DUALITY FOR p-ADIC GEOMETRIC PRO-ÉTALE COHOMOLOGY

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Abstract. We prove that p-adic geometric pro-étale cohomology of smooth partially proper rigid analytic varieties over p-adic fields seen in the category of Topological Vector Spaces satisfies a Poincaré duality as we have conjectured. This duality descends, via fully-faithfulness results of Colmez-Nizioł, from a Poincaré duality for solid quasi-coherent sheaves on the Fargues-Fontaine curve representing this cohomology. The latter duality is proved by passing, via comparison theorems, to analogous sheaves representing syntomic cohomology and then reducing to Poincaré duality for $\mathbf{B}_{\mathrm{st}}^{+}$-twisted Hyodo-Kato and filtered $\mathbf{B}_{\mathrm{dR}}^{+}$-cohomologies that, in turn, reduce to Serre duality for smooth Stein varieties – a classical result.

Contents

1. Introduction

Let p be a prime and let K be a discrete p-adic field with perfect residue field. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$.

1.1. **Duality Conjecture.** In [15, 16], the authors showed that (a part of) the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ can be realized in the p-adic (pro-)étale cohomology of the Drinfeld tower in dimension 1. It involved the duals of the unitary representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ provided by the p-adic local Langlands correspondence and not the representations themselves. In contrast, the classical local Langlands correspondence (not its dual) is realized in the compactly supported ℓ-adic cohomology rather than the usual one; this brought us to consider a possibility of duality for p-adic pro-étale cohomology.

The reason the authors of loc. cit. worked with the usual cohomology was mainly due to their newly acquired competence with comparison theorems for analytic varieties [20]. In retrospect, this was a lucky choice since the description of geometric pro-étale cohomology with compact support is not as transparent as that of usual cohomology. The compactly supported cohomology (even in dimension 1) was not defined at that time, but we did some heuristic computations assuming that it could be defined similarly to de Rham cohomology with compact support [33], and that

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(derived) comparison theorems would work for it as expected. The first results did not look very promising.

Here is the simplest of them. Let D be the open disc of dimension 1 over K. Using syntomic methods, we obtain that the only nontrivial cohomology groups are as follows:

$$\begin{aligned} &H^0_{\text{proét}}(D_C, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p(1), \quad H^1_{\text{proét}}(D_C, \mathbf{Q}_p(1)) \simeq \mathcal{O}(D_C)/C, \\ &H^2_{\text{proét},c}(D_C, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p \oplus \mathcal{O}(\partial D_C)/\mathcal{O}(D_C), \end{aligned} \tag{1.1}$$

sing4

where ∂D_C denotes the "boundary of D_C". Since we have the isomorphisms (of topological C-vector spaces)

$$\mathcal{O}(D_C)/C \xrightarrow{\sim} \Omega^1(D_C), \quad \mathcal{O}(\partial D_C)/\mathcal{O}(D_C) \xrightarrow{\sim} H^1_c(D_C, \mathcal{O}),$$

we see in (1.1) a Serre duality (of topological C-vector spaces)

$$\Omega^1(D_C) \simeq H^1_c(D_C, \mathcal{O})^*$$

as well as a simple $\mathbf{Q}_p$-duality (between $H^0_{\text{proét}}(D_C, \mathbf{Q}_p(1))$ and the $\mathbf{Q}_p$ appearing in $H^2_{\text{proét},c}(D_C, \mathbf{Q}_p(1))$) but they do not fit together into a simple Poincaré duality (since the degrees do not match, and the C-duality cannot be turned into a $\mathbf{Q}_p$-duality as $[C : \mathbf{Q}_p] = \infty$).

However, we realized (after quite a while) that, if we couple the above with the following computations[1] in the category of Banach-Colmez spaces (BC's for short) and assume that the results remain valid in the bigger category of Topological Vector Spaces (TVS's for short):

$$\begin{aligned} &\underline{\text{Hom}}_{\text{TVS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) \simeq \mathbf{Q}_p(1), \quad \underline{\text{Ext}}^1_{\text{TVS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = 0, \\ &\underline{\text{Hom}}_{\text{TVS}}(\mathbb{G}_a, \mathbf{Q}_p(1)) = 0, \quad \underline{\text{Ext}}^1_{\text{TVS}}(\mathbb{G}_a, \mathbf{Q}_p(1)) \simeq \mathbb{G}_a, \\ &\underline{\text{Ext}}^i_{\text{TVS}}(M, N) = 0, \quad i \geq 2, \ M, N \in \text{BC}, \end{aligned} \tag{1.2}$$

BC

as well as that Ext-groups of tensor products with constant objects behave as if everything was finite dimensional, we get abstract isomorphisms (with $H^i_? := H^i_?(D_C, \mathbf{Q}_p(1))$)

$$H^0_{\text{proét}} \simeq \text{Hom}_{\text{TVS}}(H^2_{\text{proét},c}, \mathbf{Q}_p(1)) \quad H^1_{\text{proét}} \simeq \text{Ext}^1_{\text{TVS}}(H^2_{\text{proét},c}, \mathbf{Q}_p(1)) \tag{1.3}$$

dua1

and an abstract exact sequence

$$0 \to \text{Ext}^1_{\text{TVS}}(H^1_{\text{proét}}, \mathbf{Q}_p(1)) \to H^2_{\text{proét},c} \to \text{Hom}_{\text{TVS}}(H^0_{\text{proét}}, \mathbf{Q}_p(1)) \to 0, \tag{1.4}$$

dua2

which suggest a (derived) duality, both ways.

These examples brought us to formulate the following conjecture (see [13], [17]):

conj-main

Conjecture 1.5. *Let X be a smooth partially proper rigid analytic variety over K of dimension d. In the category of Topological Vector Spaces we have a natural quasi-isomorphism*

$$\mathbb{R}_{\text{proét}}(X_C, \mathbf{Q}_p) \simeq \text{R}\mathcal{H}om_{\text{TVS}}(\mathbb{R}_{\text{proét},c}(X_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p).$$

Here:

(1) TVS's are $\underline{\mathbf{Q}}_p$-modules in the category of topologically enriched[2] presheaves on strictly totally disconnected spaces over C, denoted sPerf_C, with values in solid abelian groups.
(2) $\mathbb{R}_{\text{proét}}(X_C, \mathbf{Q}_p)$ is a TVS defined by $S \mapsto \text{R}\Gamma_{\text{proét}}(X_S, \mathbf{Q}_p)$; the topology on $\text{R}\Gamma_{\text{proét}}(X_S, \mathbf{Q}_p)$ is canonically inherited from the pro-étale site.

[1] In (1.2) the last nontrivial Ext group is generated by the fundamental exact sequence

$$0 \to \mathbf{Q}_p(1) \to \mathbb{B}^{+,\varphi=p}_{\text{cr}} \to \mathbb{G}_a \to 0,$$

where $\mathbb{B}^{+,\varphi=p}_{\text{cr}}$ is the TVS corresponding to $\mathbf{B}^{+,\varphi=p}_{\text{cr}}$.

[2] For the sake of the introduction we invite the reader to ignore the issue of enrichment and think of TVS's as just topological presheaves. See [19] for precise definitions.

(3) Pro-étale cohomology with compact support $\mathbb{R}_{\text{proét},c}(X_C, \mathbf{Q}_p)$ is defined by $S \mapsto \mathrm{R}\Gamma_{\text{proét},c}(X_S, \mathbf{Q}_p)$, where, for an exhaustive covering $\{U_n\}$, $U_n \Subset U_{n+1}$, by quasi-compact open spaces, we set

$$\mathrm{R}\Gamma_{\text{proét},c}(X_S, \mathbf{Q}_p) := [\mathrm{R}\Gamma_{\text{proét}}(X_S, \mathbf{Q}_p) \to \mathrm{R}\Gamma_{\text{proét}}(\partial X_S, \mathbf{Q}_p)],$$
$$\mathrm{R}\Gamma_{\text{proét}}(\partial X_S, \mathbf{Q}_p) := \operatorname{colim}_n \mathrm{R}\Gamma_{\text{proét}}(X_S \setminus U_{n,S}, \mathbf{Q}_p),$$

with the induced topology. By [1], we have $\mathrm{R}\Gamma_{\text{proét},c}(X_C, \mathbf{Q}_p) := \mathrm{R}\Gamma_{\text{ét},c,\mathrm{Hu}}(X_C, \mathbf{Q}_p)$, the Huber p-adic étale cohomology.

Remark 1.6. We state the duality only one way because of potential pathologies for extensions of Fréchet (or Banach) spaces in the condensed world, but (1.3) and (1.4) suggest strongly that there should be a duality both ways, at least in reasonnable cases.

1.2. **The main result.** The main result of this paper is the following:

main-1 **Theorem 1.7.** *Conjecture 1.5 holds.*

Our strategy for the proof of Theorem 1.7 follows the heuristic computations we have done on examples. The foundational results needed to do that mentioned in the previous section were proven in a series of papers by the authors and Piotr Achinger: Hyodo-Kato cohomology of rigid analytic and dagger varieties was defined and studied in [21], p-adic comparison theorems were proven in [21], [22], [25], compactly supported p-adic pro-étale cohomology and Hyodo-Kato dualities were studied in [1], the properties of Topological Vector Spaces were studied in [19] (that they satisfy the expected duality for BC's was derived there from a result of the same type due to Anschütz-Le Bras in the category of Vector Spaces[3] [4]).

To prove Theorem 1.7, we start with passing from pro-étale cohomology to syntomic cohomology. Recall that the latter is built from the Hyodo-Kato part, that records the mod p information, and the de filtered Rham part that records the characteristic zero information; the two parts are connected via the Hyodo-Kato morphism. It is part of the standard yoga of p-adic Hodge theory for algebraic varieties that, when dealing with syntomic cohomology one should work as long as possible separately with the Hyodo-Kato and the de Rham parts and glue them together only at the last moment.[4] In the perfectoid world this separation can be obtained geometrically by representing syntomic cohomology by a quasi-coherent sheaf on the Fargues-Fontaine curve. This sheaf will have (completed) stalks equal to twisted Hyodo-Kato cohomology at all points outside of ∞ and at ∞ it will be equal to the filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology. Now Hyodo-Kato duality (inherited itself from de Rham duality) and filtered de Rham duality (see [1]) yield a duality on the Fargues-Fontaine curve. Taking derived global sections of this duality, via fully-faithfulness results from [19], yields a duality on the level of TVS's.

before 1.3. **A Corollary.** Before reviewing our proof of Theorem 1.7 we will state an implication and sketch its proof. It contains many of the essential elements of the proof of the main theorem.

main-cor **Corollary 1.8.** *Let $i \geq 0$. There is a natural short exact sequence of* TVS *'s*

$$0 \to \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^{2d-i+1}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to \mathbb{H}^i_{\text{proét}}(X_C, \mathbf{Q}_p) \to \mathcal{H}om_{\mathrm{TVS}}(\mathbb{H}^{2d-i}_{\text{proét},c}(X_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) \to 0$$

We note that the term on right is almost constant (see the computations (1.2)). Since we have the internal RHom spectral sequence, it suffices to show that

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\text{proét},c}(X_C, \mathbf{Q}_p), \mathbf{Q}_p) = 0, \quad a \geq 2. \tag{1.9}$$ vanishing-0

[3] That is ∞-derived category of $\underline{\mathbf{Q}}_p$-modules on the site of perfectoid affinoids over C equipped with pro-étale topology.

[4] A technique inherited from Beilinson-Deligne cohomology (see [7]). For how this works for syntomic cohomology in the algebraic setting see [6], [23].

Using syntomic comparison theorems we can trivialize the TVS-structure on pro-étale cohomology. That is, for $r \geq 2d$, we have a long exact sequence of TVS's

$$\cdots \to \mathbb{DR}_c^{b-1}(X_C, r) \to \mathbb{H}^b_{\text{proét},c}(X_C, \mathbf{Q}_p(r)) \to \mathbb{HK}_c^b(X_C, r) \to \mathbb{DR}_c^b(X_C, r) \to \cdots,$$

where we set

$$\mathbb{HK}_c^b(X_C, r) := (H^b_{\text{HK},c}(X_C) \otimes^{\text{L}\square}_{\breve{C}} \mathbb{B}^+_{\text{cr}})^{\varphi=p^r} \tag{1.10}$$

$$\mathbb{DR}_c^b(X_C, r) := \lim(\cdots \to H_c^d(X, \Omega^j) \otimes^{\text{L}\square}_K (\mathbb{B}^+_{\text{dR}}/t^r) \to H_c^d(X, \Omega^{j+1}) \otimes^{\text{L}\square}_K (\mathbb{B}^+_{\text{dR}}/t^{r-1}) \to \cdots)[-d]$$

hk-dr

Here $\breve{C} = \text{Frac}(W(\overline{k}))$, where k is the residue field of K. We note that the TVS's structure in (1.10) comes solely from the period presheaves.

To show (1.9), modulo a boundary case, it suffices to show it for the Hyodo-Kato and de Rham parts separately. For the Hyodo-Kato part, passing via a limit argument to overconvergent quasi-compact opens, we may assume that the ranks of Hyodo-Kato groups are finite. But then $(H^b_{\text{HK},c}(X_C) \otimes^{\text{L}\square}_{\breve{C}} \mathbb{B}^+_{\text{cr}})^{\varphi=p^r}$ is a BC and we know that the Ext-groups for those vanish in degrees higher than 2. So far there was no functional analytic difficulties. They appear when we need to show that

$$\mathcal{E}xt^a_{\text{TVS}}(H_c^d(X, \Omega^i) \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p) = 0, \quad a \geq 2.$$

But now the space $H_c^d(X, \Omega^i)$ is of compact type thus, assuming $K = \mathbf{Q}_p$ for simplicity, we have

$$\mathcal{E}xt^a_{\text{TVS}}(H_c^d(X, \Omega^i) \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p) \simeq H_c^d(X, \Omega^i)^* \otimes^{\square}_{\mathbf{Q}_p} \mathcal{E}xt^a_{\text{TVS}}(\mathbb{G}_a, \mathbf{Q}_p),$$

which vanishes in the required range (by the BC computations), as wanted.

1.4. **Duality for p-adic geometric pro-étale cohomology on the Fargues-Fontaine curve.** We show in this paper that the p-adic geometric pro-étale cohomology seen as living on the Fargues-Fontaine curve satisfies a Poincaré duality. Recall that the p-adic geometric pro-étale cohomology of a smooth partially proper rigid analytic variety X over K can be represented by a solid quasi-coherent sheaf on the Fargues-Fontaine curve, i.e., the pro-étale cohomology can be computed as

$$\text{R}\Gamma_{\text{proét}}(X_C, \mathbf{Q}_p) \simeq \text{R}\Gamma(X_{\text{FF}}, \mathcal{E}_{\text{proét}}(X_C, \mathbf{Q}_p)),$$

for a (nuclear) solid quasi-coherent sheaf $\mathcal{E}_{\text{proét}}(X_C, \mathbf{Q}_p)$ on the Fargues-Fontaine curve $X_{\text{FF}} := X_{\text{FF},C^\flat}$ defined using relative period sheaves. Similarly, geometric compactly supported pro-étale cohomology $\text{R}\Gamma_{\text{proét},c}(X_C, \mathbf{Q}_p)$ can be represented by solid quasi-coherent sheaf $\mathcal{E}_{\text{proét},c}(X_C, \mathbf{Q}_p)$ on X_{FF}. See Section 4.1.2 for the definitions.

Via comparison theorems, we see that, if $? \in \{-, c\}$,

$$\mathcal{E}_{\text{proét},?}(X_C, \mathbf{Q}_p(r)) \simeq \mathcal{E}_{\text{syn},?}(X_C, \mathbf{Q}_p(r)), \quad r \geq 2d,$$

where d is the dimension of X and $\mathcal{E}_{\text{syn},?}(X_C, \mathbf{Q}_p(r))$ is the syntomic cohomology sheaf (a solid quasi-coherent sheaf on the Fargues-Fontaine curve representing syntomic cohomology; see Section 3.3 for a definition). This is equivalent to proving a comparison theorem between corresponding Frobenius equivariant sheaves on the Fargues-Fontaine curve Y_{FF}, which amounts to untwisting Frobenius from classical comparison theorems. Luckily for us, the proofs of comparison theorems in [21] and [1] do actually (implicitly) prove the untwisted versions we want (see Corollary 4.9 for the notation):

Theorem 1.11. ($\mathbb{B}^I$-comparison theorem) *Let X be a smooth partially proper variety over K. Let $r \geq 0$ and let $I \subset (0, \infty)$ be a compact interval with rational endpoints*[5]*. We have a natural, functorial in S, compatible with Frobenius, quasi-isomorphism in $\mathcal{D}(\mathbf{B}^I_{S^\flat,\square})$:*

$$\tau_{\leq r}\text{R}\Gamma_{\text{proét},?}(X_S, \mathbb{B}^I)(r) \simeq \tau_{\leq r}[\text{R}\Gamma^I_{\text{HK},?}(X_S, r) \xrightarrow{\iota^I_{\text{HK}}} \text{R}\Gamma^I_{\text{dR},?}(X_S, r)].$$

[5]Containing the fixed intervals from Section 2.2.2

Recall that classical syntomic cohomology is built from (φ, N)-eigenspaces of $\mathbf{B}^+_{\mathrm{st}}$-twisted Hyodo-Kato cohomology and from filtered $\mathbf{B}^+_{\mathrm{dR}}$-cohomology. Representing it (in a stable range) by the sheaf $\mathcal{E}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r))$ on the Fargues-Fontaine curve separates these terms: heuristically speaking, the (completed) stalks of $\mathcal{E}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r))$ at points outside ∞ are N-eigenspaces of $\mathbf{B}^+_{\mathrm{st}}$-twisted Hyodo-Kato cohomology and the (completed) stalk at ∞ is the r-th filtration level of $\mathbf{B}^+_{\mathrm{dR}}$-cohomology.

Now, the stalk cohomology sheaves satisfy Poincaré duality: Poincaré duality for Hyodo-Kato cohomology reduces, via the Hyodo-Kato isomorphism, to that for de Rham cohomology and Poincaré duality for filtered de Rham cohomology, in turn, reduces to Serre duality for smooth Stein varieties – a classical result (see [1] and Section 5.1 for details). These dualities are inherited by the sheaves $\mathcal{E}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r))$, for $r \geq 2d$, and then by the sheaves $\mathcal{E}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p(r))$ yielding the second main result of this paper:

first1 **Theorem 1.12.** (Poincaré duality for pro-étale sheaves) *We have a natural, Galois equivariant, quasi-isomorphim in* $\mathrm{QCoh}(X_{\mathrm{FF}})$

$$\mathcal{E}_{\mathrm{proét}}(X_C, \mathbf{Q}_p) \tilde{\to} \mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF}})}(\mathcal{E}_{\mathrm{proét},c}(X_C, \mathbf{Q}_p(d))[2d], \mathcal{O}). \tag{1.13}$$ first

The proof of the theorem does not proceed as sketched above though, due to the difficulties of passing to stalks in the theory of solid quasi-coherent sheaves. Instead we argue in a similar vein with φ-modules on the Y_{FF}-curve. In a side remark, we sketch an alternative proof of Theorem 1.12 that, instead of passing to the Y_{FF}-curve, uses dual modifications.

Analogous argument, with splitting into Hyodo-Kato and de Rham terms, yields a Künneth formula:

first2 **Theorem 1.14.** (Künneth formula) *Let X, Y be smooth partially proper varieties over K. Then the canonical map*

$$\kappa: \quad \mathcal{E}_{\mathrm{proét}}(X_C, \mathbf{Q}_p) \otimes^{\mathrm{L}}_{\mathcal{O}} \mathcal{E}_{\mathrm{proét}}(Y_C, \mathbf{Q}_p) \to \mathcal{E}_{\mathrm{proét}}((X \times_K Y)_C, \mathbf{Q}_p)$$

is a quasi-isomorphism in $\mathrm{QCoh}(X_{\mathrm{FF}})$.

Remark 1.15. In Theorem 1.12 and Theorem 1.14, we can replace C, functorially, with any affinoid perfectoid over C.

1.5. **Descend to** TVS**'s.** Finally, to prove Theorem 1.7, we need to descend the duality (1.13) to the "real" world, which for us is the world of Topological Vector Spaces. We apply the projection functor

$$\mathrm{R}\tau_* : \mathrm{QCoh}(X_{\mathrm{FF}}) \to \mathrm{TVS},$$

the derived global section functor from [19], to the duality on the Fargues-Fontaine curve (1.13) and, since

$$\mathrm{R}\tau_* \mathcal{E}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p) \simeq \mathbb{R}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p), \quad \mathrm{R}\tau_* \mathcal{O} \simeq \mathbf{Q}_p,$$

we reduce to showing that the canonical map

$$\mathrm{R}\tau_* \mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(\mathrm{X_{FF}})}(\mathcal{E}_{\mathrm{proét},c}(X_C, \mathbf{Q}_p), \mathcal{O}) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_* \mathcal{E}_{\mathrm{proét},c}, \mathrm{R}\tau_* \mathcal{O}).$$

is a quasi-isomorphism. But this fully-faithfulness result can be reduced by an argument similar to the one used in the proof of the Corollary 1.8 to fully-faithfulness for perfect complexes on the Fargues-Fontaine curve and this was proven in [19].

rks **Remark 1.16.** (1)(*Algebraic Poincaré Duality*) The duality in Conjecture (1.5) has an algebraic version in the category of Vector Spaces (see Corollary 6.19 for the statement). It is deduced from Theorem 1.7 via fully-faithfulness results from [19].

(2) (*Arithmetic Duality*) Our Conjecture 1.5 has an arithmetic version, i.e., for p-adic arithmetic pro-étale cohomology (see [13], [17]). The statement is much simpler: it is a Poincaré duality in

the category of topological spaces that yields a nonderived version working in "both directions". In [17, Th. 1.1], we have proved this conjecture for dagger curves over K via relatively down to earth techniques (p-adic comparison theorems, Serre duality, reciprocity laws via (φ,Γ)-modules). The general case was derived from Theorem 1.12 via Galois descent by Zhenghui Li in [28].

1.6. **The story of this paper and related work.** The proof of duality presented in this paper is quite simple. But a lot of foundational work went into setting up the right formalism for this to be the case. Here is the story how this developed.

We started working on this project in the spring of 2020 when the computations for the open ball suggested that there could be a Poincaré duality for p-adic geometric pro-étale cohomology of rigid analytic spaces provided one could mix the degrees of cohomology and combine $\mathbf{Q}_p$-duality and C-duality. This suggested that there could be a duality in the BC-category, by using $\mathrm{RHom}(-,\mathbf{Q}_p)$ instead of $\mathrm{Hom}(-,\mathbf{Q}_p)$; for this to work one needed the vanishing of Ext^i's for $i\geq 2$ in the BC-category (and in fact in a bigger category containing the TVS's appearing in our comparison theorems with syntomic cohomology). We discussed this with Fargues and Le Bras in Oberwolfach at the first post-covid conference in July 2020. The required vanishing in the VS-category were proved the following year by Anschütz and Le Bras [4] by a reduction to a theorem of Breen (we realized, more recently, that these vanishings are elementary in the BC-category itself). We needed a version of this result for TVS's but it was a strong indication that what we wanted could be true.

We were at the time in the middle of writing [21, 22] which contained part of the foundational tools needed for a Poincaré duality (definition of geometric Hyodo-Kato cohomology, geometrization of p-adic comparison theorems for usual cohomology), and we started considering their compact support avatars [1]. Concurrently, we did some extra computations for analytifications of algebraic curves, which again pointed strongly towards the existence of a Poincaré duality in the geometric and arithmetic cases (the computations were more involved in the arithmetic case, but the groups that were appearing looked much more manageable, belonging to the usual world of $\mathbf{Q}_p$-topological vector spaces). By the time of the workshop "Non-Archimedean Geometry and Applications" of February 2022, in Oberwolfach, we had a conjecture [13] and a strategy that seemed to work well in examples, starting from our geometrized comparison theorem, and reinterpreting syntomic cohomology as quasi-coherent φ-equivariant cohomology on the Fargues-Fontaine Y-curve. We presented these results at the workshop and were quite excited to discover that there were two other talks dealing with Poincaré duality: one by Zavyalov establishing [34] Poincaré duality for proper analytic varieties over C, and one by Mann, developing [29] a 6-functors formalism for $\mathcal{O}^+/p$-local systems from which he could also deduce Poincaré duality for proper analytic varieties over C (note that, for proper varieties, the pro-étale cohomology groups – for constant coefficients – are finite dimensional $\mathbf{Q}_p$-vector spaces, and there is no need to consider BC-duality).

