(1) $$\sum_{n=1}^{\infty} \frac{1}{\eta^2+n^2} = \frac{1}{2\eta}\left(\pi \operatorname{Cotg}\eta\pi - \frac{1}{\eta}\right)$$

Nyní vložme do identity

$$\frac{1-(-1)^n x^n}{1+x} = \sum_{\alpha=1}^{n} (-1)^{\alpha-1} x^{\alpha-1}$$

$x = \frac{\mu^2\omega^2}{\nu^2}$, a dělme $\nu^2\mu^{2n}$:

$$\frac{1}{\mu^{2n}(\mu^2\omega^2+\nu^2)} + (-1)^{n-1}\frac{\omega^{2n}}{\nu^{2n}(\mu^2\omega^2+\nu^2)}$$

$$= \sum_{\alpha=1}^{n} (-1)^{\alpha-1}\frac{\omega^{2\alpha-2}}{\mu^{2n+2-2\alpha}\,\nu^{2\alpha}}$$

Zde sečtěme pro $\mu, \nu = 1, 2, 3, \dots$; vyjde

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^{2n}} \sum_{\nu=1}^{\infty} \frac{1}{\nu^2+\mu^2\omega^2} + \cancel{(-1)^{n-1}\omega^{2n}\sum_{\nu}\frac{1}{\nu^{2n}}\sum_{\mu}\frac{1}{\dots}}$$

$$+(-1)^{n-1}\omega^{2n-2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2n}}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2+\frac{\nu^2}{\omega^2}}$$

$$= \sum_{\alpha=1}^{n}(-1)^{\alpha-1}\omega^{2\alpha-2}\zeta(2\alpha)\,\zeta(2n+2-2\alpha)$$

Levá strana =

$$\sum_{1}^{\infty}\frac{1}{2\omega\,\mu^{2n+1}}\left(\pi\operatorname{Cotg}\mu\omega\pi - \frac{1}{\mu\omega}\right)$$

$$+(-1)^{n-1}\omega^{2n-2}\sum_{1}^{\infty}\frac{\omega}{2\nu^{2n+1}}\left(\pi\operatorname{Cotg}\frac{\nu\pi}{\omega} - \frac{\omega}{\nu}\right)$$

$$= \frac{\pi}{2\omega}\sum_{1}^{\infty}\frac{1}{\mu^{2n+1}}\operatorname{Cotg}\mu\omega\pi$$

$$+(-1)^{n-1}\frac{\omega^{2n-1}}{2}\pi\sum_{1}^{\infty}\frac{\operatorname{Cotg}\frac{\nu\pi}{\omega}}{\nu^{2n+1}} - \frac{\zeta(2n+2)}{2\omega^2}$$

$$+(-1)^{n}\frac{\omega^{2n}}{2}\zeta(2n+2)$$

$$\sum_{1}^{\infty}\frac{\operatorname{Cotg}\mu\omega\pi}{\mu^{2n+1}} + (-1)^{n-1}\omega^{2n}\sum_{1}^{\infty}\frac{\operatorname{Cotg}\frac{\nu\pi}{\omega}}{\nu^{2n+1}}$$

$$= (2\pi)^{2n+1}\frac{B_{n+1}}{(2n+2)!}\left[\frac{1}{\omega}+(-1)^{n-1}\omega^{2n+1}\right]$$

$$+ (2\pi)^{2n+1}\sum_{\alpha=1}^{n}(-1)^{\alpha-1}\frac{B_\alpha B_{n+1-\alpha}}{(2\alpha)!\,(2n+2-2\alpha)!}\omega^{2\alpha-1}$$

