Integral Eisenstein cocycles on $\mathbf{GL}_n$, II: Shintani's method

Pierre Charollois
Samit Dasgupta*
Matthew Greenberg†

March 9, 2013

Abstract

We define a cocycle on $\mathbf{GL}_n(\mathbf{Q})$ using Shintani's method. This construction is closely related to earlier work of Solomon and Hill, but differs in that the cocycle property is achieved through the introduction of an auxiliary perturbation vector Q. As a corollary of our result we obtain a new proof of a theorem of Diaz y Diaz and Friedman on signed fundamental domains, and give a cohomological reformulation of Shintani's proof of the Klingen–Siegel rationality theorem on partial zeta functions of totally real fields.

Next we relate the Shintani cocycle to the Sczech cocycle by showing that the two differ by the sum of an explicit coboundary and a simple "polar" cocycle. This generalizes a result of Sczech and Solomon in the case $n = 2$.

Finally, we introduce an integral version of our cocycle by smoothing at an auxiliary prime ℓ. Applying the formalism of the first paper in this series, we prove that certain specializations of the smoothed class yield the p-adic L-functions of totally real fields. Combining our cohomological construction with a theorem of Spiess, we show that the order of vanishing of these p-adic L-functions is at least as large as the expected one.

Contents

*Partially supported by NSF CAREER grant DMS-0952251
†Partially supported by an NSERC discovery grant

Introduction

In this paper, we study a certain "Eisenstein cocycle" on $\mathbf{GL}_n(\mathbf{Q})$ defined using Shintani's method. Our construction follows previous works of Solomon, Hu, Hill, Spiess, and Steele in this direction ([So1], [HS], [Hi], [Sp2], [Stee]).

We study three main themes in this paper. First, we define an $(n-1)$-cocycle on $\mathbf{GL}_n(\mathbf{Q})$ valued in a certain space of power series denoted $\mathbf{R}((z))^{\text{hd}}$. The basic idea of defining a cocycle using Shintani's method is well-known; the value of the cocycle on a tuple of matrices is the Shintani–Solomon generating series associated to the simplicial cone whose generators are the images of a fixed vector under the action of these matrices. The difficulty in defining a cocycle stems from two issues: choosing which boundary faces to include in the definition of the cone, and dealing with degenerate situations when the generators of the cone do not lie in general position. Hill's method is to embed $\mathbf{R}^n$ into a certain ordered field with n indeterminates, and to perturb the generators of the cone using these indeterminates so that the resulting vectors are always in general position. The papers [Stee] and [Sp2] use Hill's method. Our method is related, but somewhat different. We choose an auxiliary irrational vector $Q \in \mathbf{R}^n$ and include a face of the simplicial cone if perturbing the face by this vector brings it into the interior of the cone. We learned during the writing of this paper that this perturbation idea was studied much earlier by Colmez in unpublished work for the purpose of constructing Shintani domains [Co3]. Colmez's technique was used by Diaz y Diaz and Friedman in [DDF]. However the application of this method to the cocycle property appears to be novel.

Using formulas of Shintani and Solomon, we prove that the cocycle we construct specializes under the cap product with certain homology classes to yield the special values of partial zeta functions of totally real fields of degree n at nonpositive integers. This is a cohomological reformulation of Shintani's calculation of these special values and his resulting proof of the Klingen–Siegel theorem on their rationality.

In 1993, Sczech introduced in [Sc2] an Eisenstein cocycle on $\mathbf{GL}_n(\mathbf{Q})$ that enabled him to give another proof of the Klingen–Siegel theorem. Our second main result is that the cocycles defined using Shintani's method and Sczech's method are in fact cohomologous. The fact that such a result should hold has long been suspected by experts in the field; all previous attempts were restricted to the case $n = 2$ (see for instance [Sc3], [So2, §7] or [Hi, §5]). One technicality is that the cocycles are naturally defined with values in different modules, so we first define a common module where the cocycles can be compared, and then we provide an explicit coboundary relating them.

The third and final theme explored in this paper is a smoothing process that allows for the definition of an *integral* version of the Shintani cocycle. The smoothing method was introduced in our earlier paper [CD], where we defined an integral version of the Eisenstein cocycle constructed by Sczech. The integrality property of the smoothed cocycles has strong arithmetic consequences. We showed in [CD] that one can use the smoothed Sczech–Eisenstein cocycle to construct the p-adic L-functions of totally real fields and furthermore to study the analytic behavior of these p-adic L-functions at $s = 0$. In particular, we showed using work of Spiess [Sp1] that the order of vanishing of these p-adic L-functions at $s = 0$ is at least equal to the expected one, as conjectured by Gross in [Gr]. The formal nature of our proofs implies that these arithmetic results could be deduced entirely from the integral version of the Shintani cocycle constructed in this paper. In future work, we will explore further the leading terms of these p-adic L-functions at $s = 0$ using our cohomological method [DS].

We conclude the introduction by stating our results in greater detail and indicating the direction of the proofs. Sections 3 and 4 both rely on Sections 1 and 2 but are independent from each other. Only Section 4 uses results from the earlier paper [CD].

Q-perturbation, cocycle condition and fundamental domains

Fix an integer $n \geq 2$, and let $\Gamma = \mathbf{GL}_n(\mathbf{Q})$. Let $\mathcal{K}$ denote the abelian group of functions on $\mathbf{R}^n$ generated by the characteristic functions of rational open simplicial cones, i.e. sets of the form $\mathbf{R}_{>0}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_r$ with linearly independent $v_i \in \mathbf{Q}^n$.

Let $\mathbf{R}^n_{\mathrm{Irr}} \subset \mathbf{R}^n$ denote the set of vectors with the property that their n components are linearly independent over $\mathbf{Q}$. Let $\mathcal{Q}$ denote the set of equivalence classes of $\mathbf{R}^n_{\mathrm{Irr}}$ under multiplication by $\mathbf{R}_{>0}$.

Given an n-tuple of matrices $A = (A_1, \ldots, A_n) \in \Gamma^n$, we let $\sigma_i \in \mathbf{Q}^n$ denote the leftmost column of A_i, i.e. the image under A_i of the first standard basis vector. (In fact replacing this basis vector by any nonzero vector in $\mathbf{Q}^n$ would suffice.) Fixing $Q \in \mathbf{R}^n_{\mathrm{Irr}}$, we define an element $\Phi_{\mathrm{Sh}}(A, Q) \in \mathcal{K}$ as follows. If the σ_i are linearly dependent, we simply let $\Phi_{\mathrm{Sh}}(A, Q) = 0$. If the σ_i are linearly independent, we define $\Phi_{\mathrm{Sh}}(A, Q) \in \mathcal{K}$ to be the characteristic function of the simplicial cone $C = C(\sigma_1, \dots, \sigma_n)$ and some of its boundary faces, multiplied by $\text{sgn}(\det(\sigma_1, \dots, \sigma_n))$. A boundary face is included if translation of an element of that face by a small positive multiple of Q moves the element into the interior of C. The property $Q \in \mathbf{R}^n_{\text{Irr}}$ ensures that Q does not lie in any face of the cone, and hence translation by a small multiple of Q moves any element of a face into either the interior or exterior of the cone. The definition of $\Phi_{\text{Sh}}(A, Q)$ depends on Q only up to its image in $\mathcal{Q}$.

Our first key result is the following cocycle property of Φ_{Sh} (see Theorems 1.1 and 1.6). The function

$$\sum_{i=0}^{n} (-1)^i \Phi_{\text{Sh}}(A_0, \dots, \hat{A}_i, \dots, A_n, Q) \tag{1}$$

lies in the subgroup $\mathcal{L} \subset \mathcal{K}$ generated by characteristic functions of wedges, i.e. sets of the form $\mathbf{R}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_r$ for some $r \geq 1$ and linearly independent $v_i \in \mathbf{Q}^n$. We conclude that the function Φ_{Sh} defines a homogeneous $(n-1)$-cocycle on Γ valued in the space $\mathcal{N}$ of functions $\mathcal{Q} \to \mathcal{K}/\mathcal{L}$.

Along the way we note that if the σ_i are all in the positive orthant of $\mathbf{R}^n$, then in fact the function (1) vanishes. As a result we obtain another proof of the main theorem of [DDF], which gives an explicit signed fundamental domain for the action of the group of totally positive units in a totally real field of degree n on the positive orthant. In the language of [Sp2], we show that the specialization of Φ_{Sh} to the unit group is a *Shintani cocycle* (see Theorem 1.5 below).

Using this result and Shintani's explicit formulas for the special values of zeta functions associated to simplicial cones, we recover the following classical result originally proved by Klingen and Siegel. Let F be a totally real field, and let $\mathfrak{a}$ and $\mathfrak{f}$ be relatively prime integral ideals of F. The partial zeta function of F associated to the narrow ray class of $\mathfrak{a}$ modulo $\mathfrak{f}$ is defined by

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, s) = \sum_{\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{a}} \frac{1}{\text{N}\mathfrak{b}^s}, \qquad \text{Re}(s) > 1. \tag{2}$$

Here the sum ranges over integral ideals $\mathfrak{b} \subset F$ equivalent to $\mathfrak{a}$ in the narrow ray class group modulo $\mathfrak{f}$, which we denote $G_{\mathfrak{f}}$. The function $\zeta_{\mathfrak{f}}(\mathfrak{a}, s)$ has a meromorphic continuation to $\mathbf{C}$, with only a simple pole at $s = 1$.

Theorem 1. *The values $\zeta_{\mathfrak{f}}(\mathfrak{a}, -k)$ for integers $k \geq 0$ are rational.*

We prove Theorem 1 by showing that

$$\zeta_{\mathfrak{f}}(\mathfrak{a}, -k) = \langle \Phi_{\text{Sh}}, \mathfrak{Z}_k \rangle \tag{3}$$

where $\mathfrak{Z}_k \in H_{n-1}(\Gamma, \mathcal{N}^\vee)$ is a certain homology class depending on $\mathfrak{a}, \mathfrak{f}$, and k, and the indicated pairing is the cap product

$$H^{n-1}(\Gamma, \mathcal{N}) \times H_{n-1}(\Gamma, \mathcal{N}^\vee) \longrightarrow \mathbf{R}, \qquad \mathcal{N}^\vee = \text{Hom}(\mathcal{N}, \mathbf{R}). \tag{4}$$

See Theorem 2.10 below for a precise statement. Combined with a rationality property of our cocycle (Theorem 2.9) that implies that the cap product $\langle \Phi_{\text{Sh}}, \mathfrak{Z}_k \rangle$ lies in $\mathbf{Q}$, we deduce the desired result.

Our proof of Theorem 1 is simply a cohomological reformulation of Shintani's original argument. However, our construction has the benefit that we give an explicit signed fundamental domain. This latter feature is useful for computations and served as a motivation for [DDF] as well.

Comparison with the Sczech cocycle

Sczech's proof of Theorem 1 is deduced from an identity similar to (3), but involving a different cocycle. It leads to explicit formulas in terms of Bernoulli numbers that resemble those of Shintani in [Sh]. A natural question that emerges is whether a direct comparison of the two constructions is possible. Our next result, stated precisely in Theorem 3.1, is a proof that the cocycle on Γ defined in Sections 1 and 2 using Shintani's method is cohomologous (after projecting to the $+1$-eigenspace for the action of $\{\pm 1\}$ on $\mathcal{Q}$) to the cocycle defined by Sczech, up to a simple and minor error term. Rather than describing the details of Sczech's construction in this introduction, we content ourselves with explaining the combinatorial mechanism enabling the proof, with an informal discussion in the language of [Sc1, §2.2].

For n vectors $\tau_1, \ldots, \tau_n \in \mathbf{C}^n$, define a rational function of a variable $x \in \mathbf{C}^n$ by

$$f(\tau_1, \ldots, \tau_n)(x) = \frac{\det(\tau_1, \ldots, \tau_n)}{\langle x, \tau_1 \rangle \cdots \langle x, \tau_n \rangle}.$$

Given an n-tuple of matrices $A = (A_1, \ldots, A_n) \in \Gamma^n$, denote by A_{ij} the jth column of the matrix A_i. The function f satisfies a cocycle property (see (53)) that implies that the assignment $A \mapsto \alpha(A) := f(A_{11}, A_{21}, \ldots, A_{n1})$ defines a homogeneous $(n-1)$-cocycle on Γ valued in the space of functions on Zariski open subsets of $\mathbf{C}^n$. The rational function $\alpha(A)$ is not defined on the hyperplanes $\langle x, A_{i1} \rangle = 0$.

Alternatively we consider, for each $x \in \mathbf{C}^n - \{0\}$, the index $w_i = w_i(A, x)$ giving the leftmost column of A_i not orthogonal to x. The function $\beta(A)(x) = f(A_{1w_1}, \ldots, A_{nw_n})(x)$ is then defined on $\mathbf{C}^n - \{0\}$, and the assignment $A \mapsto \beta(A)$ can also be viewed as a homogeneous $(n-1)$-cocycle on Γ.

Using an explicit computation, we show that the function α corresponds to our Shintani cocycle (Proposition 3.10), whereas the function β yields Sczech's cocycle (Proposition 3.9). A coboundary relating α and β is then given as follows. Let $A = (A_1, \ldots, A_{n-1}) \in \Gamma^{n-1}$, and define for $i = 1, \ldots, n-1$:

$$h_i(A) = \begin{cases} f(A_{1w_1}, \ldots, A_{(i-1)w_{i-1}}, A_{i1}, A_{iw_i}, A_{(i+1)1}, \ldots, A_{(n-1)1}) & \text{if } w_i > 1 \\ 0 & \text{if } w_i = 1. \end{cases}$$

Let $h = \sum_{i=1}^{n-1} (-1)^i h_i$. We show that $\beta - \alpha = dh$. In the case $n = 2$, this recovers Sczech's formula [Sc1, Page 371].

Smoothing and applications to classical and p-adic L-functions

In Section 4, we fix a prime ℓ and we introduce a smoothed version $\Phi_{\mathrm{Sh},\ell}$ of the Shintani cocycle, essentially by taking a difference between Φ_{Sh} and a version of the same shifted by a matrix of determinant ℓ. The smoothed cocycle is defined on an arithmetic subgroup $\Gamma_\ell \subset \Gamma$ and shown to satisfy an integrality property (Theorem 4.7).

Through the connection of the Shintani cocycle to zeta values given by (3), this integrality property translates as in [CD] into corresponding results about special values of zeta functions. For the interest of the reader, we have included the statements of these arithmetic results in this introduction, and sketched the proofs in Section 4.4. For the details of the proofs we refer the reader to [CD], where these applications were already presented.

Our first arithmetic application of the smoothed cocycle is the following integral refinement of Theorem 1, originally due to Pi. Cassou-Noguès [Ca] and Deligne–Ribet [DR].

Theorem 2. *Let $\mathfrak{c}$ be an integral ideal of F relatively prime to $\mathfrak{f}$ and let $\ell = \mathrm{N}\mathfrak{c}$. The smoothed zeta function*

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) = \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{a}, s)$$

assumes values in $\mathbf{Z}[1/\ell]$ at nonpositive integers s.

Cassou–Noguès' proof of Theorem 2 is a refinement of Shintani's method under the assumption that $\mathcal{O}_F/\mathfrak{c}$ is cyclic. In this paper we give another proof of Theorem 2 that is essentially a cohomological reformulation of Cassou–Noguès' argument. For simplicity we assume further that $\ell = \mathrm{N}\mathfrak{c}$ is prime. After defining a modified version of the homology class $\mathfrak{Z}_k$ denoted $\mathfrak{Z}_{k,\ell}$, we show that $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = \langle \Phi_{\mathrm{Sh},\ell}, \mathfrak{Z}_{k,\ell}\rangle$. (See Theorem 4.9 for the precise statement.) A result from [CD] restated in Theorem 4.7 below implies that the cap product $\langle \Phi_{\mathrm{Sh},\ell}, \mathfrak{Z}_{k,\ell}\rangle$ lies in $\mathbf{Z}[1/\ell]$, thereby completing our proof of Theorem 2.

The final arithmetic application of our results regards the study of the p-adic L-functions associated to abelian characters of the totally real field F. Let $\psi\colon \mathrm{Gal}(\overline{F}/F) \longrightarrow \overline{\mathbf{Q}}^*$ be a totally even finite order character. We fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, so that ψ can be viewed as taking values in $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$. Let $\omega\colon \mathrm{Gal}(\overline{F}/F) \longrightarrow \mu_{p-1} \subset \overline{\mathbf{Q}}^*$ denote† the Teichmüller character. Using the integrality properties of our cocycle $\Phi_{\mathrm{Sh},\ell}$, we recover the following theorem of Cassou-Noguès [Ca], Barsky [Bs] and Deligne–Ribet [DR].

Theorem 3. *There is a unique meromorphic p-adic L-function $L_p(\psi, s)\colon \mathbf{Z}_p \longrightarrow \mathbf{C}_p$ satisfying the interpolation property*

$$L_p(\psi, 1-k) = L^*(\psi\omega^{-k}, 1-k)$$

for integers $k \geq 1$, where L^* denotes the classical L-function with Euler factors at the primes dividing p removed. The function L_p is analytic if $\psi \neq 1$. If $\psi = 1$, there is at most a simple pole at $s = 1$ and no other poles.

Now consider the totally odd character $\chi = \psi\omega^{-1}$, and let r_χ denote the number of primes $\mathfrak{p}$ of F above p such that $\chi(\mathfrak{p}) = 1$. In [Gr], Gross proposed the following:

Conjecture 1 (Gross)**.** *We have*

$$\mathrm{ord}_{s=0} L_p(\psi, s) = r_\chi.$$

†As usual, replace μ_{p-1} by $\{\pm 1\}$ when $p = 2$.

Combining our cohomological construction of the p-adic L-function with Spiess's formalism, we prove the following partial result towards Gross's conjecture:

Theorem 4. *We have*

$$\operatorname{ord}_{s=0} L_p(\psi, s) \geq r_\chi.$$

In the case $p > 2$, the result of Theorem 4 was already known from Wiles' proof of the Iwasawa Main Conjecture [Wi]. Our method contrasts with that of Wiles in that it is purely analytic; we calculate the kth derivative of $L_{\mathfrak{c},p}(\chi\omega, s)$ at $s = 0$ and show that it equals the cap product of a cohomology class derived from $\Phi_{\mathrm{Sh},\ell}$ with a certain homology class denoted $\mathfrak{Z}_{\log^k}$. See (101) for the precise formula. Spiess' theorem that the classes $\mathfrak{Z}_{\log^k}$ vanish for $k < r_\chi$ then concludes the proof. Our method applies equally well when $p = 2$.

Spiess proved Theorem 4 as well using his formalism and his alternate construction of a Shintani cocycle [Sp2]. Note that our cocycle Φ_{Sh} is "universal" in the sense that it is defined on the group $\Gamma = \mathbf{GL}_n(\mathbf{Q})$, whereas the cocycles defined by Spiess are restricted to subgroups arising from unit groups in totally real number fields. (See Section 2.1 below, where we describe how our universal cocycle Φ_{Sh} can be specialized to yield cocycles defined on unit groups.)

We should stress that while our proofs of Theorems 1, 2 and 3 are merely cohomological reformulations of the works of Shintani [Sh] and Cassou–Noguès [Ca], the proof of Theorem 4 relies essentially on the present cohomological construction and Spiess' theorems on cohomological p-adic L-functions. In upcoming work we explore further the application of the cohomological method towards the leading terms of p-adic L-functions at $s = 0$ and their relationship to Gross–Stark units [DS].

It is a pleasure to thank Pierre Colmez, Michael Spiess, and Glenn Stevens for helpful discussions and to acknowledge the influence of their papers [Co1], [Sp2], and [Stev] on this work. The first author thanks Alin Bostan and Bruno Salvy for many related discussions that stressed the importance of power series methods. In March 2010, the second and third authors gave a course at the Arizona Winter School that discussed Eisenstein cocycles. The question of proving that the Shintani and Sczech cocycles are cohomologous was considered by the students in our group: Jonathan Cass, Veronica Ertl, Brandon Levin, Rachel Newton, Ari Shnidman, and Ying Zhang. A complete proof was given for the smoothed cocycles in the case $n = 2$. We would like to thank these students and the University of Arizona for an exciting week in which some of the ideas present in this work were fostered.