Since arithmetic duality only involved familiar objects, we decided that it would be wise to start by the proof of our conjecture for arithmetic duality for curves [17], but in the end we had to use the condensed formalism to handle functional analytic questions arising in topological dualities. For the geometric duality almost all the tools we needed were at hand: luckily for us, Andreychev [2] developed the theory of solid quasi-coherent sheaves that created a framework in which to express the duality on the Fargues-Fontaine curve that we envisaged. That was sufficient to prove a duality on the Fargues-Fontaine curve. What was missing to carry out our strategy in full was a condensed version of TVS's from [22] and the Ext-vanishing we mentioned above. That took us longer than expected mostly because we have experimented with various possible definitions settling in the end on almost verbatim translation. And, finally, in [19], we deduced the vanishing in TVS's from the one in VS's via a fully-faithfulness result. In the meantime, Zhenghui Li [28] proved in his thesis, much to our surprise[6], that the duality at the level of the Fargues-Fontaine

[6]We were rather expecting a Galois descent from TVS's.

curve was enough to deduce the existence of an arithmetic duality in general, which prompted us to put out [18], a preliminary version of this paper.

1.6.1. *Related work.* In a related work, Anschütz, Le Bras, and Mann (see [5]) followed a different path towards the proof of Conjecture 1.5 in the VS-form (see Remark 1.16). They also proceed in two steps: the first step is a duality on the Fargues-Fontaine curve, which is a byproduct of the 6-functor formalism for solid quasi-coherent sheaves on the Fargues-Fontaine curve they have developed; the second step is a descend to the world of VS's. Their first step is very different in nature and techniques from ours and includes coefficients. On the other hand, the second step, while it passes through solid sheaves of Fargues-Scholze instead of our TVS's, is very similar to ours: it relies on the Hyodo-Kato comparison theorems – the deepest part of the p-adic comparison theorems via syntomic cohomology of [20, 14, 21, 22] (or their versions using cohomology of period sheaves of [10]) – for dagger varieties and properties of their Hyodo-Kato cohomology (and filtered $\mathbf{B}_{\mathrm{dR}}^{+}$-cohomology) to control the functional analytic properties of pro-étale cohomology sheaves on the Fargues-Fontaine curve to be able to apply fully-faithfulness results (akin to the arguments sketched in Section 1.3).

Finally, we would like to mention a different approach to duality theorems developed recently by Shizhang Li, Reinecke, and Zavyalov (see [26]), which works for proper smooth rigid analytic varieties (and their relative incarnations) and, after some modifications, can be transferred to the Fargues-Fontaine curve yielding, after descending to VS's, a version of duality allowing $\mathbf{Q}_p$-local systems (see [27]).

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Notation

Notation and conventions. Let p be a prime and let K be a complete discrete valuation field with a perfect residue field, of mixed characteristic. Let $\mathcal{O}_K$ be the ring of integers in K, and k be its residue field. Let $W(k)$ be the ring of Witt vectors of k and let F be its fraction field (i.e., $W(k) = \mathcal{O}_F$).

Let $\overline{K}$ be an algebraic closure of K and let $\mathcal{O}_{\overline{K}}$ denote the integral closure of $\mathcal{O}_K$ in $\overline{K}$. Let $C = \widehat{\overline{K}}$ be the p-adic completion of $\overline{K}$. Set $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$ and let φ be the absolute Frobenius on $W(\overline{k})$. Let $\breve{C} = \mathrm{Frac}(W(\overline{k}))$.

We will denote by $\mathbf{B}_{\mathrm{cr}}, \mathbf{B}_{\mathrm{st}}, \mathbf{B}_{\mathrm{dR}}$ the crystalline, semistable, and de Rham period rings of Fontaine.

All rigid analytic spaces and dagger spaces considered will be over K or C; we assume that they are separated, taut, and countable at infinity. Huber pairs will always be sheafy. The category of affinoid perfectoid spaces over an affinoid perfectoid space S over C will be denoted by Perf_S.

We will use condensed mathematics as developed in [11], [12]. We fix an implicit cut-off cardinal κ (in the sense of [31, Sec. 4]), and assume all our perfectoid spaces, and condensed sets to be κ-small.

If $L = \mathbf{Q}_p, K, C$, we will denote by C_L the category of locally convex topological vector spaces over L.

We will use the bracket notation for certain limits: $[C_1 \xrightarrow{f} C_2]$ denotes the mapping fiber of f and we set

$$\begin{bmatrix} C_1 & \xrightarrow{f_1} & K_1 \\ \downarrow & & \downarrow \\ C_2 & \xrightarrow{f_2} & K_2 \end{bmatrix} := \big[[C_1 \xrightarrow{f_1} K_1] \to [C_2 \xrightarrow{f_2} K_2]\big].$$

2. Quasi-coherent sheaves on the Fargues-Fontaine curve

Here, we will review briefly basic facts concerning quasi-coherent sheaves on the Fargues-Fontaine curve. This is partly based on [2], [3], and [10, Sec. 6.2].

2.1. **Fargues-Fontaine curve.** Recall the definition of the relative Fargues-Fontaine curve (see [32, Lecture 12]). Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over the finite field $\mathbf{F}_p$. Let

$$Y_{\mathrm{FF},S} := \mathrm{Spa}(W(R^+), W(R^+)) \setminus V(p[p^\flat])$$

be the relative mixed characteristic punctured unit disc. It is an analytic adic space over $\mathbf{Q}_p$. The Frobenius on R^+ induces the Witt vector Frobenius and hence a Frobenius φ on $Y_{\mathrm{FF},S}$ with free and totally discontinuous action. The Fargues-Fontaine curve relative to S (and $\mathbf{Q}_p$) is defined as

$$X_{\mathrm{FF},S} := Y_{\mathrm{FF},S}/\varphi^{\mathbf{Z}}.$$

For an interval $I = [s, r] \subset (0, \infty)$ with rational endpoints, we have the open subset

$$Y^I_{\mathrm{FF},S} := \{|\cdot| : |p|^r \leq |[p^\flat]| \leq |p|^s\} \subset Y_{\mathrm{FF},S}.$$

It is a rational open subset of $\mathrm{Spa}(W(R^+), W(R^+))$ hence an affinoid space,

$$Y^I_{\mathrm{FF},S} := \mathrm{Spa}(\mathbf{B}^I_S, \mathbf{B}^{I,+}_S).$$

One can form $X_{\mathrm{FF},S}$ as the quotient of $Y^{[1,p]}_{\mathrm{FF},S}$ via the identification $\varphi : Y^{[1,1]}_{\mathrm{FF},S} \xrightarrow{\sim} Y^{[p,p]}_{\mathrm{FF},S}$. If $S = \mathrm{Spa}(C^\flat, \mathcal{O}_{C^\flat})$, we will write $Y_{\mathrm{FF}}, X_{\mathrm{FF}}, Y^I_{\mathrm{FF}}, \mathbf{B}^I, \mathbf{B}^{I,+}$.

We will denote by x_∞ the $(C, \mathcal{O}_C)$-point of the curve X_{FF} corresponding to Fontaine's map $\theta : W(\mathcal{O}_C) \to \mathcal{O}_C$, by y_∞ – the corresponding point on Y_{FF}, and by $\iota_\infty : \mathrm{Spa}(C, \mathcal{O}_C) \to T_{\mathrm{FF}}$, $T = X, Y$, the corresponding closed immersions. More generally, if S is the tilt of a perfectoid space $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p)$, there is an induced closed immersion $\theta : S^\sharp \hookrightarrow Y_{\mathrm{FF},S}$ which is locally given by Fontaine's map $\theta : W(R^+) \to R^{\sharp,+}$. We will denote by $\iota_\infty : S^\sharp \xrightarrow{\theta} T_{FF,S}$ the induced closed immersions and by y_∞, x_∞, the corresponding divisors.

We set

$$\mathbf{B}_S := \lim_{I \subset (0,\infty)} \mathbf{B}^I_S,$$

where I varies over all the compact intervals of $(0, \infty)$ with rational endpoints. We will denote by $\mathbf{B}_{S,\log}$ the log-crystalline period ring (see [24, Sec. 10.3.1]). We have $\mathbf{B}_S[U] \xrightarrow{\sim} \mathbf{B}_{S,\log}$, $U \mapsto \log([p^\flat]/p)$, with $\varphi(U) = pU, \sigma(U) = U + \log[\sigma(p^\flat)/p^\flat]$, for $\sigma \in \mathcal{G}_K$, and $N = -d/dU$. We define $\mathbf{B}^I_{S,\log}$ in a similar manner.

2.2. **Quasi-coherent sheaves on the Farges-Fontaine curve.** We will present now quasi-coherent sheaves on X_{FF} as φ-modules on a convenient chart of Y_{FF}.

2.2.1. *Solid quasi-coherent sheaves.* We start with a brief survey of solid quasi-coherent sheaves. Let Y be an analytic adic space over $\mathbf{Q}_p$. We denote by $\mathrm{QCoh}(Y)$ the ∞-category of solid quasi-coherent sheaves on Y, and by $\mathrm{Nuc}(Y)$ the full ∞-subcategory of solid nuclear sheaves on Y. See [2], [3] for the definitions of these categories and their basic properties. We will often drop the word "solid" if this does not cause confusion. If $Y = \mathrm{Spa}(R, R^+)$, then we have an equivalence [2, Th. 1.6]

$$\mathrm{QCoh}(Y) \simeq \mathcal{D}((R, R^+)_\square), \tag{2.1}$$

eq1

where the latter is the derived category of solid (R, R^+)-modules, i.e., modules over the analytic ring $(R, R^+)_\square$. In what follows, if this does not confusion, we will write

$$R_{\mathrm{an}} := (R, R^+)_\square$$

For a general Y, the category $\mathrm{QCoh}(Y)$ is obtained by gluing the categories $\mathcal{D}((R, R^+)_\square)$ in the analytic topology.

By $\mathrm{Perf}(Y)$, we denote *the full ∞-subcategory of perfect sheaves on Y*; that is, complexes which locally for the analytic topology are quasi-isomorphic to a bounded complex of finite, locally free $\mathcal{O}_Y$-modules. If $Y = \mathrm{Spa}(R, R^+)$ is affinoid, then the natural functor

$$\mathrm{Perf}(R) \to \mathrm{Perf}(Y)$$

is an equivalence, where the left-hand side denotes the ∞-category of perfect complexes of R-modules (i.e., bounded complexes of finite projective R-modules).

The categories $\mathrm{QCoh}(Y)$, $\mathrm{Nuc}(Y)$, and $\mathrm{Perf}(Y)$ are (compatibly) symmetric monoidal. In the definition of the ∞-category $\mathrm{QCoh}(Y)$ we will bound everything by a fixed uncountable cardinal so that the category is presentable; it is then also closed symmetric monoidal. The ∞-category $\mathrm{Nuc}(Y)$ is as well presentable and closed symmetric monoidal. Similarly for the ∞-category $\mathrm{Perf}(Y)$.

Remark 2.2. The categories $\mathrm{QCoh}(Y)$, $\mathrm{Nuc}(Y)$, and $\mathrm{Perf}(Y)$ can be defined in a more general setting, where $Y = \mathrm{Spa}(R, R^+)$ is a pair such that R is a complete Huber ring and $R^+ \subset R^0$ is an arbitrary subring (see [2, Sec. 3.3] for details). We will most often use the case when $R^+ = \mathbf{Z}$.

fixed

2.2.2. *Quasi-coherent φ-sheaves on Y_{FF}.* The ∞-category of quasi-coherent φ-equivariant sheaves over $Y_{\mathrm{FF},S}$ (in short: φ-sheaves over $Y_{\mathrm{FF},S}$) is the equalizer

$$\mathrm{QCoh}(Y_{\mathrm{FF},S})^\varphi := \mathrm{eq}\big(\ \mathrm{QCoh}(Y_{\mathrm{FF},S}) \underset{\mathrm{Id}}{\overset{\varphi^*}{\rightrightarrows}} \mathrm{QCoh}(Y_{\mathrm{FF},S})\ \big).$$

It is the ∞-category of pairs $(\mathcal{E}, \varphi_\mathcal{E})$, where $\mathcal{E}$ is a quasi-coherent sheaf on $Y_{\mathrm{FF},S}$ and $\varphi_\mathcal{E} : \varphi^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is a quasi-isomorphism[7]. The category $\mathrm{Nuc}(Y_{\mathrm{FF},S})^\varphi$ (resp. $\mathrm{Perf}(Y_{\mathrm{FF},S})^\varphi$) is the full ∞-subcategory of $\mathrm{QCoh}(Y_{\mathrm{FF},S})^\varphi$ spanned by the pairs $(\mathcal{E}, \varphi_\mathcal{E})$, where $\mathcal{E}$ is a nuclear (resp. perfect) sheaf on $Y_{\mathrm{FF},S}$.

In what follows we will set

$$u = (p-1)/p,\ v = p-1 \text{ if } p \neq 2; \quad \text{for } p = 2 \text{ we take } u = 3/4,\ v = 3/2.$$

If S is the tilt of a perfectoid space $S^\sharp$ over $\mathrm{Spa}(\mathbf{Q}_p)$, this choice of u, v ensures that the divisor on $Y^{[u,v]}_{\mathrm{FF},S}$ associated to t is y_∞ and t is a unit in $\mathbf{B}^{[u,v/p]}_S$, i.e., if $S^\sharp = \mathrm{Spa}(R, R^+)$, then $\mathbf{B}^{[u,v]}_S/t = R$ and $\mathbf{B}^{[u,v/p]}_S/t = 0$.

Via analytic descent, we like to describe the above categories of φ-equivariant sheaves using the chart $Y^{[u,v]}_{\mathrm{FF},S}$ (via Frobenius we have $\varphi : Y^{[u/p,v/p]}_{\mathrm{FF},S} \xrightarrow{\sim} Y^{[u,v]}_{\mathrm{FF},S}$):

$$\mathrm{QCoh}(Y_{\mathrm{FF},S})^\varphi \simeq \mathrm{eq}\big(\ \mathrm{QCoh}(Y^{[u,v]}_{\mathrm{FF},S}) \underset{j^*}{\overset{\varphi^*}{\rightrightarrows}} \mathrm{QCoh}(Y^{[u,v/p]}_{\mathrm{FF},S}\ )\big).$$

We wrote here φ, j for the Frobenius and the open embedding maps from $Y^{[u,v/p]}_{\mathrm{FF},S}$ to $Y^{[u,v]}_{\mathrm{FF},S}$, respectively. That is, $\mathrm{QCoh}(Y_{\mathrm{FF},S})^\varphi$ is the ∞-category of pairs $(\mathcal{E}, \varphi_\mathcal{E})$, where $\mathcal{E}$ is a quasi-coherent sheaf on $Y^{[u,v]}_{\mathrm{FF},S}$ and $\varphi_\mathcal{E} : \varphi^*\mathcal{E} \xrightarrow{\sim} j^*\mathcal{E}$ is a quasi-isomorphism. The categories $\mathrm{Nuc}(Y_{\mathrm{FF},S})^\varphi$, $\mathrm{Perf}(Y_{\mathrm{FF},S})^\varphi$can be described in an analogous way.

[7]We will call isomorphisms in the ∞-categories $\mathrm{QCoh}(-)$ quasi-isomorphisms to be compatible with more classical set-ups.

We note that, since we have the equivalence (2.1), we can also write[8]

$$\mathrm{QCoh}(Y_{\mathrm{FF},S})^{\varphi} \simeq \mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi} := \mathrm{eq}\big(\ \mathcal{D}(\mathbf{B}^{[u,v]}_{S,\mathrm{an}}) \underset{j^*}{\overset{\varphi^*}{\rightrightarrows}} \mathcal{D}(\mathbf{B}^{[u,v/p]}_{S,\mathrm{an}})\ \big).$$

(Frobenius φ maps $\mathbf{B}^{[u,v]}_{S,\mathrm{an}}$ to $\mathbf{B}^{[u,v/p]}_{S,\mathrm{an}}$.) It is the ∞-category of pairs $M = (M^{[u,v]}, \varphi_M)$, where $M^{[u,v]}$ is a complex of $\mathbf{B}^{[u,v]}_{S,\mathrm{an}}$-modules and the Frobenius φ_M is a quasi-isomorphism of complexes of $\mathbf{B}^{[u,v/p]}_{S,\mathrm{an}}$-modules

$$\varphi_M : \varphi^* M^{[u,v]} \xrightarrow{\sim} M^{[u,v/p]} := M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}^{[u,v]}_{S,\mathrm{an}}} \mathbf{B}^{[u,v/p]}_{S,\mathrm{an}}.$$

52 **Remark 2.3.** In what follows it will be convenient to consider the following variant $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S})^{\varphi}$ (in which we drop the "an", i.e., we consider $(R, \mathbf{Z})_{\square}$ instead of $(R, R^+)_{\square}$) of the ∞-category $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$:

$$\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S})^{\varphi} := \mathrm{eq}\big(\ \mathcal{D}(\mathbf{B}^{[u,v]}_{S,\square}) \underset{j^*}{\overset{\varphi^*}{\rightrightarrows}} \mathcal{D}(\mathbf{B}^{[u,v/p]}_{S,\square})\ \big).$$

It is the ∞-category of pairs $M = (M^{[u,v]}, \varphi_M)$, where $M^{[u,v]}$ is a complex of solid $\mathbf{B}^{[u,v]}_{S}$-modules and the Frobenius φ_M is a quasi-isomorphism of complexes of solid $\mathbf{B}^{[u,v/p]}_{S}$-modules

$$\varphi_M : \varphi^* M^{[u,v]} \xrightarrow{\sim} M^{[u,v/p]} := M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}^{[u,v]}_{S,\mathrm{an}}} \mathbf{B}^{[u,v/p]}_{S}.$$

We call $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S})^{\varphi}$ *the category of φ-complexes of $\mathbf{B}^{\mathrm{FF}}_{S}$-modules.* Since we have the equivalences of symmetric monoidal categories $\mathcal{D}((\mathbf{B}^{I}_{S}, \mathbf{Z})_{\square}) = \mathcal{D}(\mathbf{B}^{I}_{S,\mathrm{an}})$ (see [9, Lemma A.16]), this corresponds to using the analytic structure with respect to $\mathbf{Z}$ in place of $\mathbf{B}^{I,+}_{S}$. In particular, we have a canonical monoidal functor $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S})^{\varphi} \to \mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$.

monoidal

2.2.3. *Monoidal structure on quasi-coherent sheaves on Y_{FF}.* The category $\mathrm{QCoh}(Y_{\mathrm{FF},S})^{\varphi}$ is closed symmetric monoidal. We will now present how the closed symmetric monoidal structure can be seen on the level of the category $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$. In what follows we have set $\mathbf{B}_1 := \mathbf{B}^{[u,v]}_{S,\mathrm{an}}$, $\mathbf{B}_2 := \mathbf{B}^{[u,v/p]}_{S,\mathrm{an}}$.

The (derived) tensor product in $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$, denoted by $(-) \otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S,\square}} (-)$, is inherited from the one of the category $\mathcal{D}(\mathbf{B}_1)$. More precisely, for $(M, \varphi_M), (N, \varphi_N) \in \mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$, their tensor product is defined by:

$$M \otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S,\square}} N := (M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1} N^{[u,v]}, \varphi_{M\otimes N}),$$

$$\varphi_{M\otimes N} = \varphi_M \otimes \varphi_N : (M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1} N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1,\varphi} \mathbf{B}_2 \to (M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1} N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 = (M^{[u,v/p]} \otimes^{\mathrm{L}}_{\mathbf{B}_2} N^{[u,v/p]}).$$

Frobenius $\varphi_{M\otimes N}$ is a quasi-isomorphism because so are Frobeniuses φ_M and φ_N.

The internal RHom, denoted by $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^{\mathrm{FF}}_{S,\square}}(-,-)$, in the category $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})^{\varphi}$ is defined by:

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^{\mathrm{FF}}_{S,\square}}(M, N) := (\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}), \varphi_{M,N}),$$

$$\varphi_{M,N} := (\varphi_M^{-1}, \varphi_N) : \quad \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1,\varphi} \mathbf{B}_2 \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2.$$

In the definition of Frobenius $\varphi_{M,N}$ we have used the following (non-obvious[9]) fact:

fun1-kol **Lemma 2.4.** *The canonical maps*

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1,\varphi} \mathbf{B}_2 \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_2}(M^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1,\varphi} \mathbf{B}_2, N^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1,\varphi} \mathbf{B}_2),$$
$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_2}(M^{[u,v/p]}, N^{[u,v/p]})$$

are quasi-isomorphisms.

[8] We stress here that $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S,\square})$ and $\mathbf{B}^{\mathrm{FF}}_{S,\square}$ is just a notation; the ring $\mathbf{B}^{\mathrm{FF}}_{S}$ does not exist.

[9] One usually needs some finiteness condition for this kind of statements to hold.

Proof. To start, note that, since the first map is induced by the composition of the maps

$$\varphi : \mathbf{B}_{S,\mathrm{an}}^{[u,v]} \to \mathbf{B}_{S,\mathrm{an}}^{[u/p,v/p]}, \quad \mathrm{can} : \mathbf{B}_{S,\mathrm{an}}^{[u/p,v/p]} \to \mathbf{B}_{S,\mathrm{an}}^{[u,v/p]}$$

where the first map is an isomorphism, it suffices to argue for the second quasi-isomorphism in the lemma.

Write $M^{[u,v]} = \mathrm{colim}_{i\in I} M_i^{[u,v]}$ as a colimit of compact projective objects $\{M_i^{[u,v]} = \mathbf{B}_1[T_i]\}$, $i \in I$, for extremally disconnected sets T_i's. Then

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M^{[u,v]}, N^{[u,v]}) &= \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(\mathrm{colim}_{i\in I} M_i^{[u,v]}, N^{[u,v]}) \\ &\simeq \mathrm{Rlim}_I \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M_i^{[u,v]}, N^{[u,v]}) \end{aligned}$$

and similarly for $[u, v/p]$ (we set $M_i^{[u,v/p]} := M_i^{[u,v]} \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \simeq \mathbf{B}_2[T_i]$). It follows that it suffices to show that

$$(\mathrm{Rlim}_I \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M_i^{[u,v]}, N^{[u,v]})) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \xrightarrow{\sim} \mathrm{Rlim}_I \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_2}(M_i^{[u,v/p]}, N^{[u,v/p]}).$$

But, by [2, Prop. 5.38], we have

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M_i^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_2}(M_i^{[u,v/p]}, N^{[u,v/p]}).$$

Hence it suffices to show that

$$(\mathrm{Rlim}_I \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M_i^{[u,v]}, N^{[u,v]})) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \xrightarrow{\sim} \mathrm{Rlim}_I(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}_1}(M_i^{[u,v]}, N^{[u,v]}) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2).$$

That is, that the functor $(-) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2$ commutes with derived limits.

To show this write $\mathbf{B}_S^{[u,v/p]} = \mathbf{B}_S^{[u,v]}\langle f\rangle$, where $f = (p/[p^\flat]^{p/v}) \in \mathbf{B}_S^{[u,v]}$. By [2, Prop. 4.11], we have

$$(-) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2 \simeq (-) \otimes^{\mathrm{L}}_{(\mathbf{Z}[T],\mathbf{Z})_\square} (\mathbf{Z}[T], \mathbf{Z}[T])_\square, \tag{2.5}$$

fun1

where the map $(\mathbf{Z}[T], \mathbf{Z})_\square \to (\mathbf{B}_S^{[u,v]}, \mathbf{B}_S^{[u,v],+})_\square$ is induced by $T \mapsto f$. But, by [2, Prop. 3.12], for $M \in \mathcal{D}((\mathbf{Z}[T], \mathbf{Z})_\square)$, we have

$$M \otimes^{\mathrm{L}}_{(\mathbf{Z}[T],\mathbf{Z})_\square} (\mathbf{Z}[T], \mathbf{Z}[T])_\square \simeq \mathrm{R}\underline{\mathrm{Hom}}_R(R_\infty/R, M)[1],$$

where $R = \mathbf{Z}[T]$, $R_\infty = \mathbf{Z}((T^{-1}))$. It follows that the functor $(-) \otimes^{\mathrm{L}}_{\mathbf{B}_1} \mathbf{B}_2$ commutes with derived limits, as wanted. □

Finally, we note that Frobenius $\varphi_{M,N}$ is a quasi-isomorphism because so are Frobeniuses φ_M and φ_N.

zet1

Remark 2.6. (1) The tensor product computations above are valid for the category $\mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^\varphi$. For the internal Hom, they go through as well if one assumes that $(M, \varphi_M), (N, \varphi_N)$ are nuclear and so is the internal Hom between them (see [28, Lemma 4.7] for a proof of an analog of Lemma 2.4 in this setting). In this paper we will always be in this setting.

(2) Let $M, N \in \mathcal{D}((R, \mathbf{Z})_\square)$. We note that the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_{(R,\mathbf{Z})_\square}(M, N) \otimes^{\mathrm{L}}_{(R,\mathbf{Z})_\square} (R, R^+)_\square \to \mathrm{R}\underline{\mathrm{Hom}}_{(R,R^+)_\square}(M \otimes^{\mathrm{L}}_{(R,\mathbf{Z})_\square} (R, R^+)_\square, N \otimes^{\mathrm{L}}_{(R,\mathbf{Z})_\square} (R, R^+)_\square)$$

is a quasi-isomorphism in the case N is $(R, R^+)_\square$-complete. It follows that $\mathrm{R}\underline{\mathrm{Hom}}_{(R,\mathbf{Z})_\square}(M, N)$ is then also $(R, R^+)_\square$-complete. For example, this is the case when N is nuclear.

