Régulateurs et modularité

Courbes elliptiques, formes modulaires de poids un, et régulateurs de régulateurs

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Summary of Victor's lecture

Let f, g, h be p-stabilised eigenforms of weights $2, 1, 1$.

$f \leftrightarrow E/\mathbb{Q}, \quad g \leftrightarrow V_g, \quad h \leftrightarrow V_h, \qquad V_{gh} := V_g \otimes V_h.$

$\varphi_g(T_\ell) = a_\ell(g), \quad \varphi_g(U_p) = \alpha_g, \qquad I_g := \ker(\varphi_g) \subset \mathbb{T},$

$$S_1(N,\chi)[g] := S_1(N,\chi)[I_g] = S_1^{(p)}(N,\chi)[I_g],$$

$$S_1^{(p)}(N,\chi)[[g]] := \bigcup_{n\geq 1} S_1^{(p)}(N,\chi)[I_g^n].$$

Iterated integral associated to (g, f, h):

$$e_g(d^{-1} f^{[p]} \times h) \in S_1^{(p)}(N,\chi)[[g]].$$

The Bellaiche-Dimitrov condition

Definition

The eigenform g satisfies the *Bellaiche-Dimitrov condition* at p if the following equivalent conditions hold:

1. the p-adic Coleman-Mazur eigencurve is smooth, and étale over weight space, at the point attached to g;
2. the natural inclusion

$$S_1(N, \chi)[\theta_{\psi_g}] \hookrightarrow S_1^{(p)}(N, \chi)[[\theta_{\psi_g}]]$$

   is an isomorphism.

Summary of Victor's lecture, cont'd

In the "Bellaiche-Dimitrov setting", the p-adic iterated integral attached to (f, g, h) is classical, and we have the following

Conjecture (Lauder, Rotger, D)

$$e_g(d^{-1} f^{[p]} \times h) = \frac{R_p(E, V_{gh})}{\log_p(u_g)} \times g,$$

where

- $R_p(E, V_{gh})$ *is a p-adic elliptic regulator attached to* (E, V_{gh})*;*
- u_g *is a specific Stark unit in the field cut out by* $\mathrm{Ad}(V_g)$*.*

Relaxing the Bellaiche-Dimitrov conditions

Theorem (Bellaiche, Dimitrov)

The weight one form g fails to satisfy the BD condition iff

1. *it is the theta series attached to a character of a real quadratic field in which p splits, or*
2. *g is irregular at p: $x^2 - a_p(g)x + \bar{\chi}(p)$ has a double root.*

Question: What can be said about the iterated integrals in these cases?

Numerical evidence reveals that $e_g(d^{-1}f^{[p]} \times h)$ is usually *not* classical.

The structure of $S_1^{(p)}(N, \chi)[[g]]$

First problem: to better understand the generalised eigenspace $S_1^{(p)}(N, \chi)[[g]]$ to which the iterated integrals belong.

1. What is its dimension?
2. Can one write down the fourier expansions of distinguished elements of $S_1^{(p)}(N, \chi)[[g]]$?
3. Can one describe the fourier expansion of $e_g(d^{-1} f^{[p]} \times h)$?

First case: g is regular, but does not satisfy BD

By Bellaiche-Dimitrov, $g = \theta_{\psi_g}$, where

$$\psi_g : \mathrm{Gal}(H/F) \longrightarrow L^\times \subset \mathbb{C}^\times$$

is a finite order character of *mixed signature* of a real quadratic field F in which $p = \mathfrak{p}\bar{\mathfrak{p}}$.

