HODGE THEORY OF p-ADIC ANALYTIC VARIETIES: A SURVEY

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Abstract. Hodge Theory of p-adic analytic varieties was initiated by Tate in his 1967 paper on p-divisible groups, where he conjectured the existence of a Hodge-like decomposition for the p-adic étale cohomology of proper analytic varieties. Tate's conjecture was refined by Fontaine who gave the theory its definite shape. A lot of work has been done for algebraic varieties and a number of proofs of Fontaine's conjectures have been obtained between years 1985 and 2011. But the study of Hodge Theory of p-adic analytic varieties started really only in 2011 with Scholze's proof of Tate's conjecture using perfectoid methods. Methods that opened the way to an avalanche of results.

In this paper, we survey our results and conjectures (comparison theorems and their geometrization, dualities, etc.), focusing on the case of nonproper analytic varieties, where a number of new phenomena occur. We also describe the new objects that appeared along the way.

Contents

1. Introduction

1.1. Complex periods.

1.1.1. *$\mathcal{C}^\infty$-manifolds.* If M is a compact, connected, orientable $\mathcal{C}^\infty$-manifold of dimension d, the classical theorem of de Rham states that integrating a closed differential form along a cycle without boundary induces a pairing

$$H^i_{\mathrm{dR},\mathcal{C}^\infty}(M)\times H_i(M,\mathbf{Z})\to\mathbf{R}$$

between de Rham cohomology and singular homology, which identifies $H^i_{\mathrm{dR},\mathcal{C}^\infty}(M)$ with $\mathrm{Hom}(H_i(M,\mathbf{Z}),\mathbf{R})$. This can be rephrased using Betti cohomology $H^i_{\mathrm{B}}(M,\mathbf{Q})\simeq\mathrm{Hom}(H_i(M,\mathbf{Z}),\mathbf{Q})$ by saying that we have a functorial period isomorphism

$$H^i_{\mathrm{dR},\mathcal{C}^\infty}(M)\simeq\mathbf{R}\otimes_{\mathbf{Q}}H^i_{\mathrm{B}}(M,\mathbf{Q}),\tag{1.1}$$

a statements that can be proved, following Weil (see [75], Weil 18/01/47, Weil 02/02/47), by means of the $\mathcal{C}^\infty$-Poincaré Lemma.

Moreover $H^d_{\mathrm{dR},\mathcal{C}^\infty}(M)\simeq\mathrm{R}$ and $H^d_{\mathrm{B}}(M)\simeq\mathbf{Q}$, and the resulting cup-products $H^i_{\mathrm{dR},\mathcal{C}^\infty}(M)\otimes H^{d-i}_{\mathrm{dR},\mathcal{C}^\infty}(M)\to\mathrm{R}$ and $H^i_{\mathrm{B}}(M)\otimes H^{d-i}_{\mathrm{B}}(M)\to\mathbf{Q}$ are perfect dualities (Poincaré dualities).

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The isomorphism (1.1) extends to manifolds which are countable at infinity but, since the cohomology groups can now be infinite dimensional, one needs to take a completed tensor product. In this noncompact situation we have at our disposal other cohomology theories, using classes with compact support, and Poincaré duality is now a perfect duality between cohomology and cohomology with compact support.

1.1.2. *Complex algebraic and analytic varieties.* Now, if $M = X(\mathbf{C})$ is the $\mathcal{C}^\infty$-manifold defined by the $\mathbf{C}$-points of a proper smooth algebraic variety X defined over a subfield K of $\mathbf{C}$, the de Rham cohomology of M can be recovered [54] from the cohomology of the algebraic de Rham complex

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) := \mathrm{R}\Gamma(X, \mathcal{O} \to \Omega^1 \to \Omega^2 \to \cdots)$$

of X: we have functorial isomorphisms

$$\mathbf{C} \otimes_K H^i_{\mathrm{dR}}(X) \simeq \mathbf{C} \otimes_{\mathbf{R}} H^i_{\mathrm{dR},\mathcal{C}^\infty}(X(\mathbf{C})).$$

Combined with the theorem of de Rham from (1.1), this gives functorial isomorphisms

$$\mathbf{C} \otimes_K H^i_{\mathrm{dR}}(X) \simeq \mathbf{C} \otimes_{\mathbf{Q}} H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q}). \tag{1.2}$$

The coefficients of the matrix of this isomorphism in bases over K and $\mathbf{Q}$ respectively are called "periods"; for example $2\pi i$ is a period of $\mathbf{P}^1$ (for H^2) or of $\mathbf{G}_m$ (for H^1, but $\mathbf{G}_m$ is not proper), and $\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}$ is a period of the elliptic curve $y^2 = x^3 - x$ (for H^1). See [60] for general conjectures about these numbers.

The algebraic de Rham cohomology is endowed with a natural filtration (the Hodge filtration) given by the filtration of the de Rham complex:

$$\mathrm{Fil}^i\mathrm{R}\Gamma_{\mathrm{dR}}(X) := \mathrm{R}\Gamma(X, 0 \to \cdots \to 0 \to \Omega^i \to \Omega^{i+1} \to \cdots)$$

Hodge theory provides a canonical splitting of this filtration over $\mathbf{C}$ by means of harmonic forms of type (p, q), which gives a functorial (Hodge) decomposition

$$\mathbf{C} \otimes_{\mathbf{Q}} H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q}) \simeq \oplus_{p+q=i} \mathbf{C} \otimes_K H^p(X, \Omega^q) \tag{1.3}$$

This decomposition does not, in general, come from a decomposition over K.

Most of the above extends to compact complex analytic varieties (taking $K = \mathbf{C}$), but the existence of the Hodge decomposition can fail if the variety is not Kähler.

1.2. **p-adic periods.** Now, let p be a prime and let $\mathbf{C}_p$ be the completion of an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ for the p-adic valuation, and $\mathbf{Q}_p^{\mathrm{nr}} \subset \overline{\mathbf{Q}}_p$ be the maximal unramified extension of $\mathbf{Q}_p$.

Let $K \subset \mathbf{C}_p$ be a finite extension of $\mathbf{Q}_p$, and let $G_K = \mathrm{Aut}_{\mathrm{cont}}(\mathbf{C}_p/K) = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ be the absolute Galois group of K.

The Hodge theory for varieties over K was initiated by Tate in his incredibly influential paper [71] and given a definitive form by Fontaine (see [47, 48, 49]) who introduced various rings $\mathbf{B}_{\mathrm{cris}}$, $\mathbf{B}_{\mathrm{st}}$, $\mathbf{B}_{\mathrm{dR}}$, in which the periods of algebraic varieties over K should live. The situation is more complicated here than over $\mathbf{C}$: The results over $\mathbf{C}$ sketched above show that the periods of varieties defined over subfields of $\mathbf{C}$ live in $\mathbf{C}$. But in the p-adic case, as Tate showed, these periods do not, in general, live in $\mathbf{C}_p$ (for example, the p-adic avatar of $2\pi i$ does not live in $\mathbf{C}_p$ – this is a rephrasing of Tate's result [71] that[1] $H^0(G_K, \mathbf{C}_p(1)) = 0$).

If X is a smooth and proper algebraic variety defined over K, there is no direct analog of $H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$; what plays its role in the p-adic world is the étale cohomology $H^i_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$. Indeed, if X is defined over a subfield F of K that embeds into $\mathbf{C}$, and if we choose an embeding into $\mathbf{C}$ of the algebraic closure $\overline{F}$ of F in $\mathbf{C}_p$, we have natural isomorphisms (Artin [7] for the last one)

$$H^i_{\text{ét}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \simeq H^i_{\text{ét}}(X_{\overline{F}}, \mathbf{Q}_p) \simeq H^i_{\text{ét}}(X_{\mathbf{C}}, \mathbf{Q}_p) \simeq \mathbf{Q}_p \otimes_{\mathbf{Q}} H^i_{\mathrm{B}}(X(\mathbf{C}), \mathbf{Q})$$

[1]If $r \in \mathbf{Z}$, and V is a G_K-module, we let $V(r)$ be its r-th Tate twist: i.e., the module V with action of $\sigma \in G_K$ multiplied by $\chi^r(\sigma)$, where $\chi : G_K \to \mathbf{Z}_p^*$ is the cyclotomic character.

Fontaine's rings $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$ are topological rings endowed with compatible continuous actions of G_K, and containing a p-adic avatar t of $2\pi i$ (Fontaine's $2\pi i$) on which $\sigma \in G_K$ acts by $\sigma(t) = \chi(\sigma)t$. Moreover $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$ and $\mathbf{B}_{\mathrm{cris}}^{G_K} = \mathbf{B}_{\mathrm{st}}^{G_K} = K \cap \mathbf{Q}_p^{\mathrm{nr}}$, and:

• $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[\frac{1}{t}]$, where $\mathbf{B}_{\mathrm{dR}}^+$ is complete for the t-adic topology which is stronger that the natural topology (for which $\overline{K}$ is dense [26]), and we have $\mathbf{B}_{\mathrm{dR}}^+/t = \mathbf{C}_p$. The filtration $\mathbf{B}_{\mathrm{dR}}^i := t^i\mathbf{B}_{\mathrm{dR}}^+$ is stable by G_K.

• $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{cris}}[\log \tilde{p}]$ is endowed with a Frobenius φ satisfying $\varphi(t) = pt$, $\varphi(\log \tilde{p}) = p \log \tilde{p}$, and a $\mathbf{B}_{\mathrm{cris}}$-derivation N with $N(\log \tilde{p}) = -1$ (hence $N\varphi = p\varphi N$); φ and N commute with G_K.

• $\mathbf{Q}_p$ can be recovered inside $\mathbf{B}_{\mathrm{st}}$ using the above structure: we have the fundamental exact sequence

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{st}}^{N=0,\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0$$

Hence, in particular, $\mathbf{Q}_p = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{st}}^{N=0,\varphi=1})$.

Conjecture 1.4. (Fontaine [47, 49]) *We have functorial period isomorphisms commuting with all structures* (G_K, filtration, φ and N)

$$(\mathrm{C}_{\mathrm{dR}}) \quad \mathbf{B}_{\mathrm{dR}} \otimes_K H^i_{\mathrm{dR}}(X) \simeq \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$$
$$(\mathrm{C}_{\mathrm{st}}) \quad \mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} H^i_{\mathrm{HK}}(X) \simeq \mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$$

In the conjecture, $H^i_{\mathrm{HK}}(X)$ (the Hyodo-Kato cohomology) is an extra-structure on $H^i_{\mathrm{dR}}(X)$ coming from "reduction modulo p": it is a $\mathbf{Q}_p^{\mathrm{nr}}$-module equipped with a semi-linear φ, a monodromy operator N with $N\varphi = p\varphi N$, and a smooth action of G_K commuting with φ and N, and we have a (Hyodo-Kato)-isomorphism

$$H^i_{\mathrm{dR}}(X) \simeq (\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} H^i_{\mathrm{HK}}(X))^{G_K}$$

If X has good reduction, then $N = 0$, the inertia group acts trivially, and Hyodo-Kato cohomology is crystalline cohomology of the reduction mod p developed by Grothendieck and Berthelot [55, 10]; if X has a semistable model, then inertia group still acts trivially and $H^i_{\mathrm{HK}}(X)$ is the log-crystalline cohomology of the reduction modulo p as defined by Hyodo and Kato [58]. In general, $H^i_{\mathrm{HK}}(X)$ has been defined by Beilinson [9], using de Jong's alterations [44] to reduce to the semistable case.

Remark 1.5. (i) All the cohomology groups involved in Conjecture 1.4 are finite dimensional over their respective fields.

(ii) This conjecture is now a theorem (see [74], [45], [66], [9]). All these proofs construct the period map, show compatibility with cycle class maps and traces, and use Poincaré duality to conclude that the period map is an isomorphism.

(iii) By taking invariants under suitable structures, the period isomorphisms from Conjecture 1.4 give recipes to recover $(H^i_{\mathrm{HK}}(X), H^i_{\mathrm{dR}}(X))$ from $H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p)$ and vice-versa: one has isomorphisms (of filtered K-modules for the first one, of (φ, N, G_K)-modules over $\mathbf{Q}_p^{\mathrm{nr}}$ for the second one)

$$(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p))^{G_K} \simeq H^i_{\mathrm{dR}}(X),$$
$$\varinjlim_{[L:K]<\infty} (\mathbf{B}_{\mathrm{st}} \otimes H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p))^{G_L} \simeq H^i_{\mathrm{HK}}(X),$$

and an exact sequence

$$0 \to H^i_{\mathrm{\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p) \to (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p^{\mathrm{nr}}} H^i_{\mathrm{HK}}(X))^{N=0,\varphi=1} \to (\mathbf{B}_{\mathrm{dR}} \otimes_K H^i_{\mathrm{dR}}(X))/\mathrm{Fil}^0 \to 0$$

(iv) By taking Fil^0-part of both terms in C_{dR}, and moding out by t, we obtain the following result which was the initial formulation of the Hodge-Tate conjecture by Tate [71] (renamed C_{HT} by Fontaine), and which should be compared[2] to (1.3): we have a functorial G_K-equivariant

[2]There is no twist in (1.3) but, actually, if one wants cycle classes to match one needs to introduce the same powers of $2\pi i$ as for the twists in $(\mathrm{C}_{\mathrm{HT}})$: one can write $\mathbf{C}_p(-r) = t^{-r}\mathbf{C}_p$, and t is the p-adic avatar of $2\pi i$.

decomposition:

$$(\mathrm{C_{HT}})\quad \mathbf{C}_p\otimes_{\mathbf{Q}_p} H^i_{\text{ét}}(X_{\mathbf{C}_p},\mathbf{Q}_p)\simeq \oplus_{r+s=i}\mathbf{C}_p(-r)\otimes_K H^s(X,\Omega^r)$$

1.3. **p-adic analytic varieties.** The theory of p-adic analytic varieties was also initiated by Tate [72]: Tate's paper was finally published in 1971 thanks to Serre's persistence (see [70], Tate 07/10/1969), but the results were circulating as "Rigid analytic spaces, private notes of J. Tate, reproduced with(out) his permission by IHES" starting from 1962 (see [70], Tate 16/10/1961, Tate 26/10/1961, Serre 10/11/1961, Tate 14/11/1961, Serre x/04/1962).

In [71], Tate already stated his Conjecture $\mathrm{C_{HT}}$ for the p-adic étale cohomology of proper analytic varieties. In general, it is easier to consider the pro-étale cohomology rather than the étale one, but this gives the same result on quasi-compact varieties – e.g., affinoids or proper varieties. To compute both cohomologies one uses a hypercovering by affinoids; affinoids are $K(\pi,1)$-spaces hence, locally, (pro-)étale cohomology reduces to continuous cohomology of the fundamental group. The difference being that:

— for $H^i_{\text{ét}}(X_{\mathbf{C}_p},\mathbf{Q}_p)$, one uses $\mathbf{Z}_p$ as coefficients on each affinoid and inverts p at the end,

— for $H^i_{\text{proét}}(X_{\mathbf{C}_p},\mathbf{Q}_p)$, one uses $\mathbf{Q}_p$ as coefficients on each affinoid.

It is easier to work with $\mathbf{Q}_p$-coefficients since rational p-adic Hodge Theory is much easier and much more robust than the integral one.