1 The Shintani cocycle

1.1 Colmez perturbation

Consider linearly independent vectors $v_1, \ldots, v_n \in \mathbf{R}^m$. The open cone generated by the v_i is the set

$$C(v_1, \ldots, v_n) = \mathbf{R}_{>0}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_n.$$

We denote the characteristic function of this open cone by $\mathbf{1}_{C(v_1,\dots,v_n)}$. By convention, when $n = 0$, we define $C(\emptyset) = \{0\}$. Let $\mathcal{K}_{\mathbf{R}}$ denote the abelian group of functions $\mathbf{R}^m \to \mathbf{Z}$ generated by the characteristic functions of such open cones.

Fix now a subspace $V \subset \mathbf{R}^m$ spanned by arbitrary vectors $v_1,\dots,v_n \in \mathbf{R}^m$, and an auxiliary vector $Q \in \mathbf{R}^m$. We define a function $c_Q(v_1,\dots,v_n) \in \mathcal{K}_{\mathbf{R}}$ as follows. If the v_i are linearly dependent, then $c_Q(v_1,\dots,v_n) = 0$. If the v_i are linearly independent, we impose the further condition that $Q \in V$ but that Q is not in the $\mathbf{R}$-linear span of any subset of $n-1$ of the v_i. The function $c_Q(v_1,\dots,v_n)$ is defined to be the characteristic function of $C_Q(v_1,\dots,v_n)$, which is the disjoint union of the open cone $C(v_1,\dots,v_n)$ and some of its boundary faces (of all dimensions, including 0). A boundary face of the open cone C is included in C_Q if translation of an element of the face by a small positive multiple of Q sends that element into the interior of C. Formally, we have:

$$c_Q(v_1,\dots,v_n)(w) = \begin{cases} \lim_{\epsilon\to 0^+} \mathbf{1}_{C(v_1,\dots,v_n)}(w+\epsilon Q) & \text{if the } v_i \text{ are linearly independent,} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The limit in (5) is easily seen to exist and is given explicitly as follows. If $w \notin V$, then $c_Q(v_1,\dots,v_n)(w) = 0$. On the other hand if

$$w = \sum_{i=1}^n w_i v_i, \qquad Q = \sum_{i=1}^n q_i v_i \quad (\text{all } q_i \neq 0),$$

then

$$c_Q(v_1,\dots,v_n)(w) = \begin{cases} 1 & \text{if } w_i \geq 0 \text{ and } w_i = 0 \Rightarrow q_i > 0 \text{ for } i = 1,\dots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let us give one more characterization of this "Q-perturbation process" that will be useful for future calculations. For simplicity we suppose $m = n$ and that the vectors v_i are linearly independent. We denote by σ the $n \times n$ matrix whose columns are the vectors v_i. For each subset $I \subset \{1,\dots,n\}$, we have the open cone $C_I = C(v_i : i \in I)$. The weight of this cone (equal to 0 or 1) in the disjoint union C_Q is given as follows. Let $d = |I|$. The d-dimensional subspace containing the cone C_I can be expressed as the intersection of the $n-d$ codimension 1 hyperplanes determined by $v_i^* = 0$, for $i \in \overline{I} = \{1,\dots,n\} - I$. Here $\{v_i^*\}$ is the dual basis to the v_i. Under the usual inner product on $\mathbf{R}^n$, the v_i^* are the columns of the matrix σ^{-t}. Each hyperplane $v_i^* = 0$ divides its complement into a plus part and minus part, namely the half-space containing the cone $C(v_1,\dots,v_n)$ and the half-space not containing the cone (as an inequality, $\langle w, v_i^*\rangle > 0$ or < 0). The weight of C_I is equal to 1 if Q lies in the totally positive region defined by these hyperplanes, i.e. if $\langle Q, v_i^*\rangle > 0$ for all $i \in \overline{I}$. Otherwise, the weight of C_I is 0. In summary,

$$\text{weight}(C_I) = \prod_{i\in\overline{I}} \frac{1+\text{sign}(Q\sigma^{-t})_i}{2}. \tag{7}$$

Note that this formula is valid for $d = n$ as well, with the standard convention that empty products are equal to 1.

1.2 Cocycle relation

We now derive a cocycle relation satisfied by the functions c_Q. Let $v_1, \ldots, v_n \in \mathbf{R}^m$ be linearly independent vectors, with $n \geq 1$. A set of the form

$$L = \mathbf{R}v_1 + \mathbf{R}_{>0}v_2 + \cdots + \mathbf{R}_{>0}v_n \tag{8}$$

is called a *wedge*. The characteristic function $\mathbf{1}_L$ of L is an element of $\mathcal{K}_{\mathbf{R}}$ since

$$L = C(v_1, \ldots, v_n) \sqcup C(v_2, \ldots, v_n) \sqcup C(-v_1, v_2, \ldots, v_n).$$

Let $\mathcal{L}_{\mathbf{R}} = \mathcal{L}_{\mathbf{R}}(\mathbf{R}^m) \subset \mathcal{K}_{\mathbf{R}}$ be the subgroup generated by the functions $\mathbf{1}_L$ for all wedges L.

Theorem 1.1. *Let $n \geq 1$, and let $v_0, \ldots, v_n \in \mathbf{R}^m$ be nonzero vectors spanning a subspace V of dimension at most n. Let $Q \in V$ be a vector not contained in the span of any subset of $n-1$ of the v_i. Let B denote a fixed ordered basis of V and define for each i the orientation*

$$O_B(\hat{v}_i) := O_B(v_0, \ldots, \hat{v}_i, \ldots, v_n) = \operatorname{sign}\det(v_0, \ldots, \hat{v}_i, \ldots, v_n) \in \{0, \pm 1\},$$

where the written matrix gives the representation of the vectors v_j in terms of the basis B, for $j \neq i$. Then

$$\sum_{i=0}^{n} (-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n) \equiv 0 \pmod{\mathcal{L}_{\mathbf{R}}}. \tag{9}$$

Furthermore, if each v_i lies in the totally positive orthant $(\mathbf{R}_{>0})^m$, then in fact

$$\sum_{i=0}^{n} (-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n) = 0.$$

Proof. We prove the result by induction on n. For the base case $n = 1$, the argument for the "general position" case below gives the desired result; alternatively one can check the result in this case by hand.

For the inductive step, note first that the result is trivially true by the definition of c_Q unless $\dim V = n$. We therefore suppose this holds and consider two cases.

Case 1: The v_i are in general position in V, i.e. any subset of $\{v_0, \ldots, v_n\}$ of size n spans V. For any $w \in V$, it then follows from our assumption on Q that for $\epsilon > 0$ small enough, the set $\{v_0, \ldots, v_n, w + \epsilon Q\}$ is in general position in V. In view of the definition of c_Q given in (5), Proposition 2 of [Hi] therefore implies that the left side of (9) is a constant function on V taking the value $d(v_0, \ldots, v_n)$ defined as follows. Let λ_i for $i = 0, \ldots, n$ be nonzero constants such that $\sum_{i=0}^{n} \lambda_i v_i = 0$. The λ_i are well-defined up to a simultaneous scalar multiplication. Then

$$d(v_0, \ldots, v_n) = \begin{cases} (-1)^i O_B(\hat{v}_i) & \text{if the } \lambda_i \text{ all have the same sign,} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

One readily checks that right side of (10) is independent of i. Now, the characteristic function of V lies in $\mathcal{L}_{\mathbf{R}}$, giving the desired result. Furthermore, if the v_i lie in the totally positive orthant $(\mathbf{R}_{>0})^m$, then the λ_i cannot all have the same sign and hence $d(v_0, \ldots, v_n) = 0$. This completes the proof in the case where the v_i are in general position.

Case 2: The v_i are not in general position. Without loss of generality, assume that $v_0, \ldots, v_{n-1}$ are linearly dependent. Let V' denote the $(n-1)$-dimensional space spanned by these n vectors. Denote by $\pi' \colon V \to V'$ and $\pi \colon V \to \mathbf{R}$ the projections according to the direct sum decomposition $V = V' \oplus \mathbf{R}v_n$. We claim that for $i = 0, \ldots, n-1$ and $w \in V$, we have

$$c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) = c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)) \cdot g_Q(w), \tag{11}$$

where

$$g_Q(w) = \begin{cases} 1 & \text{if } \pi(w) \geq 0 \text{ and } \pi(w) = 0 \Rightarrow \pi(Q) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

First note that if $v_0, \ldots, \hat{v}_i, \ldots, v_n$ are linearly dependent, then under our conditions we necessarily have that $v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}$ are linearly dependent, and both sides of (11) are zero.

Therefore suppose that the vectors $v_0, \ldots, \hat{v}_i, \ldots, v_n$ are linearly independent, in which case $v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}$ are clearly linearly independent as well, and hence span V'. Furthermore $\pi'(Q) \in V'$ satisfies the condition that it is not contained in the span of any subset of $n-2$ of these vectors, or else Q would lie in the span of $n-1$ of the original vectors $v_0, \ldots, v_n$; hence the right side of (11) is well-defined. Equation (11) now follows directly from the interpretation of the function c_Q given in (6).

To deal with the orientations note that if B' is any other basis of V, then

$$O_B(\hat{v}_i) = O_B(B') \cdot O_{B'}(\hat{v}_i). \tag{12}$$

We therefore choose for convenience a basis B' for V whose last element is the vector v_n.

Using (11) and (12) and the fact that $c_Q(v_0, \ldots, v_{n-1}) = 0$ since $v_0, \ldots, v_{n-1}$ are linearly dependent, we calculate

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i O_B(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) &= O_B(B') \sum_{i=0}^{n-1} (-1)^i O_{B'}(\hat{v}_i) c_Q(v_0, \ldots, \hat{v}_i, \ldots, v_n)(w) \\ &= O_B(B') \ell_Q(w) g_Q(w), \end{aligned}$$

where

$$\ell_Q(w) = \sum_{i=0}^{n-1} (-1)^i O_{B'}(\hat{v}_i) c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)).$$

Now if we let B'' be the basis of V' given by the image of the first $n-1$ elements of B' under π', it is clear that

$$O_{B'}(\hat{v}_i) = O_{B''}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}).$$

Therefore the function ℓ_Q can be written

$$\ell_Q(w) = \sum_{i=0}^{n-1} (-1)^i O_{B''}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1}) c_{\pi'(Q)}(v_0, \ldots, \hat{v}_i, \ldots, v_{n-1})(\pi'(w)).$$

This is the exact form for which we can use the inductive hypothesis to conclude that $\ell_Q \in \mathcal{L}_{\mathbf{R}}(V')$ and $\ell_Q = 0$ if each v_i lies in the totally positive orthant. It is readily checked that this implies that $\ell_Q g_Q \in \mathcal{L}_{\mathbf{R}}(V)$ as desired (and $\ell_Q g_Q = 0$ if each v_i lies in the totally positive orthant). □

1.3 Signed fundamental domains

In this section we show that Theorem 1.1 can be combined with a result of Colmez to deduce a theorem of Diaz y Diaz and Friedman on the existence of signed Shintani domains. We use this result in the proof of Theorem 2.10 in order to relate our cocycle to the special values of partial zeta functions.

Consider the totally positive orthant $(\mathbf{R}_{>0})^n \subset \mathbf{R}^n$, which forms a group under the operation $*$ of componentwise multiplication. Let $D = \{x \in (\mathbf{R}_{>0})^n : x_1 x_2 \cdots x_n = 1\}$. Let $U \subset D$ denote a subgroup that is discrete and free of rank $n-1$. The goal of this section is to determine an explicit fundamental domain for the action of U on the totally positive orthant in terms of an ordered basis $\{u_1, \ldots, u_{n-1}\}$ for U.

Define the orientation

$$w_u := \operatorname{sign}\det(\log(u_{ij}))_{i,j=1}^{n-1}) = \pm 1, \tag{13}$$

where u_{ij} denotes the jth coordinate of u_i. For each permutation $\sigma \in S_{n-1}$ let

$$v_{i,\sigma} = u_{\sigma(1)} \cdots u_{\sigma(i-1)} \in U, \qquad i = 1, \ldots, n$$

(so by convention $v_{1,\sigma} = (1, 1, \ldots, 1)$ for all σ). Define

$$w_\sigma = (-1)^{n-1} w_u \operatorname{sign}(\sigma) \operatorname{sign}(\det(v_{i,\sigma})_{i=1}^{n}) \in \{0, \pm 1\}.$$

We choose for our perturbation vector the coordinate basis vector $e_n = (0, 0, \ldots, 0, 1)$, and assume that e_n satisfies the property that it does not lie in the $\mathbf{R}$-linear span of any $(n-1)$ of elements of U. Note that the action of U preserves the ray $\mathbf{R}_{>0} e_n$.

Theorem 1.2 (Colmez, [Co1], Lemme 2.2)**.** *If $w_\sigma = 1$ for all $\sigma \in S_{n-1}$, then*

$$\bigsqcup_{\sigma \in S_{n-1}} C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma}) \tag{14}$$

is a fundamental domain for the action of U on the totally positive orthant $(\mathbf{R}_{>0})^n$. In other words, we have

$$\sum_{u \in U} \sum_{\sigma \in S_{n-1}} c_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})(u * x) = 1$$

for all $x \in (\mathbf{R}_{>0})^n$.

Remark 1.3. Note that each of the vectors $v_{i,\sigma}$ lies in the positive orthant, so each open cone $C(v_{i_1,\sigma}, \ldots, v_{i_r,\sigma})$ is contained in the positive orthant when $r \geq 1$. Furthermore, e_n lies along a coordinate axis and is not contained in $C(v_{1,\sigma}, \ldots, v_{n,\sigma})$, hence $0 \notin C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})$. Therefore $C_Q(v_{1,\sigma}, \ldots, v_{n,\sigma}) \subset (\mathbf{R}_{>0})^n$.

The following generalization was recently proved by Diaz y Diaz and Friedman using topological degree theory. We will show that the cocycle property of c_Q proved in Theorem 1.1 allows one to deduce their theorem from the earlier result of Colmez. Note that our proof of the theorem relies upon Colmez's theorem, whereas the proof of Diaz y Diaz and Friedman recovers it.

Definition 1.4. A *signed fundamental domain* for the action of U on $(\mathbf{R}_{>0})^n$ is by definition a formal linear combination $D = \sum_i a_i C_i$ of open cones with $a_i \in \mathbf{Z}$ such that $\sum_{u \in U} \sum_i a_i \mathbf{1}_{C_i}(u * x) = 1$ for all $x \in (\mathbf{R}_{>0})^n$. We call $\mathbf{1}_D := \sum_i a_i \mathbf{1}_{C_i} \in \mathcal{K}_{\mathbf{R}}$ the characteristic function of D.

Note that when each $a_i = 1$ and the C_i are disjoint, the set $\sqcup_i C_i$ is a fundamental domain in the usual sense.

Theorem 1.5 (Diaz y Diaz–Friedman, [DDF], Theorem 1)**.** *The formal linear combination*

$$\sum_{\sigma \in S_{n-1}} w_\sigma C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})$$

is a signed fundamental domain for the action of U on $(\mathbf{R}_{>0})^n$, i.e.

$$\sum_{u \in U} \sum_{\sigma \in S_{n-1}} w_\sigma c_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})(u * x) = 1 \tag{15}$$

for all $x \in (\mathbf{R}_{>0})^n$.

Proof. Colmez proved the existence of a finite index subgroup $V \subset U$ such that the condition $w_\sigma = 1$ for all σ holds for some basis of V (see [Co1], Lemme 2.1). Fix such a subgroup V. Our technique is to reduce the desired result for U to the result for V, which is given by Colmez's theorem.

Endow the abelian group $\mathcal{K}_{\mathbf{R}}$ with an action of U by

$$(uf)(x) = f(u^{-1} * x). \tag{16}$$

The key point of our proof is the construction of a cohomology class $[\phi_U] \in H^{n-1}(U, \mathcal{K}_{\mathbf{R}})$ as follows. Given $v_1, \ldots, v_n \in U$, let

$$\phi_U(v_1, \ldots, v_n) = \mathrm{sign}(\det(v_i)_{i=1}^n) c_{e_n}(v_1, \ldots, v_n) \in \mathcal{K}_{\mathbf{R}}. \tag{17}$$

The U-invariance of ϕ_U follows from the definition of c_{e_n} given in (5) along with the above-noted property that the action of U preserves $\mathbf{R}_{>0} e_n$. The fact that ϕ_U satisfies the cocycle property

$$\sum_{i=0}^{n} (-1)^i \phi_U(v_0, \ldots, \hat{v}_i, \ldots, v_n) = 0$$

is given by Theorem 1.1, since the v_i lie in the positive orthant. We let $[\phi_U] \in H^{n-1}(U, \mathcal{K}_{\mathbf{R}})$ be the cohomology class represented by the homogeneous cocycle ϕ_U.

The basis $u_1, \ldots, u_{n-1}$ of U gives an explicit element $\alpha_U \in H_{n-1}(U, \mathbf{Z}) \cong \mathbf{Z}$ as follows. We represent homology classes by the standard projective resolution $C^*(U) = \mathbf{Z}[U^{*+1}]$ of $\mathbf{Z}$, and let α_U be the class represented by the cycle

$$\alpha(u_1, \ldots, u_{n-1}) = (-1)^{n-1} w_u \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma)[(v_{1,\sigma}, \ldots, v_{n,\sigma})] \in \mathbf{Z}[U^n]. \tag{18}$$

It is a standard calculation that $d\alpha(u_1, \ldots, u_{n-1}) = 0$ and that the cohomology class α_U represented by $\alpha(u_1, \ldots, u_{n-1})$ depends only on U and not the chosen basis $u_1, \ldots, u_{n-1}$ (see [Sc2, Lemma 5]).

The image of $([\phi_U], \alpha_U)$ under the cap product pairing

$$H^{n-1}(U, \mathcal{K}_{\mathbf{R}}) \times H_{n-1}(U, \mathbf{Z}) \longrightarrow \mathcal{K}_{\mathbf{R},U} := H_0(U, \mathcal{K}_{\mathbf{R}})$$

is by definition the image of the function $\sum_{\sigma \in S_{n-1}} w_\sigma c_Q(v_{1,\sigma}, \ldots, v_{n,\sigma})$ in $\mathcal{K}_{\mathbf{R},U}$.

Let $\mathcal{J}$ denote the group of functions $(\mathbf{R}_{>0})^n \longrightarrow \mathbf{Z}$, which is endowed with an action of U as in (16). Denote by $\Sigma_U : \mathcal{K}_{\mathbf{R},U} \to \mathcal{J}^U$ the map defined by

$$(\Sigma_U f)(x) = \sum_{u \in U} f(u * x). \tag{19}$$

Note that the sum (19) is locally finite by the following standard compactness argument. The action of U preserves the product of the coordinates of a vector, and applying log to the coordinates sends the surface $\{x_1 \cdots x_n = \text{constant}\}$ to a hyperplane. In this hyperplane, the image of a cone is bounded, and the action of U is translation by a lattice. Given a point x, only finitely many lattice points can translate x into the bounded region corresponding to a cone.

