Fontaine’s rings and $p$-adic $L$-functions

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These are notes from a course given at Tsinghua University during the fall of 2004. The aim of the course was to explain how to construct $p$-adic $L$-functions using the theory of $(\varphi, \Gamma)$-modules of Fontaine. This construction is an adaptation of an idea of Perrin-Riou. The content of the course is well reflected in the table of contents which is almost the only thing that I modified from the notes taken and typed by the students Wang Shanwen, Chen Miaofen, Hu Yongquan, Yin Gang, Li Yan and Hu Yong, under the supervision of Ouyang Yi, all of whom I thank heartily. The course runs in parallel to a course given by Fontaine in which the theory of $(\varphi, \Gamma)$-modules was explained as well as some topics from $p$-adic Hodge theory which are used freely in these notes, which means that they are not entirely self-contained. Also, as time runs short at the end, the last chapter is more a survey than a course. For a bibliography and further reading, the reader is referred to my Bourbaki talk of June 2003 published in Astérisque 294.
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Part I

Classical $p$-adic $L$-functions: zeta functions and modular forms
Chapter 1

The $p$-adic zeta function of Kubota-Leopoldt

1.1 The Riemann zeta function at negative integers

We first recall the definitions of Riemann zeta function and the classical Gamma function:

\[
\zeta(s) = \sum_{n=1}^{+\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}, \text{ if } \Re(s) > 1.
\]

\[
\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt, \text{ if } \Re(s) > 0.
\]

The Gamma function has the following properties:

(i) $\Gamma(s+1) = s\Gamma(s)$, which implies that $\Gamma$ has a meromorphic continuation to $\mathbb{C}$ with simple poles at negative integers and 0.

(ii) $\Gamma(n) = (n-1)!$ if $n \geq 1$.

(iii) $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$, which implies that $\frac{1}{\Gamma(s)}$ is an entire (or holomorphic) function on $\mathbb{C}$ with simple zeros at $-n$ for $n \in \mathbb{N}$.

(iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
CHAPTER 1. THE P-ADIC ZETA FUNCTION OF KUBOTA-LEOPOLDT

Then we have the following formulas:

\[ n^{-s} = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-nt}s \frac{dt}{t}, \]
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-nt}s \frac{dt}{t} = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{1}{et-1}t^{s} \frac{dt}{t}. \]

**Lemma 1.1.1.** If \( f : \mathbb{R}_+ \to \mathbb{C} \) is a \( C^\infty \)-function on \( \mathbb{R}_+ \), rapidly decreasing (i.e., \( t^n f(t) \to 0 \) when \( t \to +\infty \) for all \( n \in \mathbb{N} \)), then

\[ L(f, s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f(t) t^s \frac{dt}{t}, \quad \text{Re}(s) > 0 \]

has an analytic continuation to \( \mathbb{C} \), and

\[ L(f, -n) = (-1)^n f^{(n)}(0). \]

**Proof.** Choose a \( C^\infty \)-function \( \phi \) on \( \mathbb{R}_+ \), such that \( \phi(t) = 1 \) for \( t \in [0, 1] \) and \( \phi(t) = 0 \) for \( t \geq 2 \).

Let \( f = f_1 + f_2 \), where \( f_1 = \phi f \), \( f_2 = (1 - \phi)f \). Then \( \int_0^{+\infty} f_2(t) t^s \frac{dt}{t} \) is holomorphic on \( \mathbb{C} \), hence \( L(f_2, s) \) is also holomorphic and \( L(f_2, -n) = 0 = f_2^{(-n)}(0) \). Since, for \( \text{Re}(s) > 0 \),

\[ L(f_1, s) = \frac{1}{\Gamma(s)} \left[ f_1(t) \left( \frac{t^n}{s} \right) \right]^{+\infty}_{0} - \frac{1}{s \Gamma(s)} \int_0^{+\infty} f'_1(t) t^{s+1} \frac{dt}{t} \]

\[ = -L(f'_1(t), s + 1) = (-1)^n L(f_1^{(n)}, s + n), \]

we get analytic continuation for \( f_1 \) and hence for \( f \), moreover,

\[ L(f, -n) = L(f_1, -n) = (-1)^{n+1} L(f_1^{(n+1)}, 1) \]

\[ = (-1)^{n+1} \int_0^{+\infty} f_1^{(n+1)}(t) dt = (-1)^n f_1^{(n)}(0) = (-1)^n f^{(n)}(0). \]

\[ \square \]

We now apply the above lemma to the function \( f(t) = \frac{t}{e^t-1} \). Note that

\[ f(t) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \]
where \( B_n \in \mathbb{Q} \) is the \( n \)-th Bernoulli number with value:
\[
B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_3 = 0, \ B_4 = -\frac{1}{30}, \ B_5 = 0 \ldots
\]

Since \( f(t) - f(-t) = -t \), we have \( B_{2k+1} = 0 \) if \( k \geq 1 \). Now:
\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f(t)t^{s-1} \frac{dt}{t} = \frac{1}{s-1} L(f, s-1),
\]
so we obtain the following result.

**Theorem 1.1.2.** (i) \( \zeta \) has a meromorphic continuation to \( \mathbb{C} \). It is holomorphic except for a simple pole at \( s = 1 \) with residue \( L(f, 0) = B_0 = 1 \).

(ii) If \( n \in \mathbb{N} \), then
\[
\zeta(-n) = \frac{-1}{n+1} L(f, -n-1) = \frac{(-1)^n}{n+1} f^{(n+1)}(0)
\]
\[
= (-1)^n \frac{B_{n+1}}{n+1} \in \mathbb{Q}
\]
\[
\left( = -\frac{B_{n+1}}{n+1} \text{ if } n \geq 2 \right).
\]

**Theorem 1.1.3 (Kummer).** If \( p \) does not divide the numerators of \( \zeta(-3), \zeta(-5), \ldots, \zeta(2-p) \), then the class number of \( \mathbb{Q}(u_p) \) is prime to \( p \).

**Remark.** This theorem and a lot of extra work implies Fermat’s Last Theorem for these regular primes. We will not prove it in these notes, but we will focus on the following result, also discovered by Kummer, which plays an important role in the proof.

**Theorem 1.1.4 (Kummer’s congruences).** Let \( a \geq 2 \) be prime to \( p \). Let \( k \geq 1 \). If \( n_1, n_2 \geq k \) such that \( n_1 \equiv n_2 \mod (p-1)p^{k-1} \), then
\[
(1 - a^{1+n_1})\zeta(-n_1) \equiv (1 - a^{1+n_2})\zeta(-n_2) \mod p^k.
\]

### 1.2 \( p \)-adic Banach spaces

**Definition 1.2.1.** A \( p \)-adic Banach space \( B \) is a \( \mathbb{Q}_p \)-vector space with a lattice \( B^0(\mathbb{Z}_p\text{-module}) \) separated and complete for the \( p \)-adic topology, i.e.,
\[
B^0 \simeq \lim_{n \in \mathbb{N}} B^0/p^n B^0.
\]
For all $x \in B$, there exists $n \in \mathbb{Z}$, such that $x \in p^n B^0$. Define

$$v_B(x) = \sup_{n \in \mathbb{N} \cup \{+\infty\}} \{n : x \in p^n B^0\}.$$

It satisfies the following properties:

$$v_B(x + y) \geq \min(v_B(x), v_B(y)),$$

$$v_B(\lambda x) = v_p(\lambda) + v_B(x), \text{ if } \lambda \in \mathbb{Q}_p.$$

Then $\|x\|_B = p^{-v_B(x)}$ defines a norm on $B$, such that $B$ is complete for $\|\|_B$ and $B^0$ is the unit ball.

**Example 1.2.2.**

(i) $B = \mathbb{C}_p = \hat{\mathbb{Q}}_p$, $B^0 = \mathcal{O}_{\mathbb{C}_p}$, $v_B(x) = [v_p(x)] \in \mathbb{Z}$;

(ii) The space $B = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p)$ of continuous functions on $\mathbb{Z}_p$, $B^0 = \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Z}_p)$ is a lattice, and $v_B(f) = \inf_{x \in \mathbb{Z}} v_p(f(x)) \neq -\infty$ because $\mathbb{Z}_p$ is compact.

(iii) Let $B = \mathcal{O}_p(\mathbb{C}_p)$, $B^0 = \mathcal{O}_p(\mathbb{Z}_p, \mathbb{O}_{\mathbb{C}_p})$; $v_B(f) = \inf_{x \in \mathbb{Z}} [v_p(f(x))]$.

**Definition 1.2.3.** A Banach basis of a $p$-adic Banach space $B$ is a family $(e_i)_{i \in I}$ of elements of $B$, satisfying the following conditions:

(i) For every $x \in B$, $x = \sum_{i \in I} x_i e_i$, $x_i \in \mathbb{Q}_p$ in a unique way with $x_i \to 0$

when $i \to \infty$; equivalently for any $C$, the set $\{i \mid v_p(x_i) \leq C\}$ is a finite set.

(ii) $v_B(x) = \inf_{i \in I} v_p(x_i)$.

**Theorem 1.2.4.** A family $(e_i)_{i \in I}$ of elements of $B$ is a Banach basis if and only if

(i) $e_i \in B^0$ for all $i$;

(ii) the set $(e_i)_{i \in I}$ form a basis of $B^0/pB^0$ as a $\mathbb{F}_p$-vector space.

**Proof.** We leave the proof of the theorem as an exercise. \qed

Let $B$ and $B'$ be two $p$-adic Banach spaces with Banach basis $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ respectively, then $B \hat{\otimes} B'$ is a $p$-adic Banach space with Banach basis $(e_i \otimes f_j)_{(i,j) \in I \times J}$. Thus for all $x \in B \hat{\otimes} B'$,

$$x = \sum_{i,j} x_{i,j} e_i \otimes f_j \quad (x_{i,j} \in \mathbb{Q}_p, x_{i,j} \to 0 \text{ as } (i,j) \to \infty)$$

$$= \sum_{j} y_j \otimes f_j \quad (y_j \in B, y_j \to 0 \text{ as } j \to \infty)$$

$$= \sum_{i} e_i \otimes z_i \quad (z_i \in B', z_i \to 0 \text{ as } i \to \infty).$$
Exercise. $C^0(\mathbb{Z}_p, \mathbb{C}_p) = \mathbb{C}_p \widehat{\otimes} C^0(\mathbb{Z}_p, \mathbb{Q}_p)$.

1.3 Continuous functions on $\mathbb{Z}_p$

1.3.1 Mahler’s coefficients

We have the binomial function:

$$\binom{x}{n} = \begin{cases} 1, & \text{if } n = 0, \\ \frac{x(x-1) \cdots (x-n+1)}{n!}, & \text{if } n \geq 1. \end{cases}$$

Lemma 1.3.1. $v_{C^0}(\binom{x}{n}) = 0$.

Proof. Since $\binom{n}{n} = 1$, $v_{C^0}(\binom{x}{n}) \leq 0$.

If $x \in \mathbb{N}$, then $\binom{x}{n} \in \mathbb{N}$ implies $v_p(\binom{x}{n}) \geq 0$. Hence for all $x \in \mathbb{Z}_p$, $v_p(\binom{x}{n}) \geq 0$ because $\mathbb{N}$ is dense in $\mathbb{Z}_p$. \hfill \Box

For all $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, we write

$$f^{[0]} = f, \quad f^{[k-1]}(x) = f^{[k]}(x+1) - f^{[k]}(x)$$

and write the Mahler’s coefficient

$$a_n(f) = f^{[n]}(0).$$

Hence:

$$f^{[n]}(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + n - i),$$

$$a_n(f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n - i).$$

Theorem 1.3.2 (Mahler). If $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, then

(i) $\lim_{n \to \infty} v_p(a_n(f)) = +\infty$,

(ii) For all $x \in \mathbb{Z}_p$, $f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$,

(iii) $v_{C^0}(f) = \inf v_p(a_n(f))$. 

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Proof. Let $\ell_\infty = \{a = (a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{Q}_p$ bounded$\}$, $v_{\ell_\infty}(a) = \inf_{n \in \mathbb{N}} v_p(a_n)$. Then

- $f \mapsto a(f) = (a_n(f))_{n \in \mathbb{N}}$ is a continuous map from $C^0(\mathbb{Z}_p, \mathbb{Q}_p)$ to $\ell_\infty$.

- $v_{\ell_\infty}(a(f)) \geq v_{C^0}(f)$.

- The space $\ell_\infty^0 = \{(a_n)_{n \in \mathbb{N}} : a_n \to 0$, as $n \to \infty\}$ is a closed subspace of $\ell_\infty$ and $B = \{f : a(f) \in \ell_\infty^0\}$ is a close subspace of $C^0(\mathbb{Z}_p, \mathbb{Q}_p)$.

- For all $a \in \ell_\infty^0$,
  \[
  f_a = \sum_{n=0}^{+\infty} a_n \binom{x}{n} \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)
  \]
  because the series converges uniformly. Moreover, $v_{C^0}(f_a) \geq v_{\ell_\infty}(a)$ and as $\binom{x+1}{n+1} - \binom{x}{n+1} = \binom{x}{n}$,

  \[
  f_a^{[k]} = \sum_{n=0}^{+\infty} a_{n+k} \binom{x}{n}.
  \]

  Hence we have: $a_k(f) = f^{[k]}(0) = a_k$, which implies $a(f_a) = a$.

  - $f \mapsto a(f)$ is injective. Since $a(f) = 0$ implies $f(n) = 0$ for all $n \in \mathbb{N}$. Hence $f = 0$ by the density of $\mathbb{N}$ in $\mathbb{Z}_p$.

Now for $f \in B, a(f) \in \ell_\infty^0$ implies $f - f_a(f) = 0$ because $a(f - f_a(f)) = a(f) - a(f) = 0$ and $a$ is injective. So $f \in B$ implies that $f$ satisfies (ii). Moreover, since

  \[
  v_{\ell_\infty}(a(f)) \geq v_{C^0}(f) = v_{C^0}(f_{a(f)}) \geq v_{\ell_\infty}(a(f)),
  \]

(iii) is also true. It remains to show that:

Claim: $B = C^0(\mathbb{Z}_p, \mathbb{Q}_p)$. \(\square\)

(a) First proof. We have a lemma:

Lemma 1.3.3. If $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, then there exists $k \in \mathbb{N}$ such that

  \[
  v_{C^0}(f^{p^k}) \geq v_{C^0}(f) + 1.
  \]
1.3. CONTINUOUS FUNCTIONS ON $\mathbb{Z}_p$

Proof. We have

$$f^{[p^k]}(x) = f(x + p^k) - f(x) + \sum_{i=1}^{p^k-1} (-1)^i \binom{p^k}{i} f(x + p^k - i) + (1 + (-1)^k) f(x).$$

Now $v_p\left(\binom{p^k}{i}\right) \geq 1$, if $1 \leq i \leq p^k - 1$ et $v_p(1 + (-1)^k) \geq 1$. Since $\mathbb{Z}_p$ is compact, $f$ is uniformly continuous. For every $c$, there exists $N$, when $v_p(x - y) \geq N$, we have $v_p(f(x) - f(y)) \geq c$. It gives the result for $k = N$.

