TOPOLOGICAL VECTOR SPACES

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Abstract. Motivated by applications to duality theorems for p-adic pro-étale cohomology of rigid analytic spaces, we study the category of Topological Vector Spaces in the setting of condensed mathematics. We prove that it contains, as full subcategories, both the category of (topologically) bounded algebraic Vector Spaces and the category of perfect complexes on the Fargues-Fontaine curve. Vector Spaces coming from p-adic pro-étale cohomology of smooth partially proper rigid analytic varieties are examples of sheaves belonging to the former category.

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1. Introduction

Let p be a prime and let K be a complete discrete valuation field with a perfect residue field, of mixed characteristic. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$.

Topological Vector Spaces (TVS, for short) were introduced in [12] to geometrize p-adic period morphisms for rigid analytic varieties, i.e., to make them vary over the category of perfectoid affinoids over C. Once geometrized these period morphisms acquire rigidity that could be exploited to extract actual comparison theorems from the long exact sequence provided by the basic comparison theorems of [11]. Topological Vector Spaces were defined as enriched presheaves on sympathetic spaces over C with values in locally convex vector spaces over $\mathbf{Q}_p$ (abelianized appropriately). The enrichment condition meant that we consider only presheaves with restriction maps sensitive to the topology on the mapping spaces between sympathetic spaces.

In this paper we study the condensed version of this category. This was mostly motivated by the application to duality results in [10], where the main examples of Topological Vector Spaces come from cohomology of quasi-coherent sheaves on the Fargues-Fontaine curve hence live in the condensed universe. In the condensed version, locally convex topological vector spaces are replaced by condensed (or solid) $\mathbf{Q}_p$-modules and the sympathetic spaces are replaced by strictly totally disconnected spaces. Roughly, this means that instead of imposing that the étale cohomology of these spaces with values in $\mathbf{Q}_p$ is trivial in degrees ≥ 1 we impose this triviality for any étale sheaf. The enrichment condition puts us in the setting of topologically enriched presheaves once we equip mapping spaces with a natural condensed structure. The key point of enrichment in this language is the enriched Yoneda Lemma (see Section 3.1.5), which allows us to work homologically with

Date: October 6, 2025.
The authors's research was supported in part by the Simons Collaboration on Perfection in Algebra, Geometry, and Topology. P.C. and W.N.'s research was also supported in part by the grant ANR-19-CE40-0015-02 COLOSS.

topologically enriched presheaves in a way akin to algebraic presheaves. We note that this fails for non-enriched topological presheaves.

Topological Vector Spaces are targets of two natural functors: one from Vector Spaces[1] (VS, for short) and the other one from quasi-coherent sheaves on the Fargues-Fontaine curve. The main result of the paper is the following theorem (see Section 4 for the notation):

Theorem 1.1. *Let $S \in \mathrm{sPerf}_C$.*

(1) (Enriched fully-faithfulness) *The canonical functor from Vector Spaces to Topological Vector Spaces*
$$\mathrm{R}\pi_* : \mathcal{D}(S_{\text{proét}}, \mathbf{Q}_p) \to \underline{\mathcal{D}}(S, \mathbf{Q}_p)$$
tends to be fully faithful. More precisely, let $\mathcal{F} \in \mathcal{D}^b(S_{\text{proét}}, \mathbf{Q}_p)$ be such that $\mathrm{R}\pi_*\mathcal{F} \in \underline{\mathcal{D}}^b(S, \mathbf{Q}_p)$ and let $\mathcal{G} \in \mathcal{D}^+(S_{\text{proét}}, \mathbf{Q}_p)$. Then the canonical morphism in $\underline{\mathcal{D}}(S, \mathbf{Q}_p)$
$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\mathrm{R}\pi_*\mathcal{F}, \mathrm{R}\pi_*\mathcal{G})$$
is a quasi-isomorphism.

(2) (Fargues-Fontaine fully-faithfulness) *The functor*
$$\mathrm{R}\tau_* : \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) \to \underline{\mathcal{D}}(S, \mathbf{Q}_p)$$
is fully faithful when restricted to perfect complexes. That is, for $\mathcal{F}, \mathcal{G} \in \mathrm{Perf}(X_{\mathrm{FF},S^\flat})$, the natural map in $\mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p(S)})$
$$\mathrm{RHom}_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\underline{\mathrm{Hom}}_{S,\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}, \mathrm{R}\tau_*\mathcal{G})$$
is a quasi-isomorphism.

(3) (Compatibility of the algebraic and topological projections) *The functor*
$$\mathrm{R}\tau'_* : \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) \to \mathcal{D}(S_{\text{proét}}, \mathbf{Q}_p)$$
is compatible with the functor $\mathrm{R}\tau_*$ when restricted to nuclear sheaves. That is, the following diagram commutes
$$\begin{tikzcd}
\mathrm{Nuc}(X_{\mathrm{FF},S^\flat}) \arrow[r, "\mathrm{R}\tau'_*"] \arrow[dr, "\mathrm{R}\tau_*"'] & \mathcal{D}(S_{\text{proét}}, \mathbf{Q}_p) \arrow[d, "\mathrm{R}\pi_*"] \\
& \underline{\mathcal{D}}(S, \mathbf{Q}_p).
\end{tikzcd}$$

The first claim of the theorem states that the canonical functor from Vector Spaces to Topological Vector Spaces is fully faithful (in the enriched sense) once one imposes certain finiteness conditions. This follows from the fact that algebraic Yoneda sheaves associated to strictly totally disconnected spaces map to the topological ones and we have Yoneda Lemmas in both settings. The second claim is a topological analog of a result of Anschütz-Le Bras from [3] and follows from the latter, the third claim of the theorem, and the following immediate corollary of the first claim of the theorem:

Corollary 1.2. *Let $\mathcal{F}, \mathcal{G} \in \{\mathbb{G}_a, \underline{\mathbf{Q}}_p\}$. Then we have a natural quasi-isomorphism*
$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{S,\mathbf{Q}_p}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathcal{F}, \mathcal{G}).$$

Remark 1.3. (1) The above theorem and corollary mean that we can compute extensions of Banach-Colmez spaces or on the Fargues-Fontaine curve, or in VS's, or in TVS's. Hence, in $\underline{\mathcal{D}}(S, \mathbf{Q}_p)$:
$$\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathbf{Q}_p, \mathbf{Q}_p) \xleftarrow{\sim} \mathbf{Q}_p, \quad \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathbf{Q}_p, \mathbb{G}_a) \xleftarrow{\sim} \mathbb{G}_a,$$
$$\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathbb{G}_a \oplus \mathbb{G}_a[-1], \quad \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathbb{G}_a, \mathbf{Q}_p) \simeq \mathbb{G}_a(-1)[-1].$$

[1]That is, $\underline{\mathbf{Q}}_p$-sheaves on $\mathrm{Perf}_{C,\text{proét}}$, the pro-étale site of perfectoid affinoids over C.

More generally, the first claim of the theorem can be applied to complexes of VS's representing pro-étale cohomology of smooth partially proper rigid analytic varieties (see Corollary 4.8).

(2) The second claim of the theorem holds in greater generality than stated though we did not find a clean general statement.

(3) In the applications to geometric dualities in [10] we work mostly with the solid version of TVS's, which allows us to do computations objectwise in the category of solid $\mathbf{Q}_p$-vector spaces.

(4) There is also a solid version of VS's introduced by Fargues-Scholze in [13] (see also [4]). Certainly, there is a close relationship between these versions of VS's and TVS's but we do not study it here.

Organization of the paper. We discuss three categories of (pre)sheaves on the category of perfectoid affinoids over C that expand the category of Banach-Colmez spaces in various directions, by removing finiteness conditions:

– The VS–category (where VS stands for Vector Spaces), whose objects are sheaves of $\underline{\mathbf{Q}}_p$-vector spaces (thus without topology) on perfectoid affinoids over C equipped with pro-étale topology. This topology "encodes" a topology on the VS's.

– The NTVS–category (where NTVS stands for Naive Topological Vector Spaces), whose objects are (pre)sheaves of topological $\underline{\mathbf{Q}}_p$-vector spaces; ("topological" in the sense of Clausen-Scholze) on the category of perfectoid affinoids over C equipped with pro-étale topology. There is a canonical projection from the VS-category to the NTVS-category: the objectwise topology is induced from the topology encoded by the pro-étale site.

– The TVS–category (where TVS stands for Topological Vector Spaces), which is the topologically enriched category of presheaves NTVS's: we use the natural topology on the Hom-spaces between perfectoid spaces to impose a continuity property on the presheaf functors. To make this work we need to restrict perfectoid affinoids to the strictly totally disconnected ones. There is a canonical projection from the VS-category to the TVS-category: the objectwise topology is induced from the topology encoded by the pro-étale site and such objects are canonically enriched.

The key point to keep in mind is that the categories VS and TVS have Yoneda Lemmas and suitable generating (projective) Yoneda objects correspond under the canonical projection from VS to TVS.

The category VS was studied before; here we introduce the categories NTVS and TVS. This is done at a rather pedestrian pace allowing us to get familiar with these new objects. Section 2 is devoted to VS's and NTVS's, Section 3 to TVS's, and Section 4 to fully-faitfulness.

Acknowledgments. We are immensely grateful to Akhil Mathew for many hours of discussions on the constructions presented in this paper. We would also like to thank Johannes Anschütz for helpful comments on a draft of this paper and Ko Aoki, Guido Bosco, Juan Esteban Rodriguez Camargo, Dustin Clausen, David Hansen, Shizhang Li, Zhenghui Li, Lucas Mann, Emanuel Reinecke, Peter Scholze, and Bogdan Zavyalov for discussions concerning the content of this paper.

Parts of this paper were written during the authors' stay at the Hausdorff Research Institute for Mathematics in Bonn, in the Summer of 2023, and at IAS at Princeton in the Spring 2024. We would like to thank these Institutes for their support and hospitality.

Notation and conventions. Let p be a fixed prime, $\mathbf{Q}_p$ – the p-adic rational numbers. Let K be a complete discrete valuation field with a perfect residue field, of mixed characteristic. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. We will denote by $\mathbf{B}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the crystalline, semistable, and de Rham period rings of Fontaine.

The category of affinoid perfectoid spaces over an affinoid perfectoid space S over C will be denoted by Perf_S. The pro-étale site of S will be the category Perf_S endowed with the quasi-pro-étale topology and will be denoted by $S_{\mathrm{pro\acute{e}t}}$.

We denote by Cond, CondAb, Solid the condensed sets, condensed abelian groups, and solid abelian groups, respectively. To simplify arguments, we fix a cut-off cardinal κ and assume that perfectoid spaces and condensed sets are κ-small.

2. Naive Topological Vector Spaces

We introduce and study here the category of Naive Topological Vector Spaces. This is a category of topological sheaves, a simpler version of the category of Topological Vector Spaces that we are mostly interested in retaining however many of its properties hence allowing us a gentler introduction to our main object of study.

We start with the introduction of topological pro-étale site and examples of sheaves on it. Then we introduce the NTVS-category of sheaves with values in condensed and solid abelian groups. We prove that they form Grothendieck abelian categories, study their ∞-derived categories, the solidification functors, and monoidal structures. Finally, we extend all of this to the categories of $\underline{\mathbf{Q}}_p$-modules.

2.1. **Topological pro-étale site.** In this section, we introduce pro-étale topological sheaves.

2.1.1. *Totally disconnected spaces.* This brief review is based on [26, Sec. 7] and [20].

- A perfectoid space X is *totally disconnected* (see [26, Def. 7.1, Lemma 7.2, Lemma 7.3]) if it is qcqs and every open cover splits. Equivalently, if all connected components of X are of the form $\mathrm{Spa}(K,K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+\subset K$. Also equivalently, if for all sheaves of abelian groups $\mathcal{F}$ on X , one has $H^i(X,\mathcal{F})=0$, for $i>0$. By [26, Lemma 7.5], a totally disconnected space X is always an affinoid.

  A perfectoid space X is *w-local* (see [26, Def. 7.4]) if it is totally disconnected and the subset $X^c\subset X$ of closed points is closed.
- A perfectoid space X is *strictly totally disconnected* (see [26, Def. 7.15]) if it is qcqs and every étale cover splits. Equivalently, if every connected component is of the form $\mathrm{Spa}(K,K^+)$, where K is algebraically closed. We note that every profinite set T is strictly totally disconnected and that, by [26, Lemma 7.19], the product $X\times T$ of a strictly totally disconnected space and a profinite set is strictly totally disconnected.

  More generally (see [26, Def. 7.17], a *w-strictly local* perfectoid space is a w-local perfectoid space X such that for all $x\in X$, the completed residue field $K(x)$ is algebraically closed.
- A *w-contractible space* (see [20, Def. 1.1]) is a w-strictly local perfectoid space X such that $\pi_0(X)$ is extremally disconnected. By [20, Lemma 1.2], if X is a diamond, the w-contractible spaces in $X_{\text{proét}}$ form a basis of $X_{\text{proét}}$. Moreover, if $U\in X_{\text{proét}}$ is w-contractible then every pro-étale cover of U admits a section. In particular, for any sheaf $\mathcal{F}$ of abelian groups on $X_{\text{proét}}$ we have $H^i(U_{\text{proét}},\mathcal{F})=0$, for $i>0$, and $X_{\text{proét}}$ is replete.

2.1.2. *Topological pro-étale site.* Let S be an affinoid perfectoid over C. Let $S^{\text{top}}_{\text{proét}}:=S_{\text{proét}}\times *_{\text{proét}}$ be the topological pro-étale site of S: its objects are pairs (Y,T), where $Y\in\mathrm{Perf}_S$, and T is a profinite set, and pro-étale coverings are given by maps $(Y',T')\to(Y,T)$, where $Y'\to Y$ is a pro-étale covering and $T'\to T$ is a surjective map. We note that sheaves of sets on $S^{\text{top}}_{\text{proét}}$ form the category $\mathrm{Sh}(S_{\text{proét}},\mathrm{Cond})$ of sheaves on $S_{\text{proét}}$ with values in Cond. Similarly for sheaves of abelian groups: we get the category $\mathrm{Sh}(S_{\text{proét}},\mathrm{CondAb})$.

We have maps of sites
$$S_{\text{proét}}\xrightarrow{\pi}S^{\text{top}}_{\text{proét}}\xrightarrow{\eta}S_{\text{proét}}$$
given by $Y\times\widehat{T}\leftarrow(Y,T)$, $(Y,*)\leftarrow Y$, respectively. We note that $\eta\pi=\mathrm{Id}$. Here $\widehat{T}=\mathrm{Spa}(\mathcal{C}(T,C),\mathcal{C}(T,\mathcal{O}_C))$ is the adic space (an affinoid perfectoid) associated to T. (We will drop the hat from T if

this does not cause confusion). We define the following pushforward functors $\pi_*: \mathrm{Sh}(S_{\text{proét}}) \to \mathrm{Sh}(S^{\text{top}}_{\text{proét}}), \eta_*: \mathrm{Sh}(S^{\text{top}}_{\text{proét}}) \to \mathrm{Sh}(S_{\text{proét}})$:

$$\pi_*\mathcal{F}: \{(Y,T) \mapsto \mathcal{F}(Y \times T)\}, \quad \mathcal{F} \in \mathrm{Sh}(S_{\text{proét}});$$
$$\eta_*\mathcal{F}: \{Y \mapsto \mathcal{F}(Y,*)\}, \quad \mathcal{F} \in \mathrm{Sh}(S^{\text{top}}_{\text{proét}}).$$

We have $\eta_*\pi_* = \mathrm{Id}_*$. Thus the functor π_* is faithful.

If X is a topological space, we use the standard notation $\underline{X}$ to denote the associated condensed set, i.e., the sheaf $T \mapsto \mathcal{C}(T,X)$ on $*_{\text{proét}}$.

Example 2.1. We present some examples of topological sheaves on $S \in \mathrm{Perf}_C$.

(1) *The constant sheaf* $\underline{\mathbf{Q}}_p$. Recall that the constant sheaf $\underline{\mathbf{Q}}_p$ on $S_{\text{proét}}$ is defined via

$$\underline{\mathbf{Q}}_p : \{Y \mapsto \mathcal{C}(|Y|, \mathbf{Q}_p)\}.$$

One can see that, a priori just a presheaf, it is actually a sheaf (see [20, Lemma 3.14]). We define the presheaf on $S^{\text{top}}_{\text{proét}}$

$$\underline{\mathbf{Q}}_p : \{Y \mapsto \underline{\mathcal{C}(|Y|, \mathbf{Q}_p)}\},$$

where $\mathcal{C}(|Y|,\mathbf{Q}_p)$ is equipped with the compact open topology. We claim that $\pi_*\underline{\mathbf{Q}}_p \simeq \underline{\mathbf{Q}}_p$ (as presheaves for now). Indeed, we compute:

$$\pi_*\underline{\mathbf{Q}}_p : \{(Y,T) \mapsto \underline{\mathbf{Q}}_p(Y \times T) = \mathcal{C}(|Y \times T|, \mathbf{Q}_p)\},$$
$$\underline{\mathbf{Q}}_p : \{(Y,T) \mapsto \mathcal{C}(T, \mathcal{C}(|Y|, \mathbf{Q}_p))\}.$$

But, since $|Y| \times T \xrightarrow{\sim} |Y \times T|$, the natural map

$$\mathcal{C}(|Y \times T|, \mathbf{Q}_p) \to \mathcal{C}(T, \mathcal{C}(|Y|, \mathbf{Q}_p))$$

is a bijection[2]. It follows that $\underline{\mathbf{Q}}_p$ is a sheaf (since so is $\pi_*\underline{\mathbf{Q}}_p$).

We note that the value $\underline{\mathbf{Q}}_p(Y) \in \mathrm{Solid}$. Indeed, if Y is a strictly totally disconnected[3] perfectoid space then we have (see the proof of [20, Cor. 3.15] for an argument)

$$\underline{\mathbf{Q}}_p(Y) = \underline{\mathcal{C}(|Y|, \mathbf{Q}_p)} \xleftarrow{\sim} \underline{\mathcal{C}(\pi_0(Y), \mathbf{Q}_p)},$$

a Banach space since $\pi_0(Y)$ is profinite. Our claim now follows because solid abelian groups are closed under taking kernels.

If W is a Hausdorff locally convex topological vector space over $\mathbf{Q}_p$, we can analogously define constant sheaves $\underline{W}$ on $S^{\text{top}}_{\text{proét}}$. We have that $\underline{W}(Y) \in \mathrm{Solid}$. In what follows, if this does not cause confusion, we simply write W in place of $\underline{W}$. Similarly, on $S_{\text{proét}}$ and $S^{\text{top}}_{\text{proét}}$, if W is profinite, we have the constant sheaf $\underline{W}$; on $S_{\text{proét}}$ it is represented by the affinoid perfectoid space $\widehat{W}$.

(2) *The sheaf* $\mathbb{G}_a$. On $S_{\text{proét}}$ and $S^{\text{top}}_{\text{proét}}$ we have the presheaves

$$\mathbb{G}_a : \{Y = \mathrm{Spa}(R_Y, R_Y^+) \mapsto R_Y\}, \quad \mathbb{G}_a^{\text{top}} : \{Y = \mathrm{Spa}(R_Y, R_Y^+) \mapsto \underline{R_Y}\},$$

respectively, where R_Y is equipped with its natural Banach topology. The first presheaf is a sheaf by [26, Th. 8.7]. We claim that $\pi_*(\mathbb{G}_a) \simeq \mathbb{G}_a^{\text{top}}$ (as presheaves). Indeed, we have

$$\pi_*\mathbb{G}_a : \{(Y,T) \mapsto \mathbb{G}_a(Y \times T)\}, \quad \mathbb{G}_a^{\text{top}} : \{(Y,T) \mapsto \mathcal{C}(T, R_Y)\}.$$

But, since R_Y is a Banach space, by [22, Cor. 10.5.4], we have a natural isomorphism

$$\mathcal{C}(T, R_Y) \xleftarrow{\sim} \mathbb{G}_a(Y \times T) = \underline{R}_Y \otimes^{\square}_{\mathbf{Q}_p} \mathcal{C}(T, \mathbf{Q}_p).$$

It follows that $\mathbb{G}_a^{\text{top}}$ on $S^{\text{top}}_{\text{proét}}$ is a sheaf (as so is $\pi_*\mathbb{G}_a$). We will skip the superscript $(-)^{\text{top}}$ if this does not cause confusion.

