma). When $\chi = X^n$, we hope that the corresponding nilpotent $J^{T,e}_{D,\chi}$ can be computed.

Remark 4.3.3. — The combination of proposition 4.3.2 and theorem 3.9.1 shows that to achieve the counting of semistable Hitchin bundles (with $\deg(D) \geqslant 2g_C - 2$) it suffices to compute $J^{T=0,e}_{D,\chi}$ for $\chi = X^n$ and $\deg(D) \leqslant 0$.

5 Expansion in terms of nilpotent orbits

5.1. In this section, we will explain how to expand $J^{T=0,e}_{D,nilp}$ in terms of nilpotent orbits.

5.2. Lusztig-Spaltenstein Induction. — Let P be a standard parabolic subgroup of G with a Levi decomposition MN. For each nilpotent element $\gamma \in \mathfrak{m}$, let $\mathcal{O}_\gamma \subset \mathfrak{m}$ be the M-orbit of γ. There is a unique nilpotent G-orbit $\mathcal{O}$ in $\mathfrak{g}$ such that

$$\mathcal{O} \cap (\mathcal{O}_\gamma \oplus \mathfrak{n})$$

is open and dense in $\mathcal{O}_\gamma \oplus \mathfrak{n}$ (cf. [20]). This is the *Lusztig-Spaltenstein induced orbit* denoted by

$$I^G_P(\gamma) := \mathcal{O}.$$

In our situation $G = GL(n)$, any nilpotent orbit is a Richardson orbit i.e. is induced from the zero orbit of a Levi. For example, the regular orbit is the orbit $I^G_B(0)$ induced from the Borel subgroup. The zero orbit is never induced from a *proper* parabolic subgroup. In the following, this induction procedure will enable us to attach nilpotent orbits in Levi subgroups to a nilpotent orbit for G.

5.3. Expansion in terms of nilpotent orbits. — For each nilpotent $G(F)$-orbit $\mathcal{O}$ in $\mathfrak{g}(F)$, $g \in G(\mathbb{A})$ and $P = MN$ a p.s.g., following [7], one defines

$$k^P_{D,\mathcal{O}}(g) = \sum_{\{\gamma \in \mathfrak{m}(F)\, nilp. | I^G_P(\gamma) = \mathcal{O}\}} \int_{\mathfrak{n}(\mathbb{A})} \mathbf{1}_D(g^{-1}(\gamma + U)g)\, dU$$

and

$$k^T_{D,\mathcal{O}}(g) = \sum_{P\, p.s.g.} \varepsilon^G_P \sum_{\delta \in P(F)\backslash G(F)} \hat{\tau}_P(H_P(\delta g) - T_P) k^P_{D,\mathcal{O}}(\delta g).$$

As in §4.3, these expressions are well-defined. Moreover, we have by construction

$$k^T_{D,nilp}(g) := k^T_{D,\chi = X^n}(g) = \sum_{\mathcal{O}} k^T_{D,\mathcal{O}}(g)$$

where the sum is over the finite set of nilpotent orbits in $\mathfrak{g}(F)$.

If $\mathcal{O}=(0)$ then we have $k^T_{D,(0)}(g)=1$ for any $g\in G(\mathbb{A})$. In particular, we see that the functions $k^T_{D,\mathcal{O}}$ are in general not compactly supported.

The following theorem is one of the main results of [7].

Theorem 5.3.1. — *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}(F)$.*

1. *The following integral is convergent*

$$J^{T,e}_{D,\mathcal{O}} := \int_{G(F)\backslash G(\mathbb{A})^e} k^T_{D,\mathcal{O}}(g)\,dg.$$

2. *The map $T\in\mathbb{Z}^n \mapsto J^{T,e}_{D,\mathcal{O}}$ is quasi-polynomial.*

3. *Let $\mathcal{M}^{\leqslant T}_{\mathcal{O}}(D,n,e)\subset\mathcal{M}^{\leqslant T}(D,n,e)$ be the full subgroupoid of Hitchin bundles $(\mathcal{E},\theta)$ such that $\theta\in\mathcal{O}$ generically. Then $J^{T,e}_{D,\mathcal{O}}$ is the quasi-polynomial part of the mass of $\mathcal{M}^{\leqslant T}_{\mathcal{O}}(D,n,e)$ in the sense that*

$$\lim_{\inf(T_i-T_{i+1})\to+\infty} |\mathcal{M}^{\leqslant T}_{\mathcal{O}}(D,n,e)| - J^{T,e}_{D,\mathcal{O}} = 0.$$

The assertions 1 and 2 will be found in [7] corollary 6.2.2. The assertion 3 is a consequence of theorem 6.2.1 of [7] and the lemma 3.8.2 above.

