HARMONIC MAASS FORMS ASSOCIATED TO REAL QUADRATIC FIELDS

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Abstract. In this paper, we explicitly construct harmonic Maass forms that map to the weight one theta series associated by Hecke to odd ray class group characters of real quadratic fields. From this construction, we give precise arithmetic information contained in the Fourier coefficients of the holomorphic part of the harmonic Maass form, establishing the main part of a conjecture of the second author.

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1. Introduction.

In number theory, modular forms of weight one play an important role because of their special relationship to number fields. To each weight one eigenform $f \in S_{1,\chi}(\Gamma_0(N))$, Deligne and Serre functorially attached a 2-dimensional, odd, irreducible representation ϱ_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ [9]. This Galois representation gives rise to a finite Galois extension over $\mathbb{Q}$. Stark's conjecture [23] predicts that a certain subfield of this extension can be constructed from the special values of the L-function attached to the modular form f. It is natural to expect weight one modular forms to encode interesting arithmetic information.

In [15], Hecke gave several systematic constructions of weight one modular forms, implicitly in the vector-valued modular form setting (see [2] for such setting). One of them attached weight one cusp forms, whose Galois representations have dihedral projective images, to real quadratic fields. Specifically, suppose $\mathcal{O}$ is an order of a real quadratic field $F \subset \mathbb{R}$ and $\mathfrak{a} \subset \mathcal{O}$ is an $\mathcal{O}$-ideal. Then for any natural number $M \in \mathbb{N}$, the $\mathbb{Z}$-lattice $L = M\mathfrak{a}$ is integral, indefinite and anisotropic with respect to the quadratic form $Q = \frac{\mathrm{Nm}}{AM}$ (see Section 2.1). Hecke considered the integral

$$\vartheta(\tau, L) := \int_{\Gamma_L \backslash \mathbb{R}_+^\times} \Theta(\tau, L; t) \frac{dt}{t}, \tag{1.0.1}$$

where Γ_L and $\mathbb{R}_+^\times$ are the discriminant kernel and the symmetric space associated to L, and $\Theta(\tau, L; t)$ is a vector-valued theta function constructed from the Gaussian with a suitable polynomial. This is then a theta lift of the constant function in the orthogonal variable t to an automorphic form on SL_2 over $\mathbb{Q}$. Since L is anisotropic, the fundamental domain is compact, which we choose to be $[1, \varepsilon_L]$ with $\varepsilon_L > 1$ being a totally positive unit in $\mathcal{O}$ depending only on M. As usual, let $\{\mathfrak{e}_h : h \in L^*/L\}$ be the basis of $\mathbb{C}[L^*/L]$ and $\vartheta_h(\tau, L)$ the $\mathfrak{e}_h$ component of $\vartheta(\tau, L)$. Then it has the Fourier expansion

$$\vartheta_h(\tau, L) = \sum_{n>0} c_L(n, h) q^n, \ c_L(n, h) := \sum_{\lambda \in \Gamma_L \backslash L + h,\ Q(\lambda) = n} \mathrm{sgn}(\lambda), \ q = e^{2\pi i \tau}. \tag{1.0.2}$$

The vector-valued cusp form $\vartheta(\tau, L)$ transforms with respect to the Weil representation ρ_L on $\mathrm{SL}_2(\mathbb{Z})$, and each component $\vartheta_h(\tau, L)$ is a weight one cusp form on the principal congruence subgroup $\Gamma(DM)$ (see Section 2).

In [2], Borcherds constructed automorphic forms with singularities on orthogonal Shimura varieties via regularized theta lift. The singularities of the output are controlled by the input, which is a holomorphic modular form with poles at the cusps. Then in [3] and [4], Bruinier replaced this input with certain non-holomorphic Poincaré series, which are eigenfunctions of the hyperbolic Laplacian. He used the outputs to produce Chern classes for the Heegner divisors. Motivated by these works, Bruinier and Funke introduced in [5] the notion of *harmonic Maass forms* generalizing classical modular forms. They are annihilated by the weight-k hyperbolic Laplacian

$$\Delta_k := \xi_{2-k} \circ \xi_k,\ \xi_k := 2iv^k \overline{\frac{\partial}{\partial \overline{\tau}}},\ \tau = u + iv \in \mathcal{H} \tag{1.0.3}$$

and have poles at the cusps (see Section 2.2). Using harmonic Maass forms as the input of Borcherds' singular theta lift, Bruinier and Funke constructed an adjoint of the Kudla-Millson theta lift for orthogonal groups of arbitrary signature [5]. This theta lift then produces automorphic Green's function for special divisors on orthogonal type Shimura varieties, which enables one to study arithmetic intersection theory and leads to generalizations of the famous Gross-Zagier formula [7] and the recent proof of an averaged version of Colmez's conjecture [1].

Besides as input for the theta lift, harmonic Maass forms also have interesting Fourier coefficients. Because of the annihilation by Δ_k, harmonic Maass form naturally has a holomorphic part and non-holomorphic part in its Fourier expansion. In his ground breaking thesis [25], Zwegers completed Ramanujan's holomorphic mock theta functions and produced real-analytic modular forms of weight $\frac{1}{2}$. They turned out to be harmonic Maass forms that map to weight $\frac{3}{2}$ unary theta series under $\xi_{1/2}$. In weight $\frac{1}{2}$, there are many other important works that reveal the arithmetic nature of the Fourier coefficient of the holomorphic part (see e.g. [6], [11]).

In the self-dual case of weight $k = 1$, Kudla, Rapoport and Yang [18] constructed "incoherent Eisenstein series", which turned out to be harmonic Maass forms that map under ξ_1 to Eisenstein series associated to an imaginary quadratic field K. The Fourier coefficients of the holomorphic part are logarithms of integers, and can be interpreted as arithmetic degrees of special divisors on arithmetic curves. Later Duke and the second author studied harmonic Maass forms that map to weight one cusp forms associated to non-trivial class group characters of K [12]. The Fourier coefficients were shown to be logarithms of algebraic numbers in the Hilbert class field of K. In his thesis [13], Ehlen gave an arithmetic interpretation of the valuation of these algebraic numbers along the lines of [18]. In contrast to the incoherent Eisenstein series in [18], the harmonic Maass forms in [12] and [13] were not constructed explicitly. Also, numerical evidence suggests that given any weight one eigenform f with associated Galois representation ϱ, there exists a harmonic Maass form $\tilde{f}$ such that $\xi_1(\tilde{f}) = f$ and the Fourier coefficients of the holomorphic part of $\tilde{f}$ are $\overline{\mathbb{Q}}$ linear combinations of logarithms of algebraic numbers in the number field cut out by $\mathrm{ad}\varrho$ (see [12], [19]).

In this paper, we will *explicitly construct* a harmonic Maass form $\tilde{\vartheta}(\tau, L)$ that map under ξ_1 to Hecke's weight one cusp form $\vartheta(\tau, L)$, and study the arithmetic information contained in the Fourier coefficients of $\tilde{\vartheta}$.

Our main result is as follows.

Theorem 1.1. *Let $(L, Q) = (M\mathfrak{a}, \frac{\mathrm{Nm}}{AM})$ be as above. Then there exists a harmonic Maass form $\tilde{\vartheta}(\tau, L) = \sum_{h \in L^*/L} \tilde{\vartheta}_h(\tau, L)\mathfrak{e}_h \in H_{1,\rho_{-L}}(\mathrm{SL}_2(\mathbb{Z}))$ such that $\xi_1(\tilde{\vartheta}(\tau, L)) = \vartheta(\tau, L)$ and*

$$\tilde{\vartheta}_h(\tau, L) = \sum_{n \in \mathbb{Q}, n \gg -\infty} c_L^+(n, h) q^n - \sum_{n \in \mathbb{Q}_{>0}} c_L(n, h)\Gamma(0, 4\pi n v) q^{-n},$$

where $\Gamma(s, x) := \int_x^\infty e^{-t} t^s \frac{dt}{t}$ is the incomplete gamma function and

$$c_L^+(n, h) - \sum_{\substack{\lambda \in \Gamma_L \backslash L + h \\ -Q(\lambda) = n > 0}} \mathrm{sgn}(\lambda) \log \left| \frac{\lambda}{\lambda'} \right| \in \frac{1}{\kappa} \mathbb{Z} \cdot \log \varepsilon_L. \tag{1.0.4}$$

Here $\kappa \in \mathbb{N}$ is an explicit constant depending on $\mathcal{O}$ and M only.

Remark 1.2. When $\mathfrak{a}$ is a proper $\mathcal{O}$-ideal and $\gcd(A, M) = 1$, then one can choose κ to divide $6M^3\varphi(M)D$ in the result above, where φ is the Euler totient function.

Remark 1.3. The summation in Equation (1.0.4) is finite and the choice of the representative $\lambda \in \Gamma_L \backslash L + h$ does not affect the statement of the result.

To state the scalar-valued version of the main theorem, suppose for simplicity that F has class number one. Let φ be an odd ray class group character with conductor $\mathbf{m} = \mathfrak{m} \cdot \infty_1$. Then one can associate an eigenform

$$f_\varphi(\tau) = \sum_{n \geq 1} c_\varphi(n) q^n := \sum_{(\lambda) \subset \mathcal{O}_F} \varphi(\lambda) q^{\mathrm{Nm}((\lambda))} \in S_{1,\chi}(\Gamma_0(N)),$$

where $N := DM$, $M := \mathrm{Nm}(\mathfrak{m})$ and $\chi = \chi_D \cdot \varphi|_{\mathbb{Q}}$ (see Section 6.2 for details). The Galois representation associated to f_φ by Deligne and Serre is the induction of φ from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\varepsilon_F > 1$ be the fundamental unit of F and $\mathbb{Z}[\varphi]$ the subring of $\mathbb{C}$ obtained from $\mathbb{Z}$ by adjoining to it the values of φ. For each $n \geq 1$, we can now define an analogue of $c_\varphi(n) = \frac{1}{2} \sum_{(\lambda) \subset \mathcal{O}_F,\, \mathrm{Nm}((\lambda)) = n} (\varphi(\lambda) + \varphi(\lambda'))$ by

$$\boldsymbol{c}_\varphi(n) := \frac{1}{2} \sum_{(\lambda) \subset \mathcal{O}_F,\, \mathrm{Nm}((\lambda)) = n} \left(\varphi^{-1}(\lambda) - \varphi^{-1}(\lambda')\right) \log \left| \frac{\lambda}{\lambda'} \right| \in \mathbb{C}/(\mathbb{Z}[\varphi] \cdot \log \varepsilon_F). \tag{1.0.5}$$

It is clear that this quantity does not depend on the choice of generator for the ideal (λ), so it is well-defined. Using these quantities, we can construct the formal power series

$$\boldsymbol{f}_\varphi := \sum_{n \geq 1} \boldsymbol{c}_\varphi(n) q^n \in (\mathbb{C}/\mathbb{Z}[\varphi] \cdot \log \varepsilon_F)[\![q]\!]. \tag{1.0.6}$$

In [19], the second author conjectured that $\boldsymbol{f}_\varphi$ can be lifted to the holomorphic part of a harmonic Maass form mapping to f_φ. From Theorem 1.1 above, we can deduce the following result.

Theorem 1.4 (Theorem 6.5)**.** *In the notations above, there exists an integer $\kappa_{\mathbf{m}}$ dividing $12M^3\varphi(M)D$ and a harmonic Maass form $\tilde{f}_\varphi \in H_{1,\overline{\chi}}(\Gamma_0(N))$ with holomorphic part $\tilde{f}_\varphi^+ = \sum_{n \gg -\infty} c_\varphi^+(n)q^n$ such that $\xi_1 \tilde{f}_\varphi = f_\varphi$ and the image of $\kappa_{\mathbf{m}} \cdot f_\varphi^+$ in $\mathbb{C}/(\mathbb{Z}[\varphi] \cdot \log \varepsilon_F)$ is $\kappa_{\mathbf{m}} \cdot \boldsymbol{f}_\varphi$. In other words,*

$$c_\varphi^+(n) \in \frac{1}{\kappa_{\mathbf{m}}}\mathbb{Z}[\varphi] \cdot \log \varepsilon_F + \boldsymbol{c}_\varphi(n). \tag{1.0.7}$$

Remark 1.5. When $\kappa_{\mathbf{m}} = 1$, Theorem 1.4 answers Conjecture 7.1 in [19]. It is possible to reduce the bound $12M^3\varphi(M)D$ above with more careful analysis in some cases. Note that if $n = \ell$ or ℓ^2 with ℓ an inert prime in $F/\mathbb{Q}$, then $\kappa_{\mathbf{m}} c_\varphi^+(n) \in \mathbb{Z}[\varphi] \cdot \log \varepsilon_F$.

Remark 1.6. In [8], Darmon, Lauder and Rotger studied a non-classical, overconvergent generalized eigenform associated to f_φ. In their p-adic setting, the ℓ-th Fourier coefficient is the p-adic logarithm of a Gross-Stark ℓ-unit in a class field of F when ℓ is an inert prime in $F/\mathbb{Q}$. Otherwise, it is zero. In Section 6.2, we will rewrite $\boldsymbol{c}_\varphi(n)$ to show that the archimedean and non-archimedean settings are in some sense complementary to each other.

Remark 1.7. In [19], it is shown that certain $\mathbb{Z}[\varphi]$-linear combinations of c_φ^+ is the value of Hilbert modular functions at big CM points. Thus, getting a handle on the individual c_φ^+'s allows us to give explicit factorization formula of the CM values in the spirit of Gross and Zagier [14]. Furthermore the CM values are defined over F. We hope to pursue this line of investigation in the future.

In order to best reflect the nature of the coefficients $c_L^+(n,h)$, we have stated Theorem 1.1 as an existence result. Its proof is through *explicit construction*, which requires two ingredients. The first is the "perturbed integral" with a spectral parameter $s \in \mathbb{C}$

$$I(\tau, -L, s) := \int_1^{\varepsilon_L} t^s \Theta(\tau, -L; t) \frac{dt}{t} \tag{1.0.8}$$

and its derivative at $s = 0$, which is a real-analytic modular form in τ. It turns out that $\xi_1 I'(\tau, -L, 0)$ differs from $\vartheta(\tau, L)$ by $\log \varepsilon_L \cdot \Theta(\tau, L; 1)$ (see Equation (5.0.14) and Proposition 5.1). To prove Theorem 1.1, it then suffices to construct a preimage of $\Theta(\tau, L; 1)$ under ξ_1. We accomplish this by replacing the usual Gaussian in the theta function with a continuous function $g_\tau \in L^1(\mathbb{R})$ (see Equations (3.4.1) and (3.4.2) for definition). This is essentially the constant term in the Taylor series expansion of a closely related function $g_{\tau,z} \in L^\infty(\mathbb{R})$ defined in Equation (3.3.13). We distilled $g_{\tau,z}$ from the work of Zwegers' thesis on Ramanujan's mock theta function, and will use it to give another construction of the real-analytic Jacobi form $\hat{\mu}$ constructed by Zwegers. If one desires, it is possible to obtain a closed formula for $c_L^+(n,h)$

as a finite sum of $\mathbb{Q}$-linear combination of logarithms of algebraic numbers in F. We will do this numerically for $\eta^2(\tau)$ in the last section.

The plan of the paper is as follows. In Section 2, we will recall some basic information about vector-valued automorphic forms and theta functions following [17] and [2]. In Section 3, we will construct a continuous function $g_\tau \in L^1(\mathbb{R})$ with nice properties (see Proposition 3.7) and use it in Section 4 to construct the ξ_1-preimage of $\Theta(\tau, L; 1)$, which we denote by $\tilde{\Theta}(\tau, L)$. Also in Section 3, we will give a different proof of the modularity of the completed μ-function studied by Zwegers [25]. Finally in the last two sections, we will prove Theorem 1.1 and its scalar-valued version Theorem 6.5, and give some examples of the theorems.

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2. Theta Lift from $\mathrm{O}(1,1)$ to SL_2.

In this section, we will follow Kudla [17] to recall the work of Hecke [15] in terms of theta lift from $\mathrm{O}(1,1)$ to SL_2.

2.1. **Indefinite, anisotropic $\mathbb{Z}$-lattice of rank 2.** An even, integral lattice is a $\mathbb{Z}$-module L equipped with a quadratic form $Q : L \to \mathbb{Z}$. It is anisotropic if L does not contain an isotropic vector, i.e. $\lambda \in L$ with $Q(\lambda) = 0$. If such lattice is indefinite and has rank 2, then it can be described by $L = \mathbb{Z}^2$ and a quadratic form $Q(m,n) = Am^2 + Bmn + Cn^2$ with $A, B, C \in \mathbb{Z}$ such that the discriminant $B^2 - 4AC > 0$ is not a perfect square.

Such lattices can be related to ideals in real quadratic field. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ throughout. For a discriminant $D \geq 1$, let $F = \mathbb{Q}(\sqrt{D})$ be the corresponding real quadratic field with ring of integers $\mathcal{O}_F$. The $\mathbb{Z}$-lattice $\mathcal{O}_D := \mathbb{Z} + \mathbb{Z}\frac{D+\sqrt{D}}{2}$ is a subring of $\mathcal{O}_F$. Its dual under the quadratic form Nm is $\mathfrak{d}_D^{-1}\mathcal{O}_F$, where $\mathfrak{d}_D := \sqrt{D}\mathcal{O}_D$. If D is fundamental, then $\mathcal{O}_D = \mathcal{O}_F$ and $\mathfrak{d}_D = \mathfrak{d}_F$ is the different. The group of units $\mathcal{O}_F^\times$ is generated by ± 1 and the fundamental unit $\varepsilon_F > 1$.

For an integral ideal $\mathfrak{a} \subset \mathcal{O}_D$ with $A := [\mathcal{O}_D : \mathfrak{a}]$ and a positive integer $M \in \mathbb{N}$, consider the $\mathbb{Z}$-lattice

$$(L_{\mathfrak{a},M}, Q_{\mathfrak{a},M}) := \left(M\mathfrak{a}, \frac{\mathrm{Nm}_{F/\mathbb{Q}}}{AM} \right). \tag{2.1.1}$$

It is anisotropic and has rank 2. The induced bilinear form is given by $B_{\mathfrak{a},M}(\lambda, \mu) := \frac{\mathrm{Tr}_{F/\mathbb{Q}}(\lambda \cdot \mu')}{AM}$ for all $\lambda, \mu \in L_{\mathfrak{a},M}$ with $'$ the non-trivial automorphism in $\mathrm{Gal}(F/\mathbb{Q})$. The dual lattice $L^*_{\mathfrak{a},M}$ is given by $\mathfrak{a}\mathfrak{d}_D^{-1}$ and the finite quadratic modular $L^*_{\mathfrak{a},M}/L_{\mathfrak{a},M}$ is isomorphic to $\mathcal{O}_D/M\mathfrak{d}_D$.

