Zoology in the Hénon family: twin babies and Milnor's swallows

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Hénon Like maps

Definition (Hénon-like map)

A map f of a domain $D \subset \mathbb{R}^2$ into $\mathbb{R}^2$ is C^r-Hénon-like with parameter $(a, b) \in \mathbb{R}^2$ if it is of the form.

$$(x, y) \mapsto (x^2 + a - b^m y + \zeta(x, b^m y), x + \xi(x, b^m y)) \ ,$$

where ζ, ξ are C^r-functions from D into $\mathbb{R}$. If the C^r norms of ζ and ξ are smaller than δ, then f is δ-C^r-Hénon-like.

Definition (Hénon-like family)

For $d \leq r$, a δ-$C^{d,r}$-Hénon-like family $(f_{a\,b})_{a\,b}$ of mutilplicity $m \in \mathbb{N}$, consists of maps of the form:

$$f_{a\,b}(x, y) = (x^2 + a - b^m y + \zeta_{a\,b}(x, b^m y), x + \xi_{a\,b}(x, b^m y)) \ ,$$

where $C^{d,r}$-norm of $(\zeta_{a\,b}, \xi_{a\,b})_{a\,b}$ is at most δ.

Fact

Without lost of generality we can assume $\zeta = 0$.

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Theorem

δ-$C^{d,r}$-Hénon-like families of multiplicity m appear as a renormalized dynamics in the neighborhood of the non-degenerated unfolding of a homoclinic tangency.

Theorem

δ-$C^{d,r}$- Hénon-like families of multiplicity $m_1 \cdot m_0$ appear as a renormalized dynamics of Hénon-like families of multiplicity m_0 and period m_1.

Problem

Describe the dynamics of typical Hénon-like maps.

Problem

Describe the geometry of the parameter space.

Problem (Arnold 90')

Given a generic family of diffeomorphisms, does the number of periodic points growths at most exponentially fast with the period for Lebesgue a.e. parameter?

Problem (Palis 90', Palis-Taken 90's)

Given a generic family of surface diffeomorphisms, does the dynamics display at most finitely many attractors, for Lebesgue a.e. parameter?

Problem (Positive entropy conjecture)

Does there exist one conservative, Hénon-like maps with positive metric entropy? Given a (generic) family of conservative, Hénon-like maps, does the metric entropy is positive for a set of parameters of positive Lebesgue measure?

Theorem (B. 2016)

For any $\infty > d \leq r \leq \infty$, there are $C^{d,r}$-generic families of surface endomorphisms (or diffeomorphisms of M^n- manifold with $n \geq 3$) which displays a super exponential growth of the number of periodic points at every parameter.

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Theorem (B.-Turaev 2017)

Any conservative surface map displaying an elliptic point can be C^∞_ω-approximated by a map with positive metric entropy.

This is a solution of Herman's positive entropy conjecture.

The proof of this theorem uses renormalization nearby a chain of homoclinic tangency to construct a composition of generalized, conservative Hénon-like maps $(x, y) \mapsto (\phi_i(x) - y, x)$ with positive entropy.

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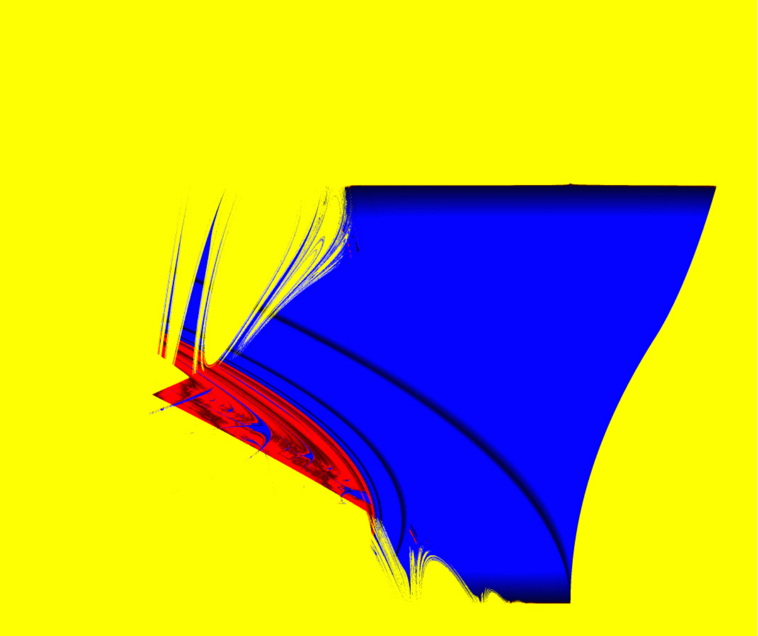
The main remaining entropic' questions are:
Does positive entropy occur with 'positive probability' in the parameter space?
Does the zero-entropy is dense in the C^∞_ω-topology?