Remark 5.3.2. — For any T, we have

$$J^{T,e}_{D,nilp} = \sum_{\mathcal{O}} J^{T,e}_{D,\mathcal{O}}$$

where the sum is over the finite set of nilpotent orbits.

To achieve the counting of semistable Hitchin bundles (with $\deg(D)\geqslant 2g_C-2$) it suffices to compute $J^{T=0,e}_{D,\mathcal{O}}$ for any $\mathcal{O}$ and $\deg(D)\leqslant 0$ (cf. remark 4.3.3). Our philosophy is that there should exist nice formulae for each $J^{T=0,e}_{D,\mathcal{O}}$. In the next section, we will explain three such computations. The reader will also find in [4] conjectures about $J^{T=0,e}_{D,\mathcal{O}}$ in the spirit of Hausel-Rodriguez-Villegas conjecture.

6 Three computations

6.1. The zero orbit. — The notations are taken from the previous section. We have

$$k^T_{D,(0)}(g)=1$$

and

$$J^{T,e}_{D,(0)} = \mathrm{vol}(G(F)\backslash G(\mathbb{A})^e).$$

This volume does not depend on T nor on e and it is given by the Siegel formula (3.3.1). Let us note that $\mathcal{M}^{\leqslant T}_{(0)}(D,n,e)$ is nothing else but the groupoid $Bun^{\leqslant T}_{n,e}$ of T-semistable vector bundle. We have indeed

$$|\mathcal{M}^{\leqslant T}_{(0)}(D,n,e)| \to_{T\to\infty} |Bun_{n,e}|.$$

6.2. The regular orbit. — This case is much more intricate. Let $\mathcal{O}_{\mathrm{reg}}\subset\mathfrak{g}(F)$ be the regular nilpotent orbit; this is also the set of nilpotent element in $\mathfrak{g}(F)$ of maximal nilpotency index n.

Let $\mathfrak{b} = Lie(B)$ and let us pick $\gamma \in \mathcal{O}_{\rm reg} \cap \mathfrak{b}(F)$. The centralizer G_γ of γ in G is included in B. We will simply note $J_{\rm reg}$ instead of $J_{\mathcal{O}_{\rm reg}}$. Let us introduce the complex torus

$$\widehat{T} = X^*(B) \otimes_{\mathbb{Z}} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^n.$$

We have a canonical pairing

$$(t, H) \in \widehat{T} \times \mathfrak{a}_B \mapsto t^H \in \mathbb{C}^\times.$$

We have a first lemma (cf. §7.4 of [7]):

Lemma 6.2.1. — *One has that $J_{D,\rm reg}^{T=0,e}$ is the limit at $t = 1$ of the following expression*

$$\int_{B(F)\backslash G(\mathbb{A})^e} t^{H_B(g)} \sum_{P\, p.s.g.} \varepsilon_P^G \hat{\tau}_P(H_P(g)) \sum_{\delta \in (M\cap G_\gamma)(F)\backslash (M\cap B)(F)} \int_{\mathfrak{n}(\mathbb{A})} \mathbf{1}_D((\delta g)^{-1}(\gamma + U)\delta g)\, dU\, dg$$

In the above lemma the expression is well-defined for t in a suitable open subset of $\widehat{T}$. After introducing t, we can permute the integral over $B(F)\backslash G(\mathbb{A})^e$ and the alternate sum and compute P by P (at least for suitable t).

The next lemma gives the computation of the main term for $P = G$. Before stating it, let us introduce more notations: we view $\widehat{T}$ as the standard maximal subtorus of $GL(n, \mathbb{C})$;

- Let $\widehat{G}' = SL(n, \mathbb{C})$, let $\widehat{T}' = \widehat{T} \cap \widehat{G}'$ be its standard maximal torus. Let $\widehat{Z}'$ be the center of $\widehat{G}'$;
- Let $\widehat{B}' \supset \widehat{T}'$ be the standard Borel subgroup of $\widehat{G}'$;
- Let $\Pi_{\widehat{B}'}$ be the set of the fundamental $\widehat{B}'$-dominant weights of $\widehat{T}'$;
- We have a pairing
$$(z, e) \in \hat{Z}' \times \mathfrak{a}_G \mapsto z^e \in \mathbb{C}^\times$$
where we identify $\mathfrak{a}_G = \mathrm{Hom}(X^*(G), \mathbb{Z}) \simeq \mathbb{Z}$ using the determinant and we identify $\hat{Z}'$ to the group of $n^{\rm th}$-roots of unity.

