A spectral expansion for the symmetric space $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$

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Abstract

In this article we state and prove the spectral expansion of theta series attached to the symmetric space $\mathrm{GL}_n(E)/\mathrm{GL}_n(F)$ where $n \geqslant 1$ and E/F is a quadratic extension of number fields. This is an important step towards the fine spectral expansion of relative trace formulas based on this symmetric space such as the Jacquet-Rallis trace formula for general linear groups. To obtain our result, we extend the work of Jacquet-Lapid-Rogawski on intertwining periods to the case of discrete automorphic representations. The expansion we get is an absolutely convergent integral of relative characters built upon Eisenstein series and intertwining periods. We also establish a crucial but technical ingredient whose interest lies beyond the focus of the article: we prove bounds for discrete Eisenstein series of GL_n on a neighborhood of the imaginary axis extending previous works of Lapid on cuspidal Eisenstein series.

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Contents

1 Introduction

1.1 Motivations

1.1.1. The problem. — Let $n \geqslant 1$ be an integer and let E/F be a quadratic extension of number fields. Let $G = G_n$ be the group $\mathrm{GL}_n(E)$ viewed by restriction of scalars as an F-group. Let ι be the Galois involution and $G' = G'_n = \mathrm{GL}_n(F)$ be its fixed point subgroup. Let S be the symmetric space G/G'. Note that there is a left action of G on S by left translation. Let $\mathbb{A}$ be the ring of the adèles of F. For any Schwartz function Φ on $S(\mathbb{A})$ and $g \in G(\mathbb{A})$ we can form the theta series

$$\theta_\Phi(g) = \sum_{\sigma \in S(F)} \Phi(g^{-1}\sigma).$$

This gives a smooth function on $[G] = G(F)\backslash G(\mathbb{A})$ and the purpose of the article is to provide a spectral decomposition of $\theta_\Phi(g)$ in terms of objects attached to the L^2-automorphic spectrum of G. This problem is raised by Jacquet in [Jac97]. We can slightly restate the problem noting that we have $S(\mathbb{A}) = G(\mathbb{A})/G'(\mathbb{A})$. Any Schwartz function Φ can be obtained from f in the space $\mathcal{S}(G(\mathbb{A}))$ of Schwartz functions on $G(\mathbb{A})$ in the following way:

$$\Phi(\sigma) = \int_{G'(\mathbb{A})} f(gh)\, dh$$

where dh is some Haar measure on $G'(\mathbb{A})$ and $\sigma \in S(\mathbb{A})$ is represented by the class $gG'(\mathbb{A})$ for some $g \in G(\mathbb{A})$. Then we introduce the automorphic kernel

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \quad x, y \in G(\mathbb{A})$$

and the so-called Flicker-Rallis period of the automorphic kernel defined by the convergent integral

$$J^G(g, f) = \int_{[G']} K_f(g, h)\, dh, \quad g \in G(\mathbb{A})$$

where $[G'] = G'(F)\backslash G'(\mathbb{A})$ and dh is the quotient measure. Since we have $S(F) = G(F)/G'(F)$ we have $\theta_\Phi(g) = J^G(g, f)$ and the problem becomes to decompose spectrally the distribution $J^G(g, f)$.

1.1.2. Jacquet-Rallis trace formula. — Let $\det : G' \to \mathbb{G}_{m,F}$ be the determinant and let η be the quadratic character of $\mathbb{A}^\times$ attached to E/F by class field theory. We can consider the twisted Flicker-Rallis period of the automorphic kernel defined by

$$J^{G,\eta}(g,f) = \int_{[G']} K_f(g,h)\eta(\det(h))^{n+1}\, dh.$$

Note that our results and methods give also the spectral decomposition of this distribution. We shall leave the reader make the obvious modifications. The Jacquet-Rallis trace formula for linear groups as stated in [Zyd20] is the regularized version of the (in general divergent) integral

$$\int_{[G_n]} J^{G_n,\eta}(g,f_n) J^{G_{n+1},\eta}(g,f_{n+1})\, dg$$

for Schwartz functions f_n and f_{n+1} respectively on $G_n(\mathbb{A})$ and $G_{n+1}(\mathbb{A})$. Here we have identified G_n as the subgroup of G_{n+1} fixing the last vector of the canonical basis. Recall that the Jacquet-Rallis trace formula plays a central role in the recent proof of Gan-Gross-Prasad and Ichino-Ikeda conjectures for unitary groups, see among others [Zha14, BPLZZ21, BPCZ22]. The more spectral contributions we understand, the more results we can extract from the (comparison of) relative trace formulae. To any cuspidal datum χ of G we can attach the χ-component K_χ of the kernel. By integrating it over $[G']$ we get the distribution $J^{G,\eta}_\chi(g,f)$. A key ingredient in [BPCZ22] is the spectral decomposition of $J^{G,\eta}_\chi(g,f)$ for some specific cuspidal data χ called $*$-regular, see [BPCZ22, theorem 4.3.3.1]. We shall provide in the article an extension of this decomposition to any cuspidal datum. Thus the article solves an important step towards the fine spectral decomposition of the Jacquet-Rallis trace formula from which we expect new applications. More generally, the results presented here should be useful for the spectral expansion of any trace formula based on the symmetric space S. For example, from our main result theorem 1.2.4.1 below, it is possible not only to get a proof of the fine spectral expansion of Jacquet's relative trace formula which is different from the proof of Lapid given in [Lap06, theorem 10.1] but also to get an explicit computation of the constants $\mathfrak{c}(M,\pi)$ in [Lap06, theorem 10.1].

1.2 Statement of the main result

1.2.1. In the following we fix $g \in G(\mathbb{A})$. As usual, all parabolic subgroups of G are assumed to be defined over F. They are standard, resp. semi-standard, if they contain the standard Borel of $\mathrm{GL}_n(E)$, resp. the group of diagonal matrices, viewed as an F-group. The Levi subgroups of G are the Levi factors defined over F of the parabolic subgroups of G.

Let $P = MN_P$ be a standard parabolic subgroup of G with its standard Levi decomposition, N_P being its unipotent radical. Let $\Pi_{\mathrm{disc}}(M)$ be the set of discrete automorphic representations of $M(\mathbb{A})$ with central character trivial on the central subgroup A_M^∞, see §§2.5.4 and 2.5.13. Let $\pi \in \Pi_{\mathrm{disc}}(M)$ and let $\mathcal{A}_{P,\pi}(G)$ be the space of (smooth) automorphic functions on the quotient $A_M^\infty M(F)N_P(\mathbb{A})\backslash G(\mathbb{A})$ that belong to the "π-component", see §2.5.13. Let λ be an element in the $\mathbb{C}$-vector space $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$ of complex unramified characters of $P(\mathbb{A})$ trivial on A_G^∞. We have a map $\varphi \mapsto \varphi_\lambda$ that identifies $\mathcal{A}_{P,\pi}(G)$ with the induced representation $I_P^G(\pi\otimes\lambda)$. Then $f \in \mathcal{S}(G(\mathbb{A}))$ acts on $I_P^G(\pi\otimes\lambda)$ and thus on $\mathcal{A}_{P,\pi}(G)$ by transport. We denote by $I(\lambda,f)$ the action we get, see §2.5.13.

1.2.2. Intertwining periods. — We define a subset denoted by $\mathcal{L}_2(M)$ of the set of Levi subgroups of G that contain M, see §2.2.3. On the space $\mathcal{A}_{P,\pi}(G)$ we shall consider several linear forms $J_L(\varphi,\lambda)$ attached to Levi subgroups $L \in \mathcal{L}_2(M)$ and $\lambda \in \mathfrak{a}^{L,*}_{M,\mathbb{C}}$. These are the so-called intertwining periods $J_L(\varphi,\lambda)$ introduced in [JLR99, section VII] when π is cuspidal. We need to extend their definition to the discrete case. For the introduction we shall give the definition only when L is also standard, for the general case see section 5. One can attach to L a permutation matrix ξ of order 2. Recall ι is the Galois involution of G. By the choice of an element $\tilde{\xi} \in G(F)$

such that $\tilde{\xi}\iota(\tilde{\xi})^{-1} = \xi$, one can identify the subgroup of P fixed by the involution $\mathrm{Int}(\xi) \circ \iota$ to a subgroup P_ξ of G' which comes with a natural Levi decomposition $P_\xi = M_\xi N_\xi$, see §5.1.2 (note that we slightly simplify the notation in the introduction). Then the intertwining period is given by

$$J_L(\varphi, \lambda) = \int_{N_\xi(\mathbb{A}) A^\infty_{M_\xi} M_\xi(F)\backslash G'(\mathbb{A})} \varphi_\lambda(\tilde{\xi}h)\, dh.$$

Here the integral is convergent for λ in some cone, see proposition 5.1.2.1, and admits a meromorphic continuation in general, see corollary 5.1.4.2.

1.2.3. Relative characters. — The relative characters we consider are distributions on $\mathcal{S}(G(\mathbb{A}))$ attached to π, $L \in \mathcal{L}_2(L)$ and $\lambda \in i\mathfrak{a}_M^{L,*}$ that is λ is a unitary character. They are given by:

$$\mathcal{J}_{L,\pi}(g, f, \lambda) = \sum_\varphi E(g, I(\lambda, f)\varphi, \lambda)\overline{J_L(\varphi, \lambda)} \tag{1.2.3.1}$$

where the sum is over some Hilbert basis of $\mathcal{A}_{P,\pi}(G)$ for the Petersson inner product (see subsection 7.2) and where $E(g, \varphi, \lambda)$ denotes an Eisenstein series, see §2.5.15. Note that whereas $E(g, I(\lambda, f)\varphi, \lambda)$ is holomorphic on $i\mathfrak{a}_M^{L,*}$ the intertwining period $J_L(\varphi, \lambda)$ may have singularities which are described through theorems 5.1.4.1 and 4.5.3.1. However it turns out that the product $E(g, I(\lambda, f)\varphi, \lambda)\overline{J_L(\varphi, \lambda)}$ is smooth on $i\mathfrak{a}_M^{L,*}$. Moreover the relative character $\mathcal{J}_{L,\pi}(g, f, \lambda)$ is smooth in $\lambda \in i\mathfrak{a}_M^{L,*}$ and continuous on $\mathcal{S}(G(\mathbb{A}))$, see proposition 7.2.3.2.

1.2.4. Main theorem. — The main result of the paper is the following theorem.

Theorem 1.2.4.1. — *(see theorem 7.3.2.1) For any $f \in \mathcal{S}(G(\mathbb{A}))$, we have*

$$J^G(g, f) = \sum_{(M,L,\pi)} c_M^L \int_{i\mathfrak{a}_M^{L,*}} \mathcal{J}_{L,\pi}(g, f, \lambda)\, d\lambda. \tag{1.2.4.2}$$

The sum above is absolutely convergent.

In (1.2.4.2) the sum is over triples (M, L, π) where M is a standard Levi subgroup, $L \in \mathcal{L}_2(M)$ and $\pi \in \Pi_{\mathrm{disc}}(M)$. We also set

$$c_M^L = |\mathcal{P}(M)|^{-1} 2^{-\dim(\mathfrak{a}_L)}$$

where $\mathcal{P}(M)$ is the finite set of parabolic subgroups admitting M as a Levi factor and $|\mathcal{P}(M)|$ is its cardinality. Let χ be a cuspidal datum of G. One obtains the spectral decomposition of $J_\chi^G(g, f)$ by restricting the sum to triples (M, L, π) such that π belongs to the subset $\Pi_\chi(M) \subset \Pi_{\mathrm{disc}}(M)$ defined in §7.1.1.

The discrete part of the spectral contribution is given by

$$J^G_{\mathrm{disc}}(g, f) = \sum_{(M,\pi)} c_M^M \mathcal{J}_{M,\pi}(g, f, 0)$$

where the sum is over pairs (M, π) where M is a standard Levi subgroup and $\pi \in \Pi_{\mathrm{disc}}(M)$. The permutation matrix is trivial in this case and the relative character takes the simple form:

$$\mathcal{J}_{M,\pi}(g, f, 0) = \sum_\varphi E(g, I(\lambda, f)\varphi, 0) \int_{N'(\mathbb{A}) A^\infty_{M'} M'(F)\backslash G'(\mathbb{A})} \overline{\varphi(h)}\, dh$$

where $P' = P \cap G'$ and $M'N'$ is its Levi decomposition. In particular, the contribution of (M, π) vanishes unless π is M'-distinguished in the sense of §4.5.1 below. The classification of M'-distinguished discrete representations in terms of distinguished cuspidal representations is known by the work of Yamana, see [Yam15, theorem 1.2]. We leave to a future paper the question of characterizing the non-vanishing of the relative characters $\mathcal{J}_{L,\pi}(g, f, \lambda)$.

1.3 Bounds of discrete Eisenstein series

1.3.1. A technical but crucial point in the proof of theorem 1.2.4.1 is the majorization of Eisenstein series for the group G in the neighborhood of the imaginary axis. For cuspidal Eisenstein series the bounds are those introduced by Lapid in [Lap13] and [Lap06]. Here we generalize the bounds to the case of discrete Eisenstein series, namely Eisenstein series built from a discrete automorphic representation.

Theorem 1.3.1.1. — *(for a stronger statement see theorem 3.4.2.1) There exists $N > 0$ such that for all $q > 0$ there is a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbb{A}))$ such that for all $f \in \mathcal{S}(G(\mathbb{A}))$, all standard parabolic subgroups $P = MN_P$, $\pi \in \Pi_{\mathrm{disc}}(M)$, $\lambda \in i\mathfrak{a}_P^{G,*}$ and $x \in G(\mathbb{A})^1$ we have*

$$\sum_{\varphi} |E(x, I(\lambda, f)\varphi, \lambda)|^2 \leqslant \frac{\|x\|_G^N \|f\|^2}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}$$

where the sum is over some Hilbert basis of $\mathcal{A}_{P,\pi}(G)$.

Note that Λ_π is the numerical invariant attached to π in §3.2.2. The other notations are explained in subsections 2.1 and 2.5. For simplicity we have stated the theorem for λ on the imaginary axis. However we shall need and prove a stronger version where the bound holds on a neighborhood of the imaginary axis depending on π.

1.3.2. To get theorem 1.3.1.1 we proceed as in [Lap13] and [Lap06], namely we majorize the Eisenstein series by the Petersson norm of a truncated Eisenstein series. Then we need a new ingredient namely the explicit computation of the truncated product of two Eisenstein series. This is provided by theorem 3.4.2.1. By explicit, we mean a combinatorial expression that involves only intertwining operators and Petersson products of discrete automorphic representations. Thus theorem 3.4.2.1 both generalizes the classical statement for cuspidal Eisenstein series and the well-known Arthur's asymptotic formula of [Art82]. The proof of the theorem is closely related to the works [JLR99] of Jacquet-Lapid-Rogawski and [Lap11a] of Lapid. More precisely we use the fact that several intricate expressions provide families of meromorphic invariant bilinear forms on some induced representations. By an observation due to Bernstein they must vanish. We also rely on the precise computation of the exponents of the discrete automorphic representations based on their description by Mœglin-Waldspurger in [MW89] from which we extract some geometric properties, see lemma 3.1.5.2.

1.4 About the proof of the main theorem

1.4.1. The starting point to get the spectral expansion of $J^G(g, f)$ is to approximate it by

(1.4.1.1) $$J^{G,T}(g, f) = \int_{[G']} (K_f \Lambda_m^T)(g, h)\, dh \quad g \in G(\mathbb{A}).$$

Here T is a parameter and Λ_m^T is the "mixed" (as opposed to Arthur's one) truncation operator introduced by Jacquet-Lapid-Rogawski in [JLR99]. When applied to the map $x \in G(\mathbb{A}) \mapsto K_f(g, x)$ we get the expression $(K_f \Lambda_m^T)(g, \cdot)$. On the one hand the limit of $J^{G,T}(g, f)$ when T goes to infinity is $J^G(g, f)$. On the other hand, we can easily get the spectral expansion of $J^{G,T}(g, f)$ using the Langlands spectral decomposition of the kernel K_f, see proposition 7.1.3.1:

$$J^{G,T}(g, f) = \frac{1}{2} \sum_{(M,\pi)} |\mathcal{P}(M)|^{-1} \int_{i\mathfrak{a}_M^{G,*}} \mathcal{E}_\pi^T(g, f, \lambda)\, d\lambda.$$

The sum above is over pairs (M, π) where M is a standard Levi subgroup and $\pi \in \Pi_{\mathrm{disc}}(M)$. The relative character here is defined by:

$$\mathcal{E}_\pi^T(g, f, \lambda) = \sum_{\varphi} E(g, I(\lambda, f)\varphi, \lambda) \overline{\int_{[G']_0} \Lambda_m^T E(\varphi, \lambda)}.$$

The main difference with (1.2.3.1) is that we have replaced the intertwining period by the Flicker-Rallis period of the mixed truncated discrete Eisenstein series. The study of such periods is undertaken is section 4. The starting point is that the Flicker-Rallis period of a truncated discrete Eisenstein series can be expressed in terms of Jacquet-Lapid-Rogawski regularized periods of Eiseinstein series, see proposition 4.4.1.1. Let's mention that the proposition is not completely formal: once again we rely on the explicit description of exponents of discrete automorphic representations. As in Arthur's asymptotic formula for the scalar product of two Eisenstein series, some of the contributions are negligible when T goes to infinity. Among the main terms, many regularized periods of Eiseinstein series vanish, see proposition 4.5.2.1. When they do not obviously vanish, they are related to the intertwining periods, see theorem 5.1.4.1. At this point, we can show that the Flicker-Rallis period of a truncated discrete Eisenstein series is asymptotic to an explicit combination of intertwining periods. Using the stronger version of theorem 1.3.1.1 one can show:

Proposition 1.4.1.1. —*(see proposition 6.2.1.1 for a stronger and more precise statement.) Let $\varepsilon > 0$. For all $q > 0$ there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbb{A}))$ such that for all pairs (M,π) as above, all $f \in \mathcal{S}(G(\mathbb{A}))$, all $\lambda \in i\mathfrak{a}_M^{G,*}$*

$$|\mathcal{E}_\pi^T(g,f,\lambda) - \sum_{Q\in\mathcal{F}_2(M)} 2^{-\dim(\mathfrak{a}_Q^G)} \mathcal{J}_{Q,\pi}(g,\varphi,\lambda)\frac{\exp(-\langle\lambda, T_Q^G\rangle)}{\theta_Q(-\lambda)}| \leqslant \frac{\exp(-\varepsilon\|T\|)\|f\|}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}$$

for all T suitably regular.

Here the set $\mathcal{F}_2(M)$ is the set of semi-standard parabolic subgroups whose semi-standard Levi factor belongs to $\mathcal{L}_2(M)$ and $\mathcal{J}_{Q,\pi}(g,\varphi,\lambda)$ is yet another relative character whose definition is given in (7.2.1.4). Moreover θ_Q is the familiar polynomial from Arthur's theory, see §2.1.8.

For any $L \in \mathcal{L}_2(M)$ the family $(\mathcal{J}_{Q,\pi}(g,\varphi,\lambda))_{Q\in\mathcal{P}(L)}$ indexed by the set of parabolic subgroups of Levi L is a (G,L)-family (in the sense of Arthur) of Schwartz functions on $i\mathfrak{a}_M^{G,*}$. This has several consequences. First the value $\mathcal{J}_{Q,\pi}(g,\varphi,\lambda)$ on $i\mathfrak{a}_M^{L,*}$ does not depend on $Q \in \mathcal{P}(L)$ and is in fact equal to $\mathcal{J}_{L,\pi}(g,\varphi,\lambda)$. Second the expression

$$\sum_{Q\in\mathcal{P}(L)} \mathcal{J}_{Q,\pi}(g,\varphi,\lambda)\frac{\exp(-\langle\lambda, T_Q^G\rangle}{\theta_Q(-\lambda)}$$

defines a Schwartz function on $i\mathfrak{a}_M^{G,*}$ and we have

$$\lim_{T\to+\infty}\int_{i\mathfrak{a}_M^{G,*}} \sum_{Q\in\mathcal{P}(L)} \mathcal{J}_{Q,\pi}(g,\varphi,\lambda)\frac{\exp(-\langle\lambda, T_Q^G\rangle}{\theta_Q(-\lambda)}\,d\lambda = \int_{i\mathfrak{a}_M^{L,*}} \mathcal{J}_{L,\pi}(g,\varphi,\lambda)\,d\lambda.$$

This gives theorem 1.2.4.1.

1.5 Organization of the paper

1.5.1. In section 2, we collect the notations and some elementary lemmas about polynomial exponential maps.

1.5.2. The section 3 is devoted to the proof of theorem 1.3.1.1 above. First in subsection 3.1 we obtain an explicit expression for the scalar product of two truncated Eisenstein series, see theorem 3.1.3.1. Then in subsection 3.2 following a work of Lapid, we use this expression to get a bound of this scalar product, see proposition 3.2.5.1. In subsection 3.3, the next step is to deduce a bound for some relative characters, see proposition 3.3.3.1. Finally in subsection 3.4 we state and prove theorems 3.4.2.1 and 3.4.3.1 which are stronger versions of theorem 1.3.1.1 above.

1.5.3. The section 4 is devoted to the study of truncated Flicker-Rallis periods of discrete Eisenstein series and their regularized versions. The definition of the mixed truncation operator of Jacquet-Lapid-Rogawski is recalled in subsection 4.2. The subsection 4.3 gives the definition (after Jacquet-Lapid-Rogawski) and the first properties of regularized periods of discrete Eisenstein series which are meromorphic functions defined in terms of truncated Flicker-Rallis periods. In subsection 4.4, the process is reversed and the truncated Flicker-Rallis period of a discrete Eisenstein series is expressed in terms of regularized periods of Eisenstein series, see proposition 4.4.1.1 and corollary 4.4.2.1. In this expansion, we can distinguish the "main part" (4.4.4.3) and we establish smoothness properties of it in proposition 4.4.4.1. This will be the main part when the parameter goes to infinity. In subsection 4.5 we show that most of the regularized periods of discrete Eisenstein series vanish. Then we determine their possible singularities on the imaginary axis, see theorem 4.5.3.1.

1.5.4. The section 5 extends the definition of intertwining periods, also due to Jacquet-Lapid-Rogawski, to the case of discrete automorphic representation. The main results are stated in subsection 5.1. For some cones, intertwining periods are given by a quite simple convergent integral see (5.1.4.4) and proposition 5.1.2.1. In theorem 5.1.4.1 we show that on the convergence domain they coincide with regularized periods of discrete Eisenstein series. As a consequence they admit a meromorphic continuation, see corollary 5.1.4.2. The bulk of the proof of theorem 5.1.4.1 is given in subsection 5.2. Note that it relies on Jacquet-Lapid-Rogawski results on intertwining periods of cuspidal representation. A crucial step is in fact to relate the intertwining period of a discrete representation to that of the cuspidal representation that is part of Mœglin-Waldspurger classification, see proposition 5.2.7.2.

1.5.5. The goal of section 6 is to introduce majorizations of various relative characters that depend on a truncation parameter and that will play a role in spectral decomposition. The first example given in subsection 6.1 is built upon Flicker-Rallis periods of truncated discrete Eisenstein series. In subsection 6.2, a second example is introduced built upon the "main part" defined in subsection 4.4. Moreover proposition 6.2.1.1 shows that the two examples coincide asymptotically, see also proposition 1.4.1.1 above.

1.5.6. The final section 7 states and proves the spectral expansion namely theorem 7.3.2.1 (and theorem 1.2.4.1 above). The starting point given in subsection 7.1 is the spectral decomposition (1.4.1.1) given by proposition 7.1.3.1 and proposition 7.1.3.2 which reduces the problem to the computation of the limit of integrals of the relative characters associated to the main part. The computation of the limit is achieved in subsection 7.2 and theorem 7.2.4.1 which gives the spectral expansion of $J^G_\chi(g,f)$ for a cuspidal datum χ. Finally in subsection 7.3 the spectral decomposition is stated and an argument based on Müller's work is given for the absolute convergence.

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2 Preliminaries

2.1 General notations

2.1.1. Let F be a field of characteristic 0.

2.1.2. Let G be an algebraic group defined over F. Let N_G be the unipotent radical of G and $X^*(G)$ be the group of algebraic morphism $G \to \mathbb{G}_{m,F}$ defined over F. Let $\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{R})$. We denote by

(2.1.2.1) $$\langle \cdot, \cdot \rangle : \mathfrak{a}_G^* \times \mathfrak{a}_G \to \mathbb{R}$$

the canonical pairing.

2.1.3. Let's assume that G is moreover reductive. We shall recall Arthur's notations. Let P_0 be a parabolic subgroup of G defined over F and minimal for these properties. Let M_0 be a Levi factor of P_0 defined over F. A parabolic (resp. and semi-standard, resp. and standard) subgroup of G is a parabolic subgroup of G defined over F (resp. which contains M_0, resp. which contains P_0). For any semi-standard parabolic subgroup P, we have a Levi decomposition $P = M_P N_P$ where M_P is the unique Levi factor that contains M_0. A Levi subgroup of G (resp. semi-standard, resp. standard) is a Levi factor defined over F of a parabolic subgroup of G (resp. semi-standard, resp. standard).

2.1.4. Let A_G be the maximal central F-split torus of G. For any semi-standard parabolic subgroup P of G, we set $A_P = A_{M_P}$. The restrictions maps $X^*(P) \to X^*(M_P) \to X^*(A_P)$ induce isomorphisms $\mathfrak{a}_P^* \simeq \mathfrak{a}_{M_P}^* \simeq \mathfrak{a}_{A_P}^*$. We set $\mathfrak{a}_0^* = \mathfrak{a}_{P_0}^*$, $\mathfrak{a}_0 = \mathfrak{a}_{P_0}$ and $A_0 = A_{P_0}$.

2.1.5. For any semi-standard parabolic subgroups $P \subset Q$ of G, the restriction map $X^*(Q) \to X^*(P)$ induces maps $\mathfrak{a}_Q^* \to \mathfrak{a}_P^*$ and $\mathfrak{a}_P \to \mathfrak{a}_Q$. The first one is injective whereas the kernel of the second one is denoted by $\mathfrak{a}_P^Q$. The restriction map $X^*(A_P) \to X^*(A_Q)$ gives a surjective map $\mathfrak{a}_P^* \to \mathfrak{a}_Q^*$ whose kernel is denoted by $\mathfrak{a}_P^{Q,*}$. We get also an injective map $\mathfrak{a}_Q \to \mathfrak{a}_P$. In this way, we get dual decompositions $\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q$ and $\mathfrak{a}_P^* = \mathfrak{a}_P^{Q,*} \oplus \mathfrak{a}_Q^*$. Thus we have projections $\mathfrak{a}_0 \to \mathfrak{a}_P^Q$ (resp. $\mathfrak{a}_P$) and $\mathfrak{a}_0^* \to \mathfrak{a}_P^{Q,*}$ (resp. $\mathfrak{a}_P^*$) denoted by $X \mapsto X_P^Q$ (resp. X_P). They depend only on the Levi factors M_P and M_Q and they will be also denoted by $X \mapsto X_{M_P}^{M_Q}$, resp. X_{M_P}.

We denote by $\mathfrak{a}_{P,\mathbb{C}}^{Q,*}$ and $\mathfrak{a}_{P,\mathbb{C}}^Q$ the $\mathbb{C}$-vector spaces obtained by extension of scalars from $\mathfrak{a}_P^{Q,*}$ and $\mathfrak{a}_P^Q$. We still denote by $\langle \cdot, \cdot \rangle$ the pairing we get from (2.1.2.1) by extension of scalars to $\mathbb{C}$. We have a decomposition

$$\mathfrak{a}_{P,\mathbb{C}}^{Q,*} = \mathfrak{a}_P^{Q,*} \oplus i\mathfrak{a}_P^{Q,*}$$

where $i^2 = -1$. We shall denote by $\Re$ and $\Im$ the real and imaginary parts associated to this decomposition and by $\bar{\lambda}$ the complex conjugate of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{Q,*}$.

2.1.6. For any parabolic subgroup P of G containing P_0, we denote by $\Delta_0^P = \Delta_{P_0}^P$ the set of simple roots of A_0 in $M_P \cap P_0$. If moreover Q is a parabolic subgroup containing P we denote by Δ_P^Q be the image of $\Delta_0^Q \setminus \Delta_0^P$ (viewed as a subset of $\mathfrak{a}_0^*$) by the projection $\mathfrak{a}_0^* \to \mathfrak{a}_P^*$. In the same way one defines the set of coroots $\Delta_P^{Q,\vee} \subset \mathfrak{a}_P^Q$. Dually, we get the set of simple weights $\hat{\Delta}_P^Q$. The sets Δ_P^Q and $\hat{\Delta}_P^Q$ determine open cones in $\mathfrak{a}_0$ whose characteristic functions are denoted respectively by τ_P^Q and $\hat{\tau}_P^Q$.

2.1.7. Let W be the Weyl group of (G, A_0) that is the quotient by M_0 of the normalizer of A_0 in $G(F)$. The group acts on $\mathfrak{a}_0$ and its dual $\mathfrak{a}_0^*$. We fix once and for all an invariant inner product on $\mathfrak{a}_0^*$. Then we can identify $\mathfrak{a}_0^*$ and $\mathfrak{a}_0$ and all the decompositions we introduced are orthogonal for the inner product. We denote by $\| \cdot \|$ the euclidean norm.

2.1.8. Let P be a semi-standard parabolic subgroup of G. We equip $\mathfrak{a}_P$ with the Haar measure that gives a covolume 1 to the lattice $\mathrm{Hom}(X^*(P), \mathbb{Z})$. The space $i\mathfrak{a}_P^*$ is then equipped with the dual Haar measure so that we have

$$\int_{i\mathfrak{a}_P^*} \int_{\mathfrak{a}_P} \phi(H) \exp(-\langle \lambda, H \rangle) \, dH d\lambda = \phi(0)$$

for all $\phi \in C_c^\infty(\mathfrak{a}_P)$.

For any basis B of $\mathfrak{a}_P^Q$ let's denote by $\mathbb{Z}(B)$ the lattice generated by B and by $\mathrm{vol}(\mathfrak{a}_P^Q/\mathbb{Z}(B))$ the covolume of this lattice where $\mathfrak{a}_P^Q \simeq \mathfrak{a}_P/\mathfrak{a}_Q$ is provided with the quotient Haar measure. We have on $\mathfrak{a}_0^*$ the polynomial functions:

$$\hat{\theta}_P^Q(\lambda) = \mathrm{vol}(\mathfrak{a}_P^Q/\mathbb{Z}(\hat{\Delta}_P^{Q,\vee}))^{-1} \prod_{\varpi^\vee \in \hat{\Delta}_P^{Q,\vee}} \langle \lambda, \varpi^\vee \rangle$$

and

$$\theta_P^Q(\lambda) = \mathrm{vol}(\mathfrak{a}_P^Q/\mathbb{Z}(\Delta_P^{Q,\vee}))^{-1} \prod_{\alpha \in \Delta_P^Q} \langle \lambda, \alpha^\vee \rangle.$$

2.1.9. Truncation parameter. — For any $T \in \mathfrak{a}_P^Q$, we set

$$d_P^Q(T) = \inf_{\alpha \in \Delta_P^Q} \langle \alpha, T \rangle.$$

Let $\mathfrak{a}_P^{Q+} = \{T \in \mathfrak{a}_P^Q \mid d_P^Q(T) \geqslant 0\}$. If $Q = G$, the exponent G is omitted. We set $d(T) = d_{P_0}^G(T)$.

Let $T \in \mathfrak{a}_0$ such that $d(T) \geqslant 0$. We shall say that T is enough positive (sufficiently regular in the terminology of [Art78, p. 937]) if $d(T)$ is large enough. Often we simply say T is a truncation parameter. Since most of the constructions we are interested in do not depend on the choice of T, the precise lower bound is irrelevant. For the constructions that really depend on T, we will be in fact interested in their asymptotic behaviour when $d(T) \to +\infty$.

For any semi-standard parabolic subgroup P and any point $T \in \mathfrak{a}_0$, we define a point $T_P \in \mathfrak{a}_P$ in the following way : we choose $w \in W$ such that $wP_0w^{-1} \subset P$, the point T_P is defined to be the orthogonal projection of $w \cdot T$ on $\mathfrak{a}_P$ that is $T_P = (w \cdot T)_{M_P}$ (with the notations of §2.1.5). One can check that this does not depend on the choice of w. If P is standard, then we have $T_P = T_{M_P}$.

2.2 Weyl groups and sets of Levi subgroups

2.2.1. Double cosets. —

Let $P = MN_P$ and $Q = LN_Q$ be standard parabolic subgroup P of G with standard Levi decomposition.

Let ${}_QW_P$ be the set of $w \in W$ such that

- $M \cap w^{-1}P_0w = M \cap P_0$;
- $L \cap wP_0w^{-1} = L \cap P_0$.

This is a set of representatives of the double quotient $W^Q \backslash W / W^P$. Moreover, $M \cap w^{-1}Lw$ is the Levi factor of the standard parabolic subgroup $P_w = (M \cap w^{-1}Qw)N_P$ included in P. In the same way, $L \cap wMw^{-1}$ is the Levi factor of the standard parabolic subgroup $Q_w = (L \cap wPw^{-1})N_Q$ with $Q_w \subset Q$. We introduce:

$$\begin{aligned} W(P;Q) &= \{w \in {}_QW_P \mid P_w = P\} = \{w \in {}_QW_P \mid M \subset w^{-1}Lw\} \\ W(P,Q) &= \{w \in {}_QW_P \mid M = w^{-1}Lw\}. \end{aligned}$$

When $P = Q$, the group $W(P,P)$ is denoted by $W(P)$. For any $w \in W(P,Q)$ one has $w\Delta_0^P = \Delta_0^Q$. Note also that $w \in W(P_w, Q_w)$ for all $w \in {}_QW_P$. We have $W(P,Q) \subset W(P;Q)$ and we can also write the set $W(P;Q)$ as the disjoint union

$$W(P;Q) = \bigcup_{R \subset Q} \{w \in W(P,R) \mid w^{-1}\Delta_0^Q > 0\} \tag{2.2.1.1}$$

where R runs over the set of standard parabolic subgroups of Q and > 0 means positive relatively to P_0. The contribution of R is empty unless R is associated to P.