First proof of the Claim. Repeat the lemma: for every $c = v_c(f) + k$, there exists an $N$, such that $v_c(f[N]) \geq c$. Hence, for all $n \geq N$, $v_p(a_n(f)) \geq c$.

1.3.2 Locally constant functions.

Choose a $z \in \mathbb{C}_p$, such that $v_p(z - 1) > 0$. Then

$$f_z(x) = \sum_{n=0}^{+\infty} \binom{x}{n} (z - 1)^n \in C^0(\mathbb{Z}_p, \mathbb{C}_p).$$

Note $k \in \mathbb{N}$, $f_z(k) = z^k$. So we write, $f_z(x) = z^x$ and we have $z^{x+y} = z^x z^y$.

Example 1.3.4. (i) $z^{\frac{1}{2}} = \sum_{n=0}^{+\infty} \binom{\frac{1}{2}}{n} (z - 1)^n$. $z = \frac{16}{9}, z - 1 = \frac{7}{9}$, the series converges in $\mathbb{R}$ to $\frac{4}{3}$, and converges in $\mathbb{Q}_7$ to $-\frac{4}{3}$.

(ii) If $z$ is a primitive $p^n$-th root of 1, then

$$v_p(z - 1) = \frac{1}{(p - 1)p^{n-1}} > 0.$$ 

Note that $z^{x+p^n} = z^x$ for all $x$, then $z^x$ is locally constant( constant mod $p^n\mathbb{Z}_p$).

The characteristic function of $i + p^n\mathbb{Z}_p$ is given by

$$1_{i+p^n\mathbb{Z}_p}(x) = \frac{1}{p^n} \sum_{z^{p^n} = 1} z^{-i} z^x$$

since

$$\sum_{z^{p^n} = 1} z^x = \begin{cases} p^n & \text{if } x \in p^n\mathbb{Z}_p; \\ 0 & \text{if not.} \end{cases}$$
Lemma 1.3.5. The set of locally constant functions \( \text{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \subset B \).

*Proof.* By compactness of \( \mathbb{Z}_p \), a locally constant function is a linear combination of \( 1_{1+p^i\mathbb{Z}_p}z^x, z \in \mu_p^\infty \), thus a linear combination of \( z^x \). But \( a_n(z^x) = (z - 1)^n \) goes to 0, when \( n \) goes to \( \infty \), hence \( z^x \in B \). \( \square \)

Lemma 1.3.6. \( \text{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \) is dense in \( C^0(\mathbb{Z}_p, \mathbb{Q}_p) \).

*Proof.* For every \( f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p) \), let

\[
f_k = \sum_{i=0}^{p^k-1} f(i)1_{1+p^i\mathbb{Z}_p},
\]

Then \( f_k \to f \) in \( C^0 \) because \( f \) is uniformly continuous. \( \square \)

*Second proof of the Claim.* By the above two lemmas, \( \text{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \subset B \subset C^0(\mathbb{Z}_p, \mathbb{Q}_p) \), \( B \) is closed and \( \text{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \) is dense in \( C^0(\mathbb{Z}_p, \mathbb{Q}_p) \), hence \( B = C^0(\mathbb{Z}_p, \mathbb{Q}_p) \). \( \square \)

1.4 Measures on \( \mathbb{Z}_p \)

1.4.1 The Amice transform

**Definition 1.4.1.** A measure \( \mu \) on \( \mathbb{Z}_p \) with values in a \( p \)-adic Banach space \( \mathcal{B} \) is a continuous linear map \( f \mapsto \int_{\mathbb{Z}_p} f(x)\mu = \int_{\mathbb{Z}_p} f(x)\mu(x) \) from \( C^0(\mathbb{Z}_p, \mathbb{Q}_p) \) to \( \mathcal{B} \).

**Remark.** (i) If \( L \subset \mathbb{C}_p \) is a closed subfield and \( \mathcal{B} \) is an \( L \)-vector space, then \( \mu \) extends by continuity and \( L \)-linearity to \( C^0(\mathbb{Z}_p, L) = L \widehat{\otimes} C^0(\mathbb{Z}_p, \mathbb{Q}_p) \).

(ii) We denote \( \mathcal{D}_0(\mathbb{Z}_p, \mathcal{B}) \) the set of the measure on \( \mathbb{Z}_p \) with values in \( \mathcal{B} \), then \( \mathcal{D}_0(\mathbb{Z}_p, \mathcal{B}) = \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} \mathcal{B} \).

**Definition 1.4.2.** The Amice transform of a measure \( \mu \) is defined to be the map:

\[
\mu \mapsto A_\mu(T) = \int_{\mathbb{Z}_p} (1 + T)^x \mu(x) = \sum_{n=0}^{+\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} \mu.
\]

**Lemma 1.4.3.** If \( v_p(z - 1) > 0 \), \( A_\mu(z - 1) = \int_{\mathbb{Z}_p} z^x \mu(x) \).
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Proof. Since $z^x = \sum_{n=0}^{+\infty} (z - 1)^n(x)_n$ with normal convergence in $C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, one can exchange $\sum$ and $\int$.

**Definition 1.4.4.** The valuation on $D_0$ is

$$v_{D_0}(\mu) = \inf_{f \neq 0} (v_p(\int_{\mathbb{Z}_p} f \mu) - v_{C^0}(f)).$$

**Theorem 1.4.5.** The map $\mu \mapsto A_{\mu}$ is an isometry from $D_0(\mathbb{Z}_p, \mathbb{Q}_p)$ to the set $\{ \sum_{n=0}^{+\infty} b_n T^n, b_n \text{ bounded, and } b_n \in \mathbb{Q}_p \}$ with the valuation $v(\sum_{n=0}^{+\infty} b_n T^n) = \inf_{n \in \mathbb{N}} v_p(b_n)$.

Proof. On one hand, for all $\mu \in D_0(\mathbb{Z}_p, \mathbb{Q}_p)$, write $A_{\mu}(T) = \sum_{n=0}^{+\infty} b_n(\mu) T^n$, then $b_n(\mu) = \int_{\mathbb{Z}_p} (x)_{n} \mu$. Since $v_{C^0}((x)_{n}) = 0$ by Lemma 1.3.1,

$$v_p(b_n(\mu)) \geq v_{D_0}(\mu) + v_{C^0}(\binom{x}{n}) \geq v_{D_0}(\mu)$$

for all $n$, hence $v(A_{\mu}) \geq v_{D_0}(\mu)$.

On the other hand, if $(b_n)_{n \in \mathbb{N}}$ is bounded, $f \mapsto \sum_{n=0}^{+\infty} b_n a_n(f)$ (by Mahler’s theorem, $a_n(f) \to 0$) gives a measure $\mu_b$ whose Amice transform is

$$A_{\mu_b}(T) = \sum_{n=0}^{+\infty} T^n \int_{\mathbb{Z}_p} \binom{x}{n} \mu_b = \sum_{n=0}^{+\infty} T^n \sum_{i=0}^{+\infty} b_i a_i(\binom{x}{n}) = \sum_{n=0}^{+\infty} b_n T^n$$

since

$$a_n(\binom{x}{i}) = \begin{cases} 1 & \text{if } i = n, \\ 0 & \text{otherwise}. \end{cases}$$

Hence

$$v_p(\sum_{n=0}^{+\infty} b_n a_n(f)) \geq \min_n (v_p(b_n) + v_p(a_n(f))) \geq \min_n (v_p(b_n)) + \min_n (a_n(f)) = v(\sum_{n=0}^{+\infty} b_n T^n) + v_{C^0}(f) = v(A_{\mu}) + v_{C^0}(f).$$

Thus $v_{D_0}(\mu_b) \geq v(A_{\mu})$. Then we have $v(A_{\mu}) = v_{D_0}(\mu)$.

$\Box$
By Lemma 1.3.6, we know that locally constant functions are dense in \( \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p) \). Explicitly, for all \( f \in \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p) \), the locally constant functions
\[
f_n = \sum_{i=0}^{p^n-1} f(i) \mathbb{1}_{i+p^n\mathbb{Z}_p} \rightarrow f
\]
in \( \mathcal{C}^0 \).

Now if \( \mu \in \mathcal{D}_0(\mathbb{Z}_p, \mathbb{Q}_p) \), set \( \mu(i+p^n\mathbb{Z}_p) = \int_{\mathbb{Z}_p} 1_{i+p^n\mathbb{Z}_p} \mu \). Then \( \int_{\mathbb{Z}_p} f \mu \) is given by the following “Riemann sums”
\[
\int_{\mathbb{Z}_p} f \mu = \lim_{n \to \infty} \sum_{i=0}^{p^n-1} f(i) \mu(i+p^n\mathbb{Z}_p)
\]
(1.1)
Note that \( v_p(\mu(i+p^n\mathbb{Z}_p)) \geq v_{\mathcal{D}_0}(\mu) \).

**Theorem 1.4.6.** If \( \mu \) is an additive bounded function on compact open subsets of \( \mathbb{Z}_p \) (by compactness of \( \mathbb{Z}_p \) is a finite disjoint union of \( i+p^n\mathbb{Z}_p \) for some \( n \)), then \( \mu \) extends uniquely as a measure on \( \mathbb{Z}_p \) via (1.1).

**Proof.** Since \( \mu \) is an additive function on compact open subsets, \( \mu \) is linear on locally constant functions. And \( \mu \) is bounded, hence \( \mu \) is continuous for \( v_{\mathcal{C}^0} \). As the locally constant functions are dense in \( \mathcal{C}^0(\mathbb{Z}_p, \mathbb{Q}_p) \), we have \( \mu \) as a measure on \( \mathbb{Z}_p \).

### 1.4.2 examples of measures on \( \mathbb{Z}_p \) and of operations on measures.

**Example 1.4.7.** *Haar measure:* \( \mu(\mathbb{Z}_p) = 1 \) and \( \mu \) is invariant by translation. We must have \( \mu(i+p^n\mathbb{Z}_p) = \frac{1}{p^n} \) which is not bounded. Hence, there exists no Haar measure on \( \mathbb{Z}_p \).

**Example 1.4.8.** *Dirac measure:* For \( a \in \mathbb{Z}_p \), we define \( \delta_a \) by \( \int_{\mathbb{Z}_p} f(x) \delta_a = f(a) \). The Amice transform of \( \delta_a \) is \( A_{\delta_a}(T) = (1+T)^a \).

**Example 1.4.9.** *Multiplication of a measure by a continuous function.* For \( \mu \in \mathcal{D}_0, f \in \mathcal{C}^0 \), we define the measure \( f \mu \) by
\[
\int_{\mathbb{Z}_p} g \cdot f \mu = \int_{\mathbb{Z}_p} f(x)g(x)\mu
\]
for all \( g \in \mathcal{C}^0 \).
1.4. MEASURES ON $\mathbb{Z}_p$

(i) Let $f(x) = x$, since

$$x\left(\frac{x}{n}\right) = (x - n + n)\left(\frac{x}{n}\right) = (n + 1)\left(\frac{x}{n + 1}\right) + n\left(\frac{x}{n}\right),$$

the Amice transform is

$$A_{x\mu} = \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \left(\frac{x}{n}\right) x\mu$$

$$= \sum_{n=0}^{\infty} T^n \left[ (n + 1) \int_{\mathbb{Z}_p} \left(\frac{x}{n + 1}\right) \mu + n \int_{\mathbb{Z}_p} \left(\frac{x}{n}\right) \mu \right]$$

$$= (1 + T) \frac{d}{dT} A_{\mu}.$$ 

(ii) Let $f(x) = z^x$, $v_p(z - 1) > 0$. For any $y$, $v_p(y - 1) > 0$, then

$$\int_{\mathbb{Z}_p} y^x (z^x) \mu = \int_{\mathbb{Z}_p} (yz)^x \mu = A_{\mu}((1 + T)z - 1),$$

which implies that

$$A_{z^x\mu}(T) = A_{\mu}((1 + T)z - 1).$$

(iii) The restriction to a compact open set $X$ of $\mathbb{Z}_p$: it is nothing but the multiplication by $1_X$. If $X = i + p^n\mathbb{Z}_p$, then $1_{i+p^n\mathbb{Z}_p}(x) = p^{-n} \sum_{z^{p^n} = 1} z^{-i}z^x$, hence

$$A_{\text{Res}_{i+p^n\mathbb{Z}_p}} \mu(T) = p^{-n} \sum_{z^{p^n} = 1} z^{-i} A_{\mu}((1 + T)z - 1).$$

Example 1.4.10. Actions of $\varphi$ and $\psi$. For $\mu \in \mathcal{D}_0$, we define the action of $\varphi$ on $\mu$ by

$$\int_{\mathbb{Z}_p} f(x) \varphi(\mu) = \int_{\mathbb{Z}_p} f(px) \mu.$$ 

Hence

$$A_{\varphi(\mu)}(T) = \sum_{n=0}^{\infty} T^n \int_{\mathbb{Z}_p} \left(\frac{px}{n}\right) \mu = A_{\mu}((1 + T)^p - 1) = \varphi(A_{\mu}(T))$$

where $\varphi : T \mapsto (1 + T)^p - 1$ (compare this formula with $(\varphi, \Gamma)$-modules). We define the action of $\psi$ by

$$\int_{\mathbb{Z}_p} f(x) \psi(\mu) = \int_{\mathbb{Z}_p} f(x/p) \mu.$$
Then $A_{\psi(\mu)} = \psi(A_\mu)$ where

$$\psi(F)((1 + T)^p - 1) = \frac{1}{p} \sum_{z^p = 1} F((1 + T)z - 1).$$

The actions $\varphi$ and $\psi$ satisfy the following properties:

(i) $\psi \circ \varphi = \text{Id}$;

(ii) $\psi(\mu) = 0 \iff \mu$ has a support in $\mathbb{Z}_p^*$;

(iii) $\text{Res}_{\mathbb{Z}_p^*}(\mu) = (1 - \varphi \psi)\mu$.

The map $\psi$ is very important in the theory of $(\varphi, \Gamma)$-modules.

**Example 1.4.11. Action of $\Gamma$.** Let $\Gamma = \text{Gal}(\mathbb{Q}_p(\mu_{p\infty})/\mathbb{Q}_p)$. Let $\chi : \Gamma \to \mathbb{Z}_p^*$ be the cyclotomic character. For $\gamma \in \Gamma$ and $\mu \in \mathcal{D}_0$, we let $\gamma \mu$ be given by

$$\int_{\mathbb{Z}_p} f(x) \gamma \mu = \int_{\mathbb{Z}_p} f(\chi(\gamma)x)\mu.$$

One can verify that $A_{\gamma \mu}(T) = A_\mu((1 + T)^{\chi(\gamma)} - 1) = \gamma(A_\mu(T))$ for $\gamma(T) = (1 + T)^{\chi(\gamma)} - 1$. (Compare this formula with $(\varphi, \Gamma)$-modules.)