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[2] Since $\mathbf{Q}_p$ is Hausdorff, the exponential law of mapping space with compact-open topology holds for locally compact (not necessary Hausdorff spaces).

[3] Recall that this means that any *étale* covering has a section.

(3) *Period sheaves.* For $i \geq 0$, on $S^{\mathrm{top}}_{\mathrm{proét}}$ we have the presheaves

$$\mathbb{B}^+_{\mathrm{dR}}/t^i : \{Y \mapsto \mathbf{B}^+_{\mathrm{dR}}(Y)/t^i\}, \quad \mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}} : \{Y \mapsto \mathbf{B}^{+,\varphi=p^i}_{\mathrm{cr}}(Y)\}, \tag{2.2}$$

where the (objectwise) period rings are equipped with their canonical topologies. These are sheaves of (solid) abelian groups. Indeed, for the first presheaf this follows, by dévissage on i, from the second example discussed above; for the second one – from the exact sequence of presheaves

$$0 \to \underline{\mathbf{Q}}_p(i) \to \mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}} \to \mathbb{B}^+_{\mathrm{dR}}/t^i \to 0 \tag{2.3}$$

and the two examples above.

Similarly, we have period sheaves $\mathbb{B}^+_{\mathrm{dR}}/t^i$ and $\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}$ on $S_{\mathrm{proét}}$ (by forgetting the topology on the period rings in (2.2)). We claim that there are natural isomorphisms

$$\pi_*\mathbb{B}^+_{\mathrm{dR}}/t^i \xleftarrow{\sim} \mathbb{B}^+_{\mathrm{dR}}/t^i, \quad \pi_*\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}} \xleftarrow{\sim} \mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}. \tag{2.4}$$

Indeed, we have

$$\pi_*(\mathbb{B}^+_{\mathrm{dR}}/t^i) = \{(Y,T) \mapsto \mathbb{B}^+_{\mathrm{dR}}(Y \times T)/t^i\}, \quad \pi_*\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}} = \{(Y,T) \mapsto \mathbb{B}^+_{\mathrm{cr}}(Y \times T)^{\varphi=p^i}\}$$

and the first map in (2.4) can be defined as the composition

$$\mathbb{B}^+_{\mathrm{dR}}(Y \times T)/t^i = \mathbb{B}^+_{\mathrm{dR}}(R_{Y\times T})/t^i \xrightarrow{\sim} \mathbb{B}^+_{\mathrm{dR}}(\mathcal{C}(T,R_Y))/t^i \xrightarrow{\sim} \mathcal{C}(T,\mathbb{B}^+_{\mathrm{dR}}(R_Y)/t^i) = \mathcal{C}(T,\mathbb{B}^+_{\mathrm{dR}}(Y)/t^i).$$

Here the third map is constructed by going back to the definitions of period rings; it is seen to be an isomorphism by dévissage reducing it to the case of $i = 1$, which was shown above. We argue similarly for $\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}$.

The following lemma shows that the sheaves $\underline{\mathbf{Q}}_p, \mathbb{G}_a$ and the period sheaves $\mathbb{B}^+_{\mathrm{dR}}/t^i$, $\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}$ are acyclic for the functor π_*:

Lemma 2.5. *Let $\mathcal{F} \in \{\underline{\mathbf{Q}}_p, \mathbb{G}_a, \mathbb{B}^+_{\mathrm{dR}}/t^i, \mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}\}$ be a sheaf on $S_{\mathrm{proét}}$. We have $\mathrm{R}^i\pi_*\mathcal{F} = 0$, for $i > 0$. In particular, we have a natural quasi-isomorphism $\pi_*\mathcal{F} \xrightarrow{\sim} \mathrm{R}\pi_*\mathcal{F}$ in $\mathcal{D}(S_{\mathrm{proét}}, \mathrm{CondAb})$.*

Proof. It suffices to show that

$$H^i_{\mathrm{proét}}(Y \times T, \mathcal{F}) = 0, \quad i > 0, \tag{2.6}$$

for a strictly totally disconnected perfectoid space Y over C and a profinite set T. (We note that $Y \times T$ is a strictly totally disconnected perfectoid.)

If $\mathcal{F} = \mathbb{G}_a$ (2.6) follows from Tate acyclicity for perfectoid affinoid spaces for pro-étale topology (see [26, Prop. 8.8]). By dévissage it yields the result for $\mathbb{B}^+_{\mathrm{dR}}/t^i$.

If $\mathcal{F} = \underline{\mathbf{Q}}_p$, we compute (set $Z := Y \times T$):

$$\mathrm{R}\Gamma_{\mathrm{proét}}(Z, \underline{\mathbf{Q}}_p) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{proét}}(Z, \underline{\mathbf{Z}}_p)[1/p], \quad \mathrm{R}\Gamma_{\mathrm{proét}}(Z, \underline{\mathbf{Z}}_p) \xrightarrow{\sim} \mathrm{R}\lim_n \mathrm{R}\Gamma_{\mathrm{proét}}(Z, \mathbf{Z}/p^n),$$
$$\mathrm{R}\Gamma_{\mathrm{proét}}(Z, \mathbf{Z}/p^n) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{ét}}(Z, \mathbf{Z}/p^n).$$

The first quasi-isomorphism holds because Y is quasi-compact; the third quasi-isomorphism follows from [26, Prop. 14.7, Prop. 14.8]. And $H^i_{\mathrm{ét}}(Z, \mathbf{Z}/p^n)$, $i > 0$, because Z is strictly totally disconnected. This yields (2.6).

Combining the above with the fundamental exact sequence (2.3), we get the claim of the lemma for $\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}$. □

2.2. **Naive Topological Vector Spaces.** We will discuss here the topological sheaves on $S_{\mathrm{proét}}$, for $S \in \mathrm{Perf}_C$, in the condensed and solid settings.

2.2.1. *Condensed and solid modules.* For the convenience of the reader we will recall briefly the basic properties of condensed and solid modules.

Proposition 2.7. (Clausen-Scholze [8, Prop. 7.5]) *Let A be an analytic ring with an underlying condensed ring $\underline{A}$.*

(1) The full subcategory of solid A-modules (inside the category of condensed $\underline{A}$-modules)

$$\mathrm{Mod}_A^{\mathrm{solid}} \subset \mathrm{Mod}_{\underline{A}}^{\mathrm{cond}} \tag{2.8}$$

is a Grothendieck abelian subcategory, stable under all limits, colimits, and extensions. The inclusion (2.8) *admits a left adjoint*

$$\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}} \to \mathrm{Mod}_A^{\mathrm{solid}} : \quad M \mapsto M \otimes_{\underline{A}} A, \tag{2.9}$$

which preserves all colimits and is symmetric monoidal.

(2) The canonical functor

$$\mathcal{D}(\mathrm{Mod}_A^{\mathrm{solid}}) \to \mathcal{D}(\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}}) \tag{2.10}$$

is fully faithful. It preserves all limits and colimits. A complex $M \in \mathcal{D}(\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}})$ is in $\mathcal{D}(\mathrm{Mod}_A^{\mathrm{solid}})$ if and only if $H^i(M)$ is in $\mathrm{Mod}_A^{\mathrm{solid}}$, for all i. The functor (2.10) *admits a left adjoint*

$$\mathcal{D}(\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}}) \to \mathcal{D}(\mathrm{Mod}_A^{\mathrm{solid}}) : M \mapsto M \otimes_{\underline{A}}^{\mathrm{L}} A, \tag{2.11}$$

which is the left derived functor of (2.9)*. It is symmetric monoidal.*

(3) For $M, N \in \mathcal{D}(\mathrm{Mod}_A^{\mathrm{solid}})$, we have the derived internal Hom

$$\mathrm{R}\underline{\mathrm{Hom}}_A(M,N) \in \mathcal{D}(\mathrm{Mod}_A^{\mathrm{solid}}).$$

The natural map $\mathrm{R}\underline{\mathrm{Hom}}_A(M,N) \to \mathrm{R}\underline{\mathrm{Hom}}_{\underline{A}}(M,N)$ is a quasi-isomorphism.

Remark 2.12. In the context of the above proposition, we set $M^{\square} := M \otimes_{\underline{A}} A$, for $M \in \mathrm{Mod}_{\underline{A}}^{\mathrm{cond}}$, and $M^{\mathrm{L}\square} := M \otimes_{\underline{A}}^{\mathrm{L}} A$, for $M \in \mathcal{D}(\mathrm{Mod}_{\underline{A}}^{\mathrm{cond}})$.

2.2.2. *Condensed sheaves.* We list here properties of topological sheaves with values in condensed abelian groups.

(•) *Sheaves of condensed abelian groups.* Consider first the category $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}}, \mathrm{CondAb}) \simeq \mathrm{Sh}(S_{\mathrm{pro\acute{e}t}}^{\mathrm{top}}, \mathrm{Ab})$ of sheaves with values in condensed abelian groups. It is a Grothendieck abelian category, which inherits a closed symmetric monoidal structure from that of CondAb: For $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(S_{\mathrm{pro\acute{e}t}}, \mathrm{CondAb})$, their tensor product $\mathcal{F} \otimes \mathcal{G}$ is defined by sheafifying the presheaf tensor product

$$\{Y \mapsto \mathcal{F}(Y) \otimes \mathcal{G}(Y) \in \mathrm{CondAb}\}, \quad Y \in S_{\mathrm{pro\acute{e}t}}.$$

The internal Hom, $\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}, \mathcal{G})$, can be defined as a presheaf by

$$\{Y \mapsto \underline{\mathrm{Hom}}_{Y^{\mathrm{top}}}(\mathcal{F}_Y, \mathcal{G}_Y) \in \mathrm{CondAb}\}, \quad Y \in S_{\mathrm{pro\acute{e}t}},$$

where we set

$$\underline{\mathrm{Hom}}_{Y^{\mathrm{top}}}(\mathcal{F}_Y, \mathcal{G}_Y) := \ker\Big(\prod_{Y_1 \to Y} \underline{\mathrm{Hom}}(\mathcal{F}(Y_1), \mathcal{G}(Y_1)) \to \prod_{Y_2 \to Y_1} \underline{\mathrm{Hom}}(\mathcal{F}(Y_1), \mathcal{G}(Y_2))\Big). \tag{2.13}$$

It is actually a sheaf (see [25, Prop. 2.2.14]). It is the (internal) right adjoint of the tensor product [25, Prop. 2.2.16].

By Remark 3.9 and Lemma 3.24 below, the category $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}}, \mathrm{CondAb})$ is generated by the set

$$\{\mathbf{Z}[h_Y^{\delta}] \otimes \mathbf{Z}[T]\}, \quad Y \in \mathrm{Perf}_S, T - \text{ profinite set.} \tag{2.14}$$

Here the sheaf h_Y^{δ} is defined by $Y_1 \mapsto \mathrm{Hom}_S(Y_1, Y)$, where the Hom is given discrete topology.

Lemma 2.15. *The set* (2.14) *for w-contractible Y's and extremally disconnected T's generates* $\mathrm{Sh}(S_{\text{proét}}, \mathrm{CondAb})$. *Moreover, these generators are compact and projective.*

Proof. Take $\mathbf{Z}[h_Y^\delta]\otimes \mathbf{Z}[T]$ as in (2.14). For compactness we need to show that the functor $\mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Y^\delta]\otimes \mathbf{Z}[T],-)$ commutes with filtered colimits. But

(2.16)
$$\begin{aligned}\mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Y^\delta]\otimes \mathbf{Z}[T],\mathcal{F}) &\simeq \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Y^\delta],[\mathbf{Z}[T],\mathcal{F}])\simeq [\mathbf{Z}[T],\mathcal{F}](Y)(*)\simeq [\mathbf{Z}[T],\mathcal{F}(Y)](*)\\ &\simeq \mathrm{Hom}(\mathbf{Z}[T],\mathcal{F}(Y)),\end{aligned}$$

where we wrote $[\mathbf{Z}[T],\mathcal{F}]$ for the sheaf $Y_1\mapsto \underline{\mathrm{Hom}}(\mathbf{Z}[T],\mathcal{F}(Y_1))$ and the second isomorphism follows from the enriched Yoneda Lemma (see Section 3.1.5). But $\Gamma(Y,-)$ commutes with colimits since both Y and $\mathbf{Z}[T]$ are quasi-compact.

For projectvity, take a surjection $\mathcal{F}_1\to\mathcal{F}_2\to 0$. We need to show that the induced map

$$\mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Y^\delta]\otimes \mathbf{Z}[T],\mathcal{F}_1)\to \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Y^\delta]\otimes \mathbf{Z}[T],\mathcal{F}_2)$$

is surjective as well. But, using (2.16), this map can be written as

$$\mathrm{Hom}(\mathbf{Z}[T],\mathcal{F}_1(Y))\to \mathrm{Hom}(\mathbf{Z}[T],\mathcal{F}_2(Y))$$

and this is surjective by projectivity of $\mathbf{Z}[T]$ since the map $\mathcal{F}_1(Y)\to\mathcal{F}_2(Y)$ is surjective as Y is w-contractible. □

(•) *Sheaves of condensed $\underline{\mathbf{Q}}_p$-modules.* We will denote by $\mathrm{Sh}(S_{\text{proét}},\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p})$ the subcategory of $\mathrm{Sh}(S_{\text{proét}},\mathrm{CondAb})$ of $\underline{\mathbf{Q}}_p$-modules. It is again a Grothendieck abelian category (as the category of modules on a ringed site). Since we have $\mathrm{Sh}(S_{\text{proét}},\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p})\simeq \mathrm{Sh}(S^{\mathrm{top}}_{\text{proét}},\underline{\mathbf{Q}}_p)$, the classical theory of tensor products and internal Homs of sheaves on ringed sites (see [29, Tag 03A4]) yields the condensed tensor product and internal Hom: For $\mathcal{F},\mathcal{G}\in \mathrm{Sh}(S_{\text{proét}},\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p})$ their tensor product $\mathcal{F}\otimes_{\underline{\mathbf{Q}}_p}\mathcal{G}$ is defined by sheafifying the presheaf tensor product

$$\{Y\mapsto \mathcal{F}(Y)\otimes_{\underline{\mathbf{Q}}_p(Y)}\mathcal{G}(Y)\in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p(Y)}\},\quad Y\in S_{\text{proét}}.$$

The internal Hom, $\mathcal{H}om_{S^{\mathrm{top}},\underline{\mathbf{Q}}_p}(\mathcal{F},\mathcal{G})$, can be defined as a presheaf by

$$\{Y\mapsto \underline{\mathrm{Hom}}_{Y^{\mathrm{top}},\underline{\mathbf{Q}}_p}(\mathcal{F}_Y,\mathcal{G}_Y)\in \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p(Y)}\},\quad Y\in S_{\text{proét}},$$

where we set

(2.17)
$$\underline{\mathrm{Hom}}_{Y^{\mathrm{top}},\underline{\mathbf{Q}}_p}(\mathcal{F}_Y,\mathcal{G}_Y):=\ker\Big(\prod_{Y_1\to Y}\underline{\mathrm{Hom}}_{\mathbf{Q}_p(Y_1)}(\mathcal{F}(Y_1),\mathcal{G}(Y_1))\to\prod_{Y_2\to Y_1}\underline{\mathrm{Hom}}_{\mathbf{Q}_p(Y_1)}(\mathcal{F}(Y_1),\mathcal{G}(Y_2))\Big).$$

It is actually a sheaf [29, Tag 03EM] and it is the (internal) right adjoint of the tensor product [29, Tag 03EO].

The category $\mathrm{Sh}(S_{\text{proét}},\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p})$ is generated by the set

$$\{\underline{\mathbf{Q}}_p[h_Y^\delta]\otimes_{\underline{\mathbf{Q}}_p}\underline{\mathbf{Q}}_p[T]\},\quad Y\in\mathrm{Perf}_S,\ T-\text{ profinite set}.$$

And we have an analog of Lemma 2.15 in this setting.

(•) *Derived picture.* We have an analogous derived picture, which, again, follows from the classical story (see [29, 01FQ]). Namely, the existence of K-flat resolutions[4] yields the derived tensor product $\mathcal{F}\otimes^{\mathrm{L}}_{\underline{\mathbf{Q}}_p}\mathcal{G}$, for $\mathcal{F},\mathcal{G}\in\mathcal{D}(S_{\text{proét}},\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p})$ (see [29, Tag 06YU]). The internal Hom,

[4]Recall that K-flatness, and the related notion of K-injectivity, for unbounded complexes (originally of modules over a ring R) were introduced by Spaltenstein in [28] as an analogue of the classical notion of flatness (and injectivity) of modules.

$\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\underline{\mathbf{Q}}_p}(\mathcal{F},\mathcal{G})$, is defined using the existence of K-injective resolutions. It is the right (internal) adjoint of the derived tensor product [29, Tag 08J9].

We have analogous definitions and properties for sheaves with values in CondAb.

Remark 2.18. For $\mathcal{F}\in\mathcal{D}(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})$, we have in $\mathcal{D}(\mathrm{Ab})$

$$\mathrm{R}\Gamma(Y^{\mathrm{top}}_{\mathrm{pro\acute{e}t}},\mathrm{R}\pi_*\mathcal{F})(T)\simeq\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y\times T,\mathcal{F}).$$

This can be easily seen for Y, which are w-contractible and by pro-étale descent in Y-variable in general. In particular, $\mathrm{R}\Gamma(Y^{\mathrm{top}}_{\mathrm{pro\acute{e}t}},\mathrm{R}\pi_*\mathcal{F})(*)\simeq\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y,\mathcal{F})$. Similarly for $\underline{\mathbf{Q}}_p$-sheaves.

2.2.3. *Solid sheaves.* We pass now to topological sheaves with values in solid abelian groups.

($\bullet$) *Sheaves of solid abelian groups.* Recall (see Proposition 2.7) that the category Solid of solid abelian groups is a Grothendieck abelian category, which is stable under limits, colimits, and extensions in CondAb. It is also compactly generated. It follows that the category $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$ behaves as expected: it is a Grothendieck abelian category. Moreover, it is generated by the set

$$\{\mathbf{Z}[h^{\delta}_Y]\otimes^{\square}\mathbf{Z}[T]\},\quad Y\in\mathrm{Perf}_S,\ T-\ \text{profinite set.}\tag{2.19}$$

And we have an analog of Lemma 2.15 in this setting.

For $\mathcal{F},\mathcal{G}\in\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$, we define $\underline{\mathrm{Hom}}^{\square}_S(\mathcal{F},\mathcal{G})\in\mathrm{Solid}$ via the end definition as in (2.13). The set $\underline{\mathrm{Hom}}^{\square}_S(\mathcal{F},\mathcal{G})(*)$ is the usual Hom-set in the category $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$. The two categories of topological sheaves $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{CondAb})$ and $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$ are related:

Lemma 2.20. *The canonical forgetful functor* $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})\to\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{CondAb})$ *is (topologically) fully faithful and has a (topological) left adjoint $\mathcal{F}\mapsto\mathcal{F}^{\square}$ given by the sheafification of the presheaf*[5]*:*

$$\mathcal{F}^{\square,\mathrm{psh}}:\quad\{Y\mapsto\mathcal{F}(Y)^{\square}\},\quad Y\in S_{\mathrm{pro\acute{e}t}}.$$

It commutes with all colimits.

