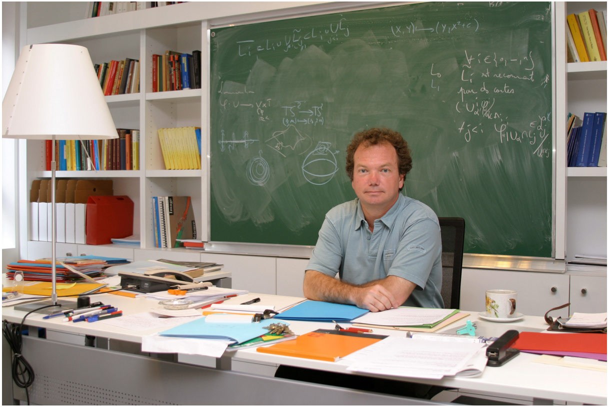
# L'École d'un Maître – The School of a Master



Jean-Christophe Yoccoz

I would like venture into one aspect of the research in Mathematics, although poorly related, but of high importance, and in which J.-C. Yoccoz was a literally a Master.

Is is about how to teach to a student to "create" or "discover" mathematics, more than how to teach already existing mathematics.

For most of us, Mathematics are at the boundary of Sciences and Arts. This makes the teaching in research in Mathematics very special.

I would not pretend to make a "theory" on his way of teaching, but merely, I would like to share some strong experiences I had with him as a master and some though he asked me to develop during the next decades.

# Toward new mathematical fields

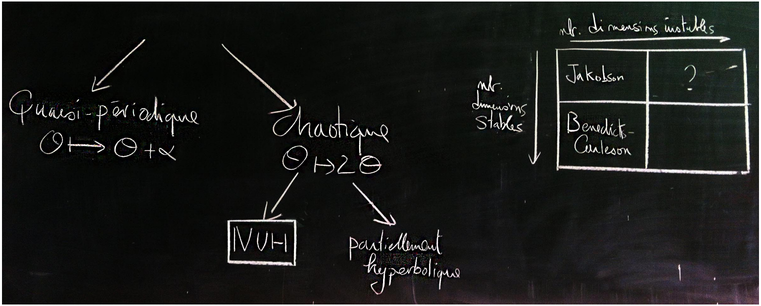
When a student starts a PhD thesis, he usually has to chose a director who proposes him a subject.

Some directors like to give an article or book to read, others suggest a specific problem.

For most of his students, the first meeting with Jean-Christophe Yoccoz was of different nature.

# Toward new mathematical fields

He drew me a panorama of the research in dynamical systems.



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There were directions that he felt too much studied to make a PhD thesis in it, directions in which he did not believe, and others that he felt unexplored, rich and fertile enough to make a PhD thesis.
I had to chose along the latter ones.

# Toward new mathematical fields

To enter in these fields, the way he proceed with with most of his students was to ask them to focus on a simple example, and read a few articles and books on related topics.

Certainly, the example was chosen by him to encompass a paradigmatic mechanism for the studied direction.

Most of the students were given a polynomial function of one or two variables (complex or real) to study. Given by a very simple analytical definition.

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A great point was the simplicity of the example. No need of studying several books before entering into the subject. Also very few works was done to study this mapping.

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He was very enthusiastic:

I am sure that such an example will show that most of the dynamics are much more complicated that we think they are. Actually I personally do not think that we will understand most of the dynamics in a bounded time (I mean before 50-100 years). This kind of example can generate a scientific revolution, by showing the high complexity of the behavior which can occur!

But at first sight, how could it be revolutionary, or just good to show that we cannot understand a phenomena? This disturbed me for years...

# Toward new mathematical fields

I interpret the fact he gave precise examples to study by his love for nature, by sailing over the seas, and hiking in the mountains.

Perhaps also by his modesty: he liked mathematics with simple applications, and which do not need years of studies to understand what are they about.

# Toward new mathematical fields

To learn to a student how to do mathematics, it is certainly clever to give a paradigmatic example displaying very rich dynamics, which do not fit into any known theory.

This enables the student to develop his "own" mathematics, following his preferences and being like this quickly independent.

# Maieutics

In the beginning of my PhD thesis, each discussion with him began by a short individual lecture on a specific theory similar to what was met in the example under study. Then he formulated questions and hint to go further.

He was always carefully listening to anything I could add, like questions, new ideas even very badly formulated or wrong.

Trying to say more than him was a beginning of a game. Eventually to understand better my topic than him, and anybody else. It is actually the aim of a PhD thesis.