For all $\gamma \in \Gamma$, $\gamma$ commutes with $\phi$ and $\psi$.

**Example 1.4.12. Convolution $\lambda \ast \mu$.** Let $\lambda$, $\mu$ be two measures, their convolution $\lambda \ast \mu$ is defined by

$$\int_{\mathbb{Z}_p} f(x) \lambda \ast \mu = \int_{\mathbb{Z}_p} \left( \int_{\mathbb{Z}_p} f(x + y)\mu(x) \right) \lambda(y).$$

Here we have to verify $y \mapsto \int_{\mathbb{Z}_p} f(x + y)\mu(x) \in C^0$, which is a direct consequence of the fact $f$ is uniformly continuous.

Let $f(x) = z^x$, $v_p(z - 1) > 0$, then

$$\int_{\mathbb{Z}_p} z^x \lambda \ast \mu = \int_{\mathbb{Z}_p} z^x \mu(x) \int_{\mathbb{Z}_p} z^y \lambda(y),$$

thus $A_{\lambda \ast \mu} = A_\lambda A_\mu$. 
1.5  The $p$-adic zeta function

1.5.1  Kummer’s congruences.

Lemma 1.5.1. For $a \in \mathbb{Z}^*_p$, there exists a measure $\lambda_a \in \mathcal{D}_0$ such that

$$A_{\lambda_a} = \int_{\mathbb{Z}_p} (1 + T)^x \lambda_a = \frac{1}{T} - \frac{a}{(1 + T)^a - 1}.$$ 

Proof. This follows from Theorem 1.4.5 and the fact

$$\frac{a}{(1 + T)^a - 1} = \sum_{n=1}^{\infty} \frac{a}{a^n} T^n = \frac{1}{T} \cdot \left(1 + \sum_{n=2}^{\infty} \frac{1}{a^n} T^{n-1}\right) \in \frac{1}{T} + \mathbb{Z}_p[[T]]$$

since $a^{-1}(a) \in \mathbb{Z}_p$. Moreover, we have $v_{D_0}(\lambda_a) = 0$. \hfill $\square$

Proposition 1.5.2. For every $n \in \mathbb{N}$, $\int_{\mathbb{Z}_p} x^n \lambda_a = (-1)^n (1 - a^{1+n}) \zeta(-n)$.

Proof. For $a \in \mathbb{R}^*_+$, for $T = e^t - 1$, let

$$f_a(t) = A_{\lambda_a}(T) = \frac{1}{e^t - 1} - \frac{a}{e^{at} - 1},$$

then $f_a$ is in $C^\infty$ on $\mathbb{R}^+$ and rapidly decreasing. Hence

$$L(f_a, s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} f_a(t) t^{s-1} \frac{dt}{t} = (1 - a^{1-s}) \zeta(s)$$

$$f_a^n(0) = (-1)^n L(f_a, -n) = (-1)^n (1 - a^{1+n}) \zeta(-n)$$

The identity $f_a^n(0) = (-1)^n (1 - a^{1+n}) \zeta(-n)$ is algebraic, so is true for all $a$, hence even on $\mathbb{Z}_p^*$. Thus

$$\int_{\mathbb{Z}_p} x^n \lambda_a = \left(\frac{d}{dt}\right)^n \left(\int_{\mathbb{Z}_p} e^{tx} \lambda_a\right)|_{t=0} = \left(\frac{d}{dt}\right)^n A_{\lambda_a}(e^t - 1)|_{t=0} = f_a^n(0).$$

\hfill $\square$

Corollary 1.5.3. For $a \in \mathbb{Z}_p^*$, $k \geq 1$ ($k \geq 2$ if $p = 2$), $n_1, n_2 \geq k$, $n_1 \equiv n_2 \mod (p - 1)p^{k-1}$, then

$$v_p((1 - a^{1+n_1}) \zeta(-n_1) - (1 - a^{1+n_2}) \zeta(-n_2)) \geq k.$$
Proof. The left hand side \( \text{LHS} = v_p((1 - a^{1+n_1})\zeta(-n_1) - (1 - a^{1+n_2})\zeta(-n_2)) \)

is

\[ v_p\left(\int_{\mathbb{Z}_p} (x^{n_1} - x^{n_2}) \lambda_a \right) \geq v_{D_0}(\lambda_a) + v_{C_0}(x^{n_1} - x^{n_2}). \]

From the proof of Lemma 1.5.1, \( v_{D_0}(\lambda_a) = 0 \), thus \( \text{LHS} \geq v_{C_0}(x^{n_1} - x^{n_2}) \). It suffices to show \( v_{C_0}(x^{n_1} - x^{n_2}) \geq k \). There are two cases:

- If \( x \in p\mathbb{Z}_p \), then \( v_p(x^{n_1}) \geq k \) and \( v_p(x^{n_2}) \geq k \) since \( n_1, n_2 \geq k \).

- If \( x \in \mathbb{Z}_p^* \), \( v_p(x^{n_1} - x^{n_2}) \geq k \) because \( (\mathbb{Z}/p^k\mathbb{Z})^* \) has order \( (p-1)p^{k-1} \) and \( n_1 - n_2 \) is divisible by \( (p-1)p^{k-1} \).

\[ \square \]

Remark. The statement is not clean because of \( x \in p\mathbb{Z}_p \).

1.5.2 Restriction to \( \mathbb{Z}_p^* \).

Lemma 1.5.4. \( \psi\left(\frac{1}{T}\right) = \frac{1}{T} \).

Proof. Let \( F(T) = \psi\left(\frac{1}{T}\right) \), then

\[ F((1 + T)^p - 1) = \frac{1}{p} \sum_{z = 1}^{+\infty} \frac{1}{(1 + T)^z - 1} \]

\[ = -\frac{1}{p} \sum_{z = 1}^{+\infty} \sum_{n = 0}^{+\infty} ((1 + T)^z)^n \]

\[ = -\sum_{n = 0}^{+\infty} (1 + T)^{pn} = \frac{1}{(1 + T)^p - 1}. \]

\[ \square \]

Proposition 1.5.5. \( \psi(\lambda_a) = \lambda_a \).

Proof. We only need to show the same thing on the Amice transform, but

\[ A_{\lambda_a}(T) = \frac{1}{T} - \frac{a}{(1 + T)^a - 1} = \frac{1}{T} - a \cdot \gamma_a\left(\frac{1}{T}\right) \]

where \( \gamma_a \in \Gamma \) is the inverse of \( a \) by \( \chi : \Gamma \rightarrow \mathbb{Z}_p^* \), i.e., \( \chi(\gamma_a) = a \). Since \( \psi \) and \( \gamma_a \) commutes and \( \psi\left(\frac{1}{T}\right) = \frac{1}{T} \), we have

\[ \psi(A_{\lambda_a}) = \frac{1}{T} - a \gamma_a\left(\frac{1}{T}\right) = A_{\lambda_a}. \]

\[ \square \]
Corollary 1.5.6. (i) \( \text{Res}_{\mathbb{Z}_p^*}(\lambda_a) = (1 - \phi \psi)\lambda_a = (1 - \phi)\lambda_a, \)

(ii) \( \int_{\mathbb{Z}_p^*} x^n\lambda_a = \int_{\mathbb{Z}_p} x^n(1 - \phi)\lambda_a = (-1)^n(1 - a^{n+1})(1 - p^n)\zeta(-n). \)

Remark. The factor \((1 - p^n)\) is the Euler factor of the zeta function at \( p \).

Theorem 1.5.7. For \( i \in \mathbb{Z}/(p - 1)\mathbb{Z} \) (or \( i \in \mathbb{Z}/2\mathbb{Z} \) if \( p = 2 \)), there exists a unique function \( \zeta_{p,i} \), analytic on \( \mathbb{Z}_p \) if \( i \neq 1 \), and \((s - 1)\zeta_{p,1}(s)\) is analytic on \( \mathbb{Z}_p \), such that \( \zeta_{p,i}(-n) = (1 - p^n)\zeta(-n) \) if \( n \equiv -i \mod p - 1 \) and \( n \in \mathbb{N} \).

Remark. (i) If \( i \equiv 0 \mod 2 \), then \( \zeta_{p,i} = 0 \) since \( \zeta(-n) = 0 \) for \( n \) even and \( \geq 2 \);

(ii) To get \( p \)-adic continuity, one has to modify \( \zeta \) by some “Euler factor at \( p \)”.

(iii) Uniqueness is trivial because \( \mathbb{N} \) is infinite and \( \mathbb{Z}_p \) is compact.

(iv) The existence is kind of a miracle. Its proof relies on Leopoldt’s \( \Gamma \)-transform.

1.5.3 Leopoldt’s \( \Gamma \)-transform.

Lemma 1.5.8. (i) Every \( x \in \mathbb{Z}_p^* \) can be written uniquely as \( x = \omega(x)\langle x \rangle \), with

\[
\omega(x) \in \mu(\mathbb{Q}_p) = \begin{cases} 
\{ \pm 1 \} & \text{if } p = 2; \\
\mu_{p-1} & \text{if } p \neq 2
\end{cases}
\text{ and } \langle x \rangle \in 1 + 2p\mathbb{Z}_p.
\]

(ii) \( \omega(xy) = \omega(x)\omega(y), \langle xy \rangle = \langle x \rangle\langle y \rangle. \)

Proof. If \( p = 2 \), it is obvious.

If \( p \neq 2 \), \( \omega(x) = \lim_{n \to \infty} x^{p^n} = [\overline{x}]. \)

Remark. (i) \( \omega \) is the so-called Teichmüller character;

(ii) \( \langle x \rangle = \exp(\log(x)) \);

(iii) \( x^n = \omega(x)^n\langle x \rangle^n \), here \( \langle x \rangle^n \) is the restriction to \( \mathbb{N} \) of \( \langle x \rangle^s \) which is continuous in \( s \), \( \omega(x)^n \) is periodic of period \( p - 1 \), which is not \( p \)-adically continuous.

Proposition 1.5.9. If \( \lambda \) is a measure on \( \mathbb{Z}_p^* \), \( u = 1 + 2p \), then there exists a measure \( \Gamma^{(i)}_{\lambda} \) on \( \mathbb{Z}_p \) (Leopoldt’s transform) such that

\[
\int_{\mathbb{Z}_p^*} \omega(x)^i\langle x \rangle^s \lambda(x) = \int_{\mathbb{Z}_p} u^{sy}\Gamma^{(i)}_{\lambda}(y) = A_{\Gamma^0}(u^s - 1).
\]
CHAPTER 1. THE P-ADIC ZETA FUNCTION OF KUBOTA-LEOPOLDT

Proof. We have
\[
\int_{\mathbb{Z}_p} \omega(x)^i \langle x \rangle^s \lambda(x) = \sum_{\varepsilon \in \mu(Q_p)} \omega(\varepsilon)^i \int_{\varepsilon + 2p\mathbb{Z}_p} \langle x \rangle^s \lambda(x) = \sum_{\varepsilon \in \mu(Q_p)} \omega(\varepsilon)^i \int_{\varepsilon + 2p\mathbb{Z}_p} \langle x \varepsilon \rangle^s \gamma_{\varepsilon^{-1}} \cdot \lambda(x),
\]
where \(\gamma_{\varepsilon} \in \Gamma\) is such that \(\chi(\gamma_{\varepsilon}) = \varepsilon\). We have an isomorphism
\[
\alpha : 1 + 2p\mathbb{Z}_p \cong \mathbb{Z}_p
\]
\[
x \mapsto y = \frac{\log(x)}{\log(u)}.
\]
Then
\[
\int_{\mathbb{Z}_p} f(y) \alpha_*(\gamma_{\varepsilon^{-1}} \lambda) = \int_{1 + 2p\mathbb{Z}_p} f(\alpha(x)) \gamma_{\varepsilon^{-1}} \lambda.
\]
Now \(\langle x \rangle^s = \exp(s \log x) = \exp(s \log uy) = u^{sy}\) and hence
\[
\sum_{\varepsilon \in \mu(Q_p)} \omega(\varepsilon)^i \int_{1 + 2p\mathbb{Z}_p} \langle x \varepsilon \rangle^s \lambda(x) = \sum_{\varepsilon \in \mu(Q_p)} \omega(\varepsilon)^i \int_{\mathbb{Z}_p} u^{sy} \alpha_*(\gamma_{\varepsilon^{-1}} \cdot \lambda),
\]
we just set \(\Gamma^{(i)}_{\lambda} = \sum_{\varepsilon \in \mu(Q_p)} \omega(\varepsilon)^i \alpha_*(\gamma_{\varepsilon^{-1}} \cdot \lambda).\)

Definition 1.5.10.
\[
\zeta_{p,i}(s) = \frac{-1}{1 - \omega(a)^{1-i} \langle a \rangle^{1-s}} \int_{\mathbb{Z}_p^*} \omega(x)^{-i} \langle x \rangle^{-s} \lambda_a(x).
\]

Proof of Theorem 1.5.7. If \(n \equiv -i \mod p - 1\), then
\[
\zeta_{p,i}(-n) = \frac{1}{1 - \omega(a)^{1-i} \langle a \rangle^{1+n}} \int_{\mathbb{Z}_p^*} \omega(x)^{-i} \langle x \rangle^{n} \lambda_a(x)
\]
\[
= \frac{1}{1 - \omega(a)^{1+n} \langle a \rangle^{1+n}} \int_{\mathbb{Z}_p^*} \omega(x)^{n} \langle x \rangle^{n} \lambda_a(x)
\]
\[
= (1 - p^{-n}) \zeta(-n).
\]
The function \(\zeta_{p,i}\) is analytic if \(\omega(a)^{1-i} \neq 1\), which can be achieved if \(i \neq 1\). If \(i = 1\), there is a pole at \(s = 1\). \(\square\)
Remark. (i) A theorem of Mazur and Wiles (originally the Main conjecture of Iwasawa theory) describes the zeros of \( \zeta_{p,i}(s) \) in terms of ideal class groups of \( \mathbb{Q}_p(\mu_{p^n}), n \in \mathbb{N} \).

(ii) Main open question: For \( i \equiv 1 \mod 2 \), can \( \zeta_{p,i}(k) = 0 \), if \( k > 1 \) and \( k \in \mathbb{N} \)?

The case \( k = 1 \) is known. In this case, \( \zeta_{p,i}(1) \) is a linear combination with coefficients in \( \bar{\mathbb{Q}}^\times \) of log of algebraic numbers, hence by transcendental number theory (Baker’s theorem), \( \zeta_{p,i}(1) \neq 0 \).