Now given $A, B, C \in \mathbb{Z}$ such that $D := B^2 - 4AC > 0$ is not a perfect square, the ideal $\mathfrak{a} := A\mathbb{Z} + \frac{B+\sqrt{D}}{2}\mathbb{Z}$ has norm $|A| \neq 0$ and

$$Q_{\mathfrak{a},1}\left(Am + \frac{B+\sqrt{D}}{2}n\right) = \operatorname{sgn}(A) \cdot \left(Am^2 + Bmn + Cn^2\right).$$

Finally, notice that $-L_{\mathfrak{a},M}$ is isometric to $L_{\mathfrak{a}\mathfrak{d}_D,M}$ via $\lambda \mapsto \lambda\sqrt{D}$.

2.2. **Weil Representation and Automorphic Forms.** Let L be an indefinite, even, integral lattice of rank 2 with bilinear form $(,) : L \times L \to \mathbb{Z}$. As usual, let $\{\mathfrak{e}_h : h \in L^*/L\}$ denote the canonical basis of the vector space $\mathbb{C}[L^*/L]$ and $\mathbf{e}(a) := e^{2\pi i a}$ for any $a \in \mathbb{C}$. Then $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}[L^*/L]$ through the Weil representation ρ_L as (see e.g. [2, §4])

$$\rho_L(T)(\mathfrak{e}_h) = \mathbf{e}(Q(h))\mathfrak{e}_h,\ \rho_L(S)(\mathfrak{e}_h) = \frac{1}{\sqrt{|L^*/L|}} \sum_{\delta \in L^*/L} \mathbf{e}(-(\delta, h))\mathfrak{e}_\delta, \tag{2.2.1}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that $\rho_{-L} = c \circ \rho_L \circ c$ on $\mathbb{C}[L^*/L]$, where $c : \mathbb{C} \to \mathbb{C}$ denotes complex conjugation.

Let d_L be the level of L and $\Gamma(d_L) \subset \Gamma$ the principal congruence subgroup of level d_L. Then ρ_L is trivial on $\Gamma(d_L)$ and can be viewed as a representation of $\mathrm{SL}_2(\mathbb{Z}/d_L\mathbb{Z})$ (see e.g. Proposition 4.5 in [21]). Let ζ_{d_L} be a primitive d_L^{th} root of unity. For $a \in (\mathbb{Z}/d_L\mathbb{Z})^\times$, let $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{d_L})/\mathbb{Q})$ be the element that sends ζ_{d_L} to $\zeta_{d_L}^a$. Then σ_a acts naturally on $W_L := \mathbb{Q}(\zeta_{d_L})[L^*/L]$. Let $\varsigma_a \in \mathrm{GL}(W_L)$ be the left action given by

$$\varsigma_a \cdot w := \sigma_a^{-1}(w),\ w \in W_L. \tag{2.2.2}$$

In [20], McGraw extended ρ_L to a unitary representation of $\mathrm{GL}_2(\mathbb{Z}/d_L\mathbb{Z})$ on W_L, where the action of

$$J_a := \left(\begin{smallmatrix} 1 & \\ & a \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/d_L\mathbb{Z}) \tag{2.2.3}$$

is ς_a. This is the main ingredient used to prove the rationality of basis of vector-valued modular forms. The result we need can be stated as follows.

Proposition 2.1. *[20, Theorem 4.3] The map that sends γ to $\rho_L(\gamma)$ when $\gamma \in \mathrm{SL}_2(\mathbb{Z}/d_L\mathbb{Z})$ and J_a to ς_a is a unitary representation of $\mathrm{GL}_2(\mathbb{Z}/d_L\mathbb{Z})$ on W_L. In other words, if we view $\rho_L(\gamma) \in \mathrm{GL}_{|L^*/L|}(\mathbb{Q}(\zeta_{d_L}))$ with respect to the standard basis of W_L, then $\sigma_a(\rho_L(\gamma)) = \rho_L(J_a^{-1}\gamma J_a)$.*

Now, we will quickly recall some facts about automorphic forms. Let $k \in \mathbb{Z}$ be an integer, V an n-dimensional $\mathbb{C}$-vector space, $\Gamma' \subset \Gamma$ a finite index subgroup and $\rho : \Gamma' \to \mathrm{GL}(V)$

a representation. Then a real-analytic function $f = (f_j)_{1\le j\le n} : \mathcal{H} \to V$ is a vector-valued automorphic form on Γ' with weight k and representation ρ if it satisfies

$$(f\mid_{k,\rho}\gamma)(\tau) := \rho(\gamma)^{-1}\cdot\left((c\tau+d)^{-k} f_j\left(\frac{a\tau+b}{c\tau+d}\right)\right)_{1\le j\le n} = f(\tau) \tag{2.2.4}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma'$ and $\tau \in \mathcal{H}$. If ρ is trivial, then we may omit it in the slash operator. We denote the space of such functions by $\mathcal{A}_{k,\rho}(\Gamma')$. The subspace of $\mathcal{A}_{k,\rho}(\Gamma')$ consisting of functions holomorphic on $\mathcal{H}$ is denoted by $M^!_{k,\rho}(\Gamma')$, which is usually called the space of weakly holomorphic modular forms. Let $M_{k,\rho}(\Gamma')$ and $S_{k,\rho}(\Gamma')$ be the usual space of modular forms and cusp forms respectively. A function $f \in \mathcal{A}_{k,\rho}(\Gamma')$ is called a harmonic weak Maass form if $\Delta_k f = 0$ and f has at most linear exponential growths at all the cusps. It satisfies $\xi_k f \in M^!_{2-k,\overline{\rho}}(\Gamma')$ by the definition of Δ_k in (1.0.3). Furthermore, if $\xi_k f$ vanishes at all the cusps, then we call f a harmonic Maass form. We use $H_{k,\rho}(\Gamma')$ to denote the space of harmonic Maass forms on Γ' of weight k and representation ρ.

Since $f \in H_{k,\rho}(\Gamma')$ satisfies $\Delta_k f = 0$, it can be written canonically as the difference of a holomorphic part f^+ and non-holomorphic part f^*. Let $\mathbf{B}$ be a basis of V. Then f^+ and f^* have the following Fourier expansions.

$$f^+(\tau) = \sum_{\mathfrak{e}\in\mathbf{B}}\left(\sum_{\substack{n\in\mathbb{Q}\\ n\gg-\infty}} c^+(n,\mathfrak{e})q^n\right)\mathfrak{e},\ f^*(\tau) = (4\pi)^{k-1}\sum_{\mathfrak{e}\in\mathbf{B}}\left(\sum_{\substack{n\in\mathbb{Q}\\ n>0}} c(n,\mathfrak{e})\Gamma(1-k,4\pi n v)q^{-n}\right)\mathfrak{e}.$$

It is readily checked that $\xi_k f(\tau) = \xi_k(-f^*(\tau)) = \sum_{\mathfrak{e}\in\mathbf{B}}\left(\sum_{\substack{n\in\mathbb{Q}\\ n>0}} n^{1-k}c(n,\mathfrak{e})q^n\right)\mathfrak{e} \in S_{2-k,\overline{\rho}}(\Gamma')$.

2.3. **Vector-Valued Theta Function.** Let $(V_\mathbb{R}, Q)$ denote the quadratic space of signature $(1,1)$ with $V_\mathbb{R} = \mathbb{R}^2$ and for $X = (x_1,x_2), Y = (y_1,y_2) \in V_\mathbb{R}$

$$Q(X) := x_1x_2,\ B(X,Y) := x_1y_2 + x_2y_1. \tag{2.3.1}$$

Note that $(V_\mathbb{R}, Q) \cong (V_\mathbb{R}, -Q)$ via the map

$$\iota((x_1,x_2)) := (x_1,-x_2). \tag{2.3.2}$$

The symmetric domain attached to $V_\mathbb{R}$ is the hyperbola $\mathcal{D} := \{Z^- \subset V_\mathbb{R} \mid (Z^-,Z^-) = -1\}$, which is parametrized by $\mathbb{R}^\times$ via

$$\begin{aligned}\Phi : \mathbb{R}^\times &\to \mathcal{D}\\ t &\mapsto Z_t^- := \left(\tfrac{t}{\sqrt{2}}, -\tfrac{t^{-1}}{\sqrt{2}}\right).\end{aligned}$$

The connected component $\mathcal{D}^+$ is parametrized by $\mathbb{R}^\times_+$ under Φ. Define

$$Z_t^+ := \frac{1}{\sqrt{2}}\left(t,t^{-1}\right) \in (Z_t^-)^\perp. \tag{2.3.3}$$

Then $d\Phi\left(t\frac{d}{dt}\right) = Z_t^+ \in \mathbb{R}^2$ and $\{Z_t^+, Z_t^-\}$ is an orthogonal basis of $V_\mathbb{R}$ with $Q(Z_t^+) = -Q(Z_t^-) = \frac{1}{2}$. We can write $X = X_t^+ - X_t^-$ with

$$X_t^\alpha := B(X, Z_t^\alpha)Z_t^\alpha = (\alpha x_1 t^{-1} + x_2 t)Z_t^\alpha \tag{2.3.4}$$

for any $X = (x_1, x_2) \in V_\mathbb{R}$ and $\alpha \in \{+, -\}$. In this basis, the quadratic space $V_\mathbb{R}$ becomes $(\mathbb{R}^{1,1}, Q_0)$ with $\mathbb{R}^{1,1} = \mathbb{R}^2$ and

$$Q_0((x, y)) := \frac{x^2 - y^2}{2}. \tag{2.3.5}$$

Let $\mathcal{F}_0$ denote the Fourier transform on $\mathbb{R}^{1,1}$ with respect to Q_0, i.e.

$$\mathcal{F}_0(\phi)(x, y) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(w, r)\mathbf{e}(xw - yr)dwdr$$

for any Schwartz function ϕ on $\mathbb{R}^{1,1}$.

An important ingredient in forming the theta kernel is the archimedean part of the Schwartz function. In the setting of Hecke, this is given by

$$\phi_\tau(x, y) := \sqrt{2v} \cdot x \cdot \mathbf{e}\left(\frac{x^2}{2}\tau - \frac{y^2}{2}\overline{\tau}\right). \tag{2.3.6}$$

As a function on the quadratic space $(\mathbb{R}^{1,1}, Q_0)$, ϕ_τ satisfies

$$\phi_{\tau+1}(W) = \mathbf{e}\left(Q_0(W)\right)\phi_\tau(W),\ \phi_{-1/\tau}(W) = -\tau\mathcal{F}_0(\phi_\tau)(W),\ W \in \mathbb{R}^{1,1}. \tag{2.3.7}$$

Now for any even, integral lattice $L \subset V_\mathbb{R}$, the vector-valued theta function $\Theta(\tau, L; t) := \sum_{h\in L^*/L} \Theta_h(\tau, L; t)\mathfrak{e}_h$ with

$$\Theta_h(\tau, L; t) := \sum_{X\in L+h} \phi_\tau(B(X, Z_t^+), B(X, Z_t^-)) \tag{2.3.8}$$

transforms on $\mathrm{SL}_2(\mathbb{Z})$ with weight 1 and representation ρ_L in the variable τ by Theorem 4.1 in [2]. Similarly, the image of $-L$ under the involution $\iota : -V_\mathbb{R} \to V_\mathbb{R}$ is an even, integral lattice in $(V_\mathbb{R}, Q)$ and define

$$\Theta(\tau, -L; t) := \Theta(\tau, \iota(-L); t). \tag{2.3.9}$$

Suppose L is the image of the embedding

$$\begin{aligned} (L_{\mathfrak{a},M}, Q_{\mathfrak{a},M}) &\hookrightarrow (V_\mathbb{R}, Q) \\ \lambda &\mapsto \boldsymbol{\lambda} := \frac{1}{\sqrt{AM}}(\lambda, \lambda'). \end{aligned} \tag{2.3.10}$$

for some $D, \mathfrak{a}, M$. Then $(L, Q) \cong (L_{\mathfrak{a},M}, Q_{\mathfrak{a},M})$ and $\Theta_h(\tau, L; t)$ becomes

$$\Theta_h(\tau, L; t) = \frac{\sqrt{v}}{\sqrt{AM}} \sum_{\substack{\lambda\in\mathfrak{a}\mathfrak{d}_D^{-1}\\ \lambda-h\in M\mathfrak{a}}} \left(\lambda' t + \lambda t^{-1}\right)\mathbf{e}\left(\frac{(\lambda t^{-1} + \lambda' t)^2\tau - (\lambda t^{-1} - \lambda' t)^2\overline{\tau}}{4AM}\right). \tag{2.3.11}$$

Similarly, the expression above becomes $\Theta_h(\tau, -L; t)$ after changing λ' to $-\lambda'$.

2.4. **Theta Integral.** As in the end of last subsection, suppose $(L, Q) = (L_{\mathfrak{a},M}, Q_{\mathfrak{a},M})$. The discriminant kernel Γ_L is the subgroup of $\mathrm{SO}^+(L) \cong \mathbb{Z}$ consisting of those units which are congruent to 1 modulo $M\sqrt{D}$. Notice that $\varepsilon \in \Gamma_L$ if and only if $\varepsilon' \in \Gamma_L$. Also, $\mathrm{Nm}(\varepsilon) = 1$ for all $\varepsilon \in \Gamma_L$. Let $\Gamma'_L \subset \Gamma_L$ be subgroup of totally positive element and $\varepsilon_L > 1$ its generator. In [15], Hecke calculated the integral

$$\vartheta(\tau, L) := \int_1^{\varepsilon_L} \Theta(\tau, L; t)\frac{dt}{t} = \int_0^{\log \varepsilon_L} \Theta(\tau, L; e^\nu) d\nu. \tag{2.4.1}$$

Since $B(\lambda\varepsilon, Z_t^\pm) = B(\lambda, Z^\pm_{\varepsilon' t})$ for any totally positive $\varepsilon \in \mathcal{O}_F^\times$, we can unfold the integral to obtain

$$\begin{aligned}\int_1^{\varepsilon_L} \Theta_h(\tau, L; t)\frac{dt}{t} &= \sum_{\lambda \in \Gamma'_L \backslash L+h,\ \lambda \neq 0} \int_0^\infty \phi_\tau(B(\lambda, Z_t^+), B(\lambda, Z_t^-))\frac{dt}{t} \\ &= \frac{\sqrt{v}}{\sqrt{AM}} \sum_{\lambda \in \Gamma'_L \backslash L+h,\ \lambda \neq 0} \mathbf{e}\left(\frac{\lambda\lambda' u}{AM}\right) \int_{-\infty}^\infty \left(\lambda' e^\nu + \lambda e^{-\nu}\right) \mathbf{e}\left(\frac{(\lambda e^{-\nu})^2 + (\lambda' e^\nu)^2}{2AM} iv\right) d\nu.\end{aligned}$$

Using the identity $e^{-2\pi y} = 2\sqrt{y}\int_{-\infty}^\infty e^\nu e^{-\pi y(e^{2\nu} + e^{-2\nu})} d\nu$, we can evaluate

$$\int_{-\infty}^\infty \lambda' e^\nu \mathbf{e}\left(\frac{(\lambda e^{-\nu})^2 + (\lambda' e^\nu)^2}{2AM} iv\right) d\nu = \frac{1}{2}\sqrt{\frac{AM}{v}} \mathrm{sgn}(\lambda') \mathbf{e}\left(\frac{|\lambda\lambda'| iv}{AM}\right), \quad \lambda \neq 0. \tag{2.4.2}$$

If $\lambda \in L + h$ has negative norm, then the integral will vanish. Thus

$$\vartheta(\tau, \pm L) = \sum_{h \in L^*/L} \mathfrak{e}_h \sum_{\lambda \in \Gamma'_L \backslash L+h,\ \pm Q(\lambda) > 0} \mathrm{sgn}(\lambda) \mathbf{e}\left(|Q(\lambda)|\tau\right) \tag{2.4.3}$$

is in $S_{1,\rho_{\pm L}}(\mathrm{SL}_2(\mathbb{Z}))$. Hecke noticed that if there exists $\varepsilon < 0$ in Γ_L, then $\vartheta(\tau, L)$ vanishes identically [15, Satz 1]. Thus, we can suppose $\Gamma'_L = \Gamma_L$.

Another way to show that $\vartheta(\tau, \pm L)$ is holomorphic without explicitly computing the integral in Equation (2.4.1) is to apply the operator ξ_τ to $\Theta(\tau, \pm L; t)\frac{dt}{t}$ and show that it is an exact form on $\mathbb{R}^\times$. Then its integral over $\Gamma_L \backslash \mathbb{R}^\times_+$ would vanish since this locally symmetry domain has no boundary. Let $\xi_t := t\frac{d}{dt}$ be the invariant vector field on $\mathbb{R}^\times_+$, i.e. an element in the Lie algebra. We have the following proposition.

Proposition 2.2. *For all $\tau \in \mathcal{H}$ and $t \in \mathbb{R}^\times$, we have*

$$2\xi_\tau \Theta(\tau, \pm L; t) = -\xi_t \Theta(\tau, \mp L; t). \tag{2.4.4}$$

Proof. We will show the equality with L on the left hand side and $-L$ on the right hand side. The other combination of signs can be proved similarly. Straightforward calculations show that

$$2\xi_\tau \phi_\tau(x,y) = \frac{x}{y}(1-4\pi v y^2)\phi_\tau(y,x),\ x\frac{\partial \phi_\tau}{\partial x} = (1+2\pi i x^2\tau)\phi_\tau,\ \frac{\partial \phi_\tau}{\partial y} = -2\pi i y\overline{\tau}\phi_\tau,\ \xi_t Z_t^{\pm} = Z_t^{\mp}.$$

This implies that for all $X \in V_{\mathbb{R}}$,

$$2\xi_\tau \phi_\tau(B(X, Z_t^+), B(X, Z_t^-)) = \xi_t \phi_\tau(B(X, Z_t^-), B(X, Z_t^+)). \tag{2.4.5}$$

Since $\phi_\tau(B(\iota(X), Z_t^+), B(\iota(X), Z_t^-)) = -\phi_\tau(B(X, Z_t^-), B(X, Z_t^+))$, Equation (2.4.4) follows immediately from the definition of $\Theta(\tau, \pm L; t)$ in Equations (2.3.8), (2.3.9) and (2.4.5). □

3. Two Special Functions.

In this section, we will introduce two special functions on $\mathbb{R}$ depending on $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$. One is non-holomorphic in τ, the other is holomorphic, and their difference behaves nicely under Fourier transform. We will use them to give a new proof of transformation formula of the completed μ-function, which was an important insight in Zwegers' groundbreaking study of Ramanujan's mock theta function [25]. In Section 4, we will use these two special functions to construct a preimage of $\Theta(\tau, L; 1)$ under ξ_τ.