$$\boxed{\begin{aligned}&\frac{(2n)!}{(2\pi)^{2n-1}}\left\{\sum_{1}^{\infty}\frac{\operatorname{Cotg}\mu\omega\pi}{\mu^{2n-1}} + (-1)^{n}\omega^{2n-2}\sum_{1}^{\infty}\frac{\operatorname{Cotg}\frac{\nu\pi}{\omega}}{\nu^{2n-1}}\right\}\\ &= B_n\left[\frac{1}{\omega}+(-1)^n\omega^{2n-1}\right]\\ &+\sum_{1}^{n-1}(-1)^{\alpha-1}\binom{2n}{2\alpha}B_\alpha B_{n-\alpha}\,\omega^{2\alpha-1} = G_n(\omega)\\ &(n>1)\end{aligned}}$$ (*)

$$\operatorname{Cotg}\mu\omega\pi = \frac{2}{e^{2\mu\omega\pi}-1}+1$$

$$\zeta(2n-1)\left[1+(-1)^n\omega^{2n-2}\right]$$

$$= \frac{(2\pi)^{2n-1}}{(2n)!}G_n(\omega) - 2\sum_{1}^{\infty}\frac{1}{(e^{2\mu\omega\pi}-1)\mu^{2n-1}}$$

(**) $$-(-1)^n\omega^{2n-2}\cdot 2\sum_{1}^{\infty}\frac{1}{\left(e^{\frac{2\nu\pi}{\omega}}-1\right)\nu^{2n-1}}$$

1) $n = 2k$, $\omega = 1$

$$\zeta(4k-1) = \frac{(2\pi)^{4k-1}}{(4k)!\,2}G_{2k}(1) - 2\sum_{1}^{\infty}\frac{1}{\nu^{4k-1}\left(e^{2\nu\pi}-1\right)}$$

(Jornal de Sc.

PRESIDIUM ZEMSKÉHO HLAVNÍHO MĚSTA BRNA.

Brno, 2. XI. 20.

Velevážený p. Kolego!

Zasílám Vám přípisy, jež mi poslal Kolega Hostinský a ~~prosím~~ a kde, abyste laskavě udal, kdy by Vám bylo libo, abychom se sešli (jak žádá kol. Hostinský)

V nejhlubší úctě

Zahradník

2) Volba $n=2k+1$, $\omega=\frac{4}{5}$ dává

$$\left(1-\frac{4^{4k}}{5^{4k}}\right)\zeta(4k+1)=\frac{(2\pi)^{4k+1}}{(4k+2)!}G_{2k+1}\left(\frac{4}{5}\right)-2\sum_{1}^{\infty}\frac{1}{\mu^{4k+1}}\left[\frac{1}{e^{\frac{8\mu\pi}{5}}-1}-\frac{\left(\frac{4}{5}\right)^{4k}}{e^{\frac{5\mu\pi}{2}}-1}\right]$$

Tyto vzorce dávají rychle hodnoty $\zeta(s)$ pro $s=3,5,7,\dots$

~~Př.~~ $\cancel{\frac{9\cdot 41}{5^4}\zeta(5)=\frac{(2\pi)^5}{720}}$

Lépe asi vyjíti z ** tj.

$$\zeta(2n-1)\left[1+(-1)^n\omega^{2n-2}\right]=\begin{cases}\frac{(2\pi)^{2n-1}}{(2n)!}G_n(\omega)\\ -2\sum_{1}^{\infty}\frac{1}{\mu^{2n-1}}\left[\frac{1}{e^{2\mu\omega\pi}-1}\right.\\ \left.+(-1)^n\frac{\omega^{2n-2}}{e^{\frac{2\mu\pi}{\omega}}-1}\right]\end{cases}$$

a vzíti derivaci na $\omega=1$ $(n=2k+1)$

$$-4k\,\zeta(4k+1)=\frac{(2\pi)^{4k+1}}{(4k+2)!}G'_{2k+1}(1)+2\sum_{1}^{\infty}\frac{1}{\mu^{4k+1}}V_\mu\,,$$

$$V_\mu=\frac{4k}{e^{2\mu\pi}-1}+\frac{4\mu\pi}{\left(e^{\mu\omega\pi}-e^{-\mu\omega\pi}\right)^2};\quad G'_{2k+1}=-(4k+2)B_{2k+1}+\sum_{1}^{2k}(-1)^{\alpha-1}(2\alpha-1)\binom{4k+2}{2\alpha}B_\alpha B_{2k+1-\alpha}$$