Replace θ_{ψ_g} by one of its (distinct) p-stabilisations:

$$U_p \theta_{\psi_g} = \alpha \theta_{\psi_g}, \qquad \alpha = \psi_g(\mathfrak{p}).$$

The Coleman-Mazur eigencurve at θ_{ψ_g}

Theorem (Cho-Vatsal, Bellaiche-Dimitrov)

The Coleman-Mazur eigencurve is smooth at the classical weight one point x_{ψ_g} attached to θ_{ψ_g}, but it is not étale *above weight space at this point.*

Proof: *Both the tangent space and the relative tangent space of the fiber above weight 1 at x_{ψ_g} are one-dimensional.* The proof uses the fact that the three irreducible constituents of

$$\mathrm{Ad}(\mathrm{Ind}_K^{\mathbb{Q}} \psi_g) = 1 \oplus \mathrm{Ad}^0(\mathrm{Ind}_K^{\mathbb{Q}} \psi_g) = 1 \oplus \chi_K \oplus \mathrm{Ind}_K^{\mathbb{Q}} \psi$$

occur with multiplicities $(0, 1, 0)$ in $\mathcal{O}_H^\times \otimes \mathbb{C}$. Here $\psi := \psi_g/\psi_g'$ is a *totally odd* ring class character of F, which plays a key role in the analysis.

Overconvergent generalised eigenforms

Recall that, in our setting, the natural inclusion

$$S_1(N,\chi)[\theta_{\psi_g}] \hookrightarrow S_1^{(p)}(N,\chi)[[\theta_{\psi_g}]]$$

is not surjective.

Definition

A modular form $\xi \in S_1^{(p)}(N,\chi)[[\theta_{\psi_g}]]$ which is not classical (i.e., not an eigenvector) is called an *overconvergent generalised eigenform* attached to θ_{ψ_g}. This generalised eigenform is said to be *normalised* if $a_1(\xi) = 0$.

The structure of $S_1^{(p)}(N, \chi)[[\theta_{\psi_g}]]$

Conjecture (Cho-Vatsal; Bellaiche-Dimitrov; Adel Betina)

The space $S_1^{(p)}(N, \chi)[[\theta_{\psi_g}]]$ is equal to $S_1^{(p)}(N, \chi)[I_g^2]$, i.e., it is two-dimensional.

If this conjecture is true, then $S_1^{(p)}(N, \chi)[[\theta_{\psi_g}]]$ is spanned by

- the classical normalised newform θ_{ψ_g};
- a normalised overconvergent generalised eigenform $\theta'_{\psi_g} \in S_1^{(p)}(N, \chi)[I_g^2]$, which is unique up to scaling.

Question: What is the fourier expansion of θ'_{ψ_g}?

Gross-Stark units

The fourier coefficients of θ'_{ψ_g} will involve $\mathfrak{p}$-adic logarithms of Gross-Stark ℓ-units for $\ell \neq p$.

These units arise in Gross's p-adic variant of the Stark conjecture on abelian L-series at $s = 0$:

Theorem (Dasgupta, Pollack, Ventullo, D)

Let $\psi : \mathrm{Gal}(H/F) \longrightarrow L^\times$ be a totally odd character of a totally real field F, and suppose that $\psi(\mathfrak{p}) = 1$ for some prime $\mathfrak{p}$ of F above p. Then there exists $u_p(\psi) \in (\mathcal{O}_H[1/p]^\times \otimes L)^\psi$ satisfying

$$L'_p(F, \psi, 0) \sim \log_p \mathrm{Norm}_{F_\mathfrak{p}/\mathbb{Q}_p}(u_p(\psi)).$$

The case of $\psi := \psi_g/\psi'_g$

The ring class character ψ is totally odd, and every prime ℓ which is inert in F splits completely in H/F.

Hence there is a non-trivial

$$u_\ell(\psi) \in (\mathcal{O}_H[1/\ell]^\times \otimes L)^\psi,$$

for all such inert primes, unique up to $L^\times$.

Using the Galois representation V_{ψ_g}, one can define *canonical* normalisations for $u_\ell(\psi)$.