1.6 \( \mathcal{C}^k \) functions

1.6.1 Definition.

Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be a given function. We define

\[
\begin{align*}
 f^{(0)}(x) &= f(x) \\
 f^{(i)}(x, h_1, \ldots, h_i) &= \frac{1}{h_1 \cdots h_i} \left( \sum_{I \subset \{1, \ldots, i\}} (-1)^{i-|I|} f(x + \sum_{j \in I} h_j) \right)
\end{align*}
\]

One notes that \( f^{(i)} \) is the analogue of the usual derivation in \( \mathcal{C}(\mathbb{R}, \mathbb{C}) \). In fact, if \( f : \mathbb{R} \to \mathbb{C} \) is in \( \mathcal{C}^k \) and \( i \leq k \), define \( f^{(i)} \) by the above formula, then

\[
 f^{(i)}(x, h_1, \ldots, h_i) = \int_{[0,1]^i} f^{(i)}(x + t_1h_1 + \cdots + t_1h_i) dt_1 \cdots dt_i,
\]

hence \( f^{(i)} \) is continuous and \( f^{(i)}(x, 0, \ldots, 0) = f^{(i)}(x) \).

Definition 1.6.1. A function \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) (or \( \mathbb{C}_p \)) is in \( \mathcal{C}^k \) if \( f^{(i)} \) can be extended as a continuous function on \( \mathbb{Z}_p^{i+1} \) for all \( i \leq k \).

Remark. If \( f \in \mathcal{C}^0 \) and \( h_1, \ldots, h_i \neq 0 \), then we have:

\[
v_p(f^{(i)}(x, h_1, \ldots, h_i)) \geq v_{\mathcal{C}^0}(f) - \sum_{j=1}^i v_p(h_j).
\]
Example 1.6.2. The definition of $\mathcal{C}^k$ here is different than the usual case. Here is an example. For all $x$ in $\mathbb{Z}_p$, $x = \sum_{n=0}^{+\infty} p^n a_n(x)$ with $a_n(x) \in \{0, 1, \cdots, p-1\}$. Let $f(x) = \sum_{n=0}^{+\infty} p^{2n} a_n(x)$, then $v_p(f(x) - f(y)) = 2v_p(x - y)$. Hence $f'(x) = 0$ for all $x \in \mathbb{Z}_p$, thus $f$ is in $\mathcal{C}^\infty$ in the usual sense. But $f$ is not $\mathcal{C}^2$ in our case. In fact, let $(x, h_1, h_2) = (0, p^n, p^n)$ and $((p-1)p^n, p^n, p^n)$, here $p \neq 2$, we have:
\[
f^{(2)}(0, p^n, p^n) = 0; \quad f^{(2)}((p-1)p^n, p^n, p^n) = p - p^2.
\]

We define a valuation on $\mathcal{C}^k$ functions by:
\[
v'_{\mathcal{C}^k}(f) = \min_{0 \leq i \leq k} \inf_{(x, h_1, \cdots, h_i) \in \mathbb{Z}_{p^{i+1}}} v_p(f^{(i)}(x, h_1, \cdots, h_i)).
\]

Let $L(n, k) = \max\{\sum_{j=1}^{i} v_p(n_j), i \leq k, \sum n_j = n, n_j \geq 1\}$

Theorem 1.6.3 (Barsky). $p^{L(n,k)} \left( \frac{x}{n} \right)$ is a Banach basis of $\mathcal{C}^k$.

Exercise. there exists a $C_k$, such that for all $n \geq 1$,
\[
k \frac{\log n}{\log p} - C_k \leq L(n, k) \leq k \frac{\log n}{\log p}.
\]

Corollary 1.6.4. The following three conditions are equivalent:

(i) $\sum_{n=0}^{+\infty} a_n(x)_n \in \mathcal{C}^k$,

(ii) $\lim_{n \to +\infty} v_p(a_n) - k \frac{\log n}{\log p} = +\infty$,

(iii) $\lim_{n \to +\infty} n^k |a_n| = 0$.

Definition 1.6.5. If $r \geq 0$, $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is in $\mathcal{C}^r$ if
\[
f = \sum_{n=0}^{+\infty} a_n(f) \left( \frac{x}{n} \right)
\]

and
\[
n^r |a_n(f)| \to 0 \text{ when } n \to +\infty.
\]

$\mathcal{C}^r$ becomes a Banach space with the valuation:
\[
v_{\mathcal{C}^r}(f) = \inf_{n \in \mathbb{N}} \{v_p(a_n) - r \frac{\log(1 + n)}{\log p}\}.
\]
1.6. \( C^K \) FUNCTIONS

1.6.2 Mahler’s coefficients of \( C^r \)-functions.

We need M"ahler’s Theorem in several variables to prove Barsky’s theorem. Let \( g(x_0, x_1, \ldots, x_i) \) be a function defined on \( \mathbb{Z}_p^{i+1} \). We define the action \( \alpha_j^{[k]} \) on \( g \) by the following formula:

\[
\alpha_j^{[1]} g(x_0, \ldots, x_i) = g(x_0, \ldots, x_j + 1, \ldots, x_i) - g(x_0, \ldots, x_i),
\]
\[
\alpha_j^{[k]} = \alpha_j^{[1]} \circ \alpha_j^{[1]} \circ \cdots \alpha_j^{[1]}, \text{ } k \text{ times}.
\]

We set

\[
a_{k_0, \ldots, k_i}(g) = \alpha_{0}^{[k_0]} \ldots \alpha_{i}^{[k_i]} g(0, \ldots, 0).
\]

Recall that

\[
C^0(\mathbb{Z}_p^{i+1}, \mathbb{Q}_p) = C^0(\mathbb{Z}_p, \mathbb{Q}_p) \otimes \cdots \otimes C^0(\mathbb{Z}_p, \mathbb{Q}_p).
\]

**Theorem 1.6.6 (M"ahler).** If \( g \) is continuous on \( \mathbb{Z}_p^{i+1} \), then \( a_{k_0, \ldots, k_i}(g) \to 0 \) when \( (k_0, \ldots, k_i) \to \infty \) and we have the following identity:

\[
g(x_0, \ldots, x_i) = \sum_{k_0, \ldots, k_i \in \mathbb{N}} a_{k_0, \ldots, k_i}(g) \binom{x_0}{k_0} \cdots \binom{x_i}{k_i} \quad (1.2)
\]

Conversely, if \( a_{k_0, \ldots, k_i} \to 0 \), then the function \( g \) via equation (1.2) is continuous on \( \mathbb{Z}_p^{i+1} \), \( a_{k_0, \ldots, k_i}(g) = a_{k_0, \ldots, k_i} \), and

\[
v_{C^0}(g) = \inf v_p(a_{k_0, \ldots, k_i}).
\]

**Proof of Theorem 1.6.3.** Let \( g_T(x) = (1 + T)^x \), then we have:

\[
g_T^{(i)}(x, h_1, \ldots, h_i) = \frac{1}{h_1 \cdots h_i} \left( \sum_{I \subset \{1, \ldots, i\}} (-1)^{|I|} g_T(x + \sum_{j \in I} h_j) \right)
\]
\[
= (1 + T)^x \prod_{j=1}^{i} \frac{(1 + T)^{h_j} - 1}{h_j}
\]

Let \( P_n = \binom{x}{n} \). Since \( \frac{1}{x} \binom{x}{n} = \frac{1}{n} \binom{x-1}{n-1} \) and \( g_T^{(i)}(x, h_1, \ldots, h_i) = \sum_{n=0}^{\infty} P_n^{(i)}(x, h_1, \ldots, h_i) T^n \), we have the following formulas:

\[
P_n^{(i)}(x_0, h_1, \ldots, h_i) = \sum_{n_0 + n_1 + \cdots + n_i = n, n_1, \ldots, n_i \geq 1} \frac{1}{n_1 \cdots n_i} \binom{x_0}{n_0} \left( \frac{h_1 - 1}{n_1 - 1} \right) \cdots \left( \frac{h_i - 1}{n_i - 1} \right).
\]
CHAPTER 1. THE $p$-ADIC ZETA FUNCTION OF KUBOTA-LEOPOLDT

Let

$$Q_{n,i}(x_0, \cdots, x_i) = P_n^{(i)}(x_0, x_1 + 1, \cdots, x_i + 1)$$

$$= \sum_{n_0+n_1+\cdots+n_i=n, n_1, \cdots, n_i \geq 1} \frac{1}{n_1 \cdots n_i} \left( \begin{array}{c} x_0 \\ n_0 \\ \end{array} \right) \left( \begin{array}{c} x_1 \\ n_1 - 1 \\ \end{array} \right) \cdots \left( \begin{array}{c} x_i \\ n_i - 1 \\ \end{array} \right).$$

For all $f \in C^0(\mathbb{Z}_p, \mathbb{Q}_p)$, we have $f(x) = \sum_{n=0}^{+\infty} a_n(f) \left( \begin{array}{c} x \\ n \end{array} \right)$. We denote

$$g_i(x_0, \cdots, x_i) = \sum_{n=0}^{+\infty} a_n(f) Q_{n,i}(x_0, \cdots, x_i)$$

if $x_j + 1 \neq 0, j \geq 1$. We have:

$$a_{n_0,n_1-1,\cdots,n_i-1}(g_i) = \sum_{n=0}^{+\infty} a_n(f) a_{n_0,n_1-1,\cdots,n_i-1}(Q_{n,i})$$

where

$$a_{n_0,n_1-1,\cdots,n_i-1}(Q_{n,i}) = \begin{cases} 0 & \text{if } n \neq \sum_{j=0}^{i} n_j, \\ \frac{1}{n_1 \cdots n_i} & \text{if } n = \sum_{j=0}^{i} n_j. \end{cases}$$

If $f$ is in $C^k$, $i \leq k$, then $g_i$ is continuous on $\mathbb{Z}_p^{i+1}$, thus

$$\frac{a_{n_0+n_1+\cdots+n_i}(f)}{n_1 \cdots n_i} \to 0.$$

Conversely, if $\frac{a_{n_0+n_1+\cdots+n_i}(f)}{n_1 \cdots n_i} \to 0$, then

$$\sum_{n=0}^{+\infty} \sum_{n_0+n_1+\cdots+n_i=n} a_{n_0,n_1,\cdots,n_i}(f) \left( \begin{array}{c} x_0 \\ n_0 \\ \end{array} \right) \left( \begin{array}{c} x_1 \\ n_1 - 1 \\ \end{array} \right) \cdots \left( \begin{array}{c} x_i \\ n_i - 1 \\ \end{array} \right)$$

defines a continuous functions $G_i$ on $\mathbb{Z}_p^{i+1}$. But $G_i = g_i$ on $\mathbb{N}^{i+1}$, hence $G_i = g_i$, $x_j + 1 \neq 0$, for all $j \geq 1$, hence $f$ is in $C^k$. \qed
1.7 Locally analytic functions

1.7.1 Analytic functions on a closed disk.

**Lemma 1.7.1.** Let \((a_n)_{n \in \mathbb{N}}\) with \(a_n \in C_p\) be a sequence such that \(v_p(a_n) \to \infty\) when \(n \to \infty\), let \(f = \sum_{n=0}^{+\infty} a_n T^n\). Then:

(i) If \(x_0 \in O_{C_p}\), then \(f^{(k)}(x_0)\) converges for all \(k\) and

\[
\lim_{n \to \infty} v_p\left(\frac{f^{(k)}}{k!}(x_0)\right) = \infty.
\]

(ii) If \(x_0, x_1\) are in \(O_{C_p}\), then

\[
f(x_1) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x_1 - x_0)^n
\]

and

\[
\inf_{n \in \mathbb{N}} v_p\left(\frac{f^{(n)}(x_0)}{n!}\right) = \inf_{n \in \mathbb{N}} v_p(a_n);
\]

(iii) \(\inf_{n \in \mathbb{N}} v_p(a_n) = \inf_{x \in O_{C_p}} v_p(f(x))\) and \(v_p(f(x)) = v_p(a_n)\) almost everywhere (i.e., outside a finite number of \(x_i + m_{C_p}\)).

**Proof.** (i) \(\frac{f^{(k)}}{k!} = \sum_{n=0}^{+\infty} a_n + k(n+k)T^n\). Let \(T = x_0^k\); since \(v_p\left(\binom{n+k}{k}\right) \geq 0\), \(v_p(x_0^n) \geq 0\), we get (1) and also

\[
v_p\left(\frac{f^{(k)}(x_0)}{k!}\right) \geq \inf_{n \in \mathbb{N}} v_p(a_n) = \inf_{n \in \mathbb{N}} v_p\left(\frac{f^{(n)}(0)}{n!}\right).
\]

(ii)

\[
f(x_1) = \sum_{n=0}^{+\infty} a_n x_1^n = \sum_{n=0}^{+\infty} a_n \left(\sum_{k=0}^{n} \binom{n}{k} (x_1 - x_0)^k x_0^{n-k}\right)
\]

\[
= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} a_n \binom{n}{k} x_0^{n-k} (x_1 - x_0)^k = \sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!} (x_1 - x_0)^n.
\]

So we can exchange the the roles of 0 and \(x_0\) to get

\[
\inf_{n \in \mathbb{N}} v_p\left(\frac{f^{(n)}(x_0)}{n!}\right) = \inf_{n \in \mathbb{N}} v_p(a_n).
\]
(iii) That \( \inf_{n \in \mathbb{N}} v_p(a_n) \leq \inf_{x \in \mathcal{O}_{\mathbb{C}_p}} v_p(f(x)) \) is clear. As \( v_p(a_n) \) goes to \( +\infty \), \( v_p(a_n) \) reaches its infimum at some \( n_0 \in \mathbb{N} \). So we can divide everything by \( a_{n_0} \) and we may assume that \( \inf_{n \in \mathbb{N}} v_p(a_n) = 0 \). Let \( \overline{f}(T) = f(T) \mod \mathfrak{m}_{\mathbb{C}_p} \in \mathbb{F}_p[T] \). If \( x \in \mathcal{O}_{\mathbb{C}_p} \) doesn’t reduce mod \( \mathfrak{m}_{\mathbb{C}_p} \) to a root of \( \overline{f} \), then \( \overline{f}(x) \neq 0 \), equivalently, \( v_p(f(x)) = 0 \).

**Corollary 1.7.2.** Let \( f = \sum_{n=0}^{+\infty} a_n T^n \), \( g = \sum_{n=0}^{+\infty} b_n T^n \), then \( fg = \sum_{n=0}^{+\infty} c_n T^n \), where \( c_n = \sum_{i=0}^{n} a_i b_{n-i} \). Suppose that \( v_p(a_n) \) and \( v_p(b_n) \) go to infinity when \( n \) goes to infinity, then \( v_p(c_n) \) goes to infinity and \( \inf_{n} v_p(c_n) = \inf_{n} (a_n) + \inf_{n}(b_n) \).

**Definition 1.7.3.** For \( x_0 \in \mathbb{C}_p \), \( r \in \mathbb{R} \), we define

\[
D(x_0, r) = \{ x \in \mathbb{C}_p, v_p(x - x_0) \geq r \}.
\]

**Definition 1.7.4.** A function \( f : D(x_0, r) \to \mathbb{C}_p \) is analytic if it is sum of its Taylor expansion at \( x_0 \) or equivalently, if

\[
\lim_{n \to +\infty} (v_p(\frac{f^{(n)}(x_0)}{n!}) + nr) = +\infty.
\]

We define \( v^{(r)}_{x_0}(f) = \inf_n(v_p(\frac{f^{(n)}(x_0)}{n!}) + nr) \).