Přejmeme základní vzorce* pro ω, předpokládajíce ω zakomplexní [illegible]

Report . . .

$$\sum_{\nu=1}^{\infty} \frac{\operatorname{cotg} \nu\omega\pi}{\nu^{2n-1}} + \omega^{2n-2} \sum_{1}^{\infty} \frac{\operatorname{cotg} \frac{\mu\pi}{\omega}}{\mu^{2n-1}} = \frac{(2\pi)^{2n-1}}{(2n)!} f(\omega)$$

$$f(\omega) = B_n \left(\frac{1}{\omega} + \omega^{2n-1} \right) - \sum_{1}^{n-1} \binom{2n}{2\alpha} B_\alpha B_{n-\alpha} \omega^{2\alpha-1}$$

Je pak $(-1)^{n-1} \frac{f(\omega)}{(2n)!}$ součinitel při x^{2n-2} ve fkci $\frac{1}{(e^x-1)(e^{\omega x}-1)}$.

Jeli x kořen algebr. rovnice s cel. souč.

$F(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots = 0$,

máme $\nu^n F\left(\frac{\mu}{\nu}\right) = a_0 \mu^n + a_1 \mu^{n-1} \nu + \dots =$ cel. číslo, abs. ≥ 1.

$F\left(\frac{\mu}{\nu}\right) \geq \frac{K}{\nu^n}$. Levá str. $= a_0 \left(\frac{\mu}{\nu} - x\right)\left(\frac{\mu}{\nu} - x'\right) \dots \left(\frac{\mu}{\nu} - x^{(n-1)}\right)$

$$\left|\frac{\mu}{\nu} - x\right| > \frac{K}{\nu^n \cdot a_0 \left(\frac{\mu}{\nu} - x'\right) \dots \left(\frac{\mu}{\nu} - x^{(n-1)}\right)}$$

Jeli zvláště $\frac{\mu}{\nu} - x \sim 0$, máme

$$\left|x - \frac{\mu}{\nu}\right| > \frac{K}{\nu^n \cdot a_0 \cdot 2 (x-x')(x-x'') \dots} = \frac{K}{2\nu^n F'(x)}$$

$$\xi = \nu x - \mu > \frac{K'}{\nu^{n-1}}, \quad \operatorname{cotg} \nu x \pi = \operatorname{cotg} \nu \xi \pi < K'' \nu^{n-1}$$

$$\sum \frac{\operatorname{cotg} \nu x \pi}{\nu^s} \sim \sum \frac{K'' \nu^{n-1}}{\nu^s} = K'' \sum \frac{1}{\nu^{s+1-n}},$$

Řada konverguje pro $s > n$, jeli x alg. číslo řádu n.

$$\Phi(\omega) = \sum_{\nu=1}^{\infty} \frac{\operatorname{cotg} \nu\omega\pi}{\nu^s}; \quad \Phi(x+iy) - \Phi(x) = \sum_{\nu=1}^{\infty} \frac{\operatorname{cotg} \nu\pi(x+iy) - \operatorname{cotg} \nu\pi x}{\nu^s}.$$

$$\sum_{p}^{\infty} \left| \frac{\operatorname{cotg}(\nu\pi x + i\nu\pi y) - \operatorname{cotg} \nu\pi x}{\nu^s} \right| < \sum \frac{\nu\pi|y|}{\nu^s} \frac{1}{|\sin^2 \nu\pi(x+i\vartheta)|} < \sum_{p}^{\infty} \frac{\pi|y|}{\nu^{s-1} \sin^2 \nu\pi x}$$

Konvergence pro $s-1-2(n-1) > 1$ t.j. pro $s > 2n$.