Parameter space of the Hénon family.



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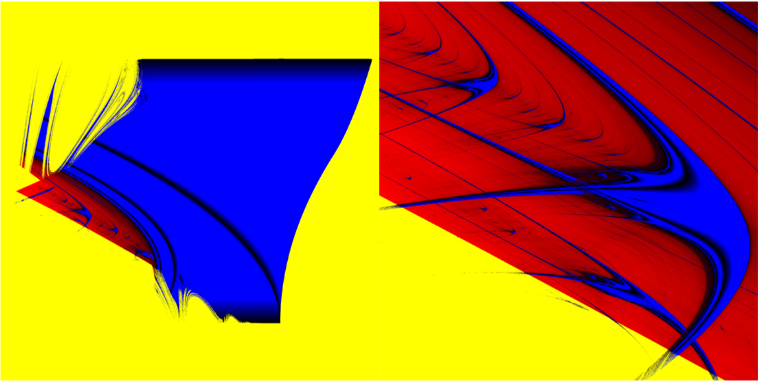
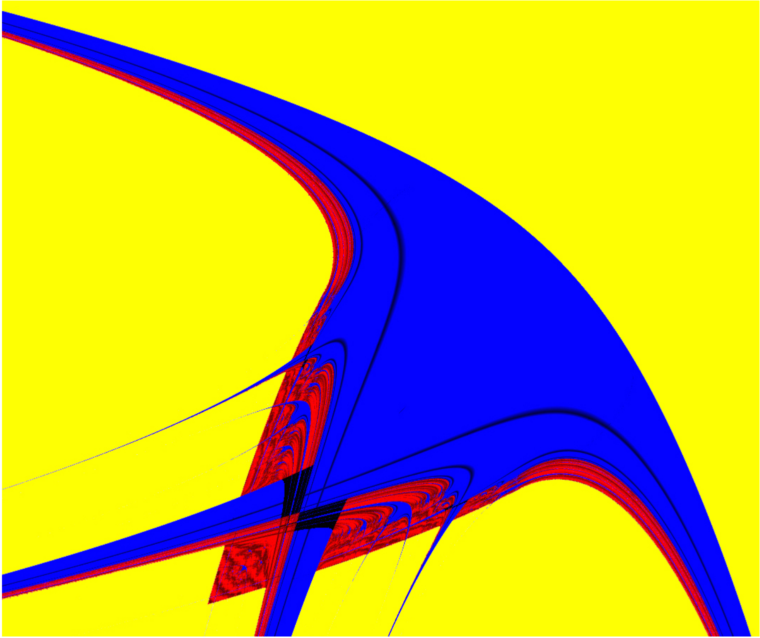
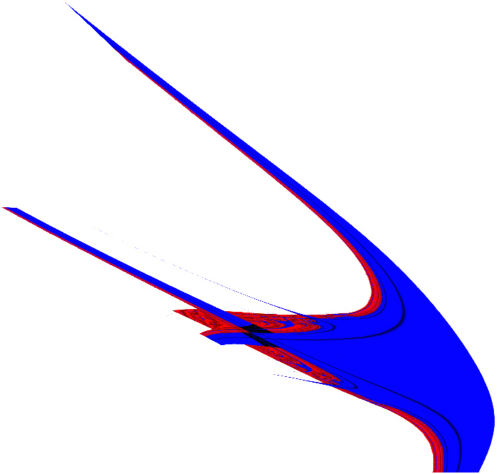


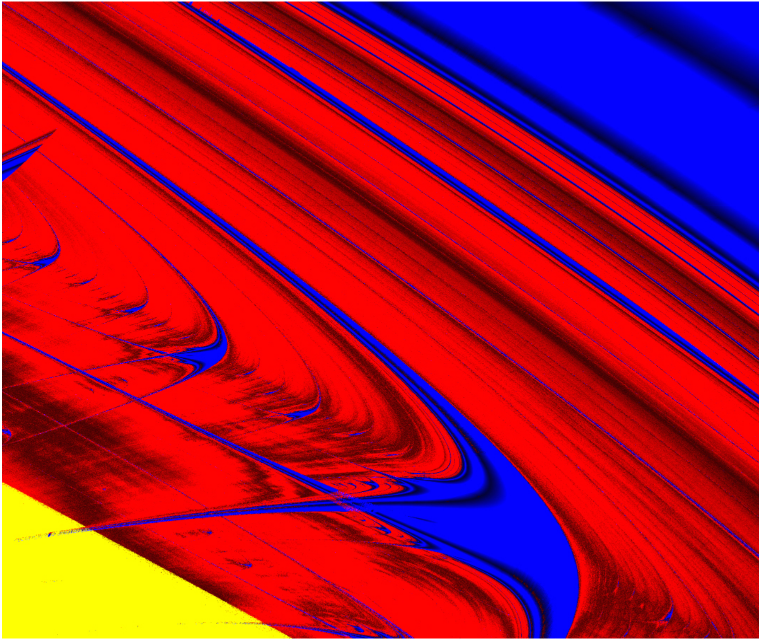
Problem

How to describe it?

