RATIONALITY OF SECANT ZETA VALUES

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Abstract. We use the Arakawa-Berndt theory of generalized η-functions to prove a conjecture of Lalín, Rodrigue and Rogers concerning the algebraic nature of special values of the secant zeta function.

1. Introduction

The cotangent and secant zeta functions attached to an algebraic irrational number α are defined, for $\mathrm{Re}(s)$ large enough, by the Dirichlet series

$$\xi(\alpha, s) = \sum_{n=1}^{\infty} \frac{\cot(\pi n\alpha)}{n^s} \qquad \text{and} \qquad \psi(\alpha, s) = \sum_{n=1}^{\infty} \frac{\sec(\pi n\alpha)}{n^s},$$

respectively. The cotangent zeta function was introduced for $s = 2k+1$, $k \geq 1$ an integer, by Lerch [7], and for general s by Berndt [4] in the course of his study of generalized Dedekind sums. Lerch stated (without proof) the following functional equation valid for algebraic irrational α and sufficiently large $k = k(\alpha)$:

$$\xi(\alpha, 2k+1) + \alpha^{2k}\xi(\tfrac{1}{\alpha}, 2k+1) = (2\pi)^{2k+1}\varphi(\alpha, 2k+1), \tag{1}$$

where

$$\varphi(\alpha, n) = \sum_{j=0}^{n+1} \frac{B_j B_{n+1-j}}{j!(n+1-j)!}\alpha^{j-1},$$

and B_k is the k-th Bernoulli number. Suppose $\alpha \neq \pm 1$ is a unit in the quadratic field $\mathbf{Q}(\sqrt{d})$ of discriminant d. Writing $\alpha = \frac{a+b\sqrt{d}}{2}$ and defining $\varepsilon = \pm 1$ by $a^2 - b^2 d = \varepsilon$, we have $\frac{1}{\alpha} = -\varepsilon(\alpha - a)$. Using the obvious identities $\xi(-\alpha, s) = -\xi(\alpha, s)$ and $\xi(\alpha - 1, s) = \xi(\alpha, s)$ together with (1), we deduce the following rationality result:

$$\frac{\xi(\alpha, 2k+1)}{(2\pi)^{2k+1}\sqrt{d}} = \frac{1}{\sqrt{d}} \frac{\varphi(\alpha, 2k+1)}{1 - \varepsilon\alpha^{2k}} \in \mathbf{Q}. \tag{2}$$

Another proof of this result was given by Berndt [4, Theorem 5.2]. More generally, if α is an arbitrary real quadratic irrationality, then Lerch uses the continued fraction expansion of α to conclude that the left hand side of (2) belongs to $\mathbf{Q}(\alpha)$ – see [7, (3)].

Lalín, Rodrigue and Rogers conjecture an analogous result for the secant zeta function:

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Conjecture 1.1 ([6, Conjecture 1])**.** *Suppose d and k are positive integers such that d is not a square. Then*

$$\frac{\psi(\sqrt{d}, 2k)}{\pi^{2k}} \in \mathbf{Q}.$$

They prove a functional equation for $\psi(\alpha, s)$ analogous to (1) and use it to deduce many instances of Conjecture 1.1 as above. In §2, we show that their functional equation is actually sufficient to prove in general that $\psi(\alpha, 2k) \in \pi^{2k}\mathbf{Q}(\alpha)$ for all real quadratic irrationalities α, not merely those of the form $\alpha = \sqrt{d}$.

In §3, we relate $\psi(\alpha, s)$ to the *generalized η-functions* studied by Arakawa [1], fascinating objects in their own right. Using this relationship, we leverage Arakawa's results to give an explicit formula for $\psi(\alpha, 2k)$ for real quadratic irrationalities α, yielding another proof of Conjecture 1.1.