The pro-étale p-adic cohomology of proper analytic varieties (which is also the étale cohomology as we mentioned above) behaves in a very similar way to the p-adic étale cohomology of algebraic varieties (note, however, that this is not the case if one considers nontrivial coefficients):

• Scholze [68, 69] proved that the $\mathbf{Q}_p$-vector spaces $H^i_{\text{proét}}(X_{\mathbf{C}_p},\mathbf{Q}_p)$ are finite dimensional. (This is not the case anymore for nontrivial coefficients as was realized some years later (see [56]): the cohomology groups are, in general, only finite Dimensional [6, 63]; i.e., they are Banach-Colmez spaces (BC's for short) see [24, 25] and section 5.1.1.)

• Scholze also proved $\mathrm{C_{dR}}$, with the same formulation as in Conjecture 1.4, but he found a way to circumvent Poincaré duality which was not available at that time.

• The authors proved $\mathrm{C_{st}}$ (see [39] in the case X has a semi-stable model[3], and [42] for the general case), using crucially the finiteness result of Scholze plus some fragments of the theory [24, 25] of BC's (as well as the basic comparison theorem, i.e., Theorem 2.4 below).

• Mann [65] and Zavyalov [76] proved Poincaré duality.

In the rest of the text, we explore what happens for pro-étale cohomology of partially proper analytic varieties (for example, analytifications of nonproper algebraic varieties). As the reader will see, it behaves very differently (the case of the open unit disk gives a good illustration of the new phenomena that appear, see Section 2.1.2). We will mainly be concerned with the geometric situation, but see Sections 2.3.6 and 6.3 for results in the arithmetic case.

A big motivation for studying p-adic étale and pro-étale cohomologies of nonproper p-adic analytic varieties comes from the potential applications to the geometrization of the hoped for p-adic local Langlands correspondence [30, 31, 33], but we will not elaborate on this in this survey, rather referring the reader to the survey [34] devoted to this topic. One reason for that is that the applications to the p-adic local Langlands correspondence use mainly the p-adic étale cohomology rather than the pro-étale one, and the results that we have about p-adic étale cohomology so far are much more fragmentary than for the pro-étale one.

Notation and conventions. Let p be a prime and let K be a complete discrete valuation field with a perfect residue field, of mixed characteristic. Let $\mathcal{O}_K$ be the ring of integers in K, and k be its residue field. Let $W(k)$ be the ring of Witt vectors of k and let F be its fraction field (i.e., $W(k)=\mathcal{O}_F$).

[3]See also [11] and [20] for a different approach.

Let $\overline{K}$ be an algebraic closure of K and let $\mathcal{O}_{\overline{K}}$ denote the integral closure of $\mathcal{O}_K$ in $\overline{K}$. Let $C = \widehat{\overline{K}}$ be the completion of $\overline{K}$ for the p-adic valuation, and let $G_K := \mathrm{Aut}_{\mathrm{cont}}(C/K) = \mathrm{Gal}(\overline{K}/K)$.

Let $\breve{C} = \mathrm{Frac}(W(\overline{k})) \subset C$ and $C^{\mathrm{nr}} = \cup_{[k':k]<\infty} W(k')[\frac{1}{p}] \subset \overline{K}$ (hence $\breve{C}$ is the completion of C^{nr}), and let φ be the absolute Frobenius on $\breve{C}$ and C^{nr}.

All rigid analytic spaces and dagger spaces considered will be over K or C; we assume that they are separated, taut, and countable at infinity.

We will use condensed mathematics as developed in [21], [22]; we will write $\mathcal{D}(\mathbf{Q}_{p,\square})$ for the ∞-derived category of solid $\mathbf{Q}_p$-vector spaces, etc.

We will use the bracket notation for certain limits: $[C_1 \xrightarrow{f} C_2]$ denotes the mapping fiber of f.

2. The basic comparison theorem

We start our survey with a discussion of a basic observation: p-adic pro-étale cohomology of analytic varieties can be computed via their refined de Rham cohomology.

2.1. **An example.** We will start with a motivating example.

2.1.1. *The complex unit disk.* Let Y be the open unit disk over $\mathbf{C}$ (field of complex numbers) viewed as a complex analytic variety of dimension 1. The de Rham and singular (Betti) cohomology groups have the familiar form:

$$H^i_{\mathrm{dR}}(Y) \simeq \begin{cases} \mathbf{C} & \text{if } i=0,\\ 0 & \text{if } i \geq 1; \end{cases} \qquad H^i_{\mathrm{B}}(Y,\mathbf{Q}) \simeq \begin{cases} \mathbf{Q} & \text{if } i=0,\\ 0 & \text{if } i \geq 1; \end{cases}$$

$$H^i_{\mathrm{dR},c}(Y) \simeq \begin{cases} \mathbf{C} & \text{if } i=2,\\ 0 & \text{if } i \neq 2; \end{cases} \qquad H^i_{\mathrm{B},c}(Y,\mathbf{Q}) \simeq \begin{cases} \mathbf{Q} & \text{if } i=2,\\ 0 & \text{if } i \neq 2. \end{cases}$$

Moreover, we have natural isomorphisms

$$H^i_{\mathrm{dR}}(Y) \simeq H^i_{\mathrm{B}}(Y,\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}, \quad H^i_{\mathrm{dR},c}(Y) \simeq H^i_{\mathrm{B},c}(Y,\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}, \quad \text{for all } i \geq 0,$$

and the natural pairings

$$H^i_{\mathrm{dR}}(Y) \times H^{2-i}_{\mathrm{dR},c}(Y) \to H^2_{\mathrm{dR},c}(Y) \simeq \mathbf{C},$$
$$H^i_{\mathrm{B}}(Y,\mathbf{Q}) \times H^{2-i}_{\mathrm{B},c}(Y,\mathbf{Q}) \to H^2_{\mathrm{B},c}(Y,\mathbf{Q}) \simeq \mathbf{Q}$$

are perfect (i.e., we have a Poincaré duality).

2.1.2. *The p-adic unit disk.* Let now $Y_{\mathbf{C}_p}$ be the unit disk over $\mathbf{C}_p$ (completion of the algebraic closure of $\mathbf{Q}_p$) viewed as a rigid analytic variety of dimension 1. Then we have the same results as above for de Rham cohomology (with $\mathbf{C}$ replaced by $\mathbf{C}_p$). The role of Betti cohomology in this setting is played by the p-adic (pro-)étale cohomology $H^*_{\mathrm{pro\acute{e}t}}(-,\mathbf{Q}_p)$ with $\mathbf{Q}_p$-coefficients. By [42, Th. 6.14], [2, Cor. 1.6], we have

$$H^i_{\mathrm{pro\acute{e}t}}(Y_{\mathbf{C}_p},\mathbf{Q}_p) \simeq \begin{cases} \mathbf{Q}_p & \text{if } i=0,\\ \mathcal{O}(Y_{\mathbf{C}_p})/\mathbf{C}_p & \text{if } i=1,\\ 0 & \text{if } i \geq 2. \end{cases}$$

$$H^i_{\mathrm{pro\acute{e}t},c}(Y_{\mathbf{C}_p},\mathbf{Q}_p) \simeq \begin{cases} 0 & \text{if } i \neq 2,\\ [\mathcal{O}(\partial Y_{\mathbf{C}_p})/\mathcal{O}(Y_{\mathbf{C}_p}) - \mathbf{Q}_p] & \text{if } i=2. \end{cases}$$

Here the bracket $[(-)-(-)]$ denotes an extension. The notation $\partial Y_{\mathbf{C}_p}$ refers to the rigid analytic ghost circle, boundary of the open unit disk; it is obtained by removing from $Y_{\mathbf{C}_p}$ all the affinoids W that it contains: we have $\mathcal{O}(\partial Y_{\mathbf{C}_p}) = \varinjlim_W \mathcal{O}(Y_{\mathbf{C}_p} \setminus W)$ and $\partial Y_{\mathbf{C}_p}$ behaves like a proper variety of "real" dimension 1 (see th. 6.11).

The main differences with the case of the complex unit disk are the following:

• $H^1_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_p)$ and $H^2_{\text{proét},c}(Y_{\mathbf{C}_p}, \mathbf{Q}_p)$ are huge groups and are not invariant by base change to a bigger complete algebraically closed extension C of $\mathbf{C}_p$ (and there are many of those: there is no p-adic Gelfand-Mazur theorem), as the "$\mathbf{C}_p$-components" $\mathcal{O}(Y_{\mathbf{C}_p})/\mathbf{C}_p$ and $\mathcal{O}(\partial Y_{\mathbf{C}_p})/\mathcal{O}(Y_{\mathbf{C}_p})$ are base changed to C.

• The groups $\mathcal{O}(Y_{\mathbf{C}_p})/\mathbf{C}_p \simeq H^0(Y_{\mathbf{C}_p}, \Omega^1)$ (via $f \mapsto df$) and $\mathcal{O}(\partial Y_{\mathbf{C}_p})/\mathcal{O}(Y_{\mathbf{C}_p}) \simeq H^1_c(Y_{\mathbf{C}_p}, \mathcal{O})$ are in Serre duality over $\mathbf{C}_p$, but this cannot be turned into a duality over $\mathbf{Q}_p$ since $[\mathbf{C}_p : \mathbf{Q}_p] = \infty$. Hence the existence of a Poincaré duality

$$H^i_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_p) \times H^{2-i}_{\text{proét},c}(Y_{\mathbf{C}_p}, \mathbf{Q}_p) \to \mathbf{Q}_p$$

looks impossible (not to mention the fact that the numerology does not cooperate either).

Remark 2.1. Note that, by contrast, for $\ell \neq p$, the results are as expected:

$$H^i_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_\ell) \simeq \begin{cases} \mathbf{Q}_\ell & \text{if } i = 0, \\ 0 & \text{if } i \geq 1; \end{cases} \quad H^i_{\text{proét},c}(Y_{\mathbf{C}_p}, \mathbf{Q}_\ell) \simeq \begin{cases} 0 & \text{if } i \neq 2, \\ \mathbf{Q}_\ell & \text{if } i = 2 \end{cases}$$

and we have a Poincaré duality

$$H^i_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_\ell) \times H^{2-i}_{\text{proét},c}(Y_{\mathbf{C}_p}, \mathbf{Q}_\ell) \to \mathbf{Q}_\ell.$$

We will explain how to remedy the two difficulties mentioned above. For the fist one, we show that $H^i_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_p)$ is actually the $\mathbf{C}_p$-points of a functor built out of copies of $\mathbf{Q}_p$ and $\mathbf{G}_a$ (it is this $\mathbf{G}_a$ part that varies with $\mathbf{C}_p$). For the second one, the clue is to consider not $\mathrm{Hom}(-, \mathbf{Q}_p)$ as what one would naturally do, but derived Hom, $\mathrm{RHom}_{\mathrm{TVS}}(-, \mathbf{Q}_p)$ in the category TVS in which the functor attached to $H^i_{\text{proét}}(Y_{\mathbf{C}_p}, \mathbf{Q}_p)$ naturally lives.

2.2. **Slogans.** We turn now to key slogans in Hodge Theory of p-adic analytic varieties. In this section Y is a smooth rigid analytic variety over K.

2.2.1. *Comparison theorems as refined Lazard isomorphisms.* (Formally) similarly to the algebraic case, pro-étale cohomology can be computed by data coming from refined de Rham cohomology

$$\mathrm{R}\Gamma_{\text{proét}}(Y_C, \mathbf{Q}_p) \longleftrightarrow \{\mathrm{R}\Gamma_{\mathrm{dR}}(Y) + \mathrm{Fil}^\bullet, \varphi, N, \rho\} \tag{2.2}$$

This data consist of the de Rham cohomology itself together with its Hodge filtration $\mathrm{Fil}^\bullet$ but also the refined structures of Frobenius φ, monodromy N, and a residual Galois action ρ that satisfy compatibility relations. The structures (φ, N, ρ) live on Hydo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C)$, which is a C^{nr}-avatar of de Rham cohomology whose existence is due to the possibility of reducing modulo p (models of) our analytic varieties, and whose cohomology groups, under favorable conditions (for example when Y is partially proper), are subgroups of de Rham cohomology groups.

The (by now standard, see [39], [52], [41], [11]) passage between the two sides of (2.2) can be briefly described as follows:

• Locally on Y, by the $K(\pi, 1)$-lemma, we can think of $\mathrm{R}\Gamma_{\text{proét}}(Y_C, \mathbf{Q}_p)$ as the continuous group cohomology $\mathrm{R}\Gamma(\pi_1(Y_C), \mathbf{Q}_p)$.

• Then, by the mixed characteristic Artin-Schreier, we can pass to group cohomology of one of the (perfectoid) period rings.

• Now, using almost purity, we can replace $\pi_1(Y_C)$ by a commutative[4] p-adic Lie group Γ, which is the Galois group of the extension obtained by extracting p^∞-roots of coordinates of Y (hence $\Gamma \simeq \mathbf{Z}_p^d$, where $d = \dim Y$), and the period ring by the (perfectoid) period ring attached to this extension.

• Then, by decompletion, we can deperfect the period ring to bring it closer to the original variety Y, and make the action of Γ locally analytic. Invoking theorems of Lazard [61] we can now

[4]In the arithmetic situation [39], this group is no longer abelian, but is the semi-direct product of an open subgroup of $\mathbf{Z}_p^*$ by $\mathbf{Z}_p^d$.

pass to Lie algebra cohomology $\mathrm{R}\Gamma(\mathrm{Lie}\,\Gamma,-)$, which is computed via a Koszul complex yielding the de Rham complex, but where the differential d is multiplied [5] by t.

The above steps can be reversed after stabilization, i.e., Tate-twisting the pro-étale cohomology by a high enough twist and reflecting this on the other side by twisting all the structures.

Remark 2.3. (1) The above passage uses most of the standard tools from the toolbox of modern p-adic Hodge theorists. It is surprising that most of these tools were developed to study Hodge Theory of p-adic algebraic varieties but they work very well in the analytic setting as well. The authors were in particular surprised by the fact that the syntomic techniques developed by Fontaine-Messing, Hyodo, Kato, Tsuji, and others to compute p-adic nearby cycles turned out to be robust enough to treat analytic varieties.

(2) There is one major exception here: the almost purity theorem that comes from the realm of "relative almost étale theory" and which Faltings was not able to prove. This theorem was proved by Kedlaya–Liu and Scholze only recently[6] and it unlocked the Hodge Theory of p-adic analytic varieties: in the absence of resolution of singularities in mixed characteristic it is the second best thing.

2.2.2. *Hyodo-Kato cohomology as an avatar of ℓ-adic cohomology.* The Hyodo-Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C)$ mentioned above should be thought of as an avatar of ℓ-adic étale cohomology. In the case it is of finite rank over C^{nr} this can be made precise: the Hyodo-Kato cohomology groups yield Weil-Deligne representations (see [59]) and it is conjectured that these representations are isomorphic to the ones arising from ℓ-adic cohomology $\mathrm{R}\Gamma_{\text{ét}}(Y_C,\mathbf{Q}_\ell)$, for $\ell\neq p$. This conjecture is known in some cases (notably in dimension 1; see [67] in the algebraic setting and [32, Th. 8.2]) and allows to bootstrap results from ℓ-adic cohomology to p-adic cohomology (as in [30]).