The fourier expansion of θ'_{ψ_g}

Theorem (Alan Lauder, Victor Rotger, D)

The normalised generalised eigenform θ'_{ψ_g} attached to θ_{ψ_g} can be scaled in such a way that, for all primes $\ell \nmid N$,

$$a_\ell(\theta'_{\psi_g}) = \begin{cases} \log_{\mathfrak{p}} u_\ell(\psi) & \text{if } \ell \text{ is inert in } F; \\ 0 & \text{if } \ell \text{ is split in } F. \end{cases}$$

More generally, for all $n \geq 2$ with $\gcd(n, N) = 1$,

$$a_n(\theta'_{\psi_g}) = \sum_{\ell \mid n} \log_{\mathfrak{p}} u_\ell(\psi) \cdot (\mathrm{ord}_\ell(n) + 1) \cdot a_{n/\ell}(\theta_{\psi_g}).$$

An example in level $5 \cdot 29$

$\chi :=$ quartic Dirichlet character of conductor $5 \cdot 29$;

$S_1(5 \cdot 29, \chi)$ is one-dimensional, spanned by

$\theta_{\psi_g} = q + iq^4 + iq^5 + (-i-1)q^7 - iq^9 + (-i+1)q^{13} - q^{16} - q^{20} + \cdots$,

ψ_g a quartic character of $F = \mathbb{Q}(\sqrt{29})$ ramified at one of the primes above (5).

θ_{ψ_g} is not a CM theta series.

(Level 145 is the smallest where this happens.)

An example in level $5 \cdot 29$, cont'd

The prime $p = 13$ is split in K, and θ_ψ is regular.

Hence the BD condition fails.

$\psi = \psi_g/\psi'_g$ cuts out the ring class field of conductor 5: a cyclic quartic extension of K

$$H = K(\sqrt{5}, \delta) \qquad \text{where } \delta^2 = \frac{\sqrt{145} - 15}{32}.$$

$$\sigma(\sqrt{5}) = -\sqrt{5}, \qquad \sigma(\delta) = -\frac{1}{4}(3\sqrt{5} + \sqrt{29})\delta.$$

An example in level $5 \cdot 29$, cont'd

For $\ell = 2, 3, 11, 17$ and 19,

$$a_\ell(\theta'_{\psi_g}) = \log_{13}(u_\ell(\psi)),$$

where (denoting the group operation in $L \otimes H^\times$ additively)

$$u_\ell(\psi) := u_\ell + i \otimes \sigma(u_\ell) - \sigma^2(u_\ell) - i \otimes \sigma^3(u_\ell),$$

for a suitable ℓ-unit u_ℓ of H. The 2-unit u_2 is given by

$$u_2 := \frac{1}{2}(-\sqrt{5} - \sqrt{29} + 6)\delta + \frac{1}{8}(\sqrt{29} - 7)\sqrt{5} + \frac{1}{8}(\sqrt{29} + 1),$$

and the others are listed in the last column of the table

An example in level $5 \cdot 29$, cont'd

ℓ	$a_\ell(\theta'_{\psi_g}) \bmod 13^{20}$	u_ℓ
3	129151967993860501500007	$(\sqrt{5}+\sqrt{29}-4)\delta + \frac{1}{4}(\sqrt{29}-4)\sqrt{5} + \frac{1}{4}(2\sqrt{29}-13)$
11	352414331862757773284 2	$\left(\frac{1}{4}\left((\sqrt{29}+1)\sqrt{5}+(-\sqrt{29}+11)\right)\delta + \frac{1}{4}\left(\sqrt{5}-1\right)\right)^4$
17	229407992393437964510	$\left((16\sqrt{29}+84)\sqrt{5}+(36\sqrt{29}+200)\right)\delta$ $+\frac{1}{4}(11\sqrt{29}+63)\sqrt{5}+\frac{1}{4}(15\sqrt{29}+83)$
19	151428348278250799655 85	$\left(\frac{1}{4}\left((3\sqrt{29}-13)\sqrt{5}+(-15\sqrt{29}+85)\right)\delta\right.$ $\left.+\frac{1}{8}(3\sqrt{29}-15)\sqrt{5}+\frac{1}{8}(7\sqrt{29}-35)\right)^2$

Digression:

Overconvergent generalised eigenforms

and the Duke-Li Conjecture

A formula of Kudla-Rapoport-Yang

Theorem (Kudla-Rapoport-Yang)