Proof. For the first claim, we need to show that, for $\mathcal{F},\mathcal{G}\in\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$, the canonical map in CondAb

$$\underline{\mathrm{Hom}}^{\square}_S(\mathcal{F},\mathcal{G})\to\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F},\mathcal{G})$$

is an isomorphism. But using the end definition of both sides this reduces to the fully-faithfulness of the category Solid in CondAb.

For the second claim, note that, from definitions, we get a natural maps (of presheaves on S_v with values in CondAb):

$$\mathcal{F}\to\mathcal{F}^{\square}.$$

We need to show that, for $\mathcal{F}\in\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{CondAb})$, $\mathcal{G}\in\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$, the induced morphism in CondAb

$$\underline{\mathrm{Hom}}^{\square}_S(\mathcal{F}^{\square},\mathcal{G})\xrightarrow{\sim}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F}^{\square},\mathcal{G})\to\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F},\mathcal{G})$$

is an isomorphism. By adjointness of the sheafification functor, we can pass from sheaves to presheaves and use the presheaf $\mathcal{F}^{\square,\mathrm{psh}}$ in place of $\mathcal{F}^{\square}$. Using the end definition, this reduces to showing that

$$\underline{\mathrm{Hom}}^{\square}(\mathcal{F}(Y_1)^{\square},\mathcal{G}(Y_2))\simeq\underline{\mathrm{Hom}}(\mathcal{F}(Y_1),\mathcal{G}(Y_2)),\quad Y_1,Y_2\in S_{\mathrm{pro\acute{e}t}},$$

which is clear. $\square$

The category $\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$ inherits a closed symmetric monoidal structure from that of Solid: For presheaves $\mathcal{F},\mathcal{G}\in\mathrm{PSh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Solid})$, their tensor product $\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G}$ is defined as the presheaf

$$\{Y\mapsto\mathcal{F}(Y)\otimes^{\square}\mathcal{G}(Y)\in\mathrm{Solid}\},\quad Y\in S_{\mathrm{pro\acute{e}t}}.$$

[5]The sheafification is necessary because the solidification functor $\square$ is not left exact.

For sheaves $\mathcal{F},\mathcal{G}\in \mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$, their tensor product $\mathcal{F}\otimes^{\square}\mathcal{G}$ is defined as the sheafification of the presheaf tensor product $\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G}$:

$$\mathcal{F}\otimes^{\square}\mathcal{G}:=(\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G})^{\sharp}.$$

We have the expected relations:

Lemma 2.21. *(1) For $\mathcal{F},\mathcal{G}\in \mathrm{PSh}(S_{\mathrm{proét}},\mathrm{Solid})$, we have a canonical isomorphism in* $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$

$$\mathcal{F}^{\sharp}\otimes^{\square}\mathcal{G}^{\sharp}\simeq(\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G})^{\sharp}.$$

(2) For $\mathcal{F},\mathcal{G}\in \mathrm{PSh}(S_{\mathrm{proét}},\mathrm{CondAb})$, we have a canonical isomorphism in $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$

$$(\mathcal{F}^{\sharp}\otimes\mathcal{G}^{\sharp})^{\square}\simeq((\mathcal{F}\otimes^{\mathrm{psh}}\mathcal{G})^{\square,\mathrm{psh}})^{\sharp}.$$

Proof. We will reduce to universal properties of the terms of the isomorphisms. For $\mathcal{H}\in \mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$, we have

$$\begin{aligned}\mathrm{Hom}^{\square}_S((\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G})^{\sharp},\mathcal{H})&\simeq \mathrm{Hom}_{\mathrm{psh}}(\mathcal{F}\otimes^{\square,\mathrm{psh}}\mathcal{G},\mathcal{H})\simeq \mathrm{Hom}_{\mathrm{psh}}(\mathcal{F},\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{G},\mathcal{H}))\\ &\simeq \mathrm{Hom}_{\mathrm{psh}}(\mathcal{F}^{\sharp},\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{G},\mathcal{H}))\simeq \mathrm{Hom}_{\mathrm{psh}}(\mathcal{F}^{\sharp}\otimes^{\square,\mathrm{psh}}\mathcal{G},\mathcal{H}).\end{aligned}$$

Here we wrote $\mathrm{Hom}_{\mathrm{psh}}(-,-)$ for the sets of morphisms in $\mathrm{PSh}(S_{\mathrm{proét}},\mathrm{Solid})$. Repeating this computation with $\mathcal{G}$ in place of $\mathcal{F}$ yields claim (1) of the lemma.

The proof of the second claim of the lemma is similar. □

The solid tensor product of sheaves inherits many properties from the solid tensor product of modules. In particular, it commutes with all colimits (since the sheafification functor does). The internal Hom, $\mathcal{H}om^{\square}_S(\mathcal{F},\mathcal{G})$, can be defined as a presheaf by

$$\{Y\mapsto \underline{\mathrm{Hom}}^{\square}_Y(\mathcal{F}_Y,\mathcal{G}_Y)\in\mathrm{Solid}\},\quad Y\in S_{\mathrm{proét}}.$$

It is actually a sheaf by [25, Prop. 2.2.14]. Moreover, the image of the sheaf $\mathcal{H}om^{\square}_S(\mathcal{F},\mathcal{G})$ by the embedding functor $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})\hookrightarrow \mathrm{Sh}(S_{\mathrm{proét}},\mathrm{CondAb})$ is the sheaf $\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F},\mathcal{G})$. We have the usual adjointness property:

Lemma 2.22. *If $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3\in \mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$, there is a canonical functorial isomorphism in* $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$

$$\mathcal{H}om^{\square}_S(\mathcal{F}_1\otimes^{\square}\mathcal{F}_2,\mathcal{F}_3)\xrightarrow{\sim}\mathcal{H}om^{\square}_S(\mathcal{F}_1,\mathcal{H}om^{\square}_S(\mathcal{F}_2,\mathcal{F}_3)).$$

In particular, we have a canonical functorial isomorphism in Solid

$$\mathrm{Hom}^{\square}_S(\mathcal{F}_1\otimes^{\square}\mathcal{F}_2,\mathcal{F}_3)\simeq \mathrm{Hom}^{\square}_S(\mathcal{F}_1,\mathcal{H}om^{\square}_S(\mathcal{F}_2,\mathcal{F}_3)).$$

Proof. This can be checked in $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{CondAb})$, where it is known. □

(•) *Sheaves of solid $\underline{\mathbf{Q}}_p$-modules.* Let $\mathrm{Sh}(S_{\mathrm{proét}},\underline{\mathbf{Q}}_{p,\square})$ denote the subcategory of $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$ of $\underline{\mathbf{Q}}_p$-modules.

Lemma 2.23. *The category $\mathrm{Sh}(S_{\mathrm{proét}},\underline{\mathbf{Q}}_{p,\square})$ is a Grothendieck abelian category.*

Proof. We need to check that the category $\mathrm{Sh}(S_{\mathrm{proét}},\underline{\mathbf{Q}}_{p,\square})$ admits a generator and small colimits such that small filtered colimits are exact. But the colimits can be inherited from $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$ and they have the required properties (use the fact that the solid tensor product of solid abelian sheaves commutes with direct sums). Moreover, if X is a generator of $\mathrm{Sh}(S_{\mathrm{proét}},\mathrm{Solid})$ then $\underline{\mathbf{Q}}_p\otimes^{\square}X$ is a generator of $\mathrm{Sh}(S_{\mathrm{proét}},\underline{\mathbf{Q}}_{p,\square})$ as easily follows, again, from the fact that the solid tensor product of solid abelian sheaves commutes with direct sums. This finishes our argument. □

The categories of $\mathbf{Q}_p(Y)_\square$-modules, $Y \in S_{\text{proét}}$, being closed symmetric monoidal, the category $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$ is equipped with tensor product and internal Hom: For $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$, their tensor product $\mathcal{F} \otimes^\square_{\underline{\mathbf{Q}}_p} \mathcal{G}$ is defined by sheafifying the presheaf tensor product

$$\{Y \mapsto \mathcal{F}(Y) \otimes^\square_{\underline{\mathbf{Q}}_p(Y)} \mathcal{G}(Y)\}, \quad Y \in S_{\text{proét}}.$$

We have an analog of Lemma 2.21 in this setting. The internal Hom, $\mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}, \mathcal{G})$, can be defined as a presheaf by

$$\{Y \mapsto \underline{\mathrm{Hom}}^\square_{Y,\underline{\mathbf{Q}}_p}(\mathcal{F}_Y, \mathcal{G}_Y) \in \mathrm{Mod}^{\text{solid}}_{\mathbf{Q}_p(Y)}\}, \quad Y \in S_{\text{proét}},$$

where we set
(2.24)
$$\underline{\mathrm{Hom}}^\square_{Y,\underline{\mathbf{Q}}_p}(\mathcal{F}_Y, \mathcal{G}_Y) := \ker\Big(\prod_{Y_1 \to Y} \underline{\mathrm{Hom}}^\square_{\mathbf{Q}_p(Y_1)}(\mathcal{F}(Y_1), \mathcal{G}(Y_1)) \to \prod_{Y_2 \to Y_1} \underline{\mathrm{Hom}}^\square_{\mathbf{Q}_p(Y_1)}(\mathcal{F}(Y_1), \mathcal{G}(Y_2))\Big).$$

It is actually a sheaf (compare with the condensed definition). Moreover, the image of the sheaf $\mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}, \mathcal{G})$ by the forgetful functor $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square}) \to \mathrm{Sh}(S^{\text{top}}_{\text{proét}}, \underline{\mathbf{Q}}_p)$ is the sheaf $\mathcal{H}om_{S^{\text{top}},\underline{\mathbf{Q}}_p}(\mathcal{F}, \mathcal{G})$. In particular, the forgetful functor $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square}) \to \mathrm{Sh}(S^{\text{top}}_{\text{proét}}, \underline{\mathbf{Q}}_p)$ is (topologically) fully faithful. We have the usual adjointness property:

Lemma 2.25. *If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$, there is a canonical functorial isomorphism in* $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$

$$\mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_1 \otimes^\square_{\underline{\mathbf{Q}}_p} \mathcal{F}_2, \mathcal{F}_3) \xrightarrow{\sim} \mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_1, \mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_2, \mathcal{F}_3)).$$

In particular, we have a canonical functorial isomorphism in $\mathrm{Mod}^{\text{solid}}_{\mathbf{Q}_p(S)}$

$$\underline{\mathrm{Hom}}^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_1 \otimes^\square_{\underline{\mathbf{Q}}_p} \mathcal{F}_2, \mathcal{F}_3) \simeq \underline{\mathrm{Hom}}^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_1, \mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}_2, \mathcal{F}_3)).$$

Proof. Reduce to the condensed adjointness as in the proof of Lemma 2.22. □

The category $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$ is generated by the set

(2.26)
$$\{\underline{\mathbf{Q}_p}[h^\delta_Y] \otimes^\square_{\mathbf{Z}} \mathbf{Z}[T]\}, \quad Y \in \mathrm{Perf}_S, T - \text{ profinite set}.$$

And we have an analog of Lemma 2.15 in this setting.

(•) *Derived picture.* Again, we have an analogous derived picture. We start with the category $\mathcal{D}(S_{\text{proét}}, \mathrm{Solid})$. Since $\mathrm{Sh}(S_{\text{proét}}, \mathrm{Solid})$ is a Grothendieck abelian category, the internal Hom, $\mathrm{R}\mathcal{H}om^\square_S(\mathcal{F}, \mathcal{G})$, for $\mathcal{F}, \mathcal{G} \in \mathcal{D}(S_{\text{proét}}, \mathrm{Solid})$, is defined using the existence of K-injective resolutions (which is a classical result, see [24], [29, Tag 079I]).

The existence of K-flat resolutions, which follows from the existence of flat generators from 2.19, yields the derived tensor product $\mathcal{F} \otimes^{\mathrm{L}\square} \mathcal{G}$, for $\mathcal{F}, \mathcal{G} \in \mathcal{D}(S_{\text{proét}}, \mathrm{Solid})$. By the usual argument ([29, Tag 08J9]), the derived tensor product is the left (interior) adjoint to the derived interior Hom.

We pass now to the category $\mathcal{D}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$. The existence of K-injective resolutions follows from the fact that the category $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$ is Grothendieck abelian (see Lemma 2.23). The existence of K-flat resolutions follows from the existence of flat generators of $\mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$ (see (2.26)). Hence we have the derived tensor product $\mathcal{F} \otimes^{\mathrm{L}\square}_{\underline{\mathbf{Q}}_p} \mathcal{G}$, for $\mathcal{F}, \mathcal{G} \in \mathcal{D}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$, the internal Hom, $\mathrm{R}\mathcal{H}om^\square_{S,\underline{\mathbf{Q}}_p}(\mathcal{F}, \mathcal{G})$, which are (internal) adjoints to each other.

2.2.4. *Constant sheaves and monoidal structures.* Let $W \in K_\square$ be a flat module and let[6] $\mathcal{F} \in \mathrm{Sh}(S_{\text{proét}}, K_\square)$. We define sheaves $W \otimes_K^{\mathrm{L}_\square} \mathcal{F}, W \otimes_K^\square \mathcal{F}$ as presheaves by setting

$$W \otimes_K^{\mathrm{L}_\square} \mathcal{F} := \{Y \mapsto W \otimes_K^{\mathrm{L}_\square} \mathcal{F}(Y)\},$$
$$W \otimes_K^{\square} \mathcal{F} := \{Y \mapsto W \otimes_K^{\square} \mathcal{F}(Y)\}.$$

These are sheaves because W is a flat module. We will need the following computations:

Lemma 2.27. *Let $W \in \mathbf{Q}_{p,\square}$ be a Fréchet or of compact type. Let $\mathcal{F} \in \mathrm{Sh}(S_{\text{proét}}, \underline{\mathbf{Q}}_{p,\square})$.*

(1) There is a canonical quasi-isomorphism in $\mathcal{D}(S_{\text{proét}}, \mathbf{Q}_{p,\square})$

$$W \otimes_{\mathbf{Q}_p}^{\mathrm{L}_\square} \mathcal{F} \xrightarrow{\sim} \underline{W} \otimes_{\mathbf{Q}_p}^{\mathrm{L}_\square} \mathcal{F}. \tag{2.28}$$

(2) There is a natural isomorphism in $\mathbf{Q}_{p,\square}$

$$\underline{\mathrm{Hom}}^\square_{S,\mathbf{Q}_p}(\underline{W}, \mathcal{F}) \xrightarrow{\sim} \underline{\mathrm{Hom}}^\square_{\mathbf{Q}_p}(W, \mathcal{F}(S)).$$

(3) There is a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\mathrm{R}\underline{\mathrm{Hom}}^\square_{S,\mathbf{Q}_p}(\underline{W}, \mathcal{F}) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(W, \mathrm{R}\Gamma(S^{\text{top}}_{\text{proét}}, \mathcal{F})).$$

(4) If W is of compact type then there is a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\mathrm{R}\underline{\mathrm{Hom}}^\square_{S,\mathbf{Q}_p}(\underline{W}, \mathcal{F}) \simeq W^* \otimes_{\mathbf{Q}_p}^{\mathrm{L}_\square} \mathrm{R}\Gamma(S^{\text{top}}_{\text{proét}}, \mathcal{F}).$$

Proof. We start with the first claim. As presheaves both sides of (2.28) are given by

$$\{Y \mapsto W \otimes_{\mathbf{Q}_p}^{\mathrm{L}_\square} \mathcal{F}(Y)\}, \quad \{Y \mapsto \underline{W}(Y) \otimes_{\mathbf{Q}_p(Y)}^{\mathrm{L}_\square} \mathcal{F}(Y)\},$$

respectively. For Y strictly totally disconnected, we compute

$$\begin{aligned}\underline{W}(Y) \otimes_{\mathbf{Q}_p(Y)}^{\mathrm{L}_\square} \mathcal{F}(Y) = \mathcal{C}(\pi_0(Y), W) \otimes_{\mathbf{Q}_p(Y)}^{\mathrm{L}_\square} \mathcal{F}(Y) &\simeq W \otimes_{\mathbf{Q}_p}^\square \mathcal{C}(\pi_0(Y), \mathbf{Q}_p) \otimes_{\mathbf{Q}_p(Y)}^{\mathrm{L}_\square} \mathcal{F}(Y)\\ &\simeq W \otimes_{\mathbf{Q}_p}^{\mathrm{L}_\square} \mathcal{F}(Y)\end{aligned}$$

Here, the second quasi-isomorphism holds by [22, Cor. 10.5.4]. Our claim follows.

For the second claim, we use the end description

$$\begin{aligned}\underline{\mathrm{Hom}}^\square_{S,\mathbf{Q}_p}(\underline{W}, \mathcal{F}) \xrightarrow{\sim} \underline{\mathrm{Hom}}^\square_{S,\mathbf{Q}_p}(W \otimes^\square_{\mathbf{Q}_p} \underline{\mathbf{Q}}_p, \mathcal{F}) &= \int_Y \underline{\mathrm{Hom}}_{\mathbf{Q}_p(Y)}(W \otimes^\square_{\mathbf{Q}_p} \mathbf{Q}_p(Y), \mathcal{F}(Y))\\ &\simeq \int_Y \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W, \mathcal{F}(Y)) \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W, \int_Y \mathcal{F}(Y)) \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W, \mathcal{F}(S)).\end{aligned}$$

Here, the first isomorphism follows from the first claim of the lemma.

The last two claims of the lemma follow from the first two ones by applying them to an injective resolution of $\mathcal{F}$. $\square$

2.2.5. *Condensed versus solid sheaves.* The following result is a sheaf version of Proposition 2.7.

Proposition 2.29. *(1) The forgetful functor*

$$\mathrm{Sh}(S_{\text{proét}}, \mathrm{Solid}) \to \mathrm{Sh}(S_{\text{proét}}, \mathrm{CondAb}) \tag{2.30}$$

is (topologically) fully faithful. The essential image is stable under all limits, colimits, and extensions. Moreover, the inclusion (2.30) *admits a (topological) left adjoint, the solidification functor,*

$$\mathrm{Sh}(S_{\text{proét}}, \mathrm{CondAb}) \to \mathrm{Sh}(S_{\text{proét}}, \mathrm{Solid}) : \quad \mathcal{F} \mapsto \mathcal{F}^\square, \tag{2.31}$$

which preserves all colimits and is symmetric monoidal. We have analogous claims for the forgetful functor $\mathrm{Sh}(S_{\text{proét}}, \mathbf{Q}_{p,\square}) \to \mathrm{Sh}(S^{\text{top}}_{\text{proét}}, \mathbf{Q}_p)$.

[6] The definition and properties of the category $\mathrm{Sh}(S_{\text{proét}}, K_\square)$ is analogous to those of the category $\mathrm{Sh}(S_{\text{proét}}, \mathbf{Q}_{p,\square})$.

(2) The forgetful functor

$$\mathcal{D}(S_{\text{proét}},\mathrm{Solid}) \to \mathcal{D}(S_{\text{proét}},\mathrm{CondAb}) \tag{2.32}$$

is (topologically) fully faithful. It preserves all limits and colimits.

(3) The functor (2.32) *admits a left adjoint*

$$\mathcal{F}\mapsto \mathcal{F}^{\mathrm{L}\square}:\mathcal{D}(S_{\text{proét}},\mathrm{CondAb})\to \mathcal{D}(S_{\text{proét}},\mathrm{Solid}),$$

which is the (topological) left derived functor of the solidification functor $(-)^{\square}$. It is symmetric monoidal. We have analogous claims for the forgetful functor $\mathcal{D}(S_{\text{proét}},\underline{\mathbf{Q}}_{p,\square})\to \mathcal{D}(S^{\mathrm{top}}_{\text{proét}},\underline{\mathbf{Q}}_p)$.