2.2.3. *Étale versus pro-étale cohomology.* There is a natural map

$$\varepsilon:\quad \mathrm{R}\Gamma_{\text{ét}}(Y_C,\mathbf{Q}_p)\to \mathrm{R}\Gamma_{\text{proét}}(Y_C,\mathbf{Q}_p)$$

from étale to pro-étale cohomology but we do not understand it. It is not always injective as one could naively expect (see Remark 2.9 for an example). Moreover, the étale cohomology itself is difficult to understand. This is because, unlike pro-étale cohomology which, via comparison theorems, uses mostly rational p-adic Hodge Theory, the étale one needs integral (or integral up to universal denominators) p-adic Hodge Theory, which is much more subtle. However, in some favorable cases the étale cohomology can be recovered from the pro-étale cohomology (see [31] for the Drinfeld space and [30] for coverings of the Drinfeld space in dimension 1).

2.3. **The basic comparison theorem.** We will discuss now in more detail the first slogan (see Section 2.2.1).

2.3.1. *Statement of the theorem.* Let Y_K be a smooth rigid analytic variety with an overconvergent (or dagger) structure. Examples of such Y_K are proper or Stein varieties, the analytifications of algebraic varieties, overconvergent affinoids, etc. We assume that Y_K is geometrically connected.

The following theorem states the basic form of the comparison theorem between p-adic pro-étale cohomology and de Rham cohomology.

Theorem 2.4. (Basic comparison theorem [41]) *Let $n\geq r$ be positive integers. We have a long exact sequence in $\mathcal{D}(\mathbf{Q}_{p,\square})$*

$$\cdots\to X^{r,n-1}\to \mathrm{DR}^{r,n-1}\to H^n_{\text{proét}}(Y_C,\mathbf{Q}_p(r))\to X^{r,n}\to \mathrm{DR}^{r,n}\to\cdots \tag{2.5}$$

[5]It is this multiplication by t that is responsible for the huge $\mathbf{C}_p$-vector spaces that appear in $H^1_{\text{proét}}$ for the unit disk in Section 2.1.2.

[6]During the Hot Topics workshop at MSRI on homological conjectures (in 2018) Faltings mentioned in his talk that he tried to prove the almost purity theorem by passing from characteristic zero to characteristic p; it did not occur to him that one could try to go the other way (the way it was proved).

where

$$X^{r,i} := (H^i_{\rm HK}(Y_C)\otimes^{\Box}_{\mathbf{Q}_p}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r},\quad {\rm DR}^{r,i} := H^i({\rm R}\Gamma_{\rm dR}(Y_K)\otimes^{\Box}_K\mathbf{B}^+_{\rm dR}/{\rm Fil}^r).$$

Here, the Hyodo-Kato cohomology $H^i_{\rm HK}(Y_C)$, just as in the algebraic setting, is an additional structure on de Rham cohomology that comes from the "reduction modulo p". More precisely, $H^i_{\rm HK}(Y_C)$ is a $C^{\rm nr}$-module, endowed with actions of G_K, φ and N, such that the action of G_K commutes with φ and N, and $N\varphi = p\varphi N$. Moreover we have a "Hyodo-Kato isomorphism"

$$\iota_{\rm HK} : H^i_{\rm HK}(Y_C)\otimes^{\Box}_{C^{\rm nr}} C \xrightarrow{\sim} H^i_{\rm dR}(Y_C).$$

The construction of this Hyodo-Kato cohomology is modeled on the one of Beilinson for algebraic varieties (see [9]): it is the classical Hyodo-Kato cohomology if Y_K has a semistable model and uses the étale local alterations of Hartl and Temkin (see [57], [73]) to reduce to that case. Topologically, $H^i_{\rm HK}(Y_C)$ is a projective limit of finite dimensional $C^{\rm nr}$-modules on which the action of G_K is smooth (so the action on $H^i_{\rm HK}(Y_C)$ itself is "pro-smooth"). This allows to control the topology on many other p-cohomologies.

Remark 2.6. (Other definitions of Hyodo-Kato cohomology.) The Beilinson-style Hyodo-Kato cohomology described above was defined in [41]. Since then two alternate constructions have appeared:

(1) The first one is due to Binda-Gallauer-Vezzani (see [13], [12], and [14] for a survey). It lives in the world of rigid analytic motives of Ayoub [8]. Its definition also uses the alterations of Hartl and Temkin to reduce to "simpler" rigid analytic varieties. But it has two major advantages over the definition à la Beilinson:
   (i) Via the theory of rigid analytic motives it blackboxes the reduction steps via alterations.
   (ii) It allows to define Hyodo-Kato cohomology for, for example, perfectoid spaces.
(2) Very recently, Anschütz-Bosco-Le Bras-Rodriguez Camargo-Scholze [4] constructed Hyodo-Kato cohomology via the theory of Gelfand stacks. This definition also allows to work in a more general setting than rigid analytic varieties.

Theorem 2.4 expresses p-adic pro-étale cohomology in terms of differential forms but in a derived fashion, i.e., it allows to recover the derived p-adic pro-étale cohomology from the refined derived de Rham cohomology. An important fact to notice here is that the period rings appear in their $+$-form, i.e., we do not invert t. This accounts for the $n\geq r$ condition. A condition that seems restrictive but it will allow us to recover the pro-étale cohomology groups (**sic !**) from the refined de Rham cohomology groups (conjecturally in general, but inconditionaly in many cases of interest; see the examples in Section 2.3.2 and the Conjecture $\mathrm{C}_{\rm st}$ in Section 4).

Remark 2.7. (Analytic varieties over C.) If we do not assume that Y_C comes from a variety over K, we have a similar result but in the definition of ${\rm DR}^{r,n}$, one has to replace ${\rm R}\Gamma_{\rm dR}(Y_K)\otimes^{{\rm L}\Box}_K\mathbf{B}^+_{\rm dR}$ with the $\mathbf{B}^+_{\rm dR}$-cohomology.

2.3.2. *Examples.* We have the following consequences of Theorem 2.4:

(i) If Y_K is proper, the sequence (2.5) splits into short exact sequences

$$0\to H^n_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))\to X^{r,n}\to {\rm DR}^{r,n}\to 0,$$

which is reminiscent of the fundamental exact sequence

$$0\to\mathbf{Q}_p(r)\to\mathbf{B}^{+,N=0,\varphi=p^r}_{\rm st}\to\mathbf{B}^+_{\rm dR}/{\rm Fil}^r\to 0. \tag{2.8}$$

Moreover $H^n_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))$ is finite dimensional over $\mathbf{Q}_p$ and $H^n_{\rm HK}(Y_C)$ is finite dimensional over $C^{\rm nr}$. Hence this case is completely analogous to what happens for p-adic étale cohomology of algebraic varieties. The finite dimensionality of $H^n_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))$ was proved by Scholze in [69]; the splitting of the exact sequence uses this finite dimensionality, the fact that the maps in the exact sequence (2.5) are induced by morphisms of Banach-Colmez spaces, plus basic properties of BC's.

(ii) If Y_K is the affine space $\mathbb{A}^d_K$ of dimension d, we have $H^n_{\rm HK}(Y_K)=0$, for $n\geq 1$, from which we deduce an isomorphism

$$H^n_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))\simeq(\Omega^{n-1}(Y_C)/\operatorname{Ker} d)(r-n),\quad \text{for } 1\leq n\leq r.$$

By contrast, one can show that $H^n_{\rm \acute{e}t}(Y_C,\mathbf{Q}_p(r))=0$, for $n\geq 1$.

(iii) More generally, if Y_K is Stein (i.e., a strictly increasing union of affinoids, which implies that $H^i(Y_C,\mathcal{F})=0$, for $i\geq 1$ and $\mathcal{F}$ a coherent sheaf on Y_C), we have a short exact sequence

$$0\to\big(\Omega^{n-1}(Y_C)/\operatorname{Ker} d\big)(r-n)\to H^n_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))\to X^{r,n}\to 0$$

Remark 2.9. (Not spherically complete puzzle.) It follows from example (iii) above that, if Y is the open unit ball in dimension 1, then $H^2_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p)=0$. On the other hand $H^2_{\rm \acute{e}t}(Y_C,\mathbf{Q}_p)=0$ if and only if C is spherically complete [32, Th. A.6]. This shows that étale cohomology does not always inject into pro-étale cohomology.

The above dichotomy between spherically complete and not spherically complete fields is a bit unsettling. But, at least in dimension 1, it can be fixed by using the *adoc generic fiber* instead of the rigid one (see [32, Rem. A.9]): if we view Y as a rigid analytic space, we need to cover Y by an increasing union of closed balls, and we miss the ghost circle at the boundary.

Another possibility would be to change slightly the definition of étale cohomology to impose that Y is a $K(\pi,1)$-space (more generally, that affine adoc spaces are $K(\pi,1)$): indeed one can show that $H^i(\pi_1(Y_C),\mathbf{Q}_p)$ coincides with $H^i_{\rm \acute{e}t}(Y_C,\mathbf{Q}_p)$ for $i\leq 1$, and is 0 if $i\geq 2$ (without imposing C to be spherically complete, see [32, Prop. A.8]).

2.3.3. *Comparison with syntomic cohomology.* The long exact sequence (2.5) is obtained from the natural period quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\alpha_{\rm BK}:\quad \tau_{\leq r}{\rm R}\Gamma_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))\simeq\tau_{\leq r}{\rm R}\Gamma^{\rm BK}_{\rm syn}(Y_C,\mathbf{Q}_p(r)),\tag{2.10}$$

where

$${\rm R}\Gamma^{\rm BK}_{\rm syn}(Y_C,\mathbf{Q}_p(r)):=\big[[{\rm R}\Gamma_{\rm HK}(Y_C)\otimes^{{\rm L}_\square}_{C^{\rm nr}}\mathbf{B}^+_{\rm st}]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\rm HK}}({\rm R}\Gamma_{\rm dR}(Y_K)\otimes^{{\rm L}_\square}_K\mathbf{B}^+_{\rm dR})/{\rm Fil}^r\big]$$

is the Bloch-Kato syntomic cohomology. The term $({\rm R}\Gamma_{\rm dR}(Y_K)\otimes^{{\rm L}_\square}_K\mathbf{B}^+_{\rm dR})/{\rm Fil}^r$ gives rise to the ${\rm DR}^{r,i}$ term in the exact sequence (2.5)) and the term $[{\rm R}\Gamma_{\rm HK}(Y_C)\otimes^{{\rm L}_\square}_{C^{\rm nr}}\mathbf{B}^+_{\rm st}]^{N=0,\varphi=p^r}$ gives rise to the $X^{r,i}$ terms in the same sequence.

The quasi-isomorphism (2.10) follows from the following two results.

Theorem 2.11. (Pro-étale vs syntomic comparison) *Let $r\geq 0$. The Fontaine-Messing period morphism:*

$$\alpha_{\rm FM}:\quad {\rm R}\Gamma^{\rm FM}_{\rm syn}(Y_C,\mathbf{Q}_p(r))\to{\rm R}\Gamma_{\rm pro\acute{e}t}(Y_C,\mathbf{Q}_p(r))$$

is a quasi-isomorphism after truncation $\tau_{\leq r}$.

Here, the syntomic cohomology (ala Fontaine-Messing) is defined as the mapping fiber

$${\rm R}\Gamma^{\rm FM}_{\rm syn}(Y_C,\mathbf{Q}_p(r)):=\big[[{\rm R}\Gamma_{\rm cris}(\mathcal{Y})]^{\varphi=p^r}\xrightarrow{\rm can}{\rm R}\Gamma_{\rm cris}(\mathcal{Y})/{\rm Fil}^r\big]\tag{2.12}$$

where $\mathcal{Y}$ is an étale hypercovering of Y_C by affinoids with a semistable model[7] (such hypercoverings can be constructed using the alterations of Hartl and Temkin) and ${\rm R}\Gamma_{\rm cris}(\mathcal{Y})$ is the absolute crystalline cohomology of the associated log-formal-schemes. The period map $\alpha_{\rm FM}$ is the one originally defined by Fontaine-Messing in [51]. It involves a version of Poincaré Lemma for the appearing period sheaves (a priori in the syntomic-étale topology).

[7]Sensu stricto, to make this definition independent of choices one has to take a colimit over all hypercoverings of such from.

The proof of Theorem 2.11 in [41] relies on local computations from [39], [52], which use (φ,Γ)-modules to express the map $\alpha_{\rm FM}$ as a serious of quasi-isomorphisms. These modification is then globalized using the approach of all-coordinates borrowed from Bhatt-Morrow-Scholze [11].

Remark 2.13. (Two types of syntomic cohomology.) The two types of syntomic cohomology mentioned above (they are quasi-isomorphic) play a different role in p-adic Hodge Theory. The Bloch-Kato syntomic cohomology (called that way because generalized Bloch-Kato Selmer groups from [15]) is useful for computations: one first tries to get a handle on de Rham cohomology, from that, via the Hyodo-Kato isomorphism on Hyodo-Kato cohomology; then all that remains is to understand the filtration, Frobenius, and monodromy (difficult in general). The Fontaine-Messing syntomic cohomology, on the other hand, was traditionally used to prove comparison theorems (see [74]).

Roughly speaking, the Fontaine-Messing period morphism $\alpha_{\rm FM}$ in Theorem 2.11 is induced by the fundamental exact sequence

$$0\to\mathbf{Q}_p(r)\to F^r\mathbb{B}^+_{\rm cr}\xrightarrow{p^r-\varphi}\mathbb{B}^+_{\rm cr}\to 0$$

of sheaves on the syntomic site. To prove that it is a quasi-isomorphism in a stable range one replaces this sequence with the exact sequence

$$0\to\mathbf{Q}_p(r)\to\mathbb{B}_I(r)\xrightarrow{1-\varphi}\mathbb{B}_{I'}(r)\to 0$$

of pro-étale sheaves and proceeds as sketched in Section 2.2.1. Here $I,I'\subset(0,\infty)$ are compact intervals with rational endpoints (conveniently chosen) and $\mathbb{B}_I,\mathbb{B}_{I'}$ are relative versions of the period rings $\mathbf{B}_I,\mathbf{B}_{I'}$ — coordinate rings of standard affinoids on the $Y_{\rm FF}$-curve (the Y-curve of Fargues-Fontaine). In particular, the choices of the intervals I,I' imply that t has one zero on ${\rm Spa}(\mathbf{B}_I)$ and is a unit in $\mathbf{B}_{I'}$.