**Proposition 1.7.5.** If the function \( f : D(x_0, r) \to \mathbb{C}_p \) is analytic, then

(i) For all \( k \in \mathbb{N} \), \( f^{(k)} \) is analytic on \( D(x_0, r) \),

\[
v^{(r)}_{x_0}(\frac{f^{(k)}(x_0)}{k!}) + kr \geq v^{(r)}_{x_0}(f)
\]

and goes to \( +\infty \) if \( k \) goes to \( +\infty \).

(ii) \( f \) is the sum of its Taylor expansion at any \( x \in D(x_0, r) \).

(iii) \( v^{(r)}_{x_0}(f) = \inf_{x \in D(x_0, r)} v_p(f(x)) \).

(iv) \( v^{(r)}_{x_0}(fg) = v^{(r)}_{x_0}(f) + v^{(r)}_{x_0}(g) \).

**Proof.** If \( r \in \mathbb{Q} \), one can choose \( \alpha \in \mathbb{C}_p \), such that \( v_p(\alpha) = r \). Let \( F(x) = f(x + \alpha x), x \in \mathcal{O}_{\mathbb{C}_p} \). Apply the previous lemma, we can get the result.

If \( r \notin \mathbb{Q} \), choose \( r_n \) decreasing with the limit \( r \), \( r_n \in \mathbb{Q} \). Use \( D(x_0, r) = \bigcup_n D(x_0, r_n) \) and the case \( r \in \mathbb{Q} \), we get the result. \( \square \)
1.7. LOCALLY ANALYTIC FUNCTIONS

1.7.2 Locally analytic functions on \( \mathbb{Z}_p \).

**Definition 1.7.6.** Let \( h \in \mathbb{N} \) be given. The space \( LA_h(\mathbb{Z}_p, \mathbb{Q}_p) \) is the space of \( f \) whose restriction to \( x_0 + p^h \mathbb{Z}_p \) is the restriction of an analytic function \( f_{x_0} \) on \( D(x_0, h) \), for all \( x_0 \in \mathbb{Z}_p \). The valuation of the space is 
\[
    v_{LA_h} = \inf_{x_0 \in S} v_f(x_0) \quad (f_{x_0}) ,
\]
where \( S \) be any set of representations of \( \mathbb{Z}_p / p^h \mathbb{Z}_p \). (Use above proposition to prove that this does not depend on \( S \).)

**Lemma 1.7.7.** \( LA_h \) is a Banach space. Moreover, let
\[
e_n = 1_{x + p^h \mathbb{Z}_p}(x + i)p^{-1}, n = mp^h - i, m \geq 1, 1 \leq i \leq p^h ,
\]
then \( e_n \)'s are a Banach basis of \( LA_h \).

**Theorem 1.7.8 (Amice).** The functions \( \left\lfloor \frac{n}{p^h} \right\rfloor ! (\frac{x}{n}) \), \( n \in \mathbb{N} \) are a Banach basis of \( LA_h \).

**Proof.** The idea is to try to relate the \( g_n = \left\lfloor \frac{n}{p^h} \right\rfloor ! (\frac{x}{n}) \) to the \( e_n \).

(i) First step: For \( 1 \leq j \leq p^h \), we denote
\[
g_{n,j}(x) = g_n(-j + p^h x) = \left\lfloor \frac{n}{p^h} \right\rfloor ! \prod_{k=0}^{n-1} (-j - k + p^h x).
\]
If \( v_p(j + k) < h \), then \( v_p(-j - k + p^h x) = v_p(j + k) \), for all \( x \in O_{\mathbb{C}_p} \). If \( v_p(j + k) \geq h \), then \( v_p(-j - k + p^h x) \geq h \) with equality if \( x \notin \mathbb{F}_p \subset \mathbb{F}_p \). So, we get
\[
v_{0}^{(g_{n,j})} = v_p(-j + p^h x) = \sum_{k=0}^{n-1} \inf(v_p(x + k), h) = \sum_{i=1}^{\infty} \# \{ k : v_p(k) \geq i, 1 \leq k \leq n \}.
\]
Since \( v_p(n!) = \sum_{k=1}^{n} v_p(k) = \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor \), we have
\[
v_p(n!) - v_p(\left\lfloor \frac{n}{p^h} \right\rfloor !) = \sum_{i=1}^{h} \# \{ k : v_p(k) \geq i, 1 \leq k \leq n \} = \sum_{k=1}^{n} \inf(v_p(k), h).
\]
Thus,
\[
v_{0}^{(g_{n,j})} = \sum_{k=1}^{n} [\inf(v_p(j + k - 1), h) - \inf(v_p(k), h)]
\]
\[= \sum_{i=1}^{h} (\lfloor \frac{n + j - 1}{p^i} \rfloor - \lfloor \frac{j - 1}{p^i} \rfloor - \lfloor \frac{n}{p^i} \rfloor).\]
As \([x + y] \geq [x] + [y]\), we have \(v_{0}^{(1)}(g_{n,j}) \geq 0\), for all \(1 \leq j \leq p^{h}\). So, we have \(v_{LA_{h}}(g_{n}) \geq 0\).

(ii) Second step: we need a lemma

**Lemma 1.7.9.** Let \(n = mp^{h} - i\), \(\overline{g_{n,j}} \in \mathbb{F}_{p}[x]\), then:

(i) \(\overline{g_{n,j}} = 0\), if \(j > i\),

(ii) \(\deg \overline{g_{n,j}} = m - 1\), if \(j = i\),

(iii) \(\deg \overline{g_{n,j}} \leq m - 1\) if \(j < i\).

The lemma implies the theorem: \(\overline{g_{n}}\) can be written in terms of the \(\overline{e_{n}}\), multiplying by an invertible upper triangular matrix. Now use the fact that \(x_{n}\) is a Banach basis if and only if \(x_{n}\) is a basis of \(LA_{h}^{0}/pLA_{h}^{0}\) over \(\mathbb{F}_{p}\).

**Proof of Lemma 1.7.9.** (i) If \(j > i\), then \(j - 1 \geq i\). Since

\[
\left\lceil \frac{n + j - 1}{p^{h}} \right\rceil - \left\lfloor \frac{j - 1}{p^{h}} \right\rfloor - \left\lfloor \frac{n}{p^{h}} \right\rfloor = m - (m - 1) = 1,
\]

we have \(v_{0}^{(1)}(g_{n,j}) \geq 1\), then \(\overline{g_{n,j}} = 0\).

(ii) and (iii): If \(j \leq i\), write

\[g_{n,j}(x) = \sum_{k=0}^{n} a_{k}x^{k}, a_{k} \in \mathbb{Z}_{p}.\]

The zeros of \(g_{n,j}\) are the \(\frac{i+k}{p^{h}}, 0 \leq k \leq n - 1\) and

\[\#\{\text{zeros in } \mathbb{Z}_{p}\} = \#\{k : v_{p}(j + k) \geq h\} = \left\lceil \frac{n + j - 1}{p^{h}} \right\rceil - \left\lfloor \frac{j - 1}{p^{h}} \right\rfloor = m - 1.\]

Let \(\{\alpha_{i} : 1 \leq i \leq m - 1\}\) be the set of the roots with \(\alpha_{1}, \cdots, \alpha_{m-1}\) in \(\mathbb{Z}_{p}\) and \(\alpha_{m}, \cdots, \alpha_{n}\) not in \(\mathbb{Z}_{p}\). Then

\[g_{n,j} = c \prod_{l=1}^{m-1} (x - \alpha_{l}) \prod_{l=m}^{n} (1 - \alpha_{l}^{-1}x), (c \text{ is a constant}).\]

Since \(v_{p}(\alpha_{l}^{-1}) > 0\) when \(l \geq m\), then \(v_{p}(a_{m-1}) = v_{p}(c) = v_{0}^{(1)}(g_{n,j})\). It implies \(c \in \mathbb{Z}_{p}\). Hence

\[\overline{g_{n,j}} = \overline{c} \prod_{l=1}^{m-1} (x - \overline{\alpha}_{l}).\]
It remains to prove $v_0^{(0)}(g_{n,i}) = 0$. Since

$$v_0^{(0)}(g_{n,i}) = \sum_{l=1}^{h} \left( \left\lfloor \frac{m^h - 1}{p^l} \right\rfloor - \left\lfloor \frac{i - 1}{p^l} \right\rfloor + \left\lfloor \frac{m^h - i}{p^l} \right\rfloor \right)$$

and $-\left\lfloor \frac{-1}{a} \right\rfloor = \left\lfloor i - 1 \right\rfloor + 1$, we get the result.

Let $LA = \{ \text{locally analytic functions on } \mathbb{Z}_p \}$. Because $\mathbb{Z}_p$ is compact, $LA = \cup LA_h$ and is an inductive limit of Banach spaces. So

(i) A function $\varphi : LA \to B$ is continuous if and only if $\varphi|_{LA_h} : LA_h \to B$ is continuous for all $h$.

(ii) A sequence $f_n \to f$ converges in $LA$ if and only if there exists $h$ such that for all $n$, $f_n \in LA_h$ and $f_n \to f$ in $LA_h$.

Since $\frac{1}{n}v_p\left(\left\lfloor \frac{n}{p} \right\rfloor \right) \sim \frac{1}{(p-1)p^r}$, we have the following theorem:

**Theorem 1.7.10.** The function $f = \sum_{n=0}^{+\infty} a_n \left( \begin{array}{c} x \\ n \end{array} \right)$ is in $LA$ if and only if there exists $r > 0$, such that $v_p(a_n) - rn \to +\infty$ when $n \to +\infty$.

### 1.8 Distributions on $\mathbb{Z}_p$

#### 1.8.1 The Amice transform of a distribution.

**Definition 1.8.1.** A distribution $\mu$ on $\mathbb{Z}_p$ with values in $B$ is a continuous linear map $f \mapsto \int_{\mathbb{Z}_p} f \mu$ from $LA$ to $B$. We denote the set of distributions from $LA$ to $B$ by $D(\mathbb{Z}_p, B)$.

**Remark.** (i) $\mu|_{LA_h}$ is continuous for all $h \in \mathbb{N}$. Set

$$v_{LA_h}(\mu) = \inf_{f \in LA_h} (v_B(\int_{\mathbb{Z}_p} f \mu) - v_{LA_h}(f)).$$

Then $v_{LA_h}$ is a valuation on $D(\mathbb{Z}_p, B)$ for all $h$, and $D(\mathbb{Z}_p, B)$ is complete for the Fréchet topology defined by $v_{LA_h}$, $h \in \mathbb{N}$ which means that $\mu_n$ goes to $\mu$ if and only if $v_{LA_h}(\mu_n - \mu) \to +\infty$ for all $h$.

(ii) $D(\mathbb{Z}_p, B) = D(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} B$. From now on, we will denote $D(\mathbb{Z}_p, \mathbb{Q}_p)$ by $D$. 
CHAPTER 1. THE $p$-ADIC ZETA FUNCTION OF KUBOTA-LEOPOLDT

Let $\mathcal{R}^+$ be the ring of analytic functions defined on $D(0,0^+) = \{ x \in \mathbb{C}_p, v_p(x) > 0 \}$. A function $f \in \mathcal{R}^+$ can be written as $f = \sum_{n=0}^{+\infty} a_n T^n$, $a_n \in \mathbb{Q}_p$ for all $n \in \mathbb{N}$.

Let $v_h = \frac{1}{(p-1)p^h} = v_p(\varepsilon - 1)$, where $\varepsilon$ is a primitive $p^{h+1}$ root of 1.

If $F(T) = \sum_{n=0}^{+\infty} b_n T^n \in \mathcal{R}^+$, we define $v^{(h)}(F)$ to be

$$v^{(h)}(F) = v_0^{(v_h)}(F) = \inf_{n \in \mathbb{N}} v_p(b_n) + nv_h.$$

Then, for $F, G \in \mathcal{R}^+$,

$$v^{(h)}(FG) = v^{(h)}(F) + v^{(h)}(G).$$

We put on $\mathcal{R}^+$ the Fréchet topology defined by the $v^{(h)}$, $h \in \mathbb{N}$.

**Definition 1.8.2.** The Amice transform of a distribution $\mu$ is the function:

$$A_\mu(T) = \sum_{n=0}^{+\infty} T^n \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) \mu = \int_{\mathbb{Z}_p} (1 + T)^x \mu.$$

Note that the last identity in the above definition is only a formal identity here. However, we have

**Lemma 1.8.3.** If $v_p(z) > 0$, then $\int_{\mathbb{Z}_p} (1 + z)^x \mu = A_\mu(z)$

**Proof.** Choose $h$ such that $v_h < v_p(z)$. Then

$$v_p(\left( \frac{z^n}{[n/p^h]} \right) \rightarrow +\infty,$$

therefore $\sum_{n=0}^{+\infty} z^n(\frac{x}{n})$ converges to $(1 + z)^x$ in $LA_h$. 

**Theorem 1.8.4.** The map $\mu \mapsto A_\mu$ is an isomorphism of Fréchet spaces from $\mathcal{D}$ to $\mathcal{R}^+$. moreover,

$$v^{(h)}(A_\mu) \geq v_{LA_h}(\mu) \geq v^{(h+1)}(A_\mu) - 1.$$
1.8. DISTRIBUTIONS ON $\mathbb{Z}_p$

Proof. Let $A_{\mu}(T) = \sum_{n=0}^{+\infty} b_n T^n$. Since $b_n = \int_{\mathbb{Z}_p} (\frac{x}{n}) \mu$ and $v_p(n!) \leq \frac{n}{p-1}$, then we have:

$$v_p(b_n) = v_p(b_n) - v_{LA_h}(\left(\frac{x}{n}\right)) + v_{LA_h}(\left(\frac{x}{n}\right))$$

$$\geq v_{LA_h}(\mu) + v_{LA_h}(\left(\frac{x}{n}\right)) = v_{LA_h}(\mu) - v_p\left(\left[\frac{n}{p^h}\right]!\right)$$

$$\geq v_{LA_h}(\mu) - \frac{n}{(p-1)p^h} = v_{LA_h}(\mu) - nv_h.$$

Hence $A_{\mu} \in \mathcal{R}^+$ and $v^{(h)}(A_{\mu}) \geq v_{LA_h}(\mu)$.

Conversely, for $F \in \mathcal{R}^+$, $F = \sum_{n=0}^{+\infty} b_n T^n$, then for all $h$,

$$v_p\left(\left[\frac{n}{p^h}\right]! b_n\right) = v_p(b_n) + \frac{n}{(p-1)p^h} \to \infty.$$

So $f \mapsto \sum_{n=0}^{+\infty} b_n a_n(f)$ is a continuous map on $LA_h$. Denote the left hand side by $\int_{\mathbb{Z}_p} f \mu$, this defines a distribution $\mu \in \mathcal{D}$. Moreover,

$$v_{LA_h}(\mu) = \inf_{n \in \mathbb{N}} v_p\left(\left[\frac{n}{p^h}\right]! b_n\right) \geq \inf_{n \in \mathbb{N}} v_p\left(\left[\frac{n}{p^{h+1}}\right]! b_n\right)$$

$$\geq \inf_{n \in \mathbb{N}} (v_p(b_n) + \frac{n}{(p-1)p^{h+1}}) - 1 = v_{LA_h}^{(h+1)}(A_{\mu}) - 1.$$

\[ \square \]

1.8.2 Examples of distributions.

(i) Measures are distributions and $\mathcal{D}_0 \subset \mathcal{D}$.