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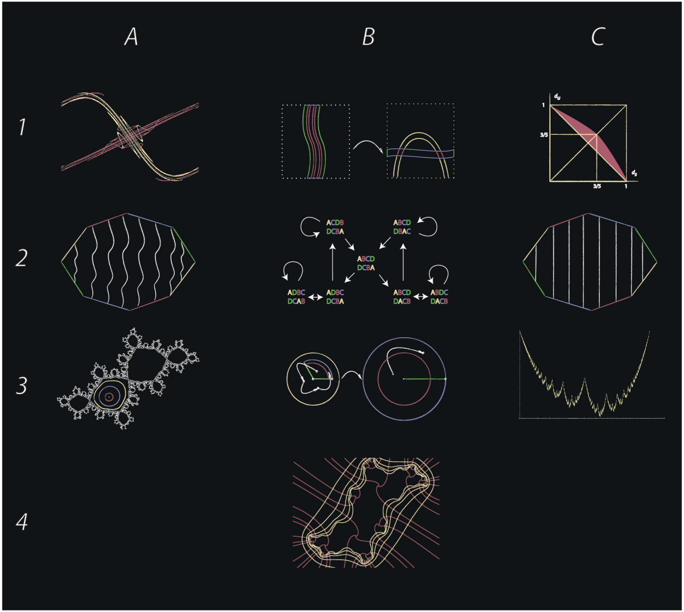
C) Extract from this a theorem.



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# Maieutics

The kind of hint could be like this:

Study the geometry of this Cantor set!

What does that mean???

This means: describe me this Cantor set.

((Reconstitution of a 'physical' conversation in 2002))

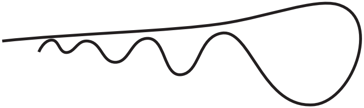
# Maieutics

One day I arrived with an idea to generalize our first result.

I think the proof can be generalized to a compact set plenty of submanifolds and of different dimensions... It is hard to say what I wanna say :/

We, mathematicians have all this problem: how to write what we see and feel.

Is it made of injectively, immersed submanifolds, like this curve?



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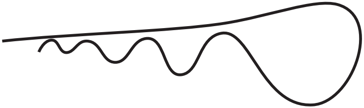


A main concern of him was to deshinibit his students. Almost every time I was confessing him I was "complexed" because of xxx he replied that the same occur to him.

Your thesis subject is so hard... I do not see how to show it in a few years.

No worry, it is just now that I think I can prove my thesis subject: more than 30 years after!

Herman asked me to show that the space of pair of circle diffeomorphisms which commute is not locally connected, to show how wild is that space. But unfortunately, I now believe that this space is locally connected, so this question is less insteresting.

I did not ask you to prove the NUH of such mappings, but to find therein a new phenomenon which shows how complicated these typical dynamics can be. This does not need to involve very hard mathematics.

# A disturbing modesty

Indeed, many times Jean-Christophe Yoccoz told me that very few examples of dynamics can understood.

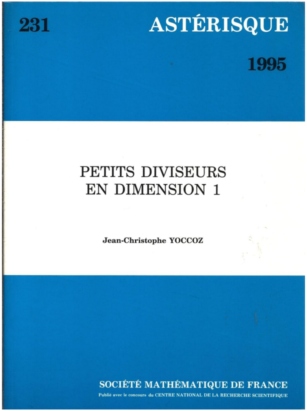
This vision certainly reminds of his own adviser Michael Herman, who began his last ICM article in 1998 with the following quote:

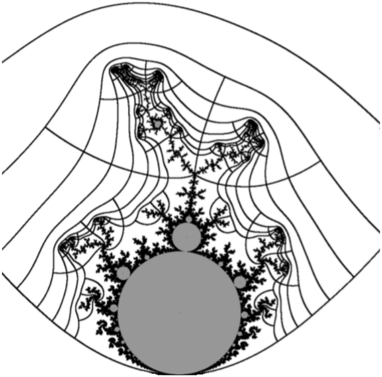
$\boxed{\text{Ce que nous savons est peu de choses ; ce que nous ignorons est immense.}}$

P.-S. de Laplace

$\boxed{\text{What we know is a few things ; what we ignore is huge.}}$

I believe that this scientific direction influenced Yoccoz' work a lot, for instance his work on the commutators of circles diffeomorphisms or his work on the local connectivity of $M$, and the direction he gave to some of his students.



Credit X. Buff.

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I believe that in these paradigms, there is the idea that the mathematical formalism is sufficiently wild to enable to 'discover' typical objects that cannot be understood, and even so wild that it leaves room for some mathematicians to 'create' theorems on the limit of this 'scientific' formalism. This is highly consistent with the following quote:

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Le plaisir que je prends à faire des mathématiques est proche de celui ressenti par un artiste qui crée une œuvre.

The pleasure I feel while doing mathematics is close to the one that an artist can feel to create an artwork.

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Chaos has much more to reveal.