Theorem 2.14. *(1)* (Künneth formula for derived de Rham cohomology) *We have a natural quasi-isomorphism:*

$$\mathrm{R}\Gamma_{\rm cris}(\mathcal{Y})/{\rm Fil}^r\simeq(\mathrm{R}\Gamma_{\rm dR}(Y_K)\otimes^{\rm L_\square}_K\mathbf{B}^+_{\rm dR})/{\rm Fil}^r$$

(2) (Hyodo-Kato rigidity) *We have a natural Frobenius equivariant quasi-isomorphism:*

$$\mathrm{R}\Gamma_{\rm cris}(\mathcal{Y})\simeq[\mathrm{R}\Gamma_{\rm HK}(Y_C)\otimes^{\rm L_\square}_{C^{\rm nr}}\mathbf{B}^+_{\rm st}]^{N=0}\tag{2.15}$$

The first claim follows from the reinterpretation of (Hodge completed) crystalline cohomology as (Hodge completed) derived de Rham cohomology, from the Künneth formula for the latter, and from the fact that $\mathbf{B}^+_{\rm dR}/{\rm Fil}^r\simeq\mathrm{R}\Gamma_{\rm dR}(\mathcal{O}_{\mathbf{C}_p})/{\rm Fil}^r$. The second claim is argueably the most difficult fact in the theory of Hyodo-Kato cohomology. It relies on the existence of a section to the canonical projection $\mathrm{R}\Gamma_{\rm cris}(\mathcal{Y})\to\mathrm{R}\Gamma_{\rm HK}(Y_C)$; a section that should be compatible both with Frobenius and monodromy and also highly functorial. Morally speaking, this follows from a version of Dwork Lemma using the fact that Frobenius is an automorphism on the Hyodo-Kato cohomology but is highly nilpotent on the kernel of the projection. In the case Hyodo-Kato cohomology is of finite rank Beilinson in [9] has written down an elegant abstract argument why this works. In general, one has to upgrade the original construction of Hyodo-Kato, which is highly nonconstructive.

2.3.4. *Pro-étale cohomology of period sheaves.* The key computation in the proof of Theorem 2.11 is that of pro-étale cohomology of the period sheaves $\mathbb{B}_I$.

Theorem 2.16. (Cohomology of $\mathbb{B}_I$) *Let $I\subset(0,\infty)$ be a compact interval with rational endpoints.*
(1) There is a natural[8] *"Frobenius equivariant" quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{I,\square})$*

$$\tau_{\leq r}\mathrm{R}\Gamma_{\rm pro\acute{e}t}(Y_C,\mathbb{B}_I)\simeq\tau_{\leq r}[[\mathrm{R}\Gamma_{\rm HK}(Y_C)\{r\}\otimes^{\rm L_\square}_{C^{\rm nr}}\mathbf{B}_{I,\log}]^{N=0}\to\oplus_{Z(I)}(\mathrm{R}\Gamma_{\rm dR}(Y_K)\otimes^{\rm L_\square}_K\mathbf{B}^+_{\rm dR})/{\rm Fil}^r](-r),\tag{2.17}$$

[8]Frobenius sends I to I/p, hence the quotation marks.

where the index set $Z(I)$ is over the zeros of t in $\mathrm{Spa}(\mathbf{B}_I)$.

(2) If t is a unit in $\mathbf{B}_I$ then there is a natural "Frobenius equivariant" quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{I,\square})$

$$\mathrm{R}\Gamma_{\text{proét}}(Y_C,\mathbb{B}_I)\simeq[\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C)\{r\}\otimes^{\mathrm{L}_\square}_{C^{\mathrm{nr}}}\mathbf{B}_{I,\log}]^{N=0}(-r).$$

Remark 2.18. Bosco in [17, Th. 4.1, Rem. 6.13] proved a version of the comparison quasi-isomorphism (2.17) for the sheaf $\mathbb{B}:=\varprojlim_I\mathbb{B}_I$, where the t-torsion on the right-hand side of (2.17) is incorporated to the left-hand side via the $\mathrm{L}\,\eta_t$-operator.

Let $I\subset(0,\infty)$ be a compact interval with rational endpoints such that t has one zero on $\mathrm{Spa}(\mathbf{B}_I)$. Tensoring both sides of (2.17) for $r\geq 2d$ ($d=\dim Y_K$) with $\mathbf{B}_I/t^n\simeq\mathbf{B}^+_{\mathrm{dR}}/t^n$, $n\in\mathbf{N}$, passing to limit over n, and dropping the twist r since we do not care about Frobenius anymore, one obtains the following:

Corollary 2.19. (Cohomology of $\mathbb{B}^+_{\mathrm{dR}}$)

$$\mathrm{R}\Gamma_{\text{proét}}(Y_C,\mathbb{B}^+_{\mathrm{dR}})\simeq F^0(\mathrm{R}\Gamma_{\mathrm{dR}}(Y_K)\otimes^{\mathrm{L}_\square}_K\mathbf{B}_{\mathrm{dR}}).\tag{2.20}$$

This result (in a more general setting) was derived in a simpler way by Bosco in [16, Th. 1.8] directly from the $\mathbb{B}_{\mathrm{dR}}$-Poincaré Lemma.

2.3.5. *Pro-étale cohomology with compact support.* It came as a surprise to us – though, in hindsight, we should have anticipated it – that compactly supported p-adic pro-étale cohomology does not behave vis a vis comparison theorems as well as the usual p-adic pro-étale cohomology. So it was a stroke of luck that in [30], for example, we started to look for a geometric realization of the p-adic local Langlands correspondence in the usual cohomology (contrary to the traditional approach to geometrization of the ℓ-adic local Langlands correspondence, where compactly supported cohomology is used).

However not all is lost: we still have the basic comparison theorem and a vestige of the fundamental diagram. See Corollary 4.12 below for the latter; we will discuss the former here.

It was not clear initially what the right definition of compactly supported p-adic pro-étale cohomology of a smooth partially proper variety Y should be. There were two immediate guesses (see [29, Sec. A.2] for a discussion):

$$\mathrm{R}\Gamma^{(1)}_{\text{proét},c}(Y,\mathbf{Q}_p):=(\mathrm{R}\lim_n\mathrm{R}\Gamma_{\text{proét},c}(Y,\mathbf{Z}/p^n))\otimes^{\mathrm{L}_\square}_{\mathbf{Z}_p}\mathbf{Q}_p;\tag{2.21}$$

$$\text{or }\mathrm{R}\Gamma^{(2)}_{\text{proét},c}(Y,\mathbf{Q}_p):=[\mathrm{R}\Gamma_{\text{proét}}(Y,\mathbf{Q}_p)\to\mathrm{R}\Gamma_{\text{proét}}(\partial Y,\mathbf{Q}_p)],$$
$$\text{with }\mathrm{R}\Gamma(\partial Y,\mathbf{Q}_p)=\mathrm{colim}_{Z\in\Phi_Y}\mathrm{R}\Gamma(Y\setminus Z,\mathbf{Q}_p),$$

where Φ_Y denotes the set of quasi-compact opens in Y and the brackets [...] denote the mapping fiber. The first definition is just a continuous version of $\mathbf{Z}/p^n$-cohomology. The heuristic for the second definition comes from the (possible) comparison with de Rham cohomology and the classical definition of van der Put of compactly supported version of the latter. Surprisingly, this definition agrees with Huber's definition of compactly supported p-adic étale (sic !) cohomology: this follows from the canonical quasi-isomorphism

$$\mathrm{R}\Gamma^{(2)}_{\text{proét},c}(Y,\mathbf{Z}_p)_{\mathbf{Q}_p}\xrightarrow{\sim}\mathrm{R}\Gamma^{(2)}_{\text{proét},c}(Y,\mathbf{Q}_p).$$

The fact that this might be the right definition comes from the work on dualities which pairs this cohomology with p-adic pro-étale cohomology (see [35], [36], [64], [6]). Hence we set

$$\mathrm{R}\Gamma_{\text{proét},c}(Y,\mathbf{Q}_p):=\mathrm{R}\Gamma^{(2)}_{\text{proét},c}(Y,\mathbf{Q}_p).$$

Theorem 2.22. (Basic comparison theorem, [2, Cor. 1.6]) *The analogs of Theorem 2.4 and Theorem 2.11 hold for the compactly supported p-adic pro-étale cohomology.*

Example 2.23. Let $Y:=\mathbb{A}^d_K$, $d\geq 1$, be the rigid analytic affine space of dimension d over K.

(1) We have the following isomorphisms in solid $\mathbf{Q}_p$-modules
$$H^i_{\text{proét},c}(Y_C,\mathbf{Q}_p)=0,\quad \text{if } i<d \text{ or } i>2d;$$
$$H^i_{\text{proét},c}(Y_C,\mathbf{Q}_p(i-d))\simeq H^d_c(Y_C,\Omega^{i-d-1})/\operatorname{Ker} d,\quad \text{if } d\leq i\leq 2d-1.$$
(2) We have an exact sequence in solid $\mathbf{Q}_p$-modules
$$0\to H^d_c(Y_C,\Omega^{d-1})/\operatorname{Ker} d\to H^{2d}_{\text{proét},c}(Y_C,\mathbf{Q}_p(d))\to\mathbf{Q}_p\to 0.$$

For $d=1$ and $i\geq 0$, this yields:

- vanishings $H^i_{\text{proét},c}(\mathbb{A}^1_C,\mathbf{Q}_p)=0$, for $i\neq 2$,
- an exact sequence of solid $\mathbf{Q}_p$-modules
$$0\to H^1_c(\mathbb{A}^1_C,\mathcal{O})\to H^2_{\text{proét},c}(\mathbb{A}^1_C,\mathbf{Q}_p(1))\to\mathbf{Q}_p\to 0.$$

2.3.6. *The arithmetic case.* The reader might wonder what can we say about the p-adic pro-étale cohomology of a smooth dagger variety Y itself (still defined over K). Turns out that in this case we do have an analog of the syntomic-pro-étale comparison (2.10) with the arithmetic syntomic cohomology
$$\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Y):=\big[[\mathrm{R}\Gamma_{\mathrm{HK}}(Y)]^{N=0,\varphi=p^r}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(Y)/\mathrm{Fil}^r\big],$$
where $\mathrm{R}\Gamma_{\mathrm{HK}}(Y)$ is the arithmetic version of Hyodo-Kato cohomology. The existence of such an analog was not obvious at all to experts as the original methods (Hyodo, Kato, Tsuji) of proving Theorem 2.11 in the algebraic case via the Milnor symbols were not easily adaptable to the arithmetic setting[9]. It is here that the (φ,Γ)-approach[10] showed its superiority (see [3] for basic facts about relative (φ,Γ)-modules).

The analog of (2.10) gives the following:

Theorem 2.24. (Basic arithmetic comparison theorem [39, 40]) *Let $n\geq r$ be positive integers. We have a long exact sequence in $\mathcal{D}(\mathbf{Q}_{p,\square})$*
$$\cdots\to X^{r,n-1}\to\mathrm{DR}^{r,n-1}\to H^n_{\text{proét}}(Y,\mathbf{Q}_p(r))\to X^{r,n}\to\mathrm{DR}^{r,n}\to\cdots\tag{2.25}$$
where
$$X^{r,i}:=H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(Y)]^{N=0,\varphi=p^r}),\quad \mathrm{DR}^{r,i}:=H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(Y)/\mathrm{Fil}^r).$$

We note that in general $X^{r,i}$ is not equal to $H^i_{\mathrm{HK}}(Y)^{N=0,\varphi=p^r}$ (see (2.26) below). The group $\mathrm{DR}^{r,i}$ tends to be huge: in the case Y is Stein we have
$$\mathrm{DR}^{r,r-1}\simeq\Omega^{r-1}(Y)/\operatorname{Im} d;\quad \mathrm{DR}^{r,i}=0,\quad i\geq r.$$

Theorem 2.24 yields the following computation ([39, Cor. 1.4],[40, Th. 1.3]):

(1) For $1\leq i\leq r-1$, the boundary map induced by the sequence (2.25)
$$\partial_r: H^{i-1}_{\mathrm{dR}}(Y)\to H^i_{\text{proét}}(Y,\mathbf{Q}_p(r))$$
is an isomorphism. In particular, the cohomology $H^i_{\text{proét}}(Y,\mathbf{Q}_p(r))$ has a natural K-structure.

(2) We have exact sequences
$$0\to\mathrm{DR}^{r-1,r}\xrightarrow{\partial_r}H^r_{\text{proét}}(Y,\mathbf{Q}_p(r))\to X^{r,r}\to\mathrm{DR}^{r,r}$$
$$0\to H^{r-1}_{\mathrm{HK}}(Y)^{\varphi=p^{r-1}}\to X^{r,r}\to H^r_{\mathrm{HK}}(Y)^{N=0,\varphi=p^r}\to 0\tag{2.26}$$

[9] One of us tried very hard to make this approach work but the torsion that appears in crystalline cohomology of ramified extension was very difficult to deal with.

[10] The starting point of our investigations was the observation that the syntomic and (φ,Γ)-module approaches to compute $H^\bullet(G_F,\mathbf{Q}_p(r))$ (for $r\geq 1$ and F a finite, unramified, extension of $\mathbf{Q}_p$) produced complexes that look so similar that there ought to be a way to go from one to the other via simple quasi-isomorphisms. Also, the syntomic approach proved to be much easier to use to do actual computations (but it only applies to very special coefficients – though more general than $\mathbf{Q}_p(r)$ with $r\geq 1$, see [1]).

Remark 2.27. It follows form the above results that, if $[K:\mathbf{Q}_p]<\infty$, and if Y is an overconvergent affinoid (or, more generally, if Y has finite dimensional de Rham cohomology), then $H^i_{\text{proét}}(Y,\mathbf{Q}_p(r))$ is finite dimensional over $\mathbf{Q}_p$ if $0\leq i\leq r-1$. This is in sharp contrast with what happens for $i=r$ where $\mathrm{DR}^{r,r-1}$ injects into $H^i_{\text{proét}}(Y,\mathbf{Q}_p(r))$, and can be huge since we have the exact sequence

$$0\to H^{r-1}_{\mathrm{dR}}(Y)\to \mathrm{DR}^{r,r-1}\to \ker\pi\to 0,$$

where $\pi:\Omega^r(Y)^{d=0}\to H^r_{\mathrm{dR}}(Y)$ is the canonical map.

3. Pro-étale cohomology of affinoids

We work in this section in the ∞-derived category $\mathcal{D}(C_{\mathbf{Q}_p})$ of locally convex topological vector spaces over $\mathbf{Q}_p$ (this is less robust than the condensed version for doing homological algebra, but is sufficient in many cases).

3.1. **Improving the comparison with syntomic cohomology.** The de Rham cohomology of affinoids is notoriously pathological (it is infinite dimensional and non separated); this is the reason for making them overconvergent and defining dagger varieties. On the other hand, if Y is an affinoid over K, its pro-étale cohomology is much better behaved: $H^r_{\text{proét}}(Y_C,\mathbf{Q}_p(r))$ is a Banach space (an elementary fact in dimension 1; the general case was proven by Bosco [18]).

Now, the methods used to prove Theorem 2.4 yield an exact sequence[11] in $\mathrm{LH}(C_{\mathbf{Q}_p})$ (the left heart of $\mathcal{D}(C_{\mathbf{Q}_p})$)

$$0\to\Omega^{r-1}(Y_C)/\mathrm{Ker}\,d\to H^r_{\text{proét}}(Y_C,\mathbf{Q}_p(r))\to(\mathbf{B}^+_{\mathrm{st}}\widehat{\otimes}_{\breve{C}}H^r_{\mathrm{HK}}(Y_C))^{N=0,\varphi=p^r}\to 0$$

This is not very satisfactory since $H^r_{\mathrm{HK}}(Y_C)$ is not separated, which makes the right-hand term not separated. We denote by $H^1_{\mathrm{HK}}(Y_C)^{\mathrm{sep}}$ its quotient by the topological closure of 0; this is a finite dimensional $\breve{C}$-module, and we still have a Hyodo-Kato isomorphism

$$C\otimes_{\breve{C}}H^1_{\mathrm{HK}}(Y_C)^{\mathrm{sep}}\simeq H^1_{\mathrm{dR}}(Y_C)^{\mathrm{sep}}.$$

If Y be an affinoid over C, let $\mathcal{O}(Y)^{**}$ be the group of $f\in\mathcal{O}(Y)^*$ such that $f-1$ is topologically nilpotent. The following theorem fixes the above mentionned problem in dimension 1.