3.1. **Tempered Distributions.** Let $\mathcal{S}(\mathbb{R})$ the spaces of complex-valued Schwartz functions on $\mathbb{R}$. Denote the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R})$ by

$$\mathcal{F}(\varphi)(x) := \int_{\mathbb{R}} \varphi(w)\mathbf{e}(wx)dw \tag{3.1.1}$$

and its inverse by $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

The space of continuous functionals on $\mathcal{S}(\mathbb{R})$ is the space of tempered distributions and denoted by $\mathcal{S}'(\mathbb{R})$. There is an injection

$$\begin{aligned} \mathcal{S}(\mathbb{R}) &\to \mathcal{S}'(\mathbb{R}) \\ \phi &\mapsto T_\phi : \varphi \mapsto \int_{\mathbb{R}} \varphi(x)\phi(x)dx. \end{aligned}$$

Using this injection, we can define the derivative $\frac{dT}{dx}$ and Fourier transform $\mathcal{F}(T)$ of $T \in \mathcal{S}'(\mathbb{R})$ to be the distribution satisfying

$$\frac{dT}{dx}(\varphi) := T\left(-\frac{d\varphi}{dx}\right),\ \mathcal{F}(T)(\overline{\varphi}) := T(\overline{\mathcal{F}^{-1}\varphi}),\ \varphi \in \mathcal{S}(\mathbb{R}).$$

Since any measurable function $\phi \in L^\infty(\mathbb{R})$ gives rise to $T_\phi \in \mathcal{S}'(\mathbb{R})$, we will use $\mathcal{F}(\phi)(x)$ to denote $\int_{\mathbb{R}} \phi(w)\mathbf{e}(wx)dw$ if this integral converges almost everywhere and is in $L^\infty(\mathbb{R})$ as a function of x. In this case, $\mathcal{F}(T_\phi) = T_{\mathcal{F}(\phi)}$. Let $\delta(x)$ be the Dirac delta function, which is defined by the property

$$T_\delta(\varphi) = \int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0),\ \varphi \in \mathcal{S}(\mathbb{R}).$$

Equivalently, we have $\delta(x) = \frac{dH(x)}{dx}$, where $H(x) := \frac{d}{dx}\max\{x, 0\} = \frac{\mathrm{sgn}(x)+1}{2}$ is the heavyside function. Now if $\phi \in L^\infty(\mathbb{R})$ is piecewise differentiable, we use $\frac{d\phi}{dx}$ to denote the function with the property $\frac{dT_\phi}{dx} = T_{\frac{d\phi}{dx}}$.

3.2. **Non-holomorphic Part.** For fixed $\tau = u + iv \in \mathcal{H}$ and $z \in \mathbb{C}$, we first recall the famous Jacobi theta function

$$\vartheta(\tau, z) := \sum_{m \in \frac{1}{2}+\mathbb{Z}} \mathbf{e}\left(\frac{m^2}{2}\tau + mz + \frac{m}{2}\right). \tag{3.2.1}$$

The Jacobi triple product formula says that

$$\vartheta(\tau, z) = i\mathbf{e}\left(\frac{\tau}{8} + \frac{z}{2}\right)(1 - \mathbf{e}(-z)) \prod_{m=1}^{\infty} (1 - \mathbf{e}(m\tau))(1 - \mathbf{e}(m\tau + z))(1 - \mathbf{e}(m\tau - z)). \tag{3.2.2}$$

Thus $\vartheta(\tau, z)$ has a simple zero whenever $z \in \mathbb{Z}\tau + \mathbb{Z}$ and $\frac{\partial \vartheta(\tau,z)}{\partial z} |_{z=0} = -2\pi\eta^3(\tau)$, where $\eta(\tau) := \mathbf{e}(\frac{\tau}{24}) \prod_{m \in \mathbb{N}} (1 - \mathbf{e}(m\tau))$ is the Dedekind eta function. It is well-known that

$$\vartheta(\tau, z) = -\vartheta(\tau, -z),\ \vartheta(\tau + 1, z) = \mathbf{e}\left(\tfrac{1}{8}\right)\vartheta(\tau, z),\ \vartheta\left(-\tfrac{1}{\tau}, -\tfrac{z}{\tau}\right) = i\alpha(\tau, z)\vartheta(\tau, z), \tag{3.2.3}$$

with $\alpha(\tau, z) := (-i\tau)^{1/2}\mathbf{e}\left(\frac{z^2}{2\tau}\right)$.

Define the function $g^*_{\tau,z} : \mathbb{R} \to \mathbb{C}$ by

$$\begin{aligned} g^*_{\tau,z}(x) &:= \frac{\vartheta(\tau, z)}{2}\mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)\left(\mathrm{sgn}(x) - \mathrm{erf}\left(\sqrt{2\pi v}\left(x + \tfrac{\mathrm{Im}(z)}{v}\right)\right)\right), \\ &= \mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)\left(\frac{\mathrm{sgn}(x)\vartheta(\tau, z)}{2} - \int_{-\mathrm{Im}(z)/v}^{x} \mathbf{e}\left(\frac{r^2}{2}\tau + rz\right) f^*_{\tau,z}(r)dr\right), \end{aligned} \tag{3.2.4}$$

where $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-r^2}dr$ is the error function and $f^*_{\tau,z} \in \mathcal{S}(\mathbb{R})$ is defined by

$$f^*_{\tau,z}(x) := \vartheta(\tau, z) \cdot \sqrt{2v} \cdot \mathbf{e}\left(-\frac{x^2}{2}\overline{\tau} - x\overline{z} + i\frac{\mathrm{Im}(z)^2}{v}\right). \tag{3.2.5}$$

The Fourier transform of $f^*_{\tau,z}$ can be calculated as follows.

Lemma 3.1. *For all $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, we have*

$$\mathcal{F}(f^*_{-1/\tau,-z/\tau}) = i \cdot f^*_{\tau,z}. \tag{3.2.6}$$

Proof. The change of variable $x' = x + \frac{\overline{z}}{\overline{\tau}}$ reduces the computation to the usual case of the Fourier transform for the Gaussian. $\square$

From the expression

$$g^*_{\tau,z}(x) = \vartheta(\tau,z)\mathrm{sgn}(x)\sqrt{2v}\mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)\left(\int_0^\infty e^{-2\pi v r^2}dr - \mathrm{sgn}(x)\int_0^{x+\frac{\mathrm{Im}(z)}{v}} e^{-2\pi v r^2}dr\right)$$

and $v > 0$, we see that $g^*_{\tau,z}(x)$ decays as a Schwartz function in the variable x. In particular, we have

$$|g^*_{\tau,z}(x)| \ll |\vartheta(\tau,z)| \cdot e^{-\pi v x^2} e^{2\pi |x\mathrm{Im}(z)|}\left(1 + \frac{\mathrm{Im}(z)}{\sqrt{v}} e^{2\pi|x\mathrm{Im}(z)|}\right). \tag{3.2.7}$$

Thus, the function $g^*_{\tau,z} \in L^1(\mathbb{R})$ has a jump discontinuity at $x = 0$ and is continuously differentiable. Let $\mathcal{F}(g^*_{\tau,z}) \in L^\infty(\mathbb{R})$ be its Fourier transform and define

$$\mathcal{D}^*_{\tau,z}(x) := g^*_{\tau,z}(x,z) - (-i\tau)^{-1}\mathcal{F}(g^*_{-1/\tau,-z/\tau})(x). \tag{3.2.8}$$

The main result of this section is that $\mathcal{D}^*_{\tau,z}$ satisfies a nice differential equation.

Proposition 3.2. *For any $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, the function $\mathcal{D}^*_{\tau,z}$ satisfies the differential equation*

$$\mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)\frac{d}{dx}\left(\mathbf{e}\left(\frac{x^2}{2}\tau + xz\right)\mathcal{D}^*_{\tau,z}(x)\right) = \vartheta(\tau,z)\delta(x) + i\vartheta(-1/\tau,-z/\tau). \tag{3.2.9}$$

Proof. First, notice that $g^*_{\tau,z}$ satisfies the differential equation

$$\frac{d}{dx}\left(\mathbf{e}\left(\frac{x^2}{2}\tau + xz\right)g^*_{\tau,z}(x)\right) = \vartheta(\tau,z)\delta(x) - \mathbf{e}\left(\frac{x^2}{2}\tau + xz\right)f^*_{\tau,z}(x). \tag{3.2.10}$$

This follows from $\frac{d}{dx}|x| = \mathrm{sgn}(x)$, $\frac{d}{dx}\mathrm{sgn}(x) = 2\delta(x)$. Applying $\mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)\delta(x) = \delta(x)$ to Equation (3.2.10) gives us

$$\frac{d}{dx}\left(g^*_{\tau,z}(x)\right) + 2\pi i(\tau x + z)g^*_{\tau,z}(x) = \vartheta(\tau,z)\delta(x) - f^*_{\tau,z}(x). \tag{3.2.11}$$

Using the standard facts of Fourier transform (see e.g. [2, Lemma 3.1]), we have

$$\frac{d}{dx}\left(\mathcal{F}(g^*_{\tau,z})(x)\right)\tau + 2\pi i(z - x)\mathcal{F}(g^*_{\tau,z})(x) = \vartheta(\tau,z) - \mathcal{F}(f^*_{\tau,z})(x).$$

After making the changes of variables $\tau \mapsto -1/\tau, z \mapsto -z/\tau$ and applying Lemma 3.1, the equation becomes

$$-\frac{d}{dx}\left(\frac{\mathcal{F}(g^*_{-1/\tau,-z/\tau})(x)}{\tau}\right) - 2\pi i(x\tau + z)\frac{\mathcal{F}(g^*_{-1/\tau,-z/\tau})(x)}{\tau} = \vartheta(-1/\tau,-z/\tau) - i\cdot f^*_{\tau,z}(x).$$

Multiplying the above equation by i and then adding it to Equation (3.2.11) gives Equation (3.2.10). $\square$

3.3. **Holomorphic Part.** Fix $\tau \in \mathcal{H},\ z \in \mathbb{C}$. We want to construct an analogue of $g^*_{\tau,z}$, such that it is holomorphic in τ, z and satisfies the same property as in Proposition 3.2. The key there input is the function $f^*_{\tau,z}$, which appears in the second expression of $g^*_{\tau,z}$ in Equation (3.2.4) and transforms nicely under Fourier transform in Lemma 3.1.

The discrete analogue of $f^*_{\tau,z}$ comes from the Jacobi theta function, which can be rewritten as

$$\vartheta(\tau, z) = \int_{-\infty}^{\infty} \mathbf{e}\left(\frac{r^2}{2}\tau + rz\right) \mathbf{e}\left(\frac{r}{2}\right) \mathcal{C}\left(r + \frac{1}{2}\right) dr, \tag{3.3.1}$$

where $\mathcal{C}(x) := \sum_{n\in\mathbb{Z}} \delta(x-n)$ is the Dirac comb function. Now we consider

$$f^+_{\tau,z}(x) := \mathbf{e}\left(\frac{x}{2}\right) \mathcal{C}\left(x + \frac{1}{2}\right). \tag{3.3.2}$$

Since $\mathcal{F}(\mathcal{C}) = \mathcal{C}$, we can combine Equations (3.2.3) and (3.3.2) to obtain the analogue of Lemma 3.1

$$\mathcal{F}(f^+_{-1/\tau,-z/\tau}) = \mathcal{F}(f^+_{\tau,z}) = i \cdot f^+_{\tau,z}. \tag{3.3.3}$$

To imitate the second expression in Equation (3.2.4), we will use the following "incomplete theta function"

$$\vartheta(x; \tau, z) := \sum_{\substack{0\le m\le |x| \\ m\in \frac{1}{2}+\mathbb{Z}}} \mathbf{e}\left(\frac{m^2}{2}\tau + \operatorname{sgn}(x) mz + \frac{m}{2}\right) = \int_0^x \mathbf{e}\left(\frac{r^2}{2}\tau + rz\right) f^+_{\tau,z}(r) dr. \tag{3.3.4}$$

It is clear that $\vartheta(x;\tau,z)$ is holomorphic in both τ and z. Furthermore, it is locally constant for $x \in \mathbb{R}\backslash(\mathbb{Z} + \frac{1}{2})$ and

$$\vartheta_\pm(\tau, z) := \lim_{x\to\pm\infty} \vartheta(x; \tau, z) = \sum_{m\in\mathbb{Z}+\frac{1}{2},\ \pm m\ge 0} \mathbf{e}\left(\tfrac{m^2\tau}{2} + mz + \tfrac{m}{2}\right) \tag{3.3.5}$$

are the two halves of $\vartheta(\tau,z)$ as $\vartheta(\tau,z) = \vartheta_+(\tau,z) - \vartheta_-(\tau,z)$ and $\vartheta_+(\tau,z) = -\vartheta_-(\tau,-z)$. For convenience, we denote $\vartheta_0(\tau,z) := \frac{\vartheta_+(\tau,z)+\vartheta_-(\tau,z)}{2}$ so that $\vartheta_{\operatorname{sgn}(x)}$ is defined for all $x \in \mathbb{R}$.

Define $g^+_{\tau,z} \in L^\infty(\mathbb{R})$ by

$$\begin{aligned} g^+_{\tau,z}(x) &:= \mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right) \left(\vartheta_{\operatorname{sgn}(x)}(\tau,z) - \vartheta(x,\tau,z)\right) \\ &= \mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right) \operatorname{sgn}(x) \sum_{m>|x|, m\in\mathbb{Z}+\frac{1}{2}} \mathbf{e}\left(\frac{m^2}{2}\tau + \operatorname{sgn}(x)mz + \frac{m}{2}\right). \end{aligned} \tag{3.3.6}$$

Then it has jump discontinuities at $(\mathbb{Z}+\frac{1}{2})\cup\{0\}$ and is continuously differentiable otherwise. Also, $g^+_{\tau,z}(x)$ is left-continuous when $x<0$ and right-continuous when $x>0$. Near $x=0$, the function $g^+_{\tau,z}$ satisfies $g^+_{\tau,z}(0)=\frac{1}{2}(\lim_{x\to 0^+} g^+_{\tau,z}(x)+\lim_{x\to 0^-} g^+_{\tau,z}(x))$.

We are interested in the Fourier transform of $g^+_{\tau,z}$ and applying Poisson summation to it. Before calculating those, it is useful to begin with the following lemma concerning the convergence of the integral defining the Fourier transform of $g^+_{\tau,z}$.

Lemma 3.3. *For fixed $\tau\in\mathcal{H}$, the improper integral*

$$\int_{-\infty}^{\infty} g^+_{\tau,z}(w)\mathbf{e}(wx)dw \tag{3.3.7}$$

converges uniformly for z in compact subset of $\mathbb{C}$ and defines a bounded, piecewise continuously differentiable function in $x\in\mathbb{R}$ with discontinuities on $\mathbb{Z}+\frac{1}{2}$.

Remark 3.4. We use $\mathcal{F}(g^+_{\tau,z})(x)$ to denote the extension of this integral to a function on $\mathbb{R}$ such that it is left-continuous when $x<0$ and right-continuous when $x>0$.

Proof. For $w\in\mathbb{R}$, define $m_w:=\lceil |w|+\frac{1}{2}\rceil-\frac{1}{2}\in\mathbb{Z}+\frac{1}{2}$ to be the least half integer greater than or equal to $|w|$. Then the function

$$g^+_{\tau,z}(w)-\mathbf{e}\left(-\tfrac{w^2}{2}\tau-wz\right)\mathbf{e}\left(\tfrac{m_w^2}{2}\tau+\operatorname{sgn}(w)m_w z+\tfrac{m_w}{2}\right) \tag{3.3.8}$$

decays as a Schwartz function. Thus, it is enough to analyze the convergence of the integral

$$\begin{aligned}&\int_{|w|\geq\frac{1}{2}}\mathbf{e}\left(-\tfrac{w^2}{2}\tau-wz\right)\mathbf{e}\left(\tfrac{m_w^2}{2}\tau+\operatorname{sgn}(w)m_w z+\tfrac{m_w}{2}\right)\mathbf{e}(wx)dw\\ &=\sum_{m\in\left(\mathbb{Z}+\frac{1}{2}\right)\backslash\{\pm\frac{1}{2}\}}\operatorname{sgn}(m)\mathbf{e}\left(m(x+\tfrac{1}{2})\right)\int_0^1\mathbf{e}\left(-\tfrac{(w')^2}{2}\tau+(m\tau+z-x)\operatorname{sgn}(m)w'\right)dw'\end{aligned}$$

for $x\in\mathbb{R}\backslash\mathbb{Z}+\frac{1}{2}$. We obtain the second equation from the first by setting $w=m-\operatorname{sgn}(w)w'$ with $w'\in[0,1)$, $m\in\mathbb{Z}+\frac{1}{2}$. It is enough to consider the sum over $|m|>M$, where $M\in\mathbb{N}$ is chosen such that $|\frac{\operatorname{Im}(z)}{v}|<M$.

After integrating by parts, one can write

$$\begin{aligned}\int_0^1\mathbf{e}\left(-\tfrac{(w')^2}{2}\tau+(m\tau+z-x)\operatorname{sgn}(m)w'\right)dw'&=-\frac{1}{2\pi i(m\tau+z-x)\operatorname{sgn}(m)}+\\ +\frac{\mathbf{e}\left(-\frac{\tau}{2}+(m\tau+z-x)\operatorname{sgn}(m)\right)}{2\pi i(-\tau+(m\tau+z-x)\operatorname{sgn}(m))}&+\int_0^1\frac{\mathbf{e}\left(-\frac{(w')^2}{2}\tau+(m\tau+z-x)\operatorname{sgn}(m)w'\right)}{4\pi^2(-\tau w'+(m\tau+z-x)\operatorname{sgn}(m))^2}dw'.\end{aligned}$$

The sum of the second and third terms over $m \in \mathbb{Z} + \frac{1}{2}$ can be bounded trivially and one only needs to consider the sum of the first term, which boils down to bounding

$$\sum_{m \in \mathbb{Z}+\frac{1}{2},\, |m|>M} \frac{\mathbf{e}\left(m(x+\frac{1}{2})\right)}{m\tau + z - x} \tag{3.3.9}$$

Given $x \in \mathbb{R} \backslash \mathbb{Z} + \frac{1}{2}$, write $x = x' + N$ uniquely with $x' \in (-\frac{1}{2}, \frac{1}{2})$ and $N \in \mathbb{Z}$. If $\frac{z-x}{\tau} \notin \mathbb{Z} + \frac{1}{2}$, then Lemma 1.19 in [25] gives us the identity

$$\sum_{m \in \mathbb{Z}+\frac{1}{2},\, |m|>M} \frac{\mathbf{e}\left(m(x+\frac{1}{2})\right)}{m\tau + z - x} = \frac{\pi i}{\tau} \frac{2\mathbf{e}\left(\frac{x-z}{\tau}x'\right)\mathbf{e}(\frac{N}{2})}{\mathbf{e}\left(\frac{z-x}{2\tau}\right) + \mathbf{e}\left(\frac{x-z}{2\tau}\right)} - \sum_{m \in \mathbb{Z}+\frac{1}{2},\, |m|<M} \frac{\mathbf{e}\left(m(x+\frac{1}{2})\right)}{m\tau + z - x}. \tag{3.3.10}$$

If there exists some $x_0 \in \mathbb{R} \backslash (\mathbb{Z} + \frac{1}{2})$ and $m' \in \mathbb{Z} + \frac{1}{2}$ such that $z = x_0 - m'\tau$, then $|m'| < M$ and the right hand side has a removable singularity at $x = x_0$. In any case, the right hand side extends to a differentiable function for $x \in \mathbb{R} \backslash (\mathbb{Z} + \frac{1}{2})$. It is also bounded as $|x| \to \infty$. Thus, the lemma holds. □

A similar, yet simpler proof gives us the following lemma.