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \pm 1$ *be an odd Dirichlet character of prime conductor* N*, let* $E_1(1,\chi)$ *be the associated weight one Eisenstein series, and let* $\tilde{E}_1(1,\chi)$ *be the derivative of its "incoherent" counterpart. For all* $n \geq 2$ *with* $\gcd(n, N) = 1$*,*

$$a_n(\tilde{E}_1(1,\chi)) = \frac{1}{2} \sum_{\ell | n} \log(\ell) \cdot (\mathrm{ord}_\ell(n) + 1) \cdot a_{n/\ell}(E_1(1,\chi)).$$

Theorem (Alan Lauder, Victor Rotger, D)

For all $n \geq 2$ *with* $\gcd(n, N) = 1$*,*

$$a_n(\theta'_{\psi_g}) = \frac{1}{2} \sum_{\ell | n} \log_{\mathfrak{p}} u_\ell(\psi) \cdot (\mathrm{ord}_\ell(n) + 1) \cdot a_{n/\ell}(\theta_{\psi_g}).$$

Generalised eigenforms and mock modular forms

Derivatives of incoherent Eisenstein series satisfy are special cases of the mock modular forms of Yingkun Li's lecture this morning.

Recall: If g is a classical weight one form, a *mock modular form* $g^\sharp$ attached to g is the holomorphic part of a WHMF having g as shadow.

Questions:

1. To what extent are overconvergent generalised eigenforms a good p-adic analogue of mock modular forms?

2. Is the fourier expansion of θ'_{ψ_g} a fragment of a "p-adic Kudla program"?

The Duke-Li conjecture

Conjecture (Bill Duke- Yingkun Li)

The fourier coefficients of the mock modular form $g^\sharp$ are simple linear combinations with algebraic coefficients of logarithms of algebraic numbers in the field which is cut out by $\mathrm{Ad}(V_g)$.

Many cases of this conjecture have been proved:

- by Duke-Li, Ehlen, Viazovska, when g is a CM theta series;

- by Li, when g is an RM theta series attached to a character ψ_g of mixed signature of a real quadratic field;

- some experimental evidence is gathered for this conjecture in the paper of Duke and Li, for an octahedral newform g of level 283.

The Duke-Li conjecture and explicit class field theory

If g is the theta series of character ψ_g of a quadratic field K, the Duke-Li conjecture expresses the fourier coefficients of $g^\sharp$ in terms of logarithms of algebraic numbers in H, where

• $H=$ the ring class field of K cut out by $\psi = \psi_g/\psi_g'$, if $\mathrm{disc}(K) < 0$;

• $H = K$, if $\mathrm{disc}(K) > 0$. This suggests that the fourier coefficients of $\theta^\sharp_{\psi_g}$ do not yield interesting class fields of K....

in contrast with what occurs when K is imaginary quadratic, or when $\theta^\sharp_{\psi_g}$ is replaced by its p-adic avatar θ'_{ψ_g}.

Remarks on [DLR] vs Duke-Li/Ehlen/Viazovska.

• The techniques in [DLR] are fundamentally p-adic in nature, relying on the theory of p-adic deformations of Galois representations, and on class field theory for H.

• These techniques are substantially simpler and less deep than those of Duke-Li, Ehlen, Viazovska: the theory of complex multiplication and singular moduli plays no role in [DLR].

• **Challenge**: Find a *more complicated* proof of [DLR], closer in spirit to the methods of Duke-Li, Ehlen, Viazovska; (eventually leading to new insights into explicit class field theory for real quadratic fields.)

• **Question**: How (if at all) are the fourier coefficients of θ'_{ψ_g} related to the real quadratic class invariants of Duke-Imamoglu-Toth and Kaneko?

End of digression

Revenons-en à nos moutons

(Back to the p-adic iterated integrals)

A conjecture in the smooth, non-étale setting

Conjecture (Lauder, Rotger, D)

$$e_{\theta_{\psi_g}}(d^{-1}f^{[p]} \times h) = \frac{R_p(E, V_{gh})}{\Omega_g} \times \theta'_{\psi_g} \pmod{S_1(N, \chi)[g]}$$

• *$R_p(E, V_{gh})$ is the same p-adic elliptic regulator attached to (E, V_{gh}) as in Victor's lecture;*
• *Ω_g is a p-adic invariant depending only on g and not on f and h.*

Special case: If $h = \theta_{\psi_h}$ attached to the same real quadratic F,

$V_{gh} = V_{\psi_1} \oplus V_{\psi_2}, \qquad \psi_1 = \psi_g\psi_h, \quad \psi_2 = \psi_g\psi'_h,$ and

$$R_p(E, V_{gh}) = \log_p(P_{E,\psi_1}) \cdot \log_p(P_{E,\psi_2}),$$

where P_{E,ψ_1} and P_{E,ψ_2} are analogous to Heegner points on E, but are defined over ring class fields of F.