2.2.4. *Quasi-coherent φ-sheaves on Y_{FF} and φ-modules.* We will now describe the categories $\mathrm{Nuc}(Y_{\mathrm{FF},S})^{\varphi}$ and $\mathrm{Perf}(Y_{\mathrm{FF},S})^{\varphi}$ using complexes of (usual) solid modules.

Recall that the natural maps of analytic rings $(\mathbf{B}_S^I, \mathbf{Z})_{\square} \to (\mathbf{B}_S^I, \mathbf{B}_S^{I,+})_{\square}$ induce base change functors

$$(-) \otimes^{\mathrm{L}}_{(\mathbf{B}_S^I, \mathbf{Z})_{\square}} (\mathbf{B}_S^I, \mathbf{B}_S^{I,+})_{\square} : \quad \mathcal{D}((\mathbf{B}_S^I, \mathbf{Z})_{\square}) \to \mathcal{D}((\mathbf{B}_S^I, \mathbf{B}_S^{I,+})_{\square}). \tag{2.7}$$ mpr1

By [10, (6.13)], the functors (2.7) induce equivalences on the full subcategories of nuclear and perfect complexes:

$$\begin{aligned} \mathrm{Nuc}(\mathbf{B}_S^I) := \mathrm{Nuc}((\mathbf{B}_S^I, \mathbf{Z})_{\square}) &\xrightarrow{\sim} \mathrm{Nuc}((\mathbf{B}_S^I, \mathbf{B}_S^{I,+})_{\square}), \\ \mathrm{Perf}(\mathbf{B}_S^I) \simeq \mathrm{Perf}((\mathbf{B}_S^I, \mathbf{Z})_{\square}) &\xrightarrow{\sim} \mathrm{Perf}((\mathbf{B}_S^I, \mathbf{B}_S^{I,+})_{\square}). \end{aligned} \tag{2.8}$$ kawa1

We define the category $\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ (resp. $\mathrm{Perf}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$) as the full ∞-subcategory of $\mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ spanned by the pairs $(M^{[u,v]}, \varphi_M)$, where $M^{[u,v]}$ is a nuclear (resp. perfect) complex over $\mathbf{B}_S^{[u,v]}$. That is, the ∞-category $\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ of nuclear φ-complexes of $\mathbf{B}_S^{\mathrm{FF}}$-modules, is defined as the equalizer:

$$\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} := \mathrm{eq}\big(\ \mathrm{Nuc}(\mathbf{B}_S^{[u,v]}) \underset{\mathrm{can}}{\overset{\varphi^*}{\rightrightarrows}} \mathrm{Nuc}(\mathbf{B}_S^{[u,v/p]})\ \big).$$

Similarly, for the category $\mathrm{Perf}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ of φ-complexes of perfect $\mathbf{B}_S^{\mathrm{FF}}$-modules.

We have the following simple fact:

simple1 **Lemma 2.9.** *The canonical functor*

$$\mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} \to \mathrm{QCoh}(Y_{\mathrm{FF},S})$$

induces equivalences of ∞-categories:

$$\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} \xrightarrow{\sim} \mathrm{Nuc}(Y_{\mathrm{FF},S})^{\varphi}, \quad \mathrm{Perf}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} \xrightarrow{\sim} \mathrm{Perf}(Y_{\mathrm{FF},S})^{\varphi}. \tag{2.10}$$ mor1

Proof. Our claim follows from equivalences (2.8). □

The categories $\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$, and $\mathrm{Perf}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ are symmetric monoidal: the (derived) tensor products (denoted by $(-) \otimes^{\mathrm{L}}_{\mathbf{B}_S^{\mathrm{FF}}} (-)$) are inherited from the ones of the categories $\mathrm{Nuc}(\mathbf{B}_S^{[u,v]})$, and $\mathrm{Perf}(\mathbf{B}_S^{[u,v]})$, respectively. The canonical functor to the category $\mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi}$ is symmetric monoidal. The functors in Lemma 2.9 are compatible with these structures.

2.2.5. *Quasi-coherent sheaves on X_{FF}.* The action of φ on $Y_{\mathrm{FF},S}$ being free and totally discontinuous, by the analytic descent for solid quasi-coherent sheaves, we obtain an equivalence of ∞-categories

$$\mathcal{E}_{\mathrm{FF},S} : \quad \mathrm{QCoh}(Y_{\mathrm{FF},S})^{\varphi} \xrightarrow{\sim} \mathrm{QCoh}(Y_{\mathrm{FF},S}/\varphi^{\mathbf{Z}}) = \mathrm{QCoh}(X_{\mathrm{FF},S}).$$

Similarly, we get equivalences of closed symmetric monoidal ∞-categories

$$\mathrm{Nuc}(Y_{\mathrm{FF},S})^{\varphi} \xrightarrow{\sim} \mathrm{Nuc}(X_{\mathrm{FF},S}), \quad \mathrm{Perf}(Y_{\mathrm{FF},S})^{\varphi} \xrightarrow{\sim} \mathrm{Perf}(X_{\mathrm{FF},S}). \tag{2.11}$$ mor2

By Lemma 2.9, this yields a functor

$$\mathcal{E}_{\mathrm{FF},S} : \mathcal{D}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} \to \mathrm{QCoh}(X_{\mathrm{FF},S}). \tag{2.12}$$ functor1

We will often skip the subscript S from $\mathcal{E}_{\mathrm{FF},S}$ if this does not cause confusion. Restricting to nuclear or perfect complexes we get the following result (see [10, Th. 6.8] for a similar statement):

leje1 **Proposition 2.13.** (1) *The functor $\mathcal{E}_{\mathrm{FF},S}$, from* (2.12)*, induces equivalences of ∞-categories*

$$\mathrm{Nuc}(\mathbf{B}_S^{\mathrm{FF}})^{\varphi} \xrightarrow{\sim} \mathrm{Nuc}(X_{\mathrm{FF},S}), \quad \mathrm{Perf}(\mathbf{B}_S)^{\varphi} \xrightarrow{\sim} \mathrm{Perf}(X_{\mathrm{FF},S}). \tag{2.14}$$ nuceq

(2) *Let $\mathcal{E} \in \mathrm{Nuc}(X_{\mathrm{FF},S})$. Let $(M(\mathcal{E})^{[u,v]}, \varphi_M)$ be the nuclear φ-complex of $\mathbf{B}_S^{\mathrm{FF}}$-modules corresponding to $\mathcal{E}$ via* (2.14)*. Then, there is a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_{\square})$*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S}, \mathcal{E}) \simeq [M(\mathcal{E})^{[u,v]} \xrightarrow{\varphi-1} M(\mathcal{E})^{[u,v/p]}].$$

Proof. The first claim is a combination of (2.10) and (2.11). For the second claim, we compute

$$\begin{aligned}\mathrm{R}\Gamma(X_{\mathrm{FF},S},\mathcal{E}) &\simeq \mathrm{R}\Gamma(\varphi^{\mathbf{Z}},\mathrm{R}\Gamma(Y_{\mathrm{FF}},\mathcal{E}_{|Y_{\mathrm{FF}}})) \simeq [\Gamma(Y_{\mathrm{FF},S}^{[u,v]},\mathcal{E}_{|Y_{\mathrm{FF}}})\xrightarrow{\varphi-1}\Gamma(Y_{\mathrm{FF},S}^{[u,v/p]},\mathcal{E}_{|Y_{\mathrm{FF}}})]\\ &\simeq [M(\mathcal{E})^{[u,v]}\xrightarrow{\varphi-1}M(\mathcal{E})^{[u,v/p]}].\end{aligned}$$

□

3. Syntomic complexes on the Fargues-Fontaine curve

In this section we define quasi-coherent sheaves on the Fargues-Fontaine curve representing various cohomologies of smooth partially proper rigid analytic varieties: de Rham (Proposition 3.1), Hyodo-Kato (Proposition 3.6), and syntomic (Formula 3.22) and Proposition 3.23). We will do the same for pro-étale cohomology in the next chapter (Proposition 4.3).

3.1. de Rham cohomology.

We start with the cohomologies of de Rham type. We use [1, Sec. 3, Sec. 4, Sec. 5] as the basic reference.

3.1.1. $\mathbf{B}_{\mathrm{dR}}^+$*-cohomology.* Let X be a partially proper rigid analytic variety over K. We have the (filtered) de Rham complexes in $\mathcal{D}(K_\square)$ and (filtered) $\mathbf{B}_{\mathrm{dR}}^+$-cohomology complexes in $\mathcal{D}(\mathbf{B}_{\mathrm{dR},\square}^+)$, respectively:

$$F^r\mathrm{R}\Gamma_{\mathrm{dR},?}(X),\quad F^r\mathrm{R}\Gamma_{\mathrm{dR},?}(X_C/\mathbf{B}_{\mathrm{dR}}^+),\quad r\in\mathbf{N},\ ?=-,c$$

as well as the quotients

$$\mathrm{R}\Gamma_{\mathrm{dR},?}(X_C,r) := \mathrm{R}\Gamma_{\mathrm{dR},?}(X/\mathbf{B}_{\mathrm{dR}}^+)/F^r.$$

The latter complexes can be represented by quasi-coherent sheaves on X_{FF}. For $r\in\mathbf{N}$, we define the *de Rham modules*

$$\mathrm{R}\Gamma_{\mathrm{dR},?}^{[u,v]}(X_C,r) := \mathrm{R}\Gamma_{\mathrm{dR},?}(X_C,r).$$

Since $\mathbf{B}_{[u,v]}/t^i=\mathbf{B}_{\mathrm{dR}}^+/t^i$, these are $\mathbf{B}_{[u,v]}$-modules. They are nuclear: for the usual cohomology, in the Stein case this follows from Section 3.1.2 below; in general case – by the fact that nuclearity is preserved by countable products. For the cohomology with compact support: we use the Stein case again and then pass to a colimit which preserves nuclearity. Since $\mathrm{R}\Gamma_{\mathrm{dR},?}^{[u,v]}(X_C,r)\otimes^{\mathrm{L}\square}_{\mathbf{B}_{[u,v]}}\mathbf{B}_{[u,v/p]}=0$ (recall that t is invertible in $\mathbf{B}_{[u,v/p]}$), these complexes taken as pairs

$$\mathrm{R}\Gamma_{\mathrm{dR},?}^{\mathbf{B}}(X_C,r)=(\mathrm{R}\Gamma_{\mathrm{dR},?}^{[u,v]}(X_C,r),0)$$

define nuclear φ-complexes over $\mathbf{B}^{\mathrm{FF}}$ (see Remark 2.3).

We denote by

$$\mathcal{E}_{\mathrm{dR},?}(X_C,r) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma_{\mathrm{dR},?}^{\mathbf{B}}(X_C,r))$$

the corresponding nuclear quasi-coherent sheaves on X_{FF}. We will call them *de Rham sheaves*. We record the following simple fact:

nyc1 **Proposition 3.1.** *Let $r\in\mathbf{N}$. We have a natural quasi-isomorphism in* $\mathrm{QCoh}(X_{\mathrm{FF}})$

$$\mathcal{E}_{\mathrm{dR},?}(X_C,r)\simeq i_{\infty,*}\mathrm{R}\Gamma_{\mathrm{dR},?}(X_C,r).$$

For $S\in\mathrm{Perf}_C$, by replacing $\mathbf{B},\mathbf{B}_{\mathrm{dR}}^+,X_{\mathrm{FF}}$ with $\mathbf{B}_{S^\flat},\mathbf{B}_{\mathrm{dR}}^+(S),X_{\mathrm{FF},S^\flat}$ in the above, we obtain de Rham modules and sheaves on $X_{\mathrm{FF},S^\flat}$: $\mathrm{R}\Gamma_{\mathrm{dR},?}^{\mathbf{B}}(X_S,r),\mathcal{E}_{\mathrm{dR},?}(X_S,r)$. These are functors on Perf_C.

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3.1.2. *Stein varieties.* Let X be a smooth Stein rigid analytic variety over K. In this case the above cohomology complexes can be made more explicit.

(•) *De Rham cohomology.* Let $r \in \mathbf{N}$. Since coherent cohomology of X is trivial in nonzero degrees and we have Serre duality, the (filtered) de Rham cohomology of X can be computed by the following complexes in $\mathcal{D}(K_\square)$:

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X) \simeq (\Omega^r(X) \to \cdots \to \Omega^d(X))[-r],$$
$$F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X) \simeq (H^d_c(X,\Omega^r) \to H^d_c(X,\Omega^{r+1}) \to \cdots \to H^d_c(X,\Omega^d))[-d-r].$$

The second quasi-isomorphism follows from the fact that $H^i_c(X,\Omega^j) = 0$, for $i \neq d$. The terms of the first complex are nuclear Fréchet over K and those of the second complex are of compact type over K (in classical terminology).

(•) $\mathbf{B}^+_{\mathrm{dR}}$*-cohomology.* Let $r \in \mathbf{N}$. The (filtered) $\mathbf{B}^+_{\mathrm{dR}}$-cohomology of X can be computed by the following complexes in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$:

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}) \simeq \mathcal{O}(X) \otimes^\square_K t^r\mathbf{B}^+_{\mathrm{dR}} \to \Omega^1(X) \otimes^\square_K t^{r-1}\mathbf{B}^+_{\mathrm{dR}} \to \cdots \to \Omega^d(X) \otimes^\square_K t^{r-d}\mathbf{B}^+_{\mathrm{dR}}, \tag{3.2}$$

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$$F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}}) \simeq (H^d_c(X,\mathcal{O}) \otimes^\square_K t^r\mathbf{B}^+_{\mathrm{dR}} \to H^d_c(X,\Omega^1) \otimes^\square_K t^{r-1}\mathbf{B}^+_{\mathrm{dR}} \to \cdots \to H^d_c(X,\Omega^d) \otimes^\square_K t^{r-d}\mathbf{B}^+_{\mathrm{dR}})[-d].$$

The tensor products are actually derived because $\mathbf{B}^+_{\mathrm{dR}}$ is Fréchet hence flat.

This yields the quasi-isomorphisms in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$:

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_C,r) \simeq \mathcal{O}(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^r) \to \Omega^1(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^{r-1}) \to \cdots \to \Omega^d(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^{r-d}), \tag{3.3}$$

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$$\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C,r) \simeq (H^d_c(X,\mathcal{O}) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^r) \to H^d_c(X,\Omega^1) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^{r-1}) \to \cdots \to H^d_c(X,\Omega^d) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^{r-d}))[-d].$$

We will denote the respective cohomology groups by $H^i_{\mathrm{dR}}(X,r)$ and $H^i_{\mathrm{dR},c}(X,r)$.

For $i \geq 0$, we have short exact sequences in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$ (see [14, Example 3.30], [1, Lemma 3.14])

$$0 \to \Omega^i(X_C)/\mathrm{Im}\, d \to H^i_{\mathrm{dR}}(X_C,r) \to H^i_{\mathrm{dR}}(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/F^{r-i-1}) \to 0, \tag{3.4}$$

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$$0 \to (H^d_c(X,\Omega^{i-d})/\mathrm{Im}\, d) \otimes^\square_K \mathrm{gr}^{r-i+d-1}_F\mathbf{B}^+_{\mathrm{dR}} \to H^i_{\mathrm{dR},c}(X_C,r) \to H^i_{\mathrm{dR},c}(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/F^{r-i+d-1}) \to 0.$$

3.2. **Hyodo-Kato cohomology.** Let X be a smooth rigid analytic variety over C. Let $\mathrm{R}\Gamma_{\mathrm{HK}}(X) \in \mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)$ be the Hyodo-Kato cohomology defined in [21, Sec. 4] (see also [10, Sec. 3]). Here $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)$ is the derived ∞-category of solid $(\varphi,N,\mathcal{G}_K)$-modules over $\breve{C}$.

3.2.1. *Hyodo-Kato cohomology on the Fargues-Fontaine curve.* Let $r \in \mathbf{Z}$. Consider the twisted Hyodo-Kato cohomology in $\mathcal{D}_{\varphi,\mathcal{G}_K}(\breve{C}_\square)$

$$\mathrm{R}\Gamma^I_{\mathrm{HK}}(X_C,r) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^I_{\log}]^{N=0},$$

where the twist $\{r\}$ means Frobenius divided by p^r and $I \subset (0,\infty)$ is a compact interval with rational endpoints. We define $\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_C,r)$ in a similar way. We claim that, for compact intervals $I \subset J \subset (0,\infty)$ with rational endpoints, we have the canonical quasi-isomorphism

$$\mathrm{R}\Gamma^J_{\mathrm{HK}}(X_C,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}^J} \mathbf{B}^I \xrightarrow{\sim} \mathrm{R}\Gamma^I_{\mathrm{HK}}(X_C,r).$$

Indeed, for that, it suffices to show that the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^J_{\log} \otimes^{\mathrm{L}\square}_{\mathbf{B}^J} \mathbf{B}^I \to \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^I_{\log}$$

is a quasi-isomorphism. But this is clear since the solid tensor product commutes with direct sums.

We define the pair[10]

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_C,r) := (\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r),\varphi),\quad \varphi : \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r) \to \mathrm{R}\Gamma^{[u,v/p]}_{\mathrm{HK}}(X_C,r),$$

where the Frobenius φ is induced from the Hyodo-Kato Frobenius and the Frobenius $\varphi : \mathbf{B}^{[u,v]} \to \mathbf{B}^{[u,v/p]}$. It yields a quasi-isomorphism in $\mathcal{D}(\mathbf{B}^{[u,v/p]}_{\square})$

$$\varphi : \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}^{[u,v]},\varphi} \mathbf{B}^{[u,v/p]} \xrightarrow{\sim} \mathrm{R}\Gamma^{[u,v/p]}_{\mathrm{HK}}(X_C,r).$$

The pair $\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_C,r)$ defines a nuclear φ-complex (actually $(\varphi,\mathcal{G}_K)$-complex) over $\mathbf{B}^{\mathrm{FF}}$, which we will call *Hyodo-Kato module*.

We define *Hyodo-Kato sheaves* on X_{FF} as

$$\mathcal{E}_{\mathrm{HK}}(X_C,r) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_C,r)).$$

By Proposition 2.13, these are nuclear quasi-coherent sheaves on X_{FF}. If the cohomology groups of $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$ are of finite rank over $\breve{C}$ then the sheaf $\mathcal{E}_{\mathrm{HK}}(X_C,r)$ is perfect. By Proposition 2.13 and [10, Th. 6.3], we have natural quasi-isomorphisms in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\begin{aligned}\mathrm{R}\Gamma(X_{\mathrm{FF}},\mathcal{E}_{\mathrm{HK}}(X_C,r)) &\simeq [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v]}_{\log}]^{N=0,\varphi=1} \\ &\xleftarrow{\sim} [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}_{\log}]^{N=0,\varphi=1}\end{aligned} \tag{3.5}$$

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where we set, for $M = \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v]}_{\log}$ or $\mathrm{R}\Gamma_{\mathrm{HK}}(X)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}_{\log}$,

$$[M]^{N=0,\varphi=1} := \begin{bmatrix} M & \xrightarrow{\varphi-1} & M \\ \downarrow N & & \downarrow N \\ M & \xrightarrow{p\varphi-1} & M \end{bmatrix}$$

For $S \in \mathrm{Perf}_C$, by changing $\mathbf{B}, \mathbf{B}^I, \mathbf{B}^I_{\log}$ to $\mathbf{B}_{S^\flat}, \mathbf{B}^I_{S^\flat}, \mathbf{B}^I_{S^\flat,\log}$, we obtain Hyodo-Kato modules and sheaves:

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_S,r),\quad \mathcal{E}_{\mathrm{HK}}(X_S,r).$$

These are functors on Perf_C. In the case X is partially proper, we have analogs $\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},c}(X_S,r)$, $\mathcal{E}_{\mathrm{HK},c}(X_S,r)$ for Hyodo-Kato cohomology with compact support[11] and the following analog of quasi-isomorphism (3.5):

sobota12

Proposition 3.6. *Let $r \in \mathbf{Z}$. We have a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat},\mathcal{E}_{\mathrm{HK},?}(X_S,r)) \simeq [\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},?}(X_S,r)]^{\varphi=1}.$$

3.2.2. *Hyodo-Kato map.* Let X be a smooth partially proper rigid analytic variety over K. Recall that we have the natural Hyodo-Kato maps (see [21, Sec. 4]) in $\mathcal{D}(\breve{C}_\square)$ and $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$, respectively:

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}),\quad \iota_{\mathrm{HK}} : \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^+_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}).$$

Combined with the canonical map $\iota : \mathbf{B}^{[u,v]}_{\log} \to \mathbf{B}^{[u,v]}/t^i$, it defines a map between complexes of solid $\mathbf{B}^{[u,v]}$-modules:

$$\iota_{\mathrm{HK}} : \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r) = [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v]}_{\log}]^{N=0} \to \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_C,r). \tag{3.7}$$

sobota11

Since we have a commutative diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r) & \xrightarrow{\varphi\otimes\varphi} & \mathrm{R}\Gamma^{[u,v/p]}_{\mathrm{HK}}(X_C,r) \\ \downarrow \iota_{\mathrm{HK}} & & \downarrow \\ \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_C,r) & \longrightarrow & 0 \end{array} \tag{3.8}$$

tiger2

[10]There is a certain doubling of notation with the previous paragraph but we hope that this will not cause confusion in what follows.

[11]See [1, Sec. 3, Sec. 4, Sec. 5] for the definition and basic properties of compactly supported Hyodo-Kato cohomology.

the map (3.7) clearly lifts to a map of φ-modules over $\mathbf{B}^{\mathrm{FF}}$:

$$\iota_{\mathrm{HK}}: \quad \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_C, r) \to \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR}}(X_C, r).$$

This Hyodo-Kato map descends to the level of nuclear quasi-coherent sheaves on X_{FF}:

$$\iota_{\mathrm{HK}}: \quad \mathcal{E}_{\mathrm{HK}}(X_C, r) \to \mathcal{E}_{\mathrm{dR}}(X_C, r).$$

Everything above has a version for compactly supported cohomologies (see [1, Sec. 3.2.2] for Hyodo-Kato morphisms), as well as for S-cohomologies, for $S \in \mathrm{Perf}_C$ (varying functorially in S).

sing2

3.3. **Syntomic cohomology.** We pass now to syntomic cohomology.

3.3.1. *Classical syntomic cohomology.* Let X be a smooth partially proper rigid analytic variety over K. Let $r \in \mathbf{N}$. Consider the classical syntomic cohomology (ala Bloch-Kato) (see [21, Sec. 5.4])

$$\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C) \otimes^{\mathrm{L}\square}_{\check{C}} \mathbf{B}^+_{\mathrm{st}}]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} \mathrm{R}\Gamma_{\mathrm{dR},?}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^r\big].$$

It satisfies the following comparison theorem:

comp0

Theorem 3.9. (Period isomorphism, [21, Th. 6.9])
Let $r \in \mathbf{N}$. There is a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\alpha_r: \quad \tau_{\le r}\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r)) \simeq \tau_{\le r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p(r)). \tag{3.10}$$

tea1

Moreover, it yields a natural quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\alpha_r: \quad \mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_C, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p(r)), \quad r \ge 2d.$$

Proof. Only the second claim requires justification. For the usual cohomology, this follows from quasi-isomorphism (3.10) and the fact that the complexes $\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn}}(X_C, \mathbf{Q}_p(r))$, $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r))$ live in the $[0, 2d]$-range. To see the latter fact in the case X is Stein, note that using (3.3) we get $H^{\mathbf{B}^+_{\mathrm{cr}},i}_{\mathrm{syn}}(X_C, \mathbf{Q}_p(r)) = 0$, for $i \ge d+1$. From this and (3.10) we get that $H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(d+j)) = 0$, for $d + j \ge i \ge d + 1, j \ge 1$, and then, by twisting, that $H^i_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) = 0$, for $i \ge d + 1$, as wanted. Now, for a general partially proper X, we need to add d for the analytic dimension of cohomology yielding the range $[0, 2d]$, as wanted.