Let

$$W_2(P) = \{w \in W(P) \mid w^2 = 1\}.$$

We will also write W^P or W^{M_P} for the Weyl group of (M_P, A_0). We view it as a subgroup of W. If $P \subset Q$, we set $W_2^Q(P) = W_2(P) \cap W^Q$. More generally, for any standard parabolic subgroup R of G, we will denote by an upperscript R an object relative to its Levi factor M_R equipped with the minimal pair $(P_0 \cap M_R, M_0)$. The notations above hold if we replace the standard parabolic subgroups by their standard Levi components, e.g. $W(M;L)$ and $W_2(M)$ respectively mean $W(P;Q)$ and $W_2(P)$ if $M = M_P$ and $L = M_Q$.

Lemma 2.2.1.1. — *Let P, Q, R be standard parabolic subgroups of G. We assume $Q \subset R$.*

1. *For any $w \in {}_RW_P$, we have ${}_QW^R_{R_w} w \subset {}_QW_P$.*
2. *For any $w_2 \in {}_QW_P$, there is a unique decomposition $w_2 = w_1 w$ with $w \in {}_RW_P$ and $w_1 \in {}_QW^R_{R_w}$. Moreover, $w_2 \in W(P;Q)$ if and only if $w_1 \in W^R(R_w;Q)$ and $w \in W(P;R)$.*

Proof. — Let $w \in {}_RW_P$ and $w_1 \in {}_QW^R_{R_w}$. First, we show that $w_2 = w_1 w$ belongs to ${}_QW_P$.

One has $M_{R_w} \cap w_1^{-1} P_0 w_1 = M_{R_w} \cap P_0$. Thus $M_{P_w} \cap w_2^{-1} P_0 w_2 = M_{P_w} \cap w P_0 w^{-1} = M_{P_w} \cap P_0$. But $M_P \cap w_2^{-1} P_0 w_2 \subset M_P \cap w^{-1} R w = M_P \cap P_w \subset P_w$ (indeed one observes that $M_P \cap w^{-1} R w$ contains $M_P \cap w^{-1} P_0 w = M_P \cap P_0$). But $P_w \cap w_2^{-1} P_0 w_2 \subset (M_{P_w} \cap w_2^{-1} P_0 w_2) N_{P_0} \subset P_0$. Hence $M_P \cap w_2^{-1} P_0 w_2 = M_P \cap P_0$.

One also has

$$\begin{aligned} M_Q \cap w_2 P_0 w_2^{-1} &\subset w_1 (M_R \cap w P_0 w^{-1}) w_1^{-1} \\ &\subset w_1 (M_R \cap P_0) w_1^{-1} \subset w_1 P_0 w_1^{-1}. \end{aligned}$$

Hence we have

$$M_Q \cap w_2 P_0 w_2^{-1} = M_Q \cap w_1 P_0 w_1^{-1} = M_Q \cap P_0.$$

Conversely, any $w_2 \in {}_QW_P$ can be uniquely written $w_2 = w_1 w$ with $w_1 \in W^R$ and $w \in {}_RW_P$. We want to show $w_1 \in {}_QW^R_{R_w}$. On the one hand, we have

$$\begin{aligned} M_{R_w} \cap w_1^{-1} P_0 w_1 &= M_R \cap w M_P w^{-1} \cap w_1^{-1} P_0 w_1 \\ &= M_R \cap w (M_P \cap w_2^{-1} P_0 w_2) w^{-1} \\ &= M_R \cap w (M_P \cap P_0) w^{-1} \\ &= M_R \cap w (M_P \cap w^{-1} P_0 w) w^{-1} \\ &\subset P_0. \end{aligned}$$

Hence $M_{R_w} \cap w_1^{-1} P_0 w_1 = M_{R_w} \cap P_0$. On the other hand,

$$\begin{aligned} M_Q \cap w_1 P_0 w_1^{-1} &\subset M_R \cap w_1 P_0 w_1^{-1} \\ &\subset w_1 (M_R \cap P_0) w_1^{-1} \\ &\subset w_1 (M_R \cap w P_0 w^{-1}) w_1^{-1} \\ &\subset w_2 P_0 w_2^{-1}. \end{aligned}$$

Hence we have $M_Q \cap w_1 P_0 w_1^{-1} = M_Q \cap w_2 P_0 w_2^{-1} = M_Q \cap P_0$.

Let's prove the last claim. Let $w_1 \in W^R(R_w;Q)$ and $w \in W(P;R)$. Set $w_2 = w_1 w$. Then

$$M_P = w^{-1} M_{R_w} w \subset w^{-1} w_1^{-1} M_Q w_1 w = w_2^{-1} M_Q w_2$$

hence $w_2 \in W(P;Q)$.

Conversely let $w \in {}_RW_P$ and $w_1 \in {}_QW^R_{R_w}$ such $w_2 = w_1 w \in W(P;Q)$. We have

$$M_P \subset w_2^{-1} M_Q w_2 \subset w_2^{-1} M_R w_2 = w^{-1} M_R w$$

hence $w \in W(P;R)$. Then

$$M_{R_w} = M_R \cap w M_P w^{-1} = w M_P w^{-1} \subset w w_2^{-1} M_Q w_2 w^{-1} = w_1^{-1} M_Q w_1$$

hence $w_1 \in W^R(R_w;Q)$. □

2.2.2. Let M a semi-standard Levi subgroup of G. We denote by $\mathcal{L}(M)$ (resp. $\mathcal{F}(M)$) the set of Levi subgroups (resp. parabolic subgroups) of G that contain M. Let $\mathcal{P}(M)$ be the set of minimal elements of $\mathcal{F}(M)$ that is the elements $P \in \mathcal{F}(M)$ such that $M_P = M$. We have a disjoint union

$$\mathcal{F}(M) = \bigcup_{L \in \mathcal{L}(M)} \mathcal{P}(L).$$

If M is moreover standard, it defines a standard parabolic subgroup $P = MN_P$. Then the map

$$(Q, w) \mapsto w^{-1} \cdot Q \tag{2.2.2.2}$$

induces a bijection from the disjoint union $\bigcup_Q W(P;Q)$ where Q runs over the set of standard parabolic subgroups of G onto $\mathcal{F}(M)$. Here $w^{-1} \cdot Q$ means the conjugate parabolic subgroup $w^{-1}Qw$. It also induces a bijection from the disjoint union $\bigcup_Q W(P,Q)$ onto $\mathcal{P}(M)$ (in this case only parabolic subgroups Q that are associated to P do contribute to the source). We denote the sets above $\mathcal{L}^G(M)$, $\mathcal{P}^G(M)$ and $\mathcal{F}^G(M)$ if we want to emphasize the dependence on G.

2.2.3. Let $P = MN_P$ be a standard parabolic subgroup of G (with its standard decomposition). Any $\xi \in W_2(M)$ induces an involution of $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$. We have a decomposition $\mathfrak{a}_M = \mathfrak{a}_M^{\xi} \oplus \mathfrak{a}_M^{-\xi}$ where $\mathfrak{a}_M^{\pm\xi}$ denotes the (± 1)-eigenspace spaces of ξ. The same notation holds for the dual space $\mathfrak{a}_M^*$. There exists a unique $L_\xi \in \mathcal{L}(M)$ such that $\mathfrak{a}_M^{L_\xi} = \mathfrak{a}_M^{-\xi}$. We get a bijective map $\xi \mapsto L_\xi$ from $W_2(M)$ onto a subset of $\mathcal{L}(M)$ denoted by $\mathcal{L}_2(M)$. We denote the inverse map $\mathcal{L}_2(M) \to W_2(M)$ by $L \mapsto \xi_M^L$. We say that the involution ξ is standard if L_ξ is standard (such a ξ is called minimal in [JLR99] section VII.16). Let $W_2^{\mathrm{st}}(M) \subset W_2(M)$ be the subset of standard ξ's. Let $\mathcal{F}_2(M) \subset \mathcal{F}(M)$ be the subset of $Q \in \mathcal{F}(M)$ such that $M_Q \in \mathcal{L}_2(M)$. We get a map

$$Q \mapsto \xi_M^Q = \xi_M^{M_Q}$$

from $\mathcal{F}_2(M)$ to $W_2(M)$.

2.3 Polynomial exponential

2.3.1. Polynomial exponential. — Let V be a real vector space of finite dimension. Let V^* be its dual. Let $V_{\mathbb{C}}^*$ and $V_{\mathbb{C}}$ be the $\mathbb{C}$-vector space obtained from V^* and V by extension of scalars to $\mathbb{C}$. We denote by $\langle \cdot, \cdot \rangle : V_{\mathbb{C}}^* \times V_{\mathbb{C}} \to \mathbb{C}$ the canonical pairing.

Let $\Omega \subset V$ be a non-empty open subset. By a polynomial exponential on Ω, we mean a map $E : \Omega \to \mathbb{C}$ which belongs to the $\mathbb{C}$-vector space generated by $T \in \Omega \mapsto p(T)\exp(\langle \lambda, T \rangle)$ where $\lambda \in V_{\mathbb{C}}^*$ and $p \in \mathbb{C}[V]$ is a polynomial. Any polynomial exponential E on Ω can be uniquely written $\sum_{\lambda \in V_{\mathbb{C}}^*} p_\lambda(T)\exp(\langle \lambda, T \rangle)$ for a unique map $\lambda \in V_{\mathbb{C}}^* \mapsto p_\lambda \in \mathbb{C}[V]$ with finite support. The finite set of $\lambda \in V_{\mathbb{C}}^*$ such that $p_\lambda \neq 0$ is the set of exponents of E. We shall say that p_λ is the polynomial part of E of exponent λ. If $\lambda = 0$, we shall simply that p_0 is the polynomial part.

Let ω be a non-empty open subset of a $\mathbb{C}$-vector space Z of finite dimension. By a meromorphic function on ω with hyperplane singularities we mean a meromorphic function f on ω such that for any $z_0 \in \omega$, there exists a finite family of linear forms $(L_i)_{i \in I}$ (not necessarily two by two distinct) such that the product $\prod_{i \in I} L_i(z - z_0) f(z)$ is holomorphic in a neighborhood of z_0 in ω. Let $\mathcal{M}(\omega)$ be the $\mathbb{C}$-algebra of meromorphic function on ω with hyperplane singularities.

The following lemma is a slight variant of a lemma used by Offen (see [Off06] lemma 8.1)

Lemma 2.3.1.1. — *Let I be a finite set. For each $i \in I$, we consider the following objects:*

- *an affine map $q_i : Z \to V_{\mathbb{C}}^*$;*
- *a polynomial $p_i \in \mathbb{C}[V]$;*
- *a meromorphic function $f_i \in \mathcal{M}(\omega)$.*

Let $z_0 \in \omega$. We assume that for each $T \in \Omega$, the map

$$z \in \omega \mapsto E(z, T) = \sum_{i \in I} p_i(T) f_i(z) \exp(\langle q_i(z), T \rangle)$$

is holomorphic in a neighborhood of z_0 in ω.

1. *Let $j \in I$ and $I_j = \{i \in I \mid q_i(z_0) = q_j(z_0)\}$. The sum*

$$z \in \omega \mapsto E_j(z,T) = \sum_{i \in I_j} p_i(T) f_i(z) \exp(\langle q_i(z) - q_j(z_0), T\rangle)$$

   is also holomorphic in a neighborhood of z_0 in ω and $E_j(z_0,T)$ is the polynomial part of $E(z_0,T)$ of exponent $q_j(z_0)$.

2. *The map $T \in \Omega \mapsto E(z_0,T)$ is a polynomial exponential whose set of exponents is included in $\{q_i(z_0), i \in I\}$.*

Proof. — Clearly assertion 2 follows from assertion 1. Let's prove assertion 1. Without loss of generality we may and shall assume $z_0 = 0$. We may also shrink ω if necessary. Let $(L_k)_{1 \leqslant k \leqslant r}$ be a finite family of non-zero linear forms on Z. Let $H_k = \mathrm{Ker}(L_k)$. We assume that hyperplanes H_k are two by two distinct for $1 \leqslant k \leqslant r$. We assume also that for each k there is an integer $n_k \geqslant 1$ such that for $L = \prod_{1 \leqslant k \leqslant r} L_k^{n_k}$ the map $z \mapsto L(z) f_i(z)$ is holomorphic in a neighborhood of 0 in ω for all $i \in I$. Let $z_k \in Z$ such that $L_k(z_k) = 1$. We denote by ∂_k the complex derivative along the vector z_k. We shall use the following elementary result: for any $f \in \mathcal{M}(\omega)$ such that the map $z \mapsto L(z) f(z)$ is holomorphic on ω, the map f is holomorphic on ω if and only if for each $1 \leqslant k \leqslant r$ and $0 \leqslant m \leqslant n_k - 1$ the map $z \mapsto \partial_k^m (Lf)(z)$ vanishes on $\omega \cap H_k$.

By assumption, we have for each $1 \leqslant k \leqslant r$ and $0 \leqslant m \leqslant n_k - 1$

$$\partial_k^m (L(z)E(z,T)) = \sum_{i \in I} p_i(T) \partial_k^m (L(z) f_i(z) \exp(\langle q_i(z), T\rangle)) \tag{2.3.1.1}$$

vanishes for all $T \in \Omega$ and $z \in \omega \cap H_k$. For all $z \in \omega \cap H_k$, the map $T \mapsto \partial_k^m (L(z) f_i(z) \exp(\langle q_i(z), T\rangle))$ is a polynomial exponential with a unique exponent namely $q_i(z)$. We may and shall shrink ω such that that for all $i, j \in I$ such $q_i(z) = q_j(z)$ for some $z \in \omega \cap H_k$ then $q_i(0) = q_j(0)$ and thus $i \in I_j$. Let $j \in I$. Identifying the summand of exponents $q_i(z)$ for $i \in I_j$ in (2.3.1.1) we get that

$$\partial_k^m (L(z) E_j(z,T)) \exp(\langle q_j(0), T\rangle) = \sum_{i \in I_j} p_i(T) \partial_k^m (L(z) f_i(z) \exp(\langle q_i(z), T\rangle)) \tag{2.3.1.2}$$

vanishes for all $T \in \Omega$ and $z \in \omega \cap H_k$. As a consequence $E_j(z,T)$ is holomorphic on ω. Let's prove that $E_j(0,T)$ is a polynomial in T. Without loss of generality, we may and shall assume that $q_j(0) = 0$. Then q_i is linear for $i \in I_j$. Let $z \in \omega$ outside the hyperplanes H_k for $1 \leqslant k \leqslant r$. Then $E_j(0,T)$ is the degree 0 contribution in the Taylor expansion at $t = 0$ of

$$t \in \mathbb{R} \mapsto \sum_{i \in I_j} p_i(T) f_i(tz) \exp(t\langle q_i(z), T\rangle)$$

which is clearly a polynomial in T. □

2.3.2. Let $n \geqslant 1$ an integer. For any $z \in \mathbb{C}^n$, we set $\|(z_1, \ldots, z_n)\|^2 = \sum_{1 \leqslant i \leqslant n} |z_i|^2$. For any $0 < r \leqslant +\infty$ we denote by B_r the set $\{z \in i\mathbb{R}^n \mid \|z\| \leqslant r\}$. The following lemma is a variant of [LW13, lemme 13.2.2]. For $k \geqslant 1$ and $c, \Lambda > 0$ we also set

$$\mathcal{R}_{\Lambda,c,k} = \{\lambda \in \mathbb{C}^n \mid \|\Re(z)\|(1 + \Lambda + \|\Im(z)\|)^k < c\}.$$

Lemma 2.3.2.1. — *Let $(L_i)_{i \in I}$ be a finite family of non-zero real linear forms on $\mathbb{C}^n$, resp. and let $k \geqslant 1$ and $\Lambda > 0$. For any holomorphic differential operator D with polynomial coefficients and any $0 < r \leqslant +\infty$, resp. any $c > 0$, there exist differential operators $D_1, \ldots, D_r$ of the same nature and $1 > \alpha > 0$ such that for any holomorphic function f on a neighborhood of $i\mathbb{R}^n$, resp. on $\mathcal{R}_{\Lambda,c,k}$, we have:*

$$\sup_{z \in B_{\alpha r}} |Df(z)| \leqslant \sum_{i=1}^{r} \sup_{z \in B_r} |D_i F(z)| \tag{2.3.2.3}$$

resp.

$$\sup_{z\in\mathcal{R}_{\Lambda,\alpha c,k}}|Df(z)|\leqslant\sum_{i=1}^{r}\sup_{z\in\mathcal{R}_{\Lambda,c,k}}|D_iF(z)| \tag{2.3.2.4}$$

where $F=f\prod_{i\in I}L_i$.

Proof. — For the convenience of the reader, we sketch a proof which is a variant of that of [LW13, lemme 13.2.2]. First, by recursion, we may assume that there is only one linear form L. There exists a real invertible matrix A of size n such that $L(Az)=z_1$ where $z=(z_1,\ldots,z_n)$. Since there exist $1>\alpha,\beta>0$ such that $B_{\alpha r}\subset A(B_{\beta r})\subset B_r$, resp. $\mathcal{R}_{\Lambda,\alpha c,k}\subset A(\mathcal{R}_{\Lambda,\beta c,k})\subset\mathcal{R}_{\Lambda,c,k}$, it suffices to bound $f\circ A$ on $B_{\beta r}$, resp. $\mathcal{R}_{\Lambda,\beta c,k}$. Replacing f by $f\circ A$, L by $L\circ A$ and r by βr, resp. c by βc we are reduced to bound Df on X in terms of $F=z_1f$ where $X=B_r$, resp. $X=\mathcal{R}_{\Lambda,c,k}$. Then we are reduced to the case $D=D'D''$ where D' depends only on z_1 and D'' depends only on the other coordinates $z_2,\ldots,z_n$. Since $D''(F)=LD''(f)$ we may assume that D itself depends only on z_1. In this way, we are reduced to the case where $Df=\frac{\partial^i}{\partial z_1^i}(z_1^jf(z))$. The case $j\geqslant 1$ is obvious. So we assume that $D=\frac{\partial^i}{\partial z_1^i}$. There exists a holomorphic differential operator D_i with polynomial coefficients such that $Df=z_1^{-(i+1)}D_iF$. For any $z\in X$ let $z'=(0,z_2,\ldots,z_n)$. We have $z'\in X$. Since D_iF vanishes at $L=0$ at the order $i+1$, the mean value theorem implies that for all $z\in X$

$$|D_iF(z)|\leqslant|z_1|^{i+1}\sup_{u\in[z;z']}|\frac{\partial^{i+1}}{\partial z_1^{i+1}}D_iF(u)|$$

hence

$$|Df(z)|\leqslant\sup_{u\in[z;z']}|\frac{\partial^{i+1}}{\partial z^{i+1}}D_iF(u)|.$$

and

$$\sup_{z\in X}|Df(z)|\leqslant\sup_{u\in X}|\frac{\partial^{i+1}}{\partial z^{i+1}}D_iF(u)|.$$

□

2.4 A result on (G,M)-families

2.4.1. Let M be a semi-standard Levi subgroup of G. A family $(c_P)_{P\in\mathcal{P}(M)}$ of functions on $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ is a Arthur's (G,M)-family (cf. [Art81]) if for each $P\in\mathcal{P}(M)$ the map c_P is smooth on $i\mathfrak{a}_P^{G,*}$ and for each pair (P,P') of adjacent parabolic subgroups in $\mathcal{P}(M)$ we have $c_P(\lambda)=c_{P'}(\lambda)$ for $\lambda\in i\mathfrak{a}_P^{G,*}$ such that $\langle\lambda,\alpha^\vee\rangle=0$ where $\{\alpha\}=\Delta_P\cap(-\Delta_{P'})$.

Proposition 2.4.1.1. — *Let M be a semi-standard Levi subgroup of G and ω be a neighborhood of $i\mathfrak{a}_M^{G,*}$ in $\mathfrak{a}_{M,\mathbb{C}}^{G,*}$. Let $(c_P)_{P\in\mathcal{F}(M)}$ be a family of holomorphic functions on ω. Assume that there exists a non-empty subset Ω of $\mathfrak{a}_0$ such that for all $T\in\Omega$ the map*

$$\lambda\mapsto E(\lambda,T)=\sum_{P\in\mathcal{F}(M)}c_P(\lambda)\frac{\exp(\langle\lambda,T_P\rangle}{\theta_P(\lambda)}$$

is holomorphic on ω. Then for any $L\in\mathcal{L}(M)$ and $\lambda_0\in i\mathfrak{a}_M^{L,*}$ the family $(c_P(\lambda_0+\lambda))_{P\in\mathcal{P}(L)}$ is a (G,L)-family of functions of the variable $\lambda\in\mathfrak{a}_{L,\mathbb{C}}^{G,*}$.

Remark 2.4.1.2. — We shall use the following reinforcement of proposition 2.4.1.1 when ω is a domain. For each pair (P, P') of adjacent parabolic subgroups in $\mathcal{P}(L)$ we have $c_P(\lambda) = c_{P'}(\lambda)$ for $\lambda \in \omega$ such that $\langle \lambda, \alpha^\vee \rangle = 0$ where $\{\alpha\} = \Delta_P \cap (-\Delta_{P'})$.

Proof. — Let $L \in \mathcal{L}(M)$. We shall assume $L \subsetneq G$ otherwise there is nothing to prove. Let (P, P') be a pair of adjacent parabolic subgroups in $\mathcal{P}(L)$. We shall assume that P is standard. There is no loss of generality because the choice of P_0 intervenes only in the definition of T_P. In general there exists $w_0 \in W$ such that $w_0 P_0 w_0^{-1} \subset P$ and T_P is the projection of $w_0^{-1} T w_0$ on $\mathfrak{a}_P$. Then we can replace P_0 by $w_0 P_0 w_0^{-1}$ and accordingly Ω by $w_0^{-1} \Omega w_0$.

The set $\Delta_P \cap (-\Delta_{P'})$ is a singleton whose unique element is denoted by α. Let $P \subset P_\alpha$ defined by $\Delta_P^{P_\alpha} = \{\alpha\}$. Let $L_\alpha = M_{P_\alpha}$. We have

$$\mathrm{Ker}(\alpha^\vee) = \{\lambda \in \mathfrak{a}_M^{G,*} \mid \langle \lambda, \alpha^\vee \rangle = 0\} = \mathfrak{a}_M^{L,*} \oplus \mathfrak{a}_{L_\alpha}^{G,*}.$$

We need to show the equality

$$c_P(\mu) = c_{P'}(\mu). \tag{2.4.1.1}$$

for all $\mu \in i\,\mathrm{Ker}(\alpha^\vee)$. Because both c_P and $c_{P'}$ are holomorphic on ω, it suffices to prove the equality for generic elements of $i\,\mathrm{Ker}(\alpha^\vee)$ in the sense that they do not belong to the finite set of proper subspaces of $i\,\mathrm{Ker}(\alpha^\vee)$ we will encounter in our discussion. From now on, μ is a generic element of $i\,\mathrm{Ker}(\alpha^\vee)$.

Let $\mathcal{F}_\alpha(M)$ be the set of $Q \in \mathcal{F}(M)$ such that $\theta_Q(\mu) = 0$ that is θ_Q vanishes identically on $\mathrm{Ker}(\alpha^\vee)$. Of course we have $P, P' \in \mathcal{F}_\alpha(M)$. More generally we have $Q \in \mathcal{F}_\alpha(M)$ if and only there is $\beta^\vee \in \Delta_Q^\vee$ such that $\mathrm{Ker}(\alpha^\vee) \subset \mathrm{Ker}(\beta^\vee)$ that is $\beta^\vee \in \mathbb{R}\alpha^\vee$. So for such a Q we have

$$\mathfrak{a}_M^{Q,*} \subset \mathrm{Ker}(\alpha^\vee). \tag{2.4.1.2}$$

We introduce:

$$E_\alpha(\lambda, T) = \sum_{Q \in \mathcal{F}_\alpha(M)} c_Q(\lambda) \frac{\exp(\langle \lambda, T_Q \rangle}{\theta_Q(\lambda)}.$$

Since $E(\lambda, T)$ is holomorphic on ω and since $E(\lambda, T) - E_\alpha(\lambda, T)$ is clearly holomorphic at $\lambda = \mu$, we deduce that $E_\alpha(\lambda, T)$ is also holomorphic at $\lambda = \mu$. First we want to determine the summand of $E_\alpha(\mu, T)$ of exponent $T \mapsto \langle \mu, T_P \rangle = \langle \mu_L, T \rangle$ (recall that P is assumed to be standard).

Let $Q \in \mathcal{F}_\alpha(M)$ such that $\langle \mu, T_Q \rangle = \langle \mu_L, T \rangle$ for all $T \in \Omega$. Let $w \in W$ be such that $w P_0 w^{-1} \subset Q$. Then $\langle \mu, T_Q \rangle = \langle w^{-1} \cdot (\mu_{M_Q}), T \rangle$. Thus we must have $\mu_L = w^{-1} \cdot (\mu_{M_Q})$. By genericity, we have $\lambda_L = w^{-1} \cdot (\lambda_{M_Q})$ for all $\lambda \in i\,\mathrm{Ker}(\alpha^\vee)$. Since $\mathfrak{a}_M^{L,*} \subset \mathrm{Ker}(\alpha^\vee)$ we deduce $\mathfrak{a}_M^{L,*} \subset \mathfrak{a}_M^{Q,*}$. But by (2.4.1.2) we have also $\mathfrak{a}_M^{Q,*} \subset \mathrm{Ker}(\alpha^\vee)$ and we deduce $\mathfrak{a}_M^{Q,*} \subset \mathfrak{a}_M^{L,*}$. As a consequence we have $\mathfrak{a}_M^{Q,*} = \mathfrak{a}_M^{L,*}$ and $L = M_Q$ that is $Q \in \mathcal{P}(L)$. Once again by genericity of μ, we deduce that w stabilizes $\mathfrak{a}_{L_\alpha}^{G,*}$ and in fact acts trivially on it. As a consequence $w \in W^{L_\alpha}$. There are only two possibilities: either $Q = P$ or $Q = P'$. By lemma 2.3.1.1, the summand of $E_\alpha(\mu, T)$ of exponent $\langle \mu_L, T \rangle$ is given by the value at $\lambda = \mu$ of the expression

$$D_{P,P'}(\lambda) = c_P(\lambda) \frac{\exp(\langle \lambda, T_P \rangle}{\theta_P(\lambda)} + c_{P'}(\lambda) \frac{\exp(\langle \lambda, T_{P'} \rangle}{\theta_{P'}(\lambda)}$$

which is holomorphic at $\lambda = \mu$. In particular, $\langle \lambda, \alpha^\vee \rangle D_{P,P'}(\lambda)$ vanishes at $\lambda = \mu$. Let d_P be the value at $\lambda = \mu$ of $\langle \lambda, \alpha^\vee \rangle \frac{\exp(\langle \lambda, T_P \rangle}{\theta_P(\lambda)}$. Let $d_{P'}$ be the value we get when P is replaced by P'. One can check that $d_{P'} = -d_P \neq 0$. Thus we must have $d_P(c_P(\mu) - c_{P'}(\mu)) = 0$ and thus $c_P(\mu) = c_{P'}(\mu)$. □

2.5 Automorphic forms

2.5.1. We continue with the notations of the previous sections. From now on we assume that the base field F is a number field. Let $\mathbb{A}$ be its adèle ring. Let $\mathbb{A}_f$ be the ring of finite adèles and $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Let V_F be the set of places of F and $V_{F,\infty} \subset V_F$ be the subset of Archimedean places. For every $v \in V_F$, we let F_v be the local field obtained by completion of F at v. We denote by $|\cdot|$ the morphism $\mathbb{A}^\times \to \mathbb{R}_+^\times$ given by the product of normalized absolute values $|\cdot|_v$ on each F_v.

2.5.2. Let $K = \prod_{v\in V_F} K_v \subset G(\mathbb{A})$ be a "good" maximal compact subgroup in good position relative to M_0 (that is it is admissible in the sense of [Art81, p.9]). Let $K_f = \prod_{v\in V_F\setminus V_{F,\infty}} K_v \subset G(\mathbb{A}_f)$ and $K_\infty = \prod_{v\in V_{F,\infty}} K_v \subset G(F_\infty)$.

2.5.3. Let P be a semi-standard parabolic subgroup. We have a canonical homomorphism

$$H_P : P(\mathbb{A}) \to \mathfrak{a}_P \tag{2.5.3.1}$$

characterized by $\langle \chi, H_P(g)\rangle = \log|\chi(g)|$ for any $g \in P(\mathbb{A})$ and $\chi \in X^*(P)$. Its kernel is denoted by $P(\mathbb{A})^1$. We extend it to the Harish-Chandra map

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_P$$

that satisfies: for every $g \in G(\mathbb{A})$ we have $H_P(g) = H_P(p)$ whenever $g \in pK$ with $p \in P(\mathbb{A})$. Let $H(g) = H_0(g) = H_{P_0}(g)$.

2.5.4. Haar measures. — For any parabolic subgroups $P \subset Q \subset G$, let ρ_P^Q be the unique element of $\mathfrak{a}_P^{Q,*}$ such that for every $m \in M_P(\mathbb{A})$

$$|\det(\mathrm{Ad}_P^Q(m))| = \exp(\langle 2\rho_P^Q, H_P(m)\rangle).$$

where Ad_P^Q is the adjoint action of M_P on the Lie algebra of $M_Q \cap N_P$. For $Q = G$, the exponent G is omitted.

Let A_P^∞ be the neutral component of $A_{P,\mathbb{Q}}(\mathbb{R})$ (for the Archimedean topology) where $A_{P,\mathbb{Q}} \subset \mathrm{Res}_{F/\mathbb{Q}}(A_P)$ is the maximal $\mathbb{Q}$-split subtorus. If $P \subset Q$ we denote by $A_P^{Q,\infty} = A_P^\infty \cap \mathrm{Ker}(H_Q)$. The group A_P^∞ is equipped with the Haar measure compatible with the isomorphism $A_P^\infty \simeq \mathfrak{a}_P$ induced by the homomorphism (2.5.3.1). Its subgroup $A_P^{Q,\infty}$ is in the same way identified to $\mathfrak{a}_P^Q$ and thus inherits the Haar measure of $\mathfrak{a}_P^Q$.

If N is a unipotent group, $N(\mathbb{A})$ is provided with the Haar measure whose quotient by the counting measure on $N(F)$ gives the total volume 1 to $[N]$. We fix Haar measures on $G(\mathbb{A})$ and K. We assume that for any semi-standard parabolic groups P of G the Haar measure on $M_P(\mathbb{A})$ is such that

$$\int_{G(\mathbb{A})} f(g)\,dg = \int_{N(\mathbb{A})\times M_P(\mathbb{A})\times K} f(nmk)\exp(-\langle 2\rho_P^Q, H_P(m)\rangle)\,dndmdk$$

for all continuous and compactly supported function f on $G(\mathbb{A})$.

We set

$$[G]_{P,0} = A_P^\infty M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A}) \text{ and } [G]_P = M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A}).$$

We equip $[G]_{P,0}$ with the "quotient measure": it is the right-invariant functional such that for all continuous and compactly supported function f on $G(\mathbb{A})$ we have

$$\int_{[G]_{P,0}}\int_{A_P^\infty}\sum_{\gamma\in M_P(F)}\int_{N_P(\mathbb{A})} \exp(-\langle 2\rho_P^Q, H_P(a)\rangle) f(a\gamma ng)\,dnda\,dg = \int_{G(\mathbb{A})} f(g)\,dg.$$

If $P=G$ we set $[G]_0=[G]_{G,0}$ and $[G]=[G]_G$. We put on $G(\mathbb{A})^1$ the Haar measure such that the natural isomorphism $G(\mathbb{A})^1\times A_G^\infty\to G(\mathbb{A})$ is compatible with the Haar measures (on the source we put the product of the Haar measures). We also set

$$[G]=G(F)\backslash G(\mathbb{A}) \text{ and } [G]^1=G(F)\backslash G(\mathbb{A})^1,$$

both being equipped with the quotient measure by the counting measure. More generally we introduce $[G]_P^1=M_P(F)N_P(\mathbb{A})\backslash P(\mathbb{A})^1K$. We have an obvious surjective map $[M_P]^1\times K\to[G]_P^1$ and we use it to push-forward the product measure on $[M_P]^1\times K$.

2.5.5. Let $\mathfrak{g}_\infty$ be the Lie algebra of $G(F_\infty)$ and $\mathcal{U}(\mathfrak{g}_\infty)$ be the enveloping algebra of its complexification and $\mathcal{Z}(\mathfrak{g}_\infty)\subset\mathcal{U}(\mathfrak{g}_\infty)$ be its center.

2.5.6. We fix a height $\|\cdot\|:G(\mathbb{A})\to\mathbb{R}_+$ as in [MW89, § I.2.2]. A function $f:G(\mathbb{A})\to\mathbb{C}$ is said to be smooth if it is right invariant by a compact-open subgroup J of $G(\mathbb{A}_f)$ and for every $g_f\in G(\mathbb{A}_f)$ the function $g_\infty\in G(F_\infty)\mapsto f(g_fg_\infty)\in V$ is smooth in the usual sense.

2.5.7. By a level J we mean a normal open compact subgroup of K_f.

2.5.8. Schwartz algebra. — This is the algebra (for the convolution product) denoted by $\mathcal{S}(G(\mathbb{A}))$ and defined as a topological vector space as the locally convex topological direct limit of the spaces $\mathcal{S}(G(\mathbb{A}),C,J)$ over all pairs (C,J) consisting of a compact subset of $G(\mathbb{A}_f)$ and a level J where $\mathcal{S}(G(\mathbb{A}),C,J)$ is the space of smooth functions $f:G(\mathbb{A})\to\mathbb{C}$ which are

- biinvariant by J and supported in the subset $G(F_\infty)\times C$
- such that

$$\|f\|_{r,X,Y}=\sup_{g\in G(\mathbb{A})}\|g\|^r|(R(X)L(Y)f)(g)|<\infty$$

  for every integer $r\geqslant 1$ and $X,Y\in\mathcal{U}(\mathfrak{g}_\infty)$.