Lemma 3.5. *For fixed $\tau \in \mathcal{H}$, the improper integral*

$$\int_{-\infty}^{\infty} \frac{\partial g^+_{\tau,z}}{\partial z}(w)\mathbf{e}(wx)dw \tag{3.3.11}$$

converges uniformly for z in compact subset of $\mathbb{C}$ and defines a bounded, continuously function in $x \in \mathbb{R}$.

Now the function $\mathcal{F}(g^+_{\tau,z}) \in L^\infty(\mathbb{R})$ can be viewed as a distribution and we can study its Fourier transform as in Section 3.2 for $\mathcal{F}(g^*_{\tau,z})$. In this case, consider the function

$$\mathcal{D}^+_{\tau,z}(x) := g^+_{\tau,z}(x) - (-i\tau)^{-1}\mathcal{F}(g^+_{-1/\tau,-z/\tau})(x) \tag{3.3.12}$$

which is the analogue of $\mathcal{D}^*_{\tau,z}$ in Equation (3.2.8). It is then no surprise that $\mathcal{D}^+_{\tau,z}$ also satisfies the differential equation (3.2.9). This gives us the following result.

Proposition 3.6. *For fixed $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, define*

$$g_{\tau,z}(x) := g^+_{\tau,z}(x) - g^*_{\tau,z}(x). \tag{3.3.13}$$

Then $\mathcal{F}(g_{\tau,z})(x) := \mathcal{F}(g^+_{\tau,z})(x) - \mathcal{F}(g^*_{\tau,z})(x)$ is a bounded, piecewise continuously differentiable function in $x \in \mathbb{R}$ with discontinuities on $\mathbb{Z} + \frac{1}{2}$. Furthermore for all $x \in \mathbb{R}$, it satisfies

$$g_{\tau,z}(x) = (-i\tau)^{-1}\mathcal{F}(g_{-1/\tau,-z/\tau})(x) \tag{3.3.14}$$

$g_{\tau,z}(-x) = -g_{\tau,-z}(x)$ and $g_{\tau+1,z}(x) = \mathbf{e}\left(-\frac{x^2}{2} + \frac{1}{8}\right) g_{\tau,z}(x)$.

Proof. The first claim follows directly from estimate (3.2.7) and Lemma 3.3. Following the same calculations as in Proposition 3.2, we see that $\mathcal{D}^+_{\tau,z}$ also satisfies the differential equation (3.2.9), i.e.

$$\frac{d}{dx}\left(\mathbf{e}\left(\frac{x^2}{2}\tau + xz\right)\left(\mathcal{D}^+_{\tau,z}(x) - \mathcal{D}^*_{\tau,z}(x)\right)\right) = 0. \tag{3.3.15}$$

This implies that $\mathcal{D}^+_{\tau,z}(x) - \mathcal{D}^*_{\tau,z}(x)$ has no jump discontinuity anywhere.

When $x \in \mathbb{R}\backslash(\{0\} \cup \mathbb{Z} + \frac{1}{2})$, the difference $\mathcal{D}^+_{\tau,z}(x) - \mathcal{D}^*_{\tau,z}(x)$ is continuously differentiable by Lemma 3.3 and the decay of $g^*_{\tau,z}$. Furthermore, the definitions of $g^+_{\tau,z}$ and $g^*_{\tau,z}$ imply that

$$2g^+_{\tau,z}(0) = \lim_{x\to 0^+} g^+_{\tau,z}(x) + \lim_{x\to 0^-} g^+_{\tau,z}(x),\ 2g^*_{\tau,z}(0) = \lim_{x\to 0^+} g^+_{\tau,z}(x) + \lim_{x\to 0^-} g^*_{\tau,z}(x).$$

Thus, $\mathcal{D}^+_{\tau,z}(x) - \mathcal{D}^*_{\tau,z}(x)$ is continuous at $x = 0$. Finally when $x \in \mathbb{Z} + \frac{1}{2}$, $\mathcal{D}^*_{\tau,z}(x)$ is continuously differentiable and $\mathcal{D}^+_{\tau,z}(x)$ is one-sided continuous. Since their difference has no jump discontinuity anywhere, we conclude that it must be continuous when $x \in \mathbb{Z} + \frac{1}{2}$. Equation (3.3.15) above then implies that $\mathcal{D}^+_{\tau,z}(x) - \mathcal{D}^*_{\tau,z}(x) = C \cdot \mathbf{e}\left(-\frac{x^2}{2}\tau - xz\right)$ for some constant C. Since $\mathcal{D}^+_{\tau,z}$ and $\mathcal{D}^*_{\tau,z}$ are both bounded by estimate (3.2.7) and Lemma 3.3, we have $C = 0$, which then implies Equation (3.3.14). The last claim follows directly from definition. □

3.4. **A special function in $L^1(\mathbb{R})$.** In this section, we will consider a special function in $L^1(\mathbb{R})$ and use it later as a replacement for the usual Schwartz function to form a real-analytic theta series. It satisfies properties similar to the function W_z in Proposition 1 of Section 2.3 of [16] (there z is in the upper half plane, as our τ here). Using W_z, Hirzebruch and Zagier constructed a non-holomorphic modular form of weight 2, which is essentially the product of the weight 3/2 non-holomorphic Eisenstein series and weight 1/2 theta series. Its holomorphic projection is then a modular form, which is the generating function of intersections of Hirzebruch-Zagier divisors. In our setting, the function g_τ will be used to produce a non-holomorphic theta series of weight 1. In special cases, this is the product of a harmonic Maass form of weight 1/2 and theta series of weight 1/2.

To construct g_τ, we consider the constant term in the Taylor series expansion of $\frac{g_{\tau,z}}{\vartheta(\tau,z)}$ around $z = 0$, which has two pieces

(3.4.1)

$$g^+_\tau(x) := \lim_{z\to 0} \frac{g^+_{\tau,z}(x) - g^+_{\tau,0}(x)}{\vartheta(\tau, z)} = \mathbf{e}\left(-\frac{x^2}{2}\tau\right)\frac{\mathrm{sgn}(x)}{\eta^3(\tau)} \sum_{\substack{m>|x| \\ m\in\mathbb{Z}+\frac{1}{2}}} (m - |x|)\,\mathbf{e}\left(\frac{m^2}{2}\tau + \frac{m}{2} - \frac{1}{4}\right),$$

$$g^*_\tau(x) := \lim_{z\to 0} \frac{g^*_{\tau,z}(x) - g^*_{\tau,0}(x)}{\vartheta(\tau, z)} = \mathbf{e}\left(-\frac{x^2}{2}\tau\right)\frac{\mathrm{sgn}(x)\mathrm{erfc}(\sqrt{2\pi v}|x|)}{2}.$$

Subtracting them gives the special function we are interested in

$$g_\tau(x) := \lim_{z\to 0} \frac{g_{\tau,z}(x) - g_{\tau,0}(x)}{\vartheta(\tau, z)} = g_\tau^+(x) - g_\tau^*(x). \tag{3.4.2}$$

As a consequence of Lemma 3.3 and Proposition 3.6, the function g_τ enjoys the following nice properties.

Proposition 3.7. *The function $g_\tau \in L^1(\mathbb{R})$ is odd, continuous on $\mathbb{R}$ and continuously differentiable on $\mathbb{R}\backslash\mathbb{Z} + \frac{1}{2}$. Furthermore for all $x \in \mathbb{R}$, it satisfies*

(1) $2iv\overline{\frac{\partial}{\partial\overline{\tau}}(g_\tau(x))} = \frac{\sqrt{2v}}{2} x\mathbf{e}\left(\frac{x^2}{2}\tau\right)$,

(2) $g_{\tau+1}(x) = \mathbf{e}\left(-\frac{x^2}{2}\right) g_\tau(x)$,

(3) $\mathcal{F}(g_{-1/\tau})(x) = -i\sqrt{-i\tau} g_\tau(x)$.

Proof. The first two claims are easy consequences of the definition. For the third, if one can show that for any $\tau \in \mathcal{H}$

$$\mathcal{F}\left(\lim_{z\to 0} \frac{g_{\tau,z} - g_{\tau,0}}{z}\right) = \lim_{z\to 0} \mathcal{F}\left(\frac{g_{\tau,z} - g_{\tau,0}}{z}\right).$$

along any path from z to 0, then applying Equation 3.3.14 will finish the proof. First, we can write $g_{\tau,z} - g_{\tau,0} = g^+_{\tau,z} - g^+_{\tau,0} - g^*_\tau \cdot \vartheta(\tau, z)$. Clearly $\mathcal{F}$ and the limit can be interchanged on $g^*_\tau \cdot \frac{\vartheta(\tau,z)}{z}$. So it suffices to show the equation above with $g_{\tau,z} - g_{\tau,0}$ replaced by $g^+_{\tau,z} - g^+_{\tau,0}$, which is implied by $\mathcal{F}\left(\frac{\partial g^+_{\tau,z}}{\partial z}\right) = \frac{\partial \mathcal{F}(g^+_{\tau,z})}{\partial z}$. This holds since the integrals defining $\mathcal{F}(g^+_{\tau,z})$ and $\mathcal{F}\left(\frac{\partial g^+_{\tau,z}}{\partial z}\right)$ converges uniformly by Lemmas 3.3 and 3.5. □

Lemma 3.8. *If $x \in \frac{1}{b}\mathbb{Z}$ for some $b \in 2\mathbb{N}$, then $b \cdot \mathbf{e}\left(\frac{x^2}{2}\tau\right) g^+_\tau(x) \in \mathbb{Z}[\![q]\!]$ with $q = \mathbf{e}(\tau)$ and $g^+_\tau(x)$ is holomorphic in the interior of the upper half plane with* $\mathrm{ord}_\infty(g^+_\tau(x)) \geq \frac{|x|}{b} + \frac{1}{2b^2} - \frac{1}{8}$.

Proof. The first claim is clear from the expression of $g^+_\tau(x)$ in Equation (3.4.1). For the second claim, notice that the inequality $m > |x|$ implies $m \geq |x| + \frac{1}{b}$ when $m \in \mathbb{Z} + \frac{1}{2}$ and $x \in \frac{1}{b}\mathbb{Z}$ with $2 \mid b$. Thus, $m^2 - x^2 \geq \frac{2|x|+1/b}{b}$ and we obtain the bound on $\mathrm{ord}_\infty(g^+_\tau(x))$. The $-\frac{1}{8}$ comes from the $\eta^3(\tau)$ in the denominator, and is responsible for a possible pole at $i\infty$. □

Proposition 3.9. *For any fixed $N \in \mathbb{N}$ and $\tau \in \mathcal{H}$, the infinite sum*

$$\sum_{n\in\frac{1}{N}\mathbb{Z}} g_\tau(n + x_0) \tag{3.4.3}$$

converges absolutely and uniformly in $x_0 \in \mathbb{R}$.

Proof. By writing $n = n' + h$ uniquely with $n' \in \mathbb{Z}$ and $h = 0, 1/N, \ldots, (N-1)/N$, we can divide the sum into N sums and consider the convergence of $\sum_{n\in\mathbb{Z}} g_\tau(n + x_0)$. Since $g_\tau = g_\tau^+ - g_\tau^*$ and the sum involving g_τ^* clearly converges absolutely by estimate (3.2.7), it suffices to prove the absolute and uniform convergence of $\sum_{n\in\mathbb{Z}} g_\tau^+(n + x_0)$. Suppose $x_0 \in [0, 1]$.

For $x \in \mathbb{R}$, define $d(x) := \lceil |x| - \frac{1}{2} \rceil - (|x| - \frac{1}{2})$ to be the distance between $|x|$ and the smallest number in $\mathbb{Z} + \frac{1}{2}$ greater than $|x|$. Then $|g_\tau^+(x)| \ll d(x) \cdot e^{-2\pi v|x|d(x)} + e^{-2\pi v|x|}$ with the implied constant depending on τ only. Notice that $d(x_0) = d(x_0 + n)$ for all $n \in \mathbb{Z}$. Then the following estimate shows that the sum converges absolutely and uniformly

$$\sum_{n\in\mathbb{Z}} |g_\tau^+(n + x_0)| \ll \sum_{n\in\mathbb{Z}} d(x_0)e^{-2\pi v|n+x_0|d(x_0)} + e^{-2\pi v|n+x_0|} \ll \frac{d(x_0)}{1 - e^{-2\pi v d(x_0)}} + \frac{1}{1 - e^{-2\pi v}}.$$

□

3.5. **New Proof of Zwegers' Result.** Now, we will return to the function $g_{\tau,z}$ defined in Equation (3.3.13). As a consequence of Proposition 3.6, averaging $g_{\tau,z}$ over a one-dimensional lattice should yield a non-holomorphic modular object of weight 1. It turns out to be the product of $\vartheta(\tau, z)$ and the completed μ-function studied by Zwegers in his groundbreaking thesis [25], where he constructed this object by averaging a nice, rapidly decaying function over a two-dimensional lattice. In comparison, we will average $g_{\tau,z}$ over a one-dimensional lattice and obtain a different proof of Zwegers' result on the modularity of the completed μ-function.

Because of delicate convergence issues, we begin by considering the following periodic function for $\epsilon, a \in \mathbb{R}$.

$$G_\epsilon(a, z; \tau) := \sum_{m\in\mathbb{Z}+\frac{1}{2}} g_{\tau,z}(m - \epsilon)\mathbf{e}(\tfrac{m}{2})\mathbf{e}(ma), \tag{3.5.1}$$

Similarly, we define G_ϵ^* and G_ϵ^+ by replacing $g_{\tau,z}$ with $g_{\tau,z}^*$ and $g_{\tau,z}^+$ respectively.

Lemma 3.10. *For fixed $\tau \in \mathcal{H}$ and $z, a \in \mathbb{C}$, the series defining $G_\epsilon^*(a, z; \tau)$ converges absolutely and uniformly for ϵ in compact subsets of $\mathbb{R}$. Furthermore,*

$$G_\epsilon^*(a, z; \tau) = \mathbf{e}\left(-\tfrac{\epsilon^2}{2}\tau + \epsilon z\right) G_0^*(a, z - \epsilon\tau; \tau) = \mathbf{e}(\epsilon a)\, G_\epsilon^*(0, z - a; \tau). \tag{3.5.2}$$

Proof. The two claims are direct consequences of estimate (3.2.7) and definition (3.5.1). □

Lemma 3.11. *For fixed $\tau \in \mathcal{H}, z \in \mathbb{C}$, the series defining $G_\epsilon^+(a, z; \tau)$ converges absolutely for any $a \in \mathbb{R}$ and uniformly for ϵ in compact subsets of $\mathbb{R}\backslash\mathbb{Z}$. At $\epsilon = 0$, the series converges absolutely for all $a \in \mathbb{C}$ satisfying $|\mathrm{Im}(a)| < \frac{\mathrm{Im}(\tau)}{2}$. Furthermore, the function $G_\epsilon^+(a, z; \tau)$*

satisfies

$$G^+_\epsilon(a, z; \tau) = \mathbf{e}(\epsilon z - \tfrac{\epsilon^2}{2}\tau)\left(\frac{1}{\mathbf{e}(\frac{\epsilon}{2}\tau + \frac{a}{2}) - \mathbf{e}(-\frac{\epsilon}{2}\tau - \frac{a}{2})} + G^+_0(a + \epsilon\tau, z; \tau)\right). \tag{3.5.3}$$

whenever both sides converges.

Proof. This is the discrete (and easier) analogue of Lemma 3.5. For ϵ in compact subsets of $[-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$, we can find $\epsilon' > 0$ uniformly in ϵ such that $\epsilon' < |m' - |m-\epsilon||$ for all $m, m' \in \mathbb{Z}+\frac{1}{2}$. One can check from definition of $g^+_{\tau,z}$ in Equation (3.3.6) that

$$|g^+_{\tau,z}(m - \epsilon)| \ll_{\tau,z} e^{-2\pi|m|\epsilon' v},\ m \in \mathbb{Z} + \tfrac{1}{2}.$$

The sum over $m \in \mathbb{Z} + \frac{1}{2}$ then converges absolutely. When $\epsilon = 0$, the bound above is even better and becomes $|g^+_{\tau,z}(m)| \ll_{\tau,z} e^{-2\pi|m|v}$ with $m \in \mathbb{Z} + \frac{1}{2}$, which implies the second claim. For any $\epsilon \in [-\frac{1}{2}, \frac{1}{2}]\backslash\{0\}$ and $m \in \mathbb{Z} + \frac{1}{2}$, we have the identity

$$g^+_{\tau,z}(m - \epsilon) = \frac{1 + \mathrm{sgn}(m\epsilon)}{2}\mathbf{e}\left(\epsilon m\tau - \tfrac{\epsilon^2}{2}\tau + \epsilon z + \tfrac{|m|}{2}\right) + \mathbf{e}\left(m\epsilon\tau - \frac{\epsilon^2}{2}\tau + \epsilon z\right) g^+_{\tau,z}(m). \tag{3.5.4}$$

Note that the first term vanishes when $\epsilon = 0$ or $\mathrm{sgn}(\epsilon)m < 0$. Using this, we conclude that

$$G^+_\epsilon(a, z; \tau) = -\mathrm{sgn}(\epsilon) \sum_{\substack{m\in\mathbb{Z}+\frac{1}{2},\\ \mathrm{sgn}(\epsilon)m>0}} \mathbf{e}\left(\epsilon m\tau - \tfrac{\epsilon^2}{2}\tau + \epsilon z + am\right) + \mathbf{e}\left(-\frac{\epsilon^2}{2}\tau + \epsilon z\right) G^+_0(a + \epsilon\tau, z; \tau).$$

The first sum is a geometric series, which can be summed to give equation (3.5.3). ☐

Proposition 3.12. *Fix $\tau \in \mathcal{H}$, $z \in \mathbb{C}$ and $a \in \mathbb{R}\backslash\mathbb{Z}$. Then $\mathbf{e}(-(a+\frac{1}{2})\epsilon)G_\epsilon(a, z; \tau)$ is bounded and 1-periodic as a function of $\epsilon \in \mathbb{R}$. When $\epsilon \in \mathbb{R}\backslash\mathbb{Z}$, G_ϵ is continuous and satisfies*

$$G_\epsilon(a, z; \tau) = \frac{\mathbf{e}(\epsilon a)}{\tau} G_a\left(-\epsilon, -\frac{z}{\tau}; -\frac{1}{\tau}\right). \tag{3.5.5}$$

Otherwise, G_ϵ has removable discontinuity at $\mathbb{Z}$.