Cena	OBNOS	
	K	h

Jde o funkci $\Phi(\omega) = \sum_1^{\infty} \frac{\cot g\, \nu\omega\pi}{\nu^s}$, $\omega = x+iy$, $y \sim 0$.

Nejprvé platí $|\cot g(x+iy)|^2 = \frac{\mathrm{Cotg}^2 y \cot^2 x + 1}{\mathrm{Cotg}^2 y + \cot^2 x} = \frac{\cot g^2 x + \mathrm{Tg}^2 y}{1 + \cot^2 x\, \mathrm{Tg}^2 y} < \cot g^2 x + \mathrm{Tg}^2 y$

tedy $|\cot g(x+iy)| < |\cot g\, x| + |\mathrm{Tg}\, y|$

Odtud plyne

$$\sum_p^{\infty} \frac{|\cot g\, \nu\pi(x+iy)|}{\nu^s} < \sum_p^{\infty} \left|\frac{\cot g\, \nu x\pi}{\nu^s}\right| + \sum_p^{\infty} \frac{|\mathrm{Tg}\, \nu y\pi|}{\nu^s} = I + II$$

$|\mathrm{Tg}\, y| < 1$, tedy $II < \sum_p^{\infty} \frac{1}{\nu^s}$ ..

Je-li x algebr. stupně n, $s > n$, máme pro dosti velké p

$$I < \frac{\delta}{3}, \quad II < \frac{\delta}{3},$$

a v identitě

$$\Phi(x+iy) - \Phi(x) = \sum_1^{p-1} \frac{\cot g\, \nu\pi(x+iy) - \cot g\, \nu\pi x}{\nu^s} + \sum_{\nu=p}^{\infty} \frac{\cot g\, \nu\pi(x+iy)}{\nu^s} - \sum_p^{\infty} \frac{\cot g\, \nu\pi x}{\nu^s}$$

jsou poslední dva členy abs. $< \frac{2\delta}{3} + \frac{\delta}{3} = \delta$ neodvislé od y,

a pro malá y součet $\sum_1^{p-1} \sim 0$, takže

$$\lim_{y=0} \Phi(x+iy) = \Phi(x).$$

Vztah III platí tedy též pro reálná ω, pokud jsou algebraické stupně $< 2n-1$.

(Nyní dvojice!)

Podaný důkaz předpokládá limitní přechod po přímce $x = \text{konst}$; v řadě $\Phi(\frac{-1}{\omega})$ probíhá argument $-\frac{1}{x+iy}$ kruh $(\perp Ox)$.

Tedy: Dokážeme nejprvé konvergenci dvojice

a z té pak vyplývá také konvergence ~~tedy~~ $\lim_{y=0} \Phi\left(\frac{-1}{x+iy}\right)$.

U dvojice $\nu\omega - \mu = \xi$, $\frac{\mu}{\omega} = \nu - \frac{\xi}{\omega}$; $\pi\left(\mathfrak{C}_{\nu} + \omega^{s-1}\mathfrak{C}_{\mu}\right) = \pi\frac{\cot g\, \nu\omega\pi}{\nu^s} + \pi\frac{\omega^{s-1}\cot g\frac{\mu\pi}{\omega}}{\mu^s} \doteq \frac{1}{\nu^s\xi} - \frac{\omega^s}{\xi\mu^s} + \text{kon.}$

$= \frac{\omega^s}{\xi}\left[\frac{1}{(\nu\omega)^s} - \frac{1}{\mu^s}\right] = \frac{\omega^s}{\xi}\cdot\frac{-s\cdot\xi}{(\nu\omega - \vartheta\xi)^{s+1}} \doteq \frac{-s\,\omega^s}{\nu^{s+1}\omega^{s+1}} = \frac{-s}{\omega\,\nu^{s+1}}$