(4) For $\mathcal{F}_1,\mathcal{F}_2\in\mathcal{D}(S_{\text{proét}},\mathrm{Solid})$, the natural map

$$\mathrm{R}\mathcal{H}om^{\square}_S(\mathcal{F}_1,\mathcal{F}_2)\to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_1,\mathcal{F}_2)$$

is a quasi-isomorphism. We have an analogous claim for the $\underline{\mathbf{Q}}_p$-sheaves.

Proof. We start with the first claim. The fully-faitfulness was shown above, in Section 2.2.3 (see Lemma 2.20). Limits of solid sheaves in the condensed setting are solid because this is true for presheaves; for colimits – this holds on the level of presheaves and condensed sheafification is compatible with solid sheafification. For extensions: If $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$ is an extension of condensed sheaves such that $\mathcal{F}_1,\mathcal{F}_3$ are solid then, on every $Y\in S_{\text{proét}}$, we have an exact sequence in CondAb

$$0\to\mathcal{F}_1(Y)\to\mathcal{F}_2(Y)\to\mathcal{F}_3(Y)\xrightarrow{f} H^1_{\text{proét}}(Y,\mathcal{F}_1).$$

The cohomology group $H^1_{\text{proét}}(Y,\mathcal{F}_1)$ is solid (because so is $\mathcal{F}_1$), hence so is the kernel of the map f. Thus, by Proposition 2.7, $\mathcal{F}_2(Y)$ is solid, as wanted. The claim about solidification follows from Lemma 2.20, Lemma 2.21, and Proposition 2.7.

We turn now to the second claim of the proposition. For fully-faitfulness, for $\mathcal{F},\mathcal{G}\in\mathcal{D}(S_{\text{proét}},\mathrm{Solid})$, we need to show that the canonical map

$$\mathrm{R}\underline{\mathrm{Hom}}^{\square}_S(\mathcal{F},\mathcal{G})\to \mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F},\mathcal{G})$$

is a quasi-isomorphism. We will need the following fact:

Lemma 2.33. *If W is a prodiscrete condensed set (i.e. a cofiltered limit of discrete condensed sets) then $\mathbf{Z}[W]^{\mathrm{L}\square}\xrightarrow{\sim}\mathbf{Z}[W]^{\square}$. Similalry with $\underline{\mathbf{Q}}_p(S)$-coefficients, for $S\in\mathrm{Perf}_C$.*

Proof. Recall that every prodiscrete set can be written as a filtered colimit of profinite sets. So let us write $W=\mathrm{colim}_i T_i$, a filtered colimit of profinite sets. We have

$$\mathbf{Z}[W]^{\mathrm{L}\square}=\mathbf{Z}[\mathrm{colim}_i T_i]^{\mathrm{L}\square}\simeq(\mathrm{colim}_i\mathbf{Z}[T_i])^{\mathrm{L}\square}\simeq\mathrm{colim}_i\mathbf{Z}[T_i]^{\mathrm{L}\square}\xrightarrow{\sim}\mathrm{colim}_i\mathbf{Z}[T_i]^{\square}\simeq\mathbf{Z}[W]^{\square},$$

which is what we wanted.

The $\mathbf{Q}_p$-case follows from the fact that $\mathbf{Z}[T]$ is a flat solid abelian group. □

This implies that if we take a resolution of $\mathcal{F}$ in $\mathrm{Sh}(S_{\text{proét}},\mathrm{CondAb})$

$$\cdots\to P_1\to P_0\to\mathcal{F}\to 0,$$

where $P_j=\oplus_j(\mathbf{Z}[h^{\delta}_{Y_j}]\otimes\mathbf{Z}[T_j])$ for a w-contractible Y_j over S, an extremally diconnected T_j, the solidification

$$\cdots\to P_1^{\square}\to P_0^{\square}\to\mathcal{F}\to 0$$

is also exact. Indeed, it suffices to show that

$$\cdots\to P_1(X)^{\square}\to P_0(X)^{\square}\to\mathcal{F}(X)^{\square}\to 0,$$

is exact for a w-contractible X over S. But this follows from the quasi-isomorphism $\mathcal{F}(X)^{\mathrm{L}\square}\xrightarrow{\sim}\mathcal{F}(X)^{\square}$, since $\mathcal{F}(X)$ is solid, and Lemma 2.33.

Using that the P_i's above are projective in $\mathrm{Sh}(S_{\mathrm{proét}}, \mathrm{CondAb})$ and their solidifications are projective in $\mathrm{Sh}(S_{\mathrm{proét}}, \mathrm{Solid})$, the fully faithfulness claim is reduced to the fact that the map

$$\mathrm{Hom}^{\square}_S(\mathbf{Z}[h^{\delta}_Y] \otimes^{\square} \mathbf{Z}[T], \mathcal{G}) \to \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h^{\delta}_Y] \otimes \mathbf{Z}[T], \mathcal{G}) \tag{2.34}$$

is an isomorphism for a w-contractible Y over S, an extremally diconnected T, and a solid sheaf $\mathcal{G}$. But this follows from the first claim of the proposition. The topological version can be derived by the same argument applied to the tensor products $\mathcal{F} \otimes^{\square} \mathbf{Z}[T]$'s.

This also proves the fourth claim of the proposition. The statement about colimits in claim (2) follows by evaluating sheaves on w-contractible perfectoids. For limits – we use the fact that in both categories we have, for Y over S w-contractible, we have $(\mathrm{R}\lim_{i\in I} \mathcal{F}_i)(Y) \simeq \mathrm{R}\lim_{i\in I} \mathcal{F}_i(Y)$ and, by Proposition 2.7, the latter limits are the same in the solid and condensed categories.

For claim (3), since we have enough projectives, the left derived functor $(-)^{\mathrm{L}\square}$ of $(-)^{\square}$ exists and it remains to show that it is the left adjoint of the map (2.30). But this can be checked on $\underline{\mathrm{Hom}}$'s and for projectives $\mathbf{Z}[h^{\delta}_Y] \otimes \mathbf{Z}[T]$ as in (2.34) and then we can use claim (1). This also implies inner left adjointness, which, in turn, implies compatibility with monoidal structures via the following computation: We want to show that, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{D}(S_{\mathrm{proét}}, \mathrm{CondAb})$, we have

$$(\mathcal{F}_1 \otimes^{\mathrm{L}} \mathcal{F}_2)^{\mathrm{L}\square} \simeq \mathcal{F}_1^{\mathrm{L}\square} \otimes^{\mathrm{L}\square} \mathcal{F}_2^{\mathrm{L}\square} \tag{2.35}$$

in $\mathcal{D}(S_{\mathrm{proét}}, \mathrm{Solid})$. But, for $\mathcal{G} \in \mathcal{D}(S_{\mathrm{proét}}, \mathrm{Solid})$, by the already proven adjointness properties and claim (4) of our proposition, we have the following quasi-isomorphims

$$\begin{aligned}
\mathrm{RHom}^{\square}_S((\mathcal{F}_1 \otimes^{\mathrm{L}} \mathcal{F}_2)^{\mathrm{L}\square}, \mathcal{G}) &\simeq \mathrm{RHom}_{S^{\mathrm{top}}}(\mathcal{F}_1 \otimes^{\mathrm{L}} \mathcal{F}_2, \mathcal{G}) \simeq \mathrm{RHom}_{S^{\mathrm{top}}}(\mathcal{F}_1, \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_2, \mathcal{G}))\\
&\xleftarrow{\sim} \mathrm{RHom}_{S^{\mathrm{top}}}(\mathcal{F}_1, \mathrm{R}\mathcal{H}om^{\square}_S(\mathcal{F}_2^{\mathrm{L}\square}, \mathcal{G})) \simeq \mathrm{RHom}^{\square}_S(\mathcal{F}_1^{\mathrm{L}\square}, \mathrm{R}\mathcal{H}om^{\square}_S(\mathcal{F}_2^{\mathrm{L}\square}, \mathcal{G}))\\
&\simeq \mathrm{RHom}^{\square}_S(\mathcal{F}_1^{\mathrm{L}\square} \otimes^{\mathrm{L}\square} \mathcal{F}_2^{\mathrm{L}\square}, \mathcal{G}),
\end{aligned}$$

as wanted.

Claim (3) for $\underline{\mathbf{Q}}_p$-coefficients follows by the same argument as in the abelian case. ∎

Remark 2.36. Recall that sheaves of $\underline{\mathbf{Q}}_p$-modules on $\mathrm{Spa}(C)_{\mathrm{proét}}$ are called Vector Spaces (VS for short). Following this, we will call sheaves from $\mathrm{Sh}(\mathrm{Spa}(C)_{\mathrm{proét}}, \underline{\mathbf{Q}}_{p,\square})$ *Naive Topological Vector Spaces* (NTVS for short).

2.2.6. *Étale Naive Topological Vector Spaces.* The constructions from Section 2.1 and Section 2.2 have étale versions though one needs to be a bit careful so that all of them go through.

Let S be a strictly totally disconnected affinoid perfectoid over C. Denote by sPerf_S the category of strictly totally disconnected affinoid perfectoid spaces over S. Proceeding as in Section 2.1.2 we can define the topological étale site $S^{\mathrm{top}}_{\mathrm{\acute{e}t}}$ based on the category sPerf_S equipped with étale topology. The constructions from Section 2.1 and Section 2.2 go through with two exceptions: we do not have the map η from Section 2.1.2 and in Lemma 2.15 we can take all Y's.

In fact, later on, we will go even further and often drop the topology altogether (but still work with the category sPerf_S).

3. Topological Vector Spaces

We pass now to our main object of study: the categories of topologically enriched presheaves. We start with the definition and properties of topology on mapping spaces between perfectoid affinoids over C. Then we define topologically enriched presheaves with values in condensed and solid abelian groups, discuss examples and the Enriched Yoneda Lemma. After that we prove that topologically enriched presheaves form Grothendieck abelian categories, study their ∞-derived categories, the solidification functors, and monoidal structures. Finally, we extend all of this to the categories of $\underline{\mathbf{Q}}_p$-modules.

3.1. **Topologically enriched presheaves.** Many of the topological presheaves we will be working with will be canonically enriched.

Remark 3.1. The main technical tool that we need that convinced us to use topologically enriched presheaves is the enriched Yoneda Lemma that is needed in the proof of Theorem 4.1 and Theorem 4.25 (used to relate topological RHom's to the algebraic ones and the ones for perfect complexes on the Fargues-Fontaine curve). The consideration of topologically enriched presheaves is not new:

(1) It appears implicitly in the original definition of Banach-Colmez spaces in [9] (Banach-Colmez spaces are presheaves valued in Banach spaces and are topologically enriched) and also explicitly[7] in the definition of qBC's in [12].
(2) Topologically enriched functors in the context of the theory of Banach spaces were studied for a while: see, for example, [15], [21]. They are called "strong functors" there. More generally, the papers [6], [7] survey the categorical approach to the theory of Banach spaces.

Our main references for enriched categories and enriched functors are [18], [14], [16, App. C].

3.1.1. *Topologized mapping space.* Let $S \in \mathrm{Perf}_C$. We will enrich the category Perf_S in Cond. For $Y_1, Y_2 \in \mathrm{Perf}_S$, we set:

$$\underline{\mathrm{Hom}}_S(Y_1,Y_2) : \{T \mapsto \mathrm{Hom}_S(Y_1 \times T, Y_2)\}.$$

For every $Y \in \mathrm{Perf}_S$, the *identity map* $i_X : \{*\} \to \underline{\mathrm{Hom}}_S(Y,Y)$ sends $\{*\}$ to Id. For every triple $Y_1, Y_2, Y_3 \in \mathrm{Perf}_S$, the *composition* map

$$\underline{\mathrm{Hom}}_S(Y_2,Y_3) \otimes \underline{\mathrm{Hom}}_S(Y_1,Y_2) \to \underline{\mathrm{Hom}}_S(Y_1,Y_3)$$

is defined by compatible compositions, for profinite sets T,

$$\mathrm{Hom}_S(Y_2 \times T, Y_3) \otimes \mathrm{Hom}_S(Y_1 \times T, Y_2) \to \mathrm{Hom}_S(Y_1 \times T, Y_3),$$

which send the pair (f_2, f_1) to the composition

$$f_2 f_1 : \quad Y_1 \times T \xrightarrow{\mathrm{Id}\times\Delta} Y_1 \times T \times T \xrightarrow{f_1 \times \mathrm{Id}} Y_2 \times T \xrightarrow{f_2} Y_3.$$

The associativity and unit axioms (see [18, (1.3), (1.4)]) are easily checked to hold.

Lemma 3.2. *The condensed set $\underline{\mathrm{Hom}}_S(Y_1,Y_2)$ is prodiscrete.*

Proof. Set $R_1^+ := R_{Y_1}^+, R_2^+ := R_{Y_2}^+, R_S^+ := R_S^+$. These are prodiscrete sets. We claim that there is an isomorphism of condensed sets

$$\underline{\mathrm{Hom}}_S(Y_1,Y_2) \simeq \underline{\mathrm{Hom}}^{\mathrm{Alg}}_{\underline{R}_S^+}(\underline{R}_2^+, \underline{R}_1^+),$$

where the second Hom is taken in $\underline{R}_S^+$-algebras. To see this we start with showing that there is an isomorphism of condensed sets

$$\underline{\mathrm{Hom}}_S(Y_1,Y_2) \simeq \underline{\mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, R_1^+)},$$

where $\mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, R_1^+)$ is equipped with the compact-open topology. For that, we compute:

$$\begin{aligned}\mathrm{Hom}_S(Y_1 \times T, Y_2) &= \mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, R_1^+ \widehat{\otimes} \mathcal{C}(T, \mathcal{O}_C)) \simeq \mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, \mathcal{C}(T, R_1^+))\\ &\simeq \mathcal{C}(T, \mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, R_1^+)),\end{aligned}$$

where the last isomorphism follows from the exponential law since T, R_1^+, R_2^+ are compactly generated (since they are colimits of profinite sets) and Hausdorff. By [8, Prop. 4.2], we have the natural isomorphism of condensed sets

$$\underline{\mathrm{Hom}}^{\mathrm{Alg}}_{\underline{R}_S^+}(\underline{R}_2^+, \underline{R}_1^+) \xrightarrow{\sim} \underline{\mathrm{Hom}^{\mathrm{Alg}}_{R_S^+}(R_2^+, R_1^+)}.$$

[7] Though at the time of writing [12] the authors were not aware that the extra continuity property they have imposed amounts to the notion of enrichment in category theory.

It suffices now to show that $\underline{\mathrm{Hom}}^{\mathrm{Alg}}_{\underline{R}^+_S}(\underline{R}^+_2,\underline{R}^+_1)$ is prodiscrete. But this is a closed condensed subset of $\underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_C}(\underline{R}^+_2,\underline{R}^+_1)$ hence, by [30], it suffices to show that $\underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_C}(\underline{R}^+_2,\underline{R}^+_1)$ is prodiscrete. For that, we compute

$$\begin{aligned}\underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_C}(\underline{R}^+_2,\underline{R}^+_1) &\simeq \lim_n \underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_{C,n}}(\underline{R}^+_{2,n},\underline{R}^+_{1,n}) \simeq \lim_n \underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_{C,n}}(\mathrm{colim}_i\, \underline{R}^+_{2,n,i},\underline{R}^+_{1,n})\\ &\simeq \lim_n \lim_i \underline{\mathrm{Hom}}_{\underline{\mathcal{O}}_{C,n}}(\underline{R}^+_{2,n,i},\underline{R}^+_{1,n}),\end{aligned}$$

where $\underline{R}^+_{2,n,i} \subset \underline{R}^+_{2,n}$ are finitely generated $\underline{\mathcal{O}}_{C,n}$-modules. Getting what we wanted. □

3.1.2. *Topologically enriched presheaves.* Let $\mathcal{C}$ be a bicomplete, locally finitely presentable, closed symmetric monoidal category (for example, Cond, CondAb, Solid). Such a category $\mathcal{C}$ is canonically $\mathcal{C}$-enriched ($\mathcal{C}$-category, for short) and we assume that it is a Cond-subcategory of Cond. We will denote by $\mathrm{Hom}_{\mathcal{C}}(-,-)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$ its Hom-set and $\mathcal{C}$-Hom, respectively. We have $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)(*) \simeq \mathrm{Hom}_{\mathcal{C}}(-,-)$. If $\mathcal{C} = \mathrm{Cond}$ we will skip the subscripts.

Let $S \in \mathrm{Perf}_C$. A *$\mathcal{C}$-enriched presheaf* (or $\mathcal{C}$-presheaf for short) $\mathcal{F} : \mathrm{Perf}_S^{\mathrm{op}} \to \mathcal{C}$ consists of a function

$$\mathcal{F} : \mathrm{Perf}_S \mapsto \mathrm{Ob}\mathcal{C}$$

together with, for $Y_1, Y_2 \in \mathrm{Perf}_S$, a map of condensed sets

$$\underline{\mathcal{F}}_{Y_1,Y_2}: \quad \underline{\mathrm{Hom}}_S(Y_1,Y_2) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}(Y_2),\mathcal{F}(Y_1)) \tag{3.3}$$

in a manner compatible with compositions and identities (see [18, (1.5), (1.6)]). For $\mathcal{C}$-presheaves $\mathcal{F},\mathcal{G}$, a $\mathcal{C}$-natural transformation $f:\mathcal{F}\to\mathcal{G}$ is a Perf_S-indexed family of maps in $\mathcal{C}$:

$$f_Y : \mathcal{F}(Y) \to \mathcal{G}(Y)$$

satisfying a $\mathcal{C}$-naturality condition (see [18, (1.7)], [14, (4.30)]): the following diagram commutes in Cond

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_S(Y,Y_1) \arrow[r, "\underline{\mathcal{F}}_{Y,Y_1}"] \arrow[d, "\underline{\mathcal{G}}_{Y,Y_1}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}(Y_1),\mathcal{F}(Y)) \arrow[d, "f_{*,Y}"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{G}(Y_1),\mathcal{G}(Y)) \arrow[r, "f^*_{Y_1}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}(Y_1),\mathcal{G}(Y)).
\end{tikzcd}$$

We will write $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})$ for the set of $\mathcal{C}$-natural transformations.

We will denote the category of such presheaves by $\underline{\mathrm{PSh}}(S,\mathcal{C})_0$ and, in the case $\mathcal{C} = \mathrm{Cond}$, we will simply write $\underline{\mathrm{PSh}}(S)_0$.

Example 3.4. (1) The topological sheaves that come from $S_{\text{proét}}$ are canonically enriched:

Lemma 3.5. *Let $\mathcal{F} \in \mathrm{Sh}(S_{\text{proét}})$. The sheaf $\pi_*\mathcal{F} \in \mathrm{Sh}(S_{\text{proét}},\mathrm{Cond})$ is canonically enriched.*

Proof. We need to define, compatible in T, maps

$$\mathrm{Hom}_S(Y_1\times T,Y_2) \to \mathrm{Hom}(\pi_*\mathcal{F}(Y_2)\times T,\pi_*\mathcal{F}(Y_1)) = \{T_1 : \mathcal{F}(Y_2\times T_1)\times \underline{T}(T_1) \to \mathcal{F}(Y_1\times T_1)\}. \tag{3.6}$$

We induce them by the following transformation of maps over S

$$(Y_1\times T \xrightarrow{f} Y_2)\times(T_1 \xrightarrow{g} T) \mapsto (Y_1\times T_1 \xrightarrow{fg} Y_2) \mapsto (Y_1\times T_1 \xrightarrow{(fg,\mathrm{Id})} Y_2\times T_1).$$

Clearly, evaluating (3.6) on $*$ yields the structure map (3.10). Commutativity of the composition and the identities diagrams [18, (1.5), (1.6)] is easy (if tedious) to check. □

(2) Lemma 3.5 combined with Example 2.1 yield that the sheaves $\mathbf{Q}_p, \mathbb{G}_a \in \mathrm{Sh}(S_{\text{proét}},\mathrm{Cond})$ are canonically enriched and so are the period sheaves $\mathbb{B}^+_{\mathrm{dR}}/t^i$ and $\mathbb{B}^{+,\varphi=p^i}_{\mathrm{cr}}$, $i \geq 0$.