The family of semi-norms $\|\cdot\|_{r,X,Y}$ defines the topology on $\mathcal{S}(G(\mathbb{A}),C,J)$. We denote by $\mathcal{S}(G(\mathbb{A}))^J$ the subalgebra of J-biinvariant function.

2.5.9. Let P be a standard parabolic subgroup of G. For any $g\in G(\mathbb{A})$, we define

$$\|g\|_P=\inf_{\delta\in M_P(F)N_P(\mathbb{A})}\|\delta g\|.$$

For all $N\in\mathbb{Z}$, $X\in\mathcal{U}(\mathfrak{g}_\infty)$ and any smooth function $\varphi:[G]_P\to\mathbb{C}$ we define

$$\|\varphi\|_{N,X}=\sup_{x\in[G]_P}\|x\|_P^N|(R(X)\varphi)(x)|\in\mathbb{R}_+\cup\{+\infty\}.$$

2.5.10. Functions of uniform moderate growth. — For any $N\geqslant 0$ and any level J let $\mathcal{T}_N([G]_P)^J$ be the space of smooth functions $\varphi:[G]_P\to\mathbb{C}$ that are right-invariant by J and such that that for every $X\in\mathcal{U}(\mathfrak{g}_\infty)$ we have $\|\varphi\|_{-N,X}<\infty$. Then $\mathcal{T}_N([G]_P)^J$ is a Fréchet space equipped with these semi-norms. We defined $\mathcal{T}_N([G]_P)$ as the locally convex topological direct limit over J of the spaces $\mathcal{T}_N([G]_P)^J$. The space $\mathcal{T}([G]_P)$ of functions of uniform moderate growth on $[G]_P$ is the locally convex topological direct limit of the spaces $\mathcal{T}_N([G]_P)$ over the integers $N\geqslant 0$.

2.5.11. Rapidly decreasing functions. — For any level J let $\mathcal{S}([G]_P)^J$ be the space of smooth functions $\varphi:[G]_P\to\mathbb{C}$ that are right-invariant by J and such that for every $N>0$ and $X\in\mathcal{U}(\mathfrak{g}_\infty)$ we have $\|\varphi\|_{-N,X}<\infty$. Then $\mathcal{S}([G]_P)^J$ is a Fréchet space equipped with these semi-norms. We define the Schwartz space $\mathcal{S}([G]_P)$ as the locally convex topological direct limit of the spaces $\mathcal{S}([G]_P)^J$ over the subgroups J. If we replace $[G]_P$ by $[G]_P^1$ we get the Schwartz space $\mathcal{S}([G]_P^1)$.

2.5.12. Automorphic forms. — The space $\mathcal{A}_P(G)$ of *automorphic forms* on $[G]_P$ is the subspace of $\mathcal{Z}(\mathfrak{g}_\infty)$-finite functions in $\mathcal{T}([G]_P)$. We denote by $\mathcal{A}_P^0(G)$ the subspace of $\varphi \in \mathcal{A}_P(G)$ such that

$$\varphi(ag) = \exp(\langle \rho_P, H_P(a) \rangle)\varphi(g)$$

for all $g \in G(\mathbb{A})$ and $a \in A_P^\infty$.

Let $\mathcal{A}_{P,\mathrm{disc}}(G) \subset \mathcal{A}_P(G)$ be the subspace of forms such that $|\varphi|$ is left-invariant by A_P^∞ and such that the Petersson norm defined by

$$\|\varphi\|_P^2 = \langle \varphi, \varphi \rangle_P = \int_{[G]_{P,0}} |\varphi(g)|^2 \, dg$$

is finite. Let $\mathcal{A}_{P,\mathrm{disc}}^0(G) = \mathcal{A}_{P,\mathrm{disc}}(G) \cap \mathcal{A}_P^0(G)$. For any ideal $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g}_\infty)$ of finite codimension, let $\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ be the subspace of $\varphi \in \mathcal{A}_{P,\mathrm{disc}}(G)$ killed by $\mathcal{J}$. There exists $N \geqslant 1$ such that $\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ is a closed subspace of $\mathcal{T}_N([G]_P)$. Then $\mathcal{A}_{P,\mathrm{disc},\mathcal{J}}(G)$ is equipped with the induced topology from $\mathcal{T}_N([G]_P)$. This topology does not depend on the choice of N. Then $\mathcal{A}_{P,\mathrm{disc}}(G)$ is provided with the locally convex direct limit topology. The group $G(\mathbb{A})$ acts on $\mathcal{A}_{P,\mathrm{disc}}(G)$ by right translation. When $P = G$, we omit the subscript P.

2.5.13. Discrete automorphic representations. — A discrete automorphic representation π of $G(\mathbb{A})$ is a topologically irreducible subrepresentation of $\mathcal{A}_{\mathrm{disc}}(G)$. Let $\Pi_{\mathrm{disc}}(G)$ be the set of discrete automorphic representations π of $G(\mathbb{A})$ that are subrepresentations of $\mathcal{A}_{\mathrm{disc}}^0(G)$. For such a π, we denote by $\mathcal{A}_\pi(G)$ the π-isotypic component of $\mathcal{A}_{\mathrm{disc}}^0(G)$. Let $\pi \in \Pi_{\mathrm{disc}}(M_P)$ and let $\mathcal{A}_{P,\pi}(G)$ be the subspace of $\varphi \in \mathcal{A}_{P,\mathrm{disc}}^0(G)$ such that the map $m \in [M_P] \mapsto \varphi(mg)$ belongs to $\mathcal{A}_\pi^0(M)$ for all $g \in G(\mathbb{A})$. Then $\mathcal{A}_{P,\pi}(G)$ is a closed subspace of $\mathcal{A}_{P,\mathrm{disc}}^0(G)$ equipped with the induced topology. For any $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we define $\pi_\lambda = \pi \otimes \exp(\langle \lambda, H_M(\cdot) \rangle)$. The map $\varphi \mapsto \varphi_\lambda$ where

$$\varphi_\lambda(g) = \exp(\langle \lambda, H_P(g) \rangle)\varphi(g)$$

identifies $\mathcal{A}_{P,\pi}(G)$ with a subspace of $\mathcal{A}_P(G)$. By transport, we denote by $I_{P,\pi}(\lambda)$, or simply $I_P(\lambda)$ if the context is clear, the action of $G(\mathbb{A})$ on $\mathcal{A}_{P,\pi}(G)$ we get from that on $\mathcal{A}_P(G)$ by right translation. The right convolution gives an action of $\mathcal{S}(G(\mathbb{A}))$ denoted by $I_{P,\pi}(\lambda, f)$, or simply $I_P(\lambda, f)$ for $f \in \mathcal{S}(G(\mathbb{A}))$.

For any $\varphi \in \mathcal{A}_P(G)$ and any standard parabolic subgroup $Q \subset P$ we define the constant term along Q by

$$\varphi_Q(g) = \int_{[N_Q]} \varphi(ng) \, dn, \ g \in G(\mathbb{A})$$

Let $\mathcal{A}_{\mathrm{cusp}}^0(G) \subset \mathcal{A}^0(G)$ be the subspace of functions whose constant terms vanish for all proper parabolic subgroups. Let $\Pi_{\mathrm{cusp}}(G)$ be the set of topologically irreducible subrepresentations of $\mathcal{A}_{\mathrm{cusp}}^0(G)$.

Let J be a level and let $\mathcal{A}_{P,\pi}(G)^J \subset \mathcal{A}_{P,\pi}(G)$ be the subspace of right-J-invariant functions. We define the subset $\Pi_{\mathrm{disc}}(M_P)^J \subset \Pi_{\mathrm{disc}}(M_P)$ by the condition $\mathcal{A}_{P,\pi}(G)^J \neq \{0\}$.

Let $\hat{K}_\infty$ be the set of isomorphism classes of irreductible unitary representations of K_∞. For any $\tau \in \hat{K}_\infty$, let $\mathcal{A}_{P,\pi}(G)^\tau$ be the subspace of functions whose right-K_∞-translate belongs to the τ-isotypic component. If J is a level we set

$$\mathcal{A}_{P,\pi}(G)^{\tau,J} = \mathcal{A}_{P,\pi}(G)^\tau \cap \mathcal{A}_{P,\pi}(G)^J.$$

2.5.14. Intertwining operators. — For any $w \in W(P,Q)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we have the intertwining operator

$$M(w, \lambda) : \mathcal{A}_P(G) \to \mathcal{A}_Q(G).$$

For $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ such that $\Re(\langle \lambda, \alpha^\vee \rangle)$ is large enough for any $\alpha \in \Delta_P$ such that $w\lambda$ is negative, it is defined by the integral

$$(M(w,\lambda)\varphi)_{w\lambda}(g) = \int_{(N_Q \cap wN_P w^{-1})(\mathbb{A})\backslash N_Q(\mathbb{A})} \varphi_\lambda(w^{-1}ng)\, dn.$$

In general, it is given on K-finite functions by analytic continuation from the previous integral. By [Lap08, corollary 2.4], it extends to a continuous operator on $\mathcal{A}_P(G)$. One has $w\Delta_0^P = \Delta_0^Q$. Let $Q' \subset Q$ and $P' \subset P$ be standard parabolic subgroups such that $w\Delta_0^{P'} = \Delta_0^{Q'}$. Using the subscript P' or Q' to denote the constant term we have

$$(M(w,\lambda)\varphi)_{Q'} = M(w,\lambda)(\varphi_{P'}).$$

2.5.15. Eisenstein series. — Let $P \subset Q$ be standard parabolic subgroups of G. For any $\varphi \in \mathcal{A}_{P,\mathrm{disc}}(G)$, $g \in G(\mathbb{A})$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, we denote by

$$E^Q(g,\varphi,\lambda) = \sum_{\delta \in P(F)\backslash Q(F)} \varphi_\lambda(\delta g)$$

the Eisenstein series where the right-hand side is convergent for $\Re(\lambda)$ in a suitable cone. For K-finite functions it admits an analytic continuation. This extends to $\mathcal{A}_{P,\mathrm{disc}}(G)$ and moreover $E^Q(\varphi,\lambda)$ has only hyperplane singularities, see §2.3.1. Moreover, for π is a discrete automorphic representation, the map $\varphi \in \mathcal{A}_{P,\pi}(G) \to E^Q(\varphi,\lambda)_{M_Q(\mathbb{A})} \in \mathcal{T}([M_Q])$ is continuous, see [Lap08, theorem 2.2].

Let E_Q^G be the constant term of E^G along Q. We have for all $g \in G(\mathbb{A})$

$$\begin{aligned} E_Q^G(g,\varphi,\lambda) &= \int_{[N_Q]} E^G(ng,\varphi,\lambda)\, dn \\ &= \sum_{w \in {}_QW_P} E^Q(g, M(w,\lambda)\varphi_{P_w}, w\lambda). \end{aligned} \tag{2.5.15.2}$$

where $M(w,\lambda)\varphi_{P_w} \in \mathcal{A}_{Q_w}$. In general, we shall omit the upper script G if the context is clear.

3 On discrete Eisenstein series

3.1 Scalar product of two truncated discrete Eisenstein series

3.1.1. In this section, the group G is $G_n = \mathrm{GL}_F(n)$ for some $n \geqslant 1$. We denote by P_0 the standard Borel subgroup and T_0 the diagonal maximal torus. The group K is the standard maximal compact subgroup of $GL(n,\mathbb{A})$. We identify naturally $\mathfrak{a}_0$ with $\mathbb{R}^n$ equipped with the canonical scalar product. Thus $\mathfrak{a}_{0,\mathbb{C}}^*$ and $\mathfrak{a}_{0,\mathbb{C}}$ are identified to $\mathbb{C}^n$ equipped with the canonical definite positive hermitian form whose associated norm is denoted by $\|\cdot\|$. We shall freely use the truncation operator $\Lambda^{T,Q}$ attached to a truncation parameter T and a parabolic subgroup Q introduced by Arthur in [Art80].

3.1.2. Let $P = MN_P$ be a standard parabolic subgroup of G. Let $\pi \in \Pi_{\mathrm{disc}}(M)$. We write $M = G_{n_1} \times \ldots \times G_{n_k}$ with $n_1 + \ldots + n_k = n$ and $\pi = \pi_1 \boxtimes \ldots \boxtimes \pi_k$ accordingly. By the classification of discrete automorphic representations of general linear groups (see [MW89]), there exist integers $r_i, d_i \geqslant 1$ such that $n_i = r_i d_i$ and $\sigma_i \in \Pi_{\mathrm{cusp}}(G_{r_i})$ such that elements of $\mathcal{A}_{\pi_i}(G_{n_i})$ are obtained as residues of Eisenstein series built from elements of $\mathcal{A}_{P_{\pi_i}, \sigma_i^{\otimes d_i}}$ where $P_{\pi_i} \subset M_i$ is the standard parabolic subgroup of Levi factor $G_{r_i}^{d_i}$. Let $P_\pi = P_{\pi_1} \times \ldots \times P_{\pi_k}$.

Let $P_\pi \subset R \subset Q \subset P$ be parabolic subgroups. We write $Q \cap M = Q_1 \times \ldots \times Q_k$ and $R \cap M = R_1 \times \ldots \times R_k$ with $R_i \subset Q_i \subset G_{n_i}$. We set

$$\nu_{R,\pi}^Q = (-\rho_{R_i}^{Q_i}/r_i)_{1 \leqslant i \leqslant r}$$

written relative to $\mathfrak{a}_R^{Q,*} = \oplus_{i=1}^k \mathfrak{a}_{R_i}^{Q_i,*}$. For convenience, we also set $\nu_{R,\pi}^Q = 0$ if $P_\pi \subsetneq R$. If $Q = P$ we omit the upperscript P that is $\nu_{R,\pi} = \nu_{R,\pi}^P$. Abusing the notation, we shall also omit the subscript π if the context is clear.

The reason to introduce ν_R is that $\varphi_{R,-\nu_R}$ belongs to $\mathcal{A}_Q^0(G)$ as one can check. Note that $\varphi_R(g) = 0$ unless $P_\pi \subset R$. Moreover $\varphi_{R,-\nu_R}$ is also square-integrable as it is shown in next lemma.

Lemma 3.1.2.1. — *Let J be a level. For any large enough $N > 0$ there exists a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that for all (P,π) as above, all $P_\pi \subset Q \subset P$ and all $\varphi \in \mathcal{A}_{P,\pi}(G)^J$ we have:*

$$\|\varphi_{Q,-\nu_Q}\|_Q \leqslant \sum_{i\in I} \|\varphi\|_{-N,X_i}.$$

Proof. — We have to consider

$$\|\varphi_{Q,-\nu_Q}\|_Q^2 = \int_{[G]_{Q,0}} |\varphi_Q(g)|^2 \exp(-\langle 2\nu_Q, H_Q(g)\rangle)\, dg$$

$$= \int_{[M_Q]_0} \int_K |\varphi_Q(mk)|^2 \exp(-\langle 2\nu_Q + 2\rho_Q, H_Q(m)\rangle)\, dkdm \tag{3.1.2.1}$$

We fix a truncation parameter T and we use the inversion formula, see [Art80, lemma 1.5]:

$$\varphi_Q(g) = \sum_{P_\pi\cap M_Q \subset R \subset M_Q} \sum_{\delta\in R(F)\backslash M_Q(F)} \tau_R^Q(H_Q(\delta g) - T)\Lambda^{T,R}\varphi_R(\delta g).$$

Let R be as in the sum above. We consider the contribution of R in (3.1.2.1) namely:

$$\int_{R(F)A_Q^\infty\backslash M_Q(\mathbb{A})} \int_K \tau_R^Q(H_Q(m) - T)\Lambda^{T,R}\varphi_R(mk)\overline{\varphi_Q(mk)} \exp(-\langle 2\nu_Q + 2\rho_Q, H_Q(m)\rangle)\, dkdm$$

$$= \int_{[M_R]^1} \int_{A_R^{Q,\infty}} \tau_R^Q(H_Q(a) - T) \exp(-\langle 2\nu_Q + 2\rho_R, H_Q(a)\rangle) \int_K \Lambda^{T,R}\varphi_R(amk)\overline{\varphi_R(amk)}\, dkdadm.$$

This last expression is the product of

$$\int_{A_R^{Q,\infty}} \tau_R^Q(H_Q(a) - T) \exp(\langle 2\nu_R^Q, H_Q(a)\rangle) da \tag{3.1.2.2}$$

and

$$\int_{[M_R]^1} \int_K \Lambda^{T,R}\varphi_R(mk)\overline{\varphi_R(mk)}\, dkdm. \tag{3.1.2.3}$$

The integral (3.1.2.2) is convergent. When the data π, Q, R vary, its values belong to a finite set and thus there is an absolute bound for (3.1.2.2). Let $r \in \mathbb{N}$ large enough such that $\int_{[M_R]^1} \|m\|^{-r} dm < \infty$. Let $N > 0$. By [Art80, lemma 1.4], there exists a finite family $(Y_i)_{i\in I'}$ of element of $\mathcal{U}(\mathfrak{m}_{R,\infty})$ such that for any smooth function ψ on $[M_R]$ that is right-invariant by $J \cap M_R(\mathbb{A})$

$$\forall m \in [M_R]^1 \quad |\Lambda^{T,R}\psi(m)| \leqslant \left(\sum_{i\in I'} \|\psi\|_{-N,Y_i}\right) \|m\|_R^{-N-r}.$$

We deduce that there exists a finite family $(Y_i)_{i\in I'}$ as above (maybe larger) such that for all ψ as above

$$\int_{[M_R]^1} |\Lambda^{T,R}\psi(m)\overline{\psi(m)}|\, dm \leqslant (\sum_{i\in I'} \|\psi\|_{-N,Y_i})^2$$

To conclude it suffices to observe that there exists $N' > 0$ and a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that for all $k \in K$

$$\sum_{i\in I'} \|\varphi_R(\cdot k)\|_{-N,Y_i} \leqslant \sum_{i\in I} \|\varphi\|_{-N',X_i}.$$

Indeed the elements kY_ik^{-1} stay in a finite dimensional space with bounded coefficients in a fixed basis. □

3.1.3. Let $\varphi, \psi \in \mathcal{A}_{P,\pi}(G)$. The goal of this subsection is to prove the following theorem which gives an exact expression in terms of intertwining operators of the following integral:

$$\langle \Lambda^T E(\varphi,\lambda), \overline{E(\psi,\lambda')}\rangle_G = \int_{[G]_0} \Lambda^T E(g,\varphi,\lambda)E(g,\psi,\lambda')\,dm \tag{3.1.3.4}$$

for all $\lambda, \lambda' \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ that are not singular for the corresponding Eisenstein series. Note that for such λ, λ' the integral is absolutely convergent by the basic properties of Arthur's truncation operator.

Theorem 3.1.3.1. — *For any λ, λ' in general position in $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$, we have:*

$$\langle \Lambda^T E(\varphi,\lambda), \overline{E(\psi,\lambda')}\rangle_G$$
$$= \sum_{R\subset G} \sum_{w,w'} \langle M(w,\lambda)\varphi_w, \overline{M(w',\lambda')\psi_{w'}}\rangle_R \frac{\exp(\langle w\lambda + w'\lambda' + w\nu^P_{P_w} + w'\nu^P_{P_{w'}}, T_R\rangle)}{\theta_R(w\lambda + w'\lambda' + w\nu^P_{P_w} + w'\nu^P_{P_{w'}})}$$

where the sum is over $w, w' \in {}_RW_P$ such that $P_\pi \subset P_w \cap P_{w'}$ and $R_w = R$ and $R_{w'} = R$. For $w \in {}_QW_P$ we set $\varphi_w = \varphi_{P_w,-\nu_{P_w}}$ and $\psi_{w'} = \psi_{P_w,-\nu_{P_w}}$.

Remark 3.1.3.2. — In the inner sum, the pairing is vanishing unless $P_\pi \subset P_w \cap P_{w'}$. In this case the corresponding denominators are non-vanishing for λ, λ' in general position, see lemma 3.1.5.1.

Proof. — It is given in §3.1.4 and it is based on lemmas given in § 3.1.5. □

3.1.4. Proof of theorem 3.1.3.1. — We consider T a truncation parameter and $T' \in \mathfrak{a}_0^+$. We start from the following formula for any function ϕ on $[G]_0$ and $g \in G(\mathbb{A})$:

$$(\Lambda^{T+T'}\phi)(g) = \sum_Q \sum_{\delta\in Q(F)\backslash G(F)} \Gamma_Q(H_Q(\delta g) - T, T')\Lambda^{T,Q}\phi.$$

where the sum is over standard parabolic subgroups Q and $\Gamma_Q(H,X)$ satifies for any parabolic subgroup $P \subset R$, see [Art81, p.13]

$$\hat{\tau}_P(H - X) = \sum_{P\subset Q} (-1)^{\dim(\mathfrak{a}^R_Q)}\hat{\tau}^Q_P(H)\Gamma_Q(H,X).$$

The formula is easily deduced from the equality above and the definition of the truncation operator in [Art80]. We apply the formula to the truncated Eisenstein in the bracket. Using the computation of the constant term of the Eisenstein series we find:

$$\langle \Lambda^{T+T'} E(\varphi,\lambda), \overline{E(\psi,\lambda')}\rangle_G =$$
$$\sum_Q \sum_{w,w'\in {}_QW_P} \langle \Lambda^{T,Q}_m E^Q(M(w,\lambda)\varphi_w, (w\lambda)^Q), \overline{E^Q(M(w',\lambda')\psi_{w'}, (w'\lambda')^Q)}\rangle_Q \times$$
$$\int_{\mathfrak{a}^G_Q} \Gamma_Q(H - T, T')\exp(\langle w(\lambda + \nu^P_{P_w}) + w'(\lambda' + \nu^P_{P_{w'}}), H\rangle)\,dH.$$

By [Art81, lemma 2.2], the last line above is equal to:

$$\exp(\langle w(\lambda+\nu_{P_w}^P)+w'(\lambda'+\nu_{P_{w'}}^P),T_Q\rangle)\cdot\sum_{Q\subset R}(-1)^{\dim(\mathfrak{a}_Q^R)}\frac{\exp(\langle w(\lambda+\nu_{P_w}^P)+w'(\lambda'+\nu_{P_{w'}}^P),T'_R\rangle}{(\hat\theta_Q^R\theta_R)(w(\lambda+\nu_{P_w}^P)+w'(\lambda'+\nu_{P_{w'}}^P))}.$$

Note that for λ,λ' in general position the right-hand side is well-defined, see lemma 3.1.5.1 below, provided $P_\pi\subset P_w\cap P_{w'}$. We shall use this expression and invert the sums over Q and over R. To do this we use lemma 2.2.1.1. In this way we get:

$$\begin{aligned}&\langle\Lambda^T E(\varphi,\lambda),\overline{E(\psi,\lambda')}\rangle_G\\ &=\sum_R\sum_{w,w'\in{}_RW_P,P_\pi\subset P_w\cap P_{w'}}\frac{\exp(\langle w(\lambda+\nu_{P_w}^P)+w'(\lambda'+\nu_{P_{w'}}^P),(T+T')_R\rangle}{\theta_R(w(\lambda+\nu_{P_w}^P)+w'(\lambda'+\nu_{P_{w'}}^P))}\mathcal{P}^{T,R}(\varphi,\psi,\lambda,\lambda',w,w')\end{aligned}$$

where we set:

$$\begin{aligned}\mathcal{P}^{T,R}(\varphi,\psi,\lambda,\lambda',w,w')&=\\ \sum_{Q\subset R}(-1)^{\dim(\mathfrak{a}_Q^R)}\sum_{w_1,w_1'}\langle\Lambda_m^{T,Q}E^Q(M(w_1,\lambda)\varphi_{w_1},(w_1\lambda)^Q),&\overline{E^Q(M(w_1',\lambda')\psi_{w_1'},(w_1'\lambda')^Q)}\rangle_Q\times\\ &\frac{\exp(\langle w_1(\lambda+\nu_{w_1}^w)+w_1'(\lambda'+\nu_{w_1'}^{w'}),T^R\rangle)}{\hat\theta_Q^R(w_1(\lambda+\nu_{w_1}^w)+w_1'(\lambda'+\nu_{w_1'}^{w'}))}\end{aligned}$$

where the inner sum is over $w_1\in{}_QW_{R_w}^R w$ and $w_1'\in{}_QW_{R_{w'}}^R w'$ and we set $\nu_{w_1}^w=\nu_{P_{w_1}}^{P_w}$. Note that the expression $\mathcal{P}^{T,R}(\varphi,\psi,\lambda,\lambda',w,w')$ is even defined for elements $\lambda\in\mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$ and $\lambda'\in\mathfrak{a}_{P_{w'},\mathbb{C}}^{G,*}$ in general position (we shall not give the proof but see lemma 4.3.3.3 below for a closely related statement). In fact it does not depend on T: indeed this is nothing else but the regularization in the sense of Jacquet-Lapid-Rogawski, see [JLR99, section 12] and also [Lap11b, § 5.2], of the (in general non-convergent) pairing:

$$\langle E^R(M(w,\lambda)\varphi_w,(w\lambda)^R),\overline{E^R(M(w',\lambda')\psi_{w'},(w'\lambda')^R)}\rangle_R. \tag{3.1.4.5}$$

The functions φ_w and $\psi_{w'}$ belong respectively to subspaces for some discrete representations $\sigma\in\Pi_{\mathrm{disc}}(M_{P_w})$ and $\sigma'\in\Pi_{\mathrm{disc}}(M_{P_{w'}})$. Then (3.1.4.5) is closely related to the regularized version of the pairing

$$(\varphi',\psi')\in\mathcal{A}_{R_w,w\sigma}(G)\times\mathcal{A}_{R_{w'},w'\sigma'}\mapsto\langle E^R(\varphi',\lambda^R),\overline{E^R(\psi',\lambda'^R)}\rangle_R$$

which is well-defined for $\lambda\in\mathfrak{a}_{R_w,\mathbb{C}}^{G,*}$ and $\lambda'\in\mathfrak{a}_{R_{w'},\mathbb{C}}^{G,*}$ in general position: moreover it gives a pairing on the product of the spaces $\mathcal{A}_{R_w,w\sigma}(G)\times\mathcal{A}_{R_{w'},w'\sigma'}$ which is invariant for the diagonal restriction to $G(\mathbb{A})$ of the action $I_{R_w,w\sigma}(\lambda^R)\times I_{R_{w'},w'\sigma'}(\lambda'^R)$. By Bernstein argument, see [JLR99, p. 208], such a pairing must vanish unless $R_w=R$ and $R_{w'}=R'$. We deduce that the pairing (3.1.4.5) must also vanish unless the same condition is satisfied: then it reduces to the (convergent) pairing:

$$\langle M(w,\lambda)\varphi_w,\overline{M(w',\lambda')\psi_{w'}}\rangle_R. \tag{3.1.4.6}$$

This finishes the proof.

3.1.5. Some lemmas. —

Lemma 3.1.5.1. — *Let $Q\subset R$ be standard parabolic subgroups and $w,w'\in{}_QW_P$ such that $P_\pi\subset P_w\cap P_{w'}$. Then the map*

$$(\lambda,\lambda')\mapsto(\hat\theta_Q^R\theta_R)(w\lambda+w'\lambda'+w\nu_{P_w}^P+w'\nu_{P_{w'}}^P)$$

does not vanish identically on $i\mathfrak{a}_P^{G,*} \times i\mathfrak{a}_P^{G,*}$.

Proof. — The statement is obvious if the map $(\lambda, \lambda') \mapsto (\hat{\theta}_Q^R \theta_R)(w\lambda + w'\lambda')$ does not vanish identically on $i\mathfrak{a}_P^{G,*} \times i\mathfrak{a}_P^{G,*}$. Otherwise there exists $\gamma^\vee \in \hat{\Delta}_Q^{R,\vee} \cup \Delta_R^\vee$ such that

$$\gamma^\vee \in w\mathfrak{a}_0^P \cap w'\mathfrak{a}_0^P.$$

One has to consider two cases:

- $\gamma^\vee \in \hat{\Delta}_Q^{R,\vee}$. In this case, there exists a maximal parabolic subgroup $Q \subset S \subsetneq R$ such that $\hat{\Delta}_S^{R,\vee} = \{\gamma^\vee\}$. Since $\dim(\mathfrak{a}_S^R) = 1$ there exists $c > 0$ such that $c\gamma^\vee \in \Delta_S^{R,\vee} \subset \Delta_S^\vee$.
- $\gamma^\vee \in \Delta_R^\vee$.

In both cases we can apply lemma 3.1.5.2 below to conclude that

$$\langle w\nu_{P_w}^P + w'\nu_{P_{w'}}^P, \gamma^\vee \rangle < 0.$$

□

Lemma 3.1.5.2. — *Let $Q \subset R$ be standard parabolic subgroups of G and let $w \in {}_QW_P$ such that $P_\pi \subset P_w$. For any $\gamma^\vee \in \Delta_R^\vee \cap w\mathfrak{a}_0^P$ we have*

$$\langle w\nu_{P_w}^P, \gamma^\vee \rangle < 0.$$

Proof. — First we observe that $w^{-1}\gamma^\vee \in \mathfrak{a}_{P_w}^P$. Indeed, for $\alpha \in \Delta_0^{P_w}$ we have

$$\langle \alpha, w^{-1}\gamma^\vee \rangle = \langle w\alpha, \gamma^\vee \rangle = 0$$

since $w\alpha \in w\Delta_0^{P_w} = \Delta_0^{Q_w} \subset \Delta_0^Q \subset \mathfrak{a}_0^{Q,*} \subset \mathfrak{a}_0^{R,*}$. In particular, we may and shall assume $P_w \subsetneq P$.

Let $R \subsetneq S \subset G$ be the standard parabolic subgroup defined by $\Delta_R^{S,\vee} = \{\gamma^\vee\}$. The Levi factor M_R is identified to a product $G_{m_1} \times \ldots \times G_{m_r}$ of general linear groups with $m_1 + \ldots + m_r = n$. There exists $1 \leqslant i < r$ such that the Levi factor M_S is identified to the product $G_{m_1} \times \ldots \times G_{m_{i-1}} \times G_{m_i+m_{i+1}} \times G_{m_{i+2}} \times \ldots \times G_{m_r}$. In the usual way, we identify $\mathfrak{a}_0$ to $\mathbb{R}^n$. Then, we have

$$\gamma^\vee = (0, \ldots, 0, 1/m_i, \ldots, 1/m_i, -1/m_{i+1}, \ldots, -1/m_{i+1}, 0, \ldots, 0) \tag{3.1.5.7}$$

where the entries are repeated respectively $m_1 + \ldots + m_{i-1}$, m_i, m_{i+1} and $m_{i+2} + \ldots + m_r$ times.

We identify M_P to the product $G_{n_1} \times \ldots \times G_{n_k}$ with $n_1 + \ldots + n_k = n$. Then $P_w \cap M_P$ is written accordingly $S_1 \times \ldots \times S_k$ where $S_j \subset G_{n_j}$ is a standard parabolic subgroup. In this way, we have

$$\mathfrak{a}_{P_w}^{P,*} = \oplus_{j=1}^k \mathfrak{a}_{S_j}^{G_{n_j},*}.$$

Then according to the computation of §3.1.2, the component of $\nu_{P_w}^P$ on $\mathfrak{a}_{S_j}^{G_{n_j},*}$ is $-\rho_{S_j}^{G_{n_j}}/r_j$ for some divisor r_j of n_j. Dually we have $\mathfrak{a}_{P_w}^P = \oplus_{j=1}^k \mathfrak{a}_{S_j}^{G_{n_j}}$. For $1 \leqslant j \leqslant k$, let $\beta_j^\vee \in \mathfrak{a}_{S_j}^{G_{n_j}}$ be the component of $w^{-1}\gamma^\vee \in \mathfrak{a}_{P_w}^P$ on $\mathfrak{a}_{S_j}^{G_{n_j}}$. Then it is enough to show the following claim: for any $\varpi \in \hat{\Delta}_0^{G_{n_j}}$ we have $\langle \varpi, \beta_j^\vee \rangle \geqslant 0$ (the subscript 0 refers to the standard Borel subgroup of G_{n_j}). Indeed with the claim we get:

$$\begin{aligned} -\langle w\nu_{P_w}^P, \gamma^\vee \rangle &= \sum_{j=1}^k \langle \rho_{S_j}^{G_{n_j}}/r_j, \beta_j^\vee \rangle \\ &= \sum_{j=1}^k \langle \rho_0^{G_{n_j}}, \beta_j^\vee \rangle / r_j \\ &= \sum_{j=1}^k \sum_{\varpi \in \hat{\Delta}_0^{G_{n_j}}} \langle \varpi, \beta_j^\vee \rangle / r_j \geqslant 0. \end{aligned}$$

Moreover the inequality is strict because otherwise we would have $\langle \varpi, \beta_j^\vee \rangle = 0$ for all j and all $\varpi \in \hat{\Delta}_0^{G_{n_j}}$ and thus $\gamma^\vee = 0$.