Theorem 3.1. ([32, Th. 0.1]) *If Y is an affinoid of dimension 1, we have a commutative diagram of Banach spaces* [12]

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Q}_p\widehat{\otimes}\mathcal{O}(Y)^{**} \arrow[r] \arrow[d, equal] & H^1_{\text{proét}}(Y,\mathbf{Q}_p(1)) \arrow[r] \arrow[d] & (\mathbf{B}^+_{\mathrm{st}}\otimes_{\breve{C}}H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}})^{N=0,\varphi=p} \arrow[r] \arrow[d, "\theta\otimes\iota_{\mathrm{HK}}"] & 0 \\
0 \arrow[r] & \mathbf{Q}_p\widehat{\otimes}\mathcal{O}(Y)^{**} \arrow[r, "\mathrm{dlog}"] & \Omega^1(Y) \arrow[r] & H^1_{\mathrm{dR}}(Y)^{\mathrm{sep}} \arrow[r] & 0
\end{tikzcd}$$

with upper row being exact and lower row being a complex and all arrows having closed image.

Remark 3.2. The group $H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}}$ has a concrete combinatorial description (see Th. 3.4), starting from a triangulation of Y (or, what amounts to the same, a semi-stable model). It is easy to obtain analogous results for Stein curves or overconvergent affinoids from the above result and this combinatorial description [32, Th. 0.5, Th. 0.7].

Remark 3.3. If Y is an affinoid of dimension d, and if $r\leq d$, the above result suggests that one could have an exact sequence

$$0\to\mathbf{Q}_p\widehat{\otimes}K^r_{\mathcal{M}}(Y)^{++}\to H^r_{\text{proét}}(Y,\mathbf{Q}_p(r))\to(\mathbf{B}^+_{\mathrm{st}}\otimes_{\breve{C}}H^r_{\mathrm{HK}}(Y)^{\mathrm{sep}})^{N=0,\varphi=p^r}\to 0$$

where $K^r_{\mathcal{M}}(Y)^{++}$ is the subgroup of the Milnor K-theory group $K^r_{\mathcal{M}}(\mathcal{O}(Y))$ generated by symbols $(f_1,\dots,f_r)$, with $f_i\in\mathcal{O}(Y)^{**}$ (this restriction makes Steinberg relation disappear since you cannot have x and $1-x$ both belonging to $\mathcal{O}(Y)^{**}$).

[11] We use here the $\breve{C}$-version of Hyodo-Kato cohomology.

[12] If M is a $\mathbf{Z}$-module, we set $\mathbf{Q}_p\widehat{\otimes}M:=\mathbf{Q}_p\otimes_{\mathbf{Z}_p}(\varprojlim_n M/p^nM)$.

The next two sections are devoted to the ingredients that come into play for the proof of Theorem 3.1. The general philosophy is inspired by the proof of Theorem 2.4, but working with curves allows some notable simplifications (in particular, the construction of the Hyodo-Kato isomorphism becomes straightforward).

3.2. **Chopping a curve into shorts and legs.** Let Y be a quasi-compact curve over C. Choose a triangulation S of Y and denote by Y_S the associated semistable model of Y. We can (and will) choose S fine enough so that the irreducible components of the special fiber $Y_S^{\rm sp}$ are smooth and intersect in at most one point.

We see $Y_S^{\rm sp}$ as a *proper curve over k_C endowed with a set A of marked points* $a = (P_a, \mu(a))$, with $A = A_c \sqcup (A \setminus A_c)$, and:

• A_c is the set of singular points and $\mu(a) \in \mathbf{Q}_{>0}$ if $a \in A_c$;

• $A \setminus A_c$ is the set of points that one has to remove to get the usual special fiber and $\mu(a) = 0^+$ if $a \in A \setminus A_c$.

The irreducible components of $Y_S^{\rm sp}$ are smooth and proper and in bijection with S (we denote by $Y_s^{\rm sp}$ the component corresponding to $s \in S$).

From this data, one builds *the adoc graph* $\Gamma^{\rm ad}(Y_S)$ of Y_S, whose vertices are S, edges A, each edge a being of length $\mu(a)$ with end points the $s \in S$ such that $P_a \in Y_s^{\rm sp}$ (hence $a \in A_c$ has two end points whereas $a \in A \setminus A_c$ has only one). The graph $\Gamma^{\rm ad}(Y_S)$ is thus the dual graph of the classical special fiber with added edges of length 0^+ originating from vertices corresponding to nonproper irreducible components, one such edge for each missing point.

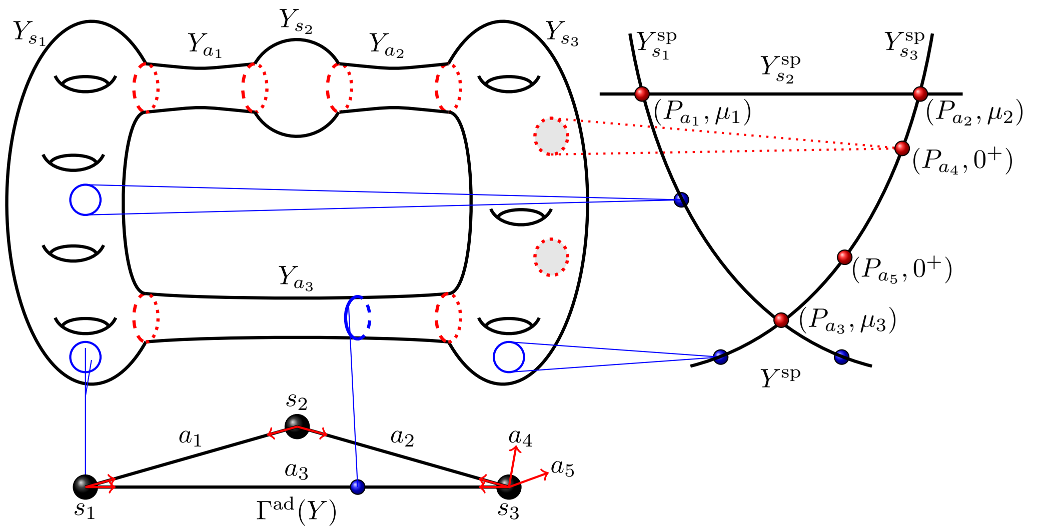
If $s \in S$, the tube Y_s of $Y_s^{\rm sp}$ with marked points removed is *a shorts* (i.e. (the formal model of) an affinoid with good reduction); if $a \in A_c$, the tube Y_a of P_a is *a leg* (i.e., an open annulus) of length $\mu(a)$ (one has $\mathcal{O}(Y_a) = \mathcal{O}_C[[T_{a,s_1}, T_{a,s_2}]]/(T_{a,s_1}T_{a,s_2} - p^{\mu(a)})$, if s_1, s_2 are the end points of a).

The Y_i, for $i \in I = S \sqcup A_c$, form a partition of Y. To reconstruct Y, one needs an additional gluing data for Y_s and Y_a if $(a, s) \in I_{2,c}$, where $I_{2,c}$ is the set of (a, s) with $a \in A_c$ and s an end point of a (such a gluing does not make sense in rigid or adic geometry, but it does in *adoc*[13] *geometry*, see [32, chap. 3]): P_a determines a rank 2 valuation on $\mathcal{O}(Y_s)$, hence *a ghost*[14] *circle* $Y_{s,a}$ (choosing a local parameter gives an isomorphism $\mathcal{O}(Y_{s,a}) \simeq \mathcal{O}_C[[T_{s,a}, T_{s,a}^{-1}\rangle$, completion of $\mathcal{O}_C[[T_{s,a}]][T_{s,a}^{-1}]$ for the p-adic topology), and Y_a contains a corresponding ghost circle $Y_{a,s}$; the gluing data is then an isomorphism $\iota_{a,s} : Y_{a,s} \simeq Y_{s,a}$, i.e. an isomorphism $\mathcal{O}_C[[T_{s,a}, T_{s,a}^{-1}\rangle \simeq \mathcal{O}_C[[T_{a,s}, T_{a,s}^{-1}\rangle$.

The above set of data (i.e. $\Gamma = (S, A, A_c, \mu)$, $(Y_i)_{i \in I}$, $(\iota_{i,j})_{(i,j) \in I_{2,c}}$), is *a pattern of curve*, see [32, §§ 3.5, 3.6]. The above discussion explains how to associate a pattern of curve to a triangulation, and how to reconstruct Y from this pattern; we refer to the figure below for a visual illustration of this procedure.

[13] An interpolation between "ad hoc" and "adic".

[14] It has no point in characteristic 0, hence the name; on the other hand, its reduction is the punctured open ball in characteristic p.



An affinoid Y built from 3 shorts and 3 legs

The above affinoid Y is obtained by gluing, along ghost circles (red dots), shorts Y_{s_1}, Y_{s_2}, Y_{s_3} of respective genus 4, 0 et 3, and legs Y_{a_1}, Y_{a_2}, Y_{a_3} of respective length μ_1, μ_2, μ_3. The special fiber $Y^{\rm sp}$ has five marked points: three singular corresponding to the specializations of the legs and two to the tubes removed from Y_{s_3}. The skeleton $\Gamma^{\rm an}(Y)$ is the triangle with vertices s_1, s_2 et s_3 (the length of edges a_1, a_2 and a_3 are μ_1, μ_2 and μ_3) and $\Gamma^{\rm ad}(Y)$ is obtained by adding the arrows a_4 and a_5 (of length 0^+).

Let $S_{\rm int}\subset S$, be the set of s such that $Y_s^{\rm sp}$ has no marked point $(a,\mu(a))$ with $\mu(a)=0^+$, and let $\Gamma_{\rm int}$ be the full subgraph of $\Gamma^{\rm ad}(Y_S)$ with set of vertices $S_{\rm int}$ (this is *the interior* of $\Gamma^{\rm ad}(Y_S)$).

Theorem 3.4. ([32, Th. 0.13]). *$H^1_{\rm HK}(Y)^{\rm sep}$ admits a natural filtration whose successive quotients are*[15]

$$H^1_{\rm HK}(Y)^{\rm sep}=\big[\ H^1(\Gamma_{\rm int},\check C)-\prod_{s\in S_{\rm int}}H^1_{\rm rig}(Y_s^{\rm sp})\oplus\prod_{s\in\partial Y}H^1_{\rm rig}(Y_s^{\rm sp})^{[1]}-H^1_c(\Gamma,\check C)^*(-1)\ \big],$$

where the exponent $[1]$ *denotes the slope* 1 *subspace* (for φ)*, and the* (-1) *twists means that the natural action of* φ *is multiplied by* p.

Remark 3.5. The operator N of footnote 15 induces $N:H^1_c(\Gamma,\check C)^*\to H^1(\Gamma,\check C)\to H^1(\Gamma_{\rm int},\check C)$ (the second map being restriction), and hence a "monodromy operator" $N:H^1_{\rm HK}(Y)^{\rm sep}\to H^1_{\rm HK}(Y)^{\rm sep}$.

3.3. **Proofs.** Choose a fine enough triangulation S of Y, let Y_S be the associated semistable model, and let $(\Gamma,(Y_i)_{i\in I},(\iota_{i,j})_{(i,j)\in I_{2,c}})$ be the associated pattern. The proof of Th. 3.1 relies on:

- the definition of a group of symbols ${\rm Symb}_p(Y)$,
- the construction of étale and syntomic regulators inducing isomorphisms:

$$H^1_{\rm ét}(Y,\mathbf{Z}_p(1))\xleftarrow{\sim}{\rm Symb}_p(Y)\xrightarrow{\sim}H^1_{\rm syn}(Y_S,\mathbf{Z}_p(1)),$$

- a description of $H^1_{\rm syn}(Y_S,\mathbf{Z}_p(1))$ in terms of the de Rham cohomology and its variations. Only this part introduces denominators (these depend on the length of the legs: the smaller the legs, the bigger the denominators).

[15] If Γ is a graph, we let $H^\bullet(\Gamma,\Lambda)$ and $H^\bullet_c(\Gamma,\Lambda)$ denote its cohomology or cohomology with compact support. These groups have a natural combinatorial description in terms of operators ∂ and ∂^* for which we refer to [32, § 1.2]. If Γ is metrized (as is the case of $\Gamma^{\rm ad}(Y_S)$), this induces $N:H^1_c(\Gamma,\Lambda)^*\to H^1(\Gamma,\Lambda)$ involving the length of the edges, see [32, 1.2.4].

3.3.1. *Symbols.* Let Y be a quasi-compact curve over C, let $C(Y)$ be the field of meromorphic functions on Y, and let $\mathrm{Div}(Y)$ be the group of finite sums $\sum_{x\in Y(C)} n_x x$, with $n_x \in \mathbf{Z}$ for all x.

One defines *the symbol group* $\mathrm{Symb}_p(Y)$ to be

$$\mathrm{Symb}_p(Y) = \frac{\{(f_n)_{n\in\mathbf{N}},\ f_n\in C(Y)^*,\ \mathrm{Div}(f_n)\in p^n\mathrm{Div}(Y),\ f_{n+1}/f_n\in (C(Y)^*)^{p^n}\}}{\{(h_n^{p^n}),\ h_n\in C(Y)^*\}}.$$

We have an exact sequence

$$0\to H^1(\Gamma,\mathbf{Z}_p(1))\to \mathrm{Symb}_p(Y)\to \mathrm{Ker}\big(\prod_{i\in I}\mathrm{Symb}_p(Y_i^{\mathrm{gen}})\to \prod_{(i,j)\in I_{2,c}}\mathrm{Symb}_p(Y_{i,j}^{\mathrm{gen}})\big),$$

where "gen" stand for "generic fiber" and $\mathrm{Symb}_p(Y_i^{\mathrm{gen}})$ is defined as above.

3.3.2. *Étale regulators.* Kummer theory gives an étale regulator $\mathrm{Symb}_p(Y)\to H^1_{\text{ét}}(Y,\mathbf{Z}_p(1))$ which can be proven to be an isomorphisme thanks to Kummer exact sequence

$$0\to (\mathbf{Z}/p^n)\otimes \mathcal{O}(Y)^*\to H^1_{\text{ét}}(Y,\mathbf{Z}/p^n(1))\to \mathrm{Pic}(Y)[p^n]\to 0$$

Via the above exact sequence, this reduces the computation of $H^1_{\text{ét}}(Y,\mathbf{Z}_p(1))$ to that of the $\mathrm{Symb}_p(Y_i^{\mathrm{gen}})$'s.