Proof. The periodicity is clear from the definition. The boundedness of G^*_ϵ and G^+_ϵ follow from Lemmas 3.10 and 3.11 respectively. Since $G_\epsilon = G^+_\epsilon - G^*_\epsilon$ by Equation (3.3.13), it is also bounded. The (dis)continuity statement follows from that of G^+_ϵ, which is a consequence of Equation (3.5.3).

Now, the transformation formula can be derived from the Poisson summation formula. Since $\mathbf{e}(-(a + \frac{1}{2})\epsilon)G_\epsilon(a, z; \tau)$ is bounded and 1-periodic, it has a Fourier expansion

$$\mathbf{e}(-(a + \tfrac{1}{2})\epsilon)G_\epsilon(a, z; \tau) = \sum_{n\in\mathbb{Z}} b_n(a, z; \tau)\mathbf{e}(n\epsilon), \tag{3.5.6}$$

where $b_n(a,z;\tau) = \int_0^1 \mathbf{e}(-(a+\frac{1}{2})\epsilon)G_\epsilon(a,z;\tau)\mathbf{e}(-n\epsilon)d\epsilon = \mathbf{e}(n/2)\mathcal{F}(g_{\tau,z})(n+a+\frac{1}{2})$. Using Proposition 3.6, we can rewrite $b_n(a,z;\tau) = (-i\tau)^{-1}\mathbf{e}(n/2)g_{-1/\tau,z/\tau}(n+a+\frac{1}{2})$. Substituting this into Equation (3.5.6) and using $G_\epsilon(a,z;\tau) = G_{-\epsilon}(-a,-z;\tau)$ finishes the proof. $\square$

For fixed $\tau \in \mathcal{H}, z \in \mathbb{C}\backslash\mathbb{Z}\tau+\mathbb{Z}$ and $a \in (-\frac{1}{2},\frac{1}{2})\backslash\{0\}$, define the function $G(a,z;\tau)$ by

$$G(a,z;\tau) := \lim_{\epsilon\to 0} G_\epsilon(a,z;\tau). \tag{3.5.7}$$

Define G^+ and G^* similarly by replacing $g_{\tau,z}$ with $g^+_{\tau,z}$ and $g^*_{\tau,z}$ above. The limit exists since G_ϵ has a removable discontinuity at 0. By Lemma 3.11, the function $G(a,z;\tau)$ is defined for all $a \in \mathbb{C}$ such that $a \notin \mathbb{Z}$ and $|\mathrm{Im}(a)| < \frac{\mathrm{Im}(\tau)}{2}$. Then combining Lemma 3.10 and Proposition 3.12 gives us the following modularity theorem.

Theorem 3.13. *The function $G(a,z;\tau)$ satisfies the following transformation property*

$$G(a,z;\tau) = \frac{\alpha(\tau,a-z)}{\alpha(\tau,z)\tau}G\left(\frac{a}{\tau},-\frac{z}{\tau};-\frac{1}{\tau}\right). \tag{3.5.8}$$

Proof. Taking the limit as ϵ goes to 0 in equation (3.5.5) gives us

$$G(a;\tau,z) = \frac{1}{\tau}\left(\lim_{\epsilon\to 0} G^+_a\left(-\epsilon,-\frac{z}{\tau};-\frac{1}{\tau}\right) - G^*_a\left(0,-\frac{z}{\tau};-\frac{1}{\tau}\right)\right).$$

Using Equations (3.5.2) and (3.5.3), we can write

$$G^*_a\left(0,-\frac{z}{\tau};-\frac{1}{\tau}\right) = \mathbf{e}\left(\tfrac{a^2}{2\tau}-\tfrac{az}{\tau}\right)G^*_0\left(-\frac{a}{\tau},\frac{z}{\tau};-\frac{1}{\tau}\right),$$
$$\lim_{\epsilon\to 0} G^+_a\left(-\epsilon,-\frac{z}{\tau};-\frac{1}{\tau}\right) = \mathbf{e}\left(\tfrac{a^2}{2\tau}-\tfrac{az}{\tau}\right)\lim_{\epsilon\to 0} G^+_\epsilon\left(-\frac{a}{\tau},-\frac{z}{\tau};-\frac{1}{\tau}\right).$$

Finally, using the identity $\alpha(\tau,z)\mathbf{e}(\frac{a^2}{2\tau}-\frac{az}{\tau}) = \alpha(\tau,a-z)$, we obtain Equation (3.5.8). $\square$

Recall that the Lerch sum is given by

$$\mu(a,z) = \mu(a,z;\tau) := \frac{\mathbf{e}(\frac{a}{2})}{\vartheta(\tau,z)}\sum_{n\in\mathbb{Z}}\frac{\mathbf{e}(\frac{n^2+n}{2}\tau+nz+\frac{n}{2})}{1-\mathbf{e}(n\tau+a)} \tag{3.5.9}$$

with $\tau \in \mathcal{H}, a,z \in \mathbb{C}\backslash(\mathbb{Z}\tau+\mathbb{Z})$. When $a \in \mathbb{R}\backslash\mathbb{Z}$, we can rewrite the Lerch sum in the following manner

$$\mu(a,z;\tau) = \frac{G^+(a,z;\tau)}{\vartheta(\tau,z)}. \tag{3.5.10}$$

Furthermore, $\frac{G(a,z;\tau)}{\vartheta(\tau,z)} = \hat{\mu}(a,z;\tau)$ is the completion of μ as in Zwegers' thesis [25]. One could derive Theorem 1.11 loc. cit. from Theorem 3.13 and the definition of $G(a,z;\tau)$.

4. Construction of $\tilde{\Theta}(\tau, L)$.

In this section, we will use the special function g_τ to construct a preimage of $\Theta(\tau, L; 1)$ under ξ_τ, which we denote by $\tilde{\Theta}(\tau, L)$. Even though it is not a harmonic Maass form, the function $\tilde{\Theta}(\tau, L)$ still breaks naturally into the sum of a holomorphic part and non-holomorphic part. The holomorphic part is a vector-valued Laurent series in q^{1/d_L}, which will have rational Fourier coefficients with explicitly bounded denominators.

Given $\tau \in \mathcal{H}$, recall that ϕ_τ was defined on $(\mathbb{R}^{1,1}, Q_0)$ in Equation (2.3.6). Now, we define (4.0.11)

$$\tilde{\phi}_\tau(x, y) := 2\mathbf{e}\left(\frac{y^2}{2}\tau\right) g_\tau(x),\ \tilde{\phi}^*_\tau(x, y) := 2\mathbf{e}\left(\frac{y^2}{2}\tau\right) g^*_\tau(x),\ \tilde{\phi}^+_\tau(x, y) := 2\mathbf{e}\left(\frac{y^2}{2}\tau\right) g^+_\tau(x).$$

Since $g_\tau = g^+_\tau - g^*_\tau$ by Equation (3.4.2), we have $\tilde{\phi}_\tau = \tilde{\phi}^+_\tau - \tilde{\phi}^*_\tau$. Recall that $\mathcal{F}_0$ denotes the Fourier transform operation on $\mathbb{R}^{1,1}$ with respect to $Q_0((x, y)) = \frac{x^2-y^2}{2}$. By Proposition 3.7, we know that $\tilde{\phi}_\tau$ satisfies

$$\tilde{\phi}_{\tau+1}(W) = \mathbf{e}\left(-Q_0(W)\right) \tilde{\phi}_\tau(W),\ \tilde{\phi}_{-1/\tau}(W) = \tau\mathcal{F}_0(\tilde{\phi}_\tau)(W),\ \xi_\tau\tilde{\phi}_\tau(W) = \phi_\tau(W) \tag{4.0.12}$$

for all $W \in \mathbb{R}^{1,1}$ and $\tau \in \mathcal{H}$.

Define the $\mathbb{C}[L^*/L]$-valued function $\tilde{\Theta}^*(\tau, L) := \sum_{h\in L^*/L} \tilde{\Theta}^*_h(\tau, L)\mathfrak{e}_h$ by

$$\tilde{\Theta}^*_h(\tau, L) := \sum_{X\in L+h} \tilde{\phi}^*_\tau(B(X, Z^+_1), B(X, Z^-_1)). \tag{4.0.13}$$

This sum converges by the estimate (3.2.7) and is the "antiderivative" of Θ with respect to ξ_τ by Equations (2.3.8) and (4.0.12). If we replace $\tilde{\phi}^*$ with $\tilde{\phi}$ and the sum still converges, then we are done with the construction. Even though this only happens for special lattices, the general case can be obtained by averaging the special cases. So we will study these special cases first, then execute the general construction from the special cases. In this process, the constant κ_L will become a bit bigger.

4.1. **Special Case.**

Suppose that $L = L_{\mathfrak{a},2AN^2}$ for some integral ideal $\mathfrak{a} \subset \mathcal{O}_D$ and positive integer N. Recall that $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ and $\boldsymbol{\lambda} := \frac{1}{AN\sqrt{2}}(\lambda, \lambda')$ for $\lambda \in L^*$. The following proposition shows that the sum we want to consider converges.

Proposition 4.1. *Write $Z^\pm := Z^\pm_1$. Then the following sum*

$$\sum_{\lambda\in L} \varphi\left(B(\boldsymbol{\lambda}, Z^+) + x_0, B(\boldsymbol{\lambda}, Z^-) + y_0\right) = \sum_{\lambda\in 2AN^2\mathfrak{a}} \varphi\left(\frac{\lambda' + \lambda}{2AN} + x_0, \frac{-\lambda' + \lambda}{2AN} + y_0\right) \tag{4.1.1}$$

converges absolutely and uniformly for all $(x_0, y_0) \in \mathbb{R}^{1,1}$ with $\varphi \in \{\tilde{\phi}_\tau, \tilde{\phi}^*_\tau, \tilde{\phi}^+_\tau\}$. When $(x_0, y_0) = \left(\frac{h+h'}{2AN}, \frac{h-h'}{2AN}\right)$ for some $h \in L^*/L$, we denote this sum by $\tilde{\Theta}_h(\tau, L), \tilde{\Theta}^*_h(\tau, L), \tilde{\Theta}^+_h(\tau, L)$

for $\varphi = \tilde{\phi}_\tau, \tilde{\phi}^*_\tau, \tilde{\phi}^+_\tau$ *respectively. Then*

$$\tilde{\Theta}(\tau, L) := \sum_{h \in L^*/L} \tilde{\Theta}_h(\tau, L)\mathfrak{e}_h \in \mathcal{A}_{1,\rho_{-L}}(\Gamma) \tag{4.1.2}$$

is a real-analytic modular form such that $\xi_\tau(\tilde{\Theta}(\tau, L)) = \Theta(\tau, L; 1)$*. Furthermore,*

$$\tilde{\Theta}^+(\tau, L) := \sum_{h \in L^*/L} \tilde{\Theta}^+_h(\tau, L)\mathfrak{e}_h \in \frac{1}{AND}\mathbb{Z}[L^*/L](\!(q^{1/d_L})\!). \tag{4.1.3}$$

Proof. Notice that $L + h \subset L^* \subset \frac{1}{2}\mathbb{Z} + \frac{\sqrt{D}}{2}\mathbb{Z}$. Thus, we have the estimate

$$\sum_{\lambda \in L+h} \left| \tilde{\phi}_\tau \left(\frac{\lambda' + \lambda}{2AN} + x_0, \frac{-\lambda' + \lambda}{2AN} + y_0 \right) \right| < \sum_{a,b \in \mathbb{Z}} \left| \tilde{\phi}_\tau \left(\frac{a}{2AN} + x_0, \frac{b\sqrt{D}}{2AN} + y_0 \right) \right|$$
$$= 2 \left(\sum_{a' \in \frac{1}{2AN}\mathbb{Z}} |g_\tau(a' + x_0)| \right) \left(\sum_{b \in \mathbb{Z}} e^{-\pi v \left(\frac{b\sqrt{D}}{2AN} + y_0 \right)^2} \right).$$

By Proposition 3.9, the sum in (4.1.1) converges absolutely with $\varphi = \tilde{\phi}_\tau$, hence also for $\varphi = \tilde{\phi}^*_\tau$ and $\tilde{\phi}^+_\tau$ since $\tilde{\phi}^*_\tau$ decays like a Gaussian and $\tilde{\phi}^+_\tau = \tilde{\phi}_\tau + \tilde{\phi}^*_\tau$. Now the modularity of $\tilde{\Theta}(\tau, L)$ and $\xi_\tau\tilde{\Theta}(\tau, L) = \Theta(\tau, L; 1)$ follows from Equations (4.0.12), Poisson summation and definitions (2.3.8) and (2.3.9). The last claim is a direct consequence of Lemma 3.8 since $\frac{\lambda+\lambda'}{2AN} \in \frac{1}{2AND}\mathbb{Z}$ for all $\lambda \in L^* = \mathfrak{a}\mathfrak{d}_D^{-1}$. $\square$

4.2. **General Case.** For any even, integral lattice (L, Q) and $N \in \mathbb{N}$, denote the scaled lattice $(NL, \frac{Q}{N})$ by NL. Note that they have the same dual lattice and there is a natural projection

$$L^*/(NL) \to L^*/L.$$

The following simple lemma then relates $\Theta(\tau, L; t)$ and $\Theta(\tau, NL; t)$.

Lemma 4.2. *Fix* $\tau \in \mathcal{H}$ *and* $t \in \mathbb{R}^\times_+$*. For any* $h \in L^*/L$*, we have*

$$\Theta_h(\tau, L; t) = \sum_{\delta \in L^*/NL,\, \delta \equiv h \bmod L} \Theta_\delta(N\tau, NL; t).$$

Equivalently, let $\mathcal{C}_{L,N}$ *be the* $|L^*/L| \times |L^*/NL|$ *matrix defined by*

$$\mathcal{C}_{L,N} := (\mathbb{1}_L(h - \delta))_{h \in L^*/L, \delta \in L^*/NL}, \tag{4.2.1}$$

where $\mathbb{1}_L$ *is the characteristic function of* L*. Then*

$$\Theta(\tau, L; t) = \mathcal{C}_{L,N} \cdot \Theta(N\tau, NL; t). \tag{4.2.2}$$

Remark 4.3. This already appeared in Hecke's work [15, Eq. III in §4].

Let $L = L_{\mathfrak{a},M}$ with $\mathfrak{a} \subset \mathcal{O}_D$ an integral ideal and $M \in \mathbb{N}$ any natural number. Define $N \in \mathbb{N}$ to be the smallest positive integer such that

$$NM = 2A(N')^2 \tag{4.2.3}$$

for some $N' \in \mathbb{N}$. In particular, we can always choose $N = 2AM$ and $N' = M$. Using the function $\tilde{\Theta}(\tau, NL)$ in Proposition 4.1, we can construct $\tilde{\Theta}(\tau, L)$ with the proposition below.

Proposition 4.4. *The function $\mathcal{C}_{L,N} \cdot \tilde{\Theta}(N\tau, NL)$ maps to $\Theta(\tau, L; 1)$ under ξ_τ and is in $\mathcal{A}_{1,\rho_{-L}}(\Gamma_0(N))$.*

Proof. The first claim follows from Equation (4.2.2) and $\xi_\tau \tilde{\Theta}(N\tau, NL) = \Theta(N\tau, NL)$. For the second claim, notice that for $\gamma \in \Gamma_0(N)$, we have

$$\left(\mathcal{C}_{L,N} \cdot \tilde{\Theta}(N\tau, NL)\right) |_{1,\rho_{-L}} \gamma = \rho_{-L}^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N) \cdot \tilde{\Theta}(N\tau, NL), \tag{4.2.4}$$

where $\gamma_N := \left(\begin{smallmatrix} N & \\ & 1 \end{smallmatrix}\right) \cdot \gamma \cdot \left(\begin{smallmatrix} 1/N & \\ & 1 \end{smallmatrix}\right) \in \Gamma$. Thus, it suffices to show the matrix identity

$$\rho_{-L}(\gamma) \cdot \mathcal{C}_{L,N} = \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N) \tag{4.2.5}$$

for all $\gamma \in \Gamma_0(N)$. Denote

$$M_\gamma := \rho_L(\gamma) \cdot \mathcal{C}_{L,N} - \mathcal{C}_{L,N} \cdot \rho_{NL}(\gamma_N).$$

It suffices to prove that $\mathfrak{e}_h$ is in the right kernel of M_γ for all $h \in L^*/NL$. For $t \in \mathbb{R}_+^\times$, consider the theta series

$$\theta(\tau, NL; t) := \sqrt{v} \sum_{h \in L^*/NL} \mathfrak{e}_h \sum_{\lambda \in NL + h} \mathbf{e}\left(\frac{(\lambda t^{-1} + \lambda' t)^2}{2AMN}\tau - \frac{(-\lambda t^{-1} + \lambda' t)^2}{2AMN}\overline{\tau}\right) \in \mathcal{A}_{0,\rho_{NL}}(\Gamma).$$

Then $\mathcal{C}_{L,N} \cdot \theta(\tau, NL; t) = \theta(\tau, L; t)$ and for all $\gamma \in \Gamma_0(N), t \in \mathbb{R}_+^\times$

$$M_\gamma \cdot \theta(N\tau, NL; t) = 0$$

as a $\mathbb{C}[L^*/L]$-valued function on $\mathcal{H}$. This follows from the same calculations that produced Equation (4.2.4). This power series identity necessarily becomes an identity between the Fourier coefficients, which are functions of v. From the asymptotic with respect to v, we can deduce that $\mathfrak{e}_h$ is in the kernel of M_γ for all $h \in L^*/NL$, i.e. M_γ vanishes identically. Applying complex conjugation then gives us Equation (4.2.5). $\square$

Now, we can average $\mathcal{C}_{L,N} \cdot \tilde{\Theta}(N\tau, NL)$ over $\Gamma_0(N)\backslash\Gamma$ to define

$$\tilde{\Theta}(\tau, L) := \frac{1}{[\Gamma : \Gamma_0(N)]} \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma} \left(\mathcal{C}_{L,N} \cdot \tilde{\Theta}(N\tau, NL)\right) |_{1,\rho_{-L}} \gamma. \tag{4.2.6}$$

The main result of this section is as follows.