(ii) One can multiply a distribution $\mu \in \mathcal{D}$ by $g \in LA$, and one gets

- $A_{x\mu} = \partial A_{\mu}$, $\partial = (1 + T) \frac{d}{dT}$;
- $A_{z\mu}(T) = A_{\mu}((1 + T)z - 1)$;
- $A_{Res_{a+p^n\mathbb{Z}_p}}(T) = p^{-n} \sum_{z^{p^n}=1} z^{-a} A_{\mu}((1 + T)z - 1)$.
(iii) one gets actions $\varphi, \psi, \Gamma$ with the same formulas than on measures.
(iv) Convolution of distributions: If $f \in LA_h$ and for all $y \in y_0 + p^h \mathbb{Z}_p$,
\[
f(x + y) = \sum_{n=0}^{+\infty} \frac{p^h f^{(n)}(x + y_0)}{n!} \left( \frac{y - y_0}{p^h} \right)^n \in LA_h(x) \boxtimes LA_h(y),
\]
and $v_{LA_h}(\frac{p^h f^{(n)}(x+y_0)}{n!})$ goes to $+\infty$, when $n \to +\infty$. Hence
\[
\int_{\mathbb{Z}_p} \left( \int_{\mathbb{Z}_p} f(x + y) \mu(x) \right) \lambda(y) = \int_{\mathbb{Z}_p} f \cdot \lambda \ast \mu
\]
is well defined, $A_{\lambda \ast \mu} = A_{\lambda} A_{\mu}$.
(v) The derived distribution: $\mu \mapsto d\mu$ given by $\int_{\mathbb{Z}_p} f d\mu = \int_{\mathbb{Z}_p} f' \mu$. Easy to check $A_{d\mu}(T) = \log(1+T) A_{\mu}(T)$. $\mu$ can’t be integrated because $\log(1+T) = 0$ if $T = \varepsilon - 1, \varepsilon \in \mu_{\mathbb{Z}_p}$. 
(vi) Division by $x$, the Amice transform $A_{x^{-1} \mu}$ of $x^{-1} \mu$ is a primitive (or called antiderivative) of $(1 + T)^{-1} A_{\mu}$, so $A_{x^{-1} \mu}$ is defined up to $\alpha \delta_0, \alpha \in \mathbb{Q}_p$ (we have $x \delta_0 = 0$).

1.8.3 Residue at $s = 1$ of the $p$-adic zeta function.

The Kubota-Leopoldt distribution $\mu_{KL}$ given by $A_{\mu_{KL}}(T) = \frac{\log(1+T)}{T}$. Then
\[
\int_{\mathbb{Z}_p} x^n \mu_{KL} = \left( \frac{d}{dt} \right)_0^n \left( \int_{\mathbb{Z}_p} e^{tx} \mu_{KL} \right) = \left( \frac{d}{dt} \right)_0^n A_{\mu_{KL}}(e^t - 1) = \left( \frac{d}{dt} \right)_0^n \left( \frac{t}{e^t - 1} \right) = (-1)^n n \zeta(1 - n), \text{ for all } n \in \mathbb{N}.
\]
Since
\[
\psi\left( \frac{1}{T} \right) = \frac{1}{T} \quad \text{and} \quad \varphi(\log(1 + T)) = \rho \log(1 + T),
\]
we get $\psi(\mu_{KL}) = \frac{1}{\rho} \mu_{KL}$ and
\[
\int_{\mathbb{Z}_p^*} x^n \mu_{KL} = (1 - p^{n-1}) \int_{\mathbb{Z}_p} x^n \mu_{KL} = (-1)^n n (1 - p^{n-1}) \zeta(1 - n);
\]
\[
\zeta_{p,i}(s) = \frac{(-1)^{i-1}}{s - 1} \int_{\mathbb{Z}_p^*} \omega(x)^{1-i}(x)^{1-s} \mu_{KL}.
\]
The integral is analytic in $s$ by the same argument as for measures.
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Proposition 1.8.5. \( \lim_{s \to 1} (s-1) \zeta_{p,1}(s) = \int_{\mathbb{Z}_p} \mu_{KL} = 1 - \frac{1}{p} \), (compare with \( \lim_{s \to 1} (s-1) \zeta(s) = 1 \)).

Proof. It follows from the following lemma. \( \square \)

Lemma 1.8.6. \( \int_{a+p^h \mathbb{Z}_p} \mu_{KL} = p^{-n} \), for all \( n \), for all \( a \in \mathbb{Z}_p \) (almost a Haar measure but \( \mu * \delta_a \neq \mu \)).

Proof.

\[
\int_{a+p^h \mathbb{Z}_p} \mu_{KL} = p^{-n} \sum_{z^{p^n}=1} z^{-a} A_{KL}(z-1) = p^{-n} \left( 1 + \sum_{z^{p^n}=1, z\neq 1} \frac{\log z}{z-1} \right),
\]

and \( \frac{\log z}{z-1} = 0 \), if \( z^{p^n} = 1, z \neq 1 \). \( \square \)

1.9 Tempered distributions

1.9.1 Analytic functions inside \( C^r \) functions

Theorem 1.9.1. For all \( r \geq 0 \), \( \text{LA} \subset C^r \). Moreover there exists a constant \( C(r) \) depending on \( r \), such that for all \( h \in \mathbb{N} \) and for all \( f \) in \( \text{LA}_h \),

\[
v_{C^r}(f) \geq v_{\text{LA}_h}(f) - rh - C(r).
\]

Proof. Since \( v_{\text{LA}_h}(f) = \inf_n (v_p(a_n(f)) - v_p([\frac{n}{p^h}]!)) \), we have

\[
v_{C^r}(f) = \inf_n (v_p(a_n(f)) - r \frac{\log(1+n)}{\log p}) \geq v_{\text{LA}_h}(f) + \inf_n (v_p([\frac{n}{p^h}]!) - r \frac{\log(1+n)}{\log p}).
\]

We have a formula for every \( a \):

\[
v_p(a!) = \left[ \frac{a}{p} \right] + \cdots + \left[ \frac{a}{p^h} \right] + \cdots \geq \frac{a}{p-1} - \frac{\log(1+a)}{\log p}
\]

Write \( n = p^h a + b, 0 \leq b \leq p^h - 1 \), then we have

\[
v_{C^r}(f) - v_{\text{LA}_h}(f) \geq \inf_n (v_p([\frac{n}{p^h}]!) - r \frac{\log(1+n)}{\log p})
\]

\[
= \inf_{a \in \mathbb{N}} \left( v_p(a!) - r \frac{\log(a p^h + b + 1)}{\log p} \right)
\]

\[
\geq \frac{a}{p-1} - (r+1) \frac{\log(a + 1)}{\log p} - rh.
\]
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The function \(-\frac{a}{p-1} + (r + 1) \frac{\log(a+1)}{\log p}\) of \(a\) is bounded above, we just let \(C(r)\) be its maximum.

Observe that the function \(\log\) is well defined on \(\mathbb{Z}_p^\times\). First if \(v_p(x-1) > 0\), let
\[
\log x = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} (x-1)^n;
\]
in general, if \(x = \omega(x)(x)\), let \(\log x = \log \omega(x)(x)\). If \(x = p\), let \(\log p = 0\). By the formula \(\log xy = \log x + \log y\), \(\log\) is well defined in \(\mathbb{Q}_p^{-\{0\}}\). This \(\log\) is the so-called Iwasawa’s log, or \(\log_0\).

However, we can define the value at \(p\) arbitrarily. For \(\mathcal{L} \in \mathbb{Q}_p\), define \(\log_\mathcal{L} p = \mathcal{L}\), then \(\log_\mathcal{L} x = \log_0 x + \mathcal{L} v_p(x)\).

**Theorem 1.9.2.** Choose a \(\mathcal{L}\) in \(\mathbb{C}_p\). Then there exists a unique \(\log_\mathcal{L} : \mathbb{C}_p^* \to \mathbb{C}_p\) satisfying:

(i) \(\log_\mathcal{L} x = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} (x-1)^n\), here \(v_p(x-1) > 0\),

(ii) \(\log_\mathcal{L} xy = \log_\mathcal{L} x + \log_\mathcal{L} y\),

(iii) \(\log_\mathcal{L} = \mathcal{L}\).

**Proposition 1.9.3.** If \(r \geq 0\), \(j > r\), then \(x^j \log_\mathcal{L} x \in \mathcal{C}^r\).

**Proof.** We have
\[
x^j \log_\mathcal{L} x = \sum_{n=0}^{+\infty} \sum_{a=1}^{p-1} 1_{p^n a + p^{n+1} \mathbb{Z}_p} x^j \log_\mathcal{L} x.
\]
Let \(f_{n,a} = 1_{p^n a + p^{n+1} \mathbb{Z}_p} x^j \log_\mathcal{L} x\). We have to prove the sum converges in \(\mathcal{C}^r\).

On \(p^n a + p^{n+1} \mathbb{Z}_p\), we have
\[
x^j \log_\mathcal{L} x = (x - p^n a + p^n a)^j \log_\mathcal{L} (p^n a + (x - p^n a))
\]
\[
= p^{nj} (a + p^{n+1} (x - p^n a)) (\log_\mathcal{L} p^n a + \log_0 (1 + p^{1/(p^{n+1} a)))).
\]
So \(f_{n,a} \in LA_{n+1}\), \(v_{LA_{n+1}}(f_{n,a}) \geq nj\). Use the previous theorem, we get \(v_{\mathcal{C}^r}(f_{n,a}) \geq nj - r(n+1) - C(r)\) and it goes to +\(\infty\). \(\Box\)
1.9. TEMPERED DISTRIBUTIONS

1.9.2 Distributions of order \( r \)

**Definition 1.9.4.** Let \( r \geq 0 \) and \( B \) be a Banach space. A distribution \( \mu \in \mathcal{D}(\mathbb{Z}_p, B) \) is a distribution of order \( r \) if \( f \mapsto \int_{\mathbb{Z}_p} f \mu \) is a continuous map from \( \mathcal{C}^r(\mathbb{Z}_p, \mathbb{Q}_p) \) to \( B \). We denote the set of distributions of order \( r \) by \( \mathcal{D}_r(\mathbb{Z}_p, B) \). We define a valuation on \( \mathcal{D}_r(\mathbb{Z}_p, B) \) by

\[
v'_{\mathcal{D}_r}(\mu) = \inf_{f \in \mathcal{C}^r} (v_p(\int_{\mathbb{Z}_p} f \mu) - v_{\mathcal{C}^r}(f)).
\]

**Remark.** (i) Under the above valuation, \( \mathcal{D}_r(\mathbb{Z}_p, B) \) is a \( p \)-adic Banach space and \( \mathcal{D}_r(\mathbb{Z}_p, B) = \mathcal{D}_r(\mathbb{Z}_p, \mathbb{Q}_p) \widehat{\otimes} B \). We denote \( \mathcal{D}_r(\mathbb{Z}_p, \mathbb{Q}_p) \) by \( \mathcal{D}_r \).

(ii) \( \mathcal{D}_{\text{temp}} = \cup \mathcal{D}_r \) = set of tempered distributions.

(iii) Since \( LA_h \subset \mathcal{C}^r \), and for \( f \in LA_h \), \( v_{\mathcal{C}^r}(f) \geq v_{LA_h}(f) - rh - C(r) \), we get, for \( \mu \in \mathcal{D}_r \subset LA_h^* \),

\[
v_{LA_h}(\mu) = \inf_{f \in LA_h} (v_p(\int_{\mathbb{Z}_p} f \mu) - v_{LA_h}(f)) \geq v'_{\mathcal{D}_r}(\mu) - rh - C(r).
\]

**Theorem 1.9.5.** \( \mu \in \mathcal{D} \), the following are equivalent: (i) \( \mu \in \mathcal{D}_r \) i.e. \( \mu \) can be extended by continuity to \( \mathcal{C}^r \).

(ii) There exists a constant \( C \), such that \( v_p(\int_{\mathbb{Z}_p} x_n \mu) \geq C - r \frac{\log(1+n)}{\log p} \), for all \( n \).

(iii) There exists a constant \( C \), such that \( v_p(\int_{a+p^h\mathbb{Z}_p} (x-a)^j \mu) \geq C + h(j-r) \), for all \( a \in \mathbb{Z}_p, j \in \mathbb{N}, h \in \mathbb{N} \).

(iv) There exists a constant \( C \), such that \( v_{LA_h}(\mu) \geq C - rh \), for all \( h \in \mathbb{N} \).

**Remark.** It follows that

\[
v_{\mathcal{D}_r}(\mu) = \inf_{a \in \mathbb{Z}_p} \inf_{j \in \mathbb{N}, n \in \mathbb{N}} (v_p(\int_{a+p^h\mathbb{Z}_p} (x-a)^j \mu) - h(j-r))
\]

is equivalent to \( v'_{\mathcal{D}_r} \).

**Proof.** (i) \( \iff \) (ii) is just the definition of \( v'_{\mathcal{D}_r} \). (iii) \( \iff \) (iv) is true by the definition of \( LA_h \) (with some \( C \)). Remains to prove (ii) \( \iff \) (iv). We have \( v^{(h)}(A_\mu) \geq v_{LA_h}(\mu) \geq v^{(h+1)}(A_\mu) - 1 \), hence the proof is reduce to the following lemma with \( F = A_\mu \). \( \Box \)
Lemma 1.9.6. Suppose $F \in \mathcal{R}^+$, $F = \sum_{n=0}^{+\infty} b_n T^n$, the following are equivalent:

(i) there exists $C$, such that $v(h)(F) \geq C - rh$, for all $h \in \mathbb{N}$,

(ii) there exists $C'$, such that $v_p(b_n) \geq C' - r \frac{\log(1+n)}{\log p}$ for all $n$.

Proof. Let

$$C_0 = \inf_{h \in \mathbb{N}} (v(h)(F) + rh) = \inf_{h \in \mathbb{N}} (\inf_{n \in \mathbb{N}} (v_p(b_n) + \frac{n}{(p-1)p^h}) + rh),$$

$$C_1 = \inf_{n \in \mathbb{N}} (v_p(b_n) + r \frac{\log(1+n)}{\log p}).$$

Let $h = \lceil \frac{\log(1+n)}{\log p} \rceil$, then

$$v_p(b_n) \geq C_0 - rh - \frac{n}{(p-1)p^h} \geq C_0 - \frac{r \log(1+n)}{\log p} - 2,$$

which implies $C_1 \geq C_0 - 2$.