3.1.3. *The* Cond-*category of topologically enriched presheaves.* Assume moreover that $\mathcal{C}$ is a bicomplete $\mathcal{C}$-category: $\mathcal{C}$ is bicomplete in the usual set-based sense and the tensors and cotensors:

$$c \otimes W, \quad [W, c], \quad c, W \in \mathcal{C},$$

satisfy the $\mathcal{C}$-adjunctions

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(c_1 \otimes W, c_2) \simeq [W, \underline{\mathrm{Hom}}_{\mathcal{C}}(c_1, c_2)] \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(c_1, [W, c_2]).$$

(Here $[-,-] = \underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$ but below we will need to separate these two operations.) *This will be our standard assumption on $\mathcal{C}$ from now on.*

The category $\underline{\mathrm{PSh}}(S, \mathcal{C})_0$ is the underlying category of the $\mathcal{C}$-category $\underline{\mathrm{PSh}}(S, \mathcal{C})$, where the Hom-object in $\underline{\mathrm{PSh}}(S, \mathcal{C})$ is defined as the enriched end:

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = \int_{Y \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}(Y), \mathcal{G}(Y)) \tag{3.7}$$

computed as an ordinary limit in $\mathcal{C}$. More precisely, it is the equalizer in $\mathcal{C}$ given by the diagram

$$\textstyle\prod_{Y \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}(Y), \mathcal{G}(Y)) \rightrightarrows \prod_{Y, Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_{\mathcal{C}}(\underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}(Y), \mathcal{G}(Y_1)),$$

where the parallel arrows are defined via the evaluation maps

$$\underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}(Y) \to \mathcal{F}(Y_1), \quad \underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{G}(Y) \to \mathcal{G}(Y_1)$$

of the $\mathcal{C}$-presheaves $\mathcal{F}, \mathcal{G}$. We have $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})(*) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})$.

The category $\underline{\mathrm{PSh}}(S, \mathcal{C})$ is a bicomplete $\mathcal{C}$-category [18, Sec. 3.3] with limits and colimits computed objectwise. Hence it is tensored and cotensored over $\mathcal{C}$: we have the tensor functor

$$\begin{aligned} (-) \otimes (-) : \quad & \mathcal{C} \times \underline{\mathrm{PSh}}(S, \mathcal{C}) \to \underline{\mathrm{PSh}}(S, \mathcal{C}), \\ & W \otimes \mathcal{F} = \{Y \mapsto W \otimes \mathcal{F}(Y)\}, \end{aligned}$$

and the cotensor functor

$$\begin{aligned} [-,-] : \quad & \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{PSh}}(S, \mathcal{C}) \to \underline{\mathrm{PSh}}(S, \mathcal{C}), \\ & [W, \mathcal{F}] = \{Y \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(W, \mathcal{F}(Y))\}, \end{aligned}$$

such that, for each $W \in \mathcal{C}$, there is a natural isomorphim

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(W, \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, [W, \mathcal{G}]). \tag{3.8}$$

Moreover, for each $W \in \mathcal{C}$, there is a natural isomorphim

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(W \otimes \mathcal{F}, \mathcal{G}) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(W, \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G})).$$

Remark 3.9. (Topological presheaves) The presheaves studied in Section 2, which we called "topological presheaves", also form a $\mathcal{C}$-category. We will call such $\mathcal{C}$-categories $\mathrm{PSh}(S, \mathcal{C})$. Their underlying categories[8] $\mathrm{PSh}(S, \mathcal{C})_0$ are defined as in Section 3.1.2 but with the trivial enrichment on the mapping spaces, i.e., in the structure maps (3.3), instead of $\underline{\mathrm{Hom}}_S(Y_1, Y_2)$ we take $\underline{\mathrm{Hom}_S(Y_1, Y_2)}$, where $\mathrm{Hom}_S(Y_1, Y_2)$ is given the discrete topology. Hence they are canonically induced by structure maps of the form

$$\mathcal{F}_{Y_1, Y_2} : \quad \mathrm{Hom}_S(Y_1, Y_2) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(Y_2), \mathcal{F}(Y_1)). \tag{3.10}$$

The Hom-object is defined as in (3.7).

We have the forgetful functor

$$(-)^{\mathrm{cl}} : \quad \underline{\mathrm{PSh}}(S, \mathcal{C}) \to \mathrm{PSh}(S, \mathcal{C})$$

given by evaluating the structure map (3.3) on $*$ to yield the structure map (3.10).

[8] For an enriched category $\mathcal{A}$, we denote by $\mathcal{A}_0$ the underlying category.

3.1.4. *Alternative description of topologically enriched presheaves.* We keep assuming that the Cond-category $\mathcal{C}$ is bicomplete. In that case the category $\underline{\mathrm{PSh}}(S,\mathcal{C})_0$ has an alternative description, which will allow us to understand it better. We want to think of a Cond-presheaf $\mathcal{F}:\mathrm{Perf}_S^{\mathrm{op}}\to\mathcal{C}$ as a function

$$\mathcal{F}:\mathrm{Perf}_S^{\mathrm{op}}\mapsto \mathrm{Ob}\mathcal{C}$$

plus a compatible collection of (structure) maps in $\mathcal{C}$

$$\underline{\mathcal{F}}_{Y,T}:\quad \mathcal{F}(Y\times T)\otimes T\to\mathcal{F}(Y),\quad Y\in\mathrm{Perf}_S,\ T\in\mathrm{ProFin}. \tag{3.11}$$

Equivalently, we can use the adjoints of the structure maps (3.11)

$$\underline{\mathcal{F}}'_{Y,T}:\quad \mathcal{F}(Y\times T)\to[T,\mathcal{F}(Y)]$$

and the compatibility condition corresponds to commutative diagrams

$$\begin{tikzcd}
\mathcal{F}(Y\times T) \arrow[r, "\underline{\mathcal{F}}'_{Y,T}"] \arrow[d, "(f\times g)^*"] & {[T,\mathcal{F}(Y)]} \arrow[d, "(g^*,f^*)"] \\
\mathcal{F}(Y_1\times T_1) \arrow[r, "\underline{\mathcal{F}}'_{Y,T}"] & {[T_1,\mathcal{F}(Y_1)],}
\end{tikzcd} \tag{3.12}$$

for all maps $f:Y_1\to Y, g:T_1\to T$, together with a composition rule.

A Cond-natural transformation $f:\mathcal{F}\to\mathcal{G}$ between two such functions $\mathcal{F}$, $\mathcal{G}$ is given by maps in $\mathcal{C}$

$$f_Y:\mathcal{F}(Y)\to\mathcal{G}(Y),\quad Y\in\mathrm{Perf}_S,$$

which are compatible with the structure maps (3.11). We will write $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})_0$ for the set of Cond-natural transformations.

We will call the category of such data $\underline{\mathrm{PSh}}'(S,\mathcal{C})_0$. We obtain the Cond-category $\underline{\mathrm{PSh}}'(S,\mathcal{C})$ (whose underlying category is $\underline{\mathrm{PSh}}'(S,\mathcal{C})_0$) by enriching in Cond the Hom-sets $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})_0$:

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G}):=\{T\mapsto\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}\otimes T,\mathcal{G})_0\},\quad \mathcal{F},\mathcal{G}\in\underline{\mathrm{PSh}}'(S,\mathcal{C})_0.$$

(We note that $\mathcal{F}\otimes T\in\underline{\mathrm{PSh}}'(S,\mathcal{C})_0$). We clearly have $\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})(*)=\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})_0$.

Lemma 3.13. *The* Cond*-categories* $\underline{\mathrm{PSh}}(S,\mathcal{C})$ *and* $\underline{\mathrm{PSh}}'(S,\mathcal{C})$ *are equivalent.*

Proof. We start with defining a Cond-functor

$$F:\underline{\mathrm{PSh}}(S,\mathcal{C})\to\underline{\mathrm{PSh}}'(S,\mathcal{C}),$$

which we will show later to be an equivalence. Let $\mathcal{F}\in\underline{\mathrm{PSh}}(S,\mathcal{C})$. We define $F(\mathcal{F})$ as being given by the same function on $\mathrm{Perf}_S^{\mathrm{op}}$ as $\mathcal{F}$. To define the structure maps $\underline{\mathcal{F}}_{Y,T}$ for $F(\mathcal{F})$, we start with the structure maps for $\mathcal{F}$

$$\underline{\mathcal{F}}_{Y,Y_1}:\underline{\mathrm{Hom}}_S(Y,Y_1)\to\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(Y_1),\mathcal{F}(Y)).$$

Evaluated at $T\in\mathrm{ProFin}$, they yield a compatible collection (indexed by T) of maps of sets

$$\underline{\mathcal{F}}_{Y,Y_1,T}:\mathrm{Hom}_S(Y\times T,Y_1)\to\mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(Y_1)\otimes T,\mathcal{F}(Y)).$$

Setting $Y_1=Y\times T$ in the above and taking the image of the identity by the map $\underline{\mathcal{F}}_{Y,Y_1,T}$, we get a compatible, in Y and T, family of maps in $\mathcal{C}$

$$\mathcal{F}(Y\times T)\otimes T\to\mathcal{F}(Y),$$

as wanted. This defines a function

$$F:\mathrm{Ob}\,\underline{\mathrm{PSh}}(S,\mathcal{C})\to\mathrm{Ob}\,\underline{\mathrm{PSh}}'(S,\mathcal{C}).$$

It remains to define, for each pair $\mathcal{F},\mathcal{G}\in\underline{\mathrm{PSh}}(S,\mathcal{C})$, a map

$$F_{\mathcal{F},\mathcal{G}}:\mathrm{Hom}_{\mathcal{C}}(\mathcal{F},\mathcal{G})\to\mathrm{Hom}_{\mathcal{C}}(F(\mathcal{F}),F(\mathcal{G}))$$

compatible with the composition and the identity (see [18, (1.5), (1.6)]). This amounts to defining, indexed by $T \in \mathrm{ProFin}$, a compatible family of maps

$$F_{\mathcal{F},\mathcal{G}}(T) : \mathrm{Hom}_{\mathcal{C}}(\mathcal{F} \otimes T, \mathcal{G})_0 \to \mathrm{Hom}_{\mathcal{C}}(F(\mathcal{F}) \otimes T, F(\mathcal{G}))_0.$$

But, by the definition of $F(\mathcal{F}), F(\mathcal{G})$, these two sets can be canonically identified.

The above proves that the Cond-functor F is in fact fully faithful. It remains thus to show that it is essentially surjective on objects. For that we will define the quasi-inverse G to F on objects: G is given by the same function on $\mathrm{Perf}_S^{\mathrm{op}}$ as F and to define the structure maps $\underline{\mathcal{F}}_{Y_1,Y_2}$ for $G(\mathcal{F})$:

$$\underline{\mathcal{F}}_{Y_1,Y_2} : \underline{\mathrm{Hom}}_S(Y_1, Y_2) \to \mathrm{Hom}_{\mathcal{C}}(G(\mathcal{F})(Y_2), G(\mathcal{F})(Y_1))$$

we need to define, a compatible in T, family of maps of sets

$$\underline{\mathcal{F}}_{Y_1,Y_2,T} : \mathrm{Hom}_S(Y_1 \times T, Y_2) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}(Y_2) \otimes T, \mathcal{F}(Y_1)).$$

To do that, we map a map $f : Y_1 \times T \to Y_2$ to the composition

$$\mathcal{F}(Y_2) \otimes T \xrightarrow{f \otimes \mathrm{Id}} \mathcal{F}(Y_1 \times T) \otimes T \xrightarrow{\mathcal{F}_{Y_1,T}} \mathcal{F}(Y_1).$$

It is easy to check that $FG = \mathrm{Id}$ and $GF = \mathrm{Id}$, hence F is essentially surjective, as wanted. □

Example 3.14. (Rigid topologically enriched presheaves) We will distinguish a subcategory of topologically enriched presheaves.

Definition 3.15. A topologically enriched presheaf $\mathcal{F} \in \underline{\mathrm{PSh}}(S, \mathcal{C})$ is called *rigid* if the structure maps from (3.12)

$$\underline{\mathcal{F}}'_{Y,T} : \quad \mathcal{F}(Y \times T) \to [T, \mathcal{F}(Y)]$$

are isomorphisms in $\mathcal{C}$, for all $Y \in \mathrm{Perf}_S$, $T \in \mathrm{ProFin}$. Equivalently, if the maps

$$\underline{\mathcal{F}}'_{Y,T}(*) : \quad \mathcal{F}(Y \times T, *) \to \mathcal{F}(Y, T)$$

are isomorphisms of sets, for all $Y \in \mathrm{Perf}_S$ and $T \in \mathrm{ProFin}$. The full subcategory of rigid presheaves in $\underline{\mathrm{PSh}}(S, \mathcal{C})$ will be denoted by $\underline{\mathrm{PSh}}^{\mathrm{rig}}(S, \mathcal{C})$. We will denote by $\underline{\mathrm{Sh}}^{\mathrm{rig}}(S, \mathcal{C})$ its full subcategory of *rigid sheaves*: a rigid presheaf $\mathcal{F}$ is a sheaf if the pro-étale presheaf $\eta_* \mathcal{F}$ is a sheaf.

We note that the equivalence in the above definition follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(Y \times T \times T_1, *) \arrow[d, "\underline{\mathcal{F}}'_{Y \times T, T_1}(*)"'] \arrow[dr, "\underline{\mathcal{F}}'_{Y, T \times T_1}(*)"] & \\
\mathcal{F}(Y \times T, T_1) \arrow[r, dashed, "\underline{\mathcal{F}}'_{Y,T}(T_1)"] & \mathcal{F}(Y, T \times T_1),
\end{tikzcd}$$

which is just a rewriting of the composition commutative diagram

$$\begin{tikzcd}
\mathcal{F}(Y \times T \times T_1)(*) \arrow[d, "\underline{\mathcal{F}}'_{Y \times T, T_1}(*)"'] \arrow[dr, "\underline{\mathcal{F}}'_{Y, T \times T_1}(*)"] & \\
{[T_1, \mathcal{F}(Y \times T)](*)} \arrow[r, dashed, "{[T_1, \underline{\mathcal{F}}'_{Y,T}]}"] & {[T_1, [T, \mathcal{F}(Y)]](*) = [T_1 \times T, \mathcal{F}(Y)](*)}.
\end{tikzcd}$$

Lemma 3.16. *Let $\mathcal{F} \in \mathrm{PSh}(S_{\text{proét}})$. The topologically enriched presheaf $\pi_* \mathcal{F} \in \underline{\mathrm{PSh}}(S)$ is a rigid sheaf. Moreover, the functor*

$$\pi_*^{\mathrm{rig}} : \quad \mathrm{Sh}(S_{\text{proét}}) \to \underline{\mathrm{Sh}}^{\mathrm{rig}}(S_{\text{proét}})_0$$

is an equivalence of categories.

Remark 3.17. The above lemma is also valid for the pair $\mathrm{Ab}, \mathrm{CondAb}$ (in place of the pair $\mathrm{Set}, \mathrm{Cond}$). With the same proof.

Proof. By Lemma 3.5, the presheaf $\pi_*\mathcal{F}$ is topologically enriched. To see that it is rigid we need to show that the structure maps

$$(\underline{\pi_*\mathcal{F}})'_{Y,T}(*): \quad \pi_*\mathcal{F}(Y\times T,*)\to \pi_*\mathcal{F}(Y,T) \tag{3.18}$$

are isomorphisms of sets. But this is clear since they are isomorphic to the identity maps

$$\mathrm{Id}:\mathcal{F}(Y\times T)\to \mathcal{F}(Y\times T).$$

Moreover, we have $\eta_*\pi_*\mathcal{F}\simeq\mathcal{F}$ hence $\pi_*\mathcal{F}$ is a sheaf.

Concerning the second claim of the lemma, let us start with essential surjectivity of the functor π_*. Take a rigid sheaf $\mathcal{F}$. We have the structure maps

$$\underline{\mathcal{F}}'_{Y,T}(*): \quad \mathcal{F}(Y\times T,*)\to \mathcal{F}(Y,T), \tag{3.19}$$

of sets satisfying certain compatibilities. In the language of Section 2.2.1, they give a natural transformation of topologically enriched presheaves[9]

$$\pi_*\eta_*\mathcal{F}\to\mathcal{F}.$$

Since these maps are isomorphisms by assumption, we get the essential surjectivity.

It remains to show that the functor π_* is fully faithful. Faithfulness is clear. Let $\mathcal{F},\mathcal{G}\in \mathrm{Sh}(S_{\text{proét}})$ and take a map of topologically enriched presheaves $f:\pi_*\mathcal{F}\to\pi_*\mathcal{G}$. Define a map $\tilde{f}:\mathcal{F}\to\mathcal{G}$ by setting $\tilde{f}_Y=f_Y(*)$. We need to show that that $f=\pi_*\tilde{f}$. But this can be seen, after unwiding the definitions, from the following commutative diagram (for all $Y\in\mathrm{Perf}_S, T\in\mathrm{ProFin}$)

$$\begin{tikzcd}
\pi_*\mathcal{F}(Y,T) \arrow[r, "f_Y(T)"] & \pi_*\mathcal{F}(Y,T) \\
\mathcal{F}(Y\times T,*) \arrow[u, "\underline{\mathcal{F}}'_{Y,T}(*)"] \arrow[r, "f_{Y\times T}(*)"] & \mathcal{G}(Y\times T,*) \arrow[u, "\underline{\mathcal{G}}'_{Y,T}(*)"']
\end{tikzcd}$$

Here, the vertical maps, after identification of the source and the target, are just identities. □

Remark 3.20. It follows from Lemma 3.16 that a rigid presheaf $\mathcal{F}$ is a sheaf if and only if the topological presheaf $\mathcal{F}^{\mathrm{cl}}$ is a sheaf.

3.1.5. *Enriched Yoneda Lemma.* Recall the following classical construction from [18, Sec. 2.4]. Let $S\in\mathrm{Perf}_C$, $Y\in\mathrm{Perf}_S$. We define the functor

$$h_Y^{\mathrm{top}}:\mathrm{Perf}_S^{\mathrm{op}}\to\mathrm{Cond},\quad X\mapsto \underline{\mathrm{Hom}}_S(X,Y).$$

It is easy to check that

$$h_Y^{\mathrm{top}}\simeq\pi_*h_Y, \tag{3.21}$$

where $\pi:S_{\text{proét}}\to S^{\mathrm{top}}_{\text{proét}}$ is the canonical projection. In particular, since the pro-étale site is subcanonical and thus the presheaf h_Y is a sheaf, same is true of h_Y^{top}. Moreover, by Lemma 3.5, this sheaf is canonically Cond-enriched.