Now $\Sigma_U(\phi_U \cap \alpha_U) \in \mathcal{J}^U$ is by definition the function on the left side of (15), namely

$$\sum_{u \in U} \sum_{\sigma \in S_{n-1}} w_\sigma c_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})(u * x).$$

It remains to analyze this picture when U is replaced by its finite index subgroup V chosen at the outset of the proof. General properties of group cohomology (see [Br, pp. 112–114]) yield a commutative diagram:

$$\begin{array}{ccccccc}
H^{n-1}(V, \mathcal{K}_{\mathbf{R}}) & \times & H_{n-1}(V, \mathbf{Z}) & \xrightarrow{\cap} & \mathcal{K}_{\mathbf{R},V} & \xrightarrow{\Sigma_V} & \mathcal{J}^V \\
{\scriptstyle \mathrm{res}}\uparrow & & \downarrow{\scriptstyle \mathrm{cores}} & & \downarrow & & \downarrow{\scriptstyle \Sigma_{U/V}} \\
H^{n-1}(U, \mathcal{K}_{\mathbf{R}}) & \times & H_{n-1}(U, \mathbf{Z}) & \xrightarrow{\cap} & \mathcal{K}_{\mathbf{R},U} & \xrightarrow{\Sigma_U} & \mathcal{J}^U.
\end{array}$$

Here $\Sigma_{U/V} : \mathcal{J}^V \to \mathcal{J}^U$ is given by

$$(\Sigma_{U/V} f)(x) = \sum_{u \in U/V} f(u * x).$$

The desired result now follows from the fact that $\Sigma_V(\phi_V \cap \alpha_V) = \mathbf{1} := \mathbf{1}_{(\mathbf{R}_{>0})^n}$ by Colmez's theorem, along with

$$\operatorname{cores}(\alpha_V) = [U:V] \cdot \alpha_U, \tag{20}$$
$$\operatorname{res} \phi_U = \phi_V, \tag{21}$$
$$\Sigma_{U/V}(\mathbf{1}) = [U:V] \cdot \mathbf{1}. \tag{22}$$

Equation (20) is proven in [Br, Sect. III, Prop. 9.5], whereas (21) and (22) are obvious. □

1.4 The Shintani cocycle on $\mathbf{GL}_n(\mathbf{Q})$

Recall the notation $\Gamma = \mathbf{GL}_n(\mathbf{Q})$. In this section we define a Shintani cocycle Φ_{Sh} on Γ. This cocycle will be directly related to the cocycles ϕ_U defined in the previous section; however, since our cocycle will be defined on the full group Γ rather than the simpler groups $U \subset D$ in the positive orthant, we will need to consider the quotient $\mathcal{K}_{\mathbf{R}}/\mathcal{L}_{\mathbf{R}}$ rather than $\mathcal{K}_{\mathbf{R}}$ (cf. definition (8) and the appearance of $\mathcal{L}_{\mathbf{R}}$ in Theorem 1.1). The relationship between Φ_{Sh} and the ϕ_U in our cases of interest will be stated precisely in Section 2.1 below.

Let $\mathbf{R}^n_{\text{Irr}}$ denote the set of elements in $\mathbf{R}^n$ (viewed as row vectors) whose components are linearly independent over $\mathbf{Q}$, i.e. the set of vectors Q such that $Q \cdot x \neq 0$ for nonzero $x \in \mathbf{Q}^n$. The set $\mathbf{R}^n_{\text{Irr}}$ is a right Γ-set by the action of right multiplication; we turn this into a left action by multiplication on the right by the transpose (i.e. $\gamma \cdot Q := Q\gamma^t$). Note that any $Q \in \mathbf{R}^n_{\text{Irr}}$ satisfies the property that it does not lie in the $\mathbf{R}$-linear span of any $n-1$ vectors in $\mathbf{Q}^n \subset \mathbf{R}^n$. The elements of $\mathbf{R}^n_{\text{Irr}}$ will therefore serve as our set of auxiliary perturbation vectors as employed in Section 1.1.† We let $\mathcal{Q} = \mathbf{R}^n_{\text{Irr}}/\mathbf{R}_{>0}$, the set of equivalence class of elements of $\mathbf{R}^n_{\text{Irr}}$ under multiplication by positive reals.

Let $\mathcal{K} \subset \mathcal{K}_{\mathbf{R}}$ denote the subgroup generated by the characteristic functions of rational open cones, i.e. by the characteristic functions of cones $C(v_1, \ldots, v_n)$ with each $v_i \in \mathbf{Q}^n$. Let $\mathcal{L} = \mathcal{L}_{\mathbf{R}} \cap \mathcal{K}$ with $\mathcal{L}_{\mathbf{R}}$ as in (8). The abelian group $\mathcal{K}$ is naturally endowed with a left Γ-module structure via

$$\gamma \cdot \varphi(x) = \operatorname{sign}(\det \gamma))\varphi(\gamma^{-1}x),$$

and $\mathcal{L}$ is a Γ-submodule of $\mathcal{K}$.

Let $\mathcal{N}$ denote the abelian group of maps $\mathcal{Q} \longrightarrow \mathcal{K}/\mathcal{L}$. This space is endowed with a Γ-action given by $(\gamma f)(Q) = \gamma f(\gamma^{-1}Q)$. We now define a homogeneous cocycle

$$\Phi_{\text{Sh}} \in Z^{n-1}(\Gamma, \mathcal{N}).$$

For $A_1, \ldots, A_n \in \Gamma$, let σ_i denote the first column of A_i. Given $Q \in \mathcal{Q}$, define

$$\Phi_{\text{Sh}}(A_1, \ldots, A_n)(Q) = \operatorname{sign}(\det(\sigma_1, \ldots, \sigma_n))c_Q(\sigma_1, \ldots, \sigma_n) \tag{23}$$

†To orient the reader who may be familiar with the notation of [Co3] or [DDF] in which one takes $Q = e_n = (0, 0, \ldots, 0, 1)$ as in Section 1.3, one goes from this vector to an element of our $\mathbf{R}^n_{\text{Irr}}$ by applying a change of basis given by the image in $\mathbf{R}^n$ of a basis of a totally real field F of degree n. Our notation allows for rational cones C and irrational perturbation vectors Q rather than the reverse. This is convenient for comparison with Sczech's cocycle, in which one also chooses a vector $Q \in \mathbf{R}^n_{\text{Irr}}$. See Section 2.1 and in particular (26) for more details.

with c_Q as in (5).

Theorem 1.6. *We have* $\Phi_{\text{Sh}} \in Z^{n-1}(\Gamma, \mathcal{N})$.

Proof. The fact that Φ_{Sh} is Γ-invariant follows directly from the definitions. In $\mathcal{K}$, the cocycle property

$$\sum_{i=0}^{n}(-1)^i \Phi_{\text{Sh}}(A_0, \ldots, \hat{A}_i, \ldots, A_n)(Q) \equiv 0 \pmod{\mathcal{L}}$$

follows from Theorem 1.1 using for B the standard basis of $\mathbf{R}^n$. □

Denote by $[\Phi_{\text{Sh}}] \in H^{n-1}(\Gamma, \mathcal{N})$ the cohomology class represented by the homogeneous cocycle Φ_{Sh}.

2 Applications to Zeta Functions

2.1 Totally real fields

Let F be a totally real field of degree n, and denote by $J_1, \ldots, J_n : F \to \mathbf{R}$ the n real embeddings of F. Write $J = (J_1, \ldots, J_n) : F \to \mathbf{R}^n$. We denote the action of F^* on $\mathbf{R}^n$ via composition with J and componentwise multiplication by $(x, v) \mapsto x * v$. Let U denote a subgroup of finite index in the group of totally positive units in $\mathcal{O}_F^*$. We can apply the discussion of Section 1.3 on fundamental domains to the group $J(U) \subset D$.

Note that $e_n = (0, 0, \ldots, 0, 1)$ satisfies the property that it does not lie in the $\mathbf{R}$-linear span of any $n-1$ elements of the form $J(u)$ for $u \in F^*$. Indeed, given $u_1, \ldots, u_{n-1} \in F^*$, there exists an $x \in F^*$ such that $\text{Tr}_{F/\mathbf{Q}}(xu_i) = 0$ for all $i = 1, \ldots, n-1$. Dot product with $J(x)$ defines an $\mathbf{R}$-linear functional on $\mathbf{R}^n$ that vanishes on the $J(u_i)$ but not on e_n, proving the claim.

In this section we explain the relationship between the class $[\Phi_{\text{Sh}}]$ and the class $[\phi_U]$ defined in Section 1.3 (where we write ϕ_U for $\phi_{J(U)}$). Choosing a $\mathbf{Z}$-basis $w = (w_1, w_2, \ldots, w_n)$ of $\mathcal{O}_F$ yields an embedding $\rho_w \colon F^* \to \Gamma$ given by

$$(w_1 u, w_2 u, \ldots, w_n u) = (w_1, w_2, \ldots, w_n)\rho_w(u). \tag{24}$$

Pullback by ρ_w (i.e. restriction) yields a class $\rho_w^* \Phi_{\text{Sh}} \in H^{n-1}(U, \mathcal{N})$.

Denote by $J(w) \in \mathbf{GL}_n(\mathbf{R})$ the matrix given by $J(w)_{ij} = J_i(w_j)$. Note that if we let $\text{diag}(J(u))$ be the diagonal matrix with diagonal entries $J_i(u)$, then

$$\rho_w(u) = J(w)^{-1} \,\text{diag}(J(u)) J(w). \tag{25}$$

Let

$$Q = (0, 0, \ldots, 1) J(w)^{-t}. \tag{26}$$

The vector Q is the image under J_n of the dual basis to w under the trace pairing $F \times F \to \mathbf{Q}$, $(x, y) \mapsto \text{Tr}_{F/\mathbf{Q}}(xy)$. In particular, Q is an element of $\mathbf{R}^n_{\text{Irr}}$. Furthermore, (25) and (26) yield

$$Q \rho_w(x)^t = J_n(x) Q$$

for $x \in F^*$, which implies that the image of Q in $\mathcal{Q}$ is invariant under the action of U. We can therefore view Q as an element of $H^0(U, \mathbf{Z}[\mathcal{Q}])$. In conjunction with the canonical map $\mathcal{N} \times \mathcal{Q} \to \mathcal{K}/\mathcal{L}$ given by $(f, Q) \mapsto f(Q)$, the cup product gives a map

$$H^{n-1}(U, \mathcal{N}) \times H^0(U, \mathbf{Z}[\mathcal{Q}]) \to H^{n-1}(U, \mathcal{K}/\mathcal{L})$$

yielding an element $\rho_w^*[\Phi_{\mathrm{Sh}}] \cup Q \in H^{n-1}(U, \mathcal{K}/\mathcal{L})$.

Now consider the map induced by $J(w)$, denoted

$$J(w)^* : \mathcal{K}_{\mathbf{R}} \longrightarrow \mathcal{K}_{\mathbf{R}},$$

given by $(J(w)^* f)(x) = f(J(w)x)$. Our desired relation is

$$J(w)^*[\phi_U] = \rho_w^*[\Phi_{\mathrm{Sh}}] \cup Q$$

in $H^{n-1}(U, \mathcal{K}_{\mathbf{R}}/\mathcal{L}_{\mathbf{R}})$. In fact, this relationship holds on the level of cocycles as follows. For any $x \in F^*$, we define a modified cocycle $\Phi_{\mathrm{Sh},x} \in Z^{n-1}(\Gamma, \mathcal{N})$ by letting $\gamma = \rho_w(x)^{-1}$ and setting

$$\Phi_{\mathrm{Sh},x}(A_1, \ldots, A_n) = \Phi_{\mathrm{Sh}}(A_1\gamma, \ldots, A_n\gamma). \tag{27}$$

It is a standard fact in group cohomology that the cohomology class represented by $\Phi_{\mathrm{Sh},x}$ is independent of x and hence equal to $[\Phi_{\mathrm{Sh}}]$ (see [Sc2, Lemma 4]). We have the following equality of cocycles:

$$J(w)^*\phi_U = \rho_w^*\Phi_{\mathrm{Sh},w_1} \cup Q$$

in $Z^{n-1}(U, \mathcal{K}_{\mathbf{R}}/\mathcal{L}_{\mathbf{R}})$. In concrete terms, this says for $u = (u_1, \ldots, u_n)$:

$$\phi_U(u)(J(w)x) = \Phi_{\mathrm{Sh},w_1}(\rho_w(u), Q)(x). \tag{28}$$

In Section 2.6 this relationship will be used along with Theorem 1.5 to relate the class $[\Phi_{\mathrm{Sh}}]$ to special values of zeta functions attached to the field F. Over the next few sections we first we recall Shintani's results on cone zeta functions.

2.2 Some bookkeeping

We will be interested in sums over the points lying in the intersection of open simplicial cones with certain lattices in $\mathbf{R}^n$. In this section we introduce a convenient way of enumerating these points. Let $\mathcal{V} = \mathbf{Q}^n/\mathbf{Z}^n$, and consider for $v \in \mathcal{V}$ the associated lattice $v + \mathbf{Z}^n \subset \mathbf{R}^n$.

Let C be a rational open cone. By scaling the generators of C, we can find $\mathbf{R}$-linearly independent vectors $\sigma_1, \ldots, \sigma_r \in \mathbf{Z}^n$ such that $C = \mathbf{R}_{>0}\sigma_1 + \cdots + \mathbf{R}_{>0}\sigma_r$. Let $\mathscr{P} = \mathscr{P}(\sigma_1, \ldots, \sigma_r)$ denote the half-open parallelpiped generated by the σ_i:

$$\mathscr{P} = \{x_1\sigma_1 + \cdots + x_r\sigma_r : 0 < x_1, \ldots, x_r \leq 1\}, \tag{29}$$

with the understanding that $\mathscr{P}(\emptyset) = \{0\}$ in the case $r = 0$. Then

$$C \cap (v + \mathbf{Z}^n) = \bigsqcup_{a \in \mathscr{P} \cap (v + \mathbf{Z}^n)} (a + \mathbf{Z}_{\geq 0}\sigma_1 + \cdots + \mathbf{Z}_{\geq 0}\sigma_r), \tag{30}$$

where the disjointness of the union follows from the linear independence of the σ_i.

Now let C be a rational open cone of maximal dimension $r = n$ in $\mathbf{R}^n$. Let $Q \in \mathcal{Q}$ and consider the set C_Q defined in Section 1.1, consisting of the disjoint union C and some of its boundary faces of all dimensions. We would like to enumerate the points in $C_Q \cap (v + \mathbf{Z}^n)$.

For each subset $I \subset \{1, \ldots, n\}$, the boundary face $C_I = C(\sigma_i : i \in I)$ is assigned a weight via the Q-perturbation process denoted weight$(C_I) \in \{0, 1\}$ and given by (7). Associated to each cone C_I is the parallelpiped $\mathscr{P}_I = \mathscr{P}(\sigma_i : i \in I)$. We have

$$C_Q \cap (v + \mathbf{Z}^n) = \bigsqcup_{\substack{I \subset \{1,\ldots,n\} \\ a \in \mathscr{P}_I \cap (v + \mathbf{Z}^n)}} \mathrm{weight}(C_I)(a + \sum_{i \in I} \mathbf{Z}_{\geq 0}\sigma_i), \tag{31}$$

where our notation means that the set $(a + \sum \mathbf{Z}_{\geq 0}\sigma_i)$ should be included if weight$(C_I) = 1$ and not included if weight$(C_I) = 0$.

Let $\sigma \in M_n(\mathbf{Z}) \cap \Gamma$ denote the matrix whose columns are the σ_i. For each $a \in \mathscr{P}_I \cap (v + \mathbf{Z}^n)$ that occurs as I ranges over all subsets of $\{1, \ldots, n\}$, we can associate the class $x = a - v \in \mathbf{Z}^n/\sigma\mathbf{Z}^n$. Conversely, given a class $x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n$, there will be at least one a giving rise to that class. To be more precise, let $J = J(x)$ denote the set of indices j for which $(\sigma^{-1}(v+x))_j \in \mathbf{Z}$. The number of points a giving rise to the class x is $2^{\#J}$. Let $\overline{J} = \{1, \ldots, n\} - J$. For each $I \supset \overline{J}$, we can write down a unique point $a_I \in \mathscr{P}_I$ such that the image of $a_I - v$ in $\mathbf{Z}^n/\sigma\mathbf{Z}^n$ is equal to x. We define a_I by letting $\sigma^{-1}(a_I)$ be congruent to $\sigma^{-1}(v + x)$ modulo $\mathbf{Z}^n$, and further requiring $\sigma^{-1}(a_I)_i \in (0, 1)$ if $i \notin J$, and

$$\sigma^{-1}(a_I)_i := \begin{cases} 0 & i \in J \cap \overline{I} = \overline{I} \\ 1 & i \in J \cap I. \end{cases} \tag{32}$$

We can then rewrite (31) as

$$C_Q \cap (v + \mathbf{Z}^n) = \bigsqcup_{\substack{x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n \\ I \supset \overline{J(x)}}} \mathrm{weight}(C_I)(a_I + \sum_{i \in I} \mathbf{Z}_{\geq 0}\sigma_i). \tag{33}$$

This decomposition will be used in Sections 3.2 and 4.2.

2.3 Cone generating functions

Let C be a rational open cone in $\mathbf{R}^n$ and let $v \in \mathbf{Q}^n$. Let $x_1, \ldots, x_n$ be variables and let $g(C, v)$ be the generating series for the set of integer points in $C - v$:

$$g(C, v)(x) = \sum_{m \in (C - v) \cap \mathbf{Z}^n} x^m \in \mathbf{Q}[[x, x^{-1}]],$$

where as usual x^m denotes $x_1^{m_1} \cdots x_n^{m_n}$. If $\mu \in \mathbf{Z}^n$, then $g(C, v + \mu) = x^{-\mu} g(C, v)$.

The series $g(C, v)$ is actually the power series expansion of a rational function. In fact, the decomposition (30) gives rise to the identity

$$g(C, v)(x) = \frac{\sum_{a \in (\mathscr{P} - v) \cap \mathbf{Z}^n} x^a}{(1 - x^{\sigma_1}) \cdots (1 - x^{\sigma_r})} \in \mathbf{Q}(x), \tag{34}$$

where σ_i are integral generators of the cone C, and $\mathscr{P} = \mathscr{P}(\sigma_1, \dots, \sigma_r)$ is the half-open parallelpiped defined in (29).

Write c for the characteristic function of C and define $g(c, v) = g(C, v)$. The following fundamental algebraic result was proved independently by Khovanskii and Pukhilov [KP] and Lawrence [La] (cf. [Bv, Theorem 2.4]).

Proposition 2.1. *There is a unique map $g : \mathcal{K} \times \mathbf{Q}^n \longrightarrow \mathbf{Q}(x)$ that is $\mathbf{Q}$-linear in the first variable such that $g(c, v) = g(C, v)$ for all rational open cones C and $g(c, v) = 0$ if $c \in \mathcal{L}$.*

Thus we may view g as a pairing

$$g : \mathcal{K}/\mathcal{L} \times \mathbf{Q}^n \longrightarrow \mathbf{Q}(x).$$

Let $\mathbf{Q}((z))$ be the field of fractions of the power series ring $\mathbf{Q}[[z]]$. In our applications, we will consider images of the functions $g(C, v)$ under the mapping $\mathbf{Q}(x) \to \mathbf{Q}((z))$ defined by $x_i \mapsto e^{z_i}$. Define

$$h(C, v)(z) = e^{v \cdot z} g(C, v)(e^{z_1}, \dots, e^{z_n}) \in \mathbf{Q}((z)).$$

With $\sigma_1, \dots, \sigma_r$ and $\mathscr{P}$ as above, we have

$$h(C, v)(z) = \frac{\sum_{a \in \mathscr{P} \cap (v + \mathbf{Z}^n)} e^{a \cdot z}}{(1 - e^{\sigma_1 \cdot z}) \cdots (1 - e^{\sigma_r \cdot z})}.$$

From the corresponding properties of the functions $g(C, v)$, it follows immediately that h may be viewed as a pairing

$$h : \mathcal{K}/\mathcal{L} \times \mathbf{Q}^n/\mathbf{Z}^n \longrightarrow \mathbf{Q}((z))$$

that is linear in the first variable. We call h the *Solomon–Hu* pairing owing to its first appearance in the works [So1, HS].

2.4 Special values of Shintani zeta functions

We now recall results relating the generating function $g(C, v)$ introduced above to special values of complex analytic *Shintani zeta functions*, whose definition we now recall.