For the cohomology with compact support, we argue similarly but using (3.4) instead of (3.3) in the case X is Stein. The case of partially proper X follows from that by a (co)-Čech argument. □

The above has a version in families. Let $S \in \mathrm{Perf}_C$ and let $r \in \mathbf{N}$. We have the classical (crystalline) syntomic cohomology in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:

$$\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C) \otimes^{\mathrm{L}\square}_{\check{C}} \mathbf{B}^+_{\mathrm{st}}(S)]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}\otimes\iota} \mathrm{R}\Gamma_{\mathrm{dR},?}(X_C/\mathbf{B}^+_{\mathrm{dR}}(S))/F^r\big] \tag{3.11}$$

kawa2

It satisfies the following comparison theorem:

comp01

Theorem 3.12. (Period isomorphism in families, [21, Cor. 7.37], [1, Prop. 6.16])
Let $r \in \mathbf{N}$. There is a natural, functorial in S, quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$

$$\alpha_r: \quad \tau_{\le r}\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) \simeq \tau_{\le r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p(r)). \tag{3.13}$$

tea2

Moreover, it yields a natural, functorial in S, quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$

$$\alpha_r: \quad \mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p(r)), \quad r \ge 2d.$$

Proof. The argument is analogous to the one used in the proof of Theorem 3.9. □

3.3.2. *Variants of syntomic cohomology.* We will need the following variant of syntomic cohomology in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:

$$\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,v]}}(X_S,\mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma_{\mathrm{HK},?}^{[u,v]}(X_S,r)]^{\varphi=1}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR},?}^{[u,v]}(X_S,r)\big],\quad r\in\mathbf{N}. \tag{3.14}$$ kawa11

grey1 **Lemma 3.15.** *Let $r\in\mathbf{N}$. There is a natural, functorial in S, quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:*

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,v]}}(X_S,\mathbf{Q}_p(r))\simeq\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{+}_{\mathrm{cr}}}(X_S,\mathbf{Q}_p(r)). \tag{3.16}$$ change1

Moreover, it yields a quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:

$$\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,v]}}(X_S,\mathbf{Q}_p(r))\simeq\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{+}_{\mathrm{cr}}}(X_S,\mathbf{Q}_p(r)),\quad r\geq d.$$

Proof. Let $\mathbf{B}_{S^\flat}^{[u,\infty]} := W(R^{\flat,+})\langle[p^\flat]/p^u\rangle[1/p]$. Define yet another variant of syntomic cohomology in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:

$$\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,\infty]}}(X_S,\mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma_{\mathrm{HK},?}^{[u,\infty]}(X_S,r)]^{\varphi=1}\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR},?}(X_S,r)\big].$$

The three different variants of syntomic cohomology introduced above are linked via maps

$$\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{+}_{\mathrm{cr}}}(X_S,\mathbf{Q}_p(r))\xrightarrow{f_1}\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,\infty]}}(X_S,\mathbf{Q}_p(r))\xrightarrow{f_2}\mathrm{R}\Gamma_{\mathrm{syn},?}^{\mathbf{B}^{[u,v]}}(X_S,\mathbf{Q}_p(r))$$

induced by canonical maps $\mathbf{B}^+_{\mathrm{cr}}(S)\to\mathbf{B}_{S^\flat}^{[u,\infty]}$, $\mathbf{B}_{S^\flat}^{[u,\infty]}\to\mathbf{B}_{S^\flat}^{[u,v]}$, and $\mathbf{B}_{S^\flat}^{[u,\infty]}\to\mathbf{B}_{S^\flat}^{[u,v/p]}$ (see [20, Sec. 2.4.2]). We claim that the map f_1 is a quasi-isomorphism and the map f_2 is a quasi-isomorphism after truncation $\tau_{\leq r}$. To show that, it suffices to prove that the related maps

$$\begin{aligned} f_1':&\quad [\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{st}}(S)]^{N=0,\varphi=p^r}\to[\mathrm{R}\Gamma_{\mathrm{HK},?}^{[u,\infty]}(X_S,r)]^{\varphi=1},\\ f_2':&\quad [\mathrm{R}\Gamma_{\mathrm{HK},?}^{[u,\infty]}(X_S,r)]^{\varphi=1}\to[\mathrm{R}\Gamma_{\mathrm{HK},?}^{[u,v]}(X_S,r)]^{\varphi=1}\end{aligned}$$

are quasi-isomorphisms in the wanted ranges. Or, first dropping (naively) $N=0$ and then log on both sides, that so are the maps

$$\begin{aligned} f_1':&\quad [\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{cr}}(S)]^{\varphi=p^j}\to[\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,\infty]}]^{\varphi=p^j},\quad j\in\mathbf{Z};\\ f_2':&\quad \tau_{\leq r}[\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,\infty]}]^{\varphi=p^s}\to\tau_{\leq r}[\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,v]}]^{\varphi=p^s},\quad s=r-1,r.\end{aligned}$$

Let us first look at the map f_1'. Taking cohomologies in degree $i\geq 0$, we get maps

$$f_1':\quad (H^i_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{cr}}(S))^{\varphi=p^j}\to(H^i_{\mathrm{HK},?}(X_C)\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,\infty]})^{\varphi=p^j}.$$

We used here [21, Prop. 5.8]. Since $H^i_{\mathrm{HK}}(X_C)$ and $H^i_{\mathrm{HK},c}(X_C)$ are a countable limit, resp. colimit, of finite rank φ-isocrystals over $\breve{C}$, we may assume that the Hyodo-Kato cohomology groups are finite rank. But then, since $\varphi(\mathbf{B}_{S^\flat}^{[u,\infty]})\subset\mathbf{B}^+_{\mathrm{cr}}(S)\subset\mathbf{B}_{S^\flat}^{[u,\infty]}$, it is clear that f_1' is an isomorphism, as wanted.

Concerning the map f_2', we first pass to cohomology in degree i and then assume that the Hyodo-Kato cohomology has finite rank as above. Let $j\in\mathbf{N}$. We then claim that the map

$$H^i_{\mathrm{HK}}(X_C)\{j\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,v]}\xrightarrow{1-\varphi}H^i_{\mathrm{HK}}(X_C)\{j\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,v/p]} \tag{3.17}$$ deszcz1

is surjective for $i\leq j$. Indeed, by Proposition 2.13, the complex (3.17) computes the cohomology of the vector bundle $\mathcal{E}_i$ on $X_{\mathrm{FF},S^\flat}$ associated to $H^i_{\mathrm{HK}}(X_C)\{j\}$. Our claim now follows from the fact that the slopes of Frobenius on $H^i_{\mathrm{HK}}(X_C)$ are $\leq i$ (see [21, proof of Prop. 5.20]) hence the slopes of $\mathcal{E}_i$ are ≥ 0 and $H^1(X_{\mathrm{FF},S^\flat},\mathcal{E}_i)=0$, as wanted.

Similarly, we see that the map

$$H^i_{\mathrm{HK},c}(X_C)\{j\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,v]}\xrightarrow{1-\varphi}H^i_{\mathrm{HK},c}(X_C)\{j\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}_{S^\flat}^{[u,v/p]} \tag{3.18}$$ deszcz11

is surjective for $j\geq d$ using the fact that the slopes of Frobenius on $H^i_{\mathrm{HK},c}(X_C)$ are in the $[i-d,d]$ range (use Poincaré duality for Hyodo-Kato cohomology to flip to the usual cohomology).

Now, it suffices to show that, for $i \in \mathbf{N}, j \geq -1$, the map

$$f_2' : \quad (H^i_{\mathrm{HK},?}(X_C) \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,\infty]}_{S^\flat})^{\varphi=p^j} \to (H^i_{\mathrm{HK},?}(X_C) \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v]}_{S^\flat})^{\varphi=p^j}$$

is an isomorphism. But in the case $S = C$ this follows from [8, Prop. 3.2] and the general case reduces to that one using the fact that all our algebras are spectral.

The above arguments prove the quasi-isomorphism in (3.16) for the usual cohomology and we get the statement for the compactly supported cohomology from the case of usual cohomology by a colim argument. Concerning the last sentence of our lemma, the above argument shows the case of compactly supported cohomology. For the usual cohomology, since the complex $\mathrm{R}\Gamma^{\mathbf{B}^+_{\mathrm{cr}}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r))$ lives in the $[0, 2d]$ range (see the proof of Theorem 3.9) it suffices to show that so does the complex $\mathrm{R}\Gamma^{\mathbf{B}^{[u,v]}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r))$. But here we can use the same argument as in the proof of Theorem 3.9. □

Remark 3.19. Bosco in [10, Th. 6.3] considered the following variant of syntomic cohomology in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:

$$\mathrm{R}\Gamma^{\mathrm{FF}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_S, r)]^{\varphi=1} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X_S, r)\big], \quad r \in \mathbf{N}.$$

Lemma 3.20. *The canonical map $\mathbf{B}_{S^\flat} \to \mathbf{B}^{[u,v]}_{S^\flat}$ induces a morphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$*

$$\mathrm{R}\Gamma^{\mathrm{FF}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma^{\mathbf{B}^{[u,v]}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)).$$

This is a quasi-isomorphism.

Proof. Arguing as in the proof of Lemma 3.15, it suffices to show that the induced morphism

$$[H^i_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}_{S^\flat} \xrightarrow{1-\varphi} H^i_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}_{S^\flat}] \to [H^i_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v]}_{S^\flat} \xrightarrow{1-\varphi} H^i_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}\square}_{\breve{C}} \mathbf{B}^{[u,v/p]}_{S^\flat}]$$

is a quasi-isomorphism in the case $H^i_{\mathrm{HK}}(X_C)$ is of finite rank. But this follows from Proposition 3.6. □

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3.3.3. *Syntomic φ-modules over $\mathbf{B}^{\mathrm{FF}}$.* Let X be a smooth partially proper rigid analytic variety over K.

Definition 3.21. Let $r \in \mathbf{N}$. Let $S \in \mathrm{Perf}_C$.

(1) Set

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},?}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},?}(X_S, r)].$$

This is a nuclear φ-module over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$. We call it a *syntomic module*. We have

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) = (\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)), \varphi),$$

where

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S, r)].$$

(2) The (nuclear) *syntomic sheaves* on $X_{\mathrm{FF},S}$ are defined by

$$\mathcal{E}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r))).$$

We have a distinguished triangle in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$

$$\mathcal{E}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) \to \mathcal{E}_{\mathrm{HK},?}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathcal{E}_{\mathrm{dR},?}(X_S, r). \tag{3.22}$$ def1

comp2 **Proposition 3.23.** *Let $r \geq 2d$. We have natural, functorial in S, quasi-isomorphisms in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathcal{E}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r))) \simeq \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)),$$
$$\mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathcal{E}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r))) \simeq \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p(r)).$$

Proof. The first quasi-isomorphism follows from Proposition 2.13. The second quasi-isomorphism follows from the first one, the quasi-isomorphism (3.16), and Theorem 3.12. □

4. Pro-étale complexes on the Fargues-Fontaine curve

In this section we define quasi-coherent sheaves on the Fargues-Fontaine curve representing p-adic (geometric) pro-étale cohomology (with $\mathbf{Q}_p(r)$-coefficients) of smooth partially proper rigid analytic varieties (Proposition 4.3) and prove a comparison theorem with the quasi-coherent sheaves representing syntomic cohomology (Proposition 4.4). This follows from a comparison theorem (Theorem 4.7 and Corollary 4.9) for the $\mathbb{B}$-period sheaf, which amounts to a comparison theorem on Y_{FF} instead of X_{FF}, i.e., to untangling comparison theorems from the action of φ (that this could be done with no much pain came to us as a surprise).

4.1. **Definitions.** We start with definitions.

4.1.1. *Twisted coefficients.* Let $S \in \mathrm{Perf}_C$. Let $n, k \geq 0$. Define the line bundle $\mathcal{O}(n,k)$ on $X_{\mathrm{FF},S^\flat}$ by the exact sequence of $\mathcal{O}_{\mathrm{FF},S^\flat}$-modules

$$0 \to \mathcal{O}(n,k) \to \mathcal{O}(n) \to i_{\infty,*}(\mathcal{O}/t^k) \to 0,$$

where the first map is an inclusion. The sheaf $\mathcal{O}(n,n)$ will be the target of our trace maps. Note that $\mathcal{O}(n,k)$ is just $\mathcal{O}(n-k)$ with (Galois-)Tate twist k; in particular, we have $H^0(X_{\mathrm{FF},S^\flat}, \mathcal{O}(n,n)) = \underline{\mathbf{Q}_p}(S)(n)$.

On the level of φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$, the sheaf $\mathcal{O}(n,k)$ is the module $\mathbf{B}_{S^\flat}\{n,k\}$ represented by the module $\mathbf{B}^{[u,v]}_{S^\flat}\{n,k\}$ defined by the exact sequence

$$0 \to \mathbf{B}^{[u,v]}_{S^\flat}\{n,k\} \to \mathbf{B}^{[u,v]}_{S^\flat}\{n\} \to \mathbf{B}^{[u,v]}_{S^\flat}\{n\}/t^k \to 0, \tag{4.1}$$ alter3

where the first map is an inclusion. We have $\mathbf{B}^{[u,v]}_{S^\flat}\{n,k\} \simeq \mathbf{B}^{[u,v]}_{S^\flat}\{n-k\}(k)$ as a Frobenius, Galois module. Note that the Frobenius map:

$$\varphi : \mathbf{B}^{[u,v]}_{S^\flat}\{n,k\} \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat},\varphi} \mathbf{B}^{[u,v/p]}_{S^\flat} \to \mathbf{B}^{[u,v]}_{S^\flat}\{n,k\} \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}} \mathbf{B}^{[u,v/p]}_{S^\flat}$$

is an isomorphism because it is isomorphic to the Frobenius on $\mathbf{B}^{[u,v/p]}_{S^\flat}\{n-k\}$.

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4.1.2. *Pro-étale modules and sheaves.* Let X be a smooth partially proper dagger variety over K. For $r \in \mathbf{N}$, $v' = v, v/p$, and $S \in \mathrm{Perf}_C$, we set

$$\mathrm{R}\Gamma^{[u,v']}_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbb{B}^{[u,v']})(r),$$

where $\mathbb{B}^{[u,v']}$ denotes the relative period sheaf corresponding to $\mathbf{B}^{[u,v']}$ (see [10, Sec. 2.3.1] for a description of condensed structure on these modules). We will need the following fact.

Lemma 4.2. *The canonical map*

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbb{B}^{[u,v]}) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}} \mathbf{B}^{[u,v/p]}_{S^\flat} \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbb{B}^{[u,v/p]})$$

is a quasi-isomorphism.

Proof. By pro-étale descent, it suffices to show that, for a set of perfectoid affinoids $\{S_i\}$, $i \in I$, the canonical map

$$(\prod_I \mathbf{B}^{[u,v]}_{S_i^\flat}) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}} \mathbf{B}^{[u,v/p]}_{S^\flat} \to \prod_I \mathbf{B}^{[u,v/p]}_{S_i^\flat}$$

is a quasi-isomorphism. (We used here the fact that $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(S_i, \mathbb{B}^J) \simeq \mathbf{B}^J_{S^\flat}$.) But this follows from the fact that this tensor product commutes with derived limits (see the proof of Lemma 2.4) and the canonical map

$$\mathbf{B}^{[u,v]}_{S_i^\flat} \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}} \mathbf{B}^{[u,v/p]}_{S^\flat} \to \mathbf{B}^{[u,v/p]}_{S_i^\flat}$$

is an isomorphism. To see the last claim, we compute

$$\mathbf{B}^{[u,v]}_{S_i^\flat} \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}} \mathbf{B}^{[u,v/p]}_{S^\flat} \simeq \mathbf{B}^{[u,v]}_{S_i^\flat} \otimes^{\mathrm{L}}_{(\mathbf{Z}[T],\mathbf{Z})_\square} (\mathbf{Z}[T], \mathbf{Z}[T])_\square \simeq \mathbf{B}^{[u,v/p]}_{S_i^\flat},$$

where we wrote $\mathbf{B}^{[u,v/p]}_{S^\flat} \simeq \mathbf{B}^{[u,v]}_{S^\flat}\langle f\rangle$ for $f = (p/[p^\flat]^{p/v}) \in \mathbf{B}^{[u,v]}_{S^\flat}$. □

We define the *pro-étale modules* as the pairs

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)) := (\mathrm{R}\Gamma^{[u,v]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)), \varphi),$$
$$\varphi : \mathrm{R}\Gamma^{[u,v]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma^{[u,v/p]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)),$$

where the Frobenius φ is induced by the Frobenius $\varphi : \mathbb{B}^{[u,v]} \to \mathbb{B}^{[u,v/p]}$. It yields a quasi-isomorphism in $\mathcal{D}(\mathbf{B}^{[u,v/p]}_{S^\flat,\square})$

$$\varphi : \mathrm{R}\Gamma^{[u,v]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat},\varphi} \mathbf{B}^{[u,v/p]}_{S^\flat} \xrightarrow{\sim} \mathrm{R}\Gamma^{[u,v/p]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)).$$

Indeed, it suffices to show that the Frobenius map

$$\varphi : \mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbb{B}^{[u,v]}) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat},\varphi} \mathbf{B}^{[u,v/p]}_{S^\flat} \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbb{B}^{[u,v']})$$

is a quasi-isomorphism. But this follows directly from [10, Lemma 4.8].

The pairs $\mathrm{R}\Gamma^{\mathbf{B}}_{\text{proét},?}(X_S, \mathbf{Q}_p(r))$ defines nuclear φ-complexes (actually $(\varphi, \mathcal{G}_K)$-complexes) over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$, which we will call *pro-étale modules*. For the nuclear property use [10, Lemma 6.15] plus preservation of nuclearity by countable products and finite limits for the usual cohomology. The case of cohomology with compact support follows since colimits preserve nuclearity. We will denote by

$$\mathcal{E}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)) := \mathcal{E}_{\mathrm{FF}}(\mathrm{R}\Gamma^{\mathbf{B}}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)))$$

the corresponding nuclear quasi-coherent sheaves on $X_{\mathrm{FF},S^\flat}$. We will call them *pro-étale sheaves*. Pro-étale modules and sheaves are functors on Perf_C.

comp2b **Proposition 4.3.** *We have a natural, functorial in S, quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$:*

$$\mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathcal{E}_{\text{proét},?}(X_S, \mathbf{Q}_p(r))) \simeq \mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbf{Q}_p(r)).$$

Proof. By Proposition 2.13 we have natural, functorial in S, quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathcal{E}_{\text{proét},?}(X_S, \mathbf{Q}_p(r))) &\simeq [\mathrm{R}\Gamma^{[u,v]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)) \xrightarrow{\varphi-1} \mathrm{R}\Gamma^{[u,v/p]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r))] \\ &\simeq [\mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbb{B}^{[u,v]})(r) \xrightarrow{\varphi-1} \mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbb{B}^{[u,v/p]})(r)] \\ &\xleftarrow{\sim} \mathrm{R}\Gamma_{\text{proét},?}(X_S, \mathbf{Q}_p(r)).\end{aligned}$$

Here, in the last quasi-isomorphism, we have used the exact sequence (see [20, Lemma 2.23])

$$0 \to \mathbf{Q}_p \to \mathbb{B}^{[u,v]} \xrightarrow{\varphi-1} \mathbb{B}^{[u,v/p]} \to 0$$ □

4.2. **Comparison theorems on the Fargues-Fontaine curve.** We move now to the comparison theorems on the two curves of Fargues-Fontaine.

4.2.1. *Comparison theorem on the X_{FF}-curve.* W start with the "bottom" curve. Let X be a smooth partially proper variety over K, of dimension d.

china1 **Proposition 4.4.** *Let $r \geq 2d$. There is a natural, functorial in S, quasi-isomorphism in* $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$*:*

$$\alpha_r : \quad \mathcal{E}_{\text{syn},?}(X_S, \mathbf{Q}_p(r)) \simeq \mathcal{E}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)). \tag{4.5}$$ hot11

Proof. It suffices to construct a natural quasi-isomorphism of φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\text{syn},?}(X_S, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma^{\mathbf{B}}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)).$$

That is, a natural quasi-isomorphism of pairs

$$(\mathrm{R}\Gamma^{[u,v]}_{\text{syn},?}(X_S, \mathbf{Q}_p(r)), \varphi) \simeq (\mathrm{R}\Gamma^{[u,v]}_{\text{proét},?}(X_S, \mathbf{Q}_p(r)), \varphi).$$

But this follows from a "Frobenius untwisted" version of Theorem 3.12 presented in Theorem 4.7 below. We just have to argue that we can drop truncations in 4.8: but this follows from the fact that both sides live in degrees $[0, 2d]$, which can be seen as in the proof of Theorem (3.9)). □

Remark 4.6. We did not list the truncated version of Theorem 3.12 in Proposition 4.4 because the issue of truncation vis a vis localization is a subtle one.

4.2.2. *Comparison theorem on the Y_{FF}-curve.* We pass now to the "top" curve.

hot1 **Theorem 4.7.** (Comparison theorem on the Y_{FF}-curve) *Let X be a smooth partially proper variety over K. Let $r \geq 0$. We have natural, functorial in S, and compatible with Frobenius quasi-isomorphisms in $\mathcal{D}(\mathbf{B}^{[u,v]}_{S^\flat,\square})$ and $\mathcal{D}(\mathbf{B}^{[u,v/p]}_{S^\flat,\square})$, respectively:*

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,\mathbb{B}^{[u,v]})(r) \simeq \tau_{\leq r}[\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r)\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r)], \tag{4.8}$$
$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,\mathbb{B}^{[u,v/p]})(r) \simeq \mathrm{R}\Gamma^{[u,v/p]}_{\mathrm{HK},?}(X_S,r).$$

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Proof. For $v' = v, v/p$, we define $F^r\mathbb{B}^{[u,v']} := t^r\mathbb{B}^{[u,v']}$. We clearly have the isomorphism $t^r : \mathbb{B}^{[u,v']}(r) \xrightarrow{\sim} F^r\mathbb{B}^{[u,v']}\{r\}$. We want to construct natural, functorial in S and compatible with Frobenius, quasi-isomorphisms in $\mathcal{D}(\mathbf{B}^{[u,v]}_{S^\flat,\square})$ and $\mathcal{D}(\mathbf{B}^{[u,v/p]}_{S^\flat,\square})$, respectively:

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,F^r\mathbb{B}^{[u,v]})\{r\} \simeq \tau_{\leq r}[\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r)\to\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r)],$$
$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,F^r\mathbb{B}^{[u,v/p]})\{r\} \simeq \mathrm{R}\Gamma^{[u,v/p]}_{\mathrm{HK},?}(X_S,r).$$

For the usual cohomology, these quasi-isomorphisms were constructed in [21, Sec. 7]. They are not explicitly stated there because we almost always carry through the constructions the eigenspaces of Frobenius but, in fact, the latter can be dropped as they are only used to pass between various period rings and here we work with one fixed period ring. For the gist of the construction the interested reader should consult the diagram (7.16) (with the top row moved a step lower and with added $[u,v]$-decoration), its refinement (7.31), Section 7.4 in general, and diagram (7.36) (with decoration changed again to $[u,v]$) in particular.

The case of compactly supported cohomology follows now easily from the case of usual cohomology by taking colimits and finite limits. □

The following result follows easily from the above theorem though it will not be used in this paper.

hot1K **Corollary 4.9.** (𝔹-comparison theorem) *Let X be a smooth partially proper variety over K. Let $r \geq 0$.*

(1) Let $I = [u,v] \subset (0,\infty)$ be a compact interval with rational endpoints containing the fixed intervals from Section 2.2.2. We have a natural, functorial in S, quasi-isomorphism in $\mathcal{D}(\mathbf{B}^I_{S^\flat,\square})$:

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,\mathbb{B}^I)(r) \simeq \tau_{\leq r}[\mathrm{R}\Gamma^I_{\mathrm{HK},?}(X_S,r)\xrightarrow{\iota^I_{\mathrm{HK}}}\mathrm{R}\Gamma^I_{\mathrm{dR},?}(X_S,r)].$$

This quasi-isomorpism is also compatible with Frobenius, i.e., the following diagram commutes (we set $u' = u/p, v' = v/p$)

$$\begin{array}{ccc}
\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,\mathbb{B}^{[u,v]})(r) & \xrightarrow{\sim} & \tau_{\leq r}[\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r)\xrightarrow{\iota^{[u,v]}_{\mathrm{HK}}}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r)] \\
\wr\downarrow\varphi & & \wr\downarrow\varphi \\
\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S,\mathbb{B}^{[u',v']})(r) & \xrightarrow{\sim} & \tau_{\leq r}[\mathrm{R}\Gamma^{[u',v']}_{\mathrm{HK},?}(X_S,r)\xrightarrow{\iota^{[u',v']}_{\mathrm{HK}}}\mathrm{R}\Gamma^{[u',v']}_{\mathrm{dR},?}(X_S,r)].
\end{array}$$

(2) We have a natural, functorial in S, and compatible with Frobenius quasi-isomorphism in $\mathcal{D}(\mathbf{B}_{S^\flat,\square})$:

$$\tau_{\leq r}\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S,\mathbb{B})(r) \simeq \tau_{\leq r}[\mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{HK}}(X_S,r)\xrightarrow{\iota^{\mathbb{B}}_{\mathrm{HK}}}\mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{dR}}(X_S,r)]$$

For $r \geq 2d$, this yields a quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S,\mathbb{B})(r) \simeq [\mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{HK}}(X_S,r)\xrightarrow{\iota^{\mathbb{B}}_{\mathrm{HK}}}\prod_I \mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{dR}}(X_S,r)]. \tag{4.10}$$

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Remark 4.11. (1) In claim (1) above, we have set

$$\mathrm{R}\Gamma^I_{\mathrm{dR},?}(X_S,r) := \bigoplus_{\mathbf{Z}(I)} (\mathrm{R}\Gamma_{\mathrm{dR},?}(X) \otimes^{\mathrm{L}_\square}_K \mathbb{B}^+_{\mathrm{dR}}(S))/F^r,$$

where $\mathbf{Z}(I) := \{n \in \mathbf{Z} | \varphi^n(y_\infty) \in Y^I_{\mathrm{FF},S^\flat}\}$. The Hyodo-Kato morphism ι^I_{HK} in degree $n \in \mathbf{Z}(I)$ is defined by precomposing the usual Hyodo-Kato morphism ι_{HK} with φ^{-n}. It is $\mathbb{B}^{[u,v]}(S)$-linear via the composition

$$\mathbb{B}^{[u,v]}(S) \xrightarrow{\varphi^{-n}} \mathbb{B}^{[p^n u, p^n v]}(S) \to \mathbb{B}^{[p^n u, p^n v]}(S)/\xi^r_\infty \simeq \mathbb{B}^+_{\mathrm{dR}}(S)/t^r,$$

where ξ_∞ is a generator of the ideal defining y_∞.