Let $\mathcal{F}\in\underline{\mathrm{PSh}}(S)$. The Cond-*enriched Yoneda Lemma* yields an existence of a Cond-natural isomorphism

$$\underline{\mathrm{Hom}}(h_Y^{\mathrm{top}},\mathcal{F})=\int_{Y_1\in\mathrm{Perf}_S}\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1,Y),\mathcal{F}(Y_1))\xleftarrow{\sim}\mathcal{F}(Y) \tag{3.22}$$

induced by the maps

$$\varphi_Y:\mathcal{F}(Y)\to\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1,Y),\mathcal{F}(Y_1))$$

obtained, via the adjunction $\mathrm{Hom}(x,\underline{\mathrm{Hom}}(y,z))\simeq\mathrm{Hom}(y,\underline{\mathrm{Hom}}(x,z))$, $x,y,z\in\mathrm{Cond}$, from the structure maps

$$\underline{\mathcal{F}}_{Y_1,Y}:\underline{\mathrm{Hom}}_S(Y_1,Y)\to\underline{\mathrm{Hom}}(\mathcal{F}(Y),\mathcal{F}(Y_1)).$$

[9]The fact that this is a natural transformation follows from the coherence of the structure maps (3.19).

The special case of (3.22) with $\mathcal{F} = h^{\mathrm{top}}_{Y_1}, Y_1 \in \mathrm{Perf}_S$, proves that the functor h^{top}_* is fully faithful. Moreover, we have $h^{\mathrm{top}}_{Y_1 \times_S Y_2} \simeq h^{\mathrm{top}}_{Y_1} \times h^{\mathrm{top}}_{Y_2}$.

Equivalently, we can formulate an enriched *co-Yoneda Lemma* as a Cond-natural isomorphism

$$\int^{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \mathcal{F}(Y_1) \xrightarrow{\sim} \mathcal{F}(Y)$$

induced by the morphisms

$$\underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \mathcal{F}(Y_1) \to \mathcal{F}(Y),$$

which are adjoints to the structure maps $\mathcal{F}_{Y,Y_1}$. Here $\int^{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \mathcal{F}(Y_1)$ is the enriched coend: the coequalizer given by the diagram

$$\coprod_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \mathcal{F}(Y_1) \leftleftarrows \coprod_{Y_1, Y_2 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \underline{\mathrm{Hom}}_S(Y_1, Y_2) \otimes \mathcal{F}(Y_2),$$

where the parallel arrows are induced by the composition and the evaluation maps

$$\underline{\mathrm{Hom}}_S(Y, Y_1) \otimes \underline{\mathrm{Hom}}_S(Y_1, Y_2) \to \underline{\mathrm{Hom}}_S(Y, Y_2), \quad \underline{\mathrm{Hom}}_S(Y_1, Y_2) \otimes \mathcal{F}(Y_2) \to \mathcal{F}(Y_1).$$

3.1.6. *Monoidal structures.* Let $S \in \mathrm{Perf}_C$ and let $\mathcal{C} = \mathrm{Cond}$. We will use the same notation for tensor products and internal Hom's of Cond-presheaves on the category Perf_S as in Section 2.2.2 and Section 2.2.3.

(•) *Tensor product.* The tensor products of presheaves are defined objectwise and the structure maps are modified in a canonical way.

(•) *Internal* Hom. For two Cond-presheaves $\mathcal{F}, \mathcal{G}$, we set

$$\begin{aligned}\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}, \mathcal{G}) &:= \{Y \mapsto \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(h^{\mathrm{top}}_Y \otimes \mathcal{F}, \mathcal{G})\} \\ &= \{Y \mapsto \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}(Y_1), \mathcal{G}(Y_1))\}.\end{aligned}$$

This is a Cond-presheaf (with values in Cond).

(•) *Adjunction.* We have the usual tensor-hom adjunction:

Lemma 3.23. *Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \underline{\mathrm{PSh}}(S)$. We have functorial isomorphisms in* Cond *and* $\underline{\mathrm{PSh}}(S)$*, respectively:*

$$\begin{aligned}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) &\simeq \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F}_1, \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_2, \mathcal{F}_3)), \\ \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) &\simeq \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_1, \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_2, \mathcal{F}_3)).\end{aligned}$$

Proof. For the first isomorphism, the computation is standard: We have the following sequence of functorial isomorphisms in Cond

$$\begin{aligned}&\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F}_1, \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_2, \mathcal{F}_3)) \\ &\quad \simeq \int_{Y \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\mathcal{F}_1(Y), \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1))) \\ &\quad \simeq \int_{Y \in \mathrm{Perf}_S} \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\mathcal{F}_1(Y), \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1))) \\ &\quad \simeq \int_{Y \in \mathrm{Perf}_S} \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\mathcal{F}_1(Y) \otimes \underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1)) \\ &\quad \simeq \int_{Y_1 \in \mathrm{Perf}_S} \int_{Y \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\mathcal{F}_1(Y) \otimes \underline{\mathrm{Hom}}_S(Y_1, Y) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1)) \\ &\quad \simeq \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}((\int^{Y \in \mathrm{Perf}_S} \mathcal{F}_1(Y) \otimes \underline{\mathrm{Hom}}_S(Y_1, Y)) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1)) \\ &\quad \simeq \int_{Y_1 \in \mathrm{Perf}_S} \underline{\mathrm{Hom}}(\mathcal{F}_1(Y_1) \otimes \mathcal{F}_2(Y_1), \mathcal{F}_3(Y_1)) \\ &\quad \simeq \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3).\end{aligned}$$

Here, the second and the fifth isomorphisms hold because internal Hom in Cond commutes with limits and colimits in the second and the first variable, respectively; the third one – by adjunction in Cond; the fourth one – by the Fubini theorem for ends; the sixth one – by co-Yoneda Lemma.

For the second isomorphism of the lemma, evaluating on $Y \in \mathrm{Perf}_S$, it suffices to show that we have a natural isomorphism in Cond

$$\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(h_Y^{\mathrm{top}} \otimes \mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) \simeq \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(h_Y^{\mathrm{top}} \otimes \mathcal{F}_1, \mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_2, \mathcal{F}_3)).$$

But this follows from the first isomorphism of the lemma. $\square$

($\bullet$) *Generators.* The above implies the following:

Lemma 3.24. *The category of* Cond*-presheaves* $\underline{\mathrm{PSh}}(S)$ *is generated by the family* $\{h_Y^{\mathrm{top}} \otimes W_i\}, i \in I$, $Y \in \mathrm{Perf}_S$*, where* $\{W_i\}, i \in I$ *is a family of generators of* Cond.

Proof. We follow the argument in the proof of [17, Th. 4.2]. Let $\alpha_1, \alpha_2 : \mathcal{F} \to \mathcal{G}$ be two maps in $\underline{\mathrm{PSh}}(S)$ such that $\alpha_1 \neq \alpha_2$. We want to show that there is $i \in I$, $Y \in \mathrm{Perf}_S$, and a map $\beta : h_Y^{\mathrm{top}} \otimes W_i \to \mathcal{F}$ such that $\alpha_1\beta \neq \alpha_2\beta$.

Since $\alpha_1 \neq \alpha_2$, there exists $Y \in \mathrm{Perf}_S$ such that $\alpha_{1,Y} \neq \alpha_{2,Y} : \mathcal{F}(Y) \to \mathcal{G}(Y)$ in Cond. We fix such a Y. Since $\{W_i\}, i \in I$, are generators of Cond, there exists a map $\overline{\beta} : W_i \to \mathcal{F}(Y)$ such that $\alpha_{1,Y}\overline{\beta} \neq \alpha_{2,Y}\overline{\beta}$. But, for any non-zero presheaf $\mathcal{F} \in \underline{\mathrm{PSh}}(S)$, we have natural isomorphisms

$$\mathrm{Hom}_{S^{\mathrm{top}}}(h_Y^{\mathrm{top}} \otimes W_i, \mathcal{F}) \simeq \mathrm{Hom}(W_i, \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(h_Y^{\mathrm{top}}, \mathcal{F})) \simeq \mathrm{Hom}(W_i, \mathcal{F}(Y));$$

the second one by the enriched Yoneda Lemma. Hence we can find a unique map $\beta : h_Y^{\mathrm{top}} \otimes W_i \to \mathcal{F}$ corresponding to $\overline{\beta}$. Now $\alpha_{1,Y}\overline{\beta} \neq \alpha_{2,Y}\overline{\beta}$ implies that $\alpha_1\beta \neq \alpha_2\beta$, as wanted. $\square$

Remark 3.25. (1) Everything in Section 3.1 outside of the definition of rigid sheaves works as well with the category Perf_C replaced by the category sPerf_C of strictly totally disconnected affinoids $S \in \mathrm{Perf}_C$. We note that we do have an analog of Lemma 3.5 by restricted the sheaves $\pi_*\mathcal{F}$ to the category sPerf_C.

(2) It is possible to define a notion of a topologically enriched sheaf as a finite-products preserving enriched functor. Such objects form a full reflective subcategory of topologically enriched presheaves in the sense of [5], [23] (the key point being that the naive sheafification of a topologically enriched presheaf is canonically enriched). In fact in the first draft of this paper we have worked in this setting only later realizing that this does not add anything (besides extra layer of complexity) to the applications we had in mind. It might be however useful in the future.

(3) We will call presheaves from $\mathrm{PSh}(\mathrm{Spa}(C), \underline{\mathbf{Q}}_{p,\square})$ *Topological Vector Spaces* (TVS for short). We will also use this term for the derived category of TVS's (see below) and sometimes, abusively, for the analogous categories with values in condensed Abelian groups, etc.

3.2. **Derived version.**

We study here the ∞-derived category of topologically enriched sheaves, show that it admits a canonical Cond-enrichment, and compare it with the category of Cond-enriched presheaves with values in corresponding ∞-derived Cond-categories.

3.2.1. *Derived categories of topologically enriched presheaves.* We start with the derived categories of topologically enriched presheaves:

$$\begin{aligned} &\underline{\mathrm{PSh}}(S, \mathrm{CondAb})_0, \quad \underline{\mathrm{PSh}}(S, \mathrm{Solid})_0, \\ &\underline{\mathrm{PSh}}(S, \underline{\mathbf{Q}}_p)_0, \quad \underline{\mathrm{PSh}}(S, \underline{\mathbf{Q}}_{p,\square})_0. \end{aligned} \tag{3.26}$$

These are Grothendieck abelian with compact, projective generators. Hence we have K-injective as well as K-flat resolutions. This yields derived (internal) Hom's, derived tensor products, and the (internal) tensor-hom adjunctions in the corresponding derived ∞-categories

$$\underline{\mathcal{D}}(S,\mathrm{CondAb}),\quad \underline{\mathcal{D}}(S,\mathrm{Solid}),$$
$$\underline{\mathcal{D}}(S,\underline{\mathbf{Q}}_p),\quad \underline{\mathcal{D}}(S,\underline{\mathbf{Q}}_{p,\square}).$$

These categories are defined by taking derived ∞-categories of the corresponding $(-)_0$ categories from (3.26) and then canonically enriching them via the global sections of the derived internal Hom's. The enriched tensor-hom adjunctions are inherited from the internal ones.

We have the following result concerning the relationship between constant sheaves and monoidal structures:

Lemma 3.27. *The enriched analog of Lemma 2.27 holds.*

Proof. Claim (1) of Lemma 2.27 has the same proof in this setting. For the second claim, we use the enriched end description

$$\begin{aligned}\underline{\mathrm{Hom}}^{\square}_{S,\mathbf{Q}_p}(\underline{W},\mathcal{F}) \xrightarrow{\sim} \underline{\mathrm{Hom}}^{\square}_{S,\mathbf{Q}_p}(W\otimes_{\mathbf{Q}_{p,\square}}\underline{\mathbf{Q}}_p,\mathcal{F}) = \int_{Y\in\mathrm{sPerf}_S}\underline{\mathrm{Hom}}_{\mathbf{Q}_p(Y)}(W\otimes_{\mathbf{Q}_{p,\square}}\mathbf{Q}_p(Y),\mathcal{F}(Y))\\ \simeq \int_{Y\in\mathrm{sPerf}_S}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W,\mathcal{F}(Y)) \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W,\int_{Y\in\mathrm{sPerf}_S}\mathcal{F}(Y)) \simeq \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(W,\mathcal{F}(S)).\end{aligned}$$

Here, we wrote $\int_{Y\in\mathrm{sPerf}_S}\mathcal{F}(Y)$ for the equalizer

$$\prod_{Y\in\mathrm{sPerf}_S}\mathcal{F}(Y) \rightrightarrows \prod_{Y,Y_1\in\mathrm{sPerf}_S}\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}_S(Y_1,Y),\mathcal{F}(Y_1))$$

and the first isomorphism follows from the first claim of the lemma.

The third claim of the lemma is the same as the second one and the fourth one follows from the third one. $\square$

3.2.2. *Condensed vs solid topologically enriched sheaves.* The following result is an enriched version of Proposition 2.29.

Proposition 3.28. *(1) The forgetful functor*

$$\underline{\mathrm{PSh}}(S,\mathrm{Solid}) \to \underline{\mathrm{PSh}}(S,\mathrm{CondAb}) \tag{3.29}$$

is (topologically) fully faithful. The essential image is stable under all limits, colimits, and extensions. Moreover, the inclusion (3.29) *admits a (topological) left adjoint, the solidification functor,*

$$\underline{\mathrm{PSh}}(S,\mathrm{CondAb}) \to \underline{\mathrm{PSh}}(S,\mathrm{Solid}):\quad \mathcal{F}\mapsto\mathcal{F}^{\square}, \tag{3.30}$$

which preserves all colimits and is symmetric monoidal. We have analogous claims for the forgetful functor $\underline{\mathrm{PSh}}(S,\mathbf{Q}_{p,\square}) \to \underline{\mathrm{PSh}}(S^{\mathrm{top}},\mathbf{Q}_p)$.

(2) The forgetful functor

$$\underline{\mathcal{D}}(S,\mathrm{Solid}) \to \underline{\mathcal{D}}(S,\mathrm{CondAb}) \tag{3.31}$$

is (topologically) fully faithful. It preserves all limits and colimits.

(3) The functor (3.31) *admits a left adjoint*

$$\mathcal{F}\mapsto\mathcal{F}^{\mathrm{L}\square}:\underline{\mathcal{D}}(S,\mathrm{CondAb})\to\underline{\mathcal{D}}(S,\mathrm{Solid}),$$

which is the (topological) left derived functor of the solidification functor $(-)^{\square}$. *It is symmetric monoidal. We have analogous claims for the forgetful functor* $\underline{\mathcal{D}}(S,\underline{\mathbf{Q}}_{p,\square}) \to \underline{\mathcal{D}}(S,\underline{\mathbf{Q}}_p)$.

(4) For $\mathcal{F}_1,\mathcal{F}_2\in\mathcal{D}(S,\mathrm{Solid})$, *the natural map*

$$\mathrm{R}\mathcal{H}om^{\square}_S(\mathcal{F}_1,\mathcal{F}_2)\to\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\mathcal{F}_1,\mathcal{F}_2)$$

is a quasi-isomorphism. We have an analogous claim for the $\underline{\mathbf{Q}}_p$*-sheaves.*

Proof. The proof of Proposition 2.29 goes through almost verbatim (we just need to replace h_Y^δ with h_Y^{top}) since we have Lemma 3.2 (for the second claim). □

4. Fully-faithfulness results

Topological Vector Spaces are closely related to Vector Spaces as well as perfect complexes on the Fargues-Fontaine curve. In this section we prove two relevant fully-faithfulness results.

4.1. Vector Spaces and Topological Vector Spaces.

The algebraic pro-étale and the topological presheaves are closely related assuming that we enrich the latter. Let $S\in \mathrm{sPerf}_C$.

Theorem 4.1. (Enriched fully-faithfulness) *Let $\mathcal{F}\in\mathcal{D}^b(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})$ be such that $\mathrm{R}\pi_*\mathcal{F}\in\underline{\mathcal{D}}^b(S,\mathrm{CondAb})_0$ and let $\mathcal{G}\in\mathcal{D}^+(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})$. The canonical morphism in $\underline{\mathcal{D}}(S,\mathrm{CondAb})$*

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\to\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\mathrm{R}\pi_*\mathcal{F},\mathrm{R}\pi_*\mathcal{G})\tag{4.2}$$

is a quasi-isomorphism. We have analogous claims for $\underline{\mathbf{Q}}_p$-sheaves.

Remark 4.3. The map (4.2) is constructed in the usual way: By adjointness, it suffices to construct a map

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\otimes^{\mathrm{L}}\mathrm{R}\pi_*\mathcal{F}\to\mathrm{R}\pi_*\mathcal{G}.$$

For this, we will use the composition

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\otimes^{\mathrm{L}}\mathrm{R}\pi_*\mathcal{F}\to\mathrm{R}\pi_*(\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\otimes^{\mathrm{L}}\mathcal{F})\to\mathrm{R}\pi_*\mathcal{G},$$

where the second arrow is $\mathrm{R}\pi_*$ applied to the canonical map

$$\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\otimes^{\mathrm{L}}\mathcal{F}\to\mathcal{G}$$

and the first arrow is the relative cup product defined in the following way. For $\mathcal{G}_1,\mathcal{G}_2\in\mathcal{D}(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})$, the relative cup product map in $\underline{\mathcal{D}}(S,\mathrm{CondAb})$

$$\mathrm{R}\pi_*\mathcal{G}_1\otimes^{\mathrm{L}}\mathrm{R}\pi_*\mathcal{G}_2\to\mathrm{R}\pi_*(\mathcal{G}_1\otimes^{\mathrm{L}}\mathcal{G}_2)$$

is induced by the functorial maps in $\mathcal{D}(\mathrm{Ab})$, for $Y\in\mathrm{sPerf}_S$, profinite set T:

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y\times T,\mathcal{G}_1)\otimes^{\mathrm{L}}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y\times T,\mathcal{G}_2)\to\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y\times T,\mathcal{G}_1\otimes^{\mathrm{L}}\mathcal{G}_2).\tag{4.4}$$

These maps are compatible with the enriched structure maps as the latter are just induced by the identity maps (see the proof of Lemma 3.16).

4.1.1. *Proof of Theorem 4.1.* We start with a reduction. For $\mathcal{F}$: we may assume that $\mathcal{F}$ is represented by a bounded complex, then we can take an injective resolution and truncate it to replace $\mathcal{F}$ with a bounded complex $\mathcal{F}'$ such that $\mathrm{R}\pi_*\mathcal{F}'\simeq\pi_*\mathcal{F}'$, and, finally, do dévissage. For $\mathcal{G}$: we replace $\mathcal{G}$ by a bounded below complex of injectives, then by a limit argument and a dévissage pass to a single injective. We end up with needing to show that

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_S(\mathcal{F},\mathcal{G})\xrightarrow{\sim}\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\pi_*\mathcal{F},\pi_*\mathcal{G})$$

for $\mathcal{F},\mathcal{G}\in\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})$ and $\mathcal{G}$ injective.