3.3.3. *Syntomic regulators.* Let $C_{\mathrm{dR}}(\widetilde{Y}_S)$ be the complex computing the absolute log-crystalline cohomology $H^\bullet_{\mathrm{dR}}(\widetilde{Y}_S)$ of Y_S; i.e.,

$$C_{\mathrm{dR}}(\widetilde{Y}_S) :=\ \prod_{i\in I}\mathcal{O}(\widetilde{Y}_i)\longrightarrow \prod_{i\in I}\Omega^1(\widetilde{Y}_i)\oplus\prod_{(i,j)\in I_{2,c}}\mathcal{O}(\widetilde{Y}_{i,j})\longrightarrow \prod_{(i,j)\in I_{2,c}}\Omega^1(\widetilde{Y}_{i,j})_{d=0}$$

where the terms of the complex are defined as follows:

• A shorts Y_s is obtained by extension of scalars from a formal scheme $\check{Y}_s$ smooth over $\mathcal{O}_{\check{C}}$, and we set $\mathcal{O}(\widetilde{Y}_s)=\mathbf{A}_{\mathrm{cris}}\widehat{\otimes}_{\mathcal{O}_{\check{C}}}\mathcal{O}(\check{Y}_s)$ (and we choose a Frobenius φ on $\mathcal{O}(\check{Y}_s)$).

• A leg Y_a is of the form $\mathcal{O}(Y_a)=\mathcal{O}_C[[T_1,T_2]]/(T_1T_2-p^r)$, where r is the length of Y_a. We chose a morphism of semi-groups $r\mapsto \tilde{p}^r$, from $\mathbf{Q}_+$ to $\mathbf{A}_{\mathrm{cris}}$, where $\tilde{p}^r$ is a Teichmüller with $\theta(\tilde{p}^r)=p^r$, and we set $\mathcal{O}(\widetilde{Y}_a)=\mathbf{A}_{\mathrm{cris}}[[T_1,T_2]]/(T_1T_2-\tilde{p}^r)$ (and we take φ defined by $\varphi(T_i)=T_i^p$, for $i=1,2$).

• If $(a,s)\in I_{2,c}$, we define $\mathcal{O}(\widetilde{Y}_{a,s})$ to be the p-adic completion of the divided power envelop of $\mathcal{O}(\widetilde{Y}_a)\widehat{\otimes}\mathcal{O}(\widetilde{Y}_s)$ with respect to the kernel of the natural map to $\mathcal{O}(Y_{a,s})$, endowed with the tensor product φ.

We define the *syntomic cohomology groups* $H^i_{\mathrm{Syn}}(Y_S,\mathbf{Z}_p(1))$ of Y_S as those of the complex

$$\mathrm{Syn}(Y_S,\mathbf{Z}_p(1)) := [\ F^1C_{\mathrm{dR}}(\widetilde{Y}_S)\xrightarrow{1-\varphi/p} C_{\mathrm{dR}}(\widetilde{Y}_S)\ ]$$

One defines a *syntomic regulator* $\mathrm{Symb}_p(Y)\to H^1_{\mathrm{syn}}(Y_S,\mathbf{Z}_p(1))$ by sending $(f_n)_{n\in\mathbf{N}}$ to the cocycle $\lim_{n\to\infty}\big(\frac{d\tilde{f}_{n,i}}{\tilde{f}_{n,i}},\frac{1}{p}\log\frac{\varphi(\tilde{f}_{n,i})}{\tilde{f}_{n,i}^p},\log(\frac{\tilde{f}_{n,i}}{\tilde{f}_{n,j}})\big)$, where $\tilde{f}_{n,i}\in\mathcal{O}(\widetilde{Y}_i)$ is a lifting of the restriction of f_n to Y_i. This regulator is also "almost" an isomorphism (on the nose if $p\neq 2$, with a cokernel killed by 2 if $p=2$). This is proven by gluing the corresponding isomorphisms for the Y_i's, the case of shorts [32, Chap. 5] being much more delicate than that of legs [32, 4.2.4].

3.3.4. *Syntomic cohomology and Hyodo-Kato cohomology.* The de Rham cohomology groups $H^i_{\mathrm{dR}}(Y)$ of Y are those of $C_{\mathrm{dR}}(Y)=C\otimes_{\mathbf{A}_{\mathrm{cris}}}C_{\mathrm{dR}}(\widetilde{Y}_S)$ whereas its Hyodo-Kato cohomology groups $H^i_{\mathrm{HK}}(Y)$ are, by definition, those of $C_{\mathrm{dR}}(\check{Y}_S):=\check{C}\otimes_{\mathbf{A}_{\mathrm{cris}}}C_{\mathrm{dR}}(\widetilde{Y}_S)$ (the extensions of scalars are through the surjective morphisms $\theta:\mathbf{A}_{\mathrm{cris}}\to\mathcal{O}_C$ et $\theta_0:\mathbf{A}_{\mathrm{cris}}\to\mathcal{O}_{\check{C}}$).

One can modify slightly (see [32, 6.3.5]) $C_{\mathrm{dR}}(\widetilde{Y}_S)$ to obtain a quasi-isomorphic complex $\overline{C}_{\mathrm{dR}}(\widetilde{Y}_S)$ such that $\mathcal{O}_{\check{C}}\hookrightarrow\mathbf{A}_{\mathrm{cris}}$ induces a morphism of complexes $\overline{C}_{\mathrm{dR}}(\check{Y}_S)\to\mathbf{Q}_p\otimes\overline{C}_{\mathrm{dR}}(\widetilde{Y}_S)$, which commutes with φ, and is a section of $\mathbf{Q}_p\otimes\overline{C}_{\mathrm{dR}}(\widetilde{Y}_S)\to\overline{C}_{\mathrm{dR}}(\check{Y}_S)$. This gives (almost for free) isomorphisms

$$\mathbf{Q}_p\otimes H^1_{\mathrm{dR}}(\widetilde{Y}_S)\simeq \mathbf{B}^+_{\mathrm{cris}}\widehat{\otimes}_{\check{C}}H^1_{\mathrm{HK}}(Y)\quad\text{and}\quad \iota_{\mathrm{HK}}: H^1_{\mathrm{dR}}(Y)\simeq C\widehat{\otimes}_{\check{C}}H^1_{\mathrm{HK}}(Y).$$

Now, playing with the different presentations of the mapping fiber defining $\mathrm{Syn}(Y_S, \mathbf{Z}_p(1))$ and using the vanishing of $H^1(Y, \mathcal{O})$, one gets an exact sequence:

$$0 \to \mathcal{O}(Y)/C \to \mathbf{Q}_p \otimes H^1_{\mathrm{syn}}(Y_S, \mathbf{Z}_p(1)) \to (\mathbf{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\breve{C}} H^1_{\mathrm{HK}}(Y))^{\varphi=p} \to 0$$

which is not independent of the choice of $r \mapsto \tilde{p}^r$. To go from $H^1_{\mathrm{HK}}(Y)$ to $H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}}$, one has to compare $(\mathbf{Q}_p \widehat{\otimes} \mathcal{O}(Y)^{**})/\exp(\mathcal{O}(Y))$ and the closure of 0 in $(\mathbf{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\breve{C}} H^1_{\mathrm{HK}}(Y))^{\varphi=p}$. This comparison is already necessary to prove that the syntomic regulator gives local isomorphisms $\mathrm{Symb}_p(Y_i^{\mathrm{gen}}) \simeq H^1_{\mathrm{syn}}(Y_i, \mathbf{Z}_p(1))$, see [32, Prop. 5.32] (the most delicate result of that paper).

Finally, to get something independent of the choice of $r \mapsto \tilde{p}^r$, one has to use the Picard-Lefshetz formula that describes the effect of changing $r \mapsto \tilde{p}^r$ in terms of the monodromy operator N on $H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}}$ and replace $\mathbf{B}^+_{\mathrm{cris}} \widehat{\otimes}_{\breve{C}} H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}}$ by $(\mathbf{B}^+_{\mathrm{st}} \widehat{\otimes}_{\breve{C}} H^1_{\mathrm{HK}}(Y)^{\mathrm{sep}})^{N=0}$, see [32, 6.4.4] for details.

4. The C_{st} conjecture

The basic comparison theorem (see Theorem 2.4) relates the derived p-adic pro-étale cohomology to the (derived) refined de Rham cohomology. For computations (see [30]) it turned out to be very useful to have also a nonderived version of this theorem. At the moment this exists only conjecturally in general but we do have it in some very important special cases (like Stein varieties, for example).

4.1. **Statement.** Let Y be a smooth, partially proper rigid analytic variety over K. Let $r \geq 0$. Recall that the Bloch-Kato syntomic cohomology fits into the top distinguished triangle in the following diagram in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\begin{tikzcd}
\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(Y_C, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & {[\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C) \otimes^{\mathrm{L}_\square}_{C^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r}} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] & (\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \otimes^{\mathrm{L}_\square}_K \mathbf{B}^+_{\mathrm{dR}})/\mathrm{Fil}^r \arrow[d, equal] \\
\mathrm{Fil}^r(\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \otimes^{\mathrm{L}_\square}_K \mathbf{B}^+_{\mathrm{dR}}) \arrow[r] & \mathrm{R}\Gamma_{\mathrm{dR}}(Y) \otimes^{\mathrm{L}_\square}_K \mathbf{B}^+_{\mathrm{dR}} \arrow[r] & (\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \otimes^{\mathrm{L}_\square}_K \mathbf{B}^+_{\mathrm{dR}})/\mathrm{Fil}^r
\end{tikzcd}$$

Clearly, the bottom row is also a distinguished triangle. The middle vertical map is induced by the Hyodo-Kato morphism ι_{HK} and the injection $\iota : \mathbf{B}^+_{\mathrm{st}} \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$. For $n \geq r$, from the above diagram and the basic comparison theorem (see Theorem 2.4), we deduce a map of long exact sequences in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{DR}^{r,n-1} \arrow[r] \arrow[d, equal] & H^n_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p(r)) \arrow[r] \arrow[d] & X^{r,n} \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}\otimes\iota"] & \mathrm{DR}^{r,n} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{DR}^{r,n-1} \arrow[r] & H^n(\mathrm{Fil}^r) \arrow[r] & H^n_{\mathrm{dR}}(Y_K) \otimes^{\square}_K \mathbf{B}^+_{\mathrm{dR}} \arrow[r] & \mathrm{DR}^{r,n} \arrow[r] & \cdots
\end{tikzcd}$$

Here (and below) we set $\mathrm{Fil}^r := \mathrm{Fil}^r(\mathrm{R}\Gamma_{\mathrm{dR}}(Y) \otimes^{\mathrm{L}_\square}_K \mathbf{B}^+_{\mathrm{dR}})$.

Conjecture 4.1. (The C_{st}-conjecture [42, Conj. 1.4, Conj. 1.6])

(i) (de Rham to étale) *The middle square of the above diagram is bicartesian: i.e., we have an exact sequence*

$$0 \to H^n_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \to H^n(\mathrm{Fil}^r) \oplus X^{r,n} \to H^n_{\mathrm{dR}}(X_K) \otimes^{\square}_K \mathbf{B}^+_{\mathrm{dR}} \to 0$$

(ii) (Étale to de Rham) *We can recover de Rham and Hyodo-Kato cohomologies from the pro-étale one*[16]*:*

$$\begin{aligned}
\mathrm{Hom}_{G_K}(H^n_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{B}^+_{\mathrm{dR}}[\tfrac{1}{t}]) &\simeq H^n_{\mathrm{dR}}(Y_K)^*, \quad \textit{as filtered } K\textit{-modules,} \\
\varinjlim_{[L:K]<\infty} \mathrm{Hom}_{G_L}(H^n_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{B}^+_{\mathrm{st}}[\tfrac{1}{t}]) &\simeq H^n_{\mathrm{HK}}(Y_C)^*, \quad \textit{as } (\varphi, N, G_K)\textit{-modules over } C^{\mathrm{nr}}.
\end{aligned}$$

[16]Recall that $(\mathbf{B}^+_{\mathrm{dR}}[\frac{1}{t}])^{G_K} = K$ and $\varinjlim_{[L:K]<\infty} (\mathbf{B}^+_{\mathrm{st}}[\frac{1}{t}])^{G_K} = C^{\mathrm{nr}}$.

Remark 4.2. The above conjecture is an extension of a classical conjecture of Fontaine for étale cohomology of algebraic varieties. Part (ii) is classically phrased in a covariant manner as in (iii) of Remark 1.5. This is equivalent to the above formulation because $H^n_{\rm ét}(Y_C, \mathbf{Q}_p)$ is finite dimensional in the case of algebraic varieties, but a similar covariant formulation for nonproper analytic varieties does not seem possible.

Theorem 4.3. (i) *In Conjecture 4.1, we have* (i)⇒(ii).

(ii) *Conjecture 4.1 is verified for proper varieties, Stein varieties, analytification of algebraic varieties, complements of a small tube of a smooth analytic subvariety of a proper variety, products of finite number of varieties of the above type with finite dimensional de Rham cohomology, products of proper and Stein varieties.*

Remark 4.4. (i) Granting point (i), point (ii) corresponds to [42, Th. 8.1, Prop. 8.17]. Point (i) is not explicitly stated in [42], but it is what is actually proved in [42, Chap. 9].

(ii) In our announcement [27], we claimed to be able to prove Conjecture 4.1 in full generality, but the reduction to the quasi-compact case that we had in mind run into R lim problems for coherent cohomology. Moreover, our proof in the quasi-compact case relied on a delicate induction on the number of affinoids needed to cover our space and one crucial lemma turned out to be false. Hence we only have been able to verify the conjecture in broad classes of examples but there is clearly room for improvements (for example, removing the "small" from complements of small tubes or proper fibrations over Stein instead of products).

Question 4.5. If Y is quasi-compact, can one compactify a model of Y in such a way that the complement of the special fiber of Y is of codimension ≥ 1 ? This is possible if $\dim Y = 1$ and it is also possible mod p. A positive answer would probably make it possible to prove $\mathrm{C}_{\rm st}$ for quasi-compact dagger varieties.

4.2. **The case of Stein varieties.** Conjecture $\mathrm{C}_{\rm st}$ is true for Stein varieties but, in fact, in that case one can show more. Let Y be a smooth Stein variety over K. From Theorem 2.4, one derives the following, very useful (see [30]), result:

Theorem 4.6. (Fundamental diagram, [42, Th. 6.14]) *Let $r \geq 0$. There is a natural map of strictly exact sequences in $\mathcal{D}(\mathbf{Q}_{p,\square})$*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{r-1}(Y_C)/\operatorname{Ker} d \arrow[r] \arrow[d, equal] & H^r_{\rm proét}(Y_C, \mathbf{Q}_p(r)) \arrow[r] \arrow[d, "\tilde{\beta}"] & (H^r_{\rm HK}(Y_C)\otimes^{\square}_{C^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^r} \arrow[r] \arrow[d, "\iota_{\rm HK}\otimes\theta"] & 0 \\
0 \arrow[r] & \Omega^{r-1}(Y_C)/\operatorname{Ker} d \arrow[r, "d"] & \Omega^r(Y_C)^{d=0} \arrow[r] & H^r_{\rm dR}(Y_C) \arrow[r] & 0
\end{tikzcd}$$

Moreover, $H^r_{\rm proét}(Y_C, \mathbf{Q}_p(r))$ is Fréchet, the vertical maps are strict and have closed images, and $\operatorname{Ker}\tilde{\beta} \simeq (H^r_{\rm HK}(Y_C)\otimes^{\square}_{C^{\rm nr}}\mathbf{B}^+_{\rm st})^{N=0,\varphi=p^{r-1}}$.