Let $\mathcal{M} \subset M_n(\mathbf{R})$ be the subset of matrices such that the entries of each column are linearly independent over $\mathbf{Q}$ (i.e. for each nonzero row vector $x \in \mathbf{Q}^n$ and $M \in \mathcal{M}$, the vector xM has no component equal to 0). Let $\mathcal{D} \subset \mathbf{SL}_n(\mathbf{R})$ be the subgroup of $n \times n$ real diagonal matrices with determinant 1. Given $M \in \mathcal{M}$, define a polynomial $f_M \in \mathbf{R}[x_1, \dots, x_n]$ by

$$\begin{aligned} f_M(x_1, \dots, x_n) &= \mathrm{N}((x_1, \dots, x_n)M) \\ &= (xM)_1 (xM)_2 \cdots (xM)_n. \end{aligned} \tag{35}$$

Note that f_M depends only on the image of the matrix M in $\mathcal{M}/\mathcal{D}$.

View the elements of the rational open cone $C = C(w_1, \dots, w_r) \subset \mathbf{R}^n$ as column vectors. Choose the w_i to have integer coordinates. We consider a matrix $M \in \mathcal{M}$ such that (C, M) satisfies the following positivity condition:

$$M^t w \subset (\mathbf{R}_{>0})^n \text{ for all } w \in C. \tag{36}$$

This positivity condition will be needed when defining analytic Shintani zeta functions. When dealing with their algebraic incarnations, i.e. the cone generating functions $h(C, v)$ introduced in the previous section, it is not required. This added flexibility in the algebraic setting is crucial for the cohomological constructions to be described in the following sections. With C and M as above and a vector $v \in \mathcal{V}$, define the Shintani zeta function

$$\zeta(C, M, v, s) = \sum_{x \in C \cap v + \mathbf{Z}^n} \frac{1}{f_M(x)^s}.$$

Using (36), it is easy to see that this series is absolutely convergent for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$. Letting $\mathscr{P} = \mathscr{P}(w_1, \ldots, w_r)$ be the parallelpiped defined in (29) and $W = (w_1, \ldots, w_r)$ the $n \times r$ matrix whose columns are the generators of the cone C, we define

$$Z(C, M, a, s) = \sum_{x \in (\mathbf{Z}_{\geq 0})^r} \frac{1}{f_M(a + Wx)^s}$$

for $a \in \mathscr{P} \cap (v + \mathbf{Z}^n)$. We obtain the finite sum decomposition

$$\zeta(C, M, v, s) = \sum_{a \in \mathscr{P} \cap v + \mathbf{Z}^n} Z(C, M, a, s).$$

Shintani [Sh] proved that each $Z(C, M, a, s)$, and hence $\zeta(C, M, v, s)$ itself, admits a meromorphic continuation to $\mathbf{C}$.

Shintani also gave a formula for the values of these zeta functions at nonpositive integers. Observe that if k is a nonnegative integer, then $f_M(x)^k$ is $(k!)^n$ times the coefficient of $\mathrm{N}(z)^k$ in the Taylor series expansion of $e^{zM^t x}$. Summing, we obtain the nonsense identity chain

$$\text{“}\zeta(C, M, v, -k) = \sum_{x \in C \cap (v + \mathbf{Z}^n)} f_M(x)^k = (k!)^n \operatorname{coeff}\left(\sum_{x \in C \cap (v + \mathbf{Z}^n)} e^{zM^t x}, \mathrm{N}(z)^k \right)$$
$$= (k!)^n \operatorname{coeff}\left(h(C, v)(zM^t), \mathrm{N}(z)^k \right).\text{”}$$

Almost nothing in the above identity chain is actually defined and in particular the given sums do not converge. Further, $h(C, v)(zM^t)$ is not holomorphic on a punctured neighborhood of $z = (0, \ldots, 0)$ if $n > 1$, making the notion of coefficient undefined. Nonetheless, via an algebraic trick—really, an algebraic version of the trick used by Shintani in his proof of the analytic continuation of $\zeta(C, M, v, s)$—we generalize the notion of coefficient to a class of functions including the $h(C, v)(zM^t)$. Remarkably, with this generalized notion of coefficient, the identity

$$\zeta(C, M, v, -k) = (k!)^n \operatorname{coeff}\left(h(C, v)(zM^t), \mathrm{N}(z)^k \right) \tag{37}$$

holds. We now define Shintani's operator and state his theorem giving a rigorous statement of (37).

Let K be a subfield of $\mathbf{C}$. For $1 \leq j \leq n$, we write

$$Z_j = (z_j z_1, \ldots, z_j z_{j-1}, z_j, z_j z_{j+1}, \ldots, z_j z_n). \tag{38}$$

The following lemma is elementary.

Lemma 2.2. *Let $g \in K[[z]]$, let $p \in K[z]$ be homogeneous of degree d with* $\operatorname{coeff}(p, z_j^d) \neq 0$, *and let $G = g/p$. Then $G(Z_j) \in z_j^{-d}K[[z]]$.*

Call a homogeneous polynomial $p \in K[z]$ of degree d *powerful* if the power monomials in p all have nonzero coefficients, i.e., if $\operatorname{coeff}(p, z_j^d) \neq 0$ for all j. The powerful polynomials of interest to us arise as follows. Call a linear form $L(z) = \ell_1 z_1 + \cdots + \ell_n z_n$ *dense* if $\ell_j \neq 0$ for all j. If $L_1, \dots, L_r$ are dense linear forms, then $p = L_1 \cdots L_r$ is powerful.

Definition 2.3. Let $K((z))^{\mathrm{hd}} \subset K((z))$ be the subalgebra consisting of $G \in K((z))$ that can be written in the form $G = g/p$ for a power series $g \in K[[z]]$ and a powerful homogeneous polynomial $p \in K[z]$.

Lemma 2.4. *Suppose C is a rational open simplicial cone in $\mathbf{R}^n$, $M \in \mathcal{M}$ and $v \in \mathbf{Q}^n$. Let $\mathbf{Q}(\{m_{ij}\})$ be the field generated by the entries of M. Then $h(C, v)(zM^t) \in \mathbf{Q}(\{m_{ij}\})((z))^{\mathrm{hd}}$.*

Proof. Write $C = C(w_1, \dots, w_r)$ and let $a \in \mathscr{P} \cap (v + \mathbf{Z}^n)$. Then $a \in \mathbf{Q}^n$, so $e^{zM^t a} \in K[[z]]$. For each $j = 1, \dots, r$, set $L_j(z) = zM^t w_j$. Then we can write $1 - e^{zM^t w_j} = L_j(z) g_j(z)$ with $g_j \in K[[z]]^\times$. Setting $f_a = e^{zM^t a} g_1^{-1} \cdots g_r^{-1}$ and $p = L_1 \cdots L_r$, we have

$$h(C, v)(zM^t) = \sum_{a \in \mathscr{P} \cap (v + \mathbf{Z}^n)} f_a / p.$$

It remains to show that p is powerful. Since $M \in \mathcal{M}$ and $w_j \in \mathbf{Q}^n$ for all j, it follows that each L_j is dense. Therefore p is powerful as desired. $\square$

By Lemma 2.2, if $G \in K((z))^{\mathrm{hd}}$, then $\operatorname{coeff}(G(Z_j), z^m)$ makes sense for any j and any $m \in \mathbf{Z}^n$. This leads to the following definition.

Definition 2.5. For $j = 1, \dots, n$, define operators $\Delta_j^{(k)} : K((z))^{\mathrm{hd}} \to K$ by

$$\Delta_j^{(k)} G = \operatorname{coeff}(G(Z_j), \mathrm{N}(Z_j)^k), \tag{39}$$

where Z_j is given in (38). Define the *Shintani operator* $\Delta^{(k)} : K((z))^{\mathrm{hd}} \to K$ by

$$\Delta^{(k)} = \frac{(k!)^n}{n} \sum_{j=1}^{n} \Delta_j^{(k)}. \tag{40}$$

Remark 2.6. If $g \in K[[z]]$, then $\Delta^{(k)} g$ is simply $(k!)^n$ times the coefficient of $(z_1 \cdots z_n)^k$ in g. Thus, the operator $\Delta^{(k)}$ extends the coefficient extraction operation from $K[[z]]$ to $K((z))^{\mathrm{hd}}$.

The Shintani operator shares the following properties with the operation of taking the $(z_1 \cdots z_n)^k$-coefficient of a regular power series. The proof is an elementary computation.

Lemma 2.7. *Let $h \in K((z))^{\mathrm{hd}}$. Then*

- *For $d_1, \dots, d_n \in K$, we have $\Delta^{(k)} h(d_1 z_1, \dots, d_n z_n) = (d_1 \cdots d_n)^k \Delta^{(k)} h(z_1, \dots, z_n)$.*

- *For any permutation σ, we have $\Delta^{(k)}h(z_{\sigma(1)},\ldots,z_{\sigma(n)}) = \Delta^{(k)}h(z_1,\ldots,z_n)$.*

Finally, we may state the following theorem of Shintani:

Theorem 2.8 ([Sh, Proposition 1]). *Let C be a rational open cone, $v \in \mathcal{V}$, and $M \in \mathcal{M}$ satisfying* (36). *The function $\zeta(C, M, v, s)$ has a meromorphic continuation to $\mathbf{C}$ and satisfies*

$$\zeta(C, M, v, -k) = \Delta^{(k)}h(C, v)(zM^t) \qquad \text{for } k \in \mathbf{Z}_{\geq 0}.$$

We observe that by Lemma 2.7, the coefficient $\Delta^k h(C, v)(zM^t)$ depends only on the image of M in $\mathcal{M}/\mathcal{D}$.

2.5 The power series-valued Shintani cocycle

In this section we define the Shintani cocycle in the form that will be most useful for our desired applications; in particular, the cocycle will take values in a module $\mathcal{F}$ for which it can be compared to the Eisenstein cocycle defined by Sczech in [Sc2] and studied in [CD].

The set $\mathcal{M}$ defined in Section 2.4 is naturally a left Γ-set via the action of left multiplication. Let $\mathcal{F}$ denote the real vector space of functions

$$f : \mathcal{M} \times \mathcal{Q} \times \mathcal{V} \longrightarrow \mathbf{R}((z))^{\mathrm{hd}}$$

satisfying the following distribution relation for each nonzero integer λ:

$$f(M, Q, v) = \operatorname{sgn}(\lambda)^n \sum_{\lambda w = v} f(\lambda M, \lambda^{-1}Q, w). \tag{41}$$

Define a left Γ-action on $\mathcal{F}$ as follows. Given $\gamma \in \Gamma$, choose a nonzero scalar multiple $A = \lambda\gamma$ with $\lambda \in \mathbf{Z}$ such that $A \in M_n(\mathbf{Z})$. For $f \in \mathcal{F}$, define

$$(\gamma f)(M, Q, v) = \sum_{r \in \mathbf{Z}^n / A\mathbf{Z}^n} \operatorname{sgn}(\det A) f(A^t M, A^{-1}Q, A^{-1}(r + v)). \tag{42}$$

The distribution relation (41) implies that (42) does not depend on the auxiliary choice of λ. Note that the action of Γ on $\mathcal{F}$ factors through $\mathbf{PGL}_n(\mathbf{Q})$. The Solomon–Hu pairing satisfies the identity

$$h(\gamma C, v)(zM^t) = \gamma h(C, v)(zM^t)$$

for any rational cone C.

We can use Φ_{Sh} to define a cocycle $\Psi_{\mathrm{Sh}} \in Z^{n-1}(\Gamma, \mathcal{F})$ by

$$\Psi_{\mathrm{Sh}}(A, M, Q, v) := h(\Phi_{\mathrm{Sh}}(A)(Q), v)(zM^t). \tag{43}$$

Here and in the sequel we simply write $\Psi_{\mathrm{Sh}}(A, M, Q, v)$ for $\Psi_{\mathrm{Sh}}(A_1, \ldots, A_n)(M, Q, v)$ with $A = (A_1, \ldots, A_n) \in \Gamma^n$. Our cocycle Ψ_{Sh} satisfies the following rationality result.

Theorem 2.9. *The value $\Delta^{(k)}\Psi_{\mathrm{Sh}}(A, M, Q, v)$ lies in the field K generated over $\mathbf{Q}$ by the coefficients of the polynomial $f_M(x)$.*

Proof. We will show that $\Delta^{(k)}(h(C,v)(zM^t))$ lies in K for any rational cone C. By the definition of $f_M(x)$, any automorphism of $\mathbf{C}$ fixing $f_M(x)$ permutes the columns of M up to scaling each column by a factor λ_i such that $\prod_{i=1}^n \lambda_i = 1$. Therefore it suffices to prove that our value is invariant under each of these operations, namely permuting the columns or scaling the columns by factors whose product is 1. Now, in the tuple zM^t, permuting the columns M has the same effect as permuting the variables z_i; and scaling the ith column of M by λ_i has the same effect as scaling z_i by λ_i. The desired result then follows from Lemma 2.7. □

2.6 Special values of zeta functions

Let F be a totally real field, and let $\mathfrak{a}$ and $\mathfrak{f}$ be relatively prime integral ideals of F. The goal of the remainder of this section is to express the special values $\zeta_{\mathfrak{f}}(\mathfrak{a}, -k)$ for integers $k \geq 0$ in terms of the cocycle Ψ_{Sh}. We invoke the notation of Section 2.1; in particular we fix an embedding $J : F \hookrightarrow \mathbf{R}^n$.

Let $\mathcal{R} = \mathbf{Z}[\mathcal{M}/\mathcal{D} \times \mathcal{Q} \times \mathcal{V}]$ denote the free abelian group on the set $\mathcal{M}/\mathcal{D} \times \mathcal{Q} \times \mathcal{V}$, which is naturally endowed with a left Γ-action by the action on the sets $\mathcal{M}/\mathcal{D}$, $\mathcal{Q}$, and $\mathcal{V}$. There is a cycle $\mathfrak{Z}_{\mathfrak{f}}(\mathfrak{a}) \in H_{n-1}(\Gamma, \mathcal{R})$ associated to our totally real field F and integral ideals $\mathfrak{a}, \mathfrak{f}$. The cycle consists of the data of elements $\mathcal{A} \in \mathbf{Z}[\Gamma^n]$, $M \in \mathcal{M}/\mathcal{D}$, $Q \in \mathcal{Q}$, and $v \in \mathcal{V}$, defined as follows.

Fix a $\mathbf{Z}$-module basis $w = (w_1, \ldots, w_n)$ for $\mathfrak{a}^{-1}\mathfrak{f}$. Let $\{\epsilon_1, \ldots, \epsilon_{n-1}\}$ denote a basis of the group U of totally positive units of F congruent to 1 modulo $\mathfrak{f}$. Following (18), define

$$\mathcal{A}(\epsilon_1, \ldots, \epsilon_{n-1}) = (-1)^{n-1} w_\epsilon \sum_{\sigma \in S_{n-1}} \mathrm{sign}(\sigma)[(\rho_w(f_{1,\sigma}), \ldots, \rho_w(f_{n,\sigma}))] \in \mathbf{Z}[\Gamma^n]. \tag{44}$$

Here ρ_w is the right regular representation of U on w defined in (24), and w_ϵ is the orientation associated to $J(\epsilon)$ as in (13).

Let $M \in \mathcal{M}/\mathcal{D}$ be represented by the matrix

$$\mathrm{N}(\mathfrak{a})^{1/n}(J_j(w_i))_{i,j=1}^n = \mathrm{N}(\mathfrak{a})^{1/n} J(w)^t. \tag{45}$$

Note that $f_M \in \mathbf{Q}[x_1, \ldots, x_n]$ is the homogeneous polynomial of degree n given by the norm:

$$f_M(x_1, \ldots, x_n) = \mathrm{N}(\mathfrak{a}) \cdot \mathrm{N}(w_1 x_1 + \cdots + w_n x_n). \tag{46}$$

Let Q be the image under the embedding $J_n : F \hookrightarrow \mathbf{R}$ of the dual basis to w under the trace pairing on F, as in (26) :

$$Q = (0, \ldots, 0, 1) J(w)^{-t} = (J_n(w_1^*), \ldots, J_n(w_n^*)), \tag{47}$$

where $\mathrm{Tr}(w_i w_j^*) = \delta_{ij}$. Define the column vector

$$v = (\mathrm{Tr}(w_1^*), \ldots, \mathrm{Tr}(w_n^*)), \qquad \text{so that} \qquad 1 = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n. \tag{48}$$

Dot product with $(w_1, \ldots, w_n)$ provides a bijection $v + \mathbf{Z}^n \longleftrightarrow 1 + \mathfrak{a}^{-1}\mathfrak{f}$.

We now define $\mathfrak{Z}_\mathfrak{f}(\mathfrak{a}) \in H_{n-1}(\Gamma, \mathcal{R})$ to be the homology class represented by the homogeneous $(n-1)$-cycle

$$\tilde{\mathfrak{Z}} = \mathcal{A} \otimes [(M, Q, v)] \in \mathbf{Z}[\Gamma^n] \otimes \mathcal{R}.$$

The fact that $\tilde{\mathfrak{Z}}$ is a cycle follows from [Sc2, Lemma 5] as in (18) using the fact that the elements M, Q, and v are invariant under the action of $\rho_w(U)$.

For each integer $k \geq 0$, the canonical Γ-invariant map $\mathcal{F} \otimes \mathcal{R} \to \mathbf{R}$ given by $f \otimes [(M, Q, v)] \mapsto \Delta^{(k)} f(M, Q, v)$ is well-defined by Lemma 2.7, and induces via cap product a pairing

$$\langle\ ,\ \rangle_k : H^{n-1}(\Gamma, \mathcal{F}) \times H_{n-1}(\Gamma, \mathcal{R}) \longrightarrow \mathbf{R}.$$

Here $\mathbf{R}$ has the trivial Γ-action.[†]

Theorem 2.10. *We have* $\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -k) = \langle \Psi_{\mathrm{Sh}}, \mathfrak{Z}_\mathfrak{f}(\mathfrak{a}) \rangle_k \in \mathbf{Q}$.

The rationality of $\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -k)$ is a celebrated theorem of Klingen and Siegel (see [IO] for a nice survey of the history of various investigations on these special values).

The proof we have outlined here is a cohomological reformulation of Shintani's original argument, with the added benefit that our definition of $\mathfrak{Z}_\mathfrak{f}(\mathfrak{a})$ gives an explicit signed fundamental domain.

Proof. Let U denote the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$, and let $D = \sum_i a_i C_i$ denote a signed fundamental domain for the action of U on the totally positive orthant of $\mathbf{R}^n$ (where as in Section 2.1, $u \in U$ acts by componentwise multiplication with $J(u)$). Then for $\mathrm{Re}(s) \gg 0$,

$$\zeta_{F,\mathfrak{f}}(\mathfrak{a}, s) = \sum_{\substack{\mathfrak{b} \subset \mathcal{O}_F \\ \mathfrak{b} \sim_\mathfrak{f} \mathfrak{a}}} \frac{1}{\mathrm{N}\mathfrak{b}^s} = \sum_{\{y \in 1 + \mathfrak{a}^{-1}\mathfrak{f},\ y \gg 0\}/U} \frac{1}{(\mathrm{N}\mathfrak{a}\mathrm{N}y)^s} \qquad (\mathfrak{a}^{-1}\mathfrak{b} = (y))$$

$$= \sum_{\substack{y \in 1 + \mathfrak{a}^{-1}\mathfrak{f} \\ J(y) \in D}} \frac{1}{(\mathrm{N}\mathfrak{a}\mathrm{N}y)^s}. \tag{49}$$

Here we use the shorthand $\sum_{J(y) \in D}$ for $\sum_i a_i \sum_{J(y) \in C_i}$. Now Theorem 1.5 implies that, using the notation of (17) and (18), the function $\phi_U(\alpha(\epsilon_1, \ldots, \epsilon_{n-1}))$ is the characteristic function $\mathbf{1}_D$ of such a signed fundamental domain D for the action of U on $(\mathbf{R}_{>0})^n$. (Recall from Definition 1.4 that if $D = \sum a_i C_i$ is a signed fundamental domain then $\mathbf{1}_D := \sum a_i \mathbf{1}_{C_i}$.) Therefore (28) implies that

$$\Phi_{\mathrm{Sh}, w_1}(\mathcal{A}, Q) = \mathbf{1}_{J(w)^{-1}D}.$$

[†]To make contact with the notation of the introduction, note that for each integer k we obtain a map $\eta_k : \mathcal{R} \to \mathcal{N}^\vee$, i.e. a pairing $\mathcal{N} \times \mathcal{R} \to \mathbf{R}$, by $(\Phi, [(M, Q, v)]) \mapsto \Delta^{(k)} h(\Phi(Q), v)(zM^t)$. The class denoted $\mathfrak{Z}_k$ in the introduction is the image of $\mathfrak{Z}_\mathfrak{f}(\mathfrak{a})$ under the map on homology induced by η_k.