Step 1. *Passage to presheaves.* We will factor the functor π_* in the following way:

$$\pi_*:\mathrm{Sh}(S_{\mathrm{pro\acute{e}t}},\mathrm{Ab})\xrightarrow{\varepsilon_*}\mathrm{Sh}(S^{\mathrm{std}}_{\mathrm{pro\acute{e}t}},\mathrm{Ab})\xrightarrow{\iota_*}\mathrm{PSh}(\mathrm{sPerf}_S,\mathrm{Ab})\xrightarrow{\pi_*^{\mathrm{psh}}}\underline{\mathrm{PSh}}(S,\mathrm{CondAb})_0.$$

Here, the site $S^{\mathrm{std}}_{\mathrm{pro\acute{e}t}}$ is the category sPerf_S equipped with the pro-étale topology. The functor ε_* is an equivalence of categories. The functor π_*^{psh} involves sheafification in the T-direction[10]; it is

[10]This amounts to forcing the additivity property for presheaves on extremally totally disconnected profinite sets and it is easy to see that this sheafification process preserves enrichment.

exact. It is well-defined because, for $Y \in \mathrm{sPerf}_S$ and a profinite set T, the affinoid perfectoid $Y \times T$ is strictly totally disconnected. For $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(S_{\text{proét}}, \mathrm{Ab})$, we have

$$\mathrm{R}\iota_* \mathrm{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_S(\mathrm{R}\iota_*\mathcal{F}, \mathrm{R}\iota_*\mathcal{G}).$$

This follows from the adjunction

$$\mathrm{R}\iota_* \mathrm{R}\mathcal{H}om_S(\iota^*\mathrm{R}\iota_*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_S(\mathrm{R}\iota_*\mathcal{F}, \mathrm{R}\iota_*\mathcal{G})$$

and the fact that $\iota^*\mathrm{R}\iota_*\mathcal{F} \simeq \mathcal{F}$. If moreover $\mathcal{G}$ is injective, since $\iota^*\iota_*\mathcal{F} \simeq \mathcal{F}$, similar adjunction yields

$$\mathrm{R}\iota_* \mathrm{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_S(\iota_*\mathcal{F}, \iota_*\mathcal{G}).$$

Step 2. *Enriched presheaves fully-faithfulness.* Hence it suffices to show that the natural map

$$\mathrm{R}\pi_*^{\mathrm{psh}} \mathrm{R}\mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F}, \pi_*^{\mathrm{psh}}\mathcal{G})$$

is a quasi-isomorphism for $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(\mathrm{sPerf}_S, \mathrm{Ab})$, with $\mathcal{G}$ an injective sheaf. Or that:

$$\pi_*^{\mathrm{psh}} \mathcal{H}om_S(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F}, \pi_*^{\mathrm{psh}}\mathcal{G}), \tag{4.5}$$
$$\mathrm{R}^i\mathcal{H}om_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F}, \pi_*^{\mathrm{psh}}\mathcal{G}) = 0, \quad i > 0.$$

We write $\mathcal{F} = \mathrm{colim}_i\, \mathbf{Z}[h_{Z_i}]$, for $Z_i \in \mathrm{sPerf}_S$. The presheaves $\mathbf{Z}[h_{Z_i}]$ are projective. To prove the first isomorphism above, since the functor π_*^{psh} commutes with limits, it suffices to show that, for $Z \in \mathrm{sPerf}_S$, we have

$$\pi_*^{\mathrm{psh}} \mathcal{H}om_S(\mathbf{Z}[h_Z], \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{S^{\mathrm{top}}}(\mathbf{Z}[h_Z^{\mathrm{top}}], \pi_*^{\mathrm{psh}}\mathcal{G}).$$

Evaluating both sides on (Y, T), we see that we need to show that

$$\mathrm{Hom}_S(\mathbf{Z}[h_Z] \otimes \mathbf{Z}[h_{Y\times T}], \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Z^{\mathrm{top}}] \otimes \mathbf{Z}[h_Y^{\mathrm{top}}] \otimes \mathbf{Z}[T], \pi_*^{\mathrm{psh}}\mathcal{G}).$$

Or, rewriting, that

$$\mathrm{Hom}_S(\mathbf{Z}[h_{Z\times_S Y}] \otimes \mathbf{Z}[h_T], \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_{Z\times_S Y}^{\mathrm{top}}] \otimes \mathbf{Z}[T], \pi_*^{\mathrm{psh}}\mathcal{G}).$$

We note that the above map is induced by the canonical isomorphism $\mathbf{Z}[h^{\mathrm{top}}_{Y\times_S Z}] \xrightarrow{\sim} \pi_*^{\mathrm{psh}}\mathbf{Z}[h_{Y\times_S Z}]$ and the morphism $\mathbf{Z}[T] \to \pi_*^{\mathrm{psh}}\mathbf{Z}[h_T]$, which factor through the canonical isomorphism $\mathbf{Z}[h_T^{\mathrm{top}}] \xrightarrow{\sim} \pi_*^{\mathrm{psh}}\mathbf{Z}[h_T]$ (and corresponds to the identity in $\mathrm{Hom}(T,T)$).

Writing $\mathbf{Z}[h_{Z\times_S Y}] = \mathrm{colim}_i\, \mathbf{Z}[h_{Z_i}]$, for $Z_i \in \mathrm{sPerf}_S$, we reduce to showing that, for $Z \in \mathrm{sPerf}_S$, we have

$$\mathrm{Hom}_S(\mathbf{Z}[h_Z] \otimes \mathbf{Z}[h_T], \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Z^{\mathrm{top}}] \otimes \mathbf{Z}[T], \pi_*^{\mathrm{psh}}\mathcal{G}) \tag{4.6}$$

But, by the classical and enriched Yoneda Lemmas, respectively, we have

$$\mathrm{Hom}_S(\mathbf{Z}[h_Z] \otimes \mathbf{Z}[h_T], \mathcal{G}) \simeq \mathrm{Hom}_S(\mathbf{Z}[h_{Z\times T}], \mathcal{G}) \simeq \mathcal{G}(Z\times T),$$
$$\mathrm{Hom}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Z^{\mathrm{top}}] \otimes \mathbf{Z}[T], \pi_*^{\mathrm{psh}}\mathcal{G}) \simeq \underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathbf{Z}[h_Z^{\mathrm{top}}], \pi_*^{\mathrm{psh}}\mathcal{G})(T) \simeq \pi_*^{\mathrm{psh}}\mathcal{G}(Z)(T) \simeq \mathcal{G}(Z\times T).$$

This yields the isomorphism (4.6).

To finish the proof of our theorem we need to show that

$$\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F}, \pi_*^{\mathrm{psh}}\mathcal{G})$$

is concentrated in degree 0. Or that so are its values on Y:

$$\mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F} \otimes^{\mathrm{L}} \mathbf{Z}[h_Y^{\mathrm{top}}], \pi_*^{\mathrm{psh}}\mathcal{G}).$$

For that, writing $\mathcal{F} = \mathrm{colim}_i\, \mathbf{Z}[h_{Z_i}]$, for $Z_i \in \mathrm{sPerf}_S$, and then each $\mathbf{Z}[h_{Z_i\times_S Y}]$ as a colimit of $\mathbf{Z}[h_Z]$, for $Z \in \mathrm{sPerf}_S$, we compute in $\mathcal{D}(\mathrm{CondAb})$:

$$\begin{aligned}\mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\pi_*^{\mathrm{psh}}\mathcal{F} \otimes^{\mathrm{L}} \mathbf{Z}[h_Y^{\mathrm{top}}], \pi_*\mathcal{G}) &\simeq \mathrm{R}\lim_i \mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathbf{Z}[h^{\mathrm{top}}_{Z_i\times_S Y}], \pi_*^{\mathrm{psh}}\mathcal{G}) \\ &\simeq \mathrm{R}\lim_j \mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}}}(\mathbf{Z}[h^{\mathrm{top}}_{Z_j}], \pi_*^{\mathrm{psh}}\mathcal{G}) \simeq \mathrm{R}\lim_j \pi_*^{\mathrm{psh}}\mathcal{G}(Z_j).\end{aligned}$$

Here the last quasi-isomorphism follows from the enriched Yoneda Lemma. Evaluating now the above on extremally disconnected T, we get

$$\begin{aligned}\mathrm{R}\lim_j \pi_*^{\mathrm{psh}}\mathcal{G}(Z_j)(T) &\simeq \mathrm{R}\lim_j \mathcal{G}(Z_j\times T)\simeq \mathrm{R}\lim_j \mathrm{RHom}_S(\mathbf{Z}[h_{Z_j\times T}],\mathcal{G})\\ &\simeq \mathrm{RHom}_S(\mathrm{colim}_i\,\mathbf{Z}[h_{Z_j\times T}],\mathcal{G})\simeq \mathrm{Hom}_S(\mathrm{colim}_i\,\mathbf{Z}[h_{Z_j\times T}],\mathcal{G}).\end{aligned}$$

Here we used that $\mathcal{G}$ is injective. Since, clearly, $\mathrm{Hom}_S(\mathrm{colim}_i\,\mathbf{Z}[h_{Z_j\times T}],\mathcal{G})$ is concentrated in degree 0, we are done.

The arguments for $\mathbf{Q}_p$-sheaves are analogous using the fact that $\pi_*^{\mathrm{psh}}\underline{\mathbf{Q}}_p[h_Y]^{\mathrm{psh}}\simeq \underline{\mathbf{Q}}_p[h_Y^{\mathrm{top}}]^{\mathrm{psh}}$.

4.1.2. *Applications of Theorem 4.1.* We list now two applications of Theorem 4.1.

Corollary 4.7. *Let $\mathcal{F},\mathcal{G}\in\{\mathbb{G}_a,\underline{\mathbf{Q}}_p\}$. Then we have a natural quasi-isomorphism*

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{S,\mathbf{Q}_p}(\mathcal{F},\mathcal{G})\xrightarrow{\sim}\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathcal{F},\mathcal{G}).$$

Proof. This follows immediately from Theorem 4.1 because both $\mathcal{F}$ and $\mathcal{G}$ are π_*-acyclic. □

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ and a perfect residue field. Let X be a smooth partially proper rigid analytic variety over K. Let $\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},*}(X_C,\mathbf{Q}_p)$, $*\in\{\varnothing,c\}$, be the Vector Space representing pro-étale cohomology:

$$S\mapsto \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},*}(X_S,\mathbf{Q}_p).$$

Corollary 4.8. *We have a natural quasi-isomorphism*

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{S,\mathbf{Q}_p}(\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},*}(X_C,\mathbf{Q}_p),\mathbf{Q}_p)\xrightarrow{\sim}\mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathrm{R}\pi_*\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},*}(X_C,\mathbf{Q}_p(r)),\mathbf{Q}_p).$$

Proof. Since the geometrized p-adic comparison theorems (see [11], [1]) imply that $\mathrm{R}\pi_*\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},*}(X_C,\mathbf{Q}_p(r))$ has a finite amplitude, this follows from Theorem 4.1. □

Remark 4.9. (1) Theorem 4.1 stays valid (with the same proof) if we take $S\in\mathrm{Perf}_C$, $\mathcal{F}$ – a bounded complex of sheaves, and replace $\mathrm{R}\pi_*\mathcal{F}$ with $\pi_*\mathcal{F}$.

(2) Corollary 4.7 is the reason why we work with strictly totally disconnected spaces in the definition of TVS's: we need vanishing of pro-étale cohomology of $\mathbb{G}_a$ and $\underline{\mathbf{Q}}_p$ in nontrivial degrees[11].

(3) We note however that we can not go "higher", that is, we can not replace strictly totally disconnected spaces Y with w-contractible spaces. This is because in the proof of Theorem 4.1 we need that Y remains strictly totally disconnected after base change to T and an analogous permanence property does not hold for w-contractible Y.

4.2. **Perfect complexes on the Fargues-Fontaine curve and Topological Vector Spaces.** Here we will prove a fully-faithfulness result between perfect complexes on the Fargues-Fontaine curve and Topological Vector Spaces. Via Theorem 4.1 we will reduce it to an analogous result for Vector Spaces (proved by Anschütz-Le Bras in [3]).

4.2.1. *Quasi-coherent sheaves on the Fargues-Fontaine curve.* Recall the definition of the relative Fargues-Fontaine curve (see [27, Lecture 12]). Let $S=\mathrm{Spa}(R,R^+)$ be an affinoid perfectoid space over the finite field $\mathbf{F}_p$. Let

$$Y_{\mathrm{FF},S}:=\mathrm{Spa}(W(R^+),W(R^+))\smallsetminus V(p[p^\flat])$$

be the relative mixed characteristic punctured unit disc. It is an analytic adic space over $\mathbf{Q}_p$. The Frobenius on R^+ induces the Witt vector Frobenius and hence a Frobenius φ on $Y_{\mathrm{FF},S}$ with free and totally discontinuous action. The Fargues-Fontaine curve relative to S (and $\mathbf{Q}_p$) is defined as

$$X_{\mathrm{FF},S}:=Y_{\mathrm{FF},S}/\varphi^{\mathbf{Z}}.$$

[11]Since the pro-étale cohomology of $\mathbb{G}_a$ vanishes in degrees ≥ 1 on any perfectoid affinoid and the pro-étale cohomology of $\underline{\mathbf{Q}}_p$ vanishes on sympathetic perfectoids, we could have also used here sympathetic spaces.

For an interval $I = [s,r] \subset (0,\infty)$ with rational endpoints, we have the open subset

$$Y_{\mathrm{FF},S,I} := \{|\cdot| : |p|^r \leq |[p^\flat]| \leq |p|^s\} \subset Y_{\mathrm{FF},S}.$$

It is a rational open subset of $\mathrm{Spa}(W(R^+), W(R^+))$ hence an affinoid space,

$$Y_{\mathrm{FF},S,I} := \mathrm{Spa}(\mathbf{B}_{S,I}, \mathbf{B}^+_{S,I}).$$

One can form $X_{\mathrm{FF},S}$ as the quotient of $Y_{\mathrm{FF},S,[1,p]}$ via the identification $\varphi : Y_{\mathrm{FF},S,[1,1]} \xrightarrow{\sim} Y_{\mathrm{FF},S,[p,p]}$.

Let Y be an analytic adic space over $\mathbf{Q}_p$. We denote by $\mathrm{QCoh}(Y)$ the ∞-category of (solid) quasi-coherent sheaves on Y, and by $\mathrm{Nuc}(Y)$ the full ∞-subcategory of solid nuclear sheaves on Y. For $Y = \mathrm{Spa}(R,R^+)$ over $\mathbf{Q}_p$, we have $\mathrm{QCoh}(Y) \simeq \mathcal{D}(R^{\mathrm{an}}_\square)$, where we wrote $R^{\mathrm{an}}_\square$ for the analytic ring $(R,R^+)_\square$. See [2] for more properties of the category $\mathrm{QCoh}(Y)$.

4.2.2. *Topological projection functor.* For $S \in \mathrm{sPerf}_C$, we have the functor

$$\begin{aligned} \mathrm{R}\overline{\tau}_* : \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) &\to \mathcal{D}(S, \mathbf{Q}_{p,\square}), \\ \mathcal{F} &\mapsto \{(f_Y : Y \to S) \to \mathrm{R}\Gamma(X_{\mathrm{FF},Y^\flat}, \mathrm{L}f_Y^*\mathcal{F})\}. \end{aligned} \tag{4.10}$$

Note that, clearly, we have $\mathrm{R}\Gamma(X_{\mathrm{FF},Y^\flat}, \mathrm{L}f_Y^*\mathcal{F}) \in \mathcal{D}(\mathbf{Q}_p(Y)_\square)$. We claim that formula (4.10) actually defines a solid enriched presheaf.

Lemma 4.11. *The functor* $\mathrm{R}\overline{\tau}_*$ *factors canonically via the* ∞*-derived category* $\underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})_0$*. That is, we have a functor*

$$\mathrm{R}\tau_* : \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) \to \underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})_0 \tag{4.12}$$

that fits into a commutative diagram

$$\begin{tikzcd} \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) \arrow[r, "\mathrm{R}\tau_*"] \arrow[dr, "\mathrm{R}\overline{\tau}_*"'] & \underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})_0 \arrow[d, "(-)^{\mathrm{cl}}"] \\ & \mathcal{D}(S, \mathbf{Q}_{p,\square}) \end{tikzcd}$$

Remark 4.13. (1) The natural definition of the functor $\mathrm{R}\tau_*$ should proceed along the following lines. We take the formula from (4.10). We need to explain why this formula can be upgraded to define an enriched presheaf. That is, we need to define structure maps

$$\underline{\mathcal{F}}'_{Y,T} : \quad \mathrm{R}\Gamma(X_{\mathrm{FF},(Y\times T)^\flat}, \mathrm{L}f^*_{Y\times T}\mathcal{F}) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(\mathbf{Q}_p[T]^\square, \mathrm{R}\Gamma(X_{\mathrm{FF},Y^\flat}, \mathrm{L}f_Y^*\mathcal{F})), \tag{4.14}$$

for $Y \in \mathrm{sPerf}_S$ and a profinite set T. That this can be done follows (working on the Y_{FF}-curve and then taking φ-eigenspaces) from the fact that, for a stably uniform analytic adic space $\mathrm{Spa}(R,R^+)$ over C and any $M \in \mathcal{D}(R^{\mathrm{an}}_\square)$, we have

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_{T,\square}}(R^{\mathrm{an}}_{T,\square}, M \otimes^{\mathrm{L}}_{R^{\mathrm{an}}_\square} R^{\mathrm{an}}_{T,\square}) &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_{T,\square}}(R^{\mathrm{an}}_{T,\square}, \mathcal{C}(T,R) \otimes^{\mathrm{L}}_{R^{\mathrm{an}}_\square} M) \\ &\xrightarrow{\delta} \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_{T,\square}}(R^{\mathrm{an}}_{T,\square}, \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_\square}(R^{\mathrm{an}}_\square[T], M)) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_\square}(R^{\mathrm{an}}_\square[T], M) \\ &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_\square}(\mathbf{Q}_p[T]^\square \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} R^{\mathrm{an}}_\square, M), \end{aligned} \tag{4.15}$$

where we wrote (R_T, R_T^+) for the cordinate Huber pair of the product $\mathrm{Spa}(R,R^+) \times T$.

However this approach would require us to work in the context of enriched ∞-presheaves and since we would like to avoid this we present a more down to earth definition in the proof below.

(2) If $\mathcal{F}$ is nuclear, by [2, Prop. 5.35], the map δ above is a quasi-isomorphism hence so are the structure maps (4.14).

Proof. (Proof of Lemma 4.11) We start with defining the functors

$$\mathrm{R}\tau_{I,*} : \mathrm{QCoh}(Y_{\mathrm{FF},S^\flat,I}) \to \underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})_0, \tag{4.16}$$

for intervals $I \subset (0,\infty)$ with rational endpoints. For $\mathcal{F}\in \mathrm{QCoh}(Y_{S^\flat,I})$, we take a K-projective resolution $\mathcal{P}(\mathcal{F})\stackrel{\sim}{\to}\mathcal{F}$ such that all terms of $\mathcal{P}(\mathcal{F})$ are direct sums of projective generators from the set $\{\mathbf{B}^{\mathrm{an}}_{S^\flat,I,\square}[T]\}$, where the T's are extremally disconnected profinite sets. We note that these resolutions are K-flat with flat terms. We define the presheaf

$$\mathrm{R}\tau_{I,*}(\mathcal{F}):=\{(f:Y\to S)\mapsto \Gamma(Y_{\mathrm{FF},Y^\flat,I},f^*\mathcal{P}(\mathcal{F}))\}. \tag{4.17}$$

Its value on $Y\to S$ is in complexes of $\underline{\mathbf{Q}}_p(Y)_\square$-modules.

We claim that the complex of presheaves (4.17) is naturally enriched. Indeed, for that we can just use a nonderived version of (4.15) (we set here $R^{\mathrm{an}}_\square=(R,R^+)_\square:=\mathbf{B}^{\mathrm{an}}_{Y^\flat,I,\square}$ to simplify the notation): for any $M\in R^{\mathrm{an}}_\square$, we have the enriching structure maps

$$\begin{aligned}\underline{M}'_{Y,T}:\quad M\otimes_{R^{\mathrm{an}}_\square}R^{\mathrm{an}}_{T,\square}\simeq \mathcal{C}(T,R)\otimes_{R^{\mathrm{an}}_\square}M &\xrightarrow{\delta_{Y,T}}\underline{\mathrm{Hom}}_{R^{\mathrm{an}}_\square}(R^{\mathrm{an}}_\square[T],M)\\ &\simeq \underline{\mathrm{Hom}}_{R^{\mathrm{an}}_\square}(\mathbf{Q}_p[T]^\square\otimes_{\mathbf{Q}_{p,\square}}R^{\mathrm{an}}_\square,M).\end{aligned} \tag{4.18}$$

These maps are clearly functorial in M and f giving us what we want. That is, we have defined the functor (4.16).