2. The Lalín-Rodrigue-Rogers functional equation

One can prove Conjecture 1.1 following Lerch's approach for the cotangent zeta function. Most of this argument was given in [6]; we complete their thought. Let $A, B \in \mathbf{PSL}_2(\mathbf{Z})$ be defined by

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The following functional equations are established in [6, (4.1), (4.2)]:

$$\psi(A\alpha, 2k) = \psi(\alpha, 2k),$$

$$\psi(B\alpha, 2k) = (2\alpha+1)^{1-2k}\psi(\alpha, 2k)$$
$$- \frac{\pi^{2k}}{(2k)!}\sum_{m=0}^{2k}(2^{m-1}-1)B_m E_{2k-m}\binom{2k}{m}(\alpha+1)^{2k-m}\Big((2\alpha+1)^{m-2k} - (2\alpha+1)^{1-2k}\Big).$$

Here, B_n is the n-th Bernoulli number and E_n is the n-th Euler number. It follows that if $C \in \langle A, B\rangle$ then there is a $\mathbf{Q}(\alpha)$-linear relation between $\psi(C\alpha, 2k)$, $\psi(\alpha, 2k)$ and π^{2k}. Thus, if there is a matrix $C \in \langle A, B\rangle$ such that $C\alpha = \alpha$ then $\psi(\alpha, 2k) \in \pi^{2k}\mathbf{Q}(\alpha)$. In [6, §4], several families of examples of such pairs (α, C) are given and the associated linear relations are worked out explicitly. We merely point out that if α is any real quadratic irrationality, then there is *always* a $C \in \langle A, B\rangle$, $C \neq 1$ such that $C\alpha = \alpha$.

To see this, let α be a real quadratic irrationality and consider the lattice $L = \mathbf{Z} + \mathbf{Z}\alpha \subset \mathbf{Q}(\alpha)$. Let $\mathcal{O}$ be the order of L:

$$\mathcal{O} = \{x \in \mathbf{Q}(\alpha) : xL \subset L\}.$$

Then $\mathcal{O}$ is an order in $\mathbf{Q}(\alpha)$. Let $u \in \mathcal{O}$. Writing $u \cdot \alpha = a\alpha + b$ and $u \cdot 1 = c\alpha + d$ with $a, b, c, d \in \mathbf{Z}$, we have

$$u\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha \\ 1 \end{pmatrix}.$$

Write $j(u)$ for the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ associated to u above. Then $j : \mathcal{O} \to \mathrm{M}_2(\mathbf{Z})$ is a ring homomorphism. Since $\left(\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right)$ is an eigenvector of $j(u)$ for all $u \in \mathcal{O}$, we have $j(u)\alpha := \frac{a\alpha+b}{c\alpha+d} = \alpha$ when $u \neq 0$. By Dirichlet's unit theorem, the group $\mathcal{O}_+^*$ of totally positive units in $\mathcal{O}$ is free of rank 1; write $\mathcal{O}_+^* = \langle\gamma\rangle$. By [9, p. 84], $\langle A, B\rangle$ is the principal congruence subgroup

$\Gamma(2) \subset \mathbf{PSL}_2(\mathbf{Z})$, this inclusion having index 6. Therefore, $C := j(\gamma^6)$ satisfies $C \neq 1$, $C \in \Gamma(2)$ and $C\alpha = \alpha$.

3. Generalized η-functions and secant zeta values

Arakawa [2] gave another proof of (2) by relating $\xi(\alpha, s)$ to generalized η-functions, the theory of which he developed in [1]. In turn, Arakawa's work has its foundations in papers of Lewittes [8] and Berndt [3]. We show that Arakawa's method can also be used to analyze the secant zeta function.

For $x \in \mathbf{R}$, define $\langle x \rangle$ (resp., $\{x\}$) by

$$0 < \langle x \rangle \leq 1 \qquad (\text{resp. } 0 \leq \{x\} < 1) \qquad \textit{and} \qquad x - \langle x \rangle \in \mathbf{Z} \qquad (\text{resp., } x - \{x\} \in \mathbf{Z}).$$

We set $e(z) = e^{2\pi i z}$.