Note that for an element $x \in F$, the vector $v = J(w)^{-1}J(x) \in \mathbf{Q}^n$ satisfies $x = w \cdot v$, where $w = (w_1, \ldots, w_n)$. Therefore $J(w)^{-1}D$ consists of rational cones and

$$\begin{aligned}
\langle \Psi_{\mathrm{Sh}}, \mathfrak{Z}_{\mathfrak{f}}(\mathfrak{a}) \rangle_k &= \Delta^{(k)} \Psi_{\mathrm{Sh}}(\mathcal{A}, M, Q, v) \\
&= \Delta^{(k)} h(\Phi_{\mathrm{Sh},w_1}(\mathcal{A}, Q), v)(zM^t) \\
&= \Delta^{(k)} h(\mathbf{1}_{J(w)^{-1}D}, v)(zM^t) \\
&= \zeta(J(w)^{-1}D, M, v, -k)
\end{aligned} \tag{50}$$

by Theorem 2.8. Here Φ_{Sh,w_1} was defined in (27), and may be substituted for Φ_{Sh} since it represents the same cohomology class. Note also that (36) is satisfied for each pair $(J(w)^{-1}C_i, M)$ by the definition of M in (45) and the fact that $C_i \subset (\mathbf{R}_{>0})^n$ (which in turn was explained in Remark 1.3). By definition, we have for $\mathrm{Re}(s)$ large enough:

$$\begin{aligned}
\zeta(J(w)^{-1}D, M, v, s) &= \sum_{x \in J(w)^{-1}D \cap v + \mathbf{Z}^n} \frac{1}{f_M(x)^s} \\
&= \sum_{\substack{y \in 1 + \mathfrak{a}^{-1}\mathfrak{f} \\ J(y) \in D}} \frac{1}{(\mathrm{N}\mathfrak{a}\mathrm{N}y)^s},
\end{aligned} \tag{51}$$

where the last equation uses the substitution $y = w \cdot x$ and (46). Comparing (49), (50), and (51) yields the desired equality $\zeta_{F,\mathfrak{f}}(\mathfrak{a}, -k) = \langle \Psi_{\mathrm{Sh}}, \mathfrak{Z}_{\mathfrak{f},\mathfrak{a}} \rangle_k$.

Finally, the rationality of $\langle \Psi_{\mathrm{Sh}}, \mathfrak{Z}_{\mathfrak{f}}(\mathfrak{a}) \rangle_k = \Delta^{(k)} \Psi_{\mathrm{Sh}}(\mathcal{A}, M, Q, v)$ follows from Theorem 2.9, since $f_M(x)$ has rational coefficients. $\square$

3 Comparison with the Sczech Cocycle

In this section we prove that the Shintani cocycle Ψ_{Sh} defined in Section 2.5 is cohomologous to the one defined by Sczech in [Sc2]. We begin by recalling the definition of Sczech's cocycle. The reader is referred to [CD] or [Sc2] for a lengthier discussion of Sczech's construction.

3.1 The Sczech cocycle

For n vectors $\tau_1, \ldots, \tau_n \in \mathbf{C}^n$, define a rational function of a variable $x \in \mathbf{C}^n$ by

$$f(\tau_1, \ldots, \tau_n)(x) = \frac{\det(\tau_1, \ldots, \tau_n)}{\langle x, \tau_1 \rangle \cdots \langle x, \tau_n \rangle}. \tag{52}$$

The function f satisfies the cocycle relation (see [Sc2, Lemma 1, pg. 586])

$$\sum_{i=0}^{n} (-1)^i f(\tau_0, \ldots, \hat{\tau}_i, \ldots, \tau_n) = 0. \tag{53}$$

Consider $A = (A_1, \ldots, A_n) \in \Gamma^n$ and $x \in \mathbf{Z}^n - \{0\}$. For $i = 1, \ldots, n$, let $\varpi_i = \varpi_i(A, x)$ denote the leftmost column of A_i that is not orthogonal to x. Let $v \in \mathcal{V} = \mathbf{Q}^n/\mathbf{Z}^n$. Sczech considers the sum

$$\sum_{x\in\mathbf{Z}^n-\{0\}} e(\langle x,v\rangle) f(\varpi_1,\ldots,\varpi_n)(x), \tag{54}$$

where $e(u) := e^{2\pi i u}$. Although the definition of ϖ_i ensures that each summand in (54) is well-defined, the sum itself is not absolutely convergent. To specify a method of summation, Sczech introduces a vector $Q \in \mathcal{Q}$ and defines the Q-summation

$$\begin{aligned}\tilde{\Psi}_{\mathrm{Z}}(A,Q,v) &= (2\pi i)^{-n} \sum_{x\in\mathbf{Z}^n-\{0\}} e(\langle x,v\rangle) f(\varpi_1,\ldots,\varpi_n)(x)|_Q \\ &:= (2\pi i)^{-n} \lim_{t\to\infty} \sum_{\substack{x\in\mathbf{Z}^n-\{0\}\\ |Q(x)|<t}} e(\langle x,v\rangle) f(\varpi_1,\ldots,\varpi_n)(x).\end{aligned} \tag{55}$$

Here the vector Q gives rise to the function $Q(x) = \langle x, Q\rangle$, and the summation over the region $|Q(x)| < t$ is absolutely convergent for each t.

More generally, given a homogeneous polynomial $P \in \mathbf{C}[x_1,\ldots,x_n]$, Sczech defines

$$\tilde{\Psi}_{\mathrm{Z}}(A,P,Q,v) = (2\pi i)^{-n-\deg P} \sum_{x\in\mathbf{Z}^n-\{0\}} e(\langle x,v\rangle) P(-\partial_{x_1},-\partial_{x_2},\ldots,-\partial_{x_n})(f(\varpi_1,\ldots,\varpi_n))(x)|_Q. \tag{56}$$

Sczech shows that the function $\tilde{\Psi}_{\mathrm{Z}}$ is a cocycle on Γ valued in the module $\tilde{\mathcal{F}}$ defined in Section 4.3 below. In order to make a comparison with our Shintani cocycle $\Psi_{\mathrm{Sh}} \in Z^{n-1}(\Gamma,\mathcal{F})$, however, we consider now an associated cocycle valued in the module $\mathcal{F}$ defined in Section 2.5. We prove in Proposition 3.9 below that there exists a power-series valued cocycle $\Psi_{\mathrm{Z}} \in Z^{n-1}(\Gamma,\mathcal{F})$ such that for each integer $k \geq 0$, we have

$$\Delta^{(k)}\Psi_{\mathrm{Z}}(A,M,Q,v) = \tilde{\Psi}_{\mathrm{Z}}(A,f_M^k,Q,v). \tag{57}$$

Our main theorem in this section is:

Theorem 3.1. *Define $\Psi_{\mathrm{Sh}}^+ \in Z^{n-1}(\Gamma,\mathcal{F})$ by*

$$\Psi_{\mathrm{Sh}}^+(A,M,Q,v) = \frac{1}{2}(\Psi_{\mathrm{Sh}}(A,M,Q,v) + \Psi_{\mathrm{Sh}}(A,M,-Q,v))$$

and let $\Psi_{\mathrm{P}} \in Z^{n-1}(\Gamma,\mathcal{F})$ be the "polar cocycle" defined by

$$\Psi_{\mathrm{P}}(A,M,Q,v) = \frac{(-1)^{n+1}\det(\sigma)}{\prod_{j=1}^n zM^t\sigma_j}, \tag{58}$$

where $\sigma = (\sigma_1,\ldots,\sigma_n)$ is the collection of the leftmost columns of the tuple $A \in \Gamma^n$. Then we have the following equality of classes in $H^{n-1}(\Gamma,\mathcal{F})$:

$$[\Psi_{\mathrm{Z}}] = [\Psi_{\mathrm{Sh}}^+] + [\Psi_{\mathrm{P}}]. \tag{59}$$

Remark 3.2. It is proven in [Sc2, Theorem 3] that the cohomology class $[\Psi_{\mathrm{P}}]$ is nontrivial. However, it clearly vanishes under application of the Shintani operator $\Delta^{(k)}$ and therefore does not intervene in arithmetic applications.

Remark 3.3. In [Sc2], Sczech considers a matrix of m vectors $Q_i \in \mathcal{Q}$ and the Q summation (56) with $Q(x) = \prod Q_i(x)$. However, the resulting cocycle is simply the average of the individual cocycles obtained from each Q_i. (This is not clear from the original definition, but follows from Sczech's explicit formulas for his cocycle.) Therefore it is sufficient to consider just one vector Q.

Remark 3.4. In view of Theorem 3.1 and (57), the evaluation of partial zeta functions of totally real fields using Sczech's cocycle given in [Sc2, Theorem 1] follows also from our Theorem 2.10. In fact, we obtain a slightly stronger result in that we obtain the evaluation using each individual vector $Q_i = J_i(w^*)$, whereas Sczech obtains the result using the matrix of all n such vectors; it would be interesting to prove this stronger result directly from the definition of Sczech's cocycle via Q-summation, rather than passing through the Shintani cocycle and Theorem 3.1.

3.2 A generalization of Sczech's construction

Let k be a positive integer, and let $A = (A_1, \ldots, A_k) \in \Gamma^k$. For each tuple $w \in \{1, \ldots, n\}^k$, let $B(A, w) \subset \mathbf{Z}^n - \{0\}$ denote the set of vectors x such that the leftmost column of A_i not orthogonal to x is the w_ith, for $i = 1, \ldots, k$. In other words,

$$B(A, w) = \bigcap_{i=1}^{k} \{x \in \mathbf{Z}^n : \langle x, A_{ij} \rangle = 0 \text{ for } j < w_i, \langle x, A_{iw_i} \rangle \neq 0\}.$$

Here A_{ij} denotes the jth column of the matrix A_i. Then

$$\mathbf{Z}^n - \{0\} = \bigsqcup_{w \in \{1,\ldots,n\}^k} B(A, w).$$

Sczech's sum (55) can be written with $k = n$ as:

$$\tilde{\Psi}_{\mathrm{Z}}(A, Q, v) = \sum_{w} \sum_{x \in B(A,w)} e(\langle x, v \rangle) f(A_{1w_1}, \ldots, A_{nw_n})(x)|_Q.$$

We now generalize this expression by replacing the columns A_{iw_i} with certain other columns of the matrices A_i.

Write $S_k = \{1, \ldots, k\}$ and for simplicity let $S = S_n$. Given $A = (A_1, \ldots, A_k) \in \Gamma^k$ and an element $t = ((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)) \in (S_k \times S)^n$, define

$$\tau(A, t) = (A_{a_1 b_1}, A_{a_2 b_2}, \ldots, A_{a_n b_n}).$$

In other words, $\tau(A, t)$ is an $n \times n$ matrix whose ith column is the b_ith column of A_{a_i}.

For any function $g : S^k \to (S_k \times S)^n$, we would like to consider the sum

$$\psi(g)(A, Q, v) = \sum_{w} \sum_{x \in B(A,w)} e(\langle x, v \rangle) f(\tau(A, g(w)))(x)|_Q. \tag{60}$$

For example, $\tilde{\Psi}_{\rm Z} = \psi(\beta)$ where $\beta(w) = ((1, w_1), (2, w_2), \ldots, (n, w_n))$. The difficulty with (60) in general, however, is that the denominators in the expression defining f may vanish; it is therefore necessary to introduce an auxiliary variable $u \in \mathbf{C}^n$ and to consider the function

$$\psi(g)(A, Q, v, u) = \sum_{w} \sum_{x \in B(A,w)} e(\langle x, v\rangle) f(\tau(A, g(w)))(x - u)|_Q. \tag{61}$$

By Sczech's analysis [Sc2, Theorem 2], this Q-summation converges for all $u \in \mathbf{C}^n$ such that the map $x \mapsto f(\tau(A, g(w)))(x - u)$ is defined on $B(A, w)$, i.e., such that the denominator of the right hand side of (52) is nonzero. Thus it converges for u in a dense open subset of $\mathbf{C}^n$ that consists of the complement of a countable union of hyperplanes. In fact, this convergence is uniform for u in sufficiently small compact subsets of $\mathbf{C}^n$.

This formalism allows for the construction of homogeneous cochains in $C^{k-1}(\Gamma, \mathcal{F})$ as follows.

Proposition 3.5. *For any function $g : S^k \to (S_k \times S)^n$ and $A = (A_1, \ldots A_k) \in \Gamma^k$, there is a unique power series*

$$\Psi(g)(A, Q, v) \in \mathbf{Q}((z))$$

such that

$$\psi(g)(A, Q, v, u) = (2\pi i)^n \Psi(g)(A, Q, v)(2\pi i u) \tag{62}$$

for any $u \in \mathbf{C}^n$ for which (61) is defined. Furthermore, for any $M \in \mathcal{M}$ we have

$$\Psi(g)(A, Q, v)(zM^t) \in \mathbf{R}((z))^{\rm hd},$$

and the assignment $(A, M, Q, v) \mapsto \Psi(g)(A, Q, v)(zM^t)$ is a homogeneous cochain in $C^{k-1}(\Gamma, \mathcal{F})$.

The following lemma is the technical heart of the proof of Proposition 3.5 and is proven by reducing to computations in [Sc2].

Lemma 3.6. *Let $H \subset \mathbf{Q}^n$ be a vector subspace and let $L = H \cap \mathbf{Z}^n$. Let $\tau = (\tau_1, \ldots, \tau_n) \in M_n(\mathbf{Z}) \cap \Gamma$. Then for every $v \in \mathbf{Q}^n$,*

$$G(u) := \sum_{x \in L} e(\langle x, v\rangle) f(\tau_1, \ldots, \tau_n)(x - u)|_Q$$

belongs to $(2\pi i)^n \mathbf{Q}((2\pi i u))$. If $M = (m_{ij}) \in \mathcal{M}$, then $G(uM^t) \in (2\pi i)^n \mathbf{Q}(\{m_{ij}\})((2\pi i u))^{\rm hd}$.

Remark 3.7. As with (61), the Q-summation defining $G(u)$ converges for u in a dense open subset of $\mathbf{C}^n$ that consists of the complement of a countable union of hyperplanes. The convergence is uniform for u in sufficiently small compact sets.

Proof. Set $x' = x\tau$, $u' = u\tau$, and $Q' = \tau^{-1}Q$. Then

$$\begin{aligned} G(u) &= \sum_{x \in L} \frac{e(\langle x, v\rangle) \det(\tau)}{\langle x - u, \tau_1\rangle \cdots \langle x - u, \tau_n\rangle}\Big|_Q \\ &= \sum_{x' \in L\tau} \frac{e(\langle x', \tau^{-1} v\rangle) \det(\tau)}{(x'_1 - u'_1)(x'_n - u'_n)}\Big|_{Q'}. \end{aligned}$$

Suppose first that $H = \mathbf{Q}^n$, so that $L = \mathbf{Z}^n$. Then $L\tau$ is a finite-index sublattice of L. Since the nontrivial characters of $L/L\tau$ are $x \mapsto e(\langle x, \tau^{-1}y\rangle)$ for $y \in L^*/\tau L^*$, we have the Fourier expansion

$$\mathbf{1}_{L\tau}(x') = \frac{1}{|\det \tau|} \sum_{y \in L^*/\tau L^*} e(\langle x', \tau^{-1}y\rangle).$$

(L^* is the dual lattice of L, with its elements naturally viewed as column vectors.) Therefore,

$$G(u) = s_\tau \sum_{y \in L^*/\tau L^*} \sum_{x' \in L} \frac{e(\langle x', \tau^{-1}(v+y)\rangle)}{(x'_1 - u'_1)\cdots(x'_n - u'_n)}\Big|_{Q'},$$

where $s_\tau = \operatorname{sgn}(\det \tau)$. Letting $p = u' - x'$, we obtain

$$G(u) = s_\tau \sum_{y \in L^*/\tau L^*} \mathscr{C}_1(u', \tau^{-1}(v+y), Q')$$

where, adopting notation from [Sc2, (3)],

$$\mathscr{C}_1(u, v, Q) = \sum_{p \in \mathbf{Z}^n\tau + u} \frac{e(\langle u - p, v\rangle)}{p_1 \cdots p_n}\Big|_Q.$$

The fact that $G(u)$ belongs to $(2\pi i)^n \mathbf{Q}((2\pi i u))$ now follows from Sczech's evaluation of $\mathscr{C}_1(u, v, Q)$ in elementary terms given in [Sc2, Theorem 2].

Now suppose $r := \dim H < n$. Choose a matrix $\lambda = (\lambda_1, \dots, \lambda_r) \in M_{n\times r}(\mathbf{Z})$ whose column space is $H^\perp$. Then $L\tau = (H \cap \mathbf{Z}^n)\tau$ has finite index in

$$K := H\tau \cap \mathbf{Z}^n = \{x \in \mathbf{Z}^n : \langle x, \lambda_1\rangle = \cdots \langle x, \lambda_r\rangle = 0\}.$$

Inserting the character relations as above, we have

$$G(u) = s_\tau \sum_{y \in K^*/\tau L^*} \sum_{x' \in K} \frac{e(\langle x', \tau^{-1}(v+y)\rangle)}{(x'_1 - u'_1)\cdots(x'_n - u'_n)}\Big|_{Q'}.$$

Computing as in [Sc2, page 599], we obtain

$$G(u) = s_\tau \sum_{y \in K^*/\tau L^*} \mathscr{C}_1(\lambda, u', \tau^{-1}(v+y), Q'),$$

where

$$\mathscr{C}_1(\lambda, u, v, Q) = \int_0^1 \cdots \int_0^1 \mathscr{C}_1(u, t_1\lambda_1 + \cdots t_r\lambda_r + v, Q)dt_1 \cdots dt_r.$$

By the proof of [Sc2, Lemma 7], we have

$$\mathscr{C}_1(\lambda, u', \tau^{-1}(v+y), Q')\prod_{i=1}^n (1 - e(u'_i)) \in (2\pi i)^n \mathbf{Q}[[2\pi i u']] = (2\pi i)^n \mathbf{Q}[[2\pi i u]].$$

for all y. Thus $G(u) \in (2\pi i)^n \mathbf{Q}((2\pi i u))$.

To prove the last statement of the lemma, let $M = (m_{ij}) \in \mathcal{M}$. Then $e(uM^t\tau_j) - 1 = 2\pi i(uM^t\tau_j)H(u)$ for an invertible power series $H(u) \in \mathbf{Q}(\{m_{ij}\})[[2\pi i u]]$. Since $M \in \mathcal{M}$, no component of $M^t\tau_j$ is equal to zero. Therefore, $uM^t\tau_j \in \mathbf{Q}(\{m_{ij}\})[u]$ is a dense linear form (see the paragraph following Lemma 2.2 for the terminology). The desired result then follows from the proof of Lemma 2.4. □

Proof of Proposition 3.5. Each $B(A, w)$ has the form $L - \bigcup_i M_i$ where L is a sublattice of $\mathbf{Z}^n$ and the M_i are finitely many distinct sublattices of L with positive codimension. The existence of the functions $\Psi(g, A, Q, v)$ now follows from inclusion-exclusion and Lemma 3.6, as does the fact that $\Psi(g)(A, Q, v)(zM^t)$ belongs to $\mathbf{R}((z))^{\mathrm{hd}}$ when $M \in \mathcal{M}$.