(2) In claim (2) above, we have set

$$\mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{dR}}(X_S,r) := \prod_{\mathbf{Z}} ((\mathrm{R}\Gamma_{\mathrm{dR},?}(X) \otimes^{\mathrm{L}_\square}_K \mathbb{B}^+_{\mathrm{dR}}(S))/F^r),$$
$$\mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{HK}}(X_S,r) := [\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\} \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbb{B}_{\log}(S)]^{N=0}.$$

Proof. The first quasi-isomorphism is proven in the same way as Theorem 4.7. The second one follows from the first one by passing to limits (we use here heavily that we may assume the Hyodo-Kato cohomology to be of finite rank) once we know that $\mathrm{R}^i \lim_{I \subset (0,\infty)} \mathbb{B}^I(S) = 0, i > 0$, where the limit is taken over compact intervals I with rational endpoints (recall that we have $\lim_{I \subset (0,\infty)} \mathbb{B}^I(S) \simeq \mathbb{B}(S)$). But this was checked in [10, proof of Lemma 2.41]. □

Remark 4.12. (1) For $r \geq 2d$, as an immediate consequence of (4.10), we get the quasi-isomorphism :

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S, \mathbb{B})(r)[1/t] \simeq \mathrm{R}\Gamma^{\mathbb{B}}_{\mathrm{HK}}(X_S,r)[1/t].$$

(2) Bosco in [10] proved a version of the comparison quasi-isomorphism (4.10), where the torsion on the right-hand side is incorporated to the left-hand side via the $\mathrm{L}\,\eta_t$ operator.

5. Poincaré dualities on the Fargues-Fontaine curve

We are now ready to state and prove pro-étale duality on the Fargues-Fontaine curve. The same techniques allow us to prove also pro-étale Künneth formula.

sing3

5.1. **Hyodo-Kato and de Rham dualities.** Let X be a smooth partially proper rigid analytic variety over K, of dimension d.

5.1.1. *De Rham dualities.* Recall the following dualities (see [1, Cor. 5.18, Th. 5.23, Cor. 5.26]).

derhamduality

Proposition 5.1. *Let $L = K, C$.*

(1) (Serre duality) *There is a trace map of solid L-modules*

$$\mathrm{Tr}_{\mathrm{coh}} : \mathrm{R}\Gamma_c(X_L, \Omega^d)[d] \to L.$$

The pairing

$$\mathrm{R}\Gamma(X_L, \Omega^j) \otimes^{\mathrm{L}_\square}_L \mathrm{R}\Gamma_c(X_L, \Omega^{d-j})[d] \to \mathrm{R}\Gamma_c(X_L, \Omega^d)[d] \xrightarrow{\mathrm{Tr}_{\mathrm{coh}}} L$$

is perfect, i.e., it yields the quasi-isomorphism in $\mathcal{D}(L_\square)$:

$$\mathrm{R}\Gamma(X_L, \Omega^j) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_c(X_L, \Omega^{d-j})[d], L).$$

(2) (Filtered de Rham duality) *There are natural trace maps in $\mathcal{D}(L_\square)$ and $L_\square$, respectively:*

$$\mathrm{Tr}_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d] \to L, \quad \mathrm{Tr}_{\mathrm{dR}} : H^{2d}_{\mathrm{dR},c}(X_L) \to L.$$

(a) The pairing in $\mathcal{D}(L_\square)$

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)\otimes^{\mathrm{L}_\square}_{L}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d]\to\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d]\xrightarrow{\mathrm{Tr}_{\mathrm{dR}}}L$$

is a perfect duality, i.e., we have induced quasi-isomorphism in $\mathcal{D}(L_\square)$

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d],L).$$

(b) More generally, let $r,r'\in\mathbf{N}, r+r'=d$. *The pairing in* $\mathcal{D}(L_\square)$

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)/F^{r'+1})\otimes^{\mathrm{L}_\square}_{L}F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d]\to\mathrm{R}\Gamma_{\mathrm{dR},c}(X_L)[2d]\xrightarrow{\mathrm{Tr}_{\mathrm{dR}}}L$$

is a perfect duality, i.e., we have induced quasi-isomorphisms in $\mathcal{D}(L_\square)$

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)/F^{r'+1}\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X)[2d],L),$$
$$F^{r'+1}\mathrm{R}\Gamma_{\mathrm{dR}}(X_L)\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{L_\square}(\mathrm{R}\Gamma_{\mathrm{dR},c}(X)/F^r[2d],L).$$

5.1.2. $\mathbf{B}^+_{\mathrm{dR}}$*-dualities.* The duality for $\mathbf{B}^+_{\mathrm{dR}}$-cohomology has a slightly different form. For $r\geq d$, a natural trace map in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$ can be defined by the composition

$$\mathrm{Tr}_{\mathbf{B}^+_{\mathrm{dR}}}:\quad F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d]\to\mathrm{R}\Gamma_c(X,\Omega^d)\otimes^{\mathrm{L}_\square}_{K}F^{r-d}\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\mathrm{Tr}_{\mathrm{coh}}\otimes\mathrm{Id}}F^{r-d}\mathbf{B}^+_{\mathrm{dR}}.$$

bdrduality **Corollary 5.2.** (Filtered $\mathbf{B}^+_{\mathrm{dR}}$-duality [1, Cor. 5.27]) *Let* $r,r'\geq d, s=r+r'-d$. *The pairing in* $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$

$$F^{r'}\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})\otimes^{\mathrm{L}_\square}_{\mathbf{B}^+_{\mathrm{dR}}}F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d]\to F^{r'+r}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d]\xrightarrow{\mathrm{Tr}_{\mathbf{B}^+_{\mathrm{dR}}}}F^s\mathbf{B}^+_{\mathrm{dR}}$$

is a perfect duality, i.e., we have an induced quasi-isomorphism in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$

$$F^{r'}\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^+_{\mathrm{dR},\square}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d],F^s\mathbf{B}^+_{\mathrm{dR}}).$$

We will need a variant of the above result. To state it, take $r,r'\geq d, s=r+r'-d$ and consider the pairing in $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^{r'})\otimes^{\mathrm{L}_\square}_{\mathbf{B}^+_{\mathrm{dR}}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/t^s)[2d-1]\to F^s\mathbf{B}^+_{\mathrm{dR}}\qquad(5.3)$$ ias223

defined as the composition

$$(\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^{r'})\otimes^{\mathrm{L}_\square}_{\mathbf{B}^+_{\mathrm{dR}}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/t^s)[2d-1]\xrightarrow{\cup}F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/t^s[2d-1]$$
$$\to\mathrm{R}\Gamma_c(X_C,\Omega^d)\otimes^{\mathrm{L}_\square}_{K}(F^{r-d}\mathbf{B}^+_{\mathrm{dR}}/t^s)[-1]\xrightarrow{\partial}\mathrm{R}\Gamma_c(X_C,\Omega^d)\otimes^{\mathrm{L}_\square}_{K}F^s\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\mathrm{Tr}_{\mathrm{coh}}\otimes\mathrm{Id}}F^s\mathbf{B}^+_{\mathrm{dR}}$$

Here the third morphism is the boundary map induced by the exact sequence

$$0\to F^s\mathbf{B}^+_{\mathrm{dR}}\xrightarrow{\mathrm{can}}F^{r-d}\mathbf{B}^+_{\mathrm{dR}}\to F^{r-d}\mathbf{B}^+_{\mathrm{dR}}/t^s\to 0$$

bdrduality1 **Corollary 5.4.** *The pairing* (5.3) *is a perfect duality, i.e., we have an induced quasi-isomorphism in* $\mathcal{D}(\mathbf{B}^+_{\mathrm{dR},\square})$

$$\gamma:\quad\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^{r'}\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^+_{\mathrm{dR},\square}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/t^s[2d-1],F^s\mathbf{B}^+_{\mathrm{dR}}).\qquad(5.5)$$ niedziela1

Proof. Consider the following map of distinguished triangles

$$\begin{array}{ccc}
F^{r'}\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}) & \xrightarrow{\sim} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^+_{\mathrm{dR},\square}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d],F^s\mathbf{B}^+_{\mathrm{dR}}) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}}) & \xrightarrow{\sim} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^+_{\mathrm{dR},\square}}(t^s\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})[2d],F^s\mathbf{B}^+_{\mathrm{dR}}) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^{r'} & \xrightarrow{\gamma} & \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^+_{\mathrm{dR},\square}}(F^r\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/t^s[2d-1],F^s\mathbf{B}^+_{\mathrm{dR}})
\end{array}$$

where the middle arrow is the de Rham duality map ($\mathbf{B}^+_{\mathrm{dR}}$-linearized) and the top arrow is the $\mathbf{B}^+_{\mathrm{dR}}$-duality map from Corollary 5.2. Both are quasi-isomorphisms (see Proposition 5.1). Hence so is the bottom duality map, as wanted. □

The duality map (5.5) can be lifted to the Fargues-Fontaine curve: the pairing (5.3) induces a pairing of $\mathbf{B}^{[u,v]}_{S^\flat}$-modules

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_S,r)\otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}}(F^{r'}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},c}(X_S)/t^s)\to\mathbf{B}^{[u,v]}_{S^\flat}\{s,s\}[-2d+1],$$

which, in turn, induces a pairing of nuclear φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR}}(X_S,r)\otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}}(F^{r'}\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},c}(X_S)/t^s)\to\mathbf{B}_{S^\flat}\{s,s\}[-2d+1],$$

where we set $F^{r'}\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},c}(X_S)/t^s:=(F^{r'}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},c}(X_S)/t^s,0)$. This descends to a pairing on $X_{\mathrm{FF},S^\flat}$:

$$\mathcal{E}_{\mathrm{dR}}(X_S,r)\otimes^{\mathrm{L}}_{\mathcal{O}} i_{\infty,*}(F^{r'}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^s)\to\mathcal{O}(s,s)[-2d+1], \tag{5.6}$$ niedziela5

where we set $\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}}):=\mathrm{R}\Gamma_{\mathrm{dR},c}(X_K)\otimes^{\mathrm{L}_\square}_K\mathbf{B}^+_{\mathrm{dR}}(S)$. The pairing (5.6) induces a duality map in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$:

$$\gamma_{X_S}:\quad \mathcal{E}_{\mathrm{dR}}(X_S,r)\to\mathbb{D}(i_{\infty,*}(F^{r'}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^s)[2d-1],\mathcal{O}(s,s)), \tag{5.7}$$ niedziela3

where we set

$$\mathbb{D}(-,-):=\mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})}(-,-).$$

kolobrzeg5a **Lemma 5.8.** *The duality map* (5.7) *is a quasi-isomorphism.*

Proof. We need to show that the duality map

$$\gamma^{\mathrm{FF}}_{X_S}:\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR}}(X_S,r)\to\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}}(F^{r'}\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},c}(X_S)/t^s[2d-1],\mathbf{B}_{S^\flat}\{s,s\})$$

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S^\flat})$. Or, passing to solid $\mathbf{B}':=\mathbf{B}^{[u,v]}_{S^\flat}$-modules, that the duality map

$$\gamma_{X_S}:\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_S,r)\to\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(F^{r'}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},c}(X_S)/t^s[2d-1],\mathbf{B}')$$

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}'_\square)$. But this is Corollary 5.4 (strictly speaking, its S-version but it holds by the same arguments). □

5.1.3. *Hyodo-Kato duality.* This is based on [1, Sec. 5.4]. There exists a natural trace map in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)$:

$$\mathrm{Tr}_X:\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\to\breve{C}\{-d\}[-2d].$$

The pairing in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)$ ($s=r+r'-d$)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\}\otimes^{\mathrm{L}_\square}_{\breve{C}}\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r'\}\to\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r+r'\}\xrightarrow{\mathrm{Tr}_X}\breve{C}\{s\}[-2d] \tag{5.9}$$ ias22

is perfect, i.e., it induces a quasi-isomorphism in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)$

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\}\simeq\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_\square)}(\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r'\},\breve{C}\{s\}[-2d]), \tag{5.10}$$ HKduality

where the internal Hom is just $\mathrm{R}\mathcal{H}om_{\breve{C}_\square}(\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C),\breve{C}[-2d])$ – the internal Hom in $\mathcal{D}(\breve{C}_\square)$ – equipped with $(\varphi,N,\mathcal{G}_K)$-actions via $\mathrm{R}\Gamma_{\mathrm{HK},c}(X_C)\{r'-s\}$.

The above duality can be lifted to the Fargues-Fontaine curve: the pairing (5.9) induces a pairing of $\mathbf{B}^{[u,v]}_{S^\flat}$-modules

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,r)\otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{S^\flat}}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r')\to\breve{C}\{s\}\otimes^{\mathrm{L}_\square}_{\breve{C}}\mathbf{B}^{[u,v]}_{S^\flat}[-2d],$$

which, in turn, induces a pairing of nuclear φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_S,r)\otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}}\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},c}(X_S,r')\to\mathbf{B}_{S^\flat}\{s\}[-2d].$$

This descends to a pairing on $X_{\mathrm{FF},S^\flat}$:

$$\mathcal{E}_{\mathrm{HK}}(X_S,r)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{E}_{\mathrm{HK}}(X_S,r')\to\mathcal{O}(s)[-2d],$$

which induces a duality map in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$:

$$\gamma_{X_S}:\quad \mathcal{E}_{\mathrm{HK}}(X_S,r)\to\mathbb{D}(\mathcal{E}_{\mathrm{HK},c}(X_S,r')[2d],\mathcal{O}(s)). \tag{5.11}$$ niedziela11

kolobrzeg4a **Lemma 5.12.** *The map γ_{X_S} above is a quasi-isomorphism in* $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$.

Proof. Since $\mathbf{B}^{[u,v]}_{S^\flat}$ is $\mathbf{B}^{[u,v]}_{S^\flat,\mathrm{an}}$-complete (see [2, Lemma 3.24]), by Remark 2.6, we may pass from $\mathbf{B}^{[u,v],+}_{S^\flat}$ to $\mathbf{Z}$, i.e., to $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$-modules. Hence we need to show that the duality map

$$\gamma^{\mathrm{FF}}_{X_S}:\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK}}(X_S,r)\to\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}}(\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},c}(X_S,r')[2d],\mathbf{B}_{S^\flat}\{s\})$$

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{S^\flat})$. Or, passing to solid $\mathbf{B}':=\mathbf{B}^{[u,v]}_{S^\flat}$-modules, that the duality map

$$\gamma_{X_S}:\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,r)\to\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r')[2d],\mathbf{B}') \tag{5.13}$$ ias23

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}'_\square)$. We claim that, for that, it suffices to check that, for $j\in\mathbf{N}$, the duality map on cohomology groups level

$$\gamma^j_{X_S}:H^{[u,v],j}_{\mathrm{HK}}(X_S,r)\to\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(H^{[u,v],2d-j}_{\mathrm{HK},c}(X_S,r'),\mathbf{B}') \tag{5.14}$$ ias231

is an isomorphism in $\mathbf{B}'_\square$. Indeed, passing to cohomology in (5.13), we need to check that the duality map

$$\gamma^j_{X_S}:H^{[u,v],j}_{\mathrm{HK}}(X_S,r)\to H^j(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r')[2d],\mathbf{B}'))$$

is an isomorphism in $\mathbf{B}'_\square$. But $H^i_{\mathrm{HK},c}(X_C)$ is a direct sum of copies of $\breve{C}$ hence we have

$$H^j(\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r')[2d],\mathbf{B}'))\simeq\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(H^{[u,v],2d-j}_{\mathrm{HK},c}(X_S,r'),\mathbf{B}'),$$

as wanted.

To prove (5.14), we observe that, for $i\in\mathbf{Z}$, we have the natural isomorphisms[12]

$$H^{[u,v],j}_{\mathrm{HK},?}(X_S,i)\simeq(H^j_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\square}_{\breve{C}}\mathbf{B}'_{\log})^{N=0}\xleftarrow{\sim}H^j_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\square}_{\breve{C}}\mathbf{B}'. \tag{5.15}$$ evening1

Here the second quasi-isomorphism is defined by the map $\exp(NU)$ (this makes sense because the monodromy operator on the Hyodo-Kato cohomology $H^j_{\mathrm{HK},?}(X_C)$ is nilpotent). For the first quasi-isomorphism

$$H^{[u,v],j}_{\mathrm{HK},?}(X_S,i)=H^j([\mathrm{R}\Gamma_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}'_{\log}]^{N=0})\simeq(H^j_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\square}_{\breve{C}}\mathbf{B}'_{\log})^{N=0}$$

we used the fact that

$$H^j(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\mathrm{L}\square}_{\breve{C}}\mathbf{B}'_{\log})\simeq H^{[u,v],j}_{\mathrm{HK},?}(X_C)\{i\}\otimes^{\square}_{\breve{C}}\mathbf{B}'_{\log},$$

that N is nilpotent on $H^{[u,v],j}_{\mathrm{HK},?}(X_C)$ (so we can do devissage by the kernels of the action of N), and that $\mathbf{B}'\xrightarrow{\sim}[\mathbf{B}'_{\log}]^{N=0}$.

It is easy to check that the maps in (5.15) are compatible with products. Hence we can write the duality map (5.14) as the Hyodo-Kato pairing

$$\gamma^j_{X_S}:H^j_{\mathrm{HK}}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}'\to\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(H^{2d-j}_{\mathrm{HK},c}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}',\mathbf{B}').$$

To show that it is an isomorphism in $\mathbf{B}'_\square$ it suffices thus to evoke the Hyodo-Kato duality (5.10) and to show that the natural map

$$\underline{\mathrm{Hom}}_{\breve{C}_\square}(H^{2d-j}_{\mathrm{HK},c}(X_C),\breve{C})\otimes^{\square}_{\breve{C}}\mathbf{B}'\to\underline{\mathrm{Hom}}_{\breve{C}_\square}(H^{2d-j}_{\mathrm{HK},c}(X_C),\mathbf{B}')$$

is an isomorphism in $\mathbf{B}'_\square$. But this is an isomorphism by [30, Th. 3.40] since $\mathbf{B}'$ is a Banach space over $\breve{C}$. □

[12]We can ignore the Galois action here.

5.2. **Syntomic duality.** Let X be a smooth partially proper rigid analytic variety over K of dimension d. Let $S \in \mathrm{Perf}_C$. Recall that syntomic φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$ are defined as (see Sec. 3.3.3)

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},?}(X_S, \mathbf{Q}_p(r)) := [\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{HK},?}(X_S, r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},?}(X_S, r)],$$

where the Hyodo-Kato map is described by diagram (3.8). The Hyodo-Kato and de Rham cup products are compatible with this diagram hence yield a cup product on the syntomic φ-modules:

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \to \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')).$$

This product can be described by an analogous product on the $\mathbf{B}' := \mathbf{B}^{[u,v]}_{S^\flat}$-chart:

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \to \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')). \tag{5.16}$$ cup11

It is compatible with the products on $\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S, i)$ and $F^i\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{dR},?}(X_S/\mathbf{B}')$. Here we defined $F^i\mathrm{R}\Gamma_{\mathrm{dR},?}(X_S/\mathbf{B}')$ as $F^i\mathrm{R}\Gamma_{\mathrm{dR},?}(X_C/\mathbf{B}^+_{\mathrm{dR}})$ with $\mathbf{B}^+_{\mathrm{dR}}$ replaced by $\mathbf{B}'$.

Let $s \geq d$. There is a trace map

$$\mathrm{Tr}_X : \quad \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s)) \to \mathbf{B}^{\mathrm{FF}}_{S^\flat}\{s-d, s-d\}[-2d]$$

defined on the $\mathbf{B}'$-chart via the trace map

$$\mathrm{Tr}^{[u,v]}_X : \quad \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s)) \to \mathbf{B}'\{s-d, s-d\}[-2d], \tag{5.17}$$ trace11

which is compatible with the Hyodo-Kato and de Rham trace maps. The map $\mathrm{Tr}^{[u,v]}_X$ is defined using the exact sequence

$$H^{[u,v],2d}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s)) \to H^{2d}_{\mathrm{HK},c}(X_S, s) \xrightarrow{\iota_{\mathrm{HK}}} H^{2d}_{\mathrm{dR},c}(X_S, s),$$

which can be written more explicitly as the exact sequence

$$H^{[u,v],2d}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s)) \to (H^{2d}_{\mathrm{HK},c}(X_C)\{s\} \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbf{B}'_{\log})^{N=0} \xrightarrow{\iota_{\mathrm{HK}}} H^{2d}_{\mathrm{dR},c}(X) \otimes^{\mathrm{L}_\square}_K (\mathbf{B}'\{s-d\}/F^{s-d}). \tag{5.18}$$ lunch1

Using the (compatible) Hyodo-Kato and de Rham trace maps

$$\mathrm{Tr}_X : H^{2d}_{\mathrm{HK},c}(X_C)\{s\} \xrightarrow{\sim} \breve{C}\{s-d\}, \quad \mathrm{Tr}_X : H^{2d}_{\mathrm{dR},c}(X) \xrightarrow{\sim} K,$$

(5.18) yields a map

$$H^{[u,v],2d}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s-d)) \to \mathrm{Ker}(\mathbf{B}'\{s-d\} \to \mathbf{B}'\{s-d\}/F^{s-d}) = \mathbf{B}'\{s-d, s-d\},$$

hence the trace (5.17), as wanted.

For $s := r + r' - d$, the above can be lifted to the Fargues-Fontaine curve: the cup product (5.16) and trace map (5.17) induce a pairing of $\mathbf{B}'$-modules

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}^{[u,v]}_X} \mathbf{B}'\{s,s\}[-2d],$$

which, in turn, induces a pairing of nuclear φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathbf{B}_{S^\flat}\{s,s\}[-2d].$$

This descends to a pairing in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$:

$$\mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}}_{\mathcal{O}} \mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathcal{O}(s,s)[-2d],$$

which induces a natural map $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$

$$\gamma_{X_S} : \quad \mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \to \mathbb{D}(\mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s,s)). \tag{5.19}$$ niedziela10

curve-duality **Theorem 5.20.** (Syntomic Poincaré duality on the Fargues-Fontaine curve)
Let $r, r' \geq 2d$, $s := r + r' - d$. The map γ_{X_S} is a quasi-isomorphism in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$.

Proof. It is enough to show this in φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$ for the corresponding map

$$\gamma_{X_S}: \quad \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn}}(X_S,\mathbf{Q}_p(r)) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}}(\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],\mathbf{B}_{S^\flat}\{s,s\}). \tag{5.21}$$ kolobrzeg3

Or in $\mathcal{D}(\mathbf{B}^{[u,v]}_{S^\flat,\square})$ for the induced map

$$\gamma^{[u,v]}_{X_S}: \quad \mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(X_S,\mathbf{Q}_p(r)) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],\mathbf{B}'(s)).$$

But for that, it is enough to check that base changes of $\gamma^{[u,v]}_{X_S}$ to both $\mathbf{B}'[1/t]$ and $\mathbf{B}'/t$ are quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_\square)$. This last claim requires a bit of justification. We have the exact sequence of solid $\mathbf{B}'$-modules

$$0 \to \mathbf{B}' \to \mathbf{B}'[1/t] \to \mathbf{B}'[1/t]/\mathbf{B}' \to 0.$$

Hence it suffices to check that base changes of $\gamma^{[u,v]}_{X_S}$ to both $\mathbf{B}'[1/t]$ and $\mathbf{B}'[1/t]/\mathbf{B}'$ are quasi-isomorphisms. Writing $\mathbf{B}'[1/t]/\mathbf{B}' = \mathrm{colim}_n(\mathbf{B}'/t^n)$ and using the fact that the tensor products commute with filtered colimits, we see that it suffices to check that base changes of $\gamma^{[u,v]}_{X_S}$ to both $\mathbf{B}'[1/t]$ and $\mathbf{B}'[1/t]/t^i$ are quasi-isomorphisms. Finally, by devissage, we can drop i to 1, as wanted.