Let's prove the claim. We identify naturally $\mathfrak{a}_{S_j}^{G_{n_j},*}$ to the hyperplane of $\mathbb{R}^{n_j}$ of tuples whose sum of coefficients vanishes. So $\beta_j^\vee$ is identified to an element of this hyperplane. According to (3.1.5.7), the possible entries of $\beta_j^\vee$ are $0, 1/m_i, -1/m_{i+1}$ with multiplicities denoted by $N_{j,0}, N_{j,+}$ and $N_{j,-}$. We have $N_{j,0} + N_{j,+} + N_{j,-} = n_j$. Because the sum of the entries is 0, we have $N_{j,+}/m_i = N_{j,-}/m_{i+1}$. Let $\varpi \in \hat{\Delta}_0^{G_{n_j}}$. We observe the pairing $\langle \varpi, \beta_j^\vee \rangle$ is equal to the sum of the first l entries of $\beta_j^\vee$ for some $1 \leqslant l \leqslant n_j - 1$ determined by ϖ. To conclude it suffices to show that the entries $1/m_i$ of $\beta_j^\vee$ are "before" the entries $-1/m_{i+1}$. If this were not true, we could find a positive root α inside G_{n_j} such that

$$\langle \alpha, \beta_j^\vee \rangle = -(1/m_i + 1/m_{i+1}).$$

We can also view α as a positive root inside M_P and we observe that $w\alpha$ is also positive since $w \in {}_QW_P$. Thus we have:

$$\langle \alpha, \beta_j^\vee \rangle = \langle \alpha, w^{-1}\gamma^\vee \rangle = \langle w\alpha, \gamma^\vee \rangle.$$

However, for a positive root β, the values of the pairing $\langle \beta, \gamma^\vee \rangle$ belong to

$$\{0, \pm 1/m_i, 1/m_i + 1/m_{i+1}, \pm 1/m_{i+1}\}$$

which does not contain $-(1/m_i + 1/m_{i+1})$. This is the contradiction we were looking for. □

3.2 Bounds for the truncated scalar product

3.2.1. We continue with the notations of subsection 3.1. The other notations are borrowed from section 2.

3.2.2. Numerical invariants. — We denote by Ω_G and Ω_{K_∞} the Casimir operator of G and K_∞ respectively associated to the standard bilinear form on $\mathfrak{g}_\infty$ associated to the trace, see e.g. [Mül98, p.323]. We set

$$\Delta = \mathrm{Id} - \Omega_G + 2\Omega_{K_\infty} \in \mathcal{U}(\mathfrak{g}_\infty).$$

For any $\tau \in \hat{K}_\infty$, let λ_τ be the eigenvalue of Ω_{K_∞}. Let $P = MN_P$ be a standard Levi subgroup. We define similarly the Casimir operator Ω_M. Let π_∞ be irreductible unitary representation of $M(F_\infty)$. We shall attach two invariants to π_∞. Let λ_{π_∞} be the eigenvalue of Ω_M. We set:

(3.2.2.1) $$\Lambda_{\pi_\infty}^M = \sqrt{\lambda_{\pi_\infty}^2 + \lambda_\tau^2}$$

where τ is a minimal $K_\infty \cap M(F_\infty)$-type of π_∞ and

(3.2.2.2) $$\Lambda_{\pi_\infty}^G = \min_\tau (\sqrt{\lambda_{\pi_\infty}^2 + \lambda_\tau^2})$$

where the minimum is taken over minimal K_∞-types of the induced representation $\mathrm{Ind}_{P(F_\infty)}^{G(F_\infty)}(\pi_\infty)$. If $M = G$, the invariants (3.2.2.1) and (3.2.2.2) are the same as the notation suggests. Let $\pi \in \Pi_{\mathrm{disc}}(M)$ and let π_∞ be the Archimedean component of π. We set

(3.2.2.3) $$\lambda_\pi = \lambda_{\pi_\infty}, \ \Lambda_\pi^M = \Lambda_{\pi_\infty}^M, \ \Lambda_\pi = \Lambda_{\pi_\infty}^G.$$

3.2.3. Pairs and triples. — For any level J, we consider a J-pair that is a pair (P, π) such that $P = MN_P$ is a standard parabolic subgroup and $\pi \in \Pi_{\mathrm{disc}}(M)^J$. A J-triple, or simply a triple if the context is clear, is a triple (P, π, τ) such that (P, π) is a J-pair, $\tau \in \hat{K}_\infty$ and $\mathcal{A}_{P,\pi}(G)^{\tau,J} \neq \{0\}$. Let e_τ be the measure supported on K_∞ given by $\deg(\tau)\,\mathrm{trace}(\tau(k))dk$ where dk is the Haar measure on K_∞ giving the total volume 1. We have $e_\tau * e_\tau = e_\tau$. In particular, for $\varphi \in \mathcal{A}_{P,\pi}(G)^J$ we have $\varphi * e_\tau$ if and only if $\varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$. For any complex function f on a group, we define $f^\vee(x) = \overline{f(x^{-1})}$.

3.2.4. Bounds for intertwinings operators. — We shall extract some bounds from the work of Lapid [Lap13, section 3]. For any standard parabolic subgroup $P = MN_P$ of G, any discrete automorphic representation π of $M(\mathbb{A})$, any $c > 0$ and and $k > 0$ we define, following [Lap13, §3.3],

$$\mathcal{R}_{\pi,c,k} = \{\lambda \in \mathfrak{a}_P^{G,*} \mid \|\Re(\lambda)\| < c(1 + \Lambda_\pi^M + \|\Im(\lambda)\|)^{-k}\}.$$

Proposition 3.2.4.1. — *There exist $k, l > 0$ such that for any level J there exist $c > 0$ and $C > 0$ such that for all J-triples (P, π, τ) and all standard parabolic subgroups R of G, all $w \in {}_RW_P$ such that $P_\pi \subset P_w$ and $R_w = R$ we have:*

$$\|M(w, \lambda)\varphi_w\|_R \leqslant C(1 + \|\lambda\|^2 + \lambda_\tau^2 + (\Lambda_\pi^M)^2)^k \|\varphi_w\|_R$$

for all $\varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$ and for all $\lambda \in \mathcal{R}_{\pi,c,l}$.

Proof. — We follow closely [Lap13, section 3]. Let's consider (P, π, τ) and (R, w) as above. We denote by σ the discrete automorphic representation of M_{P_w} such that the functions φ_w belongs to $\mathcal{A}_{P_w,\sigma}(G)$ when φ describes $\mathcal{A}_{P,\pi}(G)$. Let $M_\sigma(w, \lambda)$ be the restriction of the intertwining operator $M(w, \lambda)$ to the subspace $\mathcal{A}_{P_w,\sigma}(G)$. We have:

$$M_\sigma(w, \lambda) = n_\sigma(w, \lambda) N_\sigma(w, \lambda)$$

where $N_\sigma(w, \lambda)$ is the so-called normalized intertwining operator. The normalizing factor $n_\sigma(w, \lambda)$ is a scalar defined as

$$n_\sigma(w, \lambda) = \prod_\beta n_\sigma(\beta, \langle \lambda, \beta^\vee \rangle)$$

where the product is taken over the P_w-positive roots of A_{P_w} such that $w\beta$ is a R-negative root of A_R. We still have to define the meromorphic function $n_\sigma(\beta, s)$ of the variable $s \in \mathbb{C}$. To do this we identify M_{P_w} to a product $G_{n_1} \times \ldots \times G_{n_r}$ and σ to $\sigma_1 \boxtimes \ldots\ldots \boxtimes \sigma_r$ accordingly. Then if β is the positive root associated to the blocks G_{n_i} and G_{n_j} with $i > j$ then

$$n_\sigma(\beta, s) = \frac{L(1 - s, \sigma_1^\vee \times \sigma_2)}{L(1 + s, \sigma_1 \times \sigma_2^\vee)}$$

where the L-functions are the completed Rankin-Selberg L-functions defined in [JPSS83] and $\sigma_1^\vee$ is the contragredient of σ_1.

On the one hand, there exist $k, l > 0$ such that for any level J there exist $c > 0$ and $C > 0$ such that for any $P = MN_P$ and any $\pi \in \Pi_{\mathrm{disc}}(M)^J$ we have:

$$|n_\sigma(w, \lambda)| \leqslant C(1 + \|\lambda\|^2 + (\Lambda_\pi^M)^2)^k$$

for all $\lambda \in \mathcal{R}_{\pi,c,l}$ (where σ is associated to π as above). Note that k, l, c, C can be chosen uniformly for all (P, π, τ) and (R, w). This is essentially [Lap13, lemma 3.3 and proposition 3.4]. In fact, following [Lap13, proof of proposition 3.4], we get the same kind of result with Λ_π^M replaced by the analytic conductor of σ as defined in [Lap13, § 3.4]. But since π and σ are both residual representations obtained from the same cuspidal automorphic representation of M_{P_π} the analytic conductor of σ can be bounded by that of π and thus by Λ_π^M (up to a constant depending on the level J).

On the other hand, by [Lap13, lemma 3.11] there exist $c' > 0$ and $k > 0$ such that for any level J, there exists $C > 0$ such for all J-triples (P, π, τ)

$$|N_\sigma(w, \lambda)\varphi_w|_R \leqslant C(1 + \lambda_\tau^2)^k \|\varphi_w\|_R, \quad \forall \varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}.$$

Once again, C, k, c' can be chosen uniformly for all (P, π, τ) and (R, w). The proposition is then clear.

$\square$

3.2.5. Bounds for the scalar product of truncated Eisenstein series. — The main result is the following:

Proposition 3.2.5.1. — *There exist $k, l, r > 0$ such that for any level J there exist $c > 0$ and for any large enough $N > 0$ there exists a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that for all J-triples (P, π, τ) we have*

$$\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G$$
$$\leqslant (1 + \|\lambda\|^2 + \|\lambda'\|^2 + \lambda_\tau^2 + (\Lambda_\pi^M)^2)^k \exp(r\|T\|) \left(\sum_{i\in I} \|\varphi\|_{-N,X_i}\right) \left(\sum_{i\in I} \|\psi\|_{-N,X_i}\right)$$

for all $\varphi, \psi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$, $T \in \mathfrak{a}_0^G$ enough positive and all $\lambda, \lambda' \in \mathcal{R}_{\pi,c,l}$.

Proof. — First we observe that the expression $\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G$ is holomorphic for λ, λ' in a neighborhood of $i\mathfrak{a}_P^{G,*}$. We set

$$L_\pi(\lambda, \lambda') = \prod_{(R,\alpha,w,w')} \langle w\lambda + w'\lambda', \alpha^\vee\rangle$$

where the product is over the tuples (R, α, w, w') where:

- R is a standard parabolic subgroup of G ;
- $w, w' \in {}_RW_P$ such that $P_\pi \subset P_w \cap P_{w'}$ and $R_w = R$ and $R_{w'} = R$;
- $\alpha \in \Delta_R$ such that $\langle \alpha^\vee, w\nu_{P_w}^P + w'\nu_{P_{w'}}^P\rangle = 0$.

Note that L_π is a product of non-zero linear forms, see lemma 3.1.5.2. Note that the set of L_π for different (P, π) is in fact finite. Using theorem 3.1.3.1 and proposition 3.2.4.1, we see that there exist $c, l > 0$ such that the map $(\lambda, \lambda') \mapsto L_\pi(\lambda, \lambda')\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G)$ is holomorphic for $\lambda, \lambda' \in \mathcal{R}_{\pi,c,l}$ for some $c, l > 0$. However none of the hyperplanes defined by the linear factors of L_π can be singular for $\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G$ since the Eisenstein series hence the pairing are holomorphic on the imaginary axis. So $\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G$ is even holomorphic on $\mathcal{R}_{\pi,c,l}$. By a variant of lemma 2.3.2.1, we are reduced to majorize $D(L_\pi(\lambda, \lambda')\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G)$ on $\mathcal{R}_{\pi,c,l} \times \mathcal{R}_{\pi,c,l}$ for some constants $c, l > 0$ for a finite set which does not depend on π of holomorphic differential operators D. Using Cauchy formula, we are reduced to bound $L_\pi(\lambda, \lambda')\langle \Lambda^T E(\varphi, \lambda), \overline{E(\psi, \lambda')}\rangle_G$ on $\mathcal{R}_{\pi,c,l} \times \mathcal{R}_{\pi,c,l}$ for some constants $c, l > 0$. Using theorem 3.1.3.1, we see that this latter is a finite sum, indexed by R and $w, w' \in {}_RW_P$ such that $P_\pi \subset P_w \cap P_{w'}$ and $R_w = R$ and $R_{w'} = R$ of the product of three factors

$$\langle M(w, \lambda)\varphi_w, \overline{M(w', \lambda')\psi_{w'}}\rangle_R$$

$$\exp(\langle w\lambda + w'\lambda' + w\nu_{P_w}^P + w'\nu_{P_{w'}}^P, T_R\rangle)$$

and

$$\frac{L_\pi(\lambda,\lambda')}{\theta_R(w\lambda+w'\lambda'+w\nu^P_{P_w}+w'\nu^P_{P_{w'}})}.$$

The first factor is bounded by proposition 3.2.4.1 and lemma 3.1.2.1. The second factor, which is the only factor that depends on T, is clearly bounded on $\mathcal{R}_{\pi,c,l}\times\mathcal{R}_{\pi,c,l}$ by some power of $\exp(\|T\|)$. Finally the third factor is a rational function which, as we can assume, has no pole on $\mathcal{R}_{\pi,c,l}\times\mathcal{R}_{\pi,c,l}$. So it is bounded by some power of $1+\|\lambda\|+\|\lambda'\|$. The conclusion is then clear.

□

3.3 Bounds for some hermitian forms

3.3.1. We denote by $L^2([G]_{P,0})^{J,\infty}$ the subspace of $L^2([G]_{P,0})$ formed by right-J-invariant functions that are smooth and such that $R(X)\varphi)\in L^2([G]_{P,0})$ for all $X\in\mathcal{U}(\mathfrak{g}_\infty)$.

Lemma 3.3.1.1. — *Let J be a level and P be a parabolic subgroup of G. For any large $N>0$ and any $X\in\mathcal{U}(\mathfrak{g}_\infty)$, there exists $c>0$ and an integer $k\geqslant 0$ such that for all $\varphi\in L^2([G]_{P,0})^{J,\infty}$ we have*

$$\|\varphi\|^2_{-N,X}\leqslant c\sum_{i=0}^{k}\|R(\Delta^i)\varphi\|^2_P$$

Proof. — First we can use Sobolev inequalities (see [Ber88, key lemma]) to get that there exist $c>0$ and $Y_1,\dots,Y_r\in\mathcal{U}(\mathfrak{g}_\infty)$ such that if N is large enough we have:

$$\|\varphi\|^2_{-N,X}\leqslant c\sum_{i=1}^{r}\|R(Y_i)\varphi\|^2_P,\quad \forall\varphi\in L^2([G]_{P,0})^{J,\infty}.$$

But by [BK14, proposition 3.5], the topology on $L^2([G]_{P,0})^{J,\infty}$ is also given by the family of seminorms $(\sum_{i=0}^k\|R(\Delta^i)\varphi\|^2_P)^{1/2})_{k\in\mathbb{N}}$. The conclusion is clear. □

3.3.2. Let $\tau_1,\tau_2\in\hat{K}_\infty$. For any $f\in\mathcal{S}(G(\mathbb{A}))$ we define

$$f_{\tau_1,\tau_2}=\bar{e}^\vee_{\tau_1}*f*\bar{e}^\vee_{\tau_2}.$$

Let J be a level and T be a truncation parameter. For any J-pair (P,π) and $\lambda\in\mathfrak{a}^{G,*}_{P,\mathbb{C}}$ we define a hermitian form on $S(G(\mathbb{A}))^J$ by setting

$$B^T_{(P,\pi,\tau_1,\tau_2)}(\lambda,f)=\sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau_2,J)}\langle\Lambda^T E(I_{P,\pi}(\lambda,f_{\tau_1,\tau_2})\varphi,\lambda),E(I_{P,\pi}(\lambda,f_{\tau_1,\tau_2})\varphi,\lambda)\rangle_G \tag{3.3.2.1}$$

where $\mathcal{B}_{P,\pi}(\tau_2,J)$ is an orthonormal basis for the Petersson norm of the finite dimensional space $\mathcal{A}_{P,\pi}(G)^{\tau_2,J}$. It is well-defined if λ is non-singular for the Eisenstein series $E(\varphi,\lambda)$ for $\varphi\in\mathcal{A}_{P,\pi}(G)$. In this case, it does not depend on the choice of the orthonormal basis. Note that $B^T_{(P,\pi,\tau_1,\tau_2)}=0$ unless both (P,π,τ_1) and (P,π,τ_2) are J-triples.

Remark 3.3.2.1. — The form B does not depend on the choice of the level J, that is it induces a hermitian form on $S(G(\mathbb{A}))$. Indeed take an other level $J'\subset J$. We have $S(G(\mathbb{A}))^J\subset S(G(\mathbb{A}))^{J'}$ and $\mathcal{A}_{P,\pi}(G)^{\tau_2,J}\subset\mathcal{A}_{P,\pi}(G)^{\tau_2,J'}$. The projector p_J on $\mathcal{A}_{P,\pi}(G)^{\tau_2,J}$ is in fact an orthogonal projector. Thus we can choose $\mathcal{B}_{P,\pi}(\tau_2,J')$ to be the union of an orthonormal basis of $\ker(p_J)$ and $\mathcal{B}_{P,\pi}(\tau_2,J)$. Since the operator $I_{P,\pi}(\lambda,f_{\tau_1,\tau_2})$ factors through p_J for $f\in S(G(\mathbb{A}))^J$ we see that we can replace in (3.3.2.1) the basis $\mathcal{B}_{P,\pi}(\tau_2,J)$ by $\mathcal{B}_{P,\pi}(\tau_2,J')$.

Remark 3.3.2.2. — Since Λ^T is (in some sense) a self-adjoint projector, see [Art80, corollary 1.2 and lemma 1.3] , we have:

$$\int_{[G]_0} |\Lambda^T E(y, I_{P,\pi}(\lambda, f)\varphi, \lambda)|^2\, dy = \langle \Lambda^T E(I_{P,\pi}(\lambda, f)\varphi, \lambda), E(I_{P,\pi}(\lambda, f)\varphi, \lambda)\rangle_G.$$

As a consequence $B^T_{(P,\pi,\tau_1,\tau_2)}(\lambda, f)$ is real and non-negative.

Proposition 3.3.2.3. — *There exist $l, r > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all J-pairs (P, π), all $\tau_1, \tau_2 \in \hat{K}_\infty$, all $f \in \mathcal{S}(G(\mathbb{A}))^J$, all enough positive $T \in \mathfrak{a}_0^G$ and all $\lambda \in \mathcal{R}_{\pi,c,l}$ we have*

$$|B^T_{(P,\pi,\tau_1,\tau_2)}(f, \lambda)| \leqslant \frac{\|f_{\tau_1,\tau_2}\|^2_{\mathcal{S}} \exp(r\|T\|)}{(1 + \|\lambda\|^2)^q (1 + \lambda_\pi^2 + \lambda_{\tau_1}^2)^q (1 + \lambda_\pi^2 + \lambda_{\tau_2}^2)^q}.$$

Proof. — We may and shall assume that $f = f_{\tau_1,\tau_2}$ and that (P, π, τ_1) and (P, π, τ_2) are J-triples.

We start with the following observation. Let (P, π, τ) be a J-triple and $K_{M,\infty} = K_\infty \cap M(F_\infty)$. There exists $\sigma \in \hat{K}_{M,\infty}$ an irreducible representation that appears in the decomposition of the restriction of τ to $K_{M,\infty}$ such that σ is also a $K_{M,\infty}$-type of π_∞, in particular $\lambda_\tau \geqslant \lambda_\sigma$, see [Mül02, proof of lemma 6.1]. Hence one can bound $(\Lambda^M_\pi)^2$ by an absolute constant times $1 + \lambda_\pi^2 + \lambda_\tau^2$.

Then, using proposition 3.2.5.1, proposition 3.3.1.1 and the fact that $I_{P,\pi}(\lambda, f)\varphi \in \mathcal{A}^{\tau_1,J}_{P,\pi}$ if $\varphi \in \mathcal{A}^J_{P,\pi}$, we get the existence of $k, l > 0$ such that for any level J there exist $c, C > 0$ and $N \in \mathbb{N}$ such that for all J-pairs (P, π), all $\tau_1, \tau_2 \in \hat{K}_\infty$, all $f \in \bar{e}^\vee_{\tau_1} * \mathcal{S}(G(\mathbb{A}))^J * \bar{e}^\vee_{\tau_2}$ and all $\lambda \in \mathcal{R}_{\pi,c,l}$ we have

$$|B^T_{(P,\pi,\tau_1,\tau_2)}(f, \lambda)| \leqslant$$
$$C(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_{\tau_1}^2)^k \exp(r\|T\|) \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} \left(\sum_{i=0}^N \|R(\Delta^i) I_{P,\pi}(\lambda, f)\varphi\|_P^2 \right).$$

Before going further, we collect some general facts for a J-triple (P, π, τ).

First the operator $I_{P,\pi}(\lambda, \Delta)$ acts on the subspace $\mathcal{A}^{\tau,J}_{P,\pi}$ by the scalar $1 + (\lambda, \lambda) - \lambda_\pi + 2\lambda_\tau$, see [Mül02, eq. (6.7)] and [Mül07, proof of lemma 5.4] where $(\cdot, \cdot)$ is the standard quadratic form on $\mathfrak{a}^*_{0,\mathbb{C}}$ (whose restriction to $\mathfrak{a}^*_0$ is the standard scalar product). Moreover, we have $\lambda_\tau \geqslant \lambda_\pi$, see [Mül02, lemma 6.1], and $\lambda_\tau \geqslant 0$. So for $\varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$ we have:

$$\|R(\Delta)\varphi\|_P = (1 - \lambda_\pi + 2\lambda_\tau)\|\varphi\|_P.$$

There exists C_0 such that for all J-triples (P, π, τ) and all $\lambda \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ we have

$$(1 - \lambda_\pi + 2\lambda_\tau) \leqslant C_0(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\tau^2).$$

Hence there exists C_0' such that for all $\varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$

$$\sum_{i=0}^N \|R(\Delta^i)\varphi\|_P^2 \leqslant C_0'(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\tau^2)^{2N} \|\varphi\|_P^2. \tag{3.3.2.2}$$

On the other hand, there exists $C_1 > 0$ such that for any $\varphi \in \mathcal{A}_{P,\pi}(G)^{\tau,J}$, any $i \in \mathbb{N}$ and $\lambda \in \mathcal{R}_{\pi,c,l}$

$$(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\tau^2)^i \|\varphi\|_P \leqslant C_1^i \|I_{P,\pi}(\lambda, \Delta^{2i})\varphi\|_P. \tag{3.3.2.3}$$

Second, there exists $C_2 > 0$ and $k_1 \in \mathbb{N}$, see [Mül02, eq. (6.14)] such that

$$\dim(\mathcal{A}_{P,\pi}(G)^{\tau,J}) \leqslant C_2(1 + \lambda_\tau^2 + \lambda_\pi^2)^{k_1} \leqslant C_1(1 + \|\lambda\|^2 + \lambda_\tau^2 + \lambda_\pi^2)^{k_1} \tag{3.3.2.4}$$

Using (3.3.2.2) and (3.3.2.4), we get that there exists $C_3 > 0$ such that

$$|B^T_{(P,\pi,\tau_1,\tau_2)}(f,\lambda)| \leqslant C_3(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_{\tau_1}^2)^{k+2N}(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_{\tau_2}^2)^{k_1}\times \exp(r\|T\|)\frac{\sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau_2,J)}\|I_{P,\pi}(\lambda,f)\varphi\|_P^2}{|\mathcal{B}_{P,\pi}(\tau_2,J)|}.$$

Using (3.3.2.3), for any $q > 0$ we have:

$$|B^T_{(P,\pi,\tau_1,\tau_2)}(f,\lambda)| \leqslant C_3(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_{\tau_1}^2)^{k+2N-2q}(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_{\tau_2}^2)^{k_1-2q}\times \exp(r\|T\|)\frac{\sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau_2,J)}\|I_{P,\pi}(\lambda,L(\Delta^{2q})R(\Delta^{2q})f)\varphi\|_P^2}{|\mathcal{B}_{P,\pi}(\tau_2,J)|}.$$

Note that $I_{P,\pi}(\lambda,\Delta)I_{P,\pi}(\lambda,f_\tau)\varphi = I_{P,\pi}(\lambda,L(\Delta)f_\tau)\varphi$.

It is easy to conclude since, for every $i\in\mathbb{N}$ there is a continuous semi-norm $\|\cdot\|_i$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that for all pairs (P,π) and all $\varphi\in\mathcal{A}_{P,\pi}(G)^J$, all $\lambda\in\mathcal{R}_{\pi,c,l}$ we have

$$\|I_{P,\pi}(\lambda,L(\Delta^{2i})R(\Delta^{2i})f)\varphi\|_P \leqslant \|\varphi\|_P\|f\|_i.$$

□

3.3.3. Let J be a level and let (P,π) be a J-pair. We define for $f\in\mathcal{S}(G(\mathbb{A}))^J$

$$B^T_{(P,\pi)}(\lambda,f) = \sum_{\tau_1,\tau_2\in\hat{K}_\infty} B^T_{(P,\pi,\tau_1,\tau_2)}(\lambda,f).$$

Recall that the terms in the sum above are non-negative, see remark 3.3.2.2. The sum is convergent by the next proposition.

Proposition 3.3.3.1. — *There exist $l,r,q_0>0$ such that for all $q>q_0$ and all levels J there exist $c>0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all $f\in\mathcal{S}(G(\mathbb{A}))^J$, all enough positive $T\in\mathfrak{a}_0^G$ and all $\lambda\in\mathcal{R}_{\pi,c,l}$ we have*

$$B^T_{(P,\pi)}(f,\lambda) \leqslant \frac{\|f\|_{\mathcal{S}}^2\exp(r\|T\|)}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}.$$

Proof. — Using proposition (3.3.2.3) and its notations, we have

$$\sum_{\tau_1,\tau_2\in\hat{K}_\infty} B^T_{(P,\pi,\tau_1,\tau_2)}(\lambda,f) \leqslant \sum_{\tau_1,\tau_2}\frac{\|f_{\tau_1,\tau_2}\|_{\mathcal{S}}^2\exp(r\|T\|)}{(1+\|\lambda\|^2)^q(1+\lambda_\pi^2+\lambda_{\tau_1}^2)^q(1+\lambda_\pi^2+\lambda_{\tau_2}^2)^q}.$$

Note that in the right-hand side the sum is over $\tau_1,\tau_2\in\hat{K}_\infty$ such that (P,π,τ_1) and (P,π,τ_2) are J-triples. Using Cauchy-Schwartz inequality, we can bound the right-hand side by

$$\frac{\exp(r\|T\|)}{(1+\|\lambda\|^2)^q}\Big(\sum_{\tau_1,\tau_2\in\hat{K}_\infty}\|f_{\tau_1,\tau_2}\|_{\mathcal{S}}^4\Big)^{1/2}\times\sum_\tau(1+\lambda_\tau^2+\lambda_\pi^2)^{-2q}$$

where the last sum is over $\tau\in\hat{K}_\infty$ such that (P,π,τ) is a J-triple. We observe that $(\sum_{\tau_1,\tau_2\in\hat{K}_\infty}\|f_{\tau_1,\tau_2}\|_{\mathcal{S}}^4)^{1/4}$ is a continuous semi-norm on $\mathcal{S}(G(\mathbb{A})^J)$. Moreover, there exists $C>0$ such that for all P and $\pi\in\Pi_{\mathrm{disc}}(M)$

$$\sum_\tau(1+\lambda_\tau^2+\lambda_\pi^2)^{-2q} \leqslant C(1+\Lambda_\pi^2)^{-q}\Big(\sum_\tau(1+\lambda_\tau^2)^{-q}\Big)$$

and $\sum_{\tau\in\hat{K}_\infty}(1+\lambda_\tau^2)^{-q}<\infty$ if q is large enough. □

3.4 Bounds for Eisenstein series

3.4.1. We start with a lemma.

Lemma 3.4.1.1. — *Let $m \geqslant 0$ and $g \in C_c^m(G(\mathbb{A}))$. There exists $l > 0$ and $N > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all J-pairs (P,π), all $f \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$ and all $x \in G(\mathbb{A})^1$ we have*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}(J)} |E(x, I_{P,\pi}(\lambda, g*f)\varphi, \lambda)|^2 \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}}^2}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}$$

where $\mathcal{B}_{P,\pi}(J)$ is the union over $\tau \in \hat{K}_\infty$ of orthonormal bases of $\mathcal{A}_{P,\pi}(G)^{\tau,J}$.

Proof. — The main point is to express the square modulus of the Eisenstein series in terms of the truncated inner product and then to apply proposition 3.3.3.1. To do this we follow [Lap06, beginning of the proof of proposition 6.1].

We may and shall assume that x is in a fixed Siegel set. Let $g \in C_c^m(G(\mathbb{A}))$. According to [Lap06, lemma 6.2], there exists an absolute constant $c_0 > 0$ such that we have

$$(\Lambda^T E)(xy, I_{P,\pi}(\lambda, f)\varphi, \lambda) = E(xy, I_{P,\pi}(\lambda, f)\varphi, \lambda)$$

for all truncation parameters T and $y \in G(\mathbb{A})$ such that

$$\langle \varpi, T - H_0(xy)\rangle > c_0 \text{ for all } \varpi \in \hat{\Delta}_0. \tag{3.4.1.1}$$

We have

$$H_0(xy) = H_0(x) + H_0(k(x)y)$$

where $k(x) \in K$ is such that $xk(x)^{-1} \in B(\mathbb{A})$ (Iwasawa decomposition). In particular, there exists c_1 depending on the support $\mathrm{supp}(g)$ of g such that if T_1 is a truncation parameter such that $\langle \varpi, T_1\rangle > c_1$ for all $\varpi \in \hat{\Delta}_0$ then $T = T_1 + H_0(x)$ is enough positive and satisfies (3.4.1.1) for all $y \in \mathrm{supp}(g)$. We fix such a T_1 and we set $T = T_1 + H_0(x)$. Then we get, see [Lap06, p. 284]:

$$E(x, I_{P,\pi}(\lambda, g*f)\varphi, \lambda) = \int_{[G]_0} k_g(x,y)(\Lambda^T E)(y, I_{P,\pi}(\lambda, f)\varphi, \lambda)\, dy.$$

where we set

$$k_g(x,y) = \sum_{\gamma \in G(F)} \int_{A_G^\infty} g(x^{-1}a\gamma y)\, da.$$

According to [MW94, lemme I.2.4] there exist $N > 0$ and $c_2 > 0$ such that

$$|k_g(x,y)| \leqslant c_2 \|x\|_G^N, \quad x \in G(\mathbb{A})^1. \tag{3.4.1.2}$$

Hence we have by Cauchy-Schwartz inequality

$$|E(x, I_{P,\pi}(\lambda, g*f)\varphi, \lambda)|^2 \leqslant \mathrm{vol}([G]_0)c_2^2\|x\|_G^{2N} \int_{[G]_0} |(\Lambda^T E)(y, I_{P,\pi}(\lambda, f)\varphi, \lambda)|^2\, dy.$$

We deduce that (see remark 3.3.2.2):

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}(J)} |E(x, I_{P,\pi}(\lambda, g*f)\varphi, \lambda)|^2 \leqslant \mathrm{vol}([G]_0)c_2^2\|x\|_G^{2N} B_{P,\pi}^T(f,\lambda)$$

Using the result and the notations of proposition 3.3.3.1, we get the following bound:

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}(J)} |E(x, I_{P,\pi}(\lambda, g*f)\varphi, \lambda)|^2 \leqslant \mathrm{vol}([G]_0)c_2^2\|x\|_G^{2N} \frac{\|f\|_{\mathcal{S}}^2 \exp(r\|T\|)}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}.$$

To conclude it suffices to observe that there exists c_3 and N' such that $\exp(\|T\|) \leqslant \exp(\|T_1\| + \|H_0(x)\|) \leqslant c_3\|x\|_G^{N'}$, see [MW94, I.2.2]. $\square$

3.4.2. The following theorem is an extension of [Lap06, proposition 6.1] to the case of discrete Eisenstein series.

Theorem 3.4.2.1. — *There exists $l > 0$ and $N > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all J-pairs (P, π), all $f \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$ and all $x \in G(\mathbb{A})^1$ we have*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}(J)} |E(x, I_{P,\pi}(\lambda, f)\varphi, \lambda)|^2 \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}}^2}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}$$

where $\mathcal{B}_{P,\pi}(J)$ is the union over $\tau \in \hat{K}_\infty$ of orthonormal bases of $\mathcal{A}_{P,\pi}(G)^{\tau,J}$.

Proof. — Following [Art78, corollary 4.2], for a level J and an integer $m \geqslant 1$ large enough, we can find $Z \in \mathcal{U}(\mathfrak{g}_\infty)$, $g_1 \in C_c^\infty(G(\mathbb{A}))$ and $g_2 \in C_c^m(G(\mathbb{A}))$ such that

- Z is invariant under K_∞-conjugation ;
- g_1 and g_2 are invariant under K-conjugation and J-biinvariant;
- for any $f \in \mathcal{S}(G(\mathbb{A}))^J$ we have:

$$f = g_1 * f + g_2 * (Z * f).$$

Then the theorem is a straightforward consequence of lemma 3.4.1.1. $\square$

3.4.3. In the sequel we shall need the following slight extension of theorem 3.4.2.1 which we prefer to state separetely.

Theorem 3.4.3.1. — *There exists $l > 0$ and $N > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all standard parabolic subgroups R, for all J-pairs (P, π), all $w \in {}_RW_P$, all $f \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$ and all $x \in G(\mathbb{A})$ such that $H_R(x) = 0$ we have*

$$\sum_{\varphi \in \mathcal{B}_{P,\pi}(J)} |E^R(x, M(w,\lambda)(I_{P,\pi}(\lambda, f)\varphi)_w, (w\lambda)^R)|^2 \leqslant \frac{\|x\|_R^N \|f\|_{\mathcal{S}}^2}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}$$

where $\mathcal{B}_{P,\pi}(J)$ is the union over $\tau \in \hat{K}_\infty$ of orthonormal bases of $\mathcal{A}_{P,\pi}(G)^{\tau,J}$.