Set now $I=[1,p], I'=[1,1]$. Let $\mathcal{F}\in\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ and identify it functorially with a pair $(\mathcal{F}_I,\varphi_{\mathcal{F}})$, where $\varphi_{\mathcal{F}}:\varphi^*\mathcal{F}_I\simeq j^*\mathcal{F}_I$ is a quasi-isomorphism. Here $j:Y_{\mathrm{FF},S^\flat,I'}\hookrightarrow Y_{\mathrm{FF},S^\flat,I}$ is the canonical open immersion and

$$\varphi: Y_{\mathrm{FF},S^\flat,[1,1]}\hookrightarrow Y_{\mathrm{FF},S^\flat,[1/p,1]}\xrightarrow{\varphi}Y_{\mathrm{FF},S^\flat,[1,p]}$$

is the canonical open immersion followed by Frobenius. Set $\mathcal{F}_{I'}:=j^*\mathcal{F}_I$ and consider the induced maps

$$j^*:\mathrm{R}\tau_{I,*}(\mathcal{F}_I)\to\mathrm{R}\tau_{I',*}(\mathcal{F}_{I'}),\quad \varphi^*:\mathrm{R}\tau_{I,*}(\mathcal{F}_I)\to\mathrm{R}\tau_{I',*}(\mathcal{F}_{I'}).$$

They are easily seen to be in $\underline{\mathcal{D}}(S,\mathbf{Q}_{p,\square})_0$. Hence we get an object in $\underline{\mathcal{D}}(S,\mathbf{Q}_{p,\square})_0$

$$\mathrm{R}\tau_*(\mathcal{F}):=[\mathrm{R}\tau_{I,*}(\mathcal{F}_I)\xrightarrow{j^*-\varphi^*}\mathrm{R}\tau_{I',*}(\mathcal{F}_{I'})] \tag{4.19}$$

yielding a functor (4.12) wanted in the lemma. Also, we clearly have a natural transformation

$$\mathrm{R}\tau_*(-)^{\mathrm{cl}}\simeq\mathrm{R}\overline{\tau}_*(-),$$

as wanted. $\square$

4.2.3. *Topological vs algebraic projection functors.* Let $S\in\mathrm{sPerf}_C$. Consider the morphism of sites

$$\tau':(X_{\mathrm{FF},S^\flat})_{\text{proét}}\to S_{\text{proét}}.$$

induced by sending $Y\in\mathrm{sPerf}_S$ to $X_{\mathrm{FF},Y^\flat}$. It yields the pushforward functor

$$\mathrm{R}\tau'_*:\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})\to\mathcal{D}(S_{\text{proét}},\mathbf{Q}_p),\quad \mathcal{F}\mapsto\{Y\to\mathrm{R}\Gamma(X_{\mathrm{FF},Y^\flat},\mathrm{L}f^*\mathcal{F})|f:Y\to S\}. \tag{4.20}$$

Here the cohomology is seen in the derived ∞-category $\mathcal{D}(\mathbf{Q}_p(Y))$ of $\mathbf{Q}_p(Y)$-modules. $\mathrm{R}\tau'_*(\mathcal{F})$ is a sheaf: use the pro-étale descent for period sheaves.

We will need the following fact:

Lemma 4.21. *The following diagram commutes*

$$\begin{tikzcd} \mathrm{Nuc}(X_{\mathrm{FF},S^\flat}) \arrow[r, "\mathrm{R}\tau'_*"] \arrow[dr, "\mathrm{R}\tau_*"'] & \mathcal{D}(S_{\text{proét}},\mathbf{Q}_p) \arrow[d, "\mathrm{R}\pi_*"] \\ & \underline{\mathcal{D}}(S,\mathbf{Q}_p)_0. \end{tikzcd}$$

Proof. We have the functors from $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ to $\mathcal{D}(S,\mathbf{Q}_p)$ given by sending $\mathcal{F}\in\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ to the topological presheaves:

$$(\mathrm{R}\pi_*\mathrm{R}\tau'_*)^{\mathrm{cl}}\mathcal{F}:\{(Y,T)\mapsto \mathrm{R}\Gamma(X_{\mathrm{FF},(Y\times T)^\flat},\mathrm{L}f^*_{Y\times T}\mathcal{F})\},$$
$$\mathrm{R}\overline{\tau}_*\mathcal{F}:\{(Y,T)\mapsto \mathrm{RHom}_{\mathbf{Q}_p}(\mathbf{Q}_p[T],\mathrm{R}\Gamma(X_{\mathrm{FF},Y^\flat},\mathrm{L}f^*_Y\mathcal{F}))\}.$$

Moreover, we have a natural transformation

$$(\mathrm{R}\pi_*\mathrm{R}\tau'_*)^{\mathrm{cl}}\to\mathrm{R}\overline{\tau}_* \tag{4.22}$$

given by analytic descent from the structure morphisms $\underline{\mathcal{F}}'_{Y,T}$ described in (4.14). By Remark 4.13, they are quasi-isomorphisms when restricted to $\mathrm{Nuc}(X_{\mathrm{FF},S^\flat})$.

Both sides of (4.22) are enriched and it suffices now to show that the natural transformation (4.22) preserves this enrichment. That is, that it lifts to a natural transformation

$$\mathrm{R}\pi_*\mathrm{R}\tau'_*\to\mathrm{R}\tau_*$$

For that we can pass via the mapping fiber (4.19) and its algebraic analog to showing that the natural transformation

$$(\mathrm{R}\pi_*\mathrm{R}\tau'_{I,*})^{\mathrm{cl}}\to\mathrm{R}\overline{\tau}_{I,*}$$

is strictly compatible with enrichment on the level of a projective module $M\in R^{\mathrm{an}}_{\square}=(R_Y,R_Y^+)^{\mathrm{an}}_{\square}$. (We use the notation from (4.18).) After unwiding the definitions, this boils down to the fact that we have the identifications

$$\mathcal{C}(T',R_{Y\times T})\simeq\mathcal{C}(T',\mathcal{C}(T,R_Y))\simeq\mathcal{C}(T'\times T,R_Y)$$

and, via them, we have

$$\delta_{Y,T}\delta_{Y\times T,T'}=\delta_{Y,T\times T'}.$$

□

4.2.4. *Topological fully-faithfulness for perfect complexes.* Let $S\in\mathrm{sPerf}_C$. For $\mathcal{F}_1,\mathcal{F}_2\in\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$, we have a natural map in $\underline{\mathcal{D}}(S,\mathbf{Q}_{p,\square})$:

$$\mathrm{R}\tau_*\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1,\mathcal{F}_2)\to\mathrm{R}\mathcal{H}om^{\square}_{S,\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1,\mathrm{R}\tau_*\mathcal{F}_2), \tag{4.23}$$

which is constructed in the usual way: By adjointness, it suffices to construct a map

$$\mathrm{R}\tau_*\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1,\mathcal{F}_2)\otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p}\mathrm{R}\tau_*\mathcal{F}_1\to\mathrm{R}\tau_*\mathcal{F}_2.$$

For this, we use the composition

$$\mathrm{R}\tau_*\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1,\mathcal{F}_2)\otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p}\mathrm{R}\tau_*\mathcal{F}_1\to\mathrm{R}\tau_*(\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1,\mathcal{F}_2)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{F}_1)\to\mathrm{R}\tau_*\mathcal{F}_2,$$

where the second arrow is $\mathrm{R}\tau_*$ applied to the canonical map

$$\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1,\mathcal{F}_2)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{F}_1\to\mathcal{F}_2$$

and the first arrow is the relative cup product defined in the following way. For $\mathcal{G}_1,\mathcal{G}_2\in\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$, the relative cup product map in $\underline{\mathcal{D}}(S,\mathbf{Q}_{p,\square})$

$$\mathrm{R}\tau_*\mathcal{G}_1\otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p}\mathrm{R}\tau_*\mathcal{G}_2\to\mathrm{R}\tau_*(\mathcal{G}_1\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{G}_2) \tag{4.24}$$

is induced by the functorial maps in $\underline{\mathcal{D}}(S,\mathbf{Q}_{p,\square})$:

$$\mathrm{R}\tau_{I,*}\mathcal{G}_1\otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p}\mathrm{R}\tau_{I,*}\mathcal{G}_2\to\mathrm{R}\tau_{I,*}(\mathcal{G}_1\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{G}_2),$$

where $\mathrm{R}\tau_{I,*}$ is the functor from (4.16). And the latter comes from the identifications

$$f^*_Y\mathcal{G}_1\otimes_{\mathcal{O}}f^*_Y\mathcal{G}_2\xrightarrow{\sim}f^*_Y(\mathcal{G}_1\otimes_{\mathcal{O}}\mathcal{G}_2)$$

on the level of K-projective resolutions of the type considered in the proof of Lemma 4.11.

The following result is a topological version of [3, Cor. 3.10], which proved fully-faithfulness for $\mathrm{R}\tau'_*$ and follows fromTheorem 4.1.

Corollary 4.25. *Let $S \in \mathrm{sPerf}_C$. The functor*

$$\mathrm{R}\tau_* : \mathrm{Perf}(X_{\mathrm{FF},S^\flat}) \to \underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})$$

is fully faithful. That is, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Perf}(X_{\mathrm{FF},S^\flat})$, the natural map in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$

$$\mathrm{RHom}_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{R}\underline{\mathrm{Hom}}^{\square}_{S,\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1, \mathrm{R}\tau_*\mathcal{F}_2)$$

is a quasi-isomorphism.

Proof. We may pass to the condensed setting, that is, we want to show that the natural map in $\underline{\mathcal{D}}(\mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p(S)})$

$$\mathrm{RHom}_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{R}\underline{\mathrm{Hom}}_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1, \mathrm{R}\tau_*\mathcal{F}_2) \tag{4.26}$$

is a quasi-isomorphism. Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Perf}(X_{\mathrm{FF},S^\flat})$. The map (4.26) is constructed by taking global sections of the map (4.23) (we note that $\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Perf}(X_{\mathrm{FF},S^\flat})$):

$$\mathrm{R}\tau_*\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1, \mathcal{F}_2) \to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1, \mathrm{R}\tau_*\mathcal{F}_2).$$

It suffices thus to show that this map is a quasi-isomorphism. Consider its factorization:

$$\begin{aligned}\mathrm{R}\pi_*\mathrm{R}\tau'_*\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{F}_1, \mathcal{F}_2) &\xrightarrow{\sim} \mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{S,\mathbf{Q}_p}(\mathrm{R}\tau'_*\mathcal{F}_1, \mathrm{R}\tau'_*\mathcal{F}_2)\\ &\to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1, \mathrm{R}\tau_*\mathcal{F}_2).\end{aligned}$$

The first morphism is a quasi-isomorphism by [3, Cor. 3.10]. It remains to show that the map

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{S,\mathbf{Q}_p}(\mathrm{R}\tau'_*\mathcal{F}_1, \mathrm{R}\tau'_*\mathcal{F}_2) \to \mathrm{R}\mathcal{H}om_{S^{\mathrm{top}},\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}_1, \mathrm{R}\tau_*\mathcal{F}_2) \tag{4.27}$$

is a quasi-isomorphism.

By [3, Prop. 2.6], both $\mathcal{F}_1$ and $\mathcal{F}_2$ are strictly perfect. Hence, by dévissage, we may assume that they are both vector bundles on $X_{\mathrm{FF},S^\flat}$. We can in fact assume that both are line bundles: If $\mathcal{E}$ is a vector bundle on $X_{\mathrm{FF},S^\flat}$, up to étale localization on S (which we are allowed to do), applying [13, Prop. II.3.1] to a twist of $\mathcal{E}$ and using [13, Cor. II.2.20]), we may assume that $\mathcal{E} \in \{\mathcal{L} \otimes^{\mathrm{L}}_{\mathbf{Q}_p} \mathcal{O}(s)\}$, for a pro-étale $\mathbf{Q}_p$-local system $\mathcal{L}$ on $S^\flat$ and $s \in \mathbf{Z}$. By [19, Prop. 8.4.7], we may assume that $\mathcal{L}$ admits a $\mathbf{Z}_p$-lattice hence, by the assumption on S, is trivial.

Thus we have $\mathcal{E} \in \{\mathcal{O}(s)\}$, $s \in \mathbf{Z}$. Then we can reduce to $\mathcal{E} \in \{\mathcal{O}, \mathcal{O}(1)\}$ by using twists of the Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^2 \to \mathcal{O}(2) \to 0$$

and then, by using the exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to i_{\infty,*}R_S \to 0,$$

we may assume that $\mathcal{E} \in \{\mathcal{O}, i_{\infty,*}R_S\}$.

We claim that

$$\mathrm{R}\tau'_*(\mathcal{O}) \simeq \mathbf{Q}_p, \quad \mathrm{R}\tau'_*(i_{\infty,*}R_S) = \mathbb{G}_a.$$

Indeed, the second quasi-isomorphism is clear. The first one follows from [13, Prop. II.2.5 (ii)], which yields that $\mathrm{R}\tau'_*(\mathcal{O}) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y \times T, \mathbf{Q}_p)$ and the fact that $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y \times T, \mathbf{Q}_p) \simeq \underline{\mathbf{Q}}_p(Y \times T)$. It suffices now to invoke Corollary 4.7 to finish the proof of our corollary. $\square$

Example 4.28. Recall that, by [13, Prop. II.2.5], we have the following quasi-isomorphisms in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$:

$$\begin{aligned}&\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{O}, \mathcal{O}) \simeq \mathcal{O},\\ &\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(\mathcal{O}, i_{\infty,*}R_S) \simeq i_{\infty,*}R_S,\\ &\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(i_{\infty,*}R_S, i_{\infty,*}R_S) \simeq i_{\infty,*}R_S \oplus i_{\infty,*}R_S[-1],\\ &\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(i_{\infty,*}R_S, \mathcal{O}) \simeq i_{\infty,*}R_S(-1)[-1].\end{aligned}$$

Via Corollary 4.25, this can be transferred to the following quasi-isomorphisms in $\underline{\mathcal{D}}(S, \mathbf{Q}_{p,\square})$:

$$\begin{aligned}
&\mathrm{R}\mathcal{H}om^{\square}_{S,\mathbf{Q}_p}(\mathbf{Q}_p, \mathbf{Q}_p) \xleftarrow{\sim} \mathbf{Q}_p, \quad \mathrm{R}\mathcal{H}om^{\square}_{S,\mathbf{Q}_p}(\mathbf{Q}_p, \mathbb{G}_a) \xleftarrow{\sim} \mathbb{G}_a,\\
&\mathrm{R}\mathcal{H}om^{\square}_{S,\mathbf{Q}_p}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathbb{G}_a \oplus \mathbb{G}_a[-1], \quad \mathrm{R}\mathcal{H}om^{\square}_{S,\mathbf{Q}_p}(\mathbb{G}_a, \mathbf{Q}_p) \simeq \mathbb{G}_a(-1)[-1].
\end{aligned}$$

References

[1] P. Achinger, S. Gilles, W. Nizioł, *Compactly supported p-adic pro-étale cohomology of analytic varieties.* `arXiv:2501.13651[math.AG]`.

[2] G. Andreychev, *Pseudocoherent and perfect complexes and vector bundles on analytic adic spaces.* `arXiv:2105.12591[math.AG]`.

[3] J. Anchütz, A-C. Le Bras, *A Fourier Transform for Banach-Colmez spaces.* J. Eur. Math. Soc. (JEMS) 27 (2025), 3651–3712.

[4] J. Anchütz, A-C. Le Bras, L. Mann, *A 6-functor formalism for solid quasi-coherent sheaves on the Fargues-Fontaine curve.* `arXiv:2412.20968[math.AG]`.

[5] F. Borceux, C. Quinteiro, *A theory of enriched sheaves.* Cahiers Topologie Géom. Différentielle Catég. 37 (1996), 145–162.

[6] J.M.F. Castillo, *The Hitchhiker Guide to Categorical Banach Space Theory. Part I.* Extracta Math. 25 (2010) 103–149.

[7] J.M.F. Castillo, *The Hitchhiker guide to Categorical Banach space theory. Part II.* `arXiv:2110.06300[math.FA]`.

[8] D. Clausen, P. Scholze, *Lectures on condensed mathematics.* `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, 2019.

[9] P. Colmez, *Espaces de Banach de dimension finie.* J. Inst. Math. Jussieu 1 (2002), 331–439.

[10] P. Colmez, S. Gilles, W. Nizioł, *Duality for p-adic geometric pro-étale cohomology.* arXiv:2411.12163 [math.AG].

[11] P. Colmez, W. Nizioł, *On the cohomology of p-adic analytic spaces, I: The basic comparison theorem.* J. Algebraic Geometry **34** (2025), 1–108.

[12] P. Colmez, W. Nizioł, *On the cohomology of p-adic analytic spaces, II: The C_{st}-conjecture.* Duke Math. J. 174 (2025), 2203–2301.

[13] L. Fargues, P. Scholze, *Geometrization of the local Langlands correspondence.* `arXiv:2102.13459v3[math.RT]`, to appear in Astérique.

[14] B. J. Guillou, P. May, *Enriched model categories and presheaf categories.* New York J. Math. 26 (2020), 37–91.

[15] C. Herz, J. W. Pelletier, *Dual functors and integral operators in the category of Banach spaces.* J. Pure Appl. Algebra 8 (1976), 5–22.

[16] C. Heyer, L. Mann, 6-*Functor Formalisms and Smooth Representations.* `arXiv:2410.13038[math.CT]`.

[17] H. Al Hwaeer, G. Garkusha, *Grothendieck categories of enriched functors.* J. Algebra 450 (2016), 204–241.

[18] G.-M. Kelly, *Basic concepts of enriched category theory.* London Math. Soc. Lecture Note Ser., 64 Cambridge University Press, Cambridge-New York, 1982, 245 pp.

[19] K. S. Kedlaya, R. Liu, *Relative p-adic Hodge theory: foundations.* Astérisque 371 (2015), 239pp.

[20] L. Mann, A. Werner, *Local systems on diamonds and p-adic vector bundles.* Int. Math. Res. Not. IMRN (2023), 12785–12850.

[21] J. W. Pelletier, *Dual functors and the Radon-Nikodym property in the category of Banach spaces.* J. Austral. Math. Soc. Ser. A 27 (1979), 479–494.

[22] C. Perez-Garcia, W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields.* Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, Cambridge, 2010.

[23] A. Rosenfield, *Enriched Grothendieck topologies under change of base.* `arXiv:2405.19529[math.CT]`.

[24] C. Serpé, *Resolution of unbounded complexes in Grothendieck categories.* J. Pure Appl. Algebra 177 (2003), 103–112.

[25] J.-P. Schneiders, *Quasi-abelian categories and sheaves.* Mém. Soc. Math. Fr. 76, 1999.

[26] P. Scholze, *Étale cohomology of diamonds.* `arXiv:1709.07343v3[math.AG]`, to appear in Astérisque.

[27] P. Scholze, J. Weinstein, *Berkeley lectures on p-adic geometry.* Ann. of Math. Stud., 207, 2020.

[28] N. Spaltenstein *Resolutions of unbounded complexes.* Compositio Math. 65 (1988), 121–154.

[29] The Stacks Project Authors, Stack Project. `http://stacks.math.columbia.edu`.

[30] J. Van Name (`https://mathoverflow.net/users/22277/joseph-van-name`), "Which spaces are inverse limits of discrete spaces ?", URL (version: 2012-03-29): `https://mathoverflow.net/q/92608`

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