Following [1], let $p, q \in \mathbf{R}$ and define

$$\eta(\alpha, s, p, q) = \sum_{n=1}^{\infty} n^{s-1} \frac{e(n(p\alpha + q))}{1 - e(n\alpha)}$$
$$H(\alpha, s, p, q) = \eta(\alpha, s, \langle p \rangle, q) + e\left(\tfrac{s}{2}\right) \eta(\alpha, s, \langle -p \rangle, -q).$$

Theorem 3.1 ([1, Lemma 1 and Theorem 2])**.** *Suppose*

$$\alpha \in \mathbf{R} \cap \bar{\mathbf{Q}} \qquad \textit{and} \qquad \alpha \notin \mathbf{Q}.$$

Then $\eta(\alpha, s, p, q)$ is absolutely convergent for $\Re(s) < 0$. If, in addition,

$$[\mathbf{Q}(\alpha) : \mathbf{Q}] = 2 \qquad \textit{and} \qquad p, q \in \mathbf{Q}$$

then $H(\alpha, s, p, q)$ has analytic continuation to $\mathbf{C} - \{0\}$, and the singularity at $s = 0$ is at worst a simple pole.

Remark 3.2. The convergence of $\eta(\alpha, s, p, q)$ relies on the Thue-Siegel-Roth theorem in much the same way that the convergence of $\psi(\alpha, s)$ does – see [6, Theorem 1].

An elementary computation yields

$$\xi(\alpha, s) = -2i \left(\frac{H(\alpha, 1-s, 1, 0)}{1 + e(\frac{s}{2})} + \frac{1}{2}\zeta(s) \right),$$

so rationality statements for $H(\alpha, 1-s, 1, 0)$ for even integral s translate to rationality statements for $\xi(\alpha, s)$ at odd integral s. In contrast, $\psi(\alpha, s)$ does not seem to have a simple expression in terms of $H(\alpha, s, p, q)$. Crucially, however, we still have a relation between certain special values of H and ψ relying on the relation $\frac{3}{4} = -\frac{1}{4} + 1$:

$$\begin{aligned} \psi\left(\tfrac{\alpha}{2}, 1-s\right) &= \sum_{n=1}^{\infty} n^{s-1} \frac{2}{e(\frac{n\alpha}{4}) + e(-\frac{n\alpha}{4})} \\ &= 2\sum_{n=1}^{\infty} n^{s-1} \frac{e(\frac{n\alpha}{4})}{1 - e(n\alpha)} - 2\sum_{n=1}^{\infty} n^{s-1} \frac{e(\frac{3n\alpha}{4})}{1 - e(n\alpha)} \\ &= 2\eta(\alpha, s, \langle \tfrac{1}{4} \rangle, 0) - 2\eta(\alpha, s, \langle -\tfrac{1}{4} \rangle, 0). \end{aligned}$$

If $s = 1 - 2k$, then $e(\frac{s}{2}) = -1$ and we conclude that

$$\psi(\tfrac{\alpha}{2}, 2k) = 2H(\alpha, 1-2k, \tfrac{1}{4}, 0). \tag{3}$$

For the rest of the paper, suppose that α is a real quadratic irrationality. Formulas of Berndt and Arakawa allow us to evaluate $H(\alpha, 1-2k, \frac{1}{4}, 0)$ rather explicitly. Let