Proof. — Since the proof is very close to that of theorem 3.4.2.1, we shall be brief. For the discussion, we set $\psi = M(w,\lambda)\varphi_w$ and $\mu = w\lambda$. The starting point is that we have for $g \in C_c^m(G(\mathbb{A}))$ and x in some Siegel set and a suitable T depending on x

$$E^R(x, I_{R_w}(\mu, g * f)\psi, \mu^R) = \int_{[G]_{R,0}} k_{R,g}(x,y)\Lambda^{T,R} E^R(y, I_{R_w}(\mu, f)\psi, \mu^R)\, dy$$

where we set

$$k_{R,g}(x,y) = \int_{[N_R]} \int_{A_R^\infty} \sum_{\gamma \in R(F)} g(x^{-1} a\gamma n y)\, da.$$

There exist $N > 0$ and $c_2 > 0$ such that for all $x \in G(\mathbb{A})$ such that $H_R(g) = 0$ we have

$$|k_{R,g}(x,y)| \leqslant c_2 \|x\|_R^N.$$

In this way, we are reduced to bound $\|\Lambda^{T,R} E^R(I_{R_w}(\mu, f)\psi, \mu)\|_R$. The main point is to have a generalization of proposition 3.2.5.1 whose proof is based on the computation of the truncated scalar product given by theorem 3.1.3.1 and the majorization of intertwinings operators given by proposition 3.2.4.1. Both ingredients are also available in our situation: the computation of the scalar product in our case is a consequence of 3.1.3.1. The point we want to emphasize is that we can bound everything in terms of π, since all the other automorphic representations that appears are closely related: like π they are obtained by Mœglin-Waldspurger's description of the discrete spectrum from the same finite set of cuspidal representations. The rest of the proof is nearly identical to that theorem 3.4.2.1. $\square$

4 Flicker-Rallis periods

4.1 Notations

4.1.1. From now on E/F is a quadratic extension of number fields. Sometimes we will consider $\tau \in F^\times$ such that $E = F(\sqrt{\tau})$.

Let $n \geqslant 1$ be an integer. Let $G_n = \mathrm{Res}_{E/F}\,\mathrm{GL}_E(n)$ be the F-group obtained by restriction of scalars. Let ι be the Galois involution of G whose fixed point set is the subgroup $G'_n = \mathrm{GL}_F(n)$. The inclusion $G'_n \subset G_n$ gives an inclusion $A_{G'_n} \subset A_{G_n}$ which is in fact an equality. The restriction map $X^*(G_n) \to X^*(G'_n)$ gives an isomorphism $\mathfrak{a}^*_{G_n} \simeq \mathfrak{a}^*_{G'_n}$.

The minimal pair $(P'_{0,n}, M'_{0,n})$ for G'_n is formed by the Borel subgroup $P'_{0,n}$ of G'_n of upper triangular matrices and the diagonal maximal torus $M'_{0,n}$ of G'_n. Let $(P_{0,n}, M_{0,n})$ be the minimal pair for G_n deduced from $(P'_{0,n}, M'_{0,n})$ by extension of scalars to E and restriction to F. The words "standard" and "semi-standard" will refer to these pairs. The map $P' \mapsto P = \mathrm{Res}_{E/F}(P' \times_F E)$ induces a bijection between the sets of standard parabolic subgroups of G'_n and G_n whose inverse bijection is given by

$$P \mapsto P' = P \cap G'_n.$$

Let P be a standard parabolic subgroup of G_n. The restriction map $X^*(P) \to X^*(P')$ identifies $X^*(P)$ with a subgroup of $X(P')$ of index $2^{\dim(\mathfrak{a}_P)}$. It induces an isomorphism $\mathfrak{a}_{P'} \to \mathfrak{a}_P$ which does not preserve Haar measures: the pull-back on $\mathfrak{a}_{P'}$ of the Haar measure on $\mathfrak{a}_P$ is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $\mathfrak{a}_{P'}$. In the same way, the groups A_P^∞ and $A_{P'}^\infty$ are canonically identified but the Haar measure on A_P^∞ is $2^{\dim(\mathfrak{a}_P)}$ times the Haar measure on $A_{P'}^\infty$. For any standard parabolic subgroup $P \subset Q$, the restriction of the function τ_P^Q to $\mathfrak{a}_{P'}$ coincides with the function $\tau_{P'}^{Q'}$.

4.1.2. Let $\mathbb{A}$ be the ring of adèles of F. The groups $G_n(\mathbb{A})$ and $G'_n(\mathbb{A})$ come with their standard maximal compact subgroups respectively denoted by K_n and K'_n. We have $K'_n = K_n \cap G'_n(\mathbb{A})$. Note that for all $x \in G'(\mathbb{A})$

$$\langle \rho_P^Q, H_P(x)\rangle = 2\langle \rho_{P'}^{Q'}, H_{P'}(x)\rangle.$$

4.1.3. In most of the rest of the paper, the integer n is fixed and will be omitted in the notation ($G = G_n$, $P_0 = P_{0,n}$ etc.). As before, we identify $\mathfrak{a}_{0,\mathbb{C}}$ and its dual with $\mathbb{C}^n$ equipped with the usual non-degenerate positive definite hermitian form. We denote by $\|\cdot\|$ the associated norm. The other notations are borrowed from the previous sections.

4.1.4. In this section we denote by T a truncation parameter in $\mathfrak{a}_0$.

4.2 Mixed truncation operator

4.2.1. Let Q be a standard parabolic subgroup of G. Following Jacquet-Lapid-Rogawski, see [JLR99], we define the (mixed) truncation operator $\Lambda_m^{T,Q}$ that associates to a function φ on $[G]$ the following function of the variable $h \in [G']_{P'}$:

$$(\Lambda_m^{T,Q}\varphi)(h) = \sum_{P_0 \subset P \subset Q} (-1)^{\dim(\mathfrak{a}_P^Q)} \sum_{\delta \in P'(F)\backslash Q'(F)} \hat{\tau}_P^Q(H_P(\delta h) - T_P)\varphi_P(\delta h) \tag{4.2.1.1}$$

where φ_P is the constant term along P. If $Q = G$, the exponent G is omitted. We denote by Λ_m^{T,M_Q} the operator on the space of functions on $[M_Q]$ given by the formula 4.2.1.1 where the parabolic subgroup P is now interpreted as a standard parabolic subgroup of M_Q. By definition Λ_m^{T,M_Q} and $\Lambda_m^{T,Q}$ depend only on the projection T^Q of T on $\mathfrak{a}_0^Q$.

One of the most important property of the truncation operator is given by the following proposition whose proof is a variant of that of [JLR99, proposition 8] and is omitted.

Proposition 4.2.1.1. — *Let $J \subset K_f$ be a level and let Q be a standard parabolic subgroup. For any $N, N' > 0$ there exists a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that for any smooth and right-J-invariant function φ on $[G]_Q$, the function $\Lambda_m^{T,Q}$ is a smooth function on $[G']_{Q'}$, and we have*

$$\sup_{g\in[G']^1_{Q'}} \|g\|_Q^{N'} |\Lambda_m^{T,Q}\varphi(g)| \leqslant \sum_{i\in I} \|\varphi\|_{-N,X_i}.$$

4.3 Regularized periods of discrete Eisenstein series

4.3.1. Let $P = MN_P$ be a standard parabolic subgroup of G. Let $\pi \in \Pi_{\mathrm{disc}}(M)$.

4.3.2. Let $\varphi \in \mathcal{A}_{P,\pi}(G)$. Let Q be a standard parabolic subgroup of G and $w \in {}_QW_P$. For $\lambda \in \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$, we set

$$\mathcal{I}^{T,Q}(\varphi,\lambda,w) = \int_{[G']_{Q',0}} \Lambda_m^{T,Q} E^Q(g, M(w,\lambda)\varphi_w, (w\lambda)^Q)\, dg,$$

where we set, see §3.1.2, $\varphi_w = \varphi_{P_w,-\nu_{P_w}}$. This is what we call the truncated Flicker-Rallis period of the Eisenstein series $E^Q(g, M(w,\lambda)\varphi_w, (w\lambda)^Q)$. Note that the integrand is left-equivariant under the character

$$x \in A^\infty_{Q'} M_{Q'}(F) N_{Q'}(\mathbb{A}) \mapsto \exp(\langle \rho_Q, H_Q(x)\rangle) = \exp(\langle 2\rho_{Q'}, H_{Q'}(x)\rangle).$$

So the "integral" makes formally sense. It is in fact convergent as soon as the Eisenstein series and the intertwining operator are well-defined: this follows from the moderate growth of Eisenstein series and proposition 4.2.1.1. Note also that $\mathcal{I}^{T,Q}(\varphi,\lambda,w)$ depends only on the projection T^Q of T and it vanishes unless $P_\pi \subset P_w$. In particular, if $Q = P_0$ then the integral does not depend on T. If $Q = G$, then w must be 1 and we get:

$$\mathcal{I}^T(\varphi,\lambda) = \int_{[G']_0} \Lambda_m^T E(g,\varphi,\lambda)\, dg$$

where we omit w and G from the notation.

Lemma 4.3.2.1. — *Let $\omega \subset \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ a compact subset. Let $r(\lambda)$ be a a finite product of non-zero affine functions on $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ such that $r(\lambda)E^Q(M(w,\lambda)\varphi_w, (w\lambda)^Q)$ is holomorphic on ω.*

1. *For any $\varphi \in \mathcal{A}_{P,\pi}(G)$, the product $r(\lambda)\mathcal{I}^{T,Q}(\varphi,\lambda,w)$ is also holomorphic on ω.*
2. *There exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{A}_{P,\pi}(G)$ such that*

$$|r(\lambda)\mathcal{I}^{T,Q}(\varphi,\lambda,w)| \leqslant \|\varphi\|, \quad \varphi \in \mathcal{A}_{P,\pi}(G),\ \lambda \in \omega.$$

Proof. — This is straightforward consequence of continuity of Eisenstein series of [Lap08, theorem 2.2] and the property of the truncation operator given in proposition 4.2.1.1. □

4.3.3. Regularized periods of Eisenstein series. — Let R be a standard parabolic subgroup of G. Let $w \in {}_R W_P$. For any $w' \in {}_Q W^R_{R_w} w$ we have $P_{w'} \subset P_w \subset P$. If $P_\pi \subset P_{w'}$ we set

$$\nu^w_{w'} = \nu^{P_w}_{P_{w'}}.$$

For any $\lambda \in \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$, we introduce the Jacquet-Lapid-Rogawski regularized period

(4.3.3.1)

$$\mathcal{P}^{T,R}(\varphi,\lambda,w) = \sum_{P_0 \subset Q \subset R} (-2)^{-\dim(\mathfrak{a}^R_Q)} \sum_{w' \in {}_Q W^R_{R_w} w} \mathcal{I}^{T,Q}(\varphi,\lambda,w') \cdot \frac{\exp(\langle w'(\lambda + \nu^w_{w'}), T^R_Q \rangle)}{\hat{\theta}^R_Q(w'(\lambda+\nu^w_{w'}))}.$$

If $w = 1$ we set $\mathcal{P}^{T,R}(\varphi,\lambda) = \mathcal{P}^{T,R}(\varphi,\lambda,1)$.

Remark 4.3.3.1. — In the sum above, we tacitly assume that a summand indexed by w' such that $P_\pi \not\subset P_{w'}$ is understood to be 0 whether the numerator $\hat{\theta}^R_Q(w'(\lambda+\nu^w_{w'}))$ vanishes or not. Recall that in this case we have $\mathcal{I}^{T,Q}(\varphi,\lambda,w') = 0$. In particular, we have $\mathcal{P}^{T,R}(\varphi,\lambda,w) = 0$ unless $P_\pi \subset P_w$.

Remark 4.3.3.2. — We can replace P by R_w and φ by $M(w,\lambda)\varphi_{P_w}$. We get:

(4.3.3.2)
$$\mathcal{P}^{T,R}(\varphi,\lambda,w) = \mathcal{P}^{T,R}(M(w,\lambda)\varphi_{P_w}, w\lambda, 1).$$

We can introduce analogous objects for the Levi factor M_R in place of G or R. Then we have a parabolic descent (written for $w = 1$)

(4.3.3.3)
$$\mathcal{P}^{T,R}(\varphi,\lambda) = \mathcal{P}^{T,M_R}(\psi,\lambda^R).$$

where $\psi(m) = \int_{K'} \varphi_{-\rho_R}(mk)dk$.

Lemma 4.3.3.3. — *The expression* (4.3.3.1) *is well-defined and holomorphic for λ in a complement of hyperplanes in $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$.*

Remark 4.3.3.4. — We will in fact later show that the statement is still true if we replace $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ by $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$, see remark 4.4.1.2.

Proof. — Let $w' \in {}_Q W^R_{P_w} w$ and $P_0 \subset Q \subset R$. Using lemma 4.3.2.1 and remark 4.3.3.1, we are reduced to prove the statement for the rational maps $\lambda \mapsto \hat{\theta}^R_Q(w'(\lambda+\nu^w_{w'}))^{-1}$ where w' is such that $P_\pi \subset P_{w'}$. Let's assume that $\hat{\theta}^R_Q$ vanishes identically on $w'\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ for such a w'. More precisely let's assume that there is $\varpi^\vee \in \hat{\Delta}^{R,\vee}_Q$ such that $\varpi^\vee \in w'\mathfrak{a}^{P_w}_0$. But then by lemma 4.3.3.5 below we have $\langle w'\nu^w_{w'}, \varpi^\vee \rangle \neq 0$. In this way, we see that $\lambda \mapsto \hat{\theta}^R_Q(w'(\lambda+\nu^w_{w'}))$ does not vanish identically on $w'\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$. □

Lemma 4.3.3.5. — *Let $P_0 \subset Q \subset R$ and $w' \in {}_Q W^R_{R_w} w$ such that $P_\pi \subset P_{w'}$.*

1. *For all $\varpi^\vee \in \hat{\Delta}^{R,\vee}_Q$, we have*
$$\langle w'\nu^w_{w'}, \varpi^\vee \rangle \leqslant 0.$$
Moreover we have equality if and only if $\varpi^\vee \in w'\mathfrak{a}^G_{P_w}$.

2. *The following conditions are equivalent:*
   - *(a) $P_w = P_{w'}$*
   - *(b) $(w'\nu^w_{w'})^R_Q = 0$;*
   - *(c) $w' \in W^R(R_w; Q)w$.*

3. There exists $c_1 > 0$ such that for any $T \in \mathfrak{a}_0^+$, any $P_0 \subset Q \subset R$ and any

$$w' \in {}_QW^R_{R_w} w \setminus W^R(R_w; Q)w$$

such that $P_\pi \subset P_{w'}$ we have:

$$\langle (w'\nu^w_{w'})^R_Q, T\rangle \leqslant -c_1 d(T).$$

Proof. — The first two assertions are essentially in [Lap11a, lemma 6]. Since our setting is slightly different we give a proof for the reader's convenience.

1. Let $\varpi^\vee \in \hat{\Delta}_Q^{R,\vee}$. We have $\nu^w_{w'} = \sum_{\alpha\in\Delta^w_{w'}} c_\alpha \alpha$ where $\Delta^w_{w'} = \Delta^{P_w}_{P_{w'}}$ and $c_\alpha < 0$. Observe that $w'\Delta^w_{w'} \subset \Delta^R_{Q_{w'}}$ and $\hat{\Delta}_Q \subset \hat{\Delta}_{Q_{w'}}$. Thus $\langle w'\alpha, \varpi^\vee\rangle \geqslant 0$ for all $\alpha \in \Delta^w_{w'}$. Hence the first assertion. The condition $\langle w'\nu^w_{w'}, \varpi^\vee\rangle = 0$ is equivalent to $\langle w'\alpha, \varpi^\vee\rangle = 0$ for all $\alpha \in \Delta^w_{w'}$ that is $\varpi^\vee \in w'(\mathfrak{a}_0^{P_{w'}} \oplus \mathfrak{a}^G_{P_w})$. But we have $w'\mathfrak{a}^G_{P_w} \subset w'\mathfrak{a}_{P_{w'}} = \mathfrak{a}_{Q_{w'}}$ and $\mathfrak{a}_Q \subset \mathfrak{a}_{Q_{w'}}$. Thus $\varpi^\vee \in \mathfrak{a}_Q \cap w'(\mathfrak{a}_0^{P_{w'}} \oplus \mathfrak{a}^G_{P_w}) \subset w'\mathfrak{a}^G_{P_w}$.

2. Clearly (a) implies (b). Let's assume (b). Then by 1 we have $\mathfrak{a}^R_Q \subset w'\mathfrak{a}^G_{P_w}$. Note that $\mathfrak{a}_R \subset \mathfrak{a}_{R_w} = w\mathfrak{a}_{P_w}$ hence $\mathfrak{a}_R \subset w'\mathfrak{a}_{P_w}$ since $w \in W^R w'$. Thus we have $\mathfrak{a}_Q \subset w'\mathfrak{a}_{P_w}$ hence $w'M_{P_w}(w')^{-1} \subset M_Q$. So $M_{P_w} \subset M_P \cap (w')^{-1}M_Q w' = M_{P_{w'}}$. Hence $P_w = P_{w'}$ and (b) implies (a).

Let's prove that (a) is equivalent to (c). Let $w_1 \in {}_QW^R_{R_w}$ such that $w' = w_1 w$. We always have $w'M_{P_{w'}}(w')^{-1} \subset w'M_{P_w}(w')^{-1}$ hence $M_{Q_{w'}} \subset w_1 M_{R_w} w_1^{-1}$. If $P_w = P_{w'}$ we have $M_{Q_{w'}} = w_1 M_{R_w} w_1^{-1}$ and $M_{R_w} \subset w_1^{-1} M_Q w_1$ so $w_1 \in W^R(R_w; Q)$. Conversely if $M_{R_w} \subset w_1^{-1}M_Q w_1$ we have

$$\begin{aligned}
M_{P_{w'}} &= M_P \cap (w')^{-1} M_Q w' \\
&= w^{-1}(wM_P w^{-1} \cap w_1^{-1} M_Q w_1)w \\
&= w^{-1}(wM_P w^{-1} \cap M_R \cap w_1^{-1} M_Q w_1)w \\
&= w^{-1}(M_{R_w} \cap w_1^{-1} M_Q w_1)w \\
&= w^{-1} M_{R_w} w = M_{P_w}.
\end{aligned}$$

3. By the equivalence of 2.(b) and 2.(c) we have $(w'\nu^w_{w'})^R_Q \neq 0$. Therefore there exists $\beta \in \Delta^R_Q$ such that $\langle w'\nu^w_{w'}, \varpi^\vee_\beta\rangle \neq 0$. We write $T^R_Q = \sum_{\alpha\in\Delta^R_Q} \langle \alpha, T\rangle \varpi^\vee_\alpha$. For any $\alpha \in \Delta^R_Q$ and $T \in \mathfrak{a}_0^+$, we have $\langle \alpha, T\rangle \geqslant \langle \tilde{\alpha}, T\rangle \geqslant 0$ where $\tilde{\alpha}$ is the root in $\Delta^R_0 \setminus \Delta^Q_0$ that projects on α. We have also $\langle w'\nu^w_{w'}, \varpi^\vee_\alpha\rangle \leqslant 0$ by assertion 1. Thus we get

$$\langle (w'\nu^w_{w'})^R_Q, T\rangle = \sum_{\alpha\in\Delta^R_Q} \langle \alpha, T\rangle \langle w'\nu^w_{w'}, \varpi^\vee_\alpha\rangle \leqslant \langle w'\nu^w_{w'}, \varpi^\vee_\beta\rangle \langle \beta, T\rangle \leqslant \langle w'\nu^w_{w'}, \varpi^\vee_\beta\rangle d(T).$$

The result is clear. □

The construction (4.3.3.1) is nothing else but an explicit version of the (Flicker-Rallis) *regularized period* that we denote by

$$\mathcal{P}^R(\varphi, \lambda, w)$$

and that was introduced by Jacquet-Lapid-Rogawski in [JLR99, section 7] section 7 as a substitute of the (in general divergent) integral:

$$\begin{aligned}
&\int_{[G']_{R',0}} E^R(g, M(w,\lambda)\varphi_w, (w\lambda)^R)\, dg \\
&= \int_{[M'_R]_0} \int_{K'} \exp(-\langle 2\rho_{R'}, H_{R'}(m)\rangle) E^R(mk, M(w,\lambda)\varphi_w, (w\lambda)^R)\, dm
\end{aligned}$$

This is what we shall check among other things in the next proposition. Note we are in fact considering the obvious variant of their construction when one replaces $G'(F)\backslash G'(\mathbb{A})^1$ by the quotient $A_{R'}M'_R(F)\backslash M'_R(\mathbb{A})$.

Proposition 4.3.3.6. —

1. *Let $\lambda \in \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$. For any $\varphi \in \mathcal{A}_{P,\pi}(G)$ we have*
$$\mathcal{P}^R(\varphi,\lambda,w) = \mathcal{P}^{T,R}(\varphi,\lambda,w).$$
In particular, the regularized period is well-defined, holomorphic in a complement of hyperplanes in $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ and the right-hand side does not depend on the parameter T.

2. *Let $\omega \subset \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ a compact subset of non-empty interior. There exists $r(\lambda)$ a finite product of non-zero affine functions such that $\lambda \in \omega \mapsto r(\lambda)\mathcal{P}^R(\varphi,\lambda,w)$ is regular for all $\varphi \in \mathcal{A}_{P,\pi}(G)$. Moreover for any holomorphic differential operator D on $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$, there exists a continuous semi-norm $\|\cdot\|$ on $\mathcal{A}_{P,\pi}(G)$ such that*
$$|D(r(\lambda)\mathcal{P}^R(\varphi,\lambda,w))| \leqslant \|\varphi\|, \quad \varphi \in \mathcal{A}_{P,\pi}(G),\ \lambda \in \omega.$$

3. *Assume $w = 1$. The regularized period $\varphi \mapsto \mathcal{P}^R(\varphi,\lambda,w)$ gives, at each regular λ, a map $\mathcal{A}_{P,\pi}(G) \to \mathbb{C}$ that is $G'(\mathbb{A})$-invariant for the action $I_{P,\pi}(\lambda^R)$.*

4. *If $R = P$, the regularized period $\mathcal{P}^P(\varphi,\lambda,w)$ reduces to the following integral:*
$$\int_{[G']_{P',0}} (M(w,\lambda)\varphi_w)(g)\,dg$$
which is convergent outside the singularities of $M(w,\lambda)$.

Proof. — Set $\psi = M(w,\lambda)\varphi_w$, $\lambda' = (w\lambda)^R$ and $S = R_w$. Note that $\lambda' \in \mathfrak{a}^{S,*}_{S,\mathbb{C}}$. By definition, $\mathcal{P}^R(\varphi,\lambda,w)$ is the sum indexed by $P_0 \subset Q \subset R$ of

$$\int^*_{[G']_{Q',0}} \Lambda^{T,Q}E^R(g,\psi,\lambda')\tau^R_Q(H(g)-T)\,dg$$
$$= \sum_{w' \in {}_QW^R_S} \int^*_{[G']_{Q',0}} \Lambda^{T,Q}E^R(g,M(w',\lambda')\psi_{S_{w'}},w'\lambda')\tau^R_Q(H(g)-T)\,dg$$
$$= \sum_{w' \in {}_QW^R_S} (-2)^{-\dim(\mathfrak{a}^R_Q)} \int_{[G']_{Q',0}} \Lambda^{T,Q}E^Q(g,M(w',\lambda')\tilde{\psi}_{S_{w'}},(w'\lambda')^Q)\,dg \frac{\exp(\langle w'(\lambda'+\nu_{\psi_{S_{w'}}}),T^R_Q\rangle}{\hat{\theta}^R_Q(w'(\lambda'+\nu_{\psi_{S_{w'}}}))}$$

where we defined $\nu_{\psi_{S_{w'}}} \in \mathfrak{a}^{S,*}_{S_w}$ and $\tilde{\psi}_{S_{w'}}$ by the condition that $\tilde{\psi}_{S_{w'}} = \psi_{S_{w'},-\nu_{\psi_{S_{w'}}}} \in \mathcal{A}^0_{S_{w'}}(G)$. The reader is advised to consult [JLR99] or to take the first two lines just as a suggestive notation and the third one as a definition. The power of 2 is due to the discrepancy between the measure on $\mathfrak{a}^R_Q$ and that on $\mathfrak{a}^{R'}_{Q'}$ (see §4.1.1).

Let $w' \in {}_QW^R_S$. Set $w'' = w'w$. Then $w'' \in {}_QW_P$ by lemma 2.2.1.1. One has $P_{w''} \subset P_w \subset P$,
$$M(w',\lambda')\tilde{\psi}_{S_{w'}} = M(w'',\lambda)\varphi_{w''}$$
and $\nu_{\psi_{S_{w'}}} = w\nu^w_{w''}$. Moreover we have $w'\lambda' = (w''\lambda)^R$, $(w'\lambda')^Q = (w''\lambda)^Q$. Thus we have $w'(\lambda'+\nu_{\psi_{S_{w'}}}) = (w''\lambda)^R + w''\nu^w_{w''}$. Now the comparison with (4.3.3.1) is straightforward.

Then assertion 1 and assertion 3 come from lemma 4.3.3.3 and [JLR99, theorem 9]: either the argument of the proof of [JLR99, theorem 9] can be used in our context or one can use the parabolic descent of remark 4.3.3.2 to reduce to the case of [JLR99, theorem 9].

To prove assertion 2, the existence of $r(\lambda)$ follows from the definition of $\mathcal{P}^R(\varphi,\lambda,w)$, properties of Eisenstein series and lemma 4.3.2.1. If we add some affine factors to $r(\lambda)$ we can even assume that each factor of each term in the definition of $\mathcal{P}^R(\varphi,\lambda,w)$, see (4.3.3.1), is holomorphic. Then for $D=1$ the bound follows from lemma 4.3.2.1. Using Cauchy's integral formula, we see that the assertion holds also for any D. Now we have to remove the extra factors we add to $r(\lambda)$. By recursion, we are reduced to the case where we add only one affine factor $l(\lambda)$. So the bound holds for $g(\lambda)=l(\lambda)r(\lambda)\mathcal{P}^R(\varphi,\lambda,w)$. But then $r(\lambda)\mathcal{P}^R(\varphi,\lambda,w)$ can be expressed as an integral over a compact subset of a derivative of g. The result follows easily.

For the assertion 4, we have

$$\int_{[G']_{P',0}} (M(w,\lambda)\varphi_w)(g)\,dg = \int_{[M_{P'}]_0}\int_{K'} \exp(-\langle 2\rho_{P'}, H_{P'}(m)\rangle)(M(w,\lambda)\varphi_w)(mk)\,dkdm.$$

Since $m\mapsto \exp(-\langle 2\rho_{P'}, H_{P'}(m)\rangle)(M(w,\lambda)\varphi_w)(m)$ is square-integrable on $[M_P]_0$, its integral over $[M_{P'}]_0$ is absolutely convergent (see [Yam15, lemma 3.1]). So assertion 4 comes also from [JLR99, theorem 9]. $\square$

4.4 Truncated Flicker-Rallis periods

4.4.1. The following proposition gives an explicit expression of the truncated Flicker-Rallis period in terms of the regularized periods.

Proposition 4.4.1.1. — *For $\lambda\in\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ in general position, we have:*

$$\mathcal{I}^T(\varphi,\lambda)=\sum_{P_0\subset R} 2^{-\dim(\mathfrak{a}_R^G)}\sum_{w\in{}_RW_P}\mathcal{P}^R(\varphi,\lambda,w)\cdot\frac{\exp(\langle w(\lambda+\nu_{P_w}),T_R^G\rangle)}{\theta_R^G(w(\lambda+\nu_{P_w}))}.$$

Remark 4.4.1.2. — As we shall see later many terms in fact vanish. By definition the term $\mathcal{P}^R(\varphi,\lambda,w)$ is understood to be 0 unless $P_\pi\subset P_w$. In this case the proof shows that, for $\lambda\in\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ in general position, not only $\mathcal{P}^R(\varphi,\lambda,w)$ is well-defined (we just knew *a priori* that $\mathcal{P}^R(\varphi,\lambda,w)$ was defined for λ in general position in the *bigger* space $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$) but also the corresponding denominator is non-vanishing.

Proof. — We proceed as in the proof of theorem 3.1.3.1, see also [LR03, proof of proposition 4.1]. With notations as in the proof of theorem 3.1.3.1, for $T'\in\mathfrak{a}_0^+$, we have:

$$\mathcal{I}^{T+T',R}(\varphi,\lambda)$$
$$=\sum_{P_0\subset Q}\int_{N_{Q'}(\mathbb{A})M_{Q'}(F)A_{G'}^\infty\backslash G'(\mathbb{A})}\Gamma_Q(H_Q(g)-T,T')\Lambda_m^{T,Q}E(g,\varphi,w\lambda)\,dg.$$

Thanks to the computation of the constant term of the Eisenstein series, the expression above becomes:

$$\sum_{P_0\subset Q}\sum_{w\in{}_QW_P}\mathcal{I}^{T,Q}(\varphi,\lambda,w)\cdot 2^{-\dim(\mathfrak{a}_Q^G)}\int_{\mathfrak{a}_Q^G}\Gamma_Q(H-T,T')\exp(\langle w(\lambda+\nu_w),H\rangle)\,dH$$

the power of 2 is due to the different choices of measures on $\mathfrak{a}_{Q'}$ and $\mathfrak{a}_Q$. The expression $\mathcal{I}^{T,Q}(\varphi,\lambda,w)$ vanishes unless $P_\pi\subset P_w$. Then we have

$$\int_{\mathfrak{a}_Q^G}\Gamma_Q(H-T,T')\exp(\langle w(\lambda+\nu_w),H\rangle)\,dH$$
$$=\exp(\langle w(\lambda+\nu_w),T_Q\rangle)\cdot\sum_{Q\subset R}(-1)^{\dim(\mathfrak{a}_Q^R)}\frac{\exp(\langle w(\lambda+\nu_w),T'_R\rangle}{(\hat\theta_Q^R\theta_R)(w(\lambda+\nu_w))}.$$

Note that because we assume $P_\pi \subset P_w$, each denominator that appears is non-vanishing for λ in general position by a variant of lemma 3.1.5.1. Using this expression and inverting the sum over R and Q we find:

$$\mathcal{I}^{T+T',R}(\varphi, \lambda)$$
$$= \sum_R 2^{-\dim(\mathfrak{a}_R^G)} \sum_{w \in {}_R W_P} \exp(\langle w(\lambda + \nu_w), (T+T')_R \rangle) \times$$
$$\left[\sum_{Q \subset R} (-2)^{\dim(\mathfrak{a}_Q^R)} \sum_{w' \in {}_Q W_{R_w}^R w} \mathcal{I}^{T,Q}(\varphi, \lambda, w') \cdot \frac{\exp(\langle w'(\lambda + \nu_{w'}^w), T_Q^R \rangle)}{\hat{\theta}_Q^R(w'(\lambda + \nu_{w'}^w))}. \right]$$

But the bracket is nothing else $\mathcal{P}^R(\varphi, \lambda, w)$, see (4.3.3.1) and proposition 4.3.3.6. It suffices to take $T' = 0$ to conclude. □

4.4.2.

Corollary 4.4.2.1. — *Let R be a standard parabolic subgroup and $w \in {}_R W_P$. For $\lambda \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$ in general position, we have:*

$$\mathcal{I}^{T,R}(\varphi, \lambda, w) = \sum_{P_0 \subset Q \subset R} 2^{-\dim(\mathfrak{a}_Q^R)} \sum_{w' \in {}_Q W_{R_w}^R w} \mathcal{P}^Q(\varphi, \lambda, w') \cdot \frac{\exp(\langle w'(\lambda + \nu_{w'}^w), T_Q^R \rangle)}{\theta_Q^R(w(\lambda + \nu_{w'}^w))}.$$

Proof. — The proof is similar to that of proposition 4.4.1.1. Alternatively, by parabolic descent to M_R, see remark 4.3.3.2, one is reduced to proposition 4.4.1.1. Details are left to the reader. □

Corollary 4.4.2.2. — *We keep the notations of corollary 4.4.2.1. The map*

$$T \mapsto \mathcal{I}^{T,R}(\varphi, \lambda, w)$$

coincides with a polynomial exponential whose set of exponents is included in the following set:

$$(4.4.2.1) \qquad S_\lambda = \{(w'(\lambda + \nu_{w'}^w))_Q^R \mid Q \subset R \ ; \ w' \in {}_Q W_{R_w}^R w \ ; \ P_\pi \subset P_{w'}\}.$$

Proof. — It is a straightforward consequence of corollary 4.4.2.1, the nature of regularized period (see proposition 4.3.3.6) and lemma 2.3.1.1. □

4.4.3. Let $0 < \varepsilon < 1$. We fix $c, k > 0$ and $\Lambda \geqslant 0$ and we set

$$(4.4.3.2) \qquad \mathcal{R} = \mathcal{R}_{\Lambda,c,k} = \{\lambda \in \mathfrak{a}_{0,\mathbb{C}}^* \mid \|\Re(\lambda)\| < c(1 + \Lambda + \|\Im(\lambda)\|)^{-k}\}.$$

As follows from the methods and results of section 3, see in particular (the proof of) theorem 3.4.2.1, we may and shall assume that c, k and Λ are chosen so that for all standard parabolic subgroups R and all $w \in {}_R W_P$ the Eisenstein series $E^R(M(w,\lambda)\varphi_w, (w\lambda)^R)$ is holomorphic on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$. In this case, by lemma 4.3.2.1, $\mathcal{I}^{T,R}(\varphi, \lambda, w)$ is also holomorphic on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$.