4.3. **Cohomology with compact support.** We can also formulate a $\mathrm{C}_{\rm st}$-conjecture for compactly supported cohomology (at least an analog of part (ii) of Conjecture 4.1), but it takes a more sophisticated shape. Instead of $\operatorname{Hom}_{G_K}(-, B)$ for $B = \mathbf{B}^+_{\rm dR}[1/t], \mathbf{B}^+_{\rm st}[1/t]$, we need to take a derived Hom and also kill Galois cohomology of period rings in degrees ≥ 1. Otherwise one gets extra copies of the desired groups popping out in random degrees.

4.3.1. *Period ring* $\mathbf{B}_{\rm pFF}$. To achieve the second goal, let us introduce new period rings.

Define (as in [23]) the p-adic avatar $\log t$ of $\log 2\pi i$ as being transcendental over $\mathbf{B}^+_{\rm dR}$, with $\sigma \in G_K$ acting via $\sigma(\log t) = \log t + \log \chi(\sigma)$ (remember that $\sigma(t) = \chi(\sigma)t$). Define (after Fontaine [50])

$$\mathbf{B}^+_{\rm pdR} := \mathbf{B}^+_{\rm dR}[\log t]$$

with the filtration induced from $\mathbf{B}^+_{\rm dR}$.

Now, one can replace $\mathbf{B}_{\mathrm{st}}^+$ par $\mathbf{B}_{\mathrm{rig}}^+[\log\tilde{p}]$, with $\mathbf{B}_{\mathrm{rig}}^+ := \cap_{n\geq 0}\varphi^n(\mathbf{B}_{\mathrm{cris}}^+)$, in the statement of Conjecture 4.1 (because the relevant periods live in finite dimension spaces stable by φ). But $\mathbf{B}_{\mathrm{rig}}^+$ is the ring of analytic functions on the Fargues-Fontaine curve $\mathrm{Spa}\,\mathbf{A}_{\mathrm{inf}}\setminus\{p=0\}$. For the same reason, one can also replace $\mathbf{B}_{\mathrm{rig}}^+$ by the ring $\mathbf{B}$ of analytic functions on the Fargues-Fontaine curve $Y_{\mathrm{FF}} = \mathrm{Spa}\,\mathbf{A}_{\mathrm{inf}}\setminus\{p=0,\tilde{p}=0\}$. Define:

$$\mathbf{B}_{\mathrm{FF}}^+ := \mathbf{B}[\log\tilde{p}],\quad \mathbf{B}_{\mathrm{pFF}}^+ := \mathbf{B}_{\mathrm{FF}}^+[\log t].$$

By inverting t we get the period rings $\mathbf{B}_{\mathrm{pdR}}, \mathbf{B}_{\mathrm{FF}}, \mathbf{B}_{\mathrm{pFF}}$.

We have the following results:

Theorem 4.7. (i) (Semistable ver de Rham, [46, Prop. 10.3.15], [38, Cor. 2.3]) *The natural morphism $\mathbf{B}_{\mathrm{FF}}^+\to\mathbf{B}_{\mathrm{dR}}^+$ is injective and G_K-equivariant; the same is true for its natural extensions*

$$\mathbf{B}_{\mathrm{pFF}}^+\to\mathbf{B}_{\mathrm{pdR}}^+,\quad \mathbf{B}_{\mathrm{pFF}}\to\mathbf{B}_{\mathrm{pdR}}.$$

(ii) (Galois acyclicity, [38, Th.1.4, Th. 2.4]) *We have the isomorphisms*

$$\begin{aligned} &H^0(G_K,\mathbf{B}_{\mathrm{pdR}}) = K,\quad H^0(G_K,\mathbf{B}_{\mathrm{pFF}}) = F,\\ &H^i(G_K,\Lambda) = 0,\quad \text{if } i\geq 1,\quad \Lambda = \mathbf{B}_{\mathrm{pdR}},\mathbf{B}_{\mathrm{pFF}}.\end{aligned}$$

Remark 4.8. (Period sheaves as avatars of refined de Rham cohomologies.) Comparison quasi-isomorphisms from Corollary 2.19 and Theorem 2.16 combined with the Galois acyclicity from Theorem 4.7 yield the following result (one of our original motivations for proving Theorem 4.7):

Corollary 4.9. *(1)* ([19, Cor. 3.17]) *Let Y be a smooth quasi-compact rigid analytic variety. There are natural quasi-isomorphisms in $\mathcal{D}(K_\square)$*

$$\begin{aligned} F^0\mathrm{R}\Gamma_{\mathrm{dR}}(Y) &\simeq \mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y_C,\mathbb{B}_{\mathrm{pdR}}^+)),\\ \mathrm{R}\Gamma_{\mathrm{dR}}(Y) &\simeq \mathrm{R}\Gamma(G_K,\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y_C,\mathbb{B}_{\mathrm{pdR}})).\end{aligned}$$

(2) ([19, Cor. 3.31]) *Assume that k is algebraically closed. Let Y be a smooth partially proper rigid analytic variety with finite rank de Rham cohomology. Then there is a natural G_K-equivariant quasi-isomorphism in $\mathcal{D}_{\varphi,N}(\breve{C}_\square)$*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(Y_C)\xrightarrow{\sim}([\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(Y_C,\mathbb{B})[1/t]]^{\varphi=1}\otimes^{\mathrm{L}_\square}_{\mathbf{B}_e}\mathbf{B}_{\mathrm{pFF}})^{\mathrm{R}G_K-\mathrm{sm}}.$$

4.3.2. *Conjecture* C_{st}. The new period rings described above allowed us to formulate the following conjecture:

Conjecture 4.10. *Let K be a finite extension of $\mathbf{Q}_p$. Let Y be a smooth partially proper rigid analytic variety over K of dimension d. We can recover de Rham and Hyodo-Kato cohomologies from the compactly supported pro-étale one:*

$$\begin{aligned} \mathrm{RHom}_{G_K}(\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(Y_C,\mathbf{Q}_p(d))[2d],\mathbf{B}_{\mathrm{pdR}}) &\simeq \mathrm{R}\Gamma_{\mathrm{dR}}(Y_K),\quad \textit{as filtered } K\textit{-modules},\\ \varinjlim_{[L:K]<\infty}\mathrm{RHom}_{G_L}(\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(Y_C,\mathbf{Q}_p(d))[2d],\mathbf{B}_{\mathrm{pFF}}) &\simeq \mathrm{R}\Gamma_{\mathrm{HK}}(Y_C),\quad \textit{as } (\varphi,N,G_K)\textit{-modules over } C^{\mathrm{nr}}.\end{aligned}$$

Remark 4.11. (i) One can use Poincaré duality for de Rham cohomology to replace $H^n_{\mathrm{dR}}(Y_K)^*$ and $H^n_{\mathrm{HK}}(Y_C)^*$ by $H^{2d-n}_{\mathrm{dR},c}(Y_K)$ and $H^{2d-n}_{\mathrm{HK},c}(Y_C)$ in the statement of Conjecture 4.1, which gives it a shape closer to that of Conjecture 4.10.

(ii) Implicit in the formulation of Conjecture 4.10 is a choice of a suitable category of topological $\mathbf{Q}_p$-vector spaces with continuous action of G_K.

4.3.3. *Fundamental diagram.* Let Y be a smooth Stein rigid analytic variety over K of dimension d. In this case, under some restrictions, Theorem 2.22 yields the following result:

Corollary 4.12. (Fundamental diagram, [2, Cor. 1.6]) *Let $r\geq 0$. Then, under a strong assumption on the slopes of Frobenius on Hyodo-Kato cohomology (see* [2, Th. 8.4] *for details), there is a map of exact sequences of solid $\mathbf{Q}_p$-modules*

$$\begin{tikzcd}
0 \arrow[r] & \big(H^d_c(Y_C,\Omega^{r-d-1})/\operatorname{Ker} d\big)(d) \arrow[r] \arrow[d, equal] & H^r_{\text{proét},c}(Y_C,\mathbf{Q}_p(r)) \arrow[r] \arrow[d] & (H^r_{\text{HK},c}(Y_C)\otimes^{\Box}_{C^{\text{nr}}} t^d\mathbf{B}^+_{\text{st}})^{\varphi=p^r,N=0} \arrow[r] \arrow[d] & 0\\
0 \arrow[r] & \big(H^d_c(Y_C,\Omega^{r-d-1})/\operatorname{Ker} d\big)(d) \arrow[r] & H^d_c(Y_C,\Omega^{r-d})^{d=0}(d) \arrow[r] & H^r_{\text{dR},c}(Y_C)(d) \arrow[r] & 0
\end{tikzcd}$$

The Frobenius slope condition above is very restrictive but it holds for tori and Drinfeld spaces, for example (probably also for general p-adic period domains, but not for the higher levels of the Drinfeld tower), so we can use the fundamental diagram to compute compactly supported cohomology in these cases. Recall that for the usual cohomology we have a fundamental diagram with no restrictions on the slopes. This means that the authors of [30] got very lucky that they started computations with the usual cohomology and not with the compactly supported one as is customary in the local Langlands program.

5. Geometrization of the basic comparison theorem

A crucial ingredient in the proof of claim (ii) of Theorem 4.3 is the fact that the basic comparison theorem (see Theorem 2.4) can be geometrized, i.e., that all the terms in the exact sequence (2.5) can be turned into presheaves of topological $\mathbf{Q}_p$-vector spaces on the category of perfectoid affinoid spaces over C (TVS for short). This geometrization is also essential for the study of dualities (see Section 6.2).

5.1. **BC's, qBC's, and TVS's.** Basic examples of TVS's are:
- $\mathbf{Q}_p$, sending a perfectoid space S to $\mathcal{C}(\pi_0(S),\mathbf{Q}_p)$,
- $\mathbb{G}_a$, sending S to $\mathcal{O}(S)$.

5.1.1. *BC's.* We let BC be the smallest abelian subcategory of TVS stable under extensions and containing $\mathbf{Q}_p$ and $\mathbb{G}_a$ (this definition is taken from [62], where it is proven that it is equivalent to the original definition [24, 25]). Typical objects in this category are constructed from the period sheaves $\mathbb{B}^+_{\text{st}}$ and $\mathbb{B}^+_{\text{dR}}$: for example, $\mathbb{B}^+_{\text{dR}}/t^m$ is a BC. More general examples include:
- $X^r_{\text{dR}}(M):=(\mathbb{B}^+_{\text{dR}}\otimes_K M)/\text{Fil}^r$, if M is a finite rank filtered K-module and $r\in\mathbf{Z}$ (this is a successive extensions of $\mathbb{G}_a$'s),
- $X^r_{\text{st}}(M):=(\mathbb{B}^+_{\text{st}}\otimes_{C^{\text{nr}}} M)^{N=0,\varphi=p^r}$ if M is a finite rank (φ,N,G_K)-module over C^{nr}, and $r\in\mathbf{Z}$.

There are two functions on the category of BC's, additive in exact sequences: the *dimension* dim and the *height* ht, characterized by the fact that $\dim\mathbb{G}_a=1$ and $\dim\mathbf{Q}_p=0$, whereas $\operatorname{ht}\mathbb{G}_a=0$ and $\operatorname{ht}\mathbf{Q}_p=1$. We define *the Dimension* $\operatorname{Dim}(\mathbb{W})$ of a BC $\mathbb{W}$ as the couple $(\dim(\mathbb{W}),\operatorname{ht}(\mathbb{W}))$.

For example we have (see [39, Ex. 5.14], where the reader will find a (more complicated) formula with no restriction on r):
- $\operatorname{Dim}(X^r_{\text{dR}}(M))=(r\operatorname{rk}(M)-t_H(M),0)$ if r is $\geq$ all the jumps in the filtration, and $t_H(M)=\sum_{i\in\mathbf{Z}}i\operatorname{rk}(\operatorname{gr}^iM)$ (the endpoint of the Hodge polygon of M),
- $\operatorname{Dim}(X^r_{\text{st}}(M))=(r\operatorname{rk}(M)-t_N(M),\operatorname{rk}(M))$ if $r\geq$ all the valuations of eigenvalues of φ and $t_N(M)$ is the sum of these valuations (the endpoint of the Newton polygon of M).

5.1.2. *qBC's.* If W is a topological[17] C-vector space, $S\mapsto\mathcal{O}(S)\otimes^{\Box}_C W$ defines a TVS $\mathbb{G}_a\otimes^{\Box}_C W$. More generally, if W is a topological $\mathbf{B}^+_{\text{dR}}/t^m$-module, $S\mapsto(\mathbb{B}^+_{\text{dR}}(S)/t^m)\otimes^{\Box}_{\mathbf{B}^+_{\text{dR}}/t^m}W$ defines a TVS $(\mathbb{B}^+_{\text{dR}}/t^m)\otimes^{\Box}_{\mathbf{B}^+_{\text{dR}}/t^m}W$.

[17]We are purposefully vague here concerning the meaning of "topological". Let us just mention that, for the theory of qBC's to work well, one needs to puts a restriction of the type of topological spaces allowed.

We define a Topological $\mathbb{B}_{\mathrm{dR}}^+$-Module to be a TVS of this form. And we define [42] a qBC to be a TVS having a finite filtration whose associated graded pieces are BC's or Topological $\mathbb{B}_{\mathrm{dR}}^+$-Modules.

The height of a qBC is still defined and additive in exact sequences, but its dimension can be infinite.

5.1.3. *TVS's.* To get the right Ext-groups between BC's in the TVS category (see Section 6.1) the category TVS needs to appropriately modified (see [43]). For one, one needs to restrict the test objects to strictly totally disconnected spaces. But also all the presheaves need to be topologically enriched[18], which means that the mapping spaces between affinoid perfectoids are equipped with a topology and the presheaf functors are sensitive to this topology. This rigidifies the topological sheaves enough for them to reflect well the category of BC's, that is, the canonical functor from BC's to TVS's is fully faithful (in the derived sense).

5.2. **Geometrization.** The rings of periods are naturally TVS's $S \mapsto \mathbb{B}_{\mathrm{dR}}^+(S)$ and $S \mapsto \mathbb{B}_{\mathrm{st}}^+(S)$, which gives rise to the TVS's $\mathbb{X}^{r,i}$ and $\mathbb{DR}^{r,i}$. To geometrize $H^n_{\mathrm{proet}}(Y_C, \mathbf{Q}_p(r))$ we just have to consider[19] the presheaf $\mathbb{H}^n_{\mathrm{proet}}(Y_C, \mathbf{Q}_p(r)) : S \mapsto H^n_{\mathrm{proet}}(Y_S, \mathbf{Q}_p(r))$, where the pro-étale cohomology is equipped with its canonical topology.