To see that $(A, M, Q, v) \mapsto \Psi(g)(A, Q, v)(zM^t)$ is homogeneous $(k-1)$-cochain, we first observe that for any $C \in \Gamma$, we have $\tau(CA, g(w)) = C\tau(A, g(w))$. Furthermore, it is easy to see that $B(CA, w) = B(A, w)C^{-1}$, and a straightforward change of variables then shows that

$$\psi(g)(CA, Q, v, uM^t) = \psi(g)(A, C^{-1}Q, C^{-1}v, uM^tC). \qquad \square$$

3.3 Recovering the Sczech and Shintani cocycles

We apply the formalism of the previous section to the following functions $S^n \to (S \times S)^n$:

$$\alpha(w) = ((1,1), (2,1), \ldots, (n,1)),$$
$$\beta(w) = ((1,w_1), (2,w_2), \ldots, (n,w_n)).$$

As we now show, the power series $\Psi(\alpha)$ and $\Psi(\beta)$ associated to these functions are the Shintani and Sczech cocycles, respectively (up to an error term given by the polar cocycle in the first instance).

First we prove a lemma that evaluates the Shintani operator on a regular power series twisted by $M \in \mathcal{M}$. Let f_M be the polynomial defined in (35). Let $\sigma \in M_n(\mathbf{Z})$ and define coefficients $P_r^k(\sigma)$ indexed by tuples $r = (r_1, \ldots, r_n)$ of nonnegative integers by the formula

$$f_M(z\sigma^t)^k = \sum_r \frac{P_r^k(\sigma)}{r!} z_1^{r_1} \cdots z_n^{r_n}, \tag{63}$$

where $r! := r_1! \cdots r_n!$. When $\sigma = 1$, we simply write $P_r^k = P_r^k(1)$.

Lemma 3.8. *Let*

$$F(z) = \sum_r F_r z^r \in K[[z_1, \ldots, z_n]],$$

where r ranges over n-tuples of nonnegative integers. Let $M \in \mathcal{M}$. Then

$$\Delta^{(k)} F(zM^t) = \sum_r F_r P_r^k.$$

Proof. We have

$$\Delta^{(k)}(F(zM^t)) = \sum_r F_r \Delta^{(k)}((zM^t)^r).$$

As noted in Remark 2.6, $\Delta^{(k)}$ evaluated on a regular power series equals $(k!)^n$ times the coefficient of $z_1^k \cdots z_n^k$. Meanwhile P_r^k is $r!$ times the coefficient of z^r in $(zM)_1^k \cdots (zM)_n^k$. The desired result then follows (with $s = (k, k, \ldots, k)$) from the following general reciprocity law for any tuples r and s such that $\sum r = \sum s = m$. If we let

$$C_{r,s}(M) = s! \cdot (\text{coefficient of } z^s \text{ in } (zM)^r),$$

then

$$C_{r,s}(M) = C_{s,r}(M^t). \tag{64}$$

To see this, note that

$$C_{r,s}(M) = \frac{1}{m!} \left(\frac{\partial}{\partial z_1}\right)^{s_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{s_n} \left(\frac{\partial}{\partial y_1}\right)^{r_1} \cdots \left(\frac{\partial}{\partial y_n}\right)^{r_n} (zMy)^m \Big|_{z=y=(0,\ldots,0)}.$$

This expression is clearly invariant upon switching r and s and replacing M by M^t. ☐

Proposition 3.9. *Let $\beta(w) = ((1, w_1), (2, w_2), \ldots, (n, w_n))$. Define*

$$\Psi_{\mathrm{Z}}(A, M, Q, v) = \Psi(\beta)(A, Q, v)(zM^t). \tag{65}$$

Then Ψ_{Z} satisfies (57).

One can show directly from the definition (65) that $\Psi_{\mathrm{Z}} \in Z^{n-1}(\Gamma, \mathcal{F})$, but this follows also from our proof of Theorem 3.1 so we omit the details.

Proof. By the definition of β, the function $\psi(\beta)(A, Q, v, u)$ is well-defined for all u in an open neighborhood of 0 in $\mathbf{C}^n$. Therefore $F = \Psi(\beta)(A, Q, v)$ is a regular power series, i.e. $F(z) \in \mathbf{R}[[z_1, \ldots, z_n]]$. Hence if we write $F = \sum_r F_r z^r$ and $f_M^k(z) = \sum_r P_r^k z^r / r!$ as in (63), then Lemma 3.8 implies that

$$\Delta^{(k)} F(zM^t) = \sum_r F_r P_r^k. \tag{66}$$

On the other hand, by [Sc2, Theorem 2] the series

$$F(u) = (2\pi i)^{-n} \sum_{x \in \mathbf{Z}^n - \{0\}} e(\langle x, v \rangle) f(\varpi_1, \ldots, \varpi_n) \left(x - \frac{u}{2\pi i}\right) |_Q,$$

as well as those formed by taking partial derivatives of the general term, converge uniformly on a sufficiently small compact neighborhood of $u = 0$ in $\mathbf{C}^n$. Therefore term by term differentiation is valid for F, and after applying $f_M^k(\partial_{u_1}, \ldots, \partial_{u_n})$ and plugging in $u = 0$ we obtain

$$\sum_r F_r P_r^k = (2\pi i)^{-n(k+1)} \sum_{x \in \mathbf{Z}^n - \{0\}} e(\langle x, v \rangle) f_M^k(-\partial_{x_1}, -\partial_{x_2}, \ldots, -\partial_{x_n})(f(\varpi_1, \ldots, \varpi_n))(x)|_Q. \tag{67}$$

The right side of (67) is the definition of $\tilde{\Psi}_{\mathrm{Z}}(A, f_M^k, Q, v)$, so combining (66) and (67) gives the desired equality

$$\Delta^{(k)} F(zM^t) = \tilde{\Psi}_{\mathrm{Z}}(A, f_M^k, Q, v).$$

$\square$

Proposition 3.10. *Let* $\alpha(w) = ((1,1),(2,1),\ldots,(n,1))$. *Then*

$$\Psi^+_{\mathrm{Sh}}(A, M, Q, v) + \Psi_{\mathrm{P}}(A, M, Q, v) = \Psi(\alpha)(A, Q, v)(zM^t).$$

Proof. Attached to A and α is the square matrix $\sigma = (\sigma_1, \ldots, \sigma_n) = (A_{11}, \ldots, A_{n1})$. Arguing as in the first paragraph of the proof of Lemma 3.6, we have

$$\psi(\alpha)(A, Q, v, u) = \ s_\sigma \sum_{y \in \mathbf{Z}^n/\sigma\mathbf{Z}^n} \sum_{x' \in \mathbf{Z}^n - \{0\}} \frac{e(\langle x', \sigma^{-1}(v+y)\rangle)}{(x'_1 - u'_1)\ldots(x'_n - u'_n)} \ |_{Q'} \ . \tag{68}$$

For any given $y \in \mathbf{Z}^n$, the inner sum is identified in Sczech's notation [Sc2, (3)] as

$$(-1)^n \sum_{x \in \mathbf{Z}^n - \{0\}} \frac{e(\langle x, \sigma^{-1}(v+y)\rangle)}{(x_1 - u'_1)\ldots(x_n - u'_n)} \ |_{Q'} = \frac{-1}{u'_1 \ldots u'_n} + \mathscr{C}_1(u', \sigma^{-1}(v+y), Q'). \tag{69}$$

To express this last quantity in elementary terms, we write $v' = \sigma^{-1}(v+y)$ and let $J = J(y)$ denote the set of indices $j \in \{1, \ldots, n\}$ with $v'_j \in \mathbf{Z}$. We then invoke [Sc2, Theorem 2] to obtain for $u \in (\mathbf{C} - \mathbf{Z})^n$:

$$\begin{aligned} \mathscr{C}_1(u, v', Q') &= \frac{1}{2}(\mathscr{H}(u, v', Q') + (-1)^n \mathscr{H}(-u, -v', Q')) \\ &= \frac{1}{2}(\mathscr{H}(u, v', Q') + \mathscr{H}(u, v', -Q')), \end{aligned} \tag{70}$$

$$\text{where} \ \ \mathscr{H}(u, v', Q') = (-2\pi i)^n \prod_{j \in J} \left(\frac{e(u_j)}{1 - e(u_j)} + \frac{1 + \operatorname{sign} Q'_j}{2} \right) \prod_{j \notin J} \frac{e(u_j \{v'_j\})}{1 - e(u_j)}.$$

Fix a subset $I_0 \subset J$, select the factor $\frac{(1+\operatorname{sign} Q'_j)}{2}$ for $j \in I_0$, and expand the product for $\mathscr{H}(u, v', Q')$ accordingly. Writing $I = \overline{I_0}$ one obtains

$$\begin{aligned} \mathscr{H}(u, v', Q') =& (-2\pi i)^n \sum_{I_0 \subset J} \prod_{j \in I_0} \frac{(1 + \operatorname{sign} Q'_j)}{2} \prod_{j \in J - I_0} \frac{e(u_j)}{1 - e(u_j)} \prod_{j \notin J} \frac{e(u_j \{v'_j\})}{1 - e(u_j)} \\ &= (-2\pi i)^n \sum_{I \supset \overline{J}} \operatorname{weight}(C_I) \frac{e(u \cdot \sigma^{-1} a_I)}{\prod_{j \in I} 1 - e(u_j)}. \end{aligned} \tag{71}$$

The last line follows from the formula (7) for weight(C_I) and the definition (32) of the point $a_I \in \mathscr{P}_I$. Collecting (68), (69), (70) and (71) we arrive at

$$\psi(\alpha)(A, Q, v, u) = \frac{(-1)^{n+1} \det \sigma}{N(u\sigma)} + (2\pi i)^n s_\sigma \sum_{\substack{y \in \mathbf{Z}^n/\sigma\mathbf{Z}^n \\ I \supset \overline{J(y)}}} \operatorname{weight}^+(C_I) \frac{e(u \cdot a_I)}{\prod_{j \in I} 1 - e((u\sigma)_j)}, \tag{72}$$

where $\text{weight}^+(C_I)$ is the average of the weights for Q and $-Q$. The identity (72) holds for all $u \in \mathbf{C}^n$ as long as the vector $u\sigma$ has no component in $\mathbf{Z}$.

Unwinding the argument of Section 2.2 to go from (31) to (33), we obtain

$$\begin{aligned}\psi(\alpha)(A,Q,v,u) &= \frac{(-1)^{n+1}\det\sigma}{N(u\sigma)} + (2\pi i)^n s_\sigma \sum_{I\subset\{1,\dots,n\}} \text{weight}^+(C_I) \sum_{a\in\mathscr{P}_I\cap(v+\mathbf{Z}^n)} \frac{e(u\cdot a)}{\prod_{j\in I} 1-e((u\sigma)_j)} \\ &= \frac{(-1)^{n+1}\det\sigma}{N(u\sigma)} + (2\pi i)^n s_\sigma \sum_{I\subset\{1,\dots,n\}} \text{weight}^+(C_I) h(C_I,v)(2\pi i u) \\ &= \frac{(-1)^{n+1}\det\sigma}{N(u\sigma)} + (2\pi i)^n\, h(\Phi^+_{\text{Sh}}(A)(Q),v)(2\pi i u), \end{aligned} \tag{73}$$

where the superscript "+" again denotes the average of the contributions of Q and $-Q$. This implies the desired equality between power series using the definitions of Ψ_{Sh}, Ψ_{P}, and $\Psi(\alpha)$ given in (43), (58), and (62) respectively. □

3.4 An explicit coboundary

Recall the notation $S_k = \{1,\dots,k\}$, $S = S_n$. Extending by linearity, we can define $\Psi(g)$ for any map $g\colon S^k \to \mathbf{Z}[(S_k\times S)^n]$. In fact, more is true; if we denote by $\partial\colon \mathbf{Z}[(S_k\times S)^{n+1}] \to \mathbf{Z}[(S_k\times S)^n]$ the usual differential

$$\partial([t_0,\dots,t_n]) = \sum_{i=0}^n (-1)^i [t_0,\dots,\hat{t}_i,\dots,t_n],$$

then $\Psi(g)$ is well-defined for any map $g\colon S^k \to \mathbf{Z}[(S_k\times S)^n]/\operatorname{Image}(\partial)$. This follows from the cocycle relation (53), which implies that $f(\tau(A,t)) = 0$ for $t\in\operatorname{Image}(\partial)$.

We will show that the map $\beta-\alpha\colon S^n \to \mathbf{Z}[(S\times S)^n]/\operatorname{Image}(\partial)$ is a coboundary in the following sense. For $i = 1,\dots,n$, let $\hat{e}_i : S_{n-1}\to S$ be the unique increasing map whose image does not contain i. Given

$$h : S^{n-1} \to \mathbf{Z}[(S_{n-1}\times S)^n], \tag{74}$$

define $dh : S^n \to \mathbf{Z}[(S\times S)^n]$ by

$$(dh)(w_1,\dots,w_n) = \sum_{i=1}^n (-1)^i (\hat{e}_i\times\text{id})(h(\hat{w}_i)). \tag{75}$$

We will show that there exists an h such that $\beta-\alpha = dh \pmod{\operatorname{Image}(\partial)}$. Let us indicate why this completes the proof of Theorem 3.1. For any h as in (74), Proposition 3.5 yields a homogeneous cochain $\Psi(h)\in C^{n-2}(\Gamma,\mathcal{F})$. It is easily checked from (75) that $d(\Psi(h)) = \Psi(dh)$. Therefore, combining Propositions 3.9 and 3.10, we obtain

$$\Psi_{\text{Z}} - \Psi^+_{\text{Sh}} - \Psi_{\text{P}} = \Psi(\beta) - \Psi(\alpha) = d\Psi(h)$$

as desired. It remains to define the appropriate function h.

Proposition 3.11. *For $i = 1, \ldots, n-1$, define $h_i \colon S^{n-1} \to \mathbf{Z}[(S_{n-1} \times S)^n]$ by*

$$h_i(w) = \begin{cases} [(1, w_1), \ldots, (i-1, w_{i-1}), (i, 1), (i, w_i), (i+1, 1), \ldots, (n-1, 1)], & w_i > 1 \\ 0, & w_i = 1, \end{cases}$$

where $w = (w_1, \ldots, w_{n-1})$. Let $h = \sum_{i=1}^{n-1} (-1)^i h_i$. Then $\beta - \alpha \equiv dh \pmod{\text{Image}(\partial)}$.

Remark 3.12. For $n = 2$, the map h is given by $h(1) = 0$ and $h(2) = -[(1,1),(1,2)]$. This is the formula stated by Sczech [Sc1, Page 371].

Proof. One shows by induction on m that for $m = 1, \ldots, n$,

$$\left(\alpha + \sum_{i=1}^{m-1} dh_i\right)(w_1, \ldots, w_n)$$

is equal to

$$[(1, w_1), \ldots, (m-1, w_{m-1}), (m, 1), \ldots, (n, 1)] + \\ \sum_{i=1}^{m-1} (-1)^{i+m-1} [(1, w_1), \ldots, \widehat{(i, w_i)}, \ldots, (m-1, w_{m-1}), (m, 1), (m, w_m), (m+1, 1), \ldots, (n, 1)].$$

For $m = n$, this yields

$$\begin{aligned} (\alpha + dh)(w_1, \ldots, w_n) &= [(1, w_1), \ldots, (n-1, w_{n-1}), (n, 1)] + \\ &\quad \sum_{i=1}^{n-1} (-1)^{i+n-1} [(1, w_1), \ldots, \widehat{(i, w_i)}, \ldots, (n-1, w_{n-1}), (n, 1), (n, w_n)] \\ &\equiv [(1, w_1), \ldots, (n, w_n)] \pmod{\text{Image}(\partial)} \end{aligned}$$

as desired. □

4 Integral Shintani cocycle and applications

In this section we introduce an auxiliary prime ℓ and enact a smoothing process on our cocycle Ψ_{Sh} to define a cocycle $\Psi_{\text{Sh},\ell}$ on a certain congruence subgroup of Γ. The smoothed cocycle $\Psi_{\text{Sh},\ell}$ satisfies an integrality property refining the rationality result stated in Theorem 2.9. As a corollary, we deduce known results about the integrality properties of special values of partial zeta functions of totally real fields, and via the process of p-adic interpolation, define the associated p-adic partial zeta-functions. In the context of Shintani zeta functions, this smoothing process is known as "Cassou–Noguès' trick" ([Ca]) and was already employed in [Das] and [Sp2].

Our final result in this section is that the order of vanishing of the p-adic L-function associated to an abelian character of a totally real field is at least as large as the expected one. To prove this, we apply Spiess's theorems on cohomological p-adic L-functions, and hence our cohomological formulation of the results of Shintani and Cassou–Noguès is essential.

4.1 Definition of the smoothing

Fix a prime ℓ. Let $\mathbf{Z}_{(\ell)} = \mathbf{Z}[1/p, p \neq \ell]$ denote the localization of $\mathbf{Z}$ at the prime ideal (ℓ). Let

$$\Gamma_\ell = \Gamma_0(\ell \mathbf{Z}_{(\ell)}) = \{A \in \mathbf{GL}_n(\mathbf{Z}_{(\ell)}) : \ell \mid A_{j1} \text{ for } j > 1\}.$$

Let $\pi_\ell = \text{diag}(\ell, 1, 1, \ldots, 1)$. Note that if $A \in \Gamma_\ell$, then $\pi_\ell A \pi_\ell^{-1} \in \mathbf{GL}_n(\mathbf{Z}_{(\ell)})$.

For any $\Psi \in Z^{n-1}(\Gamma, \mathcal{F})$, define a smoothed homogeneous cocycle $\Psi_\ell \in Z^{n-1}(\Gamma_\ell, \mathcal{F})$ by

$$\Psi_\ell(A, M, Q, v) := \Psi(\pi_\ell A \pi_\ell^{-1}, \pi_\ell^{-1} M, \pi_\ell Q, \pi_\ell v) - \ell \Psi(A, M, Q, v) \tag{76}$$

for $A = (A_1, \ldots, A_n) \in \Gamma_\ell^n$. The following is a straightforward computation using the fact that Ψ is a cocycle for Γ.

Proposition 4.1. *We have $\Psi_\ell \in Z^{n-1}(\Gamma_\ell, \mathcal{F})$.*

4.2 An explicit formula

We will now give an explicit formula for $\Delta^{(k)} \circ \Psi_{\text{Sh},\ell}$ for an integer $k \geq 0$ in terms of Dedekind sums. For each integer $k \geq 0$, the Bernoulli polynomial $b_k(x)$ is defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} b_k(x) \frac{t^k}{k!}. \tag{77}$$

The following elementary lemma gives an explicit formula for the terms appearing in the definition of $h(C, v)$.

Lemma 4.2. *Consider the cone $C = C(\sigma_{i_1}, \ldots, \sigma_{i_r})$ whose generators are a subset of the columns of the matrix $\sigma \in \Gamma$. Then*

$$\frac{e^{z \cdot a}}{(1 - e^{z \cdot \sigma_{i_1}}) \cdots (1 - e^{z \cdot \sigma_{i_r}})} = (-1)^r \sum_{\substack{m_j = 0 \\ r\text{-tuples}}}^{\infty} \prod_{j=1}^{r} \frac{b_{m_j}(\sigma^{-1}(a)_{i_j})}{m_j!} (z\sigma_{i_j})^{m_j - 1} \prod_{i \notin \{i_j\}} e^{(z\sigma_i)(\sigma^{-1}(a)_i)}. \tag{78}$$

Define the periodic Bernoulli function $B_k(x) = b_k(\{x\})$, where $\{x\} \in [0, 1)$ denotes the fractional part of x. The functions B_k are continuous for $k \neq 1$, i.e. $b_k(0) = b_k(1)$. The function B_1 is not continuous at integers since $b_1(0) = -1/2$ and $b_1(1) = 1/2$. One can choose between these values by means of an auxiliary $Q \in \mathcal{Q}$ as follows.