Lemma 4.4.3.1. — *Let $Q \subset R$ be standard parabolic subgroups, $w \in {}_R W_P$, $w' \in {}_Q W_{R_w}^R w$ such that $P_\pi \subset P_{w'}$. Let $\lambda \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$ and $\mu = (w'(\lambda + \nu_{w'}^w))_Q^R$.*

1. *If $w' \in W^R(R_w; Q)w$, we have $\langle \Re(\mu), T \rangle \geqslant -c\|T\|$ for all $T \in \mathfrak{a}_0$.*
2. *If $w' \notin W^R(R_w; Q)w$, we have $\langle \Re(\mu), T \rangle \leqslant (c - c_1\varepsilon)\|T\|$ for all $T \in \mathfrak{a}_0$ such that $d(T) \geqslant \varepsilon\|T\|$ where $c_1 > 0$ appears in lemma 4.3.3.5 assertion 3.*

Proof. — We have $|\langle\Re(\mu),T\rangle| \leqslant \|\Re(\mu)\|\|T\|$. In case 1, we have then $\Re(\mu) = (w'\Re(\lambda))_Q^R$ by lemma 4.3.3.5 assertion 2. Thus $\|\Re(\mu)\| \leqslant \|\Re(\lambda)\| < c$. In case 2, by lemma 4.3.3.5 assertion 3, for any $T \in \mathfrak{a}_0^+$ we have

$$\langle\Re(\mu),T\rangle = \langle(w'\Re(\lambda))_Q^R, T\rangle + \langle(w'\nu_{w'}^w)_Q^R, T\rangle \leqslant c\|T\| - c_1 d(T).$$

The result follows. □

Let $\lambda \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$. We shall say that an exponent μ in the set S_λ defined in (4.4.2.1) is of type 1, resp. of type 2, if it satisfies the inequality 1, resp. 2, of lemma 4.4.3.1.

Remark 4.4.3.2. — We shall assume in the following that we have $c < c_1\varepsilon/2$. Then any point of $\mathfrak{a}_0^*$ that appears in S_λ for some $\lambda \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$ is either of type 1 or of type 2 but cannot be both.

We then define $S_\lambda^1 \subset S_\lambda$ as the subset of exponents of type 1. We define S_λ^2 as the complement of S_λ^1 in S_λ. As follows from remark 4.4.3.2, we have $S_\lambda^1 \cap S_{\lambda'}^2 = \emptyset$ for $\lambda, \lambda' \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$.

4.4.4. Let $w \in {}_RW_P$. We introduce the following expression (the superscript m is for "main"):

$$\mathcal{P}^{T,R,m}(\varphi,\lambda,w) = \sum_{P_0\subset Q\subset R} 2^{-\dim(\mathfrak{a}_Q^R)} \sum_{w'\in W^R(R_w;Q)w} \mathcal{P}^Q(\varphi,\lambda,w') \cdot \frac{\exp(\langle w'\lambda, T_Q^R\rangle)}{\theta_Q^R(w'\lambda)}. \tag{4.4.4.3}$$

If $w = 1$ we set $\mathcal{P}^{T,R,m}(\varphi,\lambda) = \mathcal{P}^{T,R,m}(\varphi,\lambda,1)$.

Proposition 4.4.4.1. — *Let $w \in {}_RW_P$.*

1. *The expression $\mathcal{P}^{T,R,m}(\varphi,\lambda,w)$ is well-defined and holomorphic for λ in a complement of hyperplanes in $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$, resp. in $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$.*
2. *For $\lambda \in \mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$, the map $T \mapsto \mathcal{P}^{T,R,m}(\varphi,\lambda,w)$ coincides with the polynomial exponential given by the summand of exponents of type 1 of $T \mapsto \mathcal{I}^{T,R}(\varphi,\lambda,w)$ (see corollary 4.4.2.2).*
3. *The map $\lambda \mapsto \mathcal{P}^{T,R,m}(\varphi,\lambda,w)$ is holomorphic on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$.*

Proof. — 1. The case of $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$ follows from (the proof of) proposition 4.3.3.6. The case of $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$ is observed in remark 4.4.1.2.

2. It is straightforward to identify the summand of exponents of type 1 in corollary 4.4.2.1 thanks to lemma 4.4.3.1.

3. Once we have identified $\mathcal{P}^{T,R,m}(\varphi,\lambda,w)$ with the summand of exponents of type 1 of $\mathcal{I}^{T,R}(\varphi,\lambda,w)$, the holomorphy on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$ follows from the holomorphy of $\mathcal{I}^{T,R}(\varphi,\lambda,w)$ on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*} \cap \mathcal{R}$ and lemma 2.3.1.1. □

4.5 Singularities of regularized periods

4.5.1. Distinction. — We follow the notations of §3.1.2. Recall that $P = MN_P$ is a standard parabolic subgroup and $\varphi \in \mathcal{A}_{P,\pi}(G)$. We shall say that π is M'-distinguished if the map

$$\psi \in \mathcal{A}_\pi(M) \mapsto \int_{[M']_0} \psi$$

does not vanish identically. We denote by π^* the conjugate dual of π. If π is distinguished then $\pi = \pi^*$: this is well known if π is a cuspidal representation, see the work of Flicker [Fli88], and follow from the work of Yamana [Yam15] in general.

4.5.2. Vanishing of regularized periods. — As a generalization of [JLR99, theorem 23], we show in the next proposition that the regularized period

$$\mathcal{P}(\varphi,\lambda) = \mathcal{P}^G(\varphi,\lambda,1)$$

often vanishes for $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$.

Proposition 4.5.2.1. — *Assume that the period $\mathcal{P}(\varphi, \lambda)$ is well-defined at $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$ and non-zero. Then we are in one of the following situations:*

1. *$P = G$, π is G'-distinguished and $\pi = \pi^*$.*
2. *$n = 2r$, $G = G_{2r}$, $M \simeq G_r \times G_r$ and there exists a discrete automorphic representation σ of G_r such that $\pi = \sigma \boxtimes \sigma^*$.*

Proof. — 1. It results from observations in §4.5.1.

2. Assume $P \subsetneq G$. By assumption and proposition 4.3.3.6, the period $\mathcal{P}(E(\varphi, \lambda))$ is well-defined and non-zero on a non-empty open subset Ω of $\mathfrak{a}_{P,\mathbb{C}}^{G,*}$. The representation π can be written as a restricted tensor product $\otimes_{v \in V} \pi_v$ of irreducible representations of $M(F_v)$. Let v be a finite place of F which is totally split in E. For such a place, $G'(F_v)$ is identified to the diagonal subgroup of $G(F_v) = G'(F_v) \times G'(F_v)$. Locally we get a non-zero and $G'(F_v)$-invariant linear form on the induced representation $\mathrm{Ind}_{P(F_v)}^{G(F_v)}(\pi_\lambda)$. Here $\pi_\lambda(m) = \exp(\langle \lambda, H_P(m) \rangle)\pi(m)$ for $m \in M(F_v)$. Let $\Omega_v \subset \Omega$ be a non-empty open subset such that for $\lambda \in \Omega_v$ the representation $\mathrm{Ind}_{P(F_v)}^{G(F_v)}(\pi_\lambda)$ is irreductible.

According to the identification $M(F_v) = M'(F_v) \times M'(F_v)$, one writes $\pi_v = \pi_1 \boxtimes \pi_2$. For any $\lambda \in \Omega_v$, the induced representation $\mathrm{Ind}_{P'(F_v)}^{G'(F_v)}(\pi_{1,\lambda})$ is isomorphic to the contragredient representation of $\mathrm{Ind}_{P'(F_v)}^{G'(F_v)}(\pi_{2,\lambda})$ and thus there is $w \in W^{G'}(M')$ such that $\pi_{1,\lambda+w\lambda} \simeq w\pi_2^\vee$ where $\pi_2^\vee$ is the contragredient of π_2. Let $w \in W^{G'}(M')$. We view w as an endomorphism of $\mathfrak{a}_{M'}^{G',*}$. If $\mathrm{Ker}(w+\mathrm{Id}) \subsetneq \mathfrak{a}_{M'}^{G',*}$, the set of $\lambda \in \mathfrak{a}_{M'}^{G',*}$ such that $\pi_{1,\lambda+w\lambda} \simeq w\pi_2^\vee$ is a closed subset with empty interior. So there are $\lambda \in \Omega_v$ and $w \in W^{G'}(M')$ such that $\pi_{1,\lambda+w\lambda} \simeq w\pi_2^\vee$ and $\mathrm{Ker}(w + \mathrm{Id}) = \mathfrak{a}_{M'}^{G',*}$. The latter condition implies $n = 2r$ and w switches the two factors of $M' = G'_r \times G'_r$. The relation $\pi_{1,\lambda+w\lambda} \simeq w\pi_2^\vee$ boils down to

$$\pi_1 \simeq w\pi_2^\vee. \tag{4.5.2.1}$$

Globally we have $M = G_r \times G_r$ and we can write $\pi = \sigma \boxtimes \tau$ accordingly. From (4.5.2.1), we infer that $\tau_v \simeq \sigma_v^*$ at any finite place of F which is totally split in E. Thus the automorphic representations τ and σ^* of G_r are isobaric (by the Mœglin-Waldspurger classification of the discrete spectrum, cf. [MW89]) and coincide on the set of places of E of degree 1 over F. It follows from Ramakrishnan's theorem (see [Ram18] theorem A) that $\tau = \sigma^*$ (one can also consult [Ram15] corollary B). □

4.5.3. Singularities. — We consider a set $\mathcal{R}$ as in §4.4.3. Let R be a standard parabolic subgroup and let $w \in {}_RW_P$. Let $\sigma \in \Pi_{\mathrm{disc}}(M_{P_w})$ such that the map $\varphi \mapsto \varphi_{P_w,-\nu_{P_w}}$ sends $\mathcal{A}_{P,\pi}(G)$ to $\mathcal{A}_{P_w,\sigma}(G)$. If moreover $w \in W(P; R)$, we have $P_w = P$ and $\pi = \sigma$. We assume that $\mathcal{R}$ is chosen so that all intertwining operators $M(w', \lambda)$ induces a holomorphic operator $\mathcal{A}_{P_w,\sigma}(G) \to \mathcal{A}_{Q,w'\sigma}(G)$ for all $w' \in \cup_Q W(P_w, Q)$ where Q runs over the standard parabolic subgroups.

Note that $M_{R_w} = wM_{P_w}w^{-1}$. We write $M_{R_w} = G_{n_1} \times \ldots \times G_{n_r}$ and $w\sigma = \sigma_1 \boxtimes \ldots \boxtimes \sigma_r$ accordingly. We write $M_R = G_{m_1} \times \ldots \times G_{m_s}$. Any involution $\xi \in W_2(M_{R_w})$ permutes the blocks G_{n_i}. In this way, it is identified with a permutation of $\{1, \ldots, r\}$.

Theorem 4.5.3.1. —

1. *The regularized period $\mathcal{P}^R(\varphi, \lambda, w)$ vanishes identically on $\mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$ unless there exists an involution $\xi \in W_2^R(M_{R_w})$ such that $\mathfrak{a}_{R_w}^R = \mathfrak{a}_{R_w}^{-\xi}$ and $\xi w\sigma = w\sigma^*$. If $\xi(i) = i$ then the representation σ_i is moreover G'_{n_r}-distinguished.*
2. *On $\mathcal{R} \cap \mathfrak{a}_{P_w,\mathbb{C}}^{G,*}$ the only possible poles of $\mathcal{P}^R(\varphi, \lambda, w)$ are simple and along the hyperplanes $\langle w\lambda, \alpha^\vee \rangle = 0$ where $\alpha \in \Delta_{R_w}^R$.*

3. Let $\alpha \in \Delta^R_{R_w}$. If the hyperplane $\langle w\lambda, \alpha^\vee \rangle = 0$ of $\mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$ is singular for $\mathcal{P}^R(\varphi, \lambda, w)$ then the following conditions are satified:

(a) $s_\alpha w\sigma = w\sigma$ where s_α is the elementary symmetry attached to α and $M(s_\alpha, 0)$ acts by -1 on $\mathcal{A}_{R_w,w\sigma}(G)$.

(b) If $s_\alpha(i) = i+1$ then we have $\sigma_i = \sigma_{i+1}$ and these representations are G'_{n_i}-distinguished.

Remark 4.5.3.2. — The possibility of singular hyperplanes in assertion 3 does not contradict remark 4.4.1.2. Indeed $w^{-1}\alpha^\vee$ induces a non-zero linear form on $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$ for any $\alpha \in \Delta^R_{R_w}$ (otherwise we would have $\alpha^\vee \in \mathfrak{a}_0^{wM_Pw^{-1}} \cap \mathfrak{a}^R_{R_w} = (0)$).

Proof. — By remark 4.3.3.2, we note that we can set $\varphi' = M(w,\lambda)\varphi_{P_w}$, $\lambda' = w\lambda$ and replace P_w by R_w and π by $w\sigma$. Without loss of generality, we see that we may and shall assume $w = 1$, $P \subset R$ and $\pi = \sigma$.

1. By parabolic descent, see remark 4.3.3.2, one is reduced to regularized associated to M_R and thus to proposition 4.5.2.1.

2. By assertion 1, we may and shall assume that there exist $\xi \in W_2(M)$ such that $\mathfrak{a}^R_P = \mathfrak{a}^{-\xi}_P$ and $\xi\pi = \pi^*$. We start from the fact that the expression

$$\mathcal{P}^{T,R,m}(\varphi,\lambda) = \sum_{P_0 \subset Q \subset R} 2^{-\dim(\mathfrak{a}^R_Q)} \sum_{w \in W^R(P;Q)} \mathcal{P}^Q(\varphi,\lambda,w) \cdot \frac{\exp(\langle w\lambda, T^R_Q\rangle)}{\theta^R_Q(w\lambda)}$$

is holomorphic on $\mathcal{R} \cap \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ (see proposition 4.4.4.1). For $w \in W^R(P;Q)$ the parabolic subgroups P and Q_w are R-associated. In our situation this implies $P = Q_w$ and $P \subset Q$. In this case we have $W^R(P;Q) \subset W^R(M)$. Observe also that we have here for $P \subset Q \subset R$ and $w \in W^R(M)$

$$\theta^R_Q(w\lambda) = \varepsilon^R_Q(w)\theta^R_Q(\lambda) \text{ and } \theta^R_Q(\lambda) = \theta^Q_P(\lambda)\theta^R_Q(\lambda)$$

where $\varepsilon^R_Q(w) \in \{\pm 1\}$. So we have:

$$\theta^R_P(\lambda)\mathcal{P}^R(\varphi,\lambda) = \theta^R_P(\lambda)\mathcal{P}^{T,R,m}(\varphi,\lambda)$$
$$- \sum_{P \subset Q \subsetneq R} 2^{-\dim(\mathfrak{a}^R_Q)} \sum_{w \in W^R(P;Q)} \varepsilon^R_P(w)\mathcal{P}^Q(\varphi,\lambda,w)\theta^Q_P(w\lambda)\exp(\langle w\lambda, T^R_Q\rangle).$$

By recursion, we may and shall assume that $\mathcal{P}^Q(\varphi,\lambda,w)\theta^Q_P(w\lambda)$ is holomorphic on $\mathcal{R} \cap \mathfrak{a}^{G,*}_{P,\mathbb{C}}$. It follows that $\theta^R_Q(\lambda)\mathcal{P}^G(\varphi,\lambda)$ is also holomorphic on $\mathcal{R} \cap \mathfrak{a}^{G,*}_{P,\mathbb{C}}$.

3. Let $\alpha \in \Delta^R_P$. Let $P \subset P_\alpha, R_\alpha \subset R$ defined by $\Delta^{P_\alpha}_P = \{\alpha\}$ and $\Delta^{R_\alpha}_P = \Delta^R_P \setminus \{\alpha\}$. Note that

$$\mathfrak{a}^R_P = \mathfrak{a}^{P_\alpha}_P \oplus \mathfrak{a}^{R_\alpha}_P$$

and so $\mathfrak{a}^R_{P_\alpha} = \mathfrak{a}^{R_\alpha}_P$. If one has $P \subset Q \subset R$ then either $Q \subset R_\alpha$ or $P_\alpha \subset Q$. In the former case we have

$$W^R(P;Q) = W^{R_\alpha}(P;Q)W^{P_\alpha}(M)$$

and in the latter case $W^R(P;Q) = W^{R_\alpha}(P;Q_\alpha)$ where $Q_\alpha = R_\alpha \cap Q$. The non trivial element of $W^{P_\alpha}(M)$ is s_α. Then $\mathcal{P}^{T,R,m}(\varphi,\lambda)$ is the sum of the following three contributions:

$$\sum_{P_\alpha \subset Q \subset R} 2^{-\dim(\mathfrak{a}^R_Q)} \sum_{w \in W^{R_\alpha}(P;Q_\alpha)} \mathcal{P}^Q(\varphi,\lambda,w) \cdot \frac{\exp(\langle w\lambda, T^R_Q\rangle)}{\theta^R_Q(w\lambda)} \tag{4.5.3.2}$$

$$\mathcal{P}^{T,R_\alpha,m}(\varphi,\lambda)\frac{\exp(\langle \lambda, T^{P_\alpha}_P\rangle)}{\theta^{P_\alpha}_P(\lambda)} = \tag{4.5.3.3}$$
$$\left(\sum_{P \subset Q \subset R_\alpha} 2^{-\dim(\mathfrak{a}^R_Q)} \sum_{w \in W^{R_\alpha}(P;Q)} \mathcal{P}^Q(\varphi,\lambda,w) \cdot \frac{\exp(\langle w\lambda, T^{R_\alpha}_Q\rangle)}{\theta^{R_\alpha}_Q(w\lambda)}\right) \frac{\exp(\langle \lambda, T^{P_\alpha}_P\rangle)}{\theta^{P_\alpha}_P(\lambda)}$$

$$\mathcal{P}^{T,R_\alpha,m}(M(s_\alpha,\lambda)\varphi,s_\alpha\lambda)\frac{\exp(\langle s_\alpha\lambda,T_P^{P_\alpha}\rangle)}{\theta_P^{P_\alpha}(s_\alpha\lambda)} \tag{4.5.3.4}$$
$$= -\mathcal{P}^{T,R_\alpha,m}(M(s_\alpha,\lambda)\varphi,s_\alpha\lambda)\frac{\exp(-\langle \lambda,T_P^{P_\alpha}\rangle)}{\theta_P^{P_\alpha}(\lambda)}$$

For $P_\alpha\subset Q\subset R$ and $w\in W^{R_\alpha}(P;Q)$ we set $\mathcal{P}_\alpha^Q(\varphi,\lambda,w)=\langle\lambda,\alpha^\vee\rangle\mathcal{P}^Q(\varphi,\lambda,w)$.

Recall that $\mathcal{P}^{T,R,m}(\varphi,\lambda)$ is holomorphic on $\mathcal{R}\cap\mathfrak{a}_{P,\mathbb{C}}^{G,*}$. So $\langle\lambda,\alpha^\vee\rangle\mathcal{P}^{T,R,m}(\varphi,\lambda)$ vanishes for $\lambda\in\mathrm{Ker}(\alpha^\vee)\cap\mathcal{R}\cap\mathfrak{a}_{P,\mathbb{C}}^{G,*}$. Summing the three expressions above and mutiplying them by $\langle\lambda,\alpha^\vee\rangle$ we deduce that we have for such λ:

$$\sum_{P_\alpha\subset Q\subset R}2^{-\dim(\mathfrak{a}_Q^R)}\sum_{w\in W^{R_\alpha}(P;Q_\alpha)}\mathcal{P}_\alpha^Q(\varphi,\lambda,w)\cdot\frac{\exp(\langle w\lambda,T_Q^R\rangle)}{\theta_Q^R(w\lambda)}$$
$$+\mathcal{P}^{T,R_\alpha,m}(\varphi,\lambda)-\mathcal{P}^{T,R_\alpha,m}(M(s_\alpha,0)\varphi,\lambda)=0.$$

This is a polynomial exponential in T. Extracting the part of exponent 0 (which corresponds in the double sum to $Q=R$ and thus $w=1$) we get:

$$\mathcal{P}_\alpha^R(\varphi,\lambda,1)+\mathcal{P}^{R_\alpha}(\varphi,\lambda)-\mathcal{P}^{R_\alpha}(M(s_\alpha,0)\varphi,\lambda)=0.$$

Assume $\langle\lambda,\alpha^\vee\rangle$ is singular for $\mathcal{P}^R(\varphi,\lambda,1)$. Then $\mathcal{P}_\alpha^R(\varphi,\lambda,1)\neq0$ and

$$\mathcal{P}^{R_\alpha}(\varphi,\lambda)\neq\mathcal{P}^{R_\alpha}(M(s_\alpha,0)\varphi,\lambda). \tag{4.5.3.5}$$

By assertion 1, we have $\xi\pi=\pi^*$ for $\xi\in W_2^R(P)$ such that $\mathfrak{a}_P^R=\mathfrak{a}_P^{-\xi}$. If $\mathcal{P}^{R_\alpha}(\varphi,\lambda)\neq0$, resp. $\mathcal{P}^{R_\alpha}(M(s_\alpha,0)\varphi,\lambda)\neq0$, we have $\xi'\pi=\pi^*$, resp. $\xi's_\alpha\pi=s_\alpha\pi^*$, for some $\xi'\in W^{R_\alpha}(P)$ such that $\mathfrak{a}_P^{-\xi'}=\mathfrak{a}_P^{R_\alpha}$. We have $s_\alpha=\xi'\xi$. Since $\mathcal{P}^{R_\alpha}(\varphi,\lambda)$ and $\mathcal{P}^{R_\alpha}(M(s_\alpha,0)\varphi,\lambda)$ cannot both vanish, we have in any case $s_\alpha\pi=\xi'\xi\pi=\xi'\pi^*=\pi$ and the representations $\sigma_i=\sigma_{i+1}$ where i is such that $s_\alpha(i)=i+1$, are G'_{n_i}-distinguished by proposition 4.5.2.1. Since $s_\alpha\pi=\pi$ and $s_\alpha^2=1$, the intertwining operator $M(s_\alpha,0)$ acts by a scalar which must be ±1. By (4.5.3.5), it is -1. □

5 Intertwining periods

5.1 Definition and analytic continuation

5.1.1. Let $P=MN_P$ be a standard parabolic subgroup of G and $\pi\in\Pi_{\mathrm{disc}}(M)$.

5.1.2. Let $\xi\in W_2^{\mathrm{st}}(M)$, $\tilde\xi\in G(F)$ such that $\tilde\xi\iota(\tilde\xi)^{-1}=\xi$ and

$$P_{\tilde\xi}=G'\cap\tilde\xi^{-1}P\tilde\xi.$$

This is an F-subgroup of G' with a Levi decomposition $P_{\tilde\xi}=M_{\tilde\xi}N_{\tilde\xi}$ where $N_{\tilde\xi}=G'\cap\tilde\xi^{-1}N\tilde\xi$ is the unipotent radical of $P_{\tilde\xi}$ and the Levi factor is given by

$$M_{\tilde\xi}=G'\cap\tilde\xi^{-1}M\tilde\xi.$$

The group $M_{\tilde\xi}$ is reductive whereas $N_{\tilde\xi}$ is a unipotent subgroup. Moreover $P_{\tilde\xi}=M_{\tilde\xi}N_{\tilde\xi}$ is a Levi decomposition.

For any $\lambda\in\mathfrak{a}_{P,\mathbb{C}}^{*,-\xi}$, we define the *intertwining period*

$$J(\xi,\varphi,\lambda)=\int_{N_{\tilde\xi}(\mathbb{A})A_{M_{\tilde\xi}}^\infty M_{\tilde\xi}(F)\backslash G'(\mathbb{A})}\varphi_\lambda(\tilde\xi h)\,dh. \tag{5.1.2.1}$$

We need some explanations. We equip the group $N_{\tilde{\xi}}(\mathbb{A})A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F)$ with the right-invariant Haar measure we get by transport of the product measure from the composition map

$$N_{\tilde{\xi}}(\mathbb{A}) \times A^{\infty}_{M_{\tilde{\xi}}} \times M_{\tilde{\xi}}(F) \to N_{\tilde{\xi}}(\mathbb{A})A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F).$$

This measure is not left-invariant and the modular character is given by

$$\delta_{P,\tilde{\xi}} : x \mapsto \exp(\langle \rho_P, H_P(\tilde{\xi}x\tilde{\xi}^{-1})\rangle),$$

see [JLR99, VII p.221]. Thus the integral in (5.1.2.1) has to be understood as a right-$G'(\mathbb{A})$-invariant linear form on the space of $(N_{\tilde{\xi}}(\mathbb{A})A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F), \delta_{P,\tilde{\xi}})$-equivariant functions and this space contains $\varphi_\lambda(\tilde{\xi}\cdot)$ for $\lambda \in \mathfrak{a}^{*,-\xi}_{P,\mathbb{C}}$. So the integral (5.1.2.1) makes sense at least formally. Note also that, for any ξ, the element $\tilde{\xi}$ is unique up to a right translation by an element of $G'(F)$. Such a change does not affect the definition of $J(\xi, \varphi, \lambda)$, hence the notation. We show in the next proposition that the integral (5.1.2.1) is absolutely convergent in some cone.

Proposition 5.1.2.1. — *There exists $c \in \mathbb{R}$ such that for any level $J \subset K_f$, any large $N > 0$ there exists a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that for all $b > c$ there exists $C > 0$ such that*

$$\int_{A^{\infty}_{M_{\tilde{\xi}}}M_{\tilde{\xi}}(F)N_{\tilde{\xi}}(\mathbb{A})\backslash G'(\mathbb{A})} |\varphi_\lambda|(\tilde{\xi}h)\, dh \leqslant C \sum_{i\in I} \|\varphi\|_{-N,X_i} \text{ for all } \varphi \in \mathcal{A}_{P,\pi}(G).$$

for all $\lambda \in \mathfrak{a}^{*,-\xi}_{P,\mathbb{C}}$ such that $b > \Re(\langle \lambda, \alpha^\vee\rangle) > c$ for all $\alpha \in \Delta_P$ such that $\xi\alpha = -\alpha$.

5.1.3. Proof of proposition 5.1.2.1.

— First we prove the following result.

Lemma 5.1.3.1. — *For any level $J \subset K_f$, any large $N > 0$ there exists a finite family $(X_i)_{i\in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ such that*

$$\int_{[M_{\tilde{\xi}}]_0} |\varphi_{-\rho_P}|(\tilde{\xi}mh)\, dm \leqslant \sum_{i\in I} \|\varphi\|_{-N,X_i}$$

for all $h \in G'(\mathbb{A})$ and $\varphi \in \mathcal{A}_{P,\pi}(G)$.

Proof. — Let L be the standard Levi subgroup containing M and defined by $\mathfrak{a}^L_M = \mathfrak{a}^{-\xi}_M$. We may and shall assume that $\tilde{\xi} \in L$. Then $M_{\tilde{\xi}} \subset L$. Let Q be the standard parabolic subgroup of Levi L. Using the Iwasawa decomposition $G'(\mathbb{A}) = N_{Q'}(\mathbb{A})L'(\mathbb{A})K'$, we may assume that $h \in L'(\mathbb{A})K'$. Note that $\sup_{k'\in K'} \|\varphi(\cdot k')\|_{-N,X}$ is bounded by a finite sum of norms $\|\varphi\|_{-N_i,X_i}$ we may even assume that $h \in L'(\mathbb{A})$. The group L is a product of linear factors G_r. Since the discussion is the same for each factor, for simplicity we shall assume that $L = G$. Then there are two cases. First $\xi = 1$ and $M = G$. In this case the proof is close to that of lemma 3.1.2.1: it first uses the inversion formula for the mixed truncation operator, see [JLR99, eq. (19) p.190] and then proposition 4.2.1.1. The details are left to the reader. We inspect the second case more carefully: we have then $G = G_n$ with $n = 2r$, $M = G_r \times G_r$ and ξ is the non-trivial element of $W_2(M)$. One can take (with τ as in §4.1.1)

$$\tilde{\xi} = \begin{pmatrix} I_r & \sqrt{\tau}I_r \\ I_r & -\sqrt{\tau}I_r \end{pmatrix}$$

Then the map $m \mapsto \tilde{\xi}m\tilde{\xi}^{-1}$ identifies $M_{\tilde{\xi}}$ to the embedding $G_r \hookrightarrow G_r \times G_r \subset G$ given by $g \mapsto (g, \iota(g))$. Using the Iwasawa decomposition $G(\mathbb{A}) = N_P(\mathbb{A})M(\mathbb{A})K$ applied to the element $\tilde{\xi}h$, we see that it suffices to prove the existence of $N > 0$ and a family $(X_i)_{i\in I}$ of elements of $\mathfrak{u}_{\mathfrak{m}_\infty}$ such that

$$\int_{[G_r]_0} |\varphi|(gx, \iota(g)y)\, dg \leqslant \sum_I \|\varphi\|_{-N,X_i} \qquad (5.1.3.2)$$

for all $x, y \in G_r(\mathbb{A})$ and $\varphi \in \mathcal{A}_\pi(M)^J$ where J is a fixed level in $K_f \cap M(\mathbb{A})$. By a slight variation on [MW94, proof of lemme I.4.1], for all $\mu \in \mathfrak{a}_0^{P_\pi,*}$ there exist $N > 0$ and $(X_i)_{i\in I}$ as above such that for all $\varphi \in \mathcal{A}_\pi(M)^J$ and $m \in M(\mathbb{A})$ we have

$$(5.1.3.3) \qquad |\varphi(m)| \leqslant \left(\sum_I \|\varphi\|_{-N,X_i}\right) \inf_\delta \exp(\langle \mu + \rho_{P_\pi} + \nu_{P_\pi}, H_0(\delta m)\rangle)$$

where the infimum is taken over the set of $\delta \in M(F)$ such that δm belongs to a fixed Siegel set $\mathfrak{S}^M$ of $M(\mathbb{A})$. Note that if $m \in \mathfrak{S}^M$ then this set is contained in a finite set independent of m.

We write $P_\pi = Q_1 \times Q_2$ and $\nu_{P_\pi} = (\nu_{Q_1}, \nu_{Q_2})$ accordingly. To get (5.1.3.2), we use (5.1.3.3) and Cauchy-Schwartz inequality so that we are reduced to find $\mu \in \mathfrak{a}_{P_{0,r}}^{Q_1,*}$ such that the integral

$$\int_{[G_r]_0} \left(\inf_\delta \exp(\langle \mu + \rho_{Q_1} + \nu_{Q_1}, H_0(\delta(g))\rangle)\right)^2 dg.$$

is convergent where the infimum is taken over $\delta \in M(F)$ such that δm belongs to a fixed Siegel set $\mathfrak{S}^{G_r}$ of $G_r(\mathbb{A})$. In particular we are reduced to show that the integral is finite:

$$\int_{\mathfrak{S}^{G_r}} \exp(\langle 2\mu + 2\rho_{Q_1} + \nu_{Q_1}, H_0(g)\rangle)\, dg.$$

Hence it suffices to find μ as above such that the integral

$$\int_{\mathfrak{a}_{P_{0,r}}} \tau_{P_{0,r}}(H) \exp(\langle 2\mu - 2\rho_{P_{0,r}}^{Q_1} + 2\nu_{Q_1}, H\rangle)\, dH$$

is finite. We must find $\mu \in \mathfrak{a}_{P_{0,r}}^{Q_1,*}$ so that for all $\alpha \in \Delta_{P_{0,r}}^{G_r}$ we have $\langle \mu - \rho_{P_{0,r}}^{Q_1} + \nu_{Q_1}, \varpi_\alpha^\vee\rangle < 0$. If $\alpha \notin \Delta_0^{Q_1}$ we have $\varpi_\alpha^\vee \in \mathfrak{a}_{Q_1}$ and thus

$$\langle \mu - \rho_{P_{0,r}}^{Q_1} + \nu_{Q_1}, \varpi_\alpha^\vee\rangle = \langle \nu_{Q_1}, \varpi_\alpha^\vee\rangle < 0$$

by the explicit formula for ν_{P_π}, see §3.1.2. So it is enough to take μ such that for $\alpha \in \Delta_0^{Q_1}$ we have:

$$\langle \mu, \varpi_\alpha^\vee\rangle < \langle \rho_{P_{0,r}}^{Q_1} - \nu_{Q_1}, \varpi_\alpha^\vee\rangle.$$

□

Using lemma 5.1.3.1, we are reduced to bound

$$\int_{P_{\tilde{\xi}}(\mathbb{A})\backslash G'(\mathbb{A})} \exp(\langle \Re(\lambda) + \rho_P, H_P(\tilde{\xi}h)\rangle)\, dh.$$

We follow notations of the proof of lemma 5.1.3.1. We have a parabolic subgroup Q containing P_ξ. By Iwasawa decomposition, we are reduced to bound

$$\int_{(P_{\tilde{\xi}} \cap M_{Q'})(\mathbb{A})\backslash M_{Q'}(\mathbb{A})} \exp(\langle \Re(\lambda) + \rho_P^Q, H_P(\tilde{\xi}h)\rangle)\, dh.$$

For $\lambda \in \mathfrak{a}_P^{G,*}$ such that for all $\alpha \in \Delta_P^Q$, the real part $\Re(\langle \lambda^Q, \alpha^\vee\rangle)$ is large enough, the above integral is known to be convergent (we are easily reduced to the case of [JLR99, lemma 27]).

5.1.4. Intertwining periods. — Let $Q \in \mathcal{F}_2(M)$, see §2.2.3. There exists a unique pair (R, w) such that $w \cdot Q = R$ is a standard parabolic subgroup of G and $w \in W(P; R)$. Let $L_w = wMw^{-1}$

be the standard Levi component of R_w, see §2.2.1. Then $R \in \mathcal{F}_2(L_w)$. Let $\xi = \xi^R_{L_w} \in W_2^{\mathrm{st}}(L_w)$, see §2.2.3. Then for any $\varphi \in \mathcal{A}_{P,\pi}(G)$ and $\lambda \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ we set

$$J_Q(\varphi, \lambda) = J(\xi, M(w, \lambda)\varphi, (w\lambda)^R) \tag{5.1.4.4}$$

where the right-hand side is the integral (5.1.2.1) written relatively to the standard Levi subgroup L_w and the involution $\xi \in W_2^{\mathrm{st}}(L_w)$. The right-hand side makes sense a priori only for λ in some cone given by proposition 5.1.2.1. We shall show in corollary 5.1.4.2 that the right-hand side in fact admits a meromorphic continuation to $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$.

The intertwining period is closely related to the regularized period of some Eisenstein series. This is the content of the following theorem which generalizes to the case of discrete automorphic representations some of the fundamental results of Jacquet-Lapid-Rogawski (cf. [JLR99]).