For the first base change, we have quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_\square)$

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(X_S,\mathbf{Q}_p(r))[1/t] \tilde{\to} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,r)[1/t],$$

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],\mathbf{B}')[1/t] \tilde{\leftarrow} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r')[2d],\mathbf{B}'(s))[1/t].$$

And $\gamma^{[u,v]}_{X_S}$ is just the canonical map

$$\gamma_{X_S}: \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,r)[1/t] \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r'),\mathbf{B}')[1/t]$$

induced by the Hyodo-Kato pairing (5.9). Since it is compatible with t-action, it suffices to show that the canonical map

$$\gamma_{X_S}: \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,r) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},c}(X_S,r'),\mathbf{B}')$$

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}'_\square)$. But this was shown in (5.13), in the proof of Lemma 5.12.

For the base change to $\mathbf{B}'/t$, write $S = \mathrm{Spa}(R,R^+)$; then $\mathbf{B}'/t = R$. We claim that we have a compatible with product quasi-isomorphism in $\mathcal{D}(\mathbf{B}'_\square)$

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},?}(X_S,\mathbf{Q}_p(r)) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \simeq F^r\mathrm{R}\Gamma_{\mathrm{dR},?}(X_S/\mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R. \tag{5.22}$$ morning1

To show (5.22) we compute:

$$\begin{aligned}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},?}(X_S,\mathbf{Q}_p(r)) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R &= [\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r) \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r)] \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \\ &\tilde{\to} [\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \xrightarrow{\iota_{\mathrm{HK}}\otimes\mathrm{Id}} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R]\end{aligned}$$

Then we use the following commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R & \longrightarrow & \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \\
\wr\downarrow \iota_{\mathrm{HK}} & & \| \\
\mathrm{R}\Gamma_{\mathrm{dR},?}(X_S/\mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R & \longrightarrow & \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \\
\uparrow & & \uparrow \\
F^r\mathrm{R}\Gamma_{\mathrm{dR},?}(X_S/\mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R & \longrightarrow & 0
\end{array}$$

It defines quasi-isomorphisms between the mapping fibers of the rows yielding (5.22). The quasi-isomorphism in the above diagram needs a justification: take the composition

$$(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_C)\{s\} \otimes^{\mathrm{L}\square}_{\check{C}} \mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \tilde{\to} \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK},?}(X_S,r) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR},?}(X_S/\mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R$$

It is equal to ι_{HK} hence a quasi-isomorphism, as wanted.

From (5.22), we get the quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_\square)$

$$\begin{aligned}\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],\mathbf{B}')\otimes^{\mathrm{L}\square}_{\mathbf{B}'} R &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],R)\\ &\simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(F^{r'}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}')[2d],R).\end{aligned}$$

We have quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_\square)$ compatible with products (see [1, Prop. 3.6, Prop. 3.10])

$$F^r\mathrm{R}\Gamma_{\mathrm{dR}}(X_S/\mathbf{B}')\otimes^{\mathrm{L}\square}_{\mathbf{B}'} R\xrightarrow[\beta_X]{\sim}\bigoplus_{i=0}^{d}\mathrm{R}\Gamma(X,\Omega^i)\otimes^{\mathrm{L}\square}_{K} R(r-i)[-i], \tag{5.23}$$

morning2

$$F^{r'}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}')\otimes^{\mathrm{L}\square}_{\mathbf{B}'} R\xrightarrow[\beta_X]{\sim}\bigoplus_{i=0}^{d}\mathrm{R}\Gamma_c(X,\Omega^i)\otimes^{\mathrm{L}\square}_{K} R(r'-i)[-i].$$

Putting (5.22) and (5.23) together, we get quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_\square)$ compatible with products

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(X_S,\mathbf{Q}_p(r))\otimes^{\mathrm{L}\square}_{\mathbf{B}'} R\simeq\bigoplus_{i=0}^{d}\mathrm{R}\Gamma(X,\Omega^i)\otimes^{\mathrm{L}\square}_{K} R(r-i)[-i],$$

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'_\square}(\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(r'))[2d],\mathbf{B}')\otimes^{\mathrm{L}\square}_{\mathbf{B}'} R\simeq\mathrm{R}\underline{\mathrm{Hom}}_{R_\square}(\oplus_{i=0}^{d}\mathrm{R}\Gamma_c(X,\Omega^i)\otimes^{\mathrm{L}\square}_{K} R(r'-i)[2d-i],R).$$

And our result follows from Serre duality[13] (see Proposition 5.1) which yields the quasi-isomorphisms in $\mathcal{D}(R_\square)$,

$$\begin{aligned}\mathrm{R}\Gamma(X,\Omega^i)\otimes^{\mathrm{L}\square}_{K} R &\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{K_\square}(\mathrm{R}\Gamma_c(X,\Omega^{d-i})[d],K)\otimes^{\mathrm{L}\square}_{K} R\\ &\xrightarrow{\sim}\mathrm{R}\underline{\mathrm{Hom}}_{R_\square}(\mathrm{R}\Gamma_c(X,\Omega^{d-i})\otimes^{\mathrm{L}\square}_{K} R[d],R).\end{aligned}$$

The second quasi-isomorphism holds by the same argument as the one used at the end of the proof of Lemma 5.12. □

5.3. **Syntomic duality: an alternative argument.** We present here an alternative proof of Theorem 5.20 (conditional on the unchecked tedious compatibilities in Lemma 5.27 below). It uses dual modifications to inverse the arrows in the defining syntomic distinguished triangles (3.22).

More precisely, let $j\geq i\geq 0$. We will construct a distinguished triangle in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$

$$\mathcal{E}_{\mathrm{HK},c}(X_S,i)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{O}(0,j)\to\mathcal{E}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(i))\to i_{\infty,*}F^i\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^j, \tag{5.24}$$

alter1

which is a twisted version of (3.22). To do that, consider the following map of distinguished triangles

$$\begin{array}{ccccc}\mathcal{E}_{\mathrm{HK},c}(X_S,i)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{O}(0,j) & \longrightarrow & \mathcal{E}_{\mathrm{HK},c}(X_S,i) & \xrightarrow{\iota_{\mathrm{HK}}} & i_{\infty,*}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^j\\ \downarrow & & \| & & \downarrow{\scriptstyle\mathrm{can}}\\ \mathcal{E}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(i)) & \longrightarrow & \mathcal{E}_{\mathrm{HK},c}(X_S,i) & \xrightarrow{\iota_{\mathrm{HK}}} & \mathcal{E}_{\mathrm{dR},c}(X_S,i)\end{array} \tag{5.25}$$

alter2

Here, the bottom distinguished triangle is (3.22); the top one is induced from the distinguished triangle

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,i)\otimes^{\mathrm{L}\square}_{\mathbf{B}_{S^\flat,[u,v]}}\mathbf{B}^{[u,v]}_{S^\flat}\{0,j\}\to\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,i)\xrightarrow{\iota_{\mathrm{HK}}}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_S,i)/t^j$$

obtained by tensoring the exact sequence (4.1) for $0,j$ with $\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S,i)$. (Recall that $\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C,r)=[\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\{r\}\otimes^{\mathrm{L}\square}_{\check{C}}\mathbf{B}^{[u,v]}_{S^\flat,\log}]^{N=0}$). The dashed arrow in diagram (5.25) is defined to make the diagram a map of distinguished triangles. The diagram yields quasi-isomorphisms

$$\begin{aligned}[\mathcal{E}_{\mathrm{HK},c}(X_S,i)\otimes^{\mathrm{L}}_{\mathcal{O}}\mathcal{O}(0,j)\to\mathcal{E}_{\mathrm{syn},c}(X_S,\mathbf{Q}_p(i))][1] &\xleftarrow{\sim}[i_{\infty,*}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^j\to\mathcal{E}_{\mathrm{dR},c}(X_S,i)]\\ &\xrightarrow{\sim}i_{\infty,*}F^i\mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^j.\end{aligned}$$

That is, we get a distinguished triangle (5.24), as wanted.

[13] Apply it in degree i.

Now, let $r, r' \geq 2d, s = r + r' - d$. Consider the following diagram in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$ (note that $s \geq r'$) whose columns are distinguished triangles

$$\begin{array}{ccc} \mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) & \xrightarrow{\gamma^{\mathrm{syn}}_{X_S}} & \mathbb{D}(\mathcal{E}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s,s)) \\ \downarrow & & \downarrow \\ \mathcal{E}_{\mathrm{HK}}(X_S, r) & \xrightarrow[\sim]{\gamma^{\mathrm{HK}}_{X_S}} & \mathbb{D}(\mathcal{E}_{\mathrm{HK},c}(X_S, r') \otimes^{\mathrm{L}}_{\mathcal{O}} \mathcal{O}(0,s)[2d], \mathcal{O}(s,s)) \\ \downarrow & & \downarrow \\ \mathcal{E}_{\mathrm{dR}}(X_S, r) & \xrightarrow[\sim]{\gamma^{\mathrm{dR}}_{X_S}} & \mathbb{D}(i_{\infty,*} F^{r'} \mathrm{R}\Gamma_{\mathrm{dR},c}(X_S/\mathbf{B}^+_{\mathrm{dR}})/t^s[2d-1], \mathcal{O}(s,s)) \end{array} \tag{5.26}$$ alter4

where the horizontal maps are defined by the syntomic, Hyodo-Kato, and $\mathbf{B}^+_{\mathrm{dR}}$-pairings, respectively (see (5.19), (5.11), (5.7)).

Let us assume Lemma 5.27 below. To prove that the top horizontal arrow in diagram 5.26 is a quasi-isomorphism it suffices to show that so are the two lower arrows. But this follows from Lemma 5.12 (we used the isomorphism $\mathcal{O}(0,s) \otimes^{\mathrm{L}}_{\mathcal{O}} \mathcal{O}(s) \simeq \mathcal{O}(s,s)$) and Lemma 5.8.

tedious1 **Lemma 5.27.** *Diagram* (5.26) *above is a map of distinguished triangles.*

5.4. **Pro-étale duality.** Let X be a smooth partially proper rigid analytic variety over K of dimension d. Let $S \in \mathrm{Perf}_C$. We define a cup product on the pro-étale φ-modules:

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}\square}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r')) \to \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r+r')) \tag{5.28}$$ cup11et

via the cup product on the $\mathbf{B}' := \mathbf{B}^{[u,v]}_{S^\flat}$-charts:

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S, \mathbb{B}^{[u,v]}(r)) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_S, \mathbb{B}^{[u,v]}(r')) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_S, \mathbb{B}^{[u,v]}(r+r'))$$

induced by the cup product on pro-étale cohomology. This product is compatible with the syntomic product (via the comparison quasi-isomorphism from Theorem 4.7): to see this it suffices to argue for the usual cohomology and locally, where the comparison map is known to be compatible with products.

Let $s \geq 2d$. We define a trace map

$$\mathrm{Tr}_X : \quad \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(s)) \to \mathbf{B}^{\mathrm{FF}}_{S^\flat}\{s-d, s-d\}[-2d] \tag{5.29}$$ trace11et

as the composition

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(s)) \simeq \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{syn},c}(X_S, \mathbf{Q}_p(s)) \xrightarrow{\mathrm{Tr}_X} \mathbf{B}^{\mathrm{FF}}_{S^\flat}\{s-d, s-d\}[-2d].$$

By [1, Prop. 7.17], for $S = \mathrm{Spa}(C, \mathcal{O}_C)$, this map is compatible with Huber's trace map.

For $r, r' \geq d, s := r + r' - d$, the above can be lifted to the Fargues-Fontaine curve: the cup product (5.28) and trace map (5.29) induce a pairing of nuclear φ-modules over $\mathbf{B}^{\mathrm{FF}}_{S^\flat}$

$$\mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}}_{\mathbf{B}^{\mathrm{FF}}_{S^\flat}} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathrm{R}\Gamma^{\mathbf{B}}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathbf{B}_{S^\flat}\{s,s\}[-2d].$$

This descends to a pairing in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$:

$$\mathcal{E}_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(r)) \otimes^{\mathrm{L}}_{\mathcal{O}} \mathcal{E}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r')) \xrightarrow{\cup} \mathcal{E}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r+r')) \xrightarrow{\mathrm{Tr}_X} \mathcal{O}(s,s)[-2d],$$

which induces a natural map $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$

$$\gamma_{X_S} : \quad \mathcal{E}_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(r)) \to \mathbb{D}(\mathcal{E}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(r'))[2d], \mathcal{O}(s,s)). \tag{5.30}$$ niedziela10-etale

By an abuse of notation, we will write

$$\gamma_{X_S} : \quad \mathcal{E}_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p) \to \mathbb{D}(\mathcal{E}_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(d))[2d], \mathcal{O}). \tag{5.31}$$ niedziela10-etale1

for the Tate-untwisted version of the map (5.30).

e-duality-etale **Corollary 5.32.** (Pro-étale Poincaré duality on the Fargues-Fontaine curve)
The map γ_{X_S} from (5.31) *is a quasi-isomorphism in* $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$.

Proof. Choose $r, r' \geq 2d$ and set $s := r + r' - d$. It suffices to prove that the Tate twisted map (5.30) is a quasi-isomorphism. This follows immediately from the syntomic duality from Theorem 5.20 and the comparison result from Proposition 4.4. □

5.5. **Künneth formula.** Let X, Y be smooth Stein rigid analytic varieties over K. The simple observation that we have a quasi-isomorphism in $\mathcal{D}(K_\square)$

$$\big(\Omega(X) \otimes_K^{\mathrm{L}\square} \mathcal{O}(Y)\big) \oplus \big(\mathcal{O}(X) \otimes_K^{\mathrm{L}\square} \Omega(Y)\big) \xrightarrow{\sim} \Omega(X \times_K Y), \tag{5.33}$$

tu1

which implies the Künneth formula for de Rham cohomology

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes_K^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{dR}}(Y) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_K Y)$$

leads to the syntomic Künneth formula in $\mathrm{QCoh}(X_{\mathrm{FF}})$ and hence the pro-étale as well:

Theorem 5.34. (Künneth formula) *Let X, Y be smooth partially proper rigid analytic varieties over K. Let d be larger than the dimension of $X \times_K Y$ and let $r, r' \geq 2d$. Let $S \in \mathrm{Perf}_C$. The natural maps*

$$\kappa: \quad \mathcal{E}_{\mathrm{syn}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathcal{E}_{\mathrm{syn}}(Y_S, \mathbf{Q}_p(r')) \to \mathcal{E}_{\mathrm{syn}}((X \times_K Y)_S, \mathbf{Q}_p(r+r')),$$
$$\kappa: \quad \mathcal{E}_{\mathrm{proét}}(X_S, \mathbf{Q}_p) \otimes_{\mathcal{O}}^{\mathrm{L}} \mathcal{E}_{\mathrm{proét}}(Y_S, \mathbf{Q}_p) \to \mathcal{E}_{\mathrm{proét}}((X \times_K Y)_S, \mathbf{Q}_p)$$

are quasi-isomorphisms in $\mathrm{QCoh}(X_{\mathrm{FF},S^\flat})$.

Proof. The pro-étale case follows from the syntomic one via the comparison quasi-isomorphism from Proposition 4.4.

For the syntomic case, it is enough to show that on the level of φ-modules over $\mathbf{B}_{S^\flat}^{\mathrm{FF}}$ the corresponding map

$$\kappa: \quad \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}_{S^\flat}^{\mathrm{FF}}}^{\mathrm{L}} \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}(Y_S, \mathbf{Q}_p(r')) \to \mathrm{R}\Gamma_{\mathrm{syn}}^{\mathbf{B}}((X \times_K Y)_S, \mathbf{Q}_p(r+r'))$$

is a quasi-isomorphism. Or that in $\mathcal{D}(\mathbf{B}'_\square)$, for $\mathbf{B}' := \mathbf{B}_{S^\flat}^{[u,v]}$, the induced map

$$\kappa^{[u,v]}: \quad \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r)) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(Y_S, \mathbf{Q}_p(r')) \to \mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}((X \times_K Y)_S, \mathbf{Q}_p(r+r'))$$

is a quasi-isomorphism. But for that, as in the proof of Theorem 5.20, it is enough to check that the base changes of $\kappa^{[u,v]}$ to $\mathbf{B}'[1/t]$ and to $\mathbf{B}'/t$ are quasi-isomorphisms.

For the first base change, we use the quasi-isomorphism in $\mathcal{D}(\mathbf{B}'_\square)$

$$\mathrm{R}\Gamma_{\mathrm{syn}}^{[u,v]}(X_S, \mathbf{Q}_p(r))[1/t] \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}(X_S, r)[1/t]$$

to write

$$\kappa^{[u,v]}[1/t]: \quad (\mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}(Y_S, r'))[1/t] \to \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}((X \times_K Y)_S, r+r')[1/t].$$

This map is induced by the Hyodo-Kato pairing

$$\kappa_{\mathrm{HK}}^{[u,v]}: \quad \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}(X_S, r) \otimes_{\mathbf{B}'}^{\mathrm{L}\square} \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}(Y_S, r') \to \mathrm{R}\Gamma_{\mathrm{HK}}^{[u,v]}((X \times_K Y)_S, r+r').$$

To check that this is a quasi-isomorphism we may pass to cohomology. Since $H_{\mathrm{HK}}^j(X_C)$ is Fréchet (hence flat for the solid tensor product over $\breve{C}$), this reduces to checking that the pairing

$$\bigoplus_{a=0}^{b} (H_{\mathrm{HK}}^a(X_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log})^{N=0} \otimes_{\mathbf{B}'}^{\mathrm{L}\square} (H_{\mathrm{HK}}^{j-a}(Y_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log})^{N=0} \to (H_{\mathrm{HK}}^j((X \times_K Y)_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}'_{\log})^{N=0}$$

is an isomorphism in $\mathbf{B}'_\square$.

Now, using the exponential map as in the proof of Lemma 5.12, we can reduce to proving that the pairing

$$\bigoplus_{a=0}^{b} (H_{\mathrm{HK}}^a(X_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}') \otimes_{\mathbf{B}'}^{\mathrm{L}\square} (H_{\mathrm{HK}}^{j-a}(X_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}') \to H_{\mathrm{HK}}^j(X_C \times_C Y_C) \otimes_{\breve{C}}^{\mathrm{L}\square} \mathbf{B}'$$

is an isomorphism in $\mathbf{B}'_{\square}$. Or, that so is the pairing in $\breve{C}_{\square}$

$$\bigoplus_{a=0}^{b} H^a_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}\square}_{\breve{C}} H^{j-a}_{\mathrm{HK}}(Y_C) \to H^j_{\mathrm{HK}}(X_C \times_C Y_C).$$

But this follows from the following:

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Lemma 5.35. (Hyodo-Kato Künneth formula) *Let X, Y be smooth partially proper rigid analytic varieties over C. Then the canonical pairing*

$$\kappa_{\mathrm{HK}} : \quad \mathrm{R}\Gamma_{\mathrm{HK}}(X) \otimes^{\mathrm{L}\square}_{\breve{C}} \mathrm{R}\Gamma_{\mathrm{HK}}(Y) \to \mathrm{R}\Gamma_{\mathrm{HK}}(X \times_C Y).$$

is a quasi-isomorphism in $\mathcal{D}_{\varphi,N,\mathcal{G}_K}(\breve{C}_{\square})$.

Proof. This follows from the comparison (via the Hyodo-Kato morphism) with the Künneth formula for de Rham cohomology

$$\kappa_{\mathrm{dR}} : \mathrm{R}\Gamma_{\mathrm{dR}}(X) \otimes^{\mathrm{L}\square}_{C} \mathrm{R}\Gamma_{\mathrm{dR}}(Y) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X \times_C Y).$$

The latter clearly holds if both X and Y are Stein. For a general partially proper X and Y, we use coverings by a countable number (!) of Stein varieties, the fact that all the complexes in sight are bounded complexes of Fréchet spaces, [9, Prop. 8.33], and the Stein case. □

For the base change to $R = \mathbf{B}'/t$, we get from the proof of Theorem 5.20, compatible with products, quasi-isomorphisms in $\mathcal{D}(\mathbf{B}'_{\square})$ ($T = X, Y$, $s \geq 0$)

$$\begin{aligned}\mathrm{R}\Gamma^{[u,v]}_{\mathrm{syn}}(T_S, \mathbf{Q}_p(s)) \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R &\simeq F^s \mathrm{R}\Gamma_{\mathrm{dR}}(T_S/\mathbf{B}') \otimes^{\mathrm{L}\square}_{\mathbf{B}'} R \\ &\simeq \bigoplus_{i=0}^{d_T} \mathrm{R}\Gamma(T, \Omega^i) \otimes^{\mathrm{L}\square}_{K} R(s-i)[-i].\end{aligned}$$

And the map $\kappa^{[u,v]}$ can be identified with the map

$$\begin{aligned}\Big(\bigoplus_{i=0}^{d_X} \mathrm{R}\Gamma(X, \Omega^i) \otimes^{\mathrm{L}\square}_{K} R(r-i)[-i]\Big) \otimes^{\mathrm{L}\square}_{R} \Big(\bigoplus_{i=0}^{d_Y} \mathrm{R}\Gamma(Y, \Omega^i) \otimes^{\mathrm{L}\square}_{K} R(r'-i)[-i]\Big) \\ \to \bigoplus_{i=0}^{d_X+d_Y} \mathrm{R}\Gamma(X \times_K Y, \Omega^i) \otimes^{\mathrm{L}\square}_{K} R(r+r'-i)[-i].\end{aligned}$$

If X, Y are Stein, this map in degree i is represented by the map

$$\bigoplus_{a=0}^{d_X+d_Y} \Big(\Omega^a(X) \otimes^{\mathrm{L}\square}_{K} R(r-a)\Big) \otimes^{\mathrm{L}\square}_{R} \Big(\Omega^{i-a}(Y) \otimes^{\mathrm{L}\square}_{K} R(r'-i+a)\Big) \to \Omega^i(X \times_K Y) \otimes^{\mathrm{L}\square}_{K} R(r+r'-i).$$

And the latter map is a quasi-isomorphism in $R_{\square}$ by (5.33). If X, Y are general smooth partially proper rigid analytic varieties, we can reduce to the Stein case as in the proof of Lemma 5.35. □

6. Poincaré duality for p-adic geometric pro-étale cohomology

Finally, we are ready to state and prove pro-étale duality on the level of Topological Vector Spaces (Theorem 6.8). We descend it from the analogous Poincaré duality on the Fargues-Fontaine curve (Corollary 5.32).

6.1. **Topological Vector Spaces.** In this paper, the category of Topological Vector Spaces (TVS's for short) is the ∞-category of $\underline{\mathbf{Q}}_p$-modules in the ∞-derived category $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathrm{Solid})$ of topologically enriched presheaves on sPerf_C – the category of strictly totally disconnected affinoids over C – with values in solid abelian groups. We will denote it by $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$. This category was defined and studied in [19]. We will denote by $\mathcal{D}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$ the corresponding ∞-category where we forget the enrichment; the objects of this category will be called "topological presheaves".

We list the following properties:

recall1 **Proposition 6.1.** ([19, Th. 1.1])

(1) (Enriched fully-faithfulness) *The canonical functor from Vector Spaces*[14] *to Topological Vector Spaces*

$$\mathrm{R}\pi_* : \mathcal{D}(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p) \to \underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_p)$$

tends to be fully faithful. More precisely, let $\mathcal{F} \in \mathcal{D}^b(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p)$ *be such that* $\mathrm{R}\pi_*\mathcal{F} \in \underline{\mathcal{D}}^b(\mathrm{Spa}(C), \mathbf{Q}_p)$ *and let* $\mathcal{G} \in \mathcal{D}^+(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p)$. *Then the canonical morphism in* $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_p)$

$$\mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_C(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\mathcal{H}om_{C^{\mathrm{top}}}(\mathrm{R}\pi_*\mathcal{F}, \mathrm{R}\pi_*\mathcal{G})$$

is a quasi-isomorphism.

(2) (Fargues-Fontaine fully-faithfulness) *The functor*

$$\mathrm{R}\tau_* : \mathrm{QCoh}(X_{\mathrm{FF},S^\flat}) \to \underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_p)$$

is fully faithful when restricted to perfect complexes. That is, for $\mathcal{F}, \mathcal{G} \in \mathrm{Perf}(X_{\mathrm{FF},S^\flat})$, *the natural map in* $\mathcal{D}(\mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathbf{Q}}_p(C)})$

$$\mathrm{RHom}_{\mathrm{QCoh}(X_{\mathrm{FF},C^\flat})}(\mathcal{F}, \mathcal{G}) \to \mathrm{R}\underline{\mathrm{Hom}}_{C,\mathbf{Q}_p}(\mathrm{R}\tau_*\mathcal{F}, \mathrm{R}\tau_*\mathcal{G})$$

is a quasi-isomorphism.