Definition 4.3. Let $e = (e_1, \ldots, e_n)$ be a vector of positive integers, $Q \in \mathcal{Q}$, and $v \in \mathcal{V}$. Let

$$J = \{1 \leq j \leq n : e_j = 1 \text{ and } v_j \in \mathbf{Z}\}. \tag{79}$$

Define

$$\mathbf{B}_e(v, Q) = \left(\prod_{j \in J} \frac{-\text{sgn}(Q_j)}{2} \right) \prod_{j \notin J} B_{e_j}(v_j).$$

Note that this is $\mathbf{B}_e(v, -Q)$ in the notation of [CD].

Let $\sigma \in M_n(\mathbf{Z})$ have nonzero determinant. Define the Dedekind sum

$$\mathbf{D}(\sigma, e, Q, v) = \sum_{x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n} \mathbf{B}_e(\sigma^{-1}(x+v), \sigma^{-1}Q). \tag{80}$$

Suppose that σ has the property that $\sigma_\ell := \pi_\ell\sigma/\ell \in M_n(\mathbf{Z})$. (This says that the bottom $n-1$ rows of σ are divisible by ℓ.) Write $\underline{e} = \sum e_i$. Define the ℓ-smoothed Dedekind sum

$$\mathbf{D}_\ell(\sigma, e, Q, v) = \mathbf{D}(\sigma_\ell, e, \pi_\ell Q, \pi_\ell v) - \ell^{1-n+\underline{e}}\mathbf{D}(\sigma, e, Q, v) \tag{81}$$

We can now give a formula for $\Delta^{(k)} \circ \Psi_{\mathrm{Sh},\ell}$ in terms of the smoothed Dedekind sum $\mathbf{D}_\ell$. Let $A = (A_1, \dots, A_n) \in \Gamma_\ell^n$, and let $\tilde{\sigma}$ denote the matrix consisting of the first columns of the A_i. Assume that $\det\tilde{\sigma} \neq 0$, and choose a scalar multiple $\sigma = \lambda\tilde{\sigma}$ with λ an integer coprime to ℓ such that $\sigma \in M_n(\mathbf{Z})$. Note that since each $A_i \in \Gamma_\ell$, it follows that $\sigma_\ell = \pi_\ell\sigma/\ell \in M_n(\mathbf{Z})$ as well.

Theorem 4.4. *We have*

$$\Delta^{(k)}\Psi_{\mathrm{Sh},\ell}(A, M, Q, v) = (-1)^n \mathrm{sgn}(\det\sigma) \sum_r \frac{P_r^k(\sigma)}{\ell^{\underline{r}}(r+1)!}\mathbf{D}_\ell(\sigma, r+1, Q, v),$$

where $r+1 := (r_1+1, \dots, r_n+1)$, *and the coefficients* $P_r^k(\sigma)$ *are defined in (63).*

The proof of Theorem 4.4 is involved and technical; the reader is invited to move on to the statement of Theorem 4.7 and the rest of the paper, returning to our discussion here as necessary.

The proof of Theorem 4.4 will be broken into three parts:

- Showing that the terms from (78) arising from indices $m_j = 0$ cancel under the smoothing operation; in particular, $\Psi_{\mathrm{Sh},\ell}$ takes values in $\mathbf{R}[[z_1, \dots, z_n]]$.
- Calculating the remaining terms and thereby giving a formula for $\Psi_{\mathrm{Sh},\ell}$ in terms of the Dedekind sums $\mathbf{D}_\ell$.
- Applying Lemma 3.8, which relates the values of $\Delta^{(k)}$ on a power series in $\mathbf{R}[[z_1, \dots, z_n]]$ to the coefficients appearing in (63).

Lemma 4.5. *In the evaluation of* $\Psi_{\mathrm{Sh},\ell}(A, M, Q, v)$ *using* (78), *the terms arising from tuples* m *with any component* $m_j = 0$ *in* $\Psi(\pi_\ell A\pi_\ell^{-1}, \pi_\ell^{-1}M, \pi_\ell Q, \pi_\ell v)$ *and* $\ell\Psi(A, M, Q, v)$ *cancel. In particular,* $\Psi_{\mathrm{Sh},\ell}(A, M, Q, v) \in \mathbf{R}[[z_1, \dots, z_n]]$.

Proof. This is the manifestation of Cassou–Noguès' trick in our context. Up to the factor $\mathrm{sgn}\det\sigma$, the value of $\Psi_{\mathrm{Sh}}(A, M, Q, v)$ is the right side of (78) summed over various cones $C = C(\sigma_{i_1}, \dots, \sigma_{i_r})$ and all $a \in \mathscr{P} \cap (v + \mathbf{Z}^n)$, with the subsets $\{i_j\}$ chosen according the Q-perturbation rule, $\mathscr{P}$ the parallelpiped associated to C, and z replaced by zM^t:

$$\sum_C \sum_{a \in \mathscr{P}\cap v+\mathbf{Z}^n} (-1)^r \sum_{\substack{m_j=0 \\ r\text{-tuples}}}^{\infty} \prod_{j=1}^{r} \frac{B_{m_j}(\sigma^{-1}(a)_{i_j})}{m_j!}(zM^t\sigma_{i_j})^{m_j-1} \prod_{i \notin \{i_j\}} e^{(zM^t\sigma_i)(\sigma^{-1}(a)_i)}. \tag{82}$$

Let us now fix a cone C and consider the corresponding contribution of $\pi_\ell C$ to the value $\Psi_{\mathrm{Sh}}(\pi_\ell A, \pi_\ell^{-1}M, \pi_\ell Q, \pi_\ell v)$. (Note that C will be included using perturbation via Q if and only if $\pi_\ell C$ will be included using perturbation via $\pi_\ell Q$.) In applying (78), we use the generators $\pi_\ell \sigma_{i_j}$ for the cone $\pi_\ell C$. By applying the change of variables $a \mapsto \pi_\ell^{-1}a$, we obtain the exact same expression as (82) except with $\mathbf{Z}^n$ in the second index replaced by $\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1}$:

$$\sum_C \sum_{a \in \mathscr{P} \cap v + (\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1})} \text{(same)}. \tag{83}$$

Fix a tuple $m = (m_1, \dots, m_r)$ appearing in the sum (82) such that at least one m_j is equal to zero. Fix such an index j and a point $a \in \mathscr{P} \cap v + \mathbf{Z}^n$. For each equivalence class b mod ℓ, there is a unique point of the form $a + k\sigma_{i_j}/\ell$ in $\mathscr{P} \cap v + (\frac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1})$ for an integer $k \equiv b \pmod{\ell}$. Now, the summand associated to each of these points in (83) is equal to the summand of the associated point a in (82), and in particular is independent of k. To see this, note that

$$\sigma^{-1}(a + k\sigma_{i_j}/\ell) = \sigma^{-1}(a) + (0, \dots, 0, k/\ell, 0, \dots, 0),$$

with k/ℓ in the i_jth component. Hence the only term possibly depending on k is $B_{m_j}(\sigma^{-1}(a)_{i_j})$, but $B_0(x) = 1$ is a constant. The ℓ terms $a + k\sigma_{i_j}/\ell$ in (83) therefore cancel with the term a in (82), in view of the factor ℓ in the definition (76). □

Lemma 4.6. *We have*

$$\Psi_{\mathrm{Sh},\ell}(A, M, Q, v) = (-1)^n \operatorname{sgn}\det(\sigma) \sum_r \ell^{-\underline{r}} \cdot \mathbf{D}_\ell(\sigma, r+1, Q, v) \frac{(zM^t\sigma)^r}{(r+1)!},$$

where r ranges over all n-tuples $r = (r_1, \dots, r_n)$ of nonnegative integers.

Proof. We will require the decomposition (33) for $C_Q \cap (v + \mathbf{Z}^n)$, whose notation we now recall. For each $x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n$, let $J = J(x)$ denote the set of indices j such that $\sigma^{-1}(v+x)_j \in \mathbf{Z}$. For each $I \supset \overline{J}$, consider the cone $C_I = C(\sigma_i : i \in I)$ with associated parallelpiped $\mathscr{P}_I$. The point x and subset I yield a point $a_I \in \mathscr{P}_I$ such that $a_I - v \equiv x \pmod{\sigma\mathbf{Z}^n}$, defined by (32).

We evaluate $\Psi_{\mathrm{Sh}}(A, M, Q, v)$ by employing the decomposition (33) and applying (78). By Lemma 4.5, we need only consider terms from (78) arising from $m_j \geq 1$. We write $r = (r_1, \dots, r_n) = (m_1 - 1, \dots, m_n - 1)$. Suppressing for the moment the factor of $\operatorname{sgn}\det(\sigma)$ in the definition (23) of Φ_{Sh}, we obtain that for a vector of nonnegative integers r and a class $x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n$, the contribution of the cone C_I to the coefficient of $\prod_{i=1}^n (zM^t\sigma_i)^{r_i}$ in $\Psi_{\mathrm{Sh}}(A, M, Q, v)$ for $I \supset \overline{J}$ is 0 unless $r_i = 0$ for $i \notin I$, and in that case equals

$$\operatorname{weight}(C_I)(-1)^{\#I} \prod_{i \notin J} \frac{B_{r_i+1}(\sigma^{-1}(v+x)_i)}{(r_i+1)!} \prod_{i \in J \cap I} \frac{b_{r_i+1}(1)}{(r_i+1)!}. \tag{84}$$

Therefore, let $J_r = J \cap \{i : r_i = 0\}$. The expression (84) summed over all $I \supset \overline{J_r}$ can be written

$$(-1)^n \sum_{x \in \mathbf{Z}^n/\sigma\mathbf{Z}^n} \prod_{i \notin J_r} \frac{B_{r_i+1}(\sigma^{-1}(v+x)_i)}{(r_i+1)!} 2^{-\#J_r} \sum_{I \supset \overline{J_r}} \operatorname{weight}(C_I)(-2)^{n-\#I}. \tag{85}$$

The inner sum in (85) is easily computed using (7):

$$\sum_{\overline{I}\subset J_r}(-2)^{\#\overline{I}}\prod_{i\in\overline{I}}\frac{1+\operatorname{sign}(Q\sigma^{-t})_i}{2}=(-1)^{\#J_r}\prod_{j\in J_r}\operatorname{sign}(Q\sigma^{-t})_j.$$

Therefore, we end up with the following formula for the coefficient of $\prod_{i=1}^n(zM^t\sigma_i)^{r_i}$ arising from terms with each $m_i = r_i + 1 \geq 1$:

$$\sum_{x\in\mathbf{Z}^n/\sigma\mathbf{Z}^n}\prod_{i\notin J_r}\frac{B_{r_i+1}(\sigma^{-1}(v+x)_i)}{(r_i+1)!}\prod_{j\in J_r}\frac{-\operatorname{sgn}(Q\sigma^{-t})_j}{2}=\sum_{x\in\mathbf{Z}^n/\sigma\mathbf{Z}^n}\frac{\mathbf{B}_r(\sigma^{-1}(x+v),\sigma^{-1}Q)}{(r+1)!}. \quad (86)$$

Evaluating the same expression with (A, M, Q, v) replaced by $(\pi_\ell A\pi_\ell^{-1}, \pi_\ell^{-1}M, \pi_\ell Q, \pi_\ell v)$ and using the definition of $\Psi_{\mathrm{Sh},\ell}$ gives the desired result. □

Theorem 4.4 now follows from Lemma 4.6 and Lemma 3.8 applied with

$$F=(-1)^n\operatorname{sgn}\det(\sigma)\sum_r \ell^{-\underline{r}}\cdot\mathbf{D}_\ell(\sigma,r+1,Q,v)\frac{z^r}{(r+1)!}$$

and M replaced by $\sigma^t M$.

In [CD], we show that Theorem 4.4 implies the following integrality property of $\Psi_{\mathrm{Sh},\ell}$ (see Theorem 4 and §2.7 of *loc. cit.*).

Theorem 4.7. *Suppose that M and v satisfy $f_M(v+\frac{1}{\ell}\mathbf{Z}\oplus\mathbf{Z}^{n-1})\subset\mathbf{Z}[\frac{1}{\ell}]$. Then for every nonnegative integer k, we have $\Delta^{(k)}\Psi_{\mathrm{Sh},\ell}(A,M,Q,v)\in\mathbf{Z}[\frac{1}{\ell}]$.*

4.3 A generalized cocycle

The fact that the power series $\Psi_{\mathrm{Sh},\ell}$ is regular (by Lemma 4.6) implies that its domain of definition can be expanded from matrices M and their associated polynomials f_M to arbitrary polynomials $P\in\mathbf{R}[z_1,\ldots,z_n]$. This generalization will not be used in the sequel, so the reader may choose to move ahead to the next section on zeta functions; we discuss this generalization here because it was stated without proof in [CD, Proposition 2.4].

Let $\mathcal{P}=\mathbf{R}[z_1,\ldots,z_n]$, viewed as a Γ-module via $(\gamma P)(z)=P(z\gamma)$. Let $\tilde{\mathcal{F}}$ denote the $\mathbf{R}$-vector space of functions $f\colon\mathcal{P}\times\mathcal{Q}\times\mathcal{V}\longrightarrow\mathbf{R}$ that are linear in the first variable and satisfy the distribution relation

$$f(P,Q,v)=\operatorname{sgn}(\lambda)^n\sum_{\lambda w=v}f(\lambda^{\deg P}P,\lambda^{-1}Q,w) \quad (87)$$

for each nonzero integer λ when P is homogeneous. The space $\tilde{\mathcal{F}}$ has a Γ action given by (42), with M replaced by P. Following (63), define for $P\in\mathcal{P}$ and any matrix $\sigma\in M_n(\mathbf{R})$ coefficients $P_r(\sigma)$ by

$$P(z\sigma^t)=\sum_r\frac{P_r(\sigma)}{r!}z_1^{r_1}\cdots z_n^{r_n}. \quad (88)$$

Fixing $\sigma = 1$, these coefficients define an operator $\Delta^{(P)}\colon \mathbf{R}[[z_1, \dots, z_n]] \to \mathbf{R}$ given by

$$\Delta^{(P)}\left(\sum_r F_r z^r\right) = \sum_r F_r P_r(1).$$

Proposition 4.8. *The function*

$$\tilde{\Psi}_{\mathrm{Sh},\ell}(A, P, Q, v) := \Delta^{(P)}\Psi_{\mathrm{Sh},\ell}(A, 1, Q, v) \tag{89}$$

$$= (-1)^n \mathrm{sgn}(\det\sigma) \sum_r \frac{P_r(\sigma)}{\ell^{\underline{r}}(r+1)!}\mathbf{D}_\ell(\sigma, r+1, Q, v) \tag{90}$$

is a homogeneous cocycle for Γ_ℓ *valued in* $\tilde{\mathcal{F}}$, *i.e.* $\tilde{\Psi}_{\mathrm{Sh},\ell} \in Z^{n-1}(\Gamma_\ell, \tilde{\mathcal{F}})$.

Proof. The cocycle condition $\sum_{i=0}^n (-1)^i \tilde{\Psi}_{\mathrm{Sh},\ell}(A_0, \dots, \hat{A}_i, \dots, A_n) = 0$ follows from that for $\Psi_{\mathrm{Sh},\ell}$. The fact that $\tilde{\Psi}_{\mathrm{Sh},\ell}$ is invariant under Γ_ℓ follows from the equivalent statement for $\Psi_{\mathrm{Sh},\ell}$ and the fact that for any matrix $\gamma \in M_n(\mathbf{R})$ and any $F \in \mathbf{R}[[z_1, \dots, z_n]]$, we have

$$\Delta^{(\gamma^t P)} F(z) = \Delta^{(P)} F(z\gamma). \tag{91}$$

Equation (91) is a mild generalization of Lemma 3.8 that again follows from (64). The equality between (89) and (90) follows from Lemma 4.6 and (91) applied to $\gamma = \sigma$. □

4.4 Smoothed zeta functions

Theorem 4.7 implies that the results on partial zeta functions of totally real fields derived from Sczech's cocycle in [CD] may also be deduced working entirely with our Shintani cocycle $\Psi_{\mathrm{Sh},\ell}$. We state these results here and refer the reader to [CD] for the proofs.

Let F be a totally real field of degree n, and let $\mathfrak{a}$ and $\mathfrak{f}$ be coprime integral ideals of F. Let $\mathfrak{c}$ be an integral ideal of F of prime norm ℓ. Assume that $\mathfrak{a}$ and $\mathfrak{f}$ are relatively prime to ℓ. The $\mathfrak{c}$-smoothed partial zeta functions of F are defined by

$$\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, s) = \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{c}, s) - \mathrm{N}\mathfrak{c}^{1-s}\zeta_{\mathfrak{f}}(\mathfrak{a}, s). \tag{92}$$

We will relate the special values $\zeta_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k)$ for integers $k \geq 0$ to our cocycle $\Psi_{\mathrm{Sh},\ell}$, and by applying Theorem 4.7, deduce that these values lie in $\mathbf{Z}[1/\ell]$.

We must first refine the definition of the cycle $\mathfrak{Z}_{\mathfrak{f}}(\mathfrak{a})$ given in Section 2.6 to account for the ℓ-smoothing. Choose the $\mathbf{Z}$-basis $w = (w_1, \dots, w_n)$ for $\mathfrak{a}^{-1}\mathfrak{f}$ such that $\pi_\ell^{-1} w = (\frac{1}{\ell}w_1, w_2, \dots, w_n)$ is a $\mathbf{Z}$-basis for $\mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f}$. If U denotes the group of totally positive units of F congruent to 1 modulo $\mathfrak{f}$, note that $\rho_w(U) \subset \Gamma_\ell$. Define $\mathcal{A} \in \mathbf{Z}[\Gamma_\ell]$ as in (44).

Let $M \in \mathcal{M}/\mathcal{D}$ be represented by the matrix $\mathrm{N}(\mathfrak{a}\mathfrak{c})^{1/n}(J_j(w_i))_{i,j=1}^n$ so that

$$f_M(x_1, \dots, x_n) = \mathrm{N}(\mathfrak{a}\mathfrak{c})\mathrm{N}(w_1 x_1 + \dots + w_n x_n). \tag{93}$$

Define Q and v as in (47) and (48) without change. Dot product with $(w_1, \dots, w_n)$ provides a bijection

$$v + \tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1} \longleftrightarrow 1 + \mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f}, \tag{94}$$

so P and v satisfy the key integrality property

$$f_M(v + \tfrac{1}{\ell}\mathbf{Z} \oplus \mathbf{Z}^{n-1}) \subset \mathbf{Z}[\tfrac{1}{\ell}] \tag{95}$$

of Theorem 4.7. Let $\mathfrak{Z}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a}) \in H_{n-1}(\Gamma_\ell, \mathcal{R})$ be the homology class represented by the cycle $\mathcal{A} \otimes [(M, Q, v)]$.

Theorem 4.9. *We have* $\zeta_{F,\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = \langle \Psi_{\mathrm{Sh},\ell}, \mathfrak{Z}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a})\rangle_k \in \mathbf{Z}[1/\ell]$.

The $\mathbf{Z}[1/\ell]$-integrality of the values $\zeta_{F,\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k)$ was proven by Deligne–Ribet and Cassou–Noguès. Our proof is a cohomological reformulation of Cassou–Noguès' method.

Proof. The fact that $\zeta_{F,\mathfrak{f},\mathfrak{c}}(\mathfrak{a}, -k) = \langle \Psi_{\mathrm{Sh},\ell}, \mathfrak{Z}_{\mathfrak{f},\mathfrak{a},\ell}\rangle_k$ follows directly from Theorem 2.10 and the definitions of $\Psi_{\mathrm{Sh},\ell}$ and $\mathfrak{Z}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a})$ in terms of Ψ_{Sh} and $\mathfrak{Z}_{\mathfrak{f}}(\mathfrak{a})$ (see [CD, (59)] for details). The fact that $\langle \Psi_{\mathrm{Sh},\ell}, \mathfrak{Z}_{\mathfrak{f},\mathfrak{c}}(\mathfrak{a})\rangle_k \in \mathbf{Z}[1/\ell]$ follows from Theorem 4.7 and (95). $\square$

4.5 p-adic L-functions

We summarize without proof the results of [CD] on p-adic L-functions that follow from Theorems 4.4 and 4.7. Fix a prime $p \neq \ell$. Let

$$\Gamma_{\ell,p} := \Gamma_0(\ell\mathbf{Z}[1/p]) = \Gamma_\ell \cap \mathbf{GL}_n(\mathbf{Z}[1/p]).$$

Let $\mathcal{M}_p$ denote the space of functions that assigns to each $(Q, v) \in \mathcal{Q} \times \mathcal{V}$ a $\mathbf{C}_p$-valued measure $\alpha(Q, v)$ on $\mathbf{X}_v := v + \mathbf{Z}_p^n \subset \mathbf{Q}_p^n$ such that $\alpha(Q, pv)(pU) = \alpha(Q, v)(U)$ for all compact open $U \subset \mathbf{X}_v$. The space $\mathcal{M}_p$ naturally has the structure of a $\Gamma_{\ell,p}$-module given by

$$(\gamma\alpha)(Q, v)(U) := \alpha(AQ, Av)(AU),$$

where $A = \lambda\gamma$ is chosen such that λ is a power of p and $A \in M_n(\mathbf{Z})$.