Theorem 5.1.4.1. — *Let R be a standard parabolic subgroup and $w \in W(P;R)$ such that $Q = w^{-1} \cdot R \in \mathcal{F}_2(M)$. For λ in some open cone, we have*

$$J_Q(\varphi, \lambda) = \mathcal{P}^R(\varphi, \lambda, w)$$

where the left-hand side is defined by (5.1.4.4) *and the right-hand side is defined in §4.3.3.*

Proof. — The case where π is cuspidal is due to Jacquet-Lapid-Rogawski (see the proof of lemma 32 in [JLR99]). By the definition (5.1.4.4) and the equality (4.3.3.2) of remark 4.3.3.2, we may and shall assume that $w = 1$, the parabolic subgroup Q is equal to R and thus $J_Q(\varphi, \lambda) = J(\xi, \varphi, \lambda^R)$ with $\xi = \xi^R_M$. We shall use the construction given in §5.1.2. Here on can choose $\tilde{\xi} \in M_R(F)$ so that $\tilde{\xi}\iota(\tilde{\xi})^{-1} = \xi$. Then $N_{R'} = N_R \cap G' \subset N_{\tilde{\xi}} \subset R'$ and $M_{\tilde{\xi}} \subset R'$. By Iwasawa decomposition, one can deduce that in some cone we have:

$$J(\xi, \varphi, \lambda^R) = \int_{(N_{\tilde{\xi}} \cap M_{R'})(\mathbb{A}) A^\infty_{M_{\tilde{\xi}}} M_{\tilde{\xi}}(F) \backslash M_{R'}(\mathbb{A})} \int_{K'} \varphi_{\lambda - \rho_R}(\tilde{\xi} hk)\, dk dh.$$

Using the parabolic descent (4.3.3.3), we see that we are reduced to the case $R = G$. If $M = G$ the theorem is an obvious consequence of the definitions, the right-hand side being explicitly described in proposition 4.3.3.6 assertion 4. If $M \subset G$, we are in the case $G = G_{2d}$ and $M = G_d \times G_d$ which is proved in the next section (see corollary 5.2.3.2). □

Corollary 5.1.4.2. — *The intertwining period $J_Q(\varphi, \lambda)$ which was defined on some cone admits a meromorphic continuation to $\mathfrak{a}^{G,*}_{P,\mathbb{C}}$ with hyperplane singularities.*

Proof. — It is a direct consequence of theorem 5.1.4.1 and proposition 4.3.3.6. □

Let R a standard parabolic subgroup and $w \in W(P;R)$. We have defined $\mathcal{P}^{T,R,m}(\varphi, \lambda, w)$ for $\lambda \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$, see (4.4.4.3). The following corollary gives an expression in terms of intertwining periods.

Corollary 5.1.4.3. — *Let $S = w^{-1}Rw$. We have*

$$\mathcal{P}^{T,R,m}(\varphi, \lambda, w) = \sum_{Q \in \mathcal{F}_2^S(M)} 2^{-\dim(\mathfrak{a}_Q^S)} J_Q(\varphi, \lambda) \cdot \frac{\exp(\langle \lambda, T_Q^S \rangle)}{\theta_Q^S(\lambda)}.$$

Proof. — By definition we have

$$\mathcal{P}^{T,R,m}(\varphi, \lambda, w) = \sum_{P_0 \subset Q \subset R} 2^{-\dim(\mathfrak{a}_Q^R)} \sum_{w' \in W^R(R_w;Q)w} \mathcal{P}^Q(\varphi, \lambda, w') \cdot \frac{\exp(\langle w'\lambda, T_Q^R \rangle)}{\theta_Q^R(w'\lambda)}.$$

The map $(Q, w_1) \mapsto w_1^{-1} Q w_1$ induces a bijection from the disjoint union $\bigcup_{P_0 \subset Q \subset R} W^R(R_w; Q)$ onto $\mathcal{F}^R(wMw^{-1})$. Thus we get a bijection from the disjoint union $\bigcup_{P_0 \subset Q \subset R} W^R(R_w; Q)w$ onto $\mathcal{F}^S(M)$. Let (Q, w') be in the source and let $Q' \in \mathcal{F}^S(M)$ be its image. We have

$$J_{Q'}(\varphi, \lambda) = \mathcal{P}^Q(\varphi, \lambda, w').$$

By proposition 4.5.2.1, this regularized period vanishes unless $Q \in \mathcal{F}_2(wMw^{-1})$ and thus $Q' \in \mathcal{F}_2(M)$. The formula follows easily. □

5.2 Computation of a regularized period

5.2.1. In this section, $G = G_{2n}$. Let P be a standard parabolic subgroup of G with standard Levi component $M = G_n \times G_n$.

5.2.2. Let $\pi \in \Pi_{\mathrm{disc}}(G_n)$. By Mœglin-Waldspurger classification of the discrete spectrum reviewed in §3.1.2, one can attach to π two integers $d, r \geqslant 1$ with $dr = n$ and a cuspidal automorphic representation σ of G_r.

Let Q_L be the standard parabolic subgroup of G of type the $2d$-tuple $(r, \ldots, r)$. Its standard Levi component, denoted by L, is naturally identified to G_r^{2d}. The vector space $\mathfrak{a}^*_{L,\mathbb{C}}$ is accordingly identified to $\mathbb{C}^{2d}$. Let

$$\Lambda_L^M = \left(\frac{d-1}{2}, \frac{d-3}{2}, \ldots, -\frac{d-1}{2}, \frac{d-1}{2}, \ldots, -\frac{d-1}{2}\right) \in \mathfrak{a}_L^{P,*}$$

and for any $\lambda = (\lambda_1, \ldots, \lambda_{2d}) \in \mathfrak{a}^*_{L,\mathbb{C}}$

$$D_L^M(\lambda) = \prod_{\alpha \in \Delta_{Q_L}^P} (\langle \lambda, \alpha^\vee \rangle - 1).$$

Let $\varphi \in \mathcal{A}_{Q_L, \sigma^{\otimes d} \times \sigma^{*,\otimes d}}(G)$. For any $\lambda \in \mathfrak{a}^{G,*}_{Q_L,\mathbb{C}}$, one has the Eisenstein series $E^P(\varphi, \lambda)$ defined by analytic continuation from

$$E^P(g, \varphi, \lambda) = \sum_{\delta \in Q_L(F) \backslash P(F)} \exp(\langle \lambda, H_P(\delta g) \rangle) \varphi(\delta g)$$

for $g \in G(\mathbb{A})$. Moreover, for any $g \in G(\mathbb{A})$, the map

$$D_L^M(\lambda) \cdot E^P(g, \varphi, \lambda)$$

is holomorphic at Λ_L^M. Let's denote by $E_{-1}^P(g, \varphi)$ its value at this point. Then $E_{-1}^P(\varphi) \in \mathcal{A}_{P, \pi \otimes \pi^*}(G)$.

5.2.3. Main results. —

Theorem 5.2.3.1. — *We have for all $\varphi \in \mathcal{A}_{Q_L, \sigma^{\otimes d} \times \sigma^{*,\otimes d}}(G)$ and all $\mu \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ in some cone*

(5.2.3.1) $$J_G(E_{-1}^P(\varphi), \mu) = \mathcal{P}^G(E_{-1}^P(\varphi), \mu).$$

where the left-hand side is defined by (5.1.4.4) *and the right-hand side in §4.3.3.*

Proof. — The theorem is the combination of propositions 5.2.6.1 and 5.2.7.2 below whose proofs run over §5.2.4 to §5.2.8. □

Corollary 5.2.3.2. — *For all $\psi \in \mathcal{A}_{P, \pi \times \pi^*}(G)$ and all $\mu \in \mathfrak{a}^{G,*}_{P,\mathbb{C}}$ in some cone we have*

$$J_G(\psi, \mu) = \mathcal{P}^G(\psi, \mu).$$

Proof. — This is an obvious consequence of theorem 5.2.3.1 : both members are continuous for $\mu \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$ in some cone (see proposition 5.1.2.1 and proposition 4.3.3.6) and coincides in the dense subspace generated by the functions $E_{-1}^{P}(\varphi)$. □

5.2.4. By definition, the right-hand side in (5.2.3.1) is the regularized period of the Eisenstein series $E(E_{-1}^{P}(\varphi), \mu)$ where $\mu \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$. We shall follow the same method as in [Off06] or [Yam14] to get a new expression for it. It follows from proposition 4.4.1.1 and remark 4.4.1.2, see also [LR03, proposition 8.4.1], that, at least for generic $\mu \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$, the integral

$$\int_{[G']_0} \Lambda_m^T E(g, E_{-1}^{P}(\varphi), \mu)\, dg, \tag{5.2.4.2}$$

is, as a function of T, a polynomial exponential whose purely polynomial term is in fact constant and equal to $\mathcal{P}^{G}(E_{-1}^{P}(\varphi), \mu)$. To compute this constant term, the starting point is the following lemma.

Lemma 5.2.4.1. —

$$\int_{[G']_0} \Lambda_m^T E(g, E_{-1}^{P}(\varphi), \mu)\, dg = \lim_{\lambda \to \Lambda_L^M} D_M^L(\lambda) \cdot \int_{[G']_0} \Lambda_m^T E(g, \varphi, \lambda + \mu)\, dg.$$

Proof. — First one has

$$E(E_{-1}^{P}(\varphi), \mu) = \lim_{\lambda \to \Lambda_L^M} D_L^M(\lambda) \cdot E(\varphi, \lambda + \mu).$$

One has basically to intervert the limit, the integration and the truncation. This can be easily justified by properties of the mixed truncation operator Λ_m^T (see [Art82] proof of lemma 3.1 for a similar result). □

The second step is to use the following analogue of the "Maaß-Selberg relations".

Proposition 5.2.4.2. —*(Jacquet-Lapid-Rogawski) We have for $\lambda \in \mathfrak{a}_{Q_L,\mathbb{C}}^{G,*}$ in general position*

$$\int_{[G']_0} \Lambda_m^T E(g, \varphi, \lambda)\, dg = \sum_{Q \in \mathcal{F}_2(L)} 2^{-\dim(\mathfrak{a}_Q^G)} J_Q(\varphi, \lambda) \frac{\exp(\langle \lambda, T_Q^G \rangle}{\theta_Q(\lambda)} \tag{5.2.4.3}$$

Proof. — This is just a paraphrase of theorem 40 of [JLR99]. For the sake of clarity, let's comment that the statement can be deduced from proposition 4.4.1.1. Indeed since φ is cuspidal the formula of proposition 4.4.1.1 reduces to the equality:

$$\mathcal{I}^T(\varphi, \lambda) = \mathcal{P}^{T,G,m}(\varphi, \lambda).$$

Now by definition the left-hand side of the formula above is the left-hand side of (5.2.4.3). We can conclude with corollary 5.1.4.3 that $\mathcal{P}^{T,G,m}(\varphi, \lambda)$ is the right-hand side of formula (5.2.4.3). Of course, here we implicitly use theoreme 5.1.4.1 but only for cuspidal representations for which it is due to Jacquet-Lapid-Rogawski. □

We can compute the constant term (as a function of T) in the limit in lemma 5.2.4.1 using the expression (5.2.4.3). From lemma 2.3.1.1, this constant term is obtained from the subsum restricted to the set of $Q \in \mathcal{F}_2(M)$ such that $(\Lambda_L^M + \mu)_Q^G = 0$ for all $\mu \in \mathfrak{a}_P^{G,*}$. We shall provide a description of this. First we shall identify the group $W(L)$ to the symmetric group $\mathfrak{S}_{2d}$ (in $2d$ letters). For any $\sigma \in \mathfrak{S}_d$, let $w_\sigma \in \mathfrak{S}_{2d}$ defined for any $1 \leqslant i \leqslant d$ by

$$w_\sigma(i) = 2\sigma(i) - 1 \text{ and } w_\sigma(2d + 1 - i) = 2\sigma(i).$$

Let $R \subset G$ be the unique standard parabolic subgroup with standard Levi component isomorphic to G_{2r}^d.

Lemma 5.2.4.3. — *The map $\sigma \in \mathfrak{S}_d \mapsto w_\sigma^{-1} R$ induces a bijection from $\mathfrak{S}_d$ onto the subset of $Q \in \mathcal{F}_2(M)$ such that $(\Lambda_L^M + \mu)_Q^G = 0$ for all $\mu \in \mathfrak{a}_P^{G,*}$.*

Proof. — Left to the reader. □

Let $Q = w_\sigma^{-1} \cdot R$ with $\sigma \in \mathfrak{S}_d$. Note that $w_\sigma \in W(Q_L; R)$. Let $\xi = \xi_L^R \in W_2^{\mathrm{st}}(L)$. By the definition of $J_Q(\varphi, \lambda)$, see §5.1.4, we have:

$$J_Q(\varphi, \lambda) = J(\xi, M(w_\sigma, \lambda)\varphi, (w_\sigma \lambda)^R)$$

for $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^{G,*}$. Recall that the right-hand side is given by a convergent integral if $\Re(\langle \lambda, \alpha^\vee \rangle)$ is large enough for all $\alpha \in \Delta_{Q_L}^R$. Thus we get:

Lemma 5.2.4.4. — *For $\mu \in \mathfrak{a}_{P,\mathbb{C}}^{G,*}$, the regularized period $\mathcal{P}^G(E_{-1}^P(\varphi), \mu)$ is equal to*

$$2^{1-d} \lim_{\lambda \to \Lambda_L^M} D_L^M(\lambda) \sum_{\sigma \in \mathfrak{S}_d} J(\xi, M(w_\sigma, \lambda + \mu)\varphi, (w_\sigma(\lambda + \mu))^R) \frac{1}{\theta_R(w_\sigma(\lambda + \mu))} \tag{5.2.4.4}$$

5.2.5. For any $\sigma \in \mathfrak{S}_d$, we have

$$D_L^M(\lambda) = D_{L,\sigma}^{M,+}(\lambda) D_{L,\sigma}^{M,-}(\lambda)$$

where we introduce

$$D_{L,\sigma}^{M,\pm}(\lambda) = \prod_{\alpha \in \Delta_{Q_L}^P, \pm w_\sigma \alpha < 0} (\langle \lambda, \alpha^\vee \rangle - 1).$$

Then the operator $D_{L,\sigma}^{M,+}(\lambda) M(w_\sigma, \lambda + \mu)$ has a limit at $\lambda = \Lambda_L^M$ denoted by

$$M_{-1}(w_\sigma, \mu) = \lim_{\lambda \to \Lambda_L^M} D_{L,\sigma}^{M,+}(\lambda) M(w_\sigma, \lambda + \mu).$$

However in the expression (5.2.4.4) of lemma 5.2.4.4, one cannot permute the limit and the sum. To get around this difficulty, we shall use a directional limit. To do this, we identify naturally $\mathfrak{a}_{L,\mathbb{C}}^*$ with $\mathbb{C}^{2d}$ and we take $\lambda = \Lambda_L^M + \varepsilon x$ where $\varepsilon \in \mathbb{C}$ will go to 0 and $x = (x_1, \ldots, x_{2d}) \in \mathbb{C}^{2d}$ is a fixed point in general position. For any $\sigma \in \mathfrak{S}_d$, we can write

$$D_{L,\sigma}^{M,-}(\lambda) = \varepsilon^{d-1} \prod_{i=1}^{d-1} D_{\sigma,i}^-(x)$$

with

$$D_{\sigma,i}^-(x) = \begin{cases} x_i - x_{i+1} \text{ if } \sigma(i) < \sigma(i+1) \\ x_{2d-i} - x_{2d+1-i} \text{ otherwise.} \end{cases}$$

We also have

$$\theta_R(w_\sigma(\lambda + \mu)) = \mathrm{vol}(\mathfrak{a}_R^G / \mathbb{Z}(\Delta_R^\vee))^{-1} (\varepsilon/2)^{d-1} \theta_{R,\sigma}(x)$$

with

$$\theta_{R,\sigma}(x) = \prod_{i=1}^{d-1} ((x_{\sigma^{-1}(i)} - x_{\sigma^{-1}(i+1)}) + (x_{2d+1-\sigma^{-1}(i)} - x_{2d+1-\sigma^{-1}(i+1)})).$$

Using the continuity of $J(\xi, \varphi, \lambda)$ provided by proposition 5.1.2.1, we get:

Lemma 5.2.5.1. — *Let α_0 be the unique element of Δ_P. For any $\mu \in \mathfrak{a}_P^{G,*}$ with large $\Re(\langle \mu, \alpha_0^\vee \rangle)$, the limit* (5.2.4.4) *is equal to*

$$\mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee)) \sum_{\sigma \in \mathfrak{S}_d} J(\xi, M_{-1}(w_\sigma, \mu)\varphi, (w_\sigma(\Lambda_L^M + \mu))^R) \frac{\prod_{i=1}^{d-1} D_{\sigma,i}^-(x)}{\theta_{R,\sigma}(x)}. \tag{5.2.5.5}$$

Proof. — Let $\sigma \in \mathfrak{S}_d$ and let $\tau = w_\sigma(\sigma^{\otimes d} \times \sigma^{*,\otimes d})$. By proposition 5.1.2.1, for $\Re(\langle \mu, \alpha_0^\vee \rangle)$ is large enough, the integral that defines the map

$$(\psi, \lambda) \in \mathcal{A}_{Q_L}(G) \times \mathfrak{a}_{L,\mathbb{C}}^{G,*} \mapsto J(\xi, \psi, (w_\sigma(\lambda + \mu))^R)$$

is convergent and continuous uniformly for λ in a compact neighborhood of Λ_L^M. One deduced that for $\lambda = \Lambda_L^M + \varepsilon x$, one has:

$$\lim_{\varepsilon \to 0} J(\xi, D_{L,\sigma}^{M,+}(\lambda) M(w_\sigma, \lambda + \mu)\varphi, (w_\sigma(\lambda + \mu))^R) = J(\xi, M_{-1}(w_\sigma, \mu)\varphi, (w_\sigma(\Lambda_L^M + \mu))^R).$$

The lemma follows. □

5.2.6. Of course, the expression (5.2.5.5) does not depend on a particular choice of x. It is also regular in a non-empty open subset of the subspace defined by $x_1 = x_2 = \ldots = x_d = 0$. Moreover on such a subset, all the terms in (5.2.5.5) vanish but the one corresponding to the permutation $\sigma_0 \in \mathfrak{S}_d$ that satisfies $\sigma_0(1) > \sigma_0(2) > \ldots > \sigma_0(d)$ that is $\sigma_0(i) = d + 1 - i$ for $1 \leqslant i \leqslant d$. We set

$$w_0 = w_{\sigma_0}.$$

Since we have

$$\frac{\prod_{i=1}^{d-1} D_{\sigma_0,i}^-(x)}{\theta_{R,\sigma_0}(x)} = 1,$$

on such a subset. We observe that $w_0(\Lambda_L^M + \mu) \in \mathfrak{a}_L^{R,*}$. We get:

Proposition 5.2.6.1. — *For any $\mu \in \mathfrak{a}_P^{G,*}$ with large $\Re(\langle \mu, \alpha_0^\vee \rangle)$, we have*

$$\mathcal{P}(E_{-1}^P(\varphi), \mu) = \mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee)) J(\xi, M_{-1}(w_0, \mu)\varphi, w_0(\Lambda_L^M + \mu)).$$

for any $\mu \in \mathfrak{a}_P^{G,*}$ such that both sides are regular at μ.

5.2.7. Let $w_1 \in W(L)$ be the element that corresponds to the permutation in $\mathfrak{S}_{2d}$ defined by

$$w_1(i) = \begin{cases} d + 1 - i \text{ if } 1 \leqslant i \leqslant d \\ i \text{ if } d + 1 \leqslant i \leqslant 2d. \end{cases}$$

Let $w_2 = w_0 w_1$. Since $w_1^2 = 1$, one also has $w_2 w_1 = w_0$. For $1 \leqslant i \leqslant d$ we have $w_2(i) = 2i - 1$ and $w_2(2d + 1 - i) = 2(d + 1 - i)$.

Let $S = w_2^{-1} R w_2$. Then $S \in \mathcal{F}_2(L)$ and $\xi_L^S = w_2^{-1} \xi w_2$. We can write ξ_L^S as a product of transpositions with disjoint supports:

$$\xi_L^S = (1\ d + 1) \cdot (2\ d + 2) \cdots (d\ 2d).$$

Let

$$\nu = \begin{pmatrix} I_n & \sqrt{\tau} I_n \\ I_n & -\sqrt{\tau} I_n \end{pmatrix}$$

so that $\nu \iota(\nu)^{-1} = \xi_L^S$.

Let $S_\nu = \nu^{-1} Q_L \nu \cap G'$, $L_\nu = \nu^{-1} L \nu \cap G'$, $M_\nu = \nu^{-1} M \nu \cap G'$and $N_\nu = \nu^{-1} N_{Q_L} \nu \cap G'$. Note that N_ν is the unipotent radical of S_ν and L_ν is a Levi factor of S_ν. Moreover M_ν is reductive and contains S_ν.

We define for any function $\psi \in \mathcal{A}^0_{Q_L}(G)$ which is cuspidal and $\lambda \in \mathfrak{a}^{S,*}_{L,\mathbb{C}}$,

$$(5.2.7.6) \qquad J(\xi^S_L, \psi, \lambda) = \int_{N_\nu(\mathbb{A}) A^\infty_{L_\nu} L_\nu(F)\backslash G'(\mathbb{A})} \psi_\lambda(\nu h)\, dh.$$

The next proposition shows that the integrals is convergent in some region and admits otherwise an analytic continuation.

Proposition 5.2.7.1. — *(Jacquet-Lapid-Rogawski) Let $\psi \in \mathcal{A}^0_{Q_L}(G)$ be a cuspidal automorphic form. For any $\lambda \in \mathfrak{a}^{S,*}_{L,\mathbb{C}}$ such that $\Re(\langle \lambda, \alpha_0^\vee\rangle)$ is large enough for all roots of A_L in N_P the integral in the right-hand side of* (5.2.7.6) *is convergent and satisfies:*

$$(5.2.7.7) \qquad J(\xi^S_L, \psi, \lambda) = J(\xi, M(w_2, \lambda)\psi, w_2\lambda).$$

Proof. — This is a consequence of [JLR99, theorem 31 and proposition 34]. □

One can check that

$$D^{M,+}_{L,\sigma_0}(\lambda) = \prod_{\alpha \in \Delta^P_{Q_L}, w_1\alpha < 0} (\langle \lambda, \alpha^\vee\rangle - 1).$$

In particular, we can define for $\mu \in \mathfrak{a}^{G,*}_P$

$$M_{-1}(w_1, \mu) = \lim_{\lambda \to \Lambda^M_L} D^{M,+}_{L,\sigma_0}(\lambda) M(w_1, \lambda + \mu).$$

By the functional equation of intertwining operators, we have

$$M(w_0, \lambda + \mu) = M(w_2 w_1, \lambda + \mu) = M(w_2, w_1(\lambda + \mu)) M(w_1, \lambda).$$

By multiplying by the factor $D^{M,+}_{L,\sigma_0}(\lambda)$ and taking the limit $\lambda \to \Lambda^M_L$ we get

$$M_{-1}(w_0, \mu) = M(w_2, w_1(\Lambda^M_L + \mu)) M_{-1}(w_1).$$

where $M_{-1}(w_1) = \lim_{\lambda \to \Lambda^M_L} D^{M,+}_{L,\sigma_0}(\lambda) M(w_1, \lambda)$. Note that $w_1(\Lambda^M_L + \mu)$ belongs to the cone in $\mathfrak{a}^{S,*}_{L,\mathbb{C}}$ introduced in proposition 5.2.7.1 as soon as $\Re(\langle \mu, \alpha_0^\vee\rangle)$ is large. Thus by (5.2.7.7) we have:

$$(5.2.7.8) \quad \begin{aligned} J(\xi, M_{-1}(w_0, \mu)\varphi, w_0(\Lambda^M_L + \mu)) &= J(\xi, M(w_2, w_1\Lambda^M_L + \mu) M_{-1}(w_1)\varphi, w_2 w_1(\Lambda^M_L + \mu)) \\ &= J(\xi^S_L, M_{-1}(w_1)\varphi, w_1(\Lambda^M_L + \mu)). \end{aligned}$$

Proposition 5.2.7.2. — *For any $\mu \in \mathfrak{a}^{G,*}_P$ with large $\Re(\langle \mu, \alpha_0^\vee\rangle)$ we have*

$$J(\xi, M_{-1}(w_0, \mu)\varphi, w_0(\Lambda^M_L + \mu)) = \mathrm{vol}(\mathfrak{a}^G_R/\mathbb{Z}(\Delta^\vee_R))^{-1} J_G(E^P_{-1}(\varphi), \mu).$$

Proof. — Using (5.2.7.8), we see that it amounts to show the equality:

$$J(\xi^S_L, M_{-1}(w_1)\varphi, w_1(\Lambda^M_L + \mu)) = \mathrm{vol}(\mathfrak{a}^G_R/\mathbb{Z}(\Delta^\vee_R))^{-1} J_G(E^P_{-1}(\varphi), \mu).$$

If $\Re(\langle \mu, \alpha_0^\vee\rangle)$ is large enough, the left-hand side is given by the following convergent integral:

$$\int_{N_\nu(\mathbb{A}) A^\infty_{L_\nu} L_\nu(F)\backslash G'(\mathbb{A})} \exp(\langle w_1(\Lambda^M_L + \mu), H_{Q_L}(\nu g)\rangle) M_{-1}(w_1)\varphi(\nu g)\, dg$$

$$= \int_{M_\nu(\mathbb{A})\backslash G'(\mathbb{A})} \int_{N_\nu(\mathbb{A}) A^\infty_{L_\nu} L_\nu(F)\backslash M_\nu(\mathbb{A})} \exp(\langle w_1(\Lambda^M_L + \mu), H_{Q_L}(\nu h g)\rangle) M_{-1}(w_1)\varphi(\nu h g)\, dh dg.$$

Note that $w_1\mu = \mu$ and $\langle \mu, H_{Q_L}(\nu hg)\rangle = \langle \mu, H_P(\nu hg)\rangle = \langle \mu, H_P(\nu g)\rangle$. Thus, by proposition 5.2.8.1 below, the last line is equal to (note that $w_1\mu = \mu$)

$$\begin{aligned}
\mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))^{-1} \int_{M_\nu(\mathbb{A})\backslash G'(\mathbb{A})} \exp(\langle \mu, H_P(\nu g)\rangle) \int_{[M_\nu]_0} E_{-1}^P(\varphi, \nu hg)\, dhdg \\
= \mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))^{-1} \int_{A_{M_\nu}^\infty M_\nu(F)\backslash G'(\mathbb{A})} \exp(\langle \mu, H_P(\nu g)\rangle) E_{-1}^P(\varphi, \nu g)\, dg \\
= \mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))^{-1} J_G(E_{-1}^P(\varphi), \mu).
\end{aligned}$$

□

5.2.8.

Proposition 5.2.8.1. — *For all $g \in G(\mathbb{A})$, the integral*

$$\int_{N_\nu(\mathbb{A})A_{L_\nu}^\infty L_\nu(F)\backslash M_\nu(\mathbb{A})} \exp(\langle w_1\Lambda_L^M, H_{Q_L}(\nu hg)\rangle) M_{-1}(w_1)\varphi(\nu hg)\, dh \tag{5.2.8.9}$$

is equal to

$$\mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))^{-1} \int_{[M_\nu]_0} E_{-1}^P(\varphi, \nu hg)\, dh \tag{5.2.8.10}$$

Proof. — We have $\nu M_\nu \nu^{-1} \subset M$. This embedding is identified to the embedding $\delta : G_n \to G_n \times G_n$ given by $m \mapsto \delta(m) = (m, \iota(m))$. The parabolic subgroup $\nu S_\nu \nu^{-1}$ of $\nu M_\nu \nu^{-1}$ and its Levi factor $\nu L_\nu \nu^{-1}$ are identified to corresponding subgroups of G_n respectively denoted by Q_1 and L_1. Thus the integrals (5.2.8.9) and (5.2.8.10) can be written respectively:

$$\int_{[G_n]_{Q_1,0}} \exp(\langle w_1\Lambda_L^M, H_{Q_L}(\delta(h)\nu g)\rangle) M_{-1}(w_1, \mu)\varphi(\delta(h)\nu g)\, dh$$

and

$$\mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))^{-1} \int_{[G_n]_0} E_{-1}^P(\varphi, \delta(h)\nu g)\, dh. \tag{5.2.8.11}$$

We shall show the equality of the two expressions above. From now on we may and we shall replace νg by g.

Let $T \in \mathfrak{a}_{P_{0,n},\mathbb{R}}^{G_n}$. On the group G_n, we have the Arthur's truncation operator denoted by Λ^T, see section 3. For any function ψ on $M(\mathbb{A}) = G_n(\mathbb{A}) \times G_n(\mathbb{A})$, we denote by $\Lambda_1^T\psi$ the function on $M(\mathbb{A})$ we get by applying the operator Λ^T on the first variable. When $\psi = E_{-1}^P(\cdot g, \varphi)$, resp. $\psi = E^P(\cdot g, \varphi, \lambda)$, the function $\Lambda_1^T\psi$ evaluated at $\delta(m)$ (for $m \in G_n(\mathbb{A})$) is denoted by $\Lambda_1^T E_{-1}^P(\delta(m)g, \varphi)$, resp. $\Lambda_1^T E^P)(\delta(m)g, \varphi, \lambda)$.

The integral in (5.2.8.11) (with νg replaced by g) is the constant term of the polynomial exponential in the variable T given by

$$\begin{aligned}
&\int_{[G_n]_0} \Lambda_1^T E_{-1}^P(\delta(h)g), \varphi)\, dh \\
&= \lim_{\lambda \to \Lambda_L^M} D_L^M(\lambda) \cdot \int_{[G_n]_0} (\Lambda_1^T E^P)(\delta(m)g, \varphi, \lambda)\, dm.
\end{aligned} \tag{5.2.8.12}$$

We also denote by δ the diagonal embedding of $\mathfrak{a}_{P_{0,n}}^{G_n}$ in $\mathfrak{a}_{P_{0,2n}}^{G_{2n}}$ and a dual projection δ^*. For $\lambda \in \mathfrak{a}_{L,\mathbb{C}}^*$, the theorem 3.1.3.1 takes in our situation the following form (this can also be deduced from [Art05, lemma 15.2] which gives an explicit expression of a truncated cuspidal Eisenstein series)

$$\int_{[G_n]_0} (\Lambda_1^T E^P)(\delta(m)g,\varphi,\lambda)\,dm$$

$$= \sum_{w\in W^M(L)} \int_{[G_n]_{Q_1,0}} \exp(\langle w\lambda, H_{Q_L}(k(h)g)\rangle)(M(w,\lambda)\varphi)(\delta(h)g)\,dh\cdot \frac{\exp(\langle w\lambda,\delta(T)\rangle)}{\theta_{Q_1}(w\lambda)}$$

where we choose $k\in K_n$ such that $hk^{-1}\in Q_1(\mathbb{A})$ and we set $k(h)=\delta(k)$. Note that thanks to δ^* we view θ_{Q_1} as a polynomial function on $\mathfrak{a}_{P_{0,2n}}^{G_{2n}}$. The constant term in T in the right-hand side of (5.2.8.12) corresponds in the above sum to the contributions of the elements w that satisfy $w\Lambda_L^M=0$ that is $w\in W^{G_n}(L_1)w_1$ where $W^{G_n}(L_1)$ is identified to the diagonal subgroup of $W^M(L)$. By lemma 2.3.1.1, this constant term is given by:

$$\lim_{\lambda\to\Lambda_L^M} D_L^M(\lambda) \sum_{w\in W^{G_n}(L_1)} \int_{[G_n]_{Q_1,0}} \exp(\langle ww_1\lambda, H_{Q_L}(k(h)g)\rangle)(M(ww_1,\lambda)\varphi)(\delta(h)g)\,dh\frac{1}{\theta_{Q_1}(ww_1\lambda)}.$$

As we did previously, we take $\lambda=\Lambda_L^M+\varepsilon x$ with $x=(x_1,\dots,x_d,x'_1,\dots,x'_d)$. If x is generic, one can take the limit $\varepsilon\to 0$ term by term. We identify $W^{G_n}(L_1)$ with $\mathfrak{S}_d$. Let's consider

$$\theta_w(x)=\prod_{i=1}^{d-1}(x_{w^{-1}(d+1-i)}-x_{w^{-1}(d-i)}+x'_{w^{-1}(i)}-x'_{w^{-1}(i+1)}).$$

We have $D_L^M(\lambda)=D_w^+(x)D_w^-(x)$ where we set $D_w^\pm(x)=\prod_{i=1}^{d-1}D_{w,i}^\pm(x)$ with

$$D_{w,i}^\pm(x)=\begin{cases} x_i-x_{i+1} \text{ if } \pm(w(i)-w(i+1))>0\\ x'_i-x'_{i+1} \text{ otherwise.}\end{cases}$$

We have

$$\theta_S(ww_1\lambda)=\mathrm{vol}(\mathfrak{a}_{Q_1}^{G_n}/\mathbb{Z}(\Delta_{Q_1}^\vee))^{-1}\varepsilon^{d-1}\theta_w(x).$$

If x is generic, the above limit is the product of $\mathrm{vol}(\mathfrak{a}_{Q_1}^{G_n}/\mathbb{Z}(\Delta_{Q_1}^\vee))$ and

$$\sum_{w\in W^{G_1}(M_1)} D_w^-(x)\left(\int_{[G_n]_{Q_1,0}} \exp(\langle ww_1\Lambda_L^M, H_{Q_L}(k(h)g)\rangle)(M_{-1}(ww_1)\varphi)(\delta(h)g)\,dh\right)\theta_w(x)^{-1}$$

where $M_{-1}(ww_1)=\lim_{\varepsilon\to 0}\varepsilon^{d-1}D_w^+(x)M(ww_1,\lambda)$. We can go further and take $x_1=x_2=\ldots=x_d=0$. Then all the terms vanish but the one associated to $w=1$ which gives

$$\mathrm{vol}(\mathfrak{a}_{Q_1}^{G_n}/\mathbb{Z}(\Delta_{Q_1}^\vee))\int_{[G_n]_{Q_1,0}} \exp(\langle w_1\Lambda_L^M, H_{Q_L}(k(h)g)\rangle)(M_{-1}(w_1)\varphi)(\delta(h)g)\,dh.$$

This finishes the proof since on one hand we have $\mathrm{vol}(\mathfrak{a}_{Q_1}^{G_n}/\mathbb{Z}(\Delta_{Q_1}^\vee))=\mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^\vee))$ and on the other hand we have

$$\langle w_1\Lambda_L^M, H_{Q_L}(k(h)g)\rangle=\langle w_1\Lambda_L^M, H_{Q_L}(\delta(h)g)\rangle$$

that is $\langle w_1\Lambda_L^M, H_{Q_L}(\delta(h)g)\rangle=0$ for any $h\in Q_1(\mathbb{A})$. $\square$

6 Relative characters

6.1 A first example

6.1.1. We consider the situation and notations of sections 4 and 5. So the group G is $\mathrm{Res}_{E/F}\,\mathrm{GL}_E(n)$ for some fixed quadratic extension E/F. We will also use the notations of section 3 relative to this group. Let $P = MN_P$ be a standard parabolic subgroup of G and $\pi \in \Pi_{\mathrm{disc}}(M)$.