Theorem 4.5. *Let $L = L_{\mathfrak{a},M}$. The function $\tilde{\Theta}(\tau, L) \in \mathcal{A}_{1,\rho_{-L}}(\Gamma)$ is a real-analytic automorphic form such that $\xi_\tau(\tilde{\Theta}(\tau, L)) = \Theta(\tau, L; 1)$ and $\tilde{\Theta}^+(\tau, L) := \tilde{\Theta}(\tau, L) + \tilde{\Theta}^*(\tau, L)$ is in $\frac{1}{\kappa_L}\mathbb{Z}[L^*/L]((q))$ with*

$$\kappa_L := 2A^3(N')^3 D \cdot \varphi(N), \tag{4.2.7}$$

where $\varphi(N) = N\prod_{p|N \ prime}(1 + \frac{1}{p})$ is the Euler totient function. In particular κ_L can be chosen to divide $2A^3\varphi(2A) \cdot M^3\varphi(M) \cdot D$.

Proof. Since ξ_τ commutes with $|_1$ and conjugates ρ_{-L} to ρ_L, we obtain the first claim from Proposition 4.4.

To prove the second claim, we will first show that $\tilde{\Theta}^+(\tau, L) \in \mathbb{Q}[L^*/L]((q))$, then give a bound of the denominator. For $\gamma = \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \Gamma$, we can write $N_\gamma := \gcd(N, c)$ and

$$\begin{pmatrix} N & \\ & 1 \end{pmatrix} \gamma = \gamma_N \cdot \begin{pmatrix} N_\gamma & b \\ 0 & N/N_\gamma \end{pmatrix}, \ b \in \mathbb{Z}, \ \gamma_N \in \Gamma. \tag{4.2.8}$$

Then for every $\gamma \in \Gamma, \tau \in \mathcal{H}$ and $f \in \mathcal{A}_{1,\rho}(\Gamma)$, we have

$$f(N\tau) |_1 \gamma = \frac{N_\gamma}{N}\rho(\gamma_N) \cdot f(\tau_\gamma), \ \tau_\gamma := (\gamma_N^{-1} \left(\begin{smallmatrix} N & \\ & 1 \end{smallmatrix}\right) \gamma) \cdot \tau = \frac{N_\gamma \tau + b}{N/N_\gamma} \in \mathcal{H}. \tag{4.2.9}$$

Now applying this to $f(\tau) = \tilde{\Theta}(\tau, NL) \in \mathcal{A}_{1,\rho_{-NL}}(\Gamma)$ gives us

$$\tilde{\Theta}(N\tau, NL) |_1 \gamma = \frac{N_\gamma}{N}\rho_{-NL}(\gamma_N) \cdot \tilde{\Theta}^+(\tau_\gamma, NL).$$

Substituting this into the definition of $\tilde{\Theta}(\tau, L)$ gives us

$$\tilde{\Theta}(\tau, L) = \frac{1}{[\Gamma : \Gamma_0(N)]} \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma} \frac{N_\gamma}{N}\rho_{-L}^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N)\tilde{\Theta}(\tau_\gamma, NL). \tag{4.2.10}$$

Notice that we have the following (rather cute) linear algebra identity relating the non-holomorphic parts

$$\tilde{\Theta}^*(\tau, L) = \frac{N_\gamma}{N}\rho_{-L}^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N)\tilde{\Theta}^*(\tau_\gamma, NL), \ \gamma \in \Gamma. \tag{4.2.11}$$

To prove this identity, we first apply Equation (4.2.9) to $f(\tau) = \Theta(\tau, NL; 1) \in \mathcal{A}_{1,\rho_{NL}}(\Gamma)$ to obtain

$$\Theta(\tau, L; 1) = \frac{N_\gamma}{N}\rho_L^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{NL}(\gamma_N)\Theta(\tau_\gamma, NL; 1)$$

for all $\gamma \in \Gamma$. Since $\xi_\tau(\tilde{\Theta}^*(\tau_\gamma, NL)) = \Theta(\tau_\gamma, NL; 1)$, we see that the difference between the two sides of Equation (4.2.11) vanishes under ξ_τ, implying that it is holomorphic. Furthermore, this difference is stable under $\tau \mapsto \tau + N$ and vanishing as $v \to \infty$, so has a Fourier expansion $\sum_{n\geq 1} a_n \mathbf{e}(n\tau/N)$. However, integrating this difference against $e(-nu/N)$ over $u \in [0, N]$ equals to zero for all $n \geq 1$, which means $a_n = 0$ for all $n \geq 1$. Thus the difference vanishes identically. Now substituting $\tilde{\Theta}(\tau_\gamma, NL) = \tilde{\Theta}^+(\tau_\gamma, NL) - \tilde{\Theta}^*(\tau_\gamma, NL)$ and identity (4.2.11) into Equation (4.2.10), we can write

$$\tilde{\Theta}^+(\tau, L) = \frac{1}{[\Gamma : \Gamma_0(N)]} \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma} \frac{N_\gamma}{N} \rho_{-L}^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N) \cdot \tilde{\Theta}^+(\tau_\gamma, NL). \tag{4.2.12}$$

The Weil representations ρ_{-L} and ρ_{-NL} are defined over $\mathbb{Q}(\zeta_{N^\dagger})/\mathbb{Q}$ with $N^\dagger = DAMN$ and $\zeta_{N^\dagger}$ a primitive $(N^\dagger)^{\text{th}}$ root of unity. For $a \in (\mathbb{Z}/N^\dagger\mathbb{Z})^\times$, recall that $J_a = \left(\begin{smallmatrix} 1 & \\ & a \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/N^\dagger\mathbb{Z})$ and $\sigma_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_{N^\dagger})/\mathbb{Q})$ be the corresponding element as in Section 2.2. Let $\gamma' \in \Gamma$ be any element such that its image in $\Gamma(N^\dagger)\backslash\Gamma \cong \mathrm{SL}_2(\mathbb{Z}/N^\dagger\mathbb{Z})$ is $J_a^{-1}\gamma J_a$. Then $N_\gamma = N_{\gamma'}$ and we can write

$$\begin{pmatrix} N & \\ & 1 \end{pmatrix} \gamma' = \gamma'_N \cdot \begin{pmatrix} N_\gamma & ab \\ 0 & N/N_\gamma \end{pmatrix}$$

with the image of $\gamma'_N \in \Gamma$ in $\mathrm{SL}_2(\mathbb{Z}/N^\dagger\mathbb{Z})$ being $J_a^{-1}\gamma_N J_a$.

Since $\tilde{\Theta}^+(\tau, NL) \in \mathbb{Q}[L^*/NL]((q^{1/d_{NL}}))$ and $\tau_\gamma = \frac{N_\gamma\tau + b}{N/N_\gamma}$ with $b \in \mathbb{Z}$, we have

$$\sigma_a \tilde{\Theta}^+(\tau_\gamma, NL) = \tilde{\Theta}^+\left(\frac{N_\gamma\tau + ab}{N/N_\gamma}, NL\right) = \tilde{\Theta}^+(\tau_{\gamma'}, NL), \ \tau_{\gamma'} := \left((\gamma'_N)^{-1}\begin{pmatrix} N & \\ & 1 \end{pmatrix}\gamma'\right) \cdot \tau.$$

Since $d_{-L} \mid d_{-NL} \mid N^\dagger$, the representations ρ_{-L} and ρ_{-NL} are trivial on $\Gamma(N^\dagger)$. By Proposition 2.1, we have

$$\sigma_a\left(N_\gamma \rho_{-L}^{-1}(\gamma) \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma_N) \cdot \tilde{\Theta}^+(\tau_\gamma, NL)\right) = N_{\gamma'}\rho_{-L}^{-1}(\gamma') \cdot \mathcal{C}_{L,N} \cdot \rho_{-NL}(\gamma'_N) \cdot \tilde{\Theta}^+(\tau_{\gamma'}, NL).$$

Thus, σ_a permutes the summands on the right hand side of Equation (4.2.12), which means that $\tilde{\Theta}^+(\tau, L)$ has Fourier coefficients in $\mathbb{Q}$.

From the explicit formula of Weil representation in [21, Theorem 4.7], we know that the denominator of every entry in ρ_{-L}, resp. ρ_{-NL}, is bounded by $\sqrt{AM}$, resp. $N\sqrt{AM}$. Proposition 4.1 and the choice of N in Equation (4.2.3) tells us that the denominator of $\tilde{\Theta}^+(\tau', NL)$ is bounded by $AN'D$. Thus, the denominator of $\tilde{\Theta}^+(\tau, L)$ is bounded by κ_L. □

5. Construction and Fourier Expansion of $\tilde{\vartheta}(\tau, L)$.

In this section, we will construct $\tilde{\vartheta}(\tau, L)$ with the desired property in Theorem 1.1. The starting point of the construction is the perturbed integral $I(\tau, -L, s)$ defined in Equation (1.0.8). Since $[1, \varepsilon_L]$ is compact, $I(\tau, -L, s)$ is holomorphic for $s \in \mathbb{C}$ and has the following Taylor series expansion at $s = 0$

$$I(\tau, -L, s) = \vartheta(\tau, -L) + I'(\tau, -L)s + O(s^2). \tag{5.0.13}$$

By Proposition 2.2, applying ξ_τ to $I(\tau, L, s)$ gives us

$$\xi_\tau I(\tau, -L, s) = -\frac{1}{2}\int_1^{\varepsilon_L} t^s \xi_t \Theta(\tau, L; t)\frac{dt}{t} = \frac{1 - \varepsilon_L^s}{2}\Theta(\tau, L; 1) + \frac{s}{2}I(\tau, L, s)$$

after using integration by parts, which implies

$$2\xi_\tau(I'(\tau,-L)) + \log \varepsilon_L \cdot \Theta(\tau, L; 1) = \vartheta(\tau, L). \tag{5.0.14}$$

Recall that $\tilde{\Theta}(\tau, L) \in \mathcal{A}_{1,\rho_{-L}}(\Gamma)$ is the preimage of $\Theta(\tau, L; 1)$ under ξ_τ as in Theorem 4.5. Then $2I'(\tau, -L) + \log \varepsilon_L \cdot \tilde{\Theta}(\tau, L) \in \mathcal{A}_{1,\rho_{-L}}(\Gamma)$ is a preimage of $\vartheta(\tau, L)$ under ξ_τ. In the next section, we will calculate its Fourier expansion and prove the main theorem in the introduction.

5.1. **Fourier expansion of $I'(\tau, -L)$.** Now, we will calculate the Fourier expansion of

$$I'_h(\tau, -L) := \int_1^{\varepsilon_L} \log t \cdot \Theta_h(\tau, -L; t) \frac{dt}{t} \tag{5.1.1}$$

for each $h \in L^*/L$. To state the main result, we will first setup a few notations and make some choices. For simplicity, we write $\varepsilon = \varepsilon_L$. For $\lambda \neq 0$, let

$$r(\lambda) := \left| \frac{\lambda}{\lambda'} \right|. \tag{5.1.2}$$

For each orbit $\Lambda \in \Gamma_L \backslash L + h$ with $Q(\Lambda) \neq 0$, we fix a representative $\lambda_0 \in \Lambda$ such that

$$1 \leq r(\lambda_0) < \varepsilon^2. \tag{5.1.3}$$

After such a representative has been fixed, we will set $\operatorname{sgn}(\Lambda) := \operatorname{sgn}(\lambda_0)$,

$$r(\Lambda) := \begin{cases} r(\lambda_0), & r(\lambda_0) \neq 1, \\ -\frac{\log \varepsilon}{2}, & r(\lambda_0) = 1, \end{cases} \tag{5.1.4}$$

and use the convenient notation

$$\lambda_n := \lambda_0 \varepsilon^n. \tag{5.1.5}$$

We can now state the main result of this section.

Proposition 5.1. *The function $I'_h(\tau, -L)$ has the Fourier expansion*

$$2I'_h(\tau, -L) = \sum_{\substack{\Lambda \in \Gamma_L \backslash L+h \\ Q(\Lambda) < 0}} \operatorname{sgn}(\Lambda) \log r(\Lambda) \mathbf{e}(-Q(\Lambda)\tau) - \tilde{\vartheta}^*_h(\tau, L) - \tilde{\Theta}^*_h(\tau, L), \tag{5.1.6}$$

where $\tilde{\vartheta}^*_h(\tau, L)$ and $\tilde{\Theta}^*_h(\tau, L)$ are given by

$$\tilde{\vartheta}^*_h(\tau, L) = \sum_{\substack{\lambda \in \Gamma_L \backslash L+h \\ Q(\lambda) > 0}} \operatorname{sgn}(\lambda) \Gamma(0, 4\pi Q(\lambda) v) q^{-Q(\lambda)}, \ \tilde{\Theta}^*_h(\tau, L) = \sum_{\lambda \in L+h} \tilde{\phi}^*_\tau \left(\frac{\lambda + \lambda'}{\sqrt{2AM}}, \frac{\lambda - \lambda'}{\sqrt{2AM}} \right).$$

Proof. Substituting in Equation (2.3.11) gives us

$$I_h'(\tau,-L) = \sqrt{\frac{v}{AM}} \sum_{\substack{\Lambda\in\Gamma_L\backslash L+h \\ Q(\Lambda)\neq 0}} \mathbf{e}\left(-Q(\Lambda)u\right) \sum_{\lambda\in\Lambda} \int_1^{\varepsilon} \left(\lambda t^{-1} - \lambda' t\right) \mathbf{e}\left(\frac{((\lambda t^{-1})^2 + (\lambda' t)^2)\, iv}{2AM}\right) \log t \frac{dt}{t}$$

Since $\Lambda = \{\lambda_0\varepsilon^n : n\in\mathbb{Z}\}$, each sum over Λ becomes

$$\sum_{n\in\mathbb{Z}} \int_1^{\varepsilon} \left(\lambda_n t^{-1} - \lambda_n' t\right) \mathbf{e}\left(\frac{((\lambda_n t^{-1})^2 + (\lambda_n' t)^2)\, iv}{2AM}\right) \log t \frac{dt}{t}.$$

In each summand, let $\nu := -\frac{\log r(\lambda_n)}{2} + \log t$. Then, we can write $\log t = \nu + \frac{\log r(\lambda_0)}{2} + \frac{n\log\varepsilon}{2}$ and break $I_h'(\tau,-L)$ into three pieces

$$2I_h'(\tau,-L) = \sum_{\substack{\Lambda\in\Gamma_L\backslash L+h \\ Q(\Lambda)\neq 0}} \mathrm{sgn}(\Lambda)\mathbf{e}\left(-Q(\Lambda)\tau\right)\left(J_1(\Lambda) + J_2(\Lambda) + J_3(\Lambda)\right) \tag{5.1.7}$$

where $J_1(\Lambda), J_2(\Lambda)$ and $J_3(\Lambda)$ are defined by

$$J_1(\Lambda) := \log r(\Lambda) \int_{-\infty}^{\infty} \sqrt{|Q(\Lambda)|v}\left(e^{-\nu} - \mathrm{sgn}(Q(\Lambda))e^{\nu}\right) \mathbf{e}\left(\frac{|Q(\Lambda)|\left(e^{\nu} + \mathrm{sgn}(Q(\Lambda))e^{-\nu}\right)^2 iv}{2}\right) d\nu,$$

$$J_2(\Lambda) := 2\int_{-\infty}^{\infty} \sqrt{|Q(\Lambda)v|}\left(e^{-\nu} - \mathrm{sgn}(Q(\Lambda))e^{\nu}\right) \mathbf{e}\left(\frac{|Q(\Lambda)|\left(e^{\nu} + \mathrm{sgn}(Q(\Lambda))e^{-\nu}\right)^2 iv}{2}\right) \nu d\nu,$$

$$J_3(\Lambda) := \frac{\mathrm{sgn}(\Lambda)\log\varepsilon\sqrt{v}}{\sqrt{AM}} \sum_{n\in\mathbb{Z}} n \int_1^{\varepsilon} \left(\lambda_n t^{-1} - \lambda_n' t\right) \mathbf{e}\left(\frac{(\lambda_n t^{-1} + \lambda_n' t)^2 iv}{2AM}\right) \frac{dt}{t}.$$

Like Equation (2.4.2), we can evaluate the terms $J_1(\Lambda)$ and $J_2(\Lambda)$ as

	$\mathrm{sgn}(-Q(\Lambda)) > 0$	$\mathrm{sgn}(-Q(\Lambda)) < 0$
$J_1(\Lambda)$	$\mathrm{sgn}(\Lambda)\log r(\Lambda)\mathbf{e}(-Q(\Lambda)iv)$	0
$J_2(\Lambda)$	0	$-\mathrm{sgn}(\Lambda)\mathbf{e}(2Q(\Lambda)iv)\Gamma(0, 4\pi Q(\Lambda)v)$

Using the identity $\xi_t \mathrm{erf}(at + bt^{-1}) = \frac{2}{\sqrt{\pi}}(at - bt^{-1})e^{-(at+bt^{-1})^2}$, we obtain

$$2\sqrt{\frac{v}{AM}} \int_1^{\varepsilon} \left(\lambda_n' t - \lambda_n t^{-1}\right) \mathbf{e}\left(\frac{(\lambda_n t^{-1} + \lambda_n' t)^2 iv}{2AM}\right) \frac{dt}{t} = \mathrm{erf}\left(\sqrt{\frac{\pi v}{AM}}(\lambda_n' t + \lambda_n t^{-1})\right) \Big|_1^{\varepsilon}.$$

Applying this and the identity $\mathrm{erf}(x) = \mathrm{sgn}(x) - \mathrm{sgn}(x)\mathrm{erfc}(|x|)$ to $J_3(\Lambda)$ gives us

$$\begin{aligned}\mathrm{sgn}(\Lambda)J_3(\Lambda) &= \frac{\log\varepsilon}{2}\sum_{n\in\mathbb{Z}} n\left(\mathrm{erf}\left(\sqrt{\frac{\pi v}{AM}}(\lambda'_{n-1}+\lambda_{n-1})\right) - \mathrm{erf}\left(\sqrt{\frac{\pi v}{AM}}(\lambda'_n+\lambda_n)\right)\right)\\ &= \frac{\log\varepsilon}{2}\left(\mathrm{sgn}(\lambda_0+\lambda'_0) - \mathrm{sgn}(\lambda_1+\lambda'_1)\right) - \frac{\log\varepsilon}{2}\sum_{n\in\mathbb{Z}}\mathrm{sgn}(\lambda_n+\lambda'_n)\mathrm{erfc}\left(\sqrt{\frac{\pi v}{AM}}|\lambda'_n+\lambda_n|\right).\end{aligned}$$

The first term is $-\frac{\mathrm{sgn}(\Lambda)\log\varepsilon}{2}$ if $\lambda_0 = -\lambda'_0$ and zero otherwise. After substituting these into Equation (5.1.7), we obtain Equation (5.1.6). $\square$

5.2. **Proof of Main Theorem.** Now we will prove Theorem 1.1 in the introduction. Suppose $(L, Q) = (L_{\mathfrak{a},M}, Q_{\mathfrak{a},M})$ for some $\mathfrak{a} \subset \mathcal{O}_D \subset \mathcal{O}_F$ and $M \in \mathbb{N}$. Define

$$\theta(\tau, L) := 2I'(\tau, -L) + \log\varepsilon_L\tilde{\Theta}(\tau, L). \tag{5.2.1}$$

By Theorem 4.5 and Proposition 5.1, $\theta(\tau, L) \in H_{1,\rho_{-L}}(\Gamma)$ and the holomorphic part $\theta^+(\tau, L) := \theta(\tau, L) + \tilde{\vartheta}^*(\tau, L)$ is given by

$$\theta^+(\tau, L) = \sum_{h\in L^*/L}\mathfrak{e}_h \sum_{\Lambda\in\Gamma_L\backslash L+h,\ Q(\Lambda)<0}\mathrm{sgn}(\Lambda)\log r(\Lambda)q^{-Q(\Lambda)} + \log\varepsilon_L\tilde{\Theta}^+(\tau, L).$$

Theorem 4.5 implies that $\theta(\tau, L)$ satisfies the Theorem 1.1 with $\kappa = \kappa_L$.