Now, if $h$ is fixed, then $C_1 - \frac{r \log(1+n)}{\log p} + \frac{n}{(p-1)p^h}$ is minimal for $(1+n) = (p-1)p^h$. Hence,

$$C_1 - \frac{r \log(1+n)}{\log p} + \frac{n}{(p-1)p^h} \geq C_1 - rh - \frac{\log(p-1)r}{\log p}.$$

Thus, $C_0 \geq C_1 - r \frac{\log(p-1)r}{\log p}$. \hfill \Box

For $N \geq 0$, let $LP^{[0,N]}$ be the set of the locally polynomial functions of degree no more than $N$ on $\mathbb{Z}_p$.

Theorem 1.9.7. Suppose $r \geq 0$, $N > r - 1$. If $f \mapsto \int_{\mathbb{Z}_p} f \mu$ is linear function from $LP^{[0,N]}$ to a Banach space $B$, such that there exists $C$,

$$v_p(\int_{a+p^n\mathbb{Z}_p} (x-a)^j \mu) \geq C + (j-r)n$$

for all $a \in \mathbb{Z}_p$ and $n, j \in \mathbb{N}$, then $\mu$ extends uniquely to an element of $\mathcal{D}_r$.

Remark. (i) Let $r = 0, N = 0$, we recover the construction of measures as bounded additive functions on open compact sets.
1.9. TEMPERED DISTRIBUTIONS

(ii) We define a new valuation on $D_r$

$$v_{D_r,N}(\mu) = \inf_{a \in \mathbb{Z}^p} \inf_{n \in \mathbb{N}, j \in \mathbb{N}} v_p(\int_{a+p^n\mathbb{Z}_p} (x-a)^j \mu) - n(j-r),$$

then $v_p(\int_{\mathbb{Z}_p} f \mu) \geq v_{LA_h}(f) + v_{D_r,N}(\mu) - rn$ for all $f \in LP^{[0,N]} \cap LA_h$.

(iii) The open mapping theorem in Banach spaces implies that $v_{D_r,N}$ is equivalent to $v_{D_r}$.

Proposition 1.9.8. If $f \in LA, r \geq 0, N > r - 1$, put

$$f_n = \sum_{i=0}^{p^n-1} f_n^{(i)}(x-i)^{k!} \in LP^{[0,N]},$$

then $f_n \to f$ in $C^r$. Hence $LP^{[0,N]}$ is dense in $C^r$.

Proof. There exists $h$, such that $f \in LA_h$. We assume $n \geq h$, then

$$v_{LA_h}(f - f_n) = \inf_{0 \leq i \leq p^n - 1} \inf_{k \geq N+1} v_p(p^h f^{(k)}(i) \frac{k!}{k}).$$

$f \in LA_h$ implies $v_p(p^h f^{(k)}(i) \frac{k!}{k}) \geq v_{LA_h}(f)$. Hence

$$v_{LA_h}(f - f_n) \geq v_{LA_h}(f) + (N+1)(n-h).$$

Then

$$v_{C^r}(f - f_n) \geq v_{LA_h}(f - f_n) - rn - C(r) \geq v_{LA_h}(f) - C(r) - (N+1)h + (N+1-r)n \to +\infty,$$

because $N+1-r > 0$. \hfill $\Box$

Proof of Theorem 1.9.7. The proposition implies the uniqueness in the theorem. We only need to prove the existence.

We show that if $f \in LA_h$, then $\lim_{n \to \infty} \int_{\mathbb{Z}_p} f_n \mu$ exists:

$$v_p(\int_{\mathbb{Z}_p} (f_{n+1} - f_n) \mu) \geq v_{LA_{n+1}}(f_{n+1} - f_n) + v_{D_{r,N}}(\mu) - r(n+1)$$

$$\geq \inf(v_{LA_{n+1}}(f - f_n), v_{LA_{n+1}}(f - f_{n+1}), v_{D_{r,N}}(\mu) - r(n+1))$$

$$\geq v_{D_{r,N}}(\mu) + v_{LA_h}(f) - r(h-1) + (n-h)(N+1-r) \to +\infty.$$
Set $\int_{\mathbb{Z}_p} f \mu = \lim_{n \to +\infty} \int_{\mathbb{Z}_p} f_n \mu$, then

$$v_p(\int_{\mathbb{Z}_p} f \mu) \geq \inf \left( v_p(\int_{\mathbb{Z}_p} f_n \mu), v_p(\inf_{n \geq h} \int_{\mathbb{Z}_p} (f_{n-1} - f_n) \mu) \right)$$

$$\geq v_{LA_h}(f) - rh + (v_{D,r,N}(\mu) - r).$$

This implies that $\mu \in \mathcal{D}_r$.

### 1.10 Summary

To summarize what we established:

(i) We have the inclusions:

$$C^0 \supset C^r \supset LA \supset LA_h$$

$$\mathcal{D}_0 \subset \mathcal{D}_r \subset \mathcal{D} \subset LA_h^*.$$  

Now, if $f$ is a function on $\mathbb{Z}_p$ and $\mu$ is a linear form on polynomials, then we have:

$$f \mapsto a_n(f) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(n-i)$$

$$\mu \mapsto b_n(\mu) = \int_{\mathbb{Z}_p} \binom{x}{n} \mu$$

(ii) For $f$ a function,

- $f \in C^0$ if only if $v_p(a_n(f)) \to +\infty$ and

  $$v_{C^0}(f) = \inf_{x \in \mathbb{Z}_p} v_p(f(x)) = \inf_n v_p(a_n(f)).$$

- $f \in C^r$ if only if $v_p(a_n(f)) - r \frac{\log(1+n)}{\log p} \to +\infty$ and

  $$v_{C^r}(f) = \inf_n v_p(a_n(f)) - r \frac{\log(1+n)}{\log p}.$$  

- $f \in LA$ if only if there exists $r > 0$ such that $v_p(a_n(f)) - rn \to +\infty$.  

$LA$ is not a Banach space; it is a compact inductive limit of Banach spaces.
1.10. SUMMARY

\( f \in LA_h \) if and only if \( v_p(a_n(f)) - v_p(\lfloor \frac{n}{p^h} \rfloor!) \to +\infty \) and

\[
v_{LA_h}(f) = \inf_{x \in \mathbb{Z}_p} \inf_{k \in \mathbb{N}} v_p\left(\frac{p^h f^{(k)}(x)}{h!}\right) = \inf_n (v_p(a_n(f)) - v_p(\lfloor \frac{n}{p^h} \rfloor!)).
\]

(iii) For \( \mu \) a distribution,

- \( \mu \in D_0 \) if and only if \( v_{D_0}(\mu) = \inf_n v_p(b_n(\mu)) > -\infty \).

- \( \mu \in D_r \) if and only if \( v'_{D_r}(\mu) = \inf_n v_p(b_n(\mu)) + r \log(1+n) > -\infty \).

- \( \mu \in D \) if and only if for all \( r > 0 \), \( \inf_n v_p(b_n(\mu)) + rn > -\infty \).

(iv) \( f = \sum_{n=0}^{+\infty} a_n(f) \binom{x}{n} \) and \( \int_{\mathbb{Z}_p} f \, \mu = \sum_{n=0}^{+\infty} a_n(f) b_n(\mu) \).
CHAPTER 1. THE $p$-ADIC ZETA FUNCTION OF KUBOTA-LEOPOLDT
Chapter 2

Modular forms

2.1 Generalities

2.1.1 The upper half-plane

By $SL_2$ we mean the group of $2 \times 2$ matrices with determinant 1. We write $SL_2(A)$ for those elements of $SL_2$ with entries in a ring $A$. In practice, the ring $A$ will be $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$, $z$ in $\mathbb{C} - \{-\frac{d}{c}\}$, let $\gamma z = \frac{az+b}{cz+d}$, then

$$\text{Im}(\gamma z) = \frac{(ad-bc)}{|cz+d|^2} \text{Im}(z) = \frac{\text{Im}z}{|cz+d|^2}.$$  

We denote $\mathcal{H} = \{z, \text{Im}z > 0\}$ the upper half plane. It is stable under $z \mapsto \gamma z$ and one can verify $(\gamma_1 \gamma_2)z = \gamma_1(\gamma_2 z)$.

**Proposition 2.1.1.** The transform action $z \mapsto \gamma z$ defines a group action of $SL_2(\mathbb{R})$ on $\mathcal{H}$.

**Proposition 2.1.2.** $\frac{dx \wedge dy}{y^2}$ is invariant under $SL_2(\mathbb{R})$.

(hint: $dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$ and $z \mapsto \gamma z$ is holomorphic.)

**Definition 2.1.3.** Let $f : \mathcal{H} \mapsto \mathbb{C}$ be a meromorphic function and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $SL_2(\mathbb{R})$. If $k$ in $\mathbb{Z}$, we define the weight $k$ action of $SL_2(\mathbb{R})$ by $(f|_k \gamma)(z) = (cz+d)^{-k} f(\gamma z)$.

**Exercise.** $(f|_k \gamma_1)|_k \gamma_2 = f|_k \gamma_1 \gamma_2$. 

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CHAPTER 2. MODULAR FORMS

2.1.2 Definition of modular forms

Definition 2.1.4. Let $\Gamma$ be a subgroup of $SL_2(\mathbb{Z})$ of finite index, $\chi$ is a finite order character of $\Gamma$ (i.e. $\chi(\Gamma) \subset \mu_N$). $f : \mathcal{H} \mapsto \mathbb{C}$ is a modular form of weight $k$, character $\chi$ for $\Gamma$, if:

(i) $f$ is holomorphic on $\mathcal{H}$;

(ii) $f|_{k\gamma} = \chi(\gamma)f$, if $\gamma \in \Gamma$;

(iii) $f$ is slowly increasing at infinity, i.e. for all $\gamma \in \Gamma \backslash SL_2(\mathbb{Z})$, there exists $C(\gamma)$ and $r(\gamma)$ such that $|f|_{k\gamma}(z) |\leq y^{r(\gamma)}$, if $y \geq C(\gamma)$.

Definition 2.1.5. $\Gamma$ is a congruence subgroup if $\Gamma \supset \Gamma(N) = \text{Ker } (SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/N\mathbb{Z}))$ for some $N \in \mathbb{N}$.

Example 2.1.6.

$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \} \supset \Gamma(N)$.

Any character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \rightarrow \mathbb{C}^*$ extends to a congruence character

$\chi : \Gamma_0(N) \rightarrow \mathbb{C}^* \chi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \rightarrow \chi(d)$.

Let $M_k(\Gamma, \chi)$ be the set of modular forms of weight $k$, character $\chi$ for $\Gamma$. Then $M_k(\Gamma, \chi)$ is a $\mathbb{C}$-vector space.

Remark. (i) If $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -I \in \Gamma$ and $\chi(-I) \neq (-1)^k$, then $M_k(\Gamma, \chi) = 0$;

(ii) $f \in M_k(\Gamma, \chi)$, $g \in SL_2(\mathbb{Z})$, $f|_{kg} \in M_k(g^{-1}\Gamma g, \chi_g)$ where $\chi_g(\gamma) = \chi(g\gamma g^{-1})$.

2.1.3 $q$-expansion of modular forms.

Lemma 2.1.7. If $\Gamma$ is a subgroup of finite index of $SL_2(\mathbb{Z})$ and $\chi : \Gamma \mapsto \mathbb{C}^*$ is of finite order, then there exists $M$ in $\mathbb{N} - \{0\}$, such that $\left( \begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix} \right) \in \Gamma$ and $\chi(\left( \begin{smallmatrix} 1 & M \\ 0 & 1 \end{smallmatrix} \right)) = 1$.

Proof. We can replace $\Gamma$ by $\text{Ker } \chi$ and assume $\chi = 1$. There exists $n_1 \neq n_2$, such that $\left( \begin{smallmatrix} 1 & n_1 \\ 0 & 1 \end{smallmatrix} \right)$ and $\left( \begin{smallmatrix} 1 & n_2 \\ 0 & 1 \end{smallmatrix} \right)$ have the same image in $\Gamma \backslash SL_2(\mathbb{Z})$, then $M = |n_1 - n_2|$ satisfy the condition. \hfill $\square$

For $M \in \mathbb{N} - \{0\}$, let $q_M(z) = e^{2\pi iz}$. Then $z \mapsto q_M(z)$ gives a holomorphic bijection $M\mathbb{Z} \backslash \mathcal{H} \simeq \mathcal{D}^* = \{ 0 < |q_M| < 1 \}$.
Corollary 2.1.8. If \( f \in M_k(\Gamma, \chi) \), then there exists \( M \neq 0, M \in \mathbb{N} \), such that \( f(z + M) = f(z) \). Thus there exists \( f \) holomorphic on \( \mathcal{D}^* \), such that \( f(z) = \tilde{f}(q_M) \).

Now \( \tilde{f} \) has a Laurent expansion \( \tilde{f}(q_M) = \sum_{n \in \mathbb{Z}} a_n q_M^n \) with

\[
a_n = e^{2\pi i y} \cdot \frac{1}{M} \int_{-M/2}^{M/2} f(x + iy) e^{-2\pi i x} dx
\]

for all \( y \). If \( n < 0 \), when \( y \to \infty \), the right hand side goes to 0, so \( a_n = 0 \). Hence we get the following result.

Proposition 2.1.9. If \( f \) is in \( M_k(\Gamma, \chi) \), there exists \( M \in \mathbb{N} - \{0\} \), and elements \( a_n(f) \) for each \( n \in \frac{1}{M} \mathbb{N} \), such that

\[
f = \sum_{n \in \frac{1}{M} \mathbb{N}} a_n(f) q^n,
\]

where \( q(z) = e^{2\pi i z} \), which is called the \( q \) expansion of modular forms.

2.1.4 Cusp forms.

Definition 2.1.10. (i) \( v_\infty(f) = \inf\{n \in \mathbb{Q}, a_n(f) \neq 0\} \geq 0 \) and we say that \( f \) has a zero of order \( v_\infty(f) \) at \( \infty \). We say that \( f \) has a zero at \( \infty \) if \( v_\infty(f) > 0 \).

(ii) A modular form \( f \) is a cusp form if \( f|_k \gamma \) has a zero at \( \infty \) for all \( \gamma \) in \( \Gamma \setminus SL_2(\mathbb{Z}) \). We denote \( S_k \) the set of cusp form of weight \( k \). \( S_k(\Gamma, \chi) \subset M_k(\Gamma, \chi) \).

Remark. If \( f \) is a cusp form, then \( f \) is rapidly decreasing at \( \infty \) since

\[
| (f|_k \gamma)(z) | = O(e^{-v_\infty(f|_k \gamma)2\pi y}).
\]

Theorem 2.1.11. \( S_k(\Gamma, \chi) \) and \( M_k(\Gamma, \chi) \) are finite dimensional \( \mathbb{C} \)-vector spaces with explicit formulas for the dimensions (if \( k \geq 2 \)).