The non-smooth (i.e., irregular) setting

Let g be an irregular weight one modular form. Then

$$S_1(N,\chi)[[g]] = S_1(N,\chi)[I_g^2] = \mathbb{C}_p g(q) \oplus \mathbb{C}_p g(q^p).$$

$$S_1^{(p)}(N,\chi)[[g]] = S_1(N,\chi)[[g]] \oplus S_1^{(p)}(N,\chi)[[g]]_{\mathrm{norm}}.$$

(An overconvergent generalised eigenform in $\xi \in S_1^{(p)}(N,\chi)[[g]]$ is said to be *normalised* if

$$a_1(\xi) = a_p(\xi) = 0.)$$

Conjecture (Lauder, Rotger, D)

The space $S_1^{(p)}(N,\chi)[[g]]$ is four-dimensional, i.e., $S_1^{(p)}(N,\chi)[[g]]_{\mathrm{norm}}$ is two-dimensional.

Describing $S_1^{(p)}(N, \chi)[[g]]_{\mathrm{norm}}$

Let

$$W_g = \mathrm{Ad}^0(V_g).$$

- Inner product: $\langle A, B\rangle := trace(AB)$,
- Lie bracket: $[A, B] = AB - BA$,
- Determinant function: $\det(A, B, C) := \langle A, [B, C]\rangle$.

Units and p-units

Let H be the field cut out by W_g, and $G := \mathrm{Gal}(H/\mathbb{Q})$.

Dirichlet unit theorem: $\dim_L(\mathcal{O}_H^\times \otimes W_g)^G = 1$,

$$\dim_L(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^G = \begin{cases} 2 & \text{if } g \text{ is regular at } \ell; \\ 4 & \text{if } g \text{ is irregular at } \ell \end{cases}$$

Fix a generator

$$u_g \in \log_p((\mathcal{O}_H^\times \otimes W_g)^G) \in W_g \otimes_L \mathbb{C}_p.$$

For each regular prime ℓ, the representation V_g gives an element

$$u_g(\ell) \in \log_p(\mathcal{O}_H[1/\ell]^\times \otimes W_g)^G) \in W_g \otimes_L \mathbb{C}_p,$$

which is well defined up to translation by multiples of u_g.

A conjectural description of $S_1^{(p)}(N,\chi)[[g]]_{\mathrm{norm}}$

Conjecture (Lauder, Rotger, D)

There exists an isomorphism

$$\Phi : \frac{W_g \otimes_L \mathbb{C}_p}{\mathbb{C}_p \cdot u_g} \longrightarrow S_1^{(p)}(N,\chi)[[g]]_{\mathrm{norm}}$$

satisfying, for all $\ell \nmid Np$*,*

$$a_\ell(\Phi(w)) = \begin{cases} \det(w, u_g, u_g(\ell)) & \text{if } g \text{ is regular at } \ell; \\ 0 & \text{if } g \text{ is irregular at } \ell. \end{cases}$$

The fourier expansion of $\Phi(w)$ *can be written down fully.*

The elliptic regulator $R_p(E, V_{gh})$

The elliptic regulator of Victor's lecture depends on the U_p-eigenvalue for g, and is ill-defined when g is irregular.