(3) (Compatibility of the algebraic and topological projections) *The functor*

$$\mathrm{R}\tau'_* : \mathrm{QCoh}(X_{\mathrm{FF},C^\flat}) \to \mathcal{D}(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p)$$

is compatible with the functor $\mathrm{R}\tau_*$ *when restricted to nuclear sheaves. That is, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Nuc}(X_{\mathrm{FF},C^\flat}) & \xrightarrow{\mathrm{R}\tau'_*} & \mathcal{D}(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p) \\ & \searrow^{\mathrm{R}\tau_*} & \downarrow{\scriptstyle \mathrm{R}\pi_*} \\ & & \underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_p). \end{array}$$

6.2. TVS**-version of pro-étale cohomology presheaves.** Let X be a smooth partially proper rigid analytic variety over K. For $? = -, c$, we define the presheaves on sPerf_C with values in $\mathcal{D}(\mathrm{Solid})$

$$\mathbb{R}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p) : \quad S \to \mathrm{R}\Gamma_{\mathrm{proét},?}(X_S, \mathbf{Q}_p).$$

The topology on pro-étale cohomology is induced via Čech procedure from p-adic topologies. Note that the values of these presheaves on S are actually in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$.

comp-elgin **Proposition 6.2.** *(1) The presheaf* $\mathbb{R}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p)$ *is naturally enriched:*

$$\mathbb{R}_{\mathrm{proét},?}(X_C, \mathbf{Q}_p) \in \underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square}).$$

[14]We call Vector Spaces (VS's for short) the objects in the ∞-derived category $\mathcal{D}(\mathrm{Spa}(C)_{\mathrm{proét}}, \mathbf{Q}_p)$ of $\underline{\mathbf{Q}}_p$-modules in the category of pro-étale sheaves on Perf_C, the category of perfectoid affinoids over C.

(2) There exists a natural quasi-isomorphism in $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$

$$\mathbb{R}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) \simeq \mathrm{R}\pi_* \mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p),$$

where the sheaf $\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p)$ *is the algebraic version*[15] *of* $\mathbb{R}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p)$.

(3) Let X *be a smooth Stein variety over* K. *There exists a natural quasi-isomorphism in* $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$

$$\mathrm{R}\tau_* \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) \simeq \mathbb{R}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p), \quad ? = -, c. \tag{6.3}$$ identity2

Proof. For claim (1), consider first the usual cohomology. We want to define (a straighten version of) structure maps

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_{S\times T}, \mathbf{Q}_p) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(\mathbf{Q}_{p,\square}[T], \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p)),$$

for $S \in \mathrm{sPerf}_C$ and a profinite set T. By pro-étale descent, it suffices to construct, for a set $\{S_i\}, i \in I$, $S_i \in \mathrm{sPerf}_C$, functorial structure maps

$$\prod_I \underline{\mathbf{Q}}_p(S_i \times T) \to \underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(\mathbf{Q}_{p,\square}[T], \prod_I \underline{\mathbf{Q}}_p(S_i))$$

or functorial maps

$$\underline{\mathbf{Q}}_p(S_i \times T) \to \underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(\mathbf{Q}_{p,\square}[T], \underline{\mathbf{Q}}_p(S_i))$$

But these maps can be identified with the canonical isomorphisms (see [19, Ex. 2.1])

$$\underline{\mathcal{C}(|S_i \times T|, \mathbf{Q}_p)} \xrightarrow{\sim} \underline{\mathcal{C}(T, \mathcal{C}(|S_i \times T|, \mathbf{Q}_p))}$$

To treat the compactly supported version $\mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p)$, recall that

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p) &= [\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}((\partial X)_S, \mathbf{Q}_p)], \\ \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}((\partial X)_S, \mathbf{Q}_p) &= \mathrm{colim}_{Z \in \Phi_X} \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}((X \setminus Z)_S, \mathbf{Q}_p). \end{aligned} \tag{6.4}$$ kwak-kwak1

This canonically induces the enrichment on the presheaf $\mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p)$.

Claim (2), from the above argument, is clear for the usual cohomology. Then it follows for compactly supported cohomology on the level of presheaves described by the algebraic version of (6.4) because the proof of the claim (3) below shows that these are actually sheaves.

For claim (3), by Proposition 6.1 and claim (2), it suffices to show that

$$\mathrm{R}\tau'_* \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) \simeq \mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p), \quad ? = -, c.$$

Note that, by definition, we have as presheaves

$$\begin{aligned} \mathrm{R}\tau'_* \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) &= \{S \mapsto \mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathrm{L}f_S^* \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p)\}, \\ \mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) &= \{S \mapsto \mathrm{R}\Gamma(X_{\mathrm{FF},S^\flat}, \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p)\}. \end{aligned}$$

Hence it suffices to refer to Lemma 6.5 below. □

dziecko1 **Lemma 6.5.** *On* $X_{\mathrm{FF},S^\flat}$ *we have a natural quasi-isomorphism of solid quasi-coherent sheaves*

$$\mathrm{L}f_S^* \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p) \simeq \mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_S, \mathbf{Q}_p).$$

Proof. It suffices to show that, for a compact rational interval $[u, v] \subset (0, \infty)$, the canonical map

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_C, \mathbb{B}^{[u,v]}) \otimes^{\mathrm{L}_\square}_{\mathbf{B}^{[u,v]}_{C^\flat}} \mathbf{B}^{[u,v]}_{S^\flat} \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},?}(X_S, \mathbb{B}^{[u,v]})$$

is a quasi-isomorphism. Since tensor product commutes with colimits it suffices to do this for the usual cohomology.

[15] Defined as $\mathbb{R}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p)$ but with discrete topology objectwise.

By Theorem 4.7, it suffices to show the same for the twisted Hyodo-Kato and de Rham cohomologies. Assume thus that $r \geq 2d$ and let us start with the Hyodo-Kato cohomology. We want to show that the base change map (for $\mathbf{B}'_S := \mathbf{B}^{[u,v]}_{S^\flat}$)

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_C, r) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'_C} \mathbf{B}'_S \to \mathrm{R}\Gamma^{[u,v]}_{\mathrm{HK}}(X_S, r)$$

is a quasi-isomorphism. For that, we may pass to cohomology. Since $H^j_{\mathrm{HK}}(X_C)$ is Fréchet (hence flat for the solid tensor product over $\breve{C}$), this reduces to checking that the base change, for $b \geq 0$,

$$(H^b_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbf{B}'_{C,\log})^{N=0} \otimes^{\mathrm{L}_\square}_{\mathbf{B}'_C} \mathbf{B}'_S \to (H^b_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbf{B}'_{S,\log})^{N=0}$$

is an isomorphism in $\mathbf{B}'_{S,\square}$.

Now, using the exponential map as in the proof of Lemma 5.12, we can reduce to proving that the base change

$$(H^b_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbf{B}'_C) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'_C} \mathbf{B}'_S \to H^b_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}_\square}_{\breve{C}} \mathbf{B}'_S$$

is an isomorphism in $\mathbf{B}'_{S,\square}$. But this is clear.

We pass now to the de Rham cohomology. As above it suffices to show that the base change map

$$\mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_C, r) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'_C} \mathbf{B}'_S \to \mathrm{R}\Gamma^{[u,v]}_{\mathrm{dR}}(X_S, r)$$

is a quasi-isomorphism. But this reduces to showing that the base change maps

$$(\Omega^i(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}/t^s)) \otimes^{\mathrm{L}_\square}_{\mathbf{B}'_C} \mathbf{B}'_S \to \Omega^i(X) \otimes^\square_K (\mathbf{B}^+_{\mathrm{dR}}(S)/t^s)$$

are isomorphisms. And this is clear. □

6.3. **Topological Poincaré duality.** Let X be a partially proper smooth variety over K of dimension d. Let $i, j \geq 0$. We define a pairing in $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$:

$$\mathbb{R}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(i)) \otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p} \mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(j)) \to \mathbf{Q}_p(i+j-d)[-2d] \tag{6.6}$$

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by inducing it from the compatible family of pairings in $\mathcal{D}(\mathbf{Q}_p(S)_\square)$

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_S, \mathbf{Q}_p(i)) \otimes^{\mathrm{L}_\square}_{\mathbf{Q}_p(S)} \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(j)) \overset{\cup}{\to} \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t},c}(X_S, \mathbf{Q}_p(i+j)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p(S)(i+j-d)[-2d],$$

where the trace map comes from the trace maps (5.29) via the fundamental exact sequence. The fact that this pairing is compatible with the enrichments follows from the fact that it is induced by the algebraic pairing, we have Proposition 6.2, and the projection functor $\mathrm{R}\pi_*$ is lax monoidal.

The pairing in (6.6) induces a duality map in $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$

$$\gamma_{X_C} : \quad \mathbb{R}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p). \tag{6.7}$$

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VS-duality

Theorem 6.8. (Pro-étale duality) *Let X be a smooth partially proper rigid analytic variety over K of dimension d. The duality map* (6.7) *is a quasi-isomorphism. In particular, we have a quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_{p,\square})$*

$$\gamma_{X_C} : \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \overset{\sim}{\to} \mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p).$$

Proof. (•) Assume first that de Rham cohomology of X has finite rank. It suffices to show that, for $r, r' \geq 2d$, $s = r + r' - d$, the pairing in (6.6) induces a duality map in $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$

$$\gamma_{X_C} : \quad \mathbb{R}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(r'))[2d], \mathbf{Q}_p(s)),$$

which is a quasi-isomorphism. But we have the quasi-isomorphism

$$\mathbb{R}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p(r)) \simeq \mathrm{R}\tau_*\mathcal{E}_{\mathrm{pro\acute{e}t},?}(X_C, \mathbf{Q}_p(r))$$

from Proposition 6.2 and the quasi-isomorphism $\mathrm{R}\tau_*\mathcal{O}(i,i) \simeq \mathbf{Q}_p(i)$, $i \geq 0$. Moreover, the functor $\mathrm{R}\tau_*$ is lax monoidal and compatible with pro-étale traces, hence it suffices to show that the duality map

$$\gamma_{X_C} : \quad \mathrm{R}\tau_*\mathcal{E}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(r'))[2d], \mathrm{R}\tau_*\mathcal{O}(s,s))$$

is a quasi-isomorphism.

This map factorizes as

$$\begin{array}{ccc} \mathrm{R}\tau_*\mathcal{E}_{\text{proét}}(X_C,\mathbf{Q}_p(r)) & \longrightarrow & \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\text{proét},c}(X_C,\mathbf{Q}_p(r'))[2d],\mathrm{R}\tau_*\mathcal{O}(s,s)) \\ & \overset{\mathrm{R}\tau_*\gamma_{X_C}}{\searrow}\ \ \nearrow_{\text{can}} & \\ & \mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\text{proét},c}(X_C,\mathbf{Q}_p(r'))[2d],\mathcal{O}(s,s)) & \end{array}$$

where we set

$$\mathbb{D}(-,-) := \mathrm{R}\mathcal{H}om_{\mathrm{QCoh}(\mathrm{X}_{\mathrm{FF},C^\flat})}(-,-)$$

The left slanted map is a quasi-isomorphism by the Poincaré duality on the Fargues -Fontaine curve from Theorem 5.20. Hence it suffices to show that the canonical morphism

$$\mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\text{proét},c}(X_C,\mathbf{Q}_p(r')),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\text{proét},c}(X_C,\mathbf{Q}_p(r')),\mathrm{R}\tau_*\mathcal{O}(s,s))$$

is a quasi-isomorphism. Or, by Proposition 4.4, that so is the canonical morphism

$$\mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\text{syn},c}(X_C,\mathbf{Q}_p(r')),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\text{syn},c}(X_C,\mathbf{Q}_p(r')),\mathrm{R}\tau_*\mathcal{O}(s,s)).$$

Applying $\mathrm{R}\tau_*\mathbb{D}(-,\mathcal{O}(s,s))$ and $\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*(-),\mathrm{R}\tau_*\mathcal{O}(s,s))$ to the distinguished triangle

$$\mathcal{E}_{\text{syn},c}(X,\mathbf{Q}_p(r')) \to \mathcal{E}_{\mathrm{HK},c}(X,r') \to \mathcal{E}_{\mathrm{dR},c}(X,r') \tag{6.9}$$ def11

and identifying $\mathrm{R}\tau_*\mathcal{O}(s,s) \simeq \mathbf{Q}_p(s)$, we get compatible distinguished triangles

$$\mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\text{syn},c}(X,\mathbf{Q}_p(r')),\mathcal{O}(s,s)) \leftarrow \mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\mathrm{HK},c}(X,r'),\mathcal{O}(s,s)) \leftarrow \mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\mathrm{dR},c}(X,r'),\mathcal{O}(s,s)),$$
$$\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\text{syn},c}(X,\mathbf{Q}_p(r')),\mathbf{Q}_p(s)) \leftarrow \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\mathrm{HK},c}(X,r'),\mathbf{Q}_p(s)) \leftarrow \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\mathrm{dR},c}(X,r'),\mathbf{Q}_p(s)).$$

It suffices thus to show that the canonical morphisms

$$\mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\mathrm{HK},c}(X,r'),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\mathrm{HK},c}(X,r'),\mathbf{Q}_p(s)),$$
$$\mathrm{R}\tau_*\mathbb{D}(\mathcal{E}_{\mathrm{dR},c}(X,r'),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*\mathcal{E}_{\mathrm{dR},c}(X,r'),\mathbf{Q}_p(s))$$

are quasi-isomorphisms.

Since the solid quasi-coherent complexes $\mathcal{E}_{\mathrm{HK},c}(X,r')$ and $\mathcal{O}(s,s)$ are perfect, the first quasi-isomorphism follows from Proposition 6.1. For the second quasi-isomorphism, since $\mathcal{E}_{\mathrm{dR},c}(X,r') = i_{\infty,*}\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C/\mathbf{B}^+_{\mathrm{dR}})/F^{r'}$, it suffices to show that the natural morphism

$$\mathrm{R}\tau_*\mathbb{D}(\iota_{\infty,*}(W\otimes_K^{\mathrm{L}\square}\mathbf{B}^+_{\mathrm{dR}}/t^i),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*(\iota_{\infty,*}(W\otimes_K^{\mathrm{L}\square}\mathbf{B}^+_{\mathrm{dR}}/t^i)),\mathrm{R}\tau_*\mathcal{O}(s,s)) \tag{6.10}$$ kolo241

is a quasi-isomorphism, for any space of compact type $W \in C_K$. By devissage we may assume that $i=1$. Also, if we write $W \simeq \mathrm{colim}_n W_n$ as a compact colimit of Smith spaces W_n over K we may assume that W is a Smith space over K. This is because W^* is then a compact limit of Banach spaces and such limits commute with solid tensors with Banach spaces yielding that both sides of (6.10) will change the colimit into a derived limit (see also the first claim in Lemma 6.12). Moreover, assuming that W is a Smith space over K, we can write $W = W_{\mathbf{Q}_p}\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}K$, for a Smith space $W_{\mathbf{Q}_p}$ over $\mathbf{Q}_p$. Then we have $W^* = W^*_{\mathbf{Q}_p}\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}K$. Hence we may assume that $K=\mathbf{Q}_p$ in (6.10). To sum up, we need to show that the natural morphism

$$\mathrm{R}\tau_*\mathbb{D}(\iota_{\infty,*}(W\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}C),\mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{R}\tau_*(\iota_{\infty,*}(W\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}C)),\mathrm{R}\tau_*\mathcal{O}(s,s)) \tag{6.11}$$ kolo241a

is a quasi-isomorphism, for any Smith space $W \in C_{\mathbf{Q}_p}$.

We will need the following computation:

frank2 **Lemma 6.12.** *Let $s \in \mathbf{N}$. Let $W \in C_{\mathbf{Q}_p}$ be a Smith space.*

(1) The canonical morphism in $\mathrm{QCoh}(X_{\mathrm{FF},C^\flat})$

$$f_1: \quad W^*\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}\mathbb{D}(\iota_{\infty,*}C,\mathcal{O}(s,s)) \to \mathbb{D}(\iota_{\infty,*}(W\otimes^{\mathrm{L}\square}_{\mathbf{Q}_p}C),\mathcal{O}(s,s)) \tag{6.13}$$ comp1

is a quasi-isomorphism.

(2) The canonical morphisms in $\underline{\mathcal{D}}(C, \mathbf{Q}_{p,\square})$

$$f_2 : \quad W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\tau_* \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s)) \to \mathrm{R}\tau_*(W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s))),$$
$$f_3 : \quad W \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\tau_* \iota_{\infty,*}C \to \mathrm{R}\tau_*(W \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \iota_{\infty,*}C)$$

are quasi-isomorphisms.

Proof. The case of the morphism f_3 is clear. The morphism f_2 can be written as the following composition of quasi-isomorphisms

$$\mathrm{R}\tau_*(W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s))) \simeq \mathrm{R}\tau_*(W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \iota_{\infty,*}C(s-1)[-1])$$
$$\simeq W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a(s-1)[-1] \simeq W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\tau_* \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s)).$$

For the morphism f_1, we can pass to the category $\mathcal{D}(\mathbf{B}^{\mathrm{FF}}_{C^\flat,\square})$. Set $\mathbf{B}' := \mathbf{B}^{[u,v]}_{C^\flat,\mathrm{an}}$. By Section 2.2.3, it suffices to show that the canonical map

$$W^* \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(C, \mathbf{B}') \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} C, \mathbf{B}')$$

is a quasi-isomorphism in $\mathcal{D}(\mathbf{B}')$. Or, since

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(C, \mathbf{B}') \xleftarrow{\sim} C(-1)[-1], \tag{6.14}$$ kolo242

that the composition in $\mathcal{D}(\mathbf{B}')$

$$W^* \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} C(-1)[-1] \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} C, \mathbf{B}') \tag{6.15}$$ kolo240

is a quasi-isomorphism.

For that, we write the map (6.15) as a composition of quasi-isomorphisms in $\mathcal{D}(\mathbf{B}')$

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} C, \mathbf{B}') \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} \mathbf{B}', \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(C, \mathbf{B}'))$$
$$\xleftarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} \mathbf{B}', C)(-1)[-1] \simeq (W^* \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} C)(-1)[-1].$$

Here, the first quasi-isomorphism is the internal tensor-hom adjunction; the second quasi-isomorphism follows from (6.14). The last quasi-isomorphism is induced by the following commutative diagram in $\mathcal{D}(\mathbf{Q}_{p,\square})$

$$\begin{array}{ccc} \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} \mathbf{B}', C) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathbf{B}'}(W \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} \mathbf{B}', C) \\ \wr\uparrow & & \wr\uparrow \\ \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(W, C) & \xrightarrow{\sim} & \underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(W, C), \end{array} \tag{6.16}$$ kolo243

where we have used that W is an internal projective object in solid $\mathbf{Q}_p$-modules, and the fact that $\underline{\mathrm{Hom}}_{\mathbf{Q}_{p,\square}}(W, V) \xleftarrow{\sim} W^* \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} V$, for a Fréchet space V over $\mathbf{Q}_p$ (see [30, Th. 3.40]). We note here that the arrows in diagram (6.16) are $\mathbf{B}'$-linear. This finishes the proof of the first claim of the lemma. □

By Lemma 6.12, to show that the morphism (6.11) is a quasi-isomorphism it suffices to show that so is the natural morphism

$$W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\tau_* \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s)) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(W \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a, \mathbf{Q}_p) \tag{6.17}$$ frank1

is a quasi-isomorphism. But this morphism factors as the composition

$$\begin{array}{ccc} W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\tau_* \mathbb{D}(\iota_{\infty,*}C, \mathcal{O}(s,s)) & \longrightarrow & \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(W \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a, \mathbf{Q}_p(s)) \\ & \searrow^{\sim} \quad \nearrow_{f_0} & \\ & W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p(s)), & \end{array}$$

where the left quasi-isomorphism follows from Proposition 6.1. Hence it remains to show that the morphism f_0 above is a quasi-isomorphism.

Since, by [19, Lemma 3.27], we have

$$\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(W \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a, \mathbf{Q}_p) \simeq \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\underline{W}, \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p)),$$

we can rewrite the morphism f_0 (untwisted by s) as the canonical morphism

$$W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\underline{W}, \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p)).$$

Or, since $\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) \simeq \mathbb{G}_a(-1)[-1]$, as (the shift of) the canonical morphism

$$W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a \to \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\underline{W}, \mathbb{G}_a).$$

It remains thus to show that this morphism is a quasi-isomorphism.

Let $S \in \mathrm{sPerf}_C$. We have quasi-isomorphisms

$$W^* \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbb{G}_a(S) \xrightarrow{\sim} \mathrm{R}\underline{\mathrm{Hom}}^{\square}_{\mathbf{Q}_p}(W, \mathbb{G}_a(S)),$$
$$\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\underline{W}, \mathbb{G}_a)(S) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\underline{W} \otimes^{\mathrm{L}\square}_{\mathbf{Q}_p} \mathbf{Q}_p[h_S^{\mathrm{top}}]^{\square}, \mathbb{G}_a) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\underline{W}, \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbf{Q}_p[h_S^{\mathrm{top}}]^{\square}, \mathbb{G}_a)).$$

Here the first quasi-isomorphism follows from the fact that W is a Smith space (hence an internal projective object in solid modules). This reduces us to showing that the canonical morphism

$$\mathrm{R}\underline{\mathrm{Hom}}^{\square}_{\mathbf{Q}_p}(W, \mathbb{G}_a(S)) \to \mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\underline{W}, \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbf{Q}_p[h_S^{\mathrm{top}}]^{\square}, \mathbb{G}_a))$$

is a quasi-isomorphism. But, since

$$\mathrm{R}\Gamma(\mathrm{Spa}(C)^{\mathrm{top}}, \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbf{Q}_p[h_S^{\mathrm{top}}]^{\square}, \mathbb{G}_a)) \simeq \mathrm{R}\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbf{Q}_p[h_S^{\mathrm{top}}]^{\square}, \mathbb{G}_a) \simeq \mathbb{G}_a(S),$$

this follows from [19, Lemma 3.27].

($\bullet$) For a general smooth partially proper variety X, we cover it with Stein varieties $\{X_i\}, i \in I$, such that de Rham cohomology of each X_i is of finite rank. Then we consider the associated Čech hypercovering $Y_\bullet$ of X and we compute in $\underline{\mathcal{D}}(\mathrm{Spa}(C), \mathbf{Q}_{p,\square})$

$$\begin{aligned}\mathbb{R}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) &\simeq \mathrm{Rlim}_n \mathbb{R}_{\mathrm{pro\acute{e}t}}(Y_{n,C}, \mathbf{Q}_p) \simeq \mathrm{Rlim}_n \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(Y_{n,C}, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathrm{colim}_n \mathbb{R}_{\mathrm{pro\acute{e}t},c}(Y_{n,C}, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p) \\ &\simeq \mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\mathrm{pro\acute{e}t},c}(Y_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p),\end{aligned}$$

as wanted. Here, the second quasi-isomorphism follows from the case of duality already proven. □

6.4. **Algebraic Poincaré duality.** The topological Poincaré duality from Theorem 6.8 has an algebraic version, which we will now present. Let X be a smooth partially proper rigid analytic variety over K. The algebraic analog of the pairing in (6.6) induces a duality map in $\mathcal{D}(\mathrm{Spa}(C)_{\mathrm{pro\acute{e}t}}, \mathbf{Q}_p)$

$$\gamma^{\mathrm{alg}}_{X_C}: \quad \mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \to \mathrm{R}\mathcal{H}om_{\mathrm{VS}}(\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p). \tag{6.18}$$ bonn10A

VSS-duality **Corollary 6.19.** (Algebraic pro-étale duality) *Let X be a smooth partially proper rigid analytic variety over K of dimension d. The duality map* (6.18) *is a quasi-isomorphism. In particular, we have a quasi-isomorphism in $\mathcal{D}(\mathbf{Q}_p)$*

$$\gamma^{\mathrm{alg}}_{X_C}: \quad \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p) \xrightarrow{\sim} \mathrm{RHom}_{\mathrm{VS}}(\mathbb{R}^{\mathrm{alg}}_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p(d))[2d], \mathbf{Q}_p). \tag{6.20}$$ deszcz111

Proof. Apply the projection functor $\mathrm{R}\pi_*$ to the map (6.18). We obtain the horizontal map in the following commutative diagram

$$\begin{array}{ccc}
\mathrm{R}\pi_*\mathbb{R}^{\mathrm{alg}}_{\text{proét}}(X_C,\mathbf{Q}_p) & \xrightarrow{\mathrm{R}\pi_*\gamma^{\mathrm{alg}}_{X_C}} & \mathrm{R}\pi_*\mathrm{R}\mathcal{H}om_{\mathrm{VS}}(\mathbb{R}^{\mathrm{alg}}_{\text{proét},c}(X_C,\mathbf{Q}_p(d))[2d],\mathbf{Q}_p) \\
\downarrow\wr & \searrow & \downarrow\wr \\
\mathbb{R}_{\text{proét}}(X_C,\mathbf{Q}_p) & & \mathrm{R}\mathcal{H}om_{C^{\mathrm{top}}}(\mathrm{R}\pi_*\mathbb{R}^{\mathrm{alg}}_{\text{proét},c}(X_C,\mathbf{Q}_p(d))[2d],\mathrm{R}\pi_*\mathbf{Q}_p) \\
& \xrightarrow[\sim]{\gamma_{X_C}} & \downarrow\wr \\
& & \mathrm{R}\mathcal{H}om_{C^{\mathrm{top}}}(\mathbb{R}_{\text{proét},c}(X_C,\mathbf{Q}_p(d))[2d],\mathbf{Q}_p).
\end{array}$$

The top right vertical arrow is a quasi-isomorphism by Proposition 6.1; the map γ_{X_C} is a quasi-isomorphism by Theorem 6.8. It follows that so is the map $\mathrm{R}\pi_*\gamma^{\mathrm{alg}}_{X_C}$.