Let J be a level and T be a truncation parameter. Let $\tau \in \hat{K}_\infty$ and $\mathcal{B}_{P,\pi}(\tau, J)$ be an orthonormal basis for the Petersson norm of the finite dimensional space $\mathcal{A}_{P,\pi}(G)^{\tau,J}$. Let Q be a standard parabolic subgroup of G and $w \in {}_QW_P$. For $\lambda \in \mathfrak{a}^{G,*}_{P_w,\mathbb{C}}$, $x \in G(\mathbb{A})$ and $f, f' \in \mathcal{S}(G(\mathbb{A}))^J$ we set

$$\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x, f, f', \lambda, w) = \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} E(x, I_{P,\pi}(\lambda, f)\varphi, \lambda)\overline{\mathcal{I}^{T,Q}(I_{P,\pi}(-\bar{\lambda}, f')\varphi, -\bar{\lambda}, w)}.$$

Remark 6.1.1.1. — The sum is finite and does not depend on the choice of an orthonormal basis. As in remark 3.3.2.1, we shall view $\mathcal{E}$ as a sequilinear form on $\mathcal{S}(G(\mathbb{A}))$.

We also set

$$\mathcal{E}^{T,Q}_{P,\pi}(x, f, f', \lambda, w) = \sum_{\tau \in \hat{K}_\infty} \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x, f, f', \lambda, w).$$

Proposition 6.1.1.2. — *There exist $l > 0$ and $N > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that for all J-pairs (P, π), all $f, f' \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$, all $x \in G(\mathbb{A})^1$, all standard parabolic subgroups Q and all $w \in {}_QW_P$ we have*

$$\sum_{\tau \in \hat{K}_\infty} \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} |E(x, I_{P,\pi}(\lambda, f)\varphi, \lambda)\overline{\mathcal{I}^{T,Q}(I_{P,\pi}(-\bar{\lambda}, f')\varphi, -\bar{\lambda}, w)}| \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}} \|f'\|_{\mathcal{S}}}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}.$$

Remark 6.1.1.3. — As it follows from the proof we may assume that $E(x, \varphi, \lambda)$ and $\overline{\mathcal{I}^{T,Q}(\varphi, -\bar{\lambda}, w)}$ are holomorphic on $\mathcal{R}_{\pi,c,l}$.

Proof. — Using Cauchy-Schwartz inequality it suffices to show that the following two expressions satisfy the conclusion of the theorem (with $N = 0$ for the second expression)

$$\sum_{\tau \in \hat{K}_\infty} \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} |E(x, I_{P,\pi}(\lambda, f)\varphi, \lambda)|^2 \tag{6.1.1.1}$$

and

$$\sum_{\tau \in \hat{K}_\infty} \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} |\mathcal{I}^{T,Q}(I_{P,\pi}(-\bar{\lambda}, f')\varphi, -\bar{\lambda}, w)|^2 \tag{6.1.1.2}$$

For (6.1.1.1), this is the content of theorem 3.4.2.1. To bound the expression (6.1.1.2), we shall use theorem 3.4.3.1 in combination with proposition 4.2.1.1 and the fact that the map $g \mapsto \|g\|_Q^{-N}$ is integrable for large N on $[G']^1_{Q'}$. Thus there exists $l > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$, a finite family $(X_i)_{i \in I}$ of elements of $\mathcal{U}(\mathfrak{g}_\infty)$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A})^J)$ such that for all J-pairs (P, π), all $f, f' \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$, all $x \in G(\mathbb{A})^1$, all standard parabolic subgroups Q and all $w \in {}_QW_P$ we have that (6.1.1.2) is bounded by

$$\frac{\sum_{i \in I} \|L(X_i) f\|^2_{\mathcal{S}}}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}.$$

The conclusion is clear. □

6.1.2. With notations as above, we introduce also the linear form

$$\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w) = \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} E(x, I_{P,\pi}(\lambda,f)\varphi,\lambda)\overline{\mathcal{I}^{T,Q}(\varphi,-\bar{\lambda},w)}.$$

and

$$\mathcal{E}^{T,Q}_{P,\pi}(x,f,\lambda,w) = \sum_{\tau\in\hat{K}_\infty} \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w).$$

Remark 6.1.2.1. — As before the definition of $\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w)$ does not depend on the choice of $\mathcal{B}_{P,\pi}(\tau,J)$ and we may view it as a linear form on $S(G(\mathbb{A}))$.

If $Q=G$ then $w=1$ and we remove G and w from the notation: we simply write $\mathcal{E}^{T}_{P,\pi}(x,f,\lambda)$ and $\mathcal{E}^{T}_{(P,\pi,\tau)}(x,f,\lambda)$.

Proposition 6.1.2.2. — *There exist $l>0$ and $N>0$ such that for all $q>0$ and all levels J there exist $c>0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that for all J-pairs (P,π), all $f\in\mathcal{S}(G(\mathbb{A}))^J$, all $\lambda\in\mathcal{R}_{\pi,c,l}$, all $x\in G(\mathbb{A})^1$, all standard parabolic subgroups Q and all $w\in {}_QW_P$ we have*

$$\sum_{\tau\in\hat{K}_\infty} |\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w)| \leqslant \frac{\|x\|_G^N\|f\|_{\mathcal{S}}}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}$$

Moreover $\lambda\mapsto\mathcal{E}^{T,Q}_{P,\pi}(x,f,\lambda,w)$ is holomorphic on $\mathcal{R}_{\pi,c,l}$.

Proof. — As in the proof of theorem 3.4.2.1, for a level J and an integer $m\geqslant 1$ large enough, we can find $Z\in\mathcal{U}(\mathfrak{g}_\infty)$, $g_1\in C_c^\infty(G(\mathbb{A}))$ and $g_2\in C_c^m(G(\mathbb{A}))$ such that

- Z is invariant under K_∞-conjugation ;
- g_1 and g_2 are invariant under K-conjugation and J-biinvariant;
- for any $f\in\mathcal{S}(G(\mathbb{A}))^J$ we have:

$$f = f*g_1 + f*Z*g_2.$$

For $i=1,2$ we can observe that $I(\lambda,g_i)$ preserves the subspace $\mathcal{A}^{\tau,J}_{P,\pi}$ for any $\tau\in\hat{K}_\infty$. It follows that we have:

$$\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w) = \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,g_1^\vee,\lambda,w) + \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f*Z,g_2^\vee,\lambda,w).$$

Then the proposition follows from proposition 6.1.1.2 and its mild extension to the subspace $C_c^m(G(\mathbb{A}))$ as long as m is large enough. The holomorphy of $\mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w)$ is clear on $\mathcal{R}_{\pi,c,l}$ since it is a finite sum of product of holomorphic functions (see remark 6.1.1.3). The bound shows that the convergence of $\sum_\tau \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w)$ is uniform on compact subsets of $\mathcal{R}_{\pi,c,l}$ thus it is also holomorphic. □

6.2 A second example

6.2.1. We keep the notations of § 6.1.1. For any standard parabolic subgroups R and $w \in {}_RW_P$ we set

$$\mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w) = \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} E(x, I_P(\lambda,f)\varphi,\lambda)\overline{\mathcal{P}^R(\varphi,-\bar{\lambda},w)} \tag{6.2.1.1}$$

and

$$\mathcal{P}^{T,R,m}_{(P,\pi,\tau)}(x,f,\lambda,w) = \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} E(x, I_P(\lambda,f)\varphi,\lambda)\overline{\mathcal{P}^{T,R,m}(\varphi,-\bar{\lambda},w)}. \tag{6.2.1.2}$$

Then we also set:

$$\begin{aligned}\mathcal{P}^R_{P,\pi}(x,f,\lambda,w) &= \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} \mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w)\\ \mathcal{P}^{T,R,m}_{P,\pi}(x,f,\lambda,w) &= \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} \mathcal{P}^{T,R,m}_{(P,\pi,\tau)}(x,f,\lambda,w).\end{aligned}$$

As usual if $R = G$ then $w = 1$ and we remove G and w from the notation.

Proposition 6.2.1.1. — *Let $\varepsilon > 0$. There exist $l, N, \eta > 0$ such that for all $q > 0$ and all levels J there exist $c > 0$ and a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that for all J-pairs (P,π), all $f \in \mathcal{S}(G(\mathbb{A}))^J$, all $\lambda \in \mathcal{R}_{\pi,c,l}$, all $x \in G(\mathbb{A})^1$, all standard parabolic subgroups R and all $w \in W(P;R)$ we have*

$$\sum_{\tau\in\hat{K}_\infty} |\mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w)| \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}}}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}$$

$$\sum_{\tau\in\hat{K}_\infty} |\mathcal{E}^T_{(P,\pi,\tau)}(x,f,\lambda) - \mathcal{P}^{T,m}_{(P,\pi,\tau)}(x,f,\lambda)| \leqslant \frac{\exp(-\varepsilon\|T\|)\|x\|_G^N \|f\|_{\mathcal{S}}}{(1+\|\lambda\|^2)^q(1+\Lambda_\pi^2)^q}$$

for all T such that $d(T) \geqslant \eta\|T\|$. Moreover the expressions $\mathcal{E}^T_{P,\pi}(x,f,\lambda)$, $\mathcal{P}^{T,m}_{P,\pi}(x,f,\lambda,w)$ and $\mathcal{P}^R_{P,\pi}(x,f,\lambda,w)$ are holomorphic on $\mathcal{R}_{\pi,c,l}$.

Proof. — Using remark 6.1.1.3 we get l and c such that the Eisenstein series $E(x, I_P(\lambda,f)\varphi,\lambda)$ and all $\overline{\mathcal{I}^{T,Q}(\varphi,-\bar{\lambda},w)}$ are holomorphic on $\mathcal{R}_{\pi,c,l}$. Let R be a standard parabolic subgroup and $w \in W(P;R)$. By theorem 4.5.3.1, the possible singularities of $\overline{\mathcal{P}^R(\varphi,-\bar{\lambda},w)}$ on $\mathcal{R}_{\pi,cl}$ are simple and along the hyperplanes $\langle w\lambda,\alpha^\vee\rangle = 0$ where $\alpha \in \Delta^R_{R_w}$ and for such a root α, the elementary symmetry s_α is such that $M(s_\alpha,0)$ acts by -1 on $\mathcal{A}_{R_w,w\pi}(G)$. But by the functional equation of Eisenstein series, we have:

$$E(x,\varphi,\lambda) = E(x, M(s_\alpha w,\lambda)\varphi, s_\alpha w\lambda).$$

For $\lambda \in \mathcal{R}_{\pi,c,l}$ such that $\langle w\lambda,\alpha^\vee\rangle = 0$ we have $s_\alpha w\lambda = w\lambda$ and we get:

$$\begin{aligned}E(x,\varphi,\lambda) &= E(x, M(s_\alpha,0)M(w,\lambda)\varphi, s_\alpha w\lambda)\\ &= -E(x, M(w,\lambda)\varphi, w\lambda)\\ &= -E(x,\varphi,\lambda)\end{aligned}$$

and thus $E(x,\varphi,\lambda)$ vanishes on the hyperplane $\langle w\lambda,\alpha^\vee\rangle = 0$. Thus the possible singularities cancel with the zeroes of the Eisenstein series. We deduce that $\mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w)$ is holomorphic. We want to bound it. For this we appeal to the very definition (4.3.3.1) of the regularized period to write:

$$\mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w) = \sum_{P_0\subset Q\subset R} (-2)^{-\dim(\mathfrak{a}^R_Q)} \sum_{w'\in {}_QW^R_{R_w}w} \mathcal{E}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w')\cdot\frac{\exp(\langle w'(-\lambda+\nu^w_{w'}), T^R_Q\rangle)}{\hat{\theta}^R_Q(w'(-\lambda+\nu^w_{w'}))}$$

where T is an arbitrary truncation parameter. Since we know the holomorphy of $\mathcal{P}^R_{(P,\pi,\tau)}(x,f,\lambda,w)$ we can use the same argument as in the proof of proposition 3.2.5.1 to use proposition 6.1.2.2 to get the first bound (namely we bound by Cauchy formula some derivatives of the same expression multiplied by some product of linear forms).

We have already observe that $\mathcal{E}^T_{P,\pi}(x,f,\lambda)$ is holomorphic on $\mathcal{R}_{\pi,c,l}$. This is also the case of $\mathcal{P}^{T,m}_{P,\pi}(x,f,\lambda,w)$ because it can be identified to the summand of exponent of "type 1" of $\mathcal{E}^T_{P,\pi}(x,f,\lambda)$ (see lemma 4.4.3.1 and proposition 4.4.4.1). Thus the difference $\mathcal{E}^T_{(P,\pi,\tau)}(x,f,\lambda) - \mathcal{P}^{T,m}_{(P,\pi,\tau)}(x,f,\lambda,w)$ is holomorphic and is equal to (by proposition 4.4.1.1

$$\sum_{P_0\subset Q\subset G} 2^{-\dim(\mathfrak{a}_Q^G)} \sum_{w'\in {}_QW_P^G\backslash W(P;Q)} \mathcal{P}^{T,Q}_{(P,\pi,\tau)}(x,f,\lambda,w)\cdot \frac{\exp(\langle w(-\lambda+\nu_w),T_Q\rangle)}{\theta_Q^G(w'(-\lambda+\nu^w_{w'}))}$$

By the same technics as before, the result follows from proposition 6.1.2.2 and lemma 4.4.3.1. □

7 Spectral expansion

7.1 Contribution of a spectral datum as a limit

7.1.1. Let $\mathfrak{X}(G)$ be a the set of cuspidal data of G, namely the set of equivalence classes of pairs (M,π) where M is a standard Levi subgroup of G and $\pi\in\Pi_{\mathrm{cusp}}(M)$. Two data (M,π) and (M',π') are equivalent if there exists $w\in W(M,M')$ that sends π to π'.

Let $P=MN_P$ be a standard parabolic subgroup of G. We have the coarse Langlands decomposition decomposition

$$L^2([G]_{P,0}) = \hat{\oplus}_{\chi\in\mathfrak{X}(G)} L^2_\chi([G]_{P,0}). \tag{7.1.1.1}$$

Let $\mathcal{A}_{P,\chi,\mathrm{disc}}(G)$ be the closed subspace of $\mathcal{A}^0_{P,\mathrm{disc}}(G)$ generated by the functions whose class belongs to $L^2_\chi([G]_{P,0})$. By Langlands construction of the spectral decomposition, see e.g. [MW94], and the description of the discrete spectrum obtained in [MW89], there exists a finite subset $\Pi_\chi(M)\subset\Pi_{\mathrm{disc}}(M)$ such that we have an isotypical decomposition

$$\mathcal{A}_{P,\chi,\mathrm{disc}}(G) = \bigoplus_{\pi\in\Pi_\chi(M)} \mathcal{A}_{P,\pi}(G).$$

7.1.2. Let $\chi\in\mathfrak{X}(G)$ be a cuspidal datum. Let $f\in\mathcal{S}(\mathbb{A})$ and $K^0_{f,\chi}$ be the kernel associated to the operator given by right convolution of f on $L^2_\chi([G]_0)$. By [BPCZ22, lemma 2.10.1.1], there exists $N_0>0$ such that for all $N>0$ there exists a semi-norm $\|\cdot\|$ on $\mathcal{S}(G(\mathbb{A})$ such that for all $x,y\in G(\mathbb{A})^1$ we have $|K^0_{f,\chi}(x,y)|\leqslant \|f\|\|x\|^{N+N_0}\|y\|^{-N_0}$. In particular the following integral

$$J_\chi(x,f) = \int_{[G']_0} K^0_{f,\chi}(x,y)\,dy. \tag{7.1.2.2}$$

is absolutely convergent for all $x\in G(\mathbb{A})$.

Let T be a truncation parameter. To get the spectral expansion of $J_\chi(x,f)$ it will be easier to first consider the spectral expansion of the truncated variant of $J_\chi(x,f)$ namely:

$$J^T_\chi(x,f) = \int_{[G']_0} (K^0_{f,\chi}\Lambda^T_m)(x,y)\,dy. \tag{7.1.2.3}$$

It is also absolutely convergent because of the properties of the kernel and the mixed truncation operators, see proposition 4.2.1.1. The notation $\lim_{T\to+\infty}$ means the limit when $d(T)\to+\infty$. We have:

Proposition 7.1.2.1. — *[BPCZ22, Proposition 4.3.4.1] For all $x \in G(\mathbb{A})$,*

$$\lim_{T\to+\infty} J_\chi^T(x,f) = J_\chi(x,f).$$

7.1.3. We now recall the spectral expansion of $J_\chi^T(x,f)$ in terms of the relative character $\mathcal{E}_{P,\pi}^T(x,f,\lambda)$ defined in §6.1.2.

Proposition 7.1.3.1. — *[BPCZ22, Proposition 4.2.3.3] For all $x \in G(\mathbb{A})$,*

$$J_\chi^T(x,f) = \sum_{P_0\subset P} |\mathcal{P}(M)|^{-1} \int_{i\mathfrak{a}_P^{G,*}} \sum_{\pi\in\Pi_\chi(M)} \mathcal{E}_{P,\pi}^T(x,f,\lambda)\,d\lambda$$

Note that without appealing to [BPCZ22], proposition 6.1.2.2 shows that the right-side is absolutely convergent.

Proposition 7.1.3.2. — *For all $x \in G(\mathbb{A})$, the expression*

$$\sum_{P_0\subset P} |\mathcal{P}(M)|^{-1} \int_{i\mathfrak{a}_P^{G,*}} \sum_{\pi\in\Pi_\chi(M)} \mathcal{P}_{P,\pi}^{T,m}(x,f,\lambda)\,d\lambda.$$

is absolutely convergent. For any $\eta > 0$, the limit when $T \to +\infty$ of the expression above for T such that $d(T) \geqslant \eta\|T\|$ is $J_\chi(x,f)$.

Proof. — We fix $\varepsilon > 0$ and a level J such that $f \in \mathcal{S}(G(\mathbb{A}))^J$. By proposition 6.2.1.1, we get $N, \eta > 0$ and for any (large) q a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that

$$\sum_{P_0\subset P} |\mathcal{P}(M)|^{-1} \int_{i\mathfrak{a}_P^{G,*}} \sum_{\pi\in\Pi_\chi(M)} |\mathcal{E}_{P,\pi}^T(x,f,\lambda) - \mathcal{P}_{P,\pi}^{T,m}(x,f,\lambda)|\,d\lambda$$

$$\leqslant \exp(-\varepsilon\|T\|))\|x\|_G^N\|f\|_{\mathcal{S}} \sum_{P_0\subset P} |\mathcal{P}(M)|^{-1} \left(\sum_{\pi\in\Pi_\chi(M)} (1+\Lambda_\pi^2)^{-q}\right) \int_{i\mathfrak{a}_P^{G,*}} (1+\|\lambda\|^2)^{-q}\,d\lambda$$

for all T such that $d(T) \geqslant \eta\|T\|$. It follows from the proof of proposition 6.2.1.1 that the constant η is a linear function in ε and the constant c which appears in $\mathcal{R}_{\pi,c,l}$. But here we just need to prove a majorization on the imaginary axis so we may take c arbitrary small and thus the majorization holds with η a fixed multiple of ε and thus it can be chosen abitrary small. The two statements of the theorem are then easily deduced since the set $\Pi_\chi(M)$ is finite and since we can choose q large enough so that all the integrals converge. The absolute convergence is deduced from the corresponding assertion for $\mathcal{E}_{P,\pi}^T(x,f,\lambda)$ which results also from proposition 6.2.1.1. □

7.2 Computation of the limit

7.2.1. We continue with the notations of the previous section. Let $\tau \in \hat{K}_\infty$. Let $x \in G(\mathbb{A})$. Using corollary 5.1.4.3, we get the following expression for the relative character $\mathcal{P}_{(P,\pi,\tau)}^{T,m}(x,f,\lambda)$ in terms of the intertwining periods:

$$\mathcal{P}_{(P,\pi,\tau)}^{T,m}(x,f,\lambda) = \sum_{\varphi\in\mathcal{B}_{P,\pi}(\tau,J)} E(x, I_P(\lambda,f)\varphi,\lambda)\overline{\mathcal{P}^{T,m}(\varphi,-\bar{\lambda})}$$

$$= \sum_{Q\in\mathcal{F}_2(M)} 2^{-\dim(\mathfrak{a}_Q^G)} \mathcal{J}_{(Q,\pi,\tau)}(x,f,\lambda) \frac{\exp(-\langle\lambda, T_Q\rangle)}{\theta_Q^G(-\lambda)}. \tag{7.2.1.1}$$

where for $Q \in \mathcal{F}_2(M)$ we introduce the relative character

$$\text{(7.2.1.2)} \qquad \mathcal{J}_{(Q,\pi,\tau)}(x,f,\lambda) = \sum_{\varphi \in \mathcal{B}_{P,\pi}(\tau,J)} E(x, I_P(\lambda,f)\varphi,\lambda)\overline{J_Q(\varphi,-\bar{\lambda})}.$$

Let $Q \in \mathcal{F}_2(M)$. There exist R a standard parabolic subgroup and $w \in W(P;R)$ such that $Q = w^{-1}Rw$. Then by theorem 5.1.4.1 we have

$$\text{(7.2.1.3)} \qquad \mathcal{J}_{(Q,\pi,\tau)}(x,f,\lambda) = \mathcal{P}^{T,R}_{(P,\pi,\tau)}(\varphi,\lambda,w).$$

Thus $\mathcal{J}_{(Q,\pi,\tau)}(x,f,\lambda)$ inherits all properties of $\mathcal{P}^{T,R}_{(Q,\pi,\tau)}(\varphi,\lambda,w)$ given in subsection 6.2. We can set:

$$\text{(7.2.1.4)} \qquad \mathcal{J}_{Q,\pi}(x,f,\lambda) = \sum_{\tau \in \hat{K}_\infty} \mathcal{J}_{(Q,\pi,\tau)}(x,f,\lambda).$$

Then we have

$$\text{(7.2.1.5)} \qquad \mathcal{P}^{T,m}_{P,\pi}(x,f,\lambda) = \sum_{Q \in \mathcal{F}_2(M)} 2^{-\dim(\mathfrak{a}_Q^G)} \mathcal{J}_{Q,\pi}(x,f,\lambda)\frac{\exp(-\langle\lambda,T_Q\rangle)}{\theta_Q^G(-\lambda)}$$

7.2.2. A (G,M)-family of relative characters. —

Proposition 7.2.2.1. — *Let $L \in \mathcal{L}_2(M)$. For any $f \in \mathcal{S}(G(\mathbb{A}))$, $x \in G(\mathbb{A})$ and $\lambda_0 \in i\mathfrak{a}_M^{L,*}$, the family $(\mathcal{J}_{Q,\pi}(x,f,\lambda_0+\lambda))_{Q\in\mathcal{P}(L)}$ is a (G,L)-family of Schwartz functions of the variable $\lambda \in i\mathfrak{a}_L^{G,*}$.*

Proof. — We may assume $x \in G(\mathbb{A})^1$. We fix a level J such that $f \in \mathcal{S}(G(\mathbb{A}))^J$. According to (7.2.1.3) and proposition 6.2.1.1, there exists $l > 0$ such that for all $q > 0$ there exists $c > 0$ and $C > 0$ such that we have:

$$\sum_{\tau \in \hat{K}_\infty} |\mathcal{J}_{Q,\pi}(x,f,\lambda)| \leqslant \frac{C}{(1+\|\lambda\|^2)^q}$$

for all $\lambda \in \mathcal{R}_{\pi,c,l}$ and $Q \in \mathcal{F}_2(M)$. The function $\mathcal{J}_{Q,\pi}(x,f,\lambda)$ is holomorphic on $\mathcal{R}_{\pi,c,l}$ thus smooth on $i\mathfrak{a}_P^{G,*}$. By the equality above it is also rapidly decreasing. Using Cauchy formula, all its real derivatives are also rapidly decreasing on the imaginary axis. Thus $\mathcal{J}_{Q,\pi}(x,f,\lambda)$ is a Schwartz function. On the other hand we can assume that $\mathcal{P}^{T,m}_{P,\pi}(x,f,\lambda)$ is also holomorphic on $\mathcal{R}_{\pi,c,l}$, still by proposition 6.2.1.1. We can then apply to the right-hand side of (7.2.1.5) which is smooth the proposition 2.4.1.1 to conclude. □

7.2.3. As a consequence of the definition of a (G,L)-family we have:

Corollary 7.2.3.1. — *Let $L \in \mathcal{L}_2(M)$ and $\lambda \in i\mathfrak{a}_M^{L,*}$. Then $\mathcal{J}_{Q,\pi}(x,f,\lambda)$ does not depend on the choice of $Q \in \mathcal{P}(L)$.*

As a consequence, we set

$$\text{(7.2.3.6)} \qquad \mathcal{J}_{L,\pi}(x,f,\lambda) = \mathcal{J}_{Q,\pi}(x,f,\lambda), \quad \lambda \in i\mathfrak{a}_M^{L,*}, Q \in \mathcal{P}(L).$$

Proposition 7.2.3.2. — *There exists $N > 0$ such that for all $q > 0$ there exists a continuous semi-norm $\|\cdot\|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))$ such that for all $f \in \mathcal{S}(G(\mathbb{A}))$, $x \in G(\mathbb{A})^1$, for all standard parabolic subgroup $P = MN_P$, all $\pi \in \Pi_{\mathrm{disc}}(M)$ and all $L \in \mathcal{L}_2(M)$ we have:*

$$\int_{i\mathfrak{a}_M^{L,*}} |\mathcal{J}_{L,\pi}(x,f,\lambda)|\, d\lambda \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}}}{(1+\Lambda_\pi^2)^q}$$

Proof. — According to proposition 6.2.1.1, the equality (7.2.1.3) and the definition (7.2.3.6), we see that for all $q > 0$ and all levels J there exists a continuous semi-norm $\| \cdot \|_{\mathcal{S}}$ on $\mathcal{S}(G(\mathbb{A}))^J$ such that for all $f \in \mathcal{S}(G(\mathbb{A}))^J$, $x \in G(\mathbb{A})^1$ and pairs (P, π), all $L \in \mathcal{L}_2(M)$ and all $\lambda \in i\mathfrak{a}_M^{L,*}$

$$|\mathcal{J}_{L,\pi}(x, f, \lambda)| \leqslant \frac{\|x\|_G^N \|f\|_{\mathcal{S}}}{(1 + \|\lambda\|^2)^q (1 + \Lambda_\pi^2)^q}$$

The results follow. □

7.2.4.

Theorem 7.2.4.1. — *For all $f \in \mathcal{S}(G(\mathbb{A}))$ and $x \in G(\mathbb{A})$ we have*

$$J_\chi(x, f) = \sum_{P_0 \subset P} |\mathcal{P}(M)|^{-1} \sum_{L \in \mathcal{L}_2(M)} 2^{-\dim(\mathfrak{a}_L^G)} \sum_{\pi \in \Pi_\chi(M)} \int_{i\mathfrak{a}_M^{L,*}} \mathcal{J}_{L,\pi}(x, f, \lambda) \, d\lambda$$

where the right-hand side is absolutely convergent.

Proof. — Since the sums are finite, the absolute convergence is that of the inner integral which follows from proposition 7.2.3.2. By proposition 7.1.3.2, the expression $J_\chi(x, f)$ is the limit when $T \to +\infty$ of the sum over $P_0 \subset P = MN_P$ and $\pi \in \Pi_\chi(M)$ of the constant $|\mathcal{P}(M)|^{-1}$ times the expression

$$\int_{i\mathfrak{a}_P^{G,*}} \mathcal{P}_{P,\pi}^{T,m}(x, f, \lambda) \, d\lambda.$$

Let fix (P, π). Using the expression (7.2.1.5) of $\mathcal{P}_{P,\pi}^{T,m}$, we can fix also $L \in \mathcal{L}_2(M)$ and compute for the limit when $T \to +\infty$ of

$$2^{-\dim(\mathfrak{a}_L^G)} \int_{i\mathfrak{a}_P^{G,*}} \sum_{Q \in \mathcal{P}(L)} \mathcal{J}_{Q,\pi}(x, f, \lambda) \frac{\exp(-\langle \lambda, T_Q \rangle)}{\theta_Q^G(-\lambda)} \, d\lambda.$$

Since $(\mathcal{J}_{(Q,\pi,\tau)}(x, f, \lambda))_{Q \in \mathcal{P}(L)}$ is a Schwartz (G, L)-family by proposition 7.2.2.1, not only the integrand is also a Schwartz function, see [Lap11a, corollary 3], but also, see [Lap11a, lemma 8], the limit exists and is equal to:

$$2^{-\dim(\mathfrak{a}_L^G)} \int_{i\mathfrak{a}_P^{L,*}} \mathcal{J}_{L,\pi}(x, f, \lambda) \, d\lambda.$$

□

7.3 Spectral decomposition of the Flicker-Rallis period of the automorphic kernel

7.3.1. Let $f \in \mathcal{S}(G(\mathbb{A}))$. The automorphic kernel associated to f is the kernel of the operator given by right convolution by f on $L^2([G])$ namely

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y), \quad x, y \in G(\mathbb{A}).$$

We set for $x \in G(\mathbb{A})$

(7.3.1.1)
$$J^G(x, f) = \int_{[G']} K_f(x, y) \, dy.$$

Lemma 7.3.1.1. —

1. *The integral that defines $J^G(x, f)$ is absolutely convergent and the map $f \mapsto J(x, f)$ is a continuous distribution on $\mathcal{S}(G(\mathbb{A}))$.*
2. *We have also:*

$$J^G(x, f) = \frac{1}{2} \sum_{\chi \in \mathfrak{X}(G)} J_\chi(x, f) \tag{7.3.1.2}$$

where the right-hand side is absolutely convergent.

Proof. — The assertion 1 comes from [BPCZ22, lemma 2.10.1.1]. Let $K_f^0(x, y)$ be the kernel of the operator given by right convolution by f on $L^2([G]_0)$. Then we have:

$$K_f^0(x, y) = \int_{A_G^\infty} K_f(ax, y)\, da.$$

Taking into account the discrepancy between the measures on A_G^∞ and $A_{G'}^\infty$, we get:

$$J^G(x, f) = \frac{1}{2} \int_{[G']_0} K_f^0(x, y)\, dy.$$

However we have the (coarse) spectral decomposition

$$K_f^0(x, y) = \sum_{\chi \in \mathfrak{X}(G)} K_{f,\chi}^0(x, y).$$

Integrating term by term over $y \in [G']_0$ which is possible by [BPCZ22, lemma 2.10.1.1] we get assertion 2. □

7.3.2. We can now state and prove our main theorem.

Theorem 7.3.2.1. — *For any $f \in \mathcal{S}(G(\mathbb{A}))$ and $x \in G(\mathbb{A})$ we have*

$$J(x, f) = \sum_{P_0 \subset P} |\mathcal{P}(M)|^{-1} \sum_{L \in \mathcal{L}_2(M)} 2^{-\dim(\mathfrak{a}_L)} \sum_{\pi \in \Pi_{\mathrm{disc}}(M)} \int_{i\mathfrak{a}_M^{L,*}} \mathcal{J}_{L,\pi}(x, f, \lambda)\, d\lambda$$

where the right-hand side is absolutely convergent.

Proof. — Using lemma 7.3.1.1 and then theorem 7.2.4.1 we get:

$$\begin{aligned} J(x, f) &= \frac{1}{2} \sum_{\chi \in \mathfrak{X}(G)} J_\chi(x, f) \\ &= \sum_{\chi \in \mathfrak{X}(G)} \sum_{P_0 \subset P} |\mathcal{P}(M)|^{-1} \sum_{L \in \mathcal{L}_2(M)} 2^{-\dim(\mathfrak{a}_L)} \sum_{\pi \in \Pi_\chi(M)} \int_{i\mathfrak{a}_M^{L,*}} \mathcal{J}_{L,\pi}(x, f, \lambda)\, d\lambda. \end{aligned}$$

To conclude we have to show that the expression above is absolutely convergent. Let $J \subset K_f$ be a level such that f is J-biinvariant. Then the terms attached to $\pi \in \Pi_{\mathrm{disc}}(M) = \bigcup_{\chi \in \mathfrak{X}(G)} \Pi_\chi(M)$ vanish unless $\pi \in \Pi_{\mathrm{disc}}(M)^J$. Thus by proposition 7.2.3.2 we are reduced to the following statement: for all standard Levi subgroup M and for large $q > 0$ we have

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M)^J} \frac{1}{(1 + \Lambda_\pi^2)^q} < \infty.$$

This is due to Müller [Mül02, line (6.17) p. 711 and below]. □

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