Lemma 6.2.2. — *(cf. [7] lemma 3.6.1) One has for suitable $t \in \widehat{T}'$*

$$\int_{G_\gamma(F)\backslash G(\mathbb{A})^e} t^{H_B(g)} \mathbf{1}_D(g^{-1}\gamma g)\, dg = q^{\deg(D)\frac{n(n-1)}{2}} q^{n(g_c-1)} \Phi^e_{\widehat{B}',D}(t)$$

where we have introduced the rational function of $t \in \widehat{T}'$

$$\Phi^e_{\widehat{B}',D}(t) = \frac{\zeta^*(q^{-1})}{n} \sum_{z \in \widehat{Z}'} z^e \prod_{\varpi \in \Pi_{\widehat{B}'}} (tz)^{-\deg(D)\varpi} \zeta(q^{-1}(tz)^{\varpi})$$

We observe that the function $\Phi^e_{\widehat{B}',D}$ has a pole at $t = 1$. We can also compute the other terms for $P \subsetneq G$, but the answer is rather intricate. But if we take the average over the Weyl group W of $(\widehat{G}', \widehat{T}')$ then we get a nice formula as the next theorem shows. Let us just recall that the Weyl group is just the symmetric group $\mathfrak{S}_n$ and its acts on $\widehat{T}'$ by permuting the entries.

Theorem 6.2.3. — *(cf. theorem 7.1.1 of [7])*

1. *The rational function of $t \in \widehat{T}'$*
$$q^{\deg(D)\frac{n(n-1)}{2}} q^{n(g_c-1)} \frac{1}{n!} \sum_{w \in W} \Phi^e_{\widehat{B}',D}(w \cdot t)$$
is regular at $t = 1$.

2. The degree e and the rank n are assumed to be coprime. Then its value at $t = 1$ is precisely

$$J_{D,\mathrm{reg}}^{T=0,e}.$$

Remark 6.2.4. — We have the same kind of answer as in theorem 6.2.3 for nilpotent elements that have exactly r Jordan blocks of size d with $rd = n$ (cf. [7]).

Remark 6.2.5. — The appearance of the group $SL(n,\mathbb{C})$ is not surprising. For any reductive group G, we expect the number of (semistable) G-Hitchin bundles to be expressed in terms of the complex Langlands dual $\widehat{G}$. Here $SL(n,\mathbb{C})$ is the Langlands dual of $PGL(n)$. In our situation, our counting of Hitchin bundles is essentially a $PGL(n)$-computation (we can divide by the center of $GL(n)$, cf. remark 3.2.2).

Remark 6.2.6. — The expressions $k_{D,\mathcal{O}}^T$ make sense for more general test functions than $\mathbf{1}_D$. In this case, one has a similar expression for the corresponding integral $J_{D,\mathrm{reg}}^{T=0,e}$ in which one interprets the function Φ as a regularized nilpotent integral. This may be useful if one wants to compute the number of Hitchin bundles with parabolic structures for example.

6.3. The subregular orbit in $GL(3)$. — Let us quote an other (still unpublished) result :

Theorem 6.3.1. — *For $\mathcal{O}$ the subregular orbit of $GL(3)$ and e prime to 3, $J_{D,\mathcal{O}}^{T=0,e}$ is the value at $u = 1$ of the rational function of $u \in \mathbb{C}^\times$*

$$q^{2(\deg(D)+g_C-1)}\zeta^*(q^{-1})^2\frac{\zeta(q^{-2}) - u^{-\deg(D)}\zeta(q^{-2}u)}{1-u}.$$

Here, we see that derivates of the zeta function appear.

Remark 6.3.2. — If $n = 3$ then we have three nilpotent orbits: the zero orbit, the subregular orbit and the regular orbit. So, in this case, it is possible to give a formula for

$$|\mathcal{M}^{ss}(D,3,1)|$$

for any divisor which is either canonical or of degree $> 2g_C - 2$. This is a direct combination of theorem 3.9.1, remark 5.3.2 and the three computations of §6.1, theorem 6.2.3 and theorem 6.3.1. For rank > 3, there are still lacking computations but this is a work-in-progress (see [6] for some results for number fields).

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Pierre-Henri Chaudouard
Université Paris Diderot (Paris 7) et Institut Universitaire de France
Institut de Mathématiques de Jussieu-Paris Rive Gauche
UMR 7586
Bâtiment Sophie Germain
Case 7012
75205 PARIS Cedex 13
France

E-mail: Pierre-Henri.Chaudouard@imj-prg.fr