Now, we can reduce κ as follows. Let $\mathfrak{b}$ be another ideal of $\mathcal{O}_D$ such that $\mathfrak{a} = \mathfrak{b}\cdot(\mu\mathcal{O}_D)$ with a totally positive element $\mu \in F$ satisfying $\mu - 1 \in M\mathfrak{d}_D$. This is an equivalence relation, under which there are only finitely many equivalence classes of $\mathcal{O}_D$-ideals. Let $A_1 := [\mathcal{O}_D : \mathfrak{b}]$ and $N_1 \in \mathbb{N}$ such that $N_1M = 2A(N'_1)^2$ for some $N'_1 \in \mathbb{N}$. Then $\vartheta(\tau, L) = \vartheta(\tau, L_{\mathfrak{b}})$ and $\theta_1(\tau, L) := \theta(\tau, L_{\mathfrak{b}}) + \log\left|\frac{\mu}{\mu'}\right|\vartheta(\tau, -L)$ also satisfies Theorem 1.1 with $\kappa = \kappa_{L_{\mathfrak{b}}}$. Suppose $\kappa' = \gcd(\kappa_L, \kappa_{L_{\mathfrak{b}}}) = c_0\kappa_L + c_1\kappa_{L_{\mathfrak{b}}}$ with $c_0, c_1 \in \mathbb{Z}$, then $\frac{c_0\kappa_L\cdot\theta(\tau,L)+c_1\kappa_{L_{\mathfrak{b}}}\cdot\theta_1(\tau,L)}{\kappa_2}$ will satisfy the statement in Theorem 1.1 with $\kappa = \kappa'$. Since there are only finitely many equivalence classes, we can repeated this process finitely many times to find the minimal κ, which only depends on the data $\mathcal{O}_D$ and M. This proves the last part of Theorem 1.1.

In particular when $\gcd(A, MD) = 1$, the $\mathcal{O}_D$-ideal $\mathfrak{a}$ is proper and we can choose a totally positive $\mu \in F$ such that $\mu - 1 \in M\mathfrak{d}_D$ and $\mu\mathcal{O}_D = \mathfrak{a}\mathfrak{b}^{-1}$ for some integral, proper $\mathcal{O}_D$-ideal $\mathfrak{b}$ relatively prime to $\mathfrak{a}$. Then it is possible to reduce κ to at least $12M^3\varphi(M)D$.

6. Scalar-Valued Result and Numerical Examples.

In this section, we will use the result in [24] to produce a scalar-valued version of $\vartheta(\tau, L)$, and prove the scalar-valued version of Theorem 1.1

6.1. **Reducing the level.** We first need to reduce the level of certain vector-valued automorphic forms. Fix a fundamental discriminant $D > 1$, an integral ideal $\mathfrak{m} \subset \mathcal{O}_F \subset F := \mathbb{Q}(\sqrt{D}) \subset \mathbb{R}$ and denote $M = \mathrm{Nm}(\mathfrak{m})$, $N = MD$ and $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ the quadratic Dirichlet character. Let $\mathfrak{a} \subset \mathcal{O}_F$ be an arbitrary integral ideal relatively prime to $\mathfrak{m}$ with $A := \mathrm{Nm}(\mathfrak{a})$, and denote $L = L_{\mathfrak{a}\mathfrak{d},M}$. Then $L^* = \mathfrak{a}$ and there is a canonical surjection map of finite abelian groups

$$L^*/L = \mathfrak{a}/M\mathfrak{d}\mathfrak{a} \to \mathfrak{a}/\mathfrak{m}\mathfrak{a} \overset{\cong}{\hookrightarrow} \mathcal{O}_F/\mathfrak{m}. \tag{6.1.1}$$

This induces a natural, linear map from $\mathbb{C}[L^*/L]$ to $\mathbb{C}[\mathfrak{a}/\mathfrak{m}\mathfrak{a}]$. Let $\{\mathfrak{e}_\sigma : \sigma \in \mathfrak{a}/\mathfrak{m}\mathfrak{a}\}$ be the canonical basis of $\mathbb{C}[\mathfrak{a}/\mathfrak{m}\mathfrak{a}]$. Under this and the canonical basis of $\mathbb{C}[L^*/L]$, the linear map is given by the matrix

$$\mathcal{C}_{\mathfrak{a},\mathfrak{m}} := (\mathbb{1}_{\mathfrak{m}\mathfrak{a}}(h - \sigma))_{h \in L^*/L, \sigma \in \mathfrak{a}/\mathfrak{m}\mathfrak{a}}, \tag{6.1.2}$$

where $\mathbb{1}_{\mathfrak{m}\mathfrak{a}}$ is the characteristic function of $\mathfrak{m}\mathfrak{a} \subset \mathfrak{a}$.

Define a representation $\rho_{\mathfrak{m}}$ of $\Gamma_0(N)$ on $\mathbb{C}[\mathcal{O}_F/\mathfrak{m}]$ by

$$\rho_{\mathfrak{m}}(\gamma)\mathfrak{e}_\sigma = \chi_D(d)\mathfrak{e}_{d\sigma}, \gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(N). \tag{6.1.3}$$

Since $\mathfrak{a}$ and $\mathfrak{m}$ are relatively prime to each other, there is a canonical isomorphism between $\mathbb{C}[\mathfrak{a}/\mathfrak{m}\mathfrak{a}]$ and $\mathbb{C}[\mathcal{O}_F/\mathfrak{m}]$ induced by $\mathfrak{a}/\mathfrak{m}\mathfrak{a} \overset{\cong}{\hookrightarrow} \mathcal{O}_F/\mathfrak{m}$. Conjugating $\rho_{\mathfrak{m}}$ by this isomorphism gives rise to a representation of $\Gamma_0(N)$ on $\mathbb{C}[\mathfrak{a}/\mathfrak{m}\mathfrak{a}]$, which we also denote by $\rho_{\mathfrak{m}}$. We can now use Theorem 3 in [24] to study precisely the effect of $\mathcal{C}_{\mathfrak{a},\mathfrak{m}}$ on the level of the vector-valued automorphic forms such as $\Theta(N\tau, L; t)$ in Equation (2.3.11).

Proposition 6.1. *In the notations above, we have $\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \Theta(N\tau, \pm L; t) \in \mathcal{A}_{1,\rho_{\mathfrak{m}}}(\Gamma_0(N))$ for any $t \in \mathbb{R}_+^\times$.*

Proof. For each $\sigma \in \mathfrak{a}/\mathfrak{m}\mathfrak{a}$, we denote the $\mathfrak{e}_\sigma$-component of $\mathcal{C}_{\mathfrak{a},\mathfrak{m}}\Theta(\tau, L; t)$ by $\Theta_\sigma(\tau, \mathfrak{a}, \mathfrak{m}; t)$. From the definition, it is easy to check that

$$\Theta_\sigma(AN\tau, \mathfrak{a}, \mathfrak{m}; t) = \sqrt{v} \sum_{\lambda \in \mathfrak{a}\mathfrak{m} + \sigma} (\lambda' t + \lambda t^{-1}) \mathbf{e}\left(\mathrm{Nm}(\lambda)u + \frac{1}{2}((\lambda t^{-1})^2 + (\lambda' t)^2) iv\right). \tag{6.1.4}$$

Theorem 3 in [24] with the choice of integral ideal $I = \mathfrak{a}\mathfrak{m}$ and $\sigma \in \mathfrak{a}$ then implies $\mathcal{C}_{\mathfrak{a},\mathfrak{m}}\Theta(AN\tau, L; t)) \in \mathcal{A}_{1,\rho_{\mathfrak{m}}}(\Gamma_0(AN))$. It follows readily from Equation (6.1.4) that $\mathcal{C}_{\mathfrak{a},\mathfrak{m}}\Theta(N(\tau+1), L; t) = \mathcal{C}_{\mathfrak{a},\mathfrak{m}}\Theta(N\tau, L; t)$. Since $\Gamma_0(N)$ is generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ satisfying $A \mid b$, we obtain the desired result. The same argument works for $-L$. $\square$

Remark 6.2. Integrating over t, it is easy to verify from Proposition 6.1 and Equation (2.4.1) that

$$\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \vartheta(N\tau, \pm L) = \left(\sum_{\lambda \in \Gamma_L \backslash (\mathfrak{a}\mathfrak{m}+\sigma),\ \pm \mathrm{Nm}(\lambda) > 0} \mathrm{sgn}(\lambda) q^{|\mathrm{Nm}(\lambda)|/A} \right)_{\sigma \in \mathfrak{a}/\mathfrak{m}\mathfrak{a}} \in S_{1,\rho_{\mathfrak{m}}}(\Gamma_0(N)). \tag{6.1.5}$$

Recall from Section 2.3 that $\Theta(\tau, \pm L; t) \in \mathcal{A}_{1,\rho_{\pm L}}(\mathrm{SL}_2(\mathbb{Z}))$ for all $t \in \mathbb{R}_+^\times$. It turns out that one can bootstrap a more general result out of the proposition above.

Proposition 6.3. *In the notations above, $\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot f(N\tau)$ is in $\mathcal{A}_{1,\rho_\mathfrak{m}}(\Gamma_0(N))$ for all $f \in \mathcal{A}_{1,\rho_{\pm L}}(\mathrm{SL}_2(\mathbb{Z}))$.*

Proof. The proof is analogous to that of Proposition 4.4. For $\gamma = \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right)$, recall that $\gamma_N = \left(\begin{smallmatrix} a & Nb \\ c & d \end{smallmatrix}\right)$ as in Proposition 4.4. For an arbitrary $f \in \mathcal{A}_{1,\rho_L}(\mathrm{SL}_2(\mathbb{Z}))$, we want to show that

$$(\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot f(N\tau)) \mid_1 \gamma = \rho_\mathfrak{m}(\gamma) \cdot \mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot f(N\tau).$$

Since $f(N\tau) \mid_1 \gamma = \rho_L(\gamma_N) \cdot f(N\tau)$, it suffices to prove the identity $M_\gamma \cdot f(N\tau) = 0$, where

$$M_\gamma := \mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \rho_L(\gamma_N) - \rho_\mathfrak{m}(\gamma) \cdot \mathcal{C}_{\mathfrak{a},\mathfrak{m}}.$$

By Proposition 6.1, the equality above holds with $f(\tau) = \Theta(\tau, L; t)$ for any $t \in \mathbb{R}_+^\times$. All Fourier coefficients of the corresponding identity do vanish and therefore, arguing as in Proposition 4.4, we know that the right kernel of M_γ contains $\{\mathfrak{e}_h - \mathfrak{e}_{-h} : h \in L^*/L\}$. From Equation (2.2.1), we have $\rho_L(S^2)\mathfrak{e}_h = -\mathfrak{e}_{-h}$ for all $h \in L^*/L$. Then any $f = \sum_{h \in L^*/L} f_h \mathfrak{e}_h \in \mathcal{A}_{1,\rho_L}(\mathrm{SL}_2(\mathbb{Z}))$ satisfies $f_h = -f_{-h}$. Thus, $2f(N\tau) = \sum_{h \in L^*/L} f_h(N\tau)(\mathfrak{e}_h - \mathfrak{e}_{-h})$ is in the right kernel of M_γ. The same argument works for $-L$. $\square$

6.2. **Ray class group character.** Let $\mathbf{m} = \mathfrak{m} \cdot \infty_1$ be a modulus with $M = \mathrm{Nm}(\mathfrak{m})$, $I_\mathbf{m}$ be the group of fractional ideals of $\mathcal{O}_F$ relatively prime to $\mathfrak{m}$ and $P_\mathbf{m}$ be the group of principal fractional ideals generated by elements $\mu \in F$ such that $\mu \equiv^\times 1 \bmod \mathfrak{m}$ and $\mu > 0$. Consider a ray class group character of conductor $\mathbf{m}$

$$\varphi : \mathrm{Cl}_\mathbf{m} := I_\mathbf{m}/P_\mathbf{m} \to \mathbb{C}^\times. \tag{6.2.1}$$

The class group Cl_F of F is a natural quotient of $\mathrm{Cl}_\mathbf{m}$. The presence of ∞_1 in $\mathbf{m}$ is equivalent to

$$\varphi((\mu)) = \varphi_\mathrm{f}(\mu) \cdot \mathrm{sgn}(\mu). \tag{6.2.2}$$

with φ_f a character on $(\mathcal{O}_F/\mathfrak{m})^\times$ satisfying $\varphi_\mathrm{f}(\varepsilon) = \mathrm{sgn}(\varepsilon)$ for all $\varepsilon \in \mathcal{O}_F^\times$. After extending by zero, we can view φ_f, resp. φ, as a map on F, resp. fractional ideals of $\mathcal{O}_F$. By a slight abuse of notation, we write $\varphi(\lambda) := \varphi((\lambda))$.

Let $N = DM$ and χ be the product of χ_D and the restriction of φ to $\mathbb{Q}$. Associated to φ is a Hecke eigenform $f_\varphi \in S_{1,\chi}(\Gamma_0(N))$ given by

$$f_\varphi(\tau) := \sum_{\mathfrak{b} \subset \mathcal{O}_F} \varphi(\mathfrak{b}) q^{\mathrm{Nm}(\mathfrak{b})} = \sum_{[\mathfrak{a}] \in \mathrm{Cl}_F} \varphi(\mathfrak{a}^{-1}) f_{\varphi,\mathfrak{a}}(\tau), \ f_{\varphi,\mathfrak{a}}(\tau) := \sum_{(\lambda) \subset \mathfrak{a}} \varphi(\lambda) q^{\mathrm{Nm}((\lambda)\mathfrak{a}^{-1})}. \tag{6.2.3}$$

Even though $\varphi(\mathfrak{a})$ depends on the representative of $[\mathfrak{a}] \in \mathrm{Cl}_F$, the product $\overline{\varphi}(\mathfrak{a}) f_{\varphi,\mathfrak{a}}$ is independent of such choice.

For fixed integral ideal $\mathfrak{a} \in I_{\mathbf{m}}$, let $A = \mathrm{Nm}(\mathfrak{a})$ and $L = L_{\mathfrak{a}\mathfrak{d},M}$. Then we can write $f_{\varphi,\mathfrak{a}}(\tau) = f_{\varphi,\mathfrak{a},+}(\tau) + f_{\varphi,\mathfrak{a},-}(\tau)$, where

$$f_{\varphi,\mathfrak{a},\pm}(\tau) := \frac{1}{[\mathcal{O}_F^\times : \Gamma_L]} \sum_{\lambda \in \Gamma_L \backslash \mathfrak{a},\ \pm \mathrm{Nm}(\lambda) > 0} \varphi_{\mathrm{f}}(\lambda) \mathrm{sgn}(\lambda) q^{|\mathrm{Nm}(\lambda)|/A}. \tag{6.2.4}$$

Now, we can express $f_{\varphi,\mathfrak{a}}$ as a linear combination of $\vartheta_{\sigma,\pm}$ as follows.

Proposition 6.4. *Let $\mathcal{C}_\varphi := (\varphi_{\mathrm{f}}(\sigma))_{\sigma \in \mathcal{O}_F/\mathfrak{m}}$ be a row vector. Then left multiplication by $\mathcal{C}_\varphi$ is a linear map from $\mathcal{A}_{1,\rho_{\mathfrak{m}}}(\Gamma_0(N))$ to $\mathcal{A}_{1,\chi}(\Gamma_0(N))$. In particular,*

$$f_{\varphi,\mathfrak{a},\pm}(\tau) = \mathcal{C}_\varphi \cdot \frac{\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \vartheta(N\tau, \pm L)}{[\mathcal{O}_F^\times : \Gamma_L]} \in S_{1,\chi}(\Gamma_0(N)). \tag{6.2.5}$$

Proof. This is a consequence of Remark 6.2 above and Section 5 of [24]. □

The scalar-valued version of Theorem 1.1 is as follows.

Theorem 6.5. *For each class in Cl_F, fix a representative $\mathfrak{a} \in I_{\mathbf{m}}$ with $A = \mathrm{Nm}(\mathfrak{a})$. Let φ and $f_\varphi \in S_{1,\chi}(\Gamma_0(N))$ be as above. There exists a harmonic Maass form $\tilde{f}_\varphi \in H_{1,\overline{\chi}}(\Gamma_0(N))$ such that $\xi_1 \tilde{f}_\varphi = f_\varphi$ and the holomorphic part of $\tilde{f}_\varphi$ has the Fourier expansion $\sum_{n \gg -\infty} c_\varphi^+(n) q^n$ satisfying*

$$c_\varphi^+(n) - \sum_{[\mathfrak{a}] \in \mathrm{Cl}_F} \varphi(\mathfrak{a}) \sum_{(\lambda) \subset \mathfrak{a},\ \mathrm{Nm}((\lambda)\mathfrak{a}^{-1}) = n} \overline{\varphi}(\lambda) \log \left| \frac{\lambda}{\lambda'} \right| \in \frac{1}{\kappa_{\mathbf{m}}} \mathbb{Z}[\varphi] \cdot \log \varepsilon_F \tag{6.2.6}$$

with $\kappa_{\mathbf{m}} \mid 12 M^3 \varphi(M) D$.