Theorem 5.1. (Geometrization of the basic comparison theorem [41]) *Let $r \geq 0$. Let Y be a smooth partially proper rigid analytic space over K. We have an exact sequence of TVS's (i.e., the sequence is exact, when evaluated on any perfectoid affinoid space over C)*

$$\cdots \to \mathbb{X}^{r,n-1} \to \mathbb{DR}^{r,n-1} \to \mathbb{H}^n_{\mathrm{proet}}(Y_C, \mathbf{Q}_p(r)) \to \mathbb{X}^{r,n} \to \mathbb{DR}^{r,n} \to \cdots \tag{5.2}$$

The TVS's appearing in the sequence (5.2) have special properties; indeed, we have (see [42, Cor. 7.9] for $\mathbb{H}^n_{\mathrm{proet}}$; the other results are immediate):

Proposition 5.3. *If Y is quasi-compact or, more generally, if the de Rham cohomology of Y is finite dimensional, then the exact sequence* (5.2) *is an exact sequence of qBC's in which the $\mathbb{X}^{r,i}$ are BC's and the $\mathbb{DR}^{r,i}$'s are Topological $\mathbb{B}_{\mathrm{dR}}^+$-Modules.*

Proposition 5.4. ([42, Prop. 1.17]) *If Y is small* (meaning that its de Rham cohomology is finite dimensional)*, then the* C_{st}*-conjecture for $H^n_{\mathrm{proét}}(Y_C, \mathbf{Q}_p(r))$ is equivalent to each of the following statements:*

(i) $\mathrm{ht}(\mathbb{H}^n_{\mathrm{proét}}(Y_C, \mathbf{Q}_p(r))) = \dim_K H^n_{\mathrm{dR}}(Y)$,

(ii) $H^1(X_{\mathrm{FF}}, \mathcal{E}(H^i_{\mathrm{HK}}(Y_C), H^i_{\mathrm{dR}}(Y))) = 0$, *for $i = n-1, n$, where $\mathcal{E}(H^i_{\mathrm{HK}}(Y_C), H^i_{\mathrm{dR}}(Y))$ is the vector bundle of the Fargues-Fontaine curve X_{FF} associated to the φ-module $H^i_{\mathrm{HK}}(Y_C)$ modified at ∞ by the lattice corresponding to* $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}}^+ \otimes_K^{\square} H^i_{\mathrm{dR}}(Y))$.

Criterium (i) is a bit surprising, considering how big $\mathbb{H}^n_{\mathrm{proét}}(Y_C, \mathbf{Q}_p(r))$ is in general. Criterium (ii) is very useful for checking cases of the C_{st}-conjecture.

6. Poincaré duality

As we mentioned in Section 2.1.2, the existence of a Poincaré duality for geometric p-adic pro'etale cohomology of nonproper analytic varieties looked rather problematic. Still, the computations we have done, assuming that one can extract the arithmetic cohomology from the geometric one using Hochschild-Serre spectral sequence, pointed strongly towards the existence of a usual (topological) duality for arithmetic p-adic pro-étale cohomology. It made us look harder for a duality in the geometric case. The key obstacle was to make "the $\mathbf{Q}_p$-dual of C" be C, which is

[18] Actually this is already the case for the qBC's but we ignored it.

[19] The idea is that $H^n_{\mathrm{proet}}(Y_S, \mathbf{Q}_p(r))$ "should" not depend on S (at least if $\pi_1(S)$ is trivial; that would indeed be the case over the complex numbers). This is of course already false for the open unit disk as we have mentioned but the content of Theorem 5.1 and Proposition 5.3 is that there is a constant "reasonable" object that controls $H^n_{\mathrm{proet}}(Y_S, \mathbf{Q}_p(r))$ when S varies.

clearly not possible. But we realized that, interpreting C as the C points of $\mathbb{G}_a$ (a natural thing to do, considering Theorem 5.1), we could interpret the $\mathbf{Q}_p$-dual of C as being $\mathrm{Ext}^1_{\mathrm{BC}}(\mathbb{G}_a, \mathbf{Q}_p)$ which is indeed equal to C (this computation is a crucial ingredient in the early theory of BC's [24, Prop. 9.16]). This led us to Conjecture 6.5 below [28].

6.1. Ext-**groups.** The internal Ext-groups between BC's, in the TVS category, can be computed using the following results:

$$\begin{aligned} &\mathcal{H}om_{\mathrm{TVS}}(\mathbf{Q}_p, \mathbf{Q}_p) \simeq \mathbf{Q}_p, && \mathcal{H}om_{\mathrm{TVS}}(\mathbf{Q}_p, \mathbb{G}_a) \simeq \mathbb{G}_a, \\ &\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) = 0, && \mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathbb{G}_a, \\ &\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbf{Q}_p, \mathbf{Q}_p) = 0, && \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbf{Q}_p, \mathbb{G}_a) = 0, \\ &\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) \simeq \mathbb{G}_a, && \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{G}_a, \mathbb{G}_a) \simeq \mathbb{G}_a. \end{aligned} \tag{6.1}$$

And $\mathcal{E}xt^i_{\mathrm{TVS}}(W_2, W_1) = 0$, for all $i \geq 2$ and all BC's W_1, W_2. The nontrivial $\mathcal{E}xt^1$'s above are generated by the classes of the extensions

$$\begin{aligned} &0 \to \mathbf{Q}_p t \to (\mathbb{B}^+_{\mathrm{cris}})^{\varphi=p} \to \mathbb{G}_a \to 0, \\ &0 \to \mathbb{G}_a t \to \mathbb{B}^+_{\mathrm{dR}}/t^2 \to \mathbb{G}_a \to 0 \end{aligned}$$

Remark 6.2. (1) The above computations are the same when done in the BC-category but the vanishing of $\mathcal{E}xt^i$'s, for $i \geq 2$, in the BC-category is elementary contrary to the vanishing in the TVS-category.

(2) The computation of $\mathcal{H}om$'s and $\mathcal{E}xt^1$'s in (6.1) was already done in [24]. The vanishing of the higher $\mathcal{E}xt$'s was proved by Anschütz-Le Bras in [5] in the category of pro-étale $\mathbf{Q}_p$-sheaves (with no objectwise topology) by a reduction, via MacLane-Breen resolutions, to an analogous vanishing result of Breen in characteristic p. The vanishing in the TVS-category is deduced in [43] from that via a fully-faithfulness result (for the functor from the category of pro-étale $\mathbf{Q}_p$-sheaves to TVS's).

6.2. **Geometric Duality.** If Y is a smooth proper rigid analytic variety over K, Mann [65] and Zavyalov [76] have proved that the p-adic pro-étale cohomology of Y_C satisfies Poincaré duality akin to the one for algebraic varieties. We note that in this case the cohomology groups have finite rank over $\mathbf{Q}_p$. We know now that we also have a Poincaré duality for smooth partially proper rigid analytic varieties over K but now: (1) the cohomology groups are infinite dimentional, (2) there are, in general, nontrivial Ext-groups hence we get a genuine Verdier-type duality.

6.2.1. *Heuristic: The p-adic unit disk.* The initial evidence that Poincaré duality should hold (and its precise shape) came from the following heuristic computations.

Let Y be the open unit disk over K. From the fundamental diagram (see Theorem 4.6 and Corollary 4.12) we get the exact sequence of solid $\mathbf{Q}_p$-modules

$$0 \to \mathcal{O}(\partial Y_C)/\mathcal{O}(Y_C) \to H^2_{\mathrm{proét},c}(Y_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p \to 0$$

and isomorphisms

$$H^i_{\mathrm{proét}}(Y_C, \mathbf{Q}_p) \simeq \begin{cases} \mathbf{Q}_p & \text{if } i = 0, \\ \mathcal{O}(Y_C)/\mathbf{C}_p & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

Since the groups $\mathcal{O}(Y_C)/C \simeq H^0(Y_C, \Omega^1)$ (via $f \mapsto df$) and $\mathcal{O}(\partial Y_C)/\mathcal{O}(Y_C) \simeq H^1_c(Y_C, \mathcal{O})$ are in Serre duality over C, the above computation of Ext-groups in the TVS-category suggests that

$$H^1_{\mathrm{proét},c}(Y_C, \mathbf{Q}_p) \simeq \underline{\mathrm{Ext}}^1_{\mathrm{TVS}}(\mathbb{H}^2_{\mathrm{proét},c}(Y_C, \mathbf{Q}_p(1)), \mathbf{Q}_p). \tag{6.3}$$

Similarly, one guesses that we should have an exact sequence ($\underline{\mathrm{Ext}}^i$ is the Ext^i in the TVS-category)

$$0 \to \underline{\mathrm{Ext}}^1_{\mathrm{TVS}}(\mathbb{H}^1_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{Q}_p) \to H^2_{\mathrm{pro\acute{e}t},c}(Y_C, \mathbf{Q}_p(1)) \to \underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbb{H}^0_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{Q}_p) \to 0 \tag{6.4}$$

This is because we should have

$$\mathcal{O}(\partial Y_C)/\mathcal{O}(Y_C) \simeq \underline{\mathrm{Ext}}^1_{\mathrm{TVS}}(\mathbb{H}^1_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{Q}_p),$$
$$\mathbf{Q}_p \simeq \underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbb{H}^0_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p), \mathbf{Q}_p).$$

6.2.2. *Statement.* Let Y be a smooth, partially proper rigid analytic variety over K of dimension d. Let $\mathbb{R}_{\mathrm{pro\acute{e}t},*}(Y_C, \mathbf{Q}_p(r))$ be the TVS's representing cohomology $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},*}(Y_C, \mathbf{Q}_p(r))$, i.e., we have

$$\mathbb{R}_{\mathrm{pro\acute{e}t}\,.*}(Y_C, \mathbf{Q}_p(r))(\mathrm{Spa}(C)) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},*}(Y_C, \mathbf{Q}_p(r)).$$

The above computations led to the following conjecture:

Conjecture 6.5. (Geometric duality, [37, Th. 1.7]) *There is a natural duality quasi-isomorphism in TVS-category*

$$\mathbb{R}_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p) \simeq \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(Y_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p). \tag{6.6}$$

This conjecture is now a theorem. There are two different proofs: by Anschütz-Le Bras-Mann [6] and Colmez-Gilles-Nizioł [37] (more precisely, the former replaces TVS's with pro-étale sheaves of $\mathbf{Q}_p$-modules). We will briefly discuss these proofs in Section 6.2.3.

Remark 6.7. (1) So far we do not have a setting in which a generalization of the duality (6.4) would hold. It fails in the condensed setting since there are too many extensions between Banach spaces. We phantasize that it could work in the qBC-category.

(2) Computations suggest that there should be a Poincaré duality for the boundary ∂Y of any smooth partially proper variety Y over K. But, again, it is not clear in which setting this should hold. This would follow from the duality (6.6) if we had the duality "the other way" from (1). Let us state it:

Conjecture 6.8. (Duality for boundary pro-étale cohomology) *There is a natural quasi-isomorphism in $\mathcal{D}(?)$*

$$\mathbb{R}_{\mathrm{pro\acute{e}t}}(\partial Y_C, \mathbf{Q}_p) \xrightarrow{\sim} \mathrm{R}\mathcal{H}om_?(\mathbb{R}_{\mathrm{pro\acute{e}t}}(\partial Y_C, \mathbf{Q}_p(d))[2d-1], \mathbf{Q}_p).$$

Proof. (Heuristics) Use the distinguished triangle (2.5) and dualize it. Then use the duality (6.6) and its hypothetical "inverse". ☐

This can be proved in the arithmetic setting in dimension 1 (see below).

The following pleasing corollary is implied by the duality (6.6) and the vanishing of TVS-Ext-groups in dimensions ≥ 2:

Corollary 6.9. (Verdier exact sequence, [37, Cor. 1.8] *Let Y be a smooth Stein variety over K of dimension d. Let $i \geq 0$. There exists a short exact sequence in* TVS*'s*

$$0 \to \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^{2d-i+1}_{\mathrm{pro\acute{e}t},c}(Y_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to \mathbb{H}^i_{\mathrm{pro\acute{e}t}}(Y_C, \mathbf{Q}_p) \to \mathcal{H}om_{\mathrm{TVS}}(\mathbb{H}^{2d-i}_{\mathrm{pro\acute{e}t},c}(Y_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to 0$$

6.2.3. *Proofs of Conjecture 6.5.* We start with the proof of the geometric Poincaré duality from [37]. Just as in Section 6.2.1, via the basic comparison theorem (see Theorem 2.4 and its compactly supported analog) the duality for p-adic pro-étale cohomology can be reduced to the one for Hyodo-Kato and filtered de Rham cohomologies. The first of those, in turn, can be reduced (as in [2]), via the Hyodo-Kato isomorphism, to the second one (which follows from coherent Serre duality). That should have been enough to give a proof of pro-étale duality however, at the moment, the argument goes through the Fargues-Fontaine curve that geometrically separated the Hyodo-Kato and de Rham parts (something that is more difficult to do directly in TVS's) and makes the bootstrapping of the de Rham-dualities up to pro-étale cohomology easy. The weight of the argument then moves to controlling the descent from the Fargues-Fontaine curve to TVS's, a nontrivial task.

The proof in [6] also proceeds in two steps. The duality on the Fargues-Fontaine curve is obtained from a 6-functor formalism for solid quasi-coherent sheaves. The descent is now not to TVS's but to the algebraic category of pro-étale $\mathbf{Q}_p$-sheaves. Like all the descents we have so far in this topological setting it does not work perfectly and for it to work here one needs to know functional analytic properties of the solid quasi-coherent sheaves involved in the duality theorem. That is derived from the comparison theorems for period sheaves (see Theorem 2.16 and Remark 2.18).

The method from [6] allows to prove more general dualities than the ones stated in Conjecture 6.5. For example, it yields a Poincaré duality for smooth proper rigid analytic spaces and $\mathbf{Q}_p$-local systems. This case is interesting because the pro-étale cohomology groups are BC's but not finite rank over $\mathbf{Q}_p$, in general. Another approach to this result was developed recently by Li-Nizioł-Reinecke-Zavyalov [63].

6.3. **Arithmetic duality.** Arithmetic duality for smooth partially proper analytic spaces was easier to understand than its geometric counterpart because it did not involve nontrivial Ext-groups. Moreover, in the Stein case there it has a nonderived version (both ways).

Theorem 6.10. (Arithmetic duality, [35, Th. 1.1], [64, Th. 1.1]) *Let Y be a smooth Stein rigid analytic variety over K of dimension d. There are natural duality isomorphisms*

$$H^i_{\text{proét},c}(Y,\mathbf{Q}_p) \xrightarrow{\sim} \underline{\text{Hom}}(H^{2d+2-i}_{\text{proét}}(Y,\mathbf{Q}_p(d+1)),\mathbf{Q}_p),$$
$$H^i_{\text{proét}}(Y,\mathbf{Q}_p) \xrightarrow{\sim} \underline{\text{Hom}}(H^{2d+2-i}_{\text{proét},c}(Y,\mathbf{Q}_p(d+1)),\mathbf{Q}_p).$$

A key result in the proof of arithmetic duality for analytic curves in [35] is a Poincaré duality for the ghost circle[20].

Theorem 6.11. (Arithmetic duality for ghost circle) *Let D be an open disk over K. Let $Y := \partial D$ be the "ghost circle". There is a natural duality isomorphism of solid $\mathbf{Q}_p$-vector spaces*

$$H^i_{\text{proét}}(Y,\mathbf{Q}_p(j)) \xrightarrow{\sim} H^{3-i}_{\text{proét}}(Y,\mathbf{Q}_p(2-j))^*.$$

Hence ∂D_C behaves like a proper variety of dimension $\frac{1}{2}$ (since the "arithmetic variety" ∂D behaves like a proper variety of dimension $\frac{3}{2}$), hence like a proper variety of "real" dimension 1 (as in the archimedean world!).

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