Given $A \in \Gamma_{\ell,p}^n$, $Q \in \mathcal{Q}$, and $v \in \mathbf{Q}^n$, we define a $\mathbf{Z}[1/\ell]$-valued measure $\mu_\ell = \mu_\ell(A, Q, v)$ on $\mathbf{X}_v$ as follows. Let σ denote the matrix whose columns are the first columns of the matrices in the tuple A. If $\det(\sigma) = 0$, then μ_ℓ is the 0 measure. Suppose now that $\det(\sigma) \neq 0$.

A vector $a \in \mathbf{Z}^n$ and a nonnegative integer r give rise to the compact open subset

$$a + p^r\mathbf{Z}_p^n \subset \mathbf{Z}_p^n.$$

These sets form a basis of compact open subsets of $\mathbf{Z}_p^n$, and hence their translates by v form a basis of compact open subsets of $\mathbf{X}_v$. We define μ_ℓ by applying $\Delta^{(0)}$ to $\Psi_{\mathrm{Sh},\ell}$:

$$\mu_\ell(A, Q, v)(v + a + p^r\mathbf{X}) = \Delta^{(0)}\Psi_{\mathrm{Sh},\ell}\left(A, *, Q, \frac{v + a}{p^r}\right) \in \mathbf{Z}[\tfrac{1}{\ell}] \subset \mathbf{Z}_p. \tag{96}$$

The $*$ indicates that any element of $\mathcal{M}$ may be used, as the value of $\Delta^{(0)}$ on a regular power series is simply the constant term, and hence is not affected by any linear change of variables $z \mapsto zM^t$. It is easily checked that the assignment (96) is well-defined, and that the distribution relation for Ψ_ℓ yields a corresponding distribution relation for μ_ℓ.

Proposition 4.10. *The function $\mu_\ell : \Gamma_{\ell,p}^n \longrightarrow \mathcal{M}_p$ is a homogeneous $(n-1)$-cocycle.*

Proposition 4.10 follows directly from the fact that $\Psi_{\mathrm{Sh},\ell}$ is a cocycle. The next theorem shows that values of $\Delta^{(k)}\Psi_{\mathrm{Sh},\ell}$ can be recovered from the cocycle of measures μ_ℓ.

Theorem 4.11. *For any $M \in \mathcal{M}$ and integer $k \geq 0$, we have*

$$\Delta^{(k)}\Psi_{\mathrm{Sh},\ell}(A, M, Q, v) = \int_{\mathbf{X}_v} f_M(x)^k \, d\mu_\ell(A, Q, v)(x). \tag{97}$$

This is proven in [CD, Theorem 4.2]. Theorem 4.11 implies that we can define the p-adic zeta functions associated to a totally real field F using the cocycle of measures μ_ℓ.

Write $\mathfrak{f} = \mathfrak{f}_0\mathfrak{f}_1$, where $\mathfrak{f}_0$ is the prime-to-p part of $\mathfrak{f}$ and $\mathfrak{f}_1$ is divisible only by primes above p. Define the variables $\mathcal{A}, M, Q, v$ and w as in Section 4.4 using the ideals $\mathfrak{a}$ and $\mathfrak{f}_0$. In particular, $\{w_i\}$ is a $\mathbf{Z}$-basis of $\mathfrak{a}^{-1}\mathfrak{f}_0$. Dot product with $w = (w_1, \ldots, w_n)$ gives a bijection between the spaces $\mathbf{X}_v = \mathbf{Z}_p^n$ and $\mathcal{O}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{F,\mathfrak{p}}$, which we simply denote w. For each prime ideal $\mathfrak{p} \mid p$, define $\mathcal{O}_{\mathfrak{p},\mathfrak{f}} := 1 + \mathfrak{f}\mathcal{O}_{F,\mathfrak{p}}$, and write

$$\mathcal{O}_{p,\mathfrak{f}} := 1 + \mathfrak{f}\mathcal{O}_p = \prod_{\mathfrak{p}|p} \mathcal{O}_{\mathfrak{p},\mathfrak{f}}, \qquad \mathcal{O}_{p,\mathfrak{f}}^* = \mathcal{O}_p^* \cap \mathcal{O}_{p,\mathfrak{f}}.$$

Let $X_w(\mathfrak{f}) := w^{-1}(\mathcal{O}_{p,\mathfrak{f}}^*) \subset \mathbf{Z}_p^n$. For $x \in X_w(\mathfrak{f})$, we have that $f_M(x) = \mathrm{N}(w(x)) \in \mathbf{Z}_p^*$. We may therefore define an analytic function on $\mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$ by

$$s = (t, i) \mapsto f_M(x)^s := \langle f_M(x) \rangle^t \omega(f_M(x))^i.$$

This yields a p-adic analytic family of homology classes

$$\mathfrak{Z}_\mathfrak{f}(\mathfrak{a}, s) \in H_{n-1}(\Gamma_{\ell,p}, \mathbf{Z}[C_c(\mathbf{Q}_p^n) \times \mathcal{Q} \times \mathcal{V}])$$

represented by $\mathcal{A} \otimes [f_M(x)^s \cdot \mathbf{1}_{X_w(\mathfrak{f})}, Q, v]$. Here $C_c(\mathbf{Q}_p^n)$ denotes the space of compactly supported continuous $\mathbf{C}_p$-valued functions on $\mathbf{Q}_p^n$.

The following result is deduced from Theorems 4.9 and 4.11 (see Theorem 2 and Proposition 4.4 of [CD]).

Theorem 4.12. *Define $\zeta_\mathfrak{f}^*(\mathfrak{a}, s)$ as in (2), but with the sum restricted to ideals $\mathfrak{b}$ relatively prime to p. Define $\zeta_{\mathfrak{f},\mathfrak{c}}^*(\mathfrak{a}, s)$ from $\zeta_\mathfrak{f}^*(\mathfrak{a}, s)$ as in (92). The p-adic analytic function on $\mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$ defined by*

$$\begin{aligned} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s) &= \langle \mu_\ell, \mathfrak{Z}_\mathfrak{f}(\mathfrak{a}, s) \rangle \\ &= \int_{X_w(\mathfrak{f})} f_M(x)^s d\mu_\ell(\mathcal{A}, Q, v) \end{aligned}$$

satisfies the interpolation property

$$\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s) = \zeta_{\mathfrak{f},\mathfrak{c}}^*(\mathfrak{a}, s)$$

for $s \in \mathbf{Z}_{\leq 0}$, with $\mathbf{Z}$ embedded diagonally as a dense subset of $\mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$.

Let now $\chi\colon \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}^*$ be a totally odd finite order character with conductor $\mathfrak{f}$. We define $L_{\mathfrak{c},p}(\psi, s)\colon \mathbf{Z}_p \longrightarrow \mathbf{C}_p^*$ associated to the totally even character $\psi = \chi\omega$ by

$$L_{\mathfrak{c},p}(\psi, s) := \sum_{\mathfrak{a} \in G_{\mathfrak{f}}} \chi(\mathfrak{a}\mathfrak{c})\zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}, s),$$

where we embed $\mathbf{Z}_p \subset \mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$ via $s \mapsto (s, 0)$. Let

$$L_p(\psi, s) = \frac{L_{\mathfrak{c},p}(\psi, s)}{1 - \psi(\mathfrak{c})\langle \mathrm{N}\mathfrak{c}\rangle^{s-1}}. \tag{98}$$

As our notation suggests, the right side of (98) does not depend on the choice of $\mathfrak{c}$, and Theorem 4.12 can be used to prove that $L_p(\psi, s)$ satisfies the interpolation property of Theorem 3 stated in the Introduction.

We conclude this section by stating our result on the order of vanishing of $L_p(\psi, s)$ at $s = 0$ that follows by applying Spiess's theorems to our cohomological construction of this p-adic L-function.

Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ denote the set of primes $\mathfrak{p}$ of F above p such that $\chi(\mathfrak{p}) = 1$. Let $G_{p,\chi} \subset G_{\mathfrak{f}}$ denote the subgroup generated by the images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let e_i denote the order of $\mathfrak{p}_i$ in $G_{\mathfrak{f}}$, and write $\mathfrak{p}_i^{e_i} = (\pi_i)$ for a totally positive $\pi_i \equiv 1 \pmod{\mathfrak{f}}$. Let

$$e = \left(\prod_{i}^{r} e_i\right) / \# G_{p,\chi} \in \mathbf{Z}.$$

We then have

$$\begin{aligned} L_{\mathfrak{c},p}(\psi, s) &= \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \sum_{\mathfrak{b} \in G_{p,\chi}} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s) \\ &= \frac{1}{e} \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \sum_{\mathfrak{b} \mid \prod_{i=1}^r \mathfrak{p}_i^{e_i - 1}} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s). \end{aligned} \tag{99}$$

Let

$$\mathbf{O} = \prod_{i=1}^{r} (\mathcal{O}_{\mathfrak{p}_i} - \pi_i \mathcal{O}_{\mathfrak{p}_i}) \times \prod_{\mathfrak{p} \mid p,\ \mathfrak{p} \neq \mathfrak{p}_i} \mathcal{O}_{\mathfrak{p},\mathfrak{f}}^*.$$

In [CD, Proposition 4.4], we prove a generalization of Theorem 4.12 that implies that the inner sum of (99) can be written

$$\begin{aligned} \sum_{\mathfrak{b} \mid \prod_{i=1}^r \mathfrak{p}_i^{e_i - 1}} \zeta_{\mathfrak{f},\mathfrak{c},p}(\mathfrak{a}\mathfrak{b}^{-1}, s) &= \int_{\mathbf{O}} f_M(x)^s d\mu_\ell(\mathcal{A}, Q, v) \\ &= \langle \mu_\ell, \mathfrak{Z}(s) \rangle, \end{aligned} \tag{100}$$

where $\mathfrak{Z}(s)$ is the homology class represented by $\mathcal{A} \otimes [f_M(x)^s \cdot \mathbf{1}_{w^{-1}(\mathbf{O})}, Q, v]$. For any $k \geq 0$, taking the kth derivative with respect to s and evaluating at $s = 0$ then yields

$$L_{\mathfrak{c},p}^{(k)}(\psi, 0) = \frac{1}{e} \sum_{\mathfrak{a} \in G_{\mathfrak{f}}/G_{p,\chi}} \chi(\mathfrak{a}\mathfrak{c}) \langle \mu_\ell, \mathfrak{Z}_{\log^k} \rangle, \tag{101}$$

where $\mathfrak{Z}_{\log^k}$ is the homology class represented by $\mathcal{A} \otimes [\log(f_M(x))^k \cdot \mathbf{1}_{w^{-1}(\mathbf{O})}, Q, v]$. The following theorem of Spiess [Sp1] is reproved in [CD, §5].

Theorem 4.13 (Spiess)**.** *The homology class $\mathfrak{Z}_{\log^k}$ vanishes for $k < r$.*

Let us relate the result proven in [CD] to the statement of Theorem 4.13. Let U_π denote the (free abelian) subgroup of F^* generated by the group U of totally positive units congruent to 1 modulo $\mathfrak{f}$ together with the π_i. For each $\mathfrak{p} \in S$, let $C_c^\flat(F_\mathfrak{p})$ denote the space of compactly supported continuous $\mathbf{C}_p$-valued functions on $F_\mathfrak{p}$ that are constant in a neighborhood of zero. Let $C_c^\flat(F_p)$ be the subspace of functions on $F_p = \prod_{\mathfrak{p}|p} F_\mathfrak{p}$ that can be written as a linear combination of products $\prod f_\mathfrak{p}$, where $f_\mathfrak{p} \in C_c^\flat(F_\mathfrak{p})$ for $\mathfrak{p} \in S$ and $f_\mathfrak{p} \in C_c(\mathcal{O}_\mathfrak{p}^*)$ for $\mathfrak{p} \mid p$, $\mathfrak{p} \notin S$.

We view $C_c^\flat(F_p)$ as a U_π-module by $(u \cdot f)(x) = f(x/u)$. Define $\mathfrak{L}_k \in H_{n-1}(U_\pi, C_c^\flat(F_p))$ as the class represented by the cycle

$$\alpha(\epsilon_1, \ldots, \epsilon_{n-1}) \otimes \mathbf{1}_\mathbf{O} \cdot (\log_p \mathrm{N}x)^k \in Z_{n-1}(U, C_c^\flat(F_p)),$$

with α as in (18) and $\epsilon_1, \ldots, \epsilon_{n-1}$ a basis of U. Equation (78) of [CD] states that $\mathfrak{L}_k = 0$ for $k < r$. Now, there is a map

$$\mathrm{cores}\colon H_{n-1}(U_\pi, C_c^\flat(F_p)) \to H_{n-1}(\Gamma_{\ell,p}, \mathbf{Z}[C_c(\mathbf{Q}_p^n) \times \mathcal{Q} \times \mathcal{V}])$$

induced by $\rho_w\colon U_\pi \to \Gamma_{\ell,p}$ and $w\colon \mathbf{Q}_p^n \to F_p$. The map cores sends

$$A \otimes f(x) \mapsto \rho_w(A) \otimes [(f(w \cdot x), Q, v)].$$

Under this map we have $\mathrm{cores}\,\mathfrak{L}_k = \mathfrak{Z}_{\log^k}$. Theorem 4.13 follows.

Combining (101) with Theorem 4.13 gives our cohomological proof of Theorem 4, restated below.

Theorem 4.14. *We have* $\mathrm{ord}_{s=0} L_p(\psi, s) \geq r$*.*

References

[Bs] D. Barsky. *Fonctions zêta p-adiques d'une classe de rayon des corps totalement réels.* Groupe d'étude d'analyse ultramétrique (5ème année 1977/78), exposé no. 16, 23 pp. errata 1978-1979.

[Bv] A. Barvinok. *The complexity of generating functions for integer points in polyhedra and beyond.* Proceedings of the ICM, Madrid, Spain, 2006, 763–787.

[Br] K. Brown. Cohomology of groups. Graduate Texts in Mathematics, Vol. **87**, Springer-Verlag, New-York, 1982.

[Ca] Pi. Cassou-Noguès. *Valeurs aux entiers négatifs des fonctions zêta et fonctions zêta p-adiques.* Invent. Math. **51** (1979), no. 1, 29–59.

[CD] P. Charollois, S. Dasgupta. *Integral Eisenstein cocycles on* $\mathbf{GL}_n$*, I: Sczech's cocycle and p-adic L-functions of totally real fields.* Submitted (2012), available at http://arxiv.org/abs/1206.3050.

[Co1] P. Colmez. *Résidu en* $s = 1$ *des fonctions zêta p-adiques.* Invent. Math. **91** (1988), 371–389.

[Co2] P. Colmez. *Algébricité de valeurs spéciales de fonctions L.* Invent. Math. **95** (1989), 161–205.

[Co3] P. Colmez. *La méthode de Shintani et ses variantes.* Unpublished preprint.

[Das] S. Dasgupta. *Shintani Zeta Functions and Gross-Stark units for totally real fields.* Duke Math. Journal, **143** (2008), no. 2, 225–279.

[DS] S. Dasgupta, M. Spiess. *Principal minors of Gross's p-adic regulator and the Eisenstein cocycle*, in preparation.

[DR] P. Deligne, K. Ribet. *Values of abelian L-functions at negative integers over totally real fields.* Invent. Math. **59** (1980), no. 3, 227–286.

[DDF] F. Diaz y Diaz, E. Friedman. *Signed fundamental domains for totally real number fields.* preprint.

[Gr] B. Gross, *p-adic L-series at* $s = 0$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 979–994.

[Hi] R. Hill. *Shintani cocycles on* $\mathbf{GL}_n$. Bull. of the LMS **39** (2007), 993-1004.

[HS] S. Hu, D. Solomon. *Properties of the higher-dimensional Shintani generating functions and cocycles on* $\mathbf{PGL}_3(\mathbb{Q})$. Proc. LMS **82** (2001), no. 1, 64-88.

[IO] T. Ishii, T. Oda. *A short history on investigation of the special values of zeta and L-functions of totally real number fields.* In *Automorphic forms and zeta functions. Proc. of the conf. in memory of Tsuneo Arakawa*, World Sci. Publ. (2006), 198-233.

[KP] A. G. Khovanski, G. Pukhlikov. *The Riemann-Roch theorem for integrals and sums of quasipolynomials on virtual polytopes.* Algebra i Analiz **4** (1992), no. 4, 789-812.

[La] J. Lawrence. *Rational-function-valued valuations on polyhedra.* In *Discrete and Computational Geometry*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 6, Amer. Math. Soc. Providence, RI, 1991, 199-208.

[Sc1] R. Sczech. *Eisenstein cocycles for* $\mathbf{GL}_2(\mathbf{Q})$ *and values of L-functions in real quadratic fields.* Comment. Math. Helv. **67** (1992), no. 3, 363–382.

[Sc2] R. Sczech. *Eisenstein group cocycles for* $\mathbf{GL}_n$ *and values of L-functions.* Invent. Math. **113** (1993), no. 3, 581–616.

[Sc3] R. Sczech. *A remark on the Eisenstein-Shintani cocycle.* Unpublished preprint (1998).

[Sh] T. Shintani. *On evaluation of zeta functions of totally real algebraic number fields.* J. Fac. Sci Univ. Tokyo, Sect. IA Math **23** (1976), no. 2, 393-417.

[So1] D. Solomon. *Algebraic properties of Shintani's generating functions: Dedekind sums and cocycles on* $\mathbf{PGL}_2(\mathbf{Q})$. Compositio Math. **112** (1998), no. 3, 333-362.

[So2] D. Solomon. *The Shintani cocycle II. Partial* ζ*-functions, cohomologous cocycles and* p*-adic interpolation.* Journal of Number Theory **75** (1999), 53-108.

[Sp1] M. Spiess. *On special zeros of* p*-adic* L*-functions of Hilbert modular forms.* Preprint (2012), available at http://arxiv.org/abs/1207.2289

[Sp2] M. Spiess. *Shintani cocycles and vanishing order of* p*-adic Hecke* L*-series at* $s = 0$. Preprint (2012), available at http://arxiv.org/abs/1203.6689.

[Stev] G. Stevens. *The Eisenstein measure and real quadratic fields. Théorie des nombres (Québec, PQ, 1987),* 887-927, de Gruyter, Berlin, 1989.

[Stee] A. Steele. *The* p*-adic Shintani cocycle.* Preprint (2012), available at http://arxiv.org/pdf/1209.5018.pdf.

[We] U. Weselmann. *An elementary identity in the theory of Hecke* L*-functions.* Invent. Math. **95** (1989), 207-214.

[Wi] A. Wiles. *The Iwasawa conjecture for totally real fields.* Ann. of Math. (2) **131** (1990), no. 3, 493-540.

P. C. : Institut de Mathématiques de Jussieu, Université Paris 6, France
E-mail **charollois (at) math (dot) jussieu (dot) fr**

S. D. : Dept. of Mathematics, University of California Santa Cruz, USA
E-mail **sdasgup2 (at) ucsc (dot) edu**

M. G. : Dept. of Mathematics & Statistics, University of Calgary, Canada
E-mail **mgreenbe (at) ucalgary (dot) ca**