Proof. For each representative $\mathfrak{a} \in I_{\mathbf{m}}$, let $L = L_{\mathfrak{a}\mathfrak{d},M}$ and $\tilde{\vartheta}(\tau, \pm L) \in H_{1,\rho_{\pm L}}(\mathrm{SL}_2(\mathbb{Z}))$ be the harmonic Maass form constructed in Theorem 1.1. Consider

$$\tilde{f}_{\varphi,\mathfrak{a}}(\tau) := \frac{1}{[\mathcal{O}_F^\times : \Gamma_L]} \mathcal{C}_{\overline{\varphi}} \cdot \left(\mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \tilde{\vartheta}(N\tau, L) + \mathcal{C}_{\mathfrak{a},\mathfrak{m}} \cdot \tilde{\vartheta}(N\tau, -L) \right). \tag{6.2.7}$$

By Propositions 6.3 and 6.4, $\tilde{f}_{\varphi,\mathfrak{a}} \in H_{1,\overline{\chi}}(\Gamma_0(N))$ and $\xi_1 \tilde{f}_{\varphi,\mathfrak{a}} = f_{\varphi,\mathfrak{a}}$. Its holomorphic part $\tilde{f}^+_{\varphi,\mathfrak{a}}$ has the Fourier expansion

$$\tilde{f}^+_{\varphi,\mathfrak{a}}(\tau) = \sum_{n \gg 0} c^+_{\varphi,\mathfrak{a}}(n) q^n.$$

As a consequence of Theorem 1.1, the coefficient $c^+_{\varphi,\mathfrak{a}}(n)$ satisfies

$$c^+_{\varphi,\mathfrak{a}}(n) - \sum_{(\lambda) \subset \mathfrak{a},\ \mathrm{Nm}((\lambda)\mathfrak{a}^{-1}) = n} \overline{\varphi}((\lambda)) \log \left| \frac{\lambda}{\lambda'} \right| \in \frac{1}{\kappa \cdot [\mathcal{O}_F^\times : \Gamma_L]} \mathbb{Z}[\varphi] \cdot \log \varepsilon_L.$$

Since $\mathcal{O}_F^\times = \{\pm \varepsilon_F^n : n \in \mathbb{Z}\}$, we have $\frac{\log \varepsilon_L}{[\mathcal{O}_F^\times : \Gamma_L]} = \frac{1}{2} \log \varepsilon_F$. By Remark 1.2, we can choose $\kappa = 6 M^3 \varphi(M) D$. Summing over the representatives of all the classes in Cl_F with respect to $\overline{\varphi}(\mathfrak{a}^{-1}) = \varphi(\mathfrak{a})$ finishes the proof. □

When F has class number one, we can choose $\mathfrak{a} = \mathcal{O}_F$ as the only summand in the sum over Cl_F and arrange to have

$$\sum_{\substack{(\lambda)\subset\mathcal{O}_F\\ \mathrm{Nm}((\lambda))=n}} \overline{\varphi}(\lambda)\log\left|\frac{\lambda}{\lambda'}\right| = \frac{1}{2}\left(\sum_{\substack{(\lambda)\subset\mathcal{O}_F\\ \mathrm{Nm}((\lambda))=n}} \overline{\varphi}(\lambda)\log\left|\frac{\lambda}{\lambda'}\right| + \sum_{\substack{(\lambda)\subset\mathcal{O}_F\\ \mathrm{Nm}((\lambda))=n}} \overline{\varphi}(\lambda')\log\left|\frac{\lambda'}{\lambda}\right|\right)$$
$$= \boldsymbol{c}_\varphi(n) \in \mathbb{C}/(\mathbb{Z}[\varphi]\log\varepsilon_F).$$

Theorem 1.4 then follows from Theorem 6.5.

To compare this result with the p-adic version in [8], we first define the character $\varphi_\heartsuit := \varphi/\varphi'$ with φ' the composition of φ and the conjugation on F. The kernel of $\varphi_\heartsuit$ fixes a number field H, which is a class field of F. When ℓ is an inert prime in $F/\mathbb{Q}$, Darmon, Lauder and Rotger defined in [8] an element $u(\varphi_\heartsuit, \ell) \in \mathbb{Z}[\varphi] \otimes \mathcal{O}_H[1/\ell]^\times$ and showed that its p-adic logarithm is the ℓ^{th} Fourier coefficient of a generalized overconvergent eigenform of weight one.

When $\ell = \lambda\lambda'$ is a split prime in $F/\mathbb{Q}$ with $\sigma_\lambda, \sigma_{\lambda'} \in \mathrm{Gal}(H/F)$ the respective Frobenius elements, the ℓ^{th} Fourier coefficient in the p-adic setting is zero, whereas we can define an element $u(\varphi_\heartsuit, \ell) \in \mathbb{Z}[\varphi] \otimes (F^\times/\mathcal{O}_F^\times)$ by

$$u(\varphi_\heartsuit, \ell) := \sum_{\sigma\in\mathrm{Gal}(F/\mathbb{Q})} \varphi_\heartsuit^{-1}(\sigma\sigma_\lambda\sigma^{-1}) \otimes \sigma(\lambda) \in \mathbb{Z}[\varphi] \otimes (\mathcal{O}_F[1/\ell]^\times/\mathcal{O}_F^\times). \tag{6.2.8}$$

After extending the complex logarithm by $\mathbb{Z}[\varphi]$-linearity, we have

$$\boldsymbol{c}_\varphi(\ell) = \log|u(\varphi_\heartsuit, \ell)| \in \mathbb{C}/(\mathbb{Z}[\varphi]\log\varepsilon_F). \tag{6.2.9}$$

This might be a complicated way to write $\boldsymbol{c}_\varphi(\ell)$ since there are only two elements in $\mathrm{Gal}(F/\mathbb{Q})$. Nevertheless, it shows that the archimedean and non-archimedean situations complement each other. In the latter, the element $u(\varphi_\heartsuit, \ell)$ is well-defined in $\mathbb{Z}[\varphi]\otimes\mathcal{O}_H[1/\ell]^\times$ and can be used to generate class fields of F. In the former, the ℓ-unit $u(\varphi_\heartsuit, \ell)$ lies in the ground field F and is only well-defined up to $\mathbb{Z}[\varphi]\log\varepsilon_F$. This reflects the difficulty in choosing a canonical harmonic Maass form with a given ξ-image.

6.3. **Examples.** The first example was studied in detail in Section 7 of [19], where $F = \mathbb{Q}(\sqrt{29})$, $\mathfrak{m} = \left(\frac{3+\sqrt{29}}{2}\right)$ and $\varphi(2) = i$. The cusp form f_φ is the unique normalized eigenform in $S_{1,\chi}(\Gamma_0(145))$ with $\chi = \left(\frac{\cdot}{29}\right)\cdot\varphi\,|_{\mathbb{Q}}$. We showed loc. cit. that there exists a unique harmonic Maass form $\mathcal{F}_\varphi$ in $H_{1,\overline{\chi}}(\Gamma_0(145))$ that maps to f_φ and has only a simple pole at $i\infty$. Using a modularity-based algorithm, we numerically calculated the Fourier coefficients $c_\varphi^+(n)$ of the holomorphic part of $\mathcal{F}_\varphi$ as complex number. Then we identified them as $\mathbb{Z}[\varphi]$-linear combinations of logarithm of numbers in $F^\times$. When $n = \ell = \lambda\lambda'$ is a split prime in F, we even give a precise conjecture about the shape of $c_\varphi^+(\ell)$, which is implied by Theorem 1.4 if $\kappa_{\mathbf{m}} = 1$ there.

The second example is about a rather classical weight one cusp form studied by Hecke [15]. Set $D = 12, F = \mathbb{Q}(\sqrt{D}), L = \mathfrak{a} = \mathfrak{d}_F = 2\sqrt{3}\mathcal{O}_F$, $M = 1$, and $L^* = \mathcal{O}_F$. The discriminant kernel Γ_L is generated by $\varepsilon_L = 7+2\sqrt{3}$, which is the square of the fundamental unit $\varepsilon_F = 2+\sqrt{3}$. In this case, the theta series $\vartheta(\tau, L)$ is a 12-dimensional, holomorphic vector-valued weight one cusp form on $\mathrm{SL}_2(\mathbb{Z})$. The components correspond to $L^*/L = \mathcal{O}_F/\mathfrak{d}_F$. It turns out 8 of the 12 components vanish identically. The other 4 components correspond to $h = \pm 1, \pm(2+\sqrt{3}) \in \mathcal{O}_F/\mathfrak{d}_F$ and satisfies $\vartheta_1 = -\vartheta_{-1} = \pm\vartheta_{\pm(2+\sqrt{3})}$. So $\mathfrak{e} := \mathfrak{e}_1 + \mathfrak{e}_{2+\sqrt{3}} - \mathfrak{e}_{-1} - \mathfrak{e}_{-(2+\sqrt{3})}$ is an eigenvector of $\rho_L(T)$ and $\rho_L(S)$ with eigenvalues $\mathbf{e}(1/12)$ and $-i$ respectively. This implies

$$\vartheta_1\left(-\frac{1}{\tau}, L\right) = -i\tau\vartheta_1(\tau, L),\ \vartheta_1(\tau+1, L) = \mathbf{e}\left(\frac{1}{12}\right)\vartheta_1(\tau, L).$$

Since $\eta(\tau)^2 := q^{1/12}\prod_{n=1}^\infty(1-q^n)^2$ also satisfies this transformation property and has zero only at the cusps, they are equal up to a multiplicative constant. By comparing the first non-vanishing Fourier coefficient, we obtain

$$\vartheta_1(\tau, L) = \sum_{\substack{\lambda\in\Gamma_L\backslash\mathcal{O}_F \\ \lambda\equiv 1 \bmod 2\sqrt{3} \\ \mathrm{Nm}(\lambda)>0}} \mathrm{sgn}(\lambda) q^{\mathrm{Nm}(\lambda)/12} = \eta^2(\tau) \in S_{1,\rho_L^{\mathrm{sc}}}(\Gamma),$$

with $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and ρ_L^{sc} the restriction of ρ_L on the eigenspace spanned by $\mathfrak{e}$.

The representation ρ_L^{sc} is a multiplier system of Γ and can be expressed in terms of Dedekind sums. Similarly, ρ_{-L}^{sc} is the restriction of ρ_{-L} on $\mathfrak{e}$, and the conjugate of ρ_L^{sc}. The space $S_{1,\rho_L^{\mathrm{sc}}}(\Gamma)$ is spanned by $\vartheta_1(\tau, L)$ since the 12^{th} power of any form in that space is in the one dimensional space $S_1(\Gamma)$. On the other hand, the space $S_{1,\rho_{-L}^{\mathrm{sc}}}(\Gamma)$ is trivial and $\vartheta(\tau, -L)$ vanishes identically. Therefore the elements $\lambda \in \mathcal{O}_F$ with negative norm will never contribute to any holomorphic modular object. It is worth noting that $\eta^2(\tau)$ can also be constructed from lattices of signature $(2,0)$, or equivalently from characters associated to the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ (see e.g. [15, 22]).

Before constructing the harmonic Maass form $\tilde{\vartheta}(\tau, L)$, notice that $\vartheta(\tau, L) = \vartheta(\tau, L_{\mathfrak{d}_F/2,1})$. So we can suppose that $L = L_{\mathfrak{d}/2,1}$. First, we need to construct $\tilde{\Theta}(\tau, L)$. This can be done as in Section 4.2 with $N = 6$ in Equation (4.2.3). Then $|L^*/6L| = 432 = 6^2|L^*/L|$. Using the procedures in Section 4.1, we can construct $\tilde{\Theta}(\tau, 6L)$. Its holomorphic part $\tilde{\Theta}^+(\tau, 6L)$ has rational Fourier coefficients with denominators bounded by 6. Then Equation (4.2.6) defines $\tilde{\Theta}(\tau, L)$, whose holomorphic part is given by Equation (4.2.12). There, the sum over $\Gamma_0(6)\backslash\Gamma$ has 12 summands, each of which is the product of a 12×432 matrix $\frac{N_\gamma}{N}\rho_{-L}^{-1}(\gamma)\cdot\mathcal{C}_{L,N}\cdot\rho_{-NL}(\gamma_N)$ and a vector $\tilde{\Theta}^+(\tau_\gamma, NL)$ of size 432. By Theorem 4.5, all the components of $\tilde{\Theta}^+(\tau, L)$ have rational Fourier coefficients with bounded denominator. Using SAGE [10], we have numerically implemented this procedure and calculated the Fourier coefficients of $\tilde{\Theta}_h^+(\tau, L)$

for each $h \in L^*/L$. Since $\rho_{-L}(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right))\mathfrak{e}_h = \mathfrak{e}_{-h}$ and the weight is odd, we have $\tilde{\Theta}_h(\tau, L) = -\tilde{\Theta}_{-h}(\tau, L)$ for all $h \in L^*/L$. When $h = 0, \sqrt{3}, 3$ and $3+\sqrt{3}$, $h = -h$ and $\tilde{\Theta}_h(\tau, L) = 0$. For the other h, the Fourier expansions are listed in the table below.

h	$[\Gamma : \Gamma_0(6)] \cdot \tilde{\Theta}_h^+(\tau, L)$
1	$\frac{2}{3}q^{-1/12} - \frac{292}{3}q^{11/12} - \frac{4882}{3}q^{23/12} - \frac{44012}{3}q^{35/12} + O(q^{47/12})$
$1+\sqrt{3}$	$\frac{4}{3}q^{2/12} - \frac{592}{3}q^{14/12} - \frac{8576}{3}q^{26/12} - \frac{71456}{3}q^{38/12} + O(q^{50/12})$
2	$\frac{1}{6}q^{-4/12} + 32q^{8/12} + \frac{2368}{3}q^{20/12} + \frac{25526}{3}q^{32/12} + 60939q^{44/12} + O(q^{56/12})$
$2+\sqrt{3}$	$\frac{1}{3}q^{-1/12} + \frac{190}{3}q^{11/12} + \frac{4357}{3}q^{23/12} + \frac{42530}{3}q^{35/12} + O(q^{47/12})$

Now as in Equation 5.2.1, we define $\tilde{\vartheta}(\tau, L) := 2I'(\tau, -L) + \log \varepsilon_L \cdot \tilde{\Theta}(\tau, L) \in H_{1,\rho_{-L}}(\Gamma)$, which satisfies $\xi_1 \tilde{\vartheta}(\tau, L) = \vartheta(\tau, L)$. Then the eigen-component

$$\tilde{f}(\tau) := \frac{1}{4}\left(\tilde{\vartheta}_1(\tau, L) + \tilde{\vartheta}_{2+\sqrt{3}}(\tau, L) - \tilde{\vartheta}_{-1}(\tau, L) - \tilde{\vartheta}_{-(2+\sqrt{3})}(\tau, L)\right) \in H_{1,\rho^{\mathrm{sc}}_{-L}}(\Gamma) \tag{6.3.1}$$

maps to $\eta^2(\tau)$ under ξ_1. The Fourier coefficients of its holomorphic part

$$\tilde{f}^+ = \sum_{n \geq -1, n \equiv 11 \bmod 12} c^+(n) q^{n/12}$$

have the shape

$$c^+(n) = \log\left|\frac{u(n)}{u(n)'}\right| + \frac{b(n)}{24} \log\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right), \tag{6.3.2}$$

where $b(n) \in \mathbb{Z}$ and $u(n) \in \mathcal{O}_F$ is a unit outside primes dividing n. They are listed in the table below for $n \leq 35$.

n	$u(n)$	$b(n)$
-1	1	1
11	$1+2\sqrt{3}$	-34
23	$2+3\sqrt{3}$	-175
35	1	-494

Finally, an application of Stokes' theorem tells us that the Petersson norm of $\eta^2(\tau)$ is $\frac{\log(2+\sqrt{3})}{6}$.

References

1. Fabrizio Andreatta, Eyal Z. Goren, Benjamin Howard, and Keerthi Madapusi Pera, *Faltings heights of abelian varieties with complex multiplication*, (2015).
2. Richard E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562. MR 1625724 (99c:11049)
3. Jan H. Bruinier, *Borcherds products and Chern classes of Hirzebruch-Zagier divisors*, Invent. Math. **138** (1999), no. 1, 51–83. MR 1714336
4. ———, *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780, Springer-Verlag, Berlin, 2002. MR 1903920 (2003h:11052)

5. Jan H. Bruinier and Jens Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90. MR 2097357 (2005m:11089)
6. Jan H. Bruinier and Ken Ono, *Heegner divisors, L-functions and harmonic weak Maass forms*, Ann. of Math. (2) **172** (2010), no. 3, 2135–2181. MR 2726107
7. Jan H. Bruinier and Tonghai Yang, *Faltings heights of CM cycles and derivatives of L-functions*, Invent. Math. **177** (2009), no. 3, 631–681. MR 2534103
8. Henri Darmon, Alan Lauder, and Victor Rotger, *Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields*, Adv. Math. **283** (2015), 130–142. MR 3383798
9. Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975). MR 0379379 (52 #284)
10. The Sage Developers, *Sagemath, the Sage Mathematics Software System (Version 6.9)*, 2015, `http://www.sagemath.org`.
11. William Duke, Özlem. Imamoḡlu, and Árpád. Tóth, *Cycle integrals of the j-function and mock modular forms*, Ann. of Math. (2) **173** (2011), no. 2, 947–981. MR 2776366
12. William Duke and Yingkun Li, *Harmonic Maass forms of weight 1*, Duke Math. J. **164** (2015), no. 1, 39–113. MR 3299102
13. Stephan Ehlen, *CM values of regularized theta lifts*, 2013, Thesis (Ph.D.)–T.U. Darmstadt.
14. Benedict H. Gross and Don B. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220. MR 772491 (86j:11041)
15. Erich Hecke, *Zur Theorie der elliptischen Modulfunktionen (=Werke, no. 23)*, Math. Ann. **97** (1926), no. 1, 210–242.
16. Friedrich Hirzebruch and Don B. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113. MR 0453649 (56 #11909)
17. Stephen S. Kudla, *Holomorphic Siegel modular forms associated to* $\mathrm{SO}(n,1)$, Math. Ann. **256** (1981), no. 4, 517–534. MR 628232 (83g:10021)
18. Stephen S. Kudla, Michael Rapoport, and Tonghai Yang, *On the derivative of an Eisenstein series of weight one*, Internat. Math. Res. Notices (1999), no. 7, 347–385. MR 1683308 (2000b:11057)
19. Yingkun Li, *Real-dihedral harmonic Maass forms and CM-values of Hilbert modular functions*, Compos. Math. **152** (2016), no. 6, 1159–1197. MR 3518308
20. William J. McGraw, *The rationality of vector valued modular forms associated with the Weil representation*, Math. Ann. **326** (2003), no. 1, 105–122. MR 1981614
21. Nils R. Scheithauer, *The Weil representation of* $\mathrm{SL}_2(\mathbb{Z})$ *and some applications*, Int. Math. Res. Not. IMRN (2009), no. 8, 1488–1545. MR 2496771
22. Bruno Schoeneberg, *Über den Zusammenhang der Eisensteinschen Reihen und Thetareihen mit der Diskriminante der elliptischen Funktionen*, Math. Ann. **126** (1953), 177–184. MR 0056639
23. Harold M. Stark, *Class fields and modular forms of weight one*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), Springer, Berlin, 1977, pp. 277–287. Lecture Notes in Math., Vol. 601. MR 0450243
24. ______, *On modular forms of weight one from real quadratic fields and theta functions*, J. Ramanujan Math. Soc. **3** (1988), no. 1, 63–79. MR 975837
25. Sander P. Zwegers, *Mock theta functions*, Proefschrift Universiteit Utrecht, 2002, Thesis (Ph.D.)–Universiteit Utrecht.

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