Parameter space of $(x \mapsto (x^2 + a)^2 + b))_{a\, b}$







Théorème (B. 17, numerical conjecture by J.Milnor "Remarks on iterated cubic maps" Exp.Math. (1992))

For all $R > 0$, $\delta > 0$, $r \geq 1$, there exists a domain Δ of parameters (a, b) of the Hénon family $(h_{a\,b})_{a,b}$, there exists $n \geq 1$, such that for every $(a, b) \in \Delta$, there exist a domain $D_{a,b} \subset \mathbb{R}^2$, and embeddings $\phi_{a\,b} \in C^r(D_{a,b}, \mathbb{R}^2)$ and $\psi \in Diff^r(\Delta, [-R, R]^2)$ such that:

- *the domain of definition of $\mathcal{R}h_{a\,b} := \phi_{a\,b}^{-1} \circ h_{a\,b}^n \circ \phi_{a\,b}$ contains $[-R, R]^2$,*
- *$\mathcal{R}h_{a\,b}$ is δ-C^r close to the map $(x, y) \mapsto ((x^2 + c_2 + y)^2 + c_1, 0)$, with $(c_1, c_2) = \psi(a, b)$.*

This implies that the hyperbolic continuation of the attracting cycles of a quadratic mapping cannot explain the coexistence of infinitely many sinks of any Hénon maps (a negative answer to a question by Van Strien).

Pre-existing Zoo

Every quadratic map has at most one attractor. If the quadractic map has a periodic sink then a Hénon map nearby has also a unique sink.

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Question (Lyubich)

Is there a Hénon map which has exactly 2 *periodic sinks?*

Positive answer to questions of Lyubich

Théorème (B.)

There exist parameters (a, b) of the Hénon family $h_{ab}(x, y) = (x^2 + a + y, -bx)$, for which there exist exactly two periodic sinks, both attracting Lebesgue almost every point not escaping to ∞.

Théorème (B.)

There exist parameters (a, b) of the Hénon family $h_{ab}(x, y) = (x^2 + a + y, -bx)$, for which there exist exactly one Hénon like attractor (with any parameter $a \in [2 + \epsilon, 1/4 - \epsilon]$) and one periodic sink, both attracting Lebesgue almost every point not escaping to ∞.

Tools of the proof

- A generalization of Yoccoz puzzle piece for Hénon-like maps.
- A developement of the bounds of Shilnikov cross map/Palis-Yoccoz affine-like representation.
- two renormalizations Theorems using these bounds.

Definition (Distortion bound $\mathcal{B}^{d,r}$)

Let $\mathfrak{d}$ be a piece for a family of maps $(f_p)_p$ and with affine-like representation $(A_p, B_p)_p$. We define for $1 \leq d \leq r < \infty$:

$$\begin{aligned} \mathcal{B}_0^{d,r}(\mathfrak{d}, p) &:= \|(A_p, B_p)_p\|_{C^{d,r}} . \\ \mathcal{B}_1^{d,r}(\mathfrak{d}, p) &:= \|(D \log |\partial_x A_p|, D \log |\partial_y B_p|)_p\|_{C^{\min(r-2,d),r-2}} . \end{aligned}$$

In the latter definition D denotes the differential w.r.t. x and y.
In the case of surface diffeomorphism family far to be degenerated, the derivatives ∂_p^k is bounded by:

$$\mathcal{B}_0^{d}(\mathfrak{d}, p) := \frac{1}{n_{\mathfrak{d}}} \max_{1 \leq k \leq \min(d,r-1)} \left\| \partial_p^k \log |\partial_x A_p|, \partial_p^k \log |\partial_y B_p| \right\|_{C^0} .$$

In the Hénon-like context, with $p = (a, b)$:

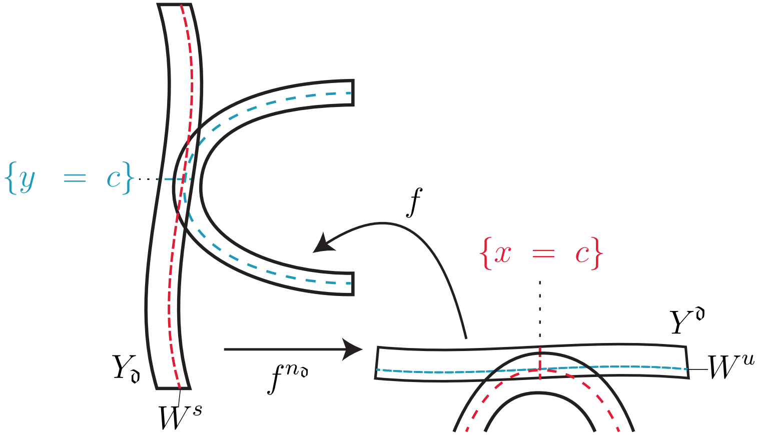
$$\mathcal{B}_m^{d}(\mathfrak{d}, p) := \frac{1}{n_{\mathfrak{d}}} \max_{1 \leq k \leq \min(d,r-1)} \left\| \partial_p^k \log |\partial_x A_p|, \partial_p^k \log |\partial_y B_p / b^{m n_{\mathfrak{d}}}| \right\|_{C^0} .$$

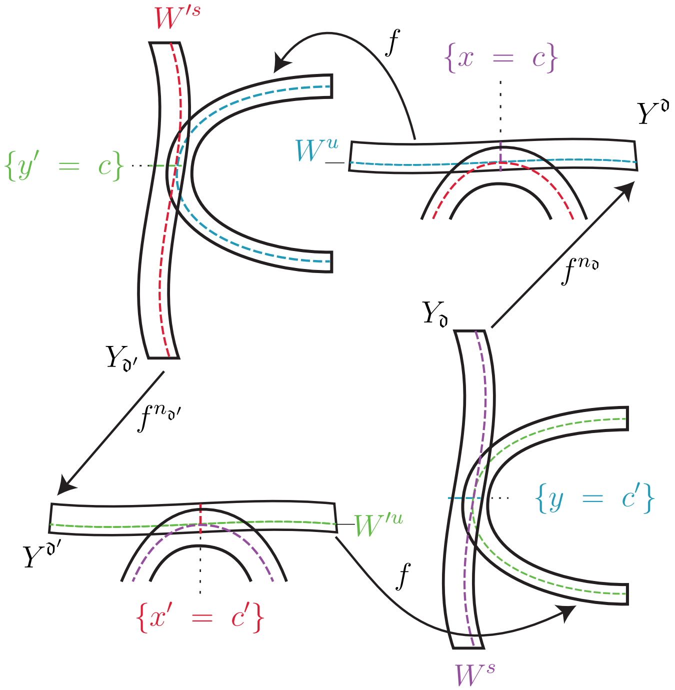
Theorem

For every K_0, there exists K_1 so that for every $N \geq 1$, for every hyperbolic puzzles pieces $\mathfrak{d}_1, \cdots, \mathfrak{d}_N$ of a Hénon-like family of multiplicity m, if the affine-like representations of each $\mathfrak{d}_i$ has its bounds $\mathcal{B}_0^{d,r}, \mathcal{B}_1^{d,r}, \mathcal{B}_m^d$ bounded by K_0, then the bounds $\mathcal{B}_0^{d,r}, \mathcal{B}_1^{d,r}, \mathcal{B}_m^d$ of $\mathfrak{d} = \mathfrak{d}_1 \star \cdots \mathfrak{d}_N$ is bounded by K_1.

IMPORTANT: K_1 is indenpendant of N and independent of b small (or large).

Let $d \leq r-3$ and let $(f_p)_p$ is Hénon like of multiplicity m. If the piece $\mathfrak{d}$ display a quadratic tangency at $(x_1, y_0) = (c, c)$, which undold in a degenerate way, and its horizontal expansion is large compare to the bounds $\mathcal{B}_0^{d,r}, \mathcal{B}_1^{d,r}, \mathcal{B}_2^{d,r}$ of $\mathfrak{d}$, then the renormalization is a $\delta - C^{d,r}$-Hénon like map of multiplicity $m \cdot (n_{\mathfrak{d}} + 1)$.





W'^s
f
$\{x = c\}$
$Y^{\mathfrak{d}}$
$\{y' = c\}$
W^u
$f^{n_{\mathfrak{d}}}$
$Y_{\mathfrak{d}}$
$Y_{\mathfrak{d}'}$
$f^{n_{\mathfrak{d}'}}$
$\{y = c'\}$
W'^u
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