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{Z})$$

be a matrix such that

$$c > 0 \qquad \text{and} \qquad \beta := c\alpha + d > 0.$$

Set

$$(p', q') = (p, q)V \qquad \text{and} \qquad \varrho = \{q'\}c - \{p'\}d.$$

Theorem 3.3 ([1, Theorem 1 and Eq. (1.19)])**.** *Suppose that p and p' are not in $\mathbf{Z}$. Then the following transformation formula holds:*

$$\beta^{-s} H(V\alpha, s, p, q) = H(\alpha, s, p', q') + (2\pi)^{-s} e(-\frac{s}{4}) L(\alpha, s, p', q', c, d),$$

where $L(\alpha, s, p', q', c, d)$ is as in (2) of [3]*. If $s = -m$ is a negative integer, then*

$$L(\alpha, -m, p', q', c, d) = 2\pi i \sum_{j=1}^{c} \sum_{\ell=0}^{m+2} b_\ell \left(\frac{j - \{p'\}}{c} \right) b_{m+2-\ell} \left(\left\{ \frac{jd + \varrho}{c} \right\} \right) (-\beta)^{\ell-1}.$$

Here, b_ℓ is the (normalized) ℓ-th Bernoulli polynomial defined by the generating series

$$\frac{ue^{ux}}{e^u - 1} = \sum_{\ell \geq 0} b_\ell(x) u^\ell.$$

By [1, Lemma 4], for any rational numbers p and q there is a totally positive unit β of $\mathbf{Q}(\alpha)$ and a matrix $V \in \mathbf{SL}_2(\mathbf{Z})$ such that

$$c > 0, \qquad (p', q') := (p, q)V \equiv (p, q) \pmod{\mathbf{Z}^2}, \qquad \text{and} \qquad V \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \beta \begin{pmatrix} \alpha \\ 1 \end{pmatrix}.$$

The last condition implies that $\beta = c\alpha + d$, consistently with the notation introduced above. Suppose $p \notin \mathbf{Z}$. Then $p' \notin \mathbf{Z}$, too, as $p \equiv p' \pmod{\mathbf{Z}}$. Applying Theorem 3.3, observing that H and L only depend on the class of $(p, q) \equiv (p', q')$ modulo $\mathbf{Z}^2$, and rearranging terms, we get

$$(\beta^{-s} - 1) H(\alpha, s, p, q) = (2\pi)^{-s} e(-\tfrac{s}{4}) L(\alpha, s, p, q, c, d).$$

By the second part of Theorem 3.3, if $s = 1 - 2k$ then

$$\text{(4)} \qquad \frac{H(\alpha, 1-2k, p, q)}{\pi^{2k}} = \frac{2^{2k}(-1)^k}{(\beta^{2k-1} - 1)} \sum_{j=1}^{c} \sum_{\ell=0}^{2k+1} b_\ell \left(\frac{j - \{p\}}{c} \right) b_{2k+1-\ell} \left(\left\{ \frac{jd + \varrho}{c} \right\} \right) (-\beta)^{\ell-1}.$$

Setting $(p, q) = (\frac{1}{4}, 0)$ and using (3), this formula specializes to

$$\text{(5)} \qquad \frac{\psi(\frac{\alpha}{2}, 2k)}{\pi^{2k}} = \frac{2^{2k+1}(-1)^k}{(\beta^{2k-1} - 1)} \sum_{j=1}^{c} \sum_{\ell=0}^{2k+1} b_\ell \left(\frac{j - \frac{1}{4}}{c} \right) b_{2k+1-\ell} \left(\left\{ \frac{d(j - \frac{1}{4})}{c} \right\} \right) (-\beta)^{\ell-1}.$$

We conclude :

Theorem 3.4. *Suppose α is a real quadratic irrationality and k is a positive integer. Then*

$$\frac{\psi(\alpha, 2k)}{\pi^{2k}} \in \mathbf{Q}(\alpha).$$

Moreover, if $x \mapsto x'$ is the nontrivial automorphism of $\mathbf{Q}(\alpha)$, then

$$\left(\frac{\psi(\alpha, 2k)}{\pi^{2k}}\right)' = \frac{\psi(\alpha', 2k)}{\pi^{2k}}.$$

Conjecture 1.1 follows from Theorem 3.4 and the evenness of the secant function.

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