Now, we have a functor $\mathrm{R}\eta_* : \mathcal{D}^{\mathrm{psh}}(\mathrm{sPerf}_C, \mathrm{Mod}^{\mathrm{cond}}_{\mathbf{Q}_p}) \to \mathcal{D}^{\mathrm{psh}}(\mathrm{sPerf}_C,\mathbf{Q}_p)$ from topological presheaves to algebraic presheaves ("evaluation at $*$") such that $\mathrm{R}\eta_*\mathrm{R}\pi_* \simeq \mathrm{R}\iota_*$, the canonical forgetful functor from sheaves to presheaves (see [19, Sec. 2.1.2] for details). Applying $\mathrm{R}\eta_*$ to $\mathrm{R}\pi_*\gamma^{\mathrm{alg}}_{X_C}$ we get that the map $\mathrm{R}\iota_*\gamma^{\mathrm{alg}}_{X_C}$ is a quasi-isomorphism and hence so is the map $\gamma^{\mathrm{alg}}_{X_C}$ (after applying the sheafification functor), as wanted. □

6.5. **Verdier exact sequence.** In the Stein case, the duality (6.20) takes a simple form.

rainy-day

Corollary 6.21. *Let X be a smooth Stein variety over K. Let $i \geq 0$. There exists a short exact sequence in* TVS*'s*

$$0 \to \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^{2d-i+1}_{\text{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p) \to \mathbb{H}^i_{\text{proét}}(X_C,\mathbf{Q}_p) \to \mathcal{H}om_{\mathrm{TVS}}(\mathbb{H}^{2d-i}_{\text{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p) \to 0$$

In particular, there exists a short exact sequence in $\mathbf{Q}_{p,\square}$

$$0 \to \underline{\mathrm{Ext}}^1_{\mathrm{TVS}}(\mathbb{H}^{2d-i+1}_{\text{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p) \to H^i_{\text{proét}}(X_C,\mathbf{Q}_p) \to \underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbb{H}^{2d-i}_{\text{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p) \to 0$$

Proof. The second claim follows easily from the first claim. For the first claim, having Theorem 6.8 and the spectral sequence

$$E_2^{a,b} = \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^{-b}_{\text{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p) \Rightarrow H^{a+b}(\mathrm{R}\mathcal{H}om_{\mathrm{TVS}}(\mathbb{R}_{\text{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p))$$

it suffices to show that

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\text{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p) = 0, \quad a \geq 2. \tag{6.22}$$

vanishing

Let $r \geq 2d$. By Proposition 4.4, we have the quasi-isomorphism

$$\mathcal{E}_{\text{proét},c}(X_C,\mathbf{Q}_p(r)) \simeq \mathcal{E}_{\mathrm{syn},c}(X_C,\mathbf{Q}_p(r)),$$

which, by (3.11) and (3.3), yields an exact sequence

$$\cdots \xrightarrow{\iota_{b-1}} \mathbb{DR}^{b-1}_c(X_C,r) \to \mathbb{H}^b_{\text{proét},c}(X_C,\mathbf{Q}_p(r)) \to \mathbb{HK}^b_c(X_C,r) \xrightarrow{\iota_b} \mathbb{DR}^b_c(X_C,r) \to \cdots \tag{6.23}$$

presentation1

where we set

$$\mathbb{HK}^b_c(X_C,r) := (H^r_{\mathrm{HK},c}(X_C) \otimes^{\square}_{\breve{C}} \mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r}$$

$$\mathbb{DR}^b_c(X_C,r) := H^{b-d}(H^d_c(X,\mathcal{O}) \otimes^{\square}_K (\mathbb{B}^+_{\mathrm{dR}}/t^r) \to H^d_c(X,\Omega^1) \otimes^{\square}_K (\mathbb{B}^+_{\mathrm{dR}}/t^{r-1}) \to \cdots \to H^d_c(X,\Omega^d) \otimes^{\square}_K (\mathbb{B}^+_{\mathrm{dR}}/t^{r-d}))$$

We used here the natural isomorphisms ($? = -, c$)

$$H^b\mathrm{R}\tau_*\mathcal{E}_{\mathrm{HK},?}(X_C,r) \simeq \mathbb{HK}^b_?(X_C,r), \quad H^b\mathrm{R}\tau_*\mathcal{E}_{\mathrm{dR},?}(X_C,r) \simeq \mathbb{DR}^b_?(X_C,r),$$

that are compatible with the Hyodo-Kato map and which follow immediately from the computations in the proof of Proposition 6.2.

Let $i \geq 0$. We will need to understand the maps ι_i from (6.23) better. We have a commutative diagram in TVS, where we set $s = r - i + d - 1$:

$$\begin{array}{ccccccccc}
0 & \to & (H^i_{\mathrm{HK},c}(X_C) \otimes^{\square}_{\check{C}} t^s\mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} & \to & \mathbb{HK}^i_c(X_C,r) & \to & H^i_{\mathrm{dR},c}(X) \otimes^{\square}_K (\mathbb{B}^+_{\mathrm{dR}}/F^s) & & \\
 & & \downarrow{\scriptstyle \iota_{\mathrm{HK}}} & & \downarrow{\scriptstyle \iota_i} & & \| & & \\
0 & \to & (H^d_c(X,\Omega^{i-d})/\mathrm{Im}\, d) \otimes^{\square}_K t^s\mathbb{G}_a & \to & \mathbb{DR}^i_c(X_C,r) & \to & H^i_{\mathrm{dR},c}(X) \otimes^{\square}_K (\mathbb{B}^+_{\mathrm{dR}}/F^s) & \to & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_3} & & \\
0 & \to & \mathbb{E}^i_1 & \xrightarrow{f_2} & \mathbb{E}^i & \to & \mathbb{E}^i_2 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{6.24}$$

air1

Here, The middle row comes from (3.4); the top row from [1, Lemma 8.1]. We have defined $\mathbb{E}^i := \mathrm{coker}(\iota_i)$; $\mathbb{E}^i_1$ is the image of $(H^d_c(X,\Omega^{i-d})/\mathrm{Im}\, d) \otimes^{\square}_K t^s\mathbb{G}_a$ under the map f_1, and $\mathbb{E}^i_2$ is the cokernel of the map f_2. The rows are exact; so are the first and the second columns. Moreover, the map $\iota_{\mathrm{HK}} : (H^i_{\mathrm{HK},c}(X_C) \otimes^{\square}_{\check{C}} t^s\mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \to (H^d_c(X,\Omega^{i-d})/\mathrm{Im}\, d) \otimes^{\square}_K t^s\mathbb{G}_a$ factors as

$$(H^i_{\mathrm{HK},c}(X_C) \otimes^{\square}_{\check{C}} t^s\mathbb{B}^+_{\mathrm{st}})^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} H^i_{\mathrm{dR},c}(X) \otimes^{\square}_K t^s\mathbb{G}_a \xrightarrow{\mathrm{can}} (H^d_c(X,\Omega^{i-d})/\mathrm{Im}\, d) \otimes^{\square}_K t^s\mathbb{G}_a, \tag{6.25}$$

wreszcie0

where the second map is an injection with quotient $(H^d_c(X,\Omega^{i-d})/\mathrm{Ker}\, d) \otimes^{\square}_K t^s\mathbb{G}_a$. It follows that we have an exact sequence

$$0 \to \mathbb{V} \to \mathbb{E}^i_1 \to (H^d_c(X,\Omega^{i-d})/\mathrm{Ker}\, d) \otimes^{\square}_K t^s\mathbb{G}_a \to 0, \tag{6.26}$$

wreszcie

where $\mathbb{V}$ is a (topological) BC.

(•) *Finite rank case.* Assume first that the de Rham cohomology has finite rank. Let $\mathbb{A}^i := \ker(\iota_i)$. It is a BC: this is because the map ι_i factors through de Rham cohomology, which is a BC (see (6.25)). We have the exact sequence

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{A}^b, \mathbf{Q}_p) \to \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{pro\acute{e}t},c}(X_C, \mathbf{Q}_p), \mathbf{Q}_p) \to \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{E}^{b-1}, \mathbf{Q}_p). \tag{6.27}$$

presentation2

Hence it suffices to show that, for $a \geq 2$, we have

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{A}^b, \mathbf{Q}_p) = 0, \quad \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{E}^{b-1}, \mathbf{Q}_p) = 0. \tag{6.28}$$

presentation3

This is clear for $\mathbb{A}^b$ because it is a BC and we have [19, Ex. 4.29]. For $\mathbb{E}^{b-1}$, we use the exact sequence (6.26). It follows that it suffices to show that, for $a \geq 2$, we have

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{V}, \mathbf{Q}_p) = 0, \quad \mathcal{E}xt^a_{\mathrm{TVS}}(W \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p) = 0,$$

where we set $W = H^d_c(X,\Omega^{b-1-d})/\mathrm{Ker}\, d$. Since $\mathbb{V}$ is a BC, the first equality is clear. For the second one, if $K = \mathbf{Q}_p$, we have in $\mathbf{Q}_{p,\square}$ (see [19, Lemma 3.27])

$$\mathcal{E}xt^a_{\mathrm{TVS}}(W \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p) \simeq W^* \otimes^{\square}_{\mathbf{Q}_p} \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) = 0,$$

as wanted. We used here that W is of compact type.

For a general K, assume first that W is a Smith space. Then we can write $W = W_{\mathbf{Q}_p} \otimes^{\mathrm{L}}_{\mathbf{Q}_{p,\square}} K$, for a Smith space $W_{\mathbf{Q}_p}$ over $\mathbf{Q}_p$. It follows that the canonical morphism

$$W^* \otimes^{\square}_K \mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{G}_a, \mathbf{Q}_p) \to \mathcal{E}xt^a_{\mathrm{TVS}}(W \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p)$$

is an isomorphism. For a general W, write $W \simeq \mathrm{colim}_n W_n$ as a compact colimit of Smith spaces W_n over K and note that

$$\mathcal{E}xt^a_{\mathrm{TVS}}(W \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p) \simeq \lim_n \mathcal{E}xt^a_{\mathrm{TVS}}(W_n \otimes^{\square}_K \mathbb{G}_a, \mathbf{Q}_p).$$

This is because $\mathrm{R}^1\lim_n \mathcal{E}xt^{a-1}_{\mathrm{TVS}}(W_n\otimes^{\square}_K \mathbb{G}_a,\mathbf{Q}_p)=0$: this follows from the Smith case for $a\geq 3$; for $a=2$, we have

$$\mathrm{R}^1\lim_n \mathcal{E}xt^1_{\mathrm{TVS}}(W_n\otimes^{\square}_K \mathbb{G}_a,\mathbf{Q}_p)\overset{\sim}{\leftarrow}\mathrm{R}^1\lim_n(W_n^*\otimes^{\square}_K \mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{G}_a,\mathbf{Q}_p))$$
$$\simeq \mathrm{R}^1\lim_n(W_n^*\otimes^{\square}_K \mathbb{G}_a)\simeq(\mathrm{R}^1\lim_n W_n^*)\otimes^{\square}_K \mathbb{G}_a=0.$$

The last isomorphism follows from the fact that $\{W_n^*\}$ is a pro-system of Banach spaces with dense transition maps. The penultimate isomorphism follows from the following fact[16]:

Lemma 6.29. *Let $\{V_n\}$ be a set of Banach spaces over K and let V be a Banach space over K. Then the canonical map*

$$V\otimes^{\square}_K\prod_n V_n\to\prod_n(V\otimes^{\square}_K V_n)$$

is an isomorphism.

Proof. Write $V=\underline{\mathrm{Hom}}(\mathbf{Z}[T],K)$, for a profinite set T. Then

$$V\otimes^{\square}_K\prod_n V_n\simeq\underline{\mathrm{Hom}}(\mathbf{Z}[T],K)\otimes^{\square}_K\prod_n V_n\simeq\underline{\mathrm{Hom}}(\mathbf{Z}[T],\prod_n V_n)$$
$$\simeq\prod_n\underline{\mathrm{Hom}}(\mathbf{Z}[T],V_n)\simeq\prod_n(\underline{\mathrm{Hom}}(\mathbf{Z}[T],K)\otimes^{\square}_K V_n)\simeq\prod_n(V\otimes^{\square}_K V_n).$$

The second isomorphism follows from the fact that the product $\prod_n V_n$ is a nuclear K-vector space and the penultimate one from the fact that so is every V_n. □

(•) *General case.* For a general smooth Stein variety X over K, we cover X with an exhaustive sequence $\{X_n\}$ of Stein varieties with finite dimensional de Rham cohomologies. We have

$$\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p)\simeq\mathcal{E}xt^a_{\mathrm{TVS}}(\mathrm{colim}_n\,\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p).$$

This yields the exact sequence

$$0\to\mathrm{R}^1\lim_n\mathcal{E}xt^{a-1}_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)\to\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p)\to\lim_n\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)-$$

Hence, by the above, $\mathcal{E}xt^a_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p)=0$, for $a\geq 3$. For $a=2$, we have

$$\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)\overset{\sim}{\to}\mathcal{E}xt^2_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_C,\mathbf{Q}_p),\mathbf{Q}_p).$$

It suffices thus to show that

$$\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)=0.$$

From the exact sequence (6.27) and isomorphisms (6.28), we get the exact sequence (for X_n)

$$\mathcal{H}om_{\mathrm{TVS}}(\mathbb{E}_n^{b-1},\mathbf{Q}_p)\to\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{A}_n^b,\mathbf{Q}_p)\to\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)\to\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{E}_n^{b-1},\mathbf{Q}_p)\to 0$$

This yields the exact sequence

$$\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{A}_n^b,\mathbf{Q}_p)\to\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{H}^b_{\mathrm{proét},c}(X_{n,C},\mathbf{Q}_p),\mathbf{Q}_p)\to\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{E}_n^{b-1},\mathbf{Q}_p)\to 0.$$

Since $\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{A}_n^b,\mathbf{Q}_p)$ is a BC we have $\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{A}_n^b,\mathbf{Q}_p)=0$ as we have Mittag-Leffler in this setting. It suffices thus to show that $\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{E}_n^{b-1},\mathbf{Q}_p)=0$.

But from diagram (6.24), we see that it suffices to show that

$$\mathrm{R}^1\lim_n\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{E}_{i,n}^{b-1},\mathbf{Q}_p)=0,\quad i=1,2.$$

[16]Probably well-known but we did not find a reference.

For $i=2$ this is clear because $\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{E}^{b-1}_{2,n},\mathbf{Q}_p)$ is a BC (because $\mathbb{E}^{b-1}_{2,n}$ is the cokernel of the map $\mathbb{HK}^{b-1}_c(X_C,r)\to H^{b-1}_{\mathrm{dR},c}(X)\otimes^{\square}_K(\mathbb{B}^+_{\mathrm{dR}}/F^s)$, hence a BC). It remains to treat the case $i=1$. For that we use the exact sequence (6.26). Since $\mathbb{V}_n$ is a BC, it suffices to show that

$$\mathrm{R}^1\lim_n \mathcal{E}xt^1_{\mathrm{TVS}}(W_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p)=0,$$

where $W_n=H^d_c(X_n,\Omega^{b-1-d})/\mathrm{Ker}\,d$. Note that $W_n\hookrightarrow H^d_c(X_n,\Omega^{b-d})=:W'_n$. Hence we have the surjection

$$\mathcal{E}xt^1_{\mathrm{TVS}}(W'_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p)\twoheadrightarrow\mathcal{E}xt^1_{\mathrm{TVS}}(W_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p).$$

It suffices thus to show that

$$\mathrm{R}^1\lim_n \mathcal{E}xt^1_{\mathrm{TVS}}(W'_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p)=0. \tag{6.30}$$ air2

We claim that the canonical morphism

$$W'^{,*}_n\otimes^{\square}_K\mathcal{E}xt^1_{\mathrm{TVS}}(\mathbb{G}_a,\mathbf{Q}_p)\to\mathcal{E}xt^1_{\mathrm{TVS}}(W'_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p)$$

is an isomorphism. Indeed, we can write $W'_n=\mathrm{colim}_m W'_{n,m}$ as a compact colimit of Smith spaces and argue as above using the fact that

$$\mathcal{H}om_{\mathrm{TVS}}(W'_{n,m}\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p)\simeq W'_{n,m,*}\otimes^{\square}_K\mathcal{H}om_{\mathrm{TVS}}(\mathbb{G}_a,\mathbf{Q}_p)=0.$$

Hence we have a functorial in n isomorphism

$$W'^{,*}_n\otimes^{\square}_K\mathbb{G}_a\simeq\mathcal{E}xt^1_{\mathrm{TVS}}(W'_n\otimes^{\square}_K\mathbb{G}_a,\mathbf{Q}_p).$$

Since we have $W'^{,*}_n\simeq\Omega^{2d-b}(X_n)$, (6.30) holds because the pro-system $\{\Omega^{2d-b}(X_n)\}$, $n\in\mathbf{N}$, is equivalent to a pro-system of Banach spaces $\{V_n\}$ with dense transition maps. □

6.6. **Examples.** We will discuss in some detail here dualities for Stein curves and Drinfeld spaces.

6.6.1. *Proper varieties.* Let X be a smooth proper rigid analytic variety over K. Then, since pro-étale cohomology of X_C is finite dimensional over $\mathbf{Q}_p$, Corollary 6.21 yields a duality isomorphism

$$H^i_{\mathrm{proét}}(X_C,\mathbf{Q}_p)\xrightarrow{\sim}H^{2d-i}_{\mathrm{proét}}(X_C,\mathbf{Q}_p(d))^*.$$

A result known by the work of Zavyalov [34] and Mann [29].

6.6.2. *Stein curves.* Let X be a geometrically connected smooth Stein curve over K. From comparison theorems (see [22, Th. 6.14]) we get the following isomorphism and a short exact sequence in $\mathbf{Q}_{p,\square}$

$$H^0_{\mathrm{proét}}(X_C,\mathbf{Q}_p)\simeq\mathbf{Q}_p, \tag{6.31}$$ leje1-25

$$0\to\mathcal{O}(X_C)/C\to H^1_{\mathrm{proét}}(X_C,\mathbf{Q}_p(1))\to(H^1_{\mathrm{HK}}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p}\to 0$$

They lift to TVS's. Similarly, for compactly supported cohomology, by [1, Sec. 7.2], we get the following isomorphism and a short exact sequence in $\mathbf{Q}_{p,\square}$

$$H^1_{\mathrm{proét},c}(X_C,\mathbf{Q}_p(1))\simeq(H^1_{\mathrm{HK},c}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=1},$$
$$\to(H^1_{\mathrm{HK},c}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=p^2}\xrightarrow{\iota_{\mathrm{HK}}}H^1\mathrm{DR}_c(X_C,2)\to H^2_{\mathrm{proét},c}(X_C,\mathbf{Q}_p(2))\to\mathbf{Q}_p(1)\to 0,$$

where we set $\mathrm{DR}_c(X_C,2):=\mathrm{R}\Gamma_{\mathrm{dR},c}(X_C,\mathbf{B}^+_{\mathrm{dR}})/F^2$. We note that, if the de Rham cohomology of X is of finite rank over K, then $(H^1_{\mathrm{HK},c}(X_C)\otimes^{\square}_{\breve{C}}\mathbf{B}^+_{\mathrm{st}})^{N=0,\varphi=1}$ is a finite rank $\mathbf{Q}_p$-vector space because the slopes of Frobenius on $H^1_{\mathrm{HK},c}(X_C)$ are ≥ 0 (see [1, Remark 7.10]). Again, everything lifts to TVS's.

By Corollary 6.21 we get a short exact sequence in $\mathbf{Q}_{p,\square}$

$$0\to\underline{\mathrm{Ext}}^1_{\mathrm{TVS}}(\mathbb{H}^{2d-i+1}_{\mathrm{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p)\to H^i_{\mathrm{proét}}(X_C,\mathbf{Q}_p)\to\underline{\mathrm{Hom}}_{\mathrm{TVS}}(\mathbb{H}^{2d-i}_{\mathrm{proét},c}(X_C,\mathbf{Q}_p(d)),\mathbf{Q}_p)\to 0$$

Hence, using the above computations and $d = 1$ since X is a curve, we get the following isomorphism and a short exact sequence in $\mathbf{Q}_{p,\square}$

$$H^0_{\text{proét}}(X_C, \mathbf{Q}_p) \simeq \underline{\text{Hom}}_{\text{TVS}}(\mathbb{H}^2_{\text{proét},c}(X_C, \mathbf{Q}_p(1)), \mathbf{Q}_p) \tag{6.32}$$ leje2-25

$$0 \to \underline{\text{Ext}}^1_{\text{TVS}}(\mathbb{H}^2_{\text{proét},c}(X_C,\mathbf{Q}_p(1)), \mathbf{Q}_p) \to H^1_{\text{proét}}(X_C, \mathbf{Q}_p) \to \underline{\text{Hom}}_{\text{TVS}}(\mathbb{H}^1_{\text{proét},c}(X_C, \mathbf{Q}_p(1)), \mathbf{Q}_p) \to 0$$

It is tempting to think that the exact sequence in (6.32) recovers the exact sequence in (6.31) but this is not the case: if the de Rham cohomology of X is of finite rank over K, the term on the right in (6.32) is a finite rank $\mathbf{Q}_p$-vector space while the term on the right in (6.31) will, in general, have a nontrivial C-part.

6.6.3. *Drinfeld space.* Let K be a finite extension of $\mathbf{Q}_p$ and let $d \geq 1$. Let $\mathcal{H}^d_K$ be the Drinfeld space of dimension d over K. By [14, Th. 1.3], [1, Lemma 8.13], we have exact sequences in $\mathbf{Q}_{p,\square}$ ($i \geq 0$)

$$\begin{aligned} 0 \to \Omega^{i-1}(\mathcal{H}^d_C)/\ker d \to & H^i_{\text{proét}}(\mathcal{H}^d_C, \mathbf{Q}_p(i)) \to \text{Sp}_i(\mathbf{Q}_p)^* \to 0 \\ 0 \to H^d_c(\mathcal{H}^d_C, \Omega^{i-d-1})/\ker d \to & H^i_{\text{proét},c}(\mathcal{H}^d_C, \mathbf{Q}_p(i-d)) \to \text{Sp}_{2d-i}(\mathbf{Q}_p) \to 0 \end{aligned} \tag{6.33}$$ leje3-25

Here $\text{Sp}_i(\mathbf{Q}_p)$ denotes the generalized locally constant Steinberg $\mathbf{Q}_p$-representation of $\text{GL}_{d+1}(K)$ (see [14, Sec. 5.2.1] for a definition). Hence the terms on the right in (6.33) are nuclear Fréchet and of compact type over $\mathbf{Q}_p$, respectively. It follows that

$$\begin{aligned} \underline{\text{Hom}}_{\text{TVS}}(\mathbb{H}^{2d-i}_{\text{proét},c}(\mathcal{H}_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) &\simeq \underline{\text{Hom}}_{\text{TVS}}(\text{Sp}_i(\mathbf{Q}_p)(i), \mathbf{Q}_p) \simeq \text{Sp}_i(\mathbf{Q}_p)^*(-i), \\ \underline{\text{Ext}}^1_{\text{TVS}}(\mathbb{H}^{2d-i+1}_{\text{proét},c}(\mathcal{H}_C, \mathbf{Q}_p(d)), \mathbf{Q}_p) &\simeq \underline{\text{Ext}}^1_{\text{TVS}}((H^d_c(\mathcal{H}^d_C, \Omega^{d-i})/\ker d)(i-1), \mathbf{Q}_p) \\ &\simeq (H^d_c(\mathcal{H}^d_C, \Omega^{d-i})/\ker d)^*(-i) \simeq (\Omega^{i-1}(\mathcal{H}^d_C)/\ker d)(-i). \end{aligned}$$

The last quasi-isomorphism uses Serre's duality (see [1, Remark 8.11] for details).

Hence, in this example, the duality sequence from Corollary 6.21 does transfer the compact support fundamental exact sequence into the usual fundamental exact sequence.

Remark 6.34. The case of affine spaces and tori is similar to the case of Drinfeld space but simpler since the Hyodo-Kato terms on the right in the fundamental exact sequences are actually finite dimensional over $\mathbf{Q}_p$. See [14, Prop. 4.17], [1, Sec. 7.1] for the shape of these fundamental exact sequences.

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