Instead we set $R_p(E, V_{gh}) = 0$ if $\dim_L((E(H_{gh}) \otimes V_{gh})^G) \neq 2$, and consider the sequence of maps

$$\bigwedge^2((E(H_{gh}) \otimes V_{gh})^G) \longrightarrow (\mathrm{Sym}^2 E(H_{gh}) \otimes \bigwedge^2 V_{gh})^G$$

$$\xrightarrow{\quad p_g \quad} (\mathrm{Sym}^2 E(H_{gh}) \otimes W_g)^G$$

$$\xrightarrow{\quad \log_{\mathfrak{p}}^{\otimes 2} \quad} W_g \otimes \mathbb{C}_p$$

Elliptic regulator: $R_p(E, V_{gh}) := \log_{\mathfrak{p}}^{\otimes 2} \circ p_g(P \wedge Q) \in W_g \otimes \mathbb{C}_p$.

A conjectural conjecture in the irregular setting

Conjecture (Lauder, Rotger, D)

For all irregular g,

$$e_g(d^{-1}f^{[p]} \times h) = \frac{1}{\Omega_g} \times \Phi(R_p(E, V_{gh})) \pmod{S_1(N,\chi)[[g]]},$$

where Ω_g is a p-adic invariant depending only on g and p, but not on f and h.

A conjectural conjecture on regulators of regulators

This conjecture implies that, for all primes ℓ that are regular for V_g,

$$\Omega_g \cdot a_\ell(e_g(d^{-1} f^{[p]} \times h)) \sim_{L^\times}$$

$$\det \begin{pmatrix} \begin{vmatrix} \log_{\mathfrak{p}} P_1 & \log_{\mathfrak{p}} P_2 \\ \log_{\mathfrak{p}} Q_1 & \log_{\mathfrak{p}} Q_2 \end{vmatrix} & \begin{vmatrix} \log_{\mathfrak{p}} P_3 & \log_{\mathfrak{p}} P_4 \\ \log_{\mathfrak{p}} Q_3 & \log_{\mathfrak{p}} Q_4 \end{vmatrix} & \begin{vmatrix} \log_{\mathfrak{p}} P_5 & \log_{\mathfrak{p}} P_6 \\ \log_{\mathfrak{p}} Q_5 & \log_{\mathfrak{p}} Q_6 \end{vmatrix} \\ \log_{\mathfrak{p}} u_1 & \log_{\mathfrak{p}} u_2 & \log_{\mathfrak{p}} u_3 \\ \log_{\mathfrak{p}} u_1(\ell) & \log_{\mathfrak{p}} u_2(\ell) & \log_{\mathfrak{p}} u_3(\ell) \end{pmatrix},$$

with $\quad P_i, Q_j \in E(H_{gh}), \qquad u_j \in \mathcal{O}_H^\times, \qquad u_j(\ell) \in \mathcal{O}_H[1/\ell]^\times.$

Theoretical evidence

Suppose that $g = h$ is induced from a quartic ring class character ψ_g of an imaginary quadratic field K in which p splits.

Then $\psi = \psi_g/\psi'_g = \psi_g^2$ is a genus character.

$H = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$.

Theorem (Lauder, Rotger, D, in progress)

The conjectural conjecture is true, with

$$\Omega_g = \log_{\mathfrak{p}} u \times (\log_{\mathfrak{p}} v_1(p) - \log_{\mathfrak{p}} v_2(p)),$$

where

- *u is the fundamental unit of the real quadratic subfield of H;*
- *$v_1(p)$ and $v_2(p)$ are fundamental p-units of the two imaginary quadratic subfields of H.*

Theoretical evidence, cont'd

Theorem (Lauder, Rotger, D, in progress)

The conjectural conjecture is true, with

$$\Omega_g = \log_{\mathfrak{p}} u \times (\log_{\mathfrak{p}} v_1(p) - \log_{\mathfrak{p}} v_2(p))$$

Ingredients in the proof:

- Explicit p-adic deformations of g;
- The p-adic Gross-Zagier/Waldspurger formula of Bertolini, D, Prasanna;
- Katz's p-adic Kronecker limit formula;
- An "exceptional zero formula" for the Katz L-function, due to Ralph Greenberg.

Experimental evidence

A lot of experimental evidence for the conjecture has been gathered, using Alan Lauder's fast algorithm for computing the ordinary projection on a space of overconvergent modular forms.

Thank you for your attention!!