Remark. \( \oplus_{k, \chi} M_k(\Gamma, \chi) = M(\Gamma) \) is an algebra.

The study of \( M_k(\Gamma, \chi) \) for congruence subgroup and congruence characters \( (\text{Ker } \chi \text{ congruence subgroup}) \) can be reduced to the study of \( M_k(\Gamma_0(N), \chi) \) for a simple group theoretic reason. From now on, we write

\[
M_k(N, \chi) = M_k(\Gamma_0(N), \chi), \quad S_k(N, \chi) = S_k(\Gamma_0(N), \chi).
\]
2.2 The case $\Gamma = \text{SL}_2(\mathbb{Z})$

2.2.1 The generators $S$ and $T$ of $\text{SL}_2(\mathbb{Z})$.

Let $M_k(1) = M_k(\text{SL}_2(\mathbb{Z}), 1)$, $S_k(1) = S_k(\text{SL}_2(\mathbb{Z}), 1)$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

It is easy to verify

$$T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \text{ for any } n \in \mathbb{Z}.$$  

So $Sz = -\frac{1}{z}$, $T^n z = z + n$.

**Proposition 2.2.1.** (i) If $(a, b) = 1$, then there exists $n = n(a, b)$, $(a_0, b_0) = (1, 0)$, $(a_1, b_1) = (0, 1)$, $\cdots$ $(a_n, b_n) = (a, b)$, such that

$$\begin{pmatrix} a_l & a_{l+1} \\ b_l & b_{l+1} \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ for any } l.$$  

(ii) $\text{SL}_2(\mathbb{Z}) = \langle S, T \rangle$.

**Proof.** (i) We prove it by induction on $|a| + |b|$.  

If $|a| + |b| = 1$, one can do it by hand:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

If $|a| + |b| \geq 2$, there exists $\mu, \nu \in \mathbb{Z}$, such that $b\mu - a\nu = 1$, and $|\nu| < |b|$, which implies $|\mu| \leq |a|$. Then we have $\begin{pmatrix} \mu & a \\ \nu & b \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ and $|\mu| + |\nu| < |a| + |b|$. Therefore the conclusion is obtained by the inductive assumption.

(ii) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$, there exists $n = n(a, b)$, $(a_0, b_0) = (1, 0)$, $(a_1, b_1) = (0, 1)$, $\cdots$ $(a_n, b_n) = (a, b)$, such that

$$\gamma_l = \begin{pmatrix} a_l & a_{l+1} \\ b_l & b_{l+1} \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \text{ for any } l.$$  

As $\gamma_1 = I$ and

$$\gamma_{l+1}^{-1} \gamma_l = \begin{pmatrix} n_l & 1 \\ -1 & 0 \end{pmatrix} = T^{-n_l} S^3,$$

then $\gamma = \prod (\gamma_{l+1}^{-1} \gamma_l)^{-1} \in \langle T, S \rangle$. \hfill $\Box$
2.2. THE CASE $\Gamma = \text{SL}_2(\mathbb{Z})$

Corollary 2.2.2. Let $f = \sum_{n=0}^{+\infty} a_nq^n$, where $q = e^{2\pi iz}$, then $f \in M_k(1)$ if and only if the following two conditions hold:

(i) $\sum_{n=0}^{+\infty} a_nq^n$ converges if $|q| < 1$.

(ii) $f(-\frac{1}{z}) = z^k f(z)$.

2.2.2 Eisenstein series

Proposition 2.2.3. If $k \geq 3$, then $G_k \in M_k(1)$, where

$$G_k(z) = \frac{1}{2} \frac{\Gamma(k)}{(-2\pi i)^k} \sum' \frac{1}{(mz + n)^k} \in M_k(1),$$

and $\sum'$ means the summation runs over all pairs of integers $(m, n)$ distinct from $(0, 0)$.

Proof. As $|mz + n| \geq \min(y, y/|z|) \sup(|m|, |n|)$, the series converges uniformly on compact subsets of $\mathcal{H}$ and is bounded at $\infty$.

Let $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, since

$$(cz + d)^{-k} \sum_{m,n} \frac{1}{(m\frac{az+b}{cz+d} + n)^k} = \sum_{m,n} \frac{1}{((am + cn)z + (bm + dn))^k},$$

and

$$(m, n) \mapsto (am + cn, bm + dn)$$

is a bijection of $\mathbb{Z}^2 - \{(0, 0)\}$, it follows that $G_k|k\gamma = G_k$.

Proposition 2.2.4.

$$G_k(z) = \frac{\Gamma(k)}{(-2\pi i)^k} \zeta(k) + \sum_{n=1}^{+\infty} \sigma_{k-1}(n)q^n,$$

where $\sigma_s(n) = \sum_{d|n, d \geq 1} d^s$, and $k$ is even (if $k$ is odd, $M_k(1) = 0$, since $-1 \in \text{SL}_2(\mathbb{Z})$).

Proof. $G_k(z) = \frac{\Gamma(k)}{(-2\pi i)^k} \zeta(k) + \frac{\Gamma(k)}{(-2\pi i)^k} \sum_{m=1}^{+\infty} A_k(mz),$
where
\[ A_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z + n)^k} = \sum_{l \in \mathbb{Z}} \hat{\phi}(l) q^l \]
for the last identity given by the Poisson summation formula of Fourier transforms, and (by residue computation)
\[ \hat{\phi}(l) = \int_{-\infty}^{+\infty} \frac{e^{-2\pi ilx}}{(x + iy)^k} dx = \begin{cases} 0, & \text{if } l \leq 0, \\ \frac{(-2\pi i)^k l^{k-1}}{(k-1)!}, & \text{if } l \geq 0. \end{cases} \]
It follows that
\[ G_k(z) = \frac{\Gamma(k)}{(-2\pi i)^k} \zeta(k) + \sum_{m=1}^{+\infty} \sum_{l=1}^{+\infty} l^{-k} q^l = \frac{\Gamma(k)}{(-2\pi i)^k} \zeta(k) + \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n. \]

Remark. (i) \( G_2(z) = \frac{\Gamma(2)}{(-2\pi i)^2} \zeta(2) + \sum_{n=1}^{+\infty} \sigma_1(n) q^n \) is not a modular form, but it is almost one. Let
\[ G_2^*(z) = G_2(z) + \frac{1}{8\pi y} = \frac{1}{2} \frac{\Gamma(2)}{(-2\pi i)^2} \lim_{s \to 0} \sum_{m,n} \frac{1}{(mz + n)^2} \frac{y^s}{(mz + n)^2}; \]
\( G_2^* \) is not holomorphic, but \( G_2^*|_{2\gamma} = G_2^* \), for any \( \gamma \in \text{SL}_2(\mathbb{Z}) \).

(ii) Let \( E_k = \frac{G_k}{\text{a}_0(G_k)} \), so that \( \text{a}_0(E_k) = 1 \).

2.2.3 The fundamental domain for \( \text{SL}_2(\mathbb{Z}) \)

Theorem 2.2.5. Let \( D \) denotes the shadows in Figure 1.1. Then it is a fundamental domain for \( \text{PSL}_2(\mathbb{Z}) \). Moreover, the stabilizer of \( z \in D \) is

- \( \{I\} \) if \( z \neq i, \rho \);
- \( \{I, S\} \) if \( z = i \);
- \( \{I, TS, (TS)^2\} \) if \( z = \rho \).
2.2. **THE CASE** $\Gamma = \text{SL}_2(\mathbb{Z})$

Figure 2.1: The Fundamental Domain.

**Proof.** Let $z_0 \in \mathcal{H},$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

Since $\text{Im} (\gamma z_0) = \frac{z_0}{|cz_0+d|^2}$ tends to zero, as $(c,d)$ tends to infinity, there exists $\gamma_0$ such that $\text{Im} (\gamma_0 z_0)$ is maximal. There exists a unique $n$ such that:

$$-\frac{1}{2} < \text{Re} (\gamma_0 z_0) + n \leq \frac{1}{2}.$$

Let $\gamma_1 = T^n \gamma_0$, then

$$\text{Im} (\gamma_1 z_0) = \text{Im} (\gamma_0 z_0) \geq \text{Im} (S \gamma_1 z_0) = \frac{\text{Im} (\gamma_1 z_0)}{|\gamma_1 z_0|^2}$$

which implies $|\gamma_1 z_0| \geq 1$. Therefore $D$ contains a fundamental domain.

If $z_1, z_2 \in D$, and there exists $\gamma \in \text{SL}_2(\mathbb{Z})$, such that $z_1 = \gamma z_2$, we want to show $z_1 = z_2$. By symmetry, we may assume $\text{Im} (z_2) \geq \text{Im} (z_1)$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\text{Im} (z_2) \geq \frac{\text{Im} (z_2)}{|cz_2+d|^2}$ implies $|cz_2+d|^2 \leq 1$. As $\text{Im} (z_2) \geq \frac{\sqrt{3}}{2}$, we have $c \leq 1, d \leq 1$. It remains only finite number of cases to check.

If $c = 0$, then $d = \pm 1$, and $\gamma$ is the translation by $\pm b$. Since

$$-\frac{1}{2} < \text{Re} (z_1), \quad \text{Re} (z_2) \leq \frac{1}{2},$$

this implies $b = 0$, and $\gamma = \pm I$.

If $c = 1$, the fact $|z_2 + d| \leq 1$ implies $d = 0$ except if $z_2 = \rho$, in which case we can have $d = 0, -1$. The case $d = 0$ gives $|z_2| \leq 1$, hence $|z_2| = 1$; on the other hand, $\gamma \in \text{SL}_2(\mathbb{Z})$ implies $b = -1$, hence $z_1 = \gamma z_2 = a - 1/z_2 \in D,$
which implies $a = 0$, and $z_1 = z_2 = i$. The case $z_2 = \rho$, and $d = -1$ gives $a + b + 1 = 0$ and $z_1 = \gamma z_2 = a - \frac{1}{\rho - 1} = a + \rho \in D$, which implies $a = 0$ and $z_1 = z_2 = \rho$.

If $c = -1$, we have similar argument as $c = 1$.

This completes the proof of the Theorem. \hfill \square

### 2.2.4 The $\frac{k}{12}$ formula.

The following proposition is usually called “the $\frac{k}{12}$ formula”.

**Proposition 2.2.6.** Let $f \in M_k - \{0\}$, then

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) + \sum_{z \in D - \{i, \rho\}} v_z(f) = \frac{k}{12}.$$ 

**Proof.** Apply Cauchy residue formula to $d \log f$ over the path showed in Figure 1.2. As $M \to +\infty$, and $\varepsilon \to 0$, we have:

$$\lim_{M \to +\infty} \frac{1}{2\pi i} \int_{C(M, \varepsilon)} \ d \log f = \sum_{z \in D - \{i, \rho\}} v_z(f),$$

$$\lim_{M \to +\infty} \frac{1}{2\pi i} \int_{C_M} \ d \log f = \lim_{M \to +\infty} -\frac{1}{2\pi i} \int_{|z| = e^{-2\pi M}} \ d \log \sum a_n(f) z^n = -v_\infty(f),$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C(i, \varepsilon)} \ d \log f = -\frac{1}{2} v_i(f),$$
2.2. **THE CASE** $\Gamma = \text{SL}_2(\mathbb{Z})$

\[
\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C(\rho,\varepsilon)} d\log f = -\frac{1}{6} v_\rho(f) = -\frac{1}{6} v_{\rho^2}(f) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C(\rho^2,\varepsilon)} d\log f
\]

\[
\frac{1}{2\pi i} (\int_i^i d\log f + \int_i^\rho d\log f) = \frac{1}{2\pi i} \int_{\rho^2}^i (d\log f - d\log f(-\frac{1}{z}))
\]

\[
= -\frac{1}{2\pi i} \int_{\rho^2}^i (d\log f - d\log z^k f(z))
\]

\[
= -\frac{k}{2\pi i} \int_{\rho^2}^i \frac{dz}{z} = -\frac{k}{2\pi i} (\log i - \log \rho^2) = \frac{k}{12}
\]

Putting all these equations together, we get the required formula. \hfill \Box

**Corollary 2.2.7.** $G_4$ has its only zero on $D$ at $z = \rho$, $G_6$ has its only zero on $D$ at $z = i$.

\[
\Delta = \left(\frac{G_4}{a_0(G_4)}\right)^3 - \left(\frac{G_6}{a_0(G_6)}\right)^2 \frac{1}{3a_0^{-1}(G_4) - 2a_0^{-1}(G_6)} = q + \cdots \in M_{12}(1)
\]

does not vanish on $D$ ($v_\infty(\Delta) = 1$).

**Remark.** One can prove $\Delta = q \prod_{n=1}^{+\infty} (1 - q^n)^{24}$.

### 2.2.5 Dimension of spaces of modular forms.

**Theorem 2.2.8.** (i) $M_k(1) = 0$, if $k$ is odd or $k = 2$.

(ii) $\dim M_k(1) = 1$, if $k = 0$ or $k$ is even and $2 < k \leq 10$. In this case $M_k(1) = \mathbb{C} \cdot G_k$ (We have $G_0 = 1$).

(iii) $M_{k+12}(1) = \mathbb{C} \cdot G_{k+12} \oplus \Delta \cdot M_k(1)$.

**Proof.** If $f \in M_{k+12}$, then

\[
f = \frac{a_0(f)}{a_0(G_{k+12})} G_{k+12} + \Delta g,
\]

where $g \in M_k(1)$, because $\Delta$ does not vanish on $\mathcal{H}$, $v_\infty(\Delta) = 1$ and $v_\infty(f - \frac{a_0(f)}{a_0(G_{k+12})} G_{k+12}) \geq 1$. \hfill \Box
Corollary 2.2.9. If $k$ is even, $\dim C M_k(1) = \begin{cases} \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \mod 12, \\ \lfloor \frac{k}{12} \rfloor + 1, & \text{if not}. \end{cases}$

**Remark.** Finite dimensionality of spaces of modular forms has many combinatorial applications. For example, let

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z},$$

$$\Gamma_\theta = \{ \gamma \in \text{SL}_2(\mathbb{Z}), \gamma \equiv I \text{ or } \gamma \equiv S \mod 2 \},$$

$$\chi_\theta : \Gamma_\theta \to \{ \pm 1 \}. \chi_\theta(\gamma) = \begin{cases} 1 & \text{if } \gamma \equiv I \\ -1 & \text{if } \gamma \equiv S \end{cases}$$

One can check that $\dim M_2(\Gamma_\theta, \chi_\theta) \leq 1$, $\theta^4 \in M_2(\Gamma_\theta, \chi_\theta)$, and $4G_2^*(2z) - G_2^*(\frac{z}{2}) \in M_2(\Gamma_\theta, \chi_\theta)$, so we have

$$4G_2^*(2z) - G_2^*(\frac{z}{2}) = \frac{3\zeta(2)\Gamma(2)}{(-2\pi i)^2}\theta^4,$$

hence

$$\left| \{(a, b, c, d) \in \mathbb{Z}^4 : a^2 + b^2 + c^2 + d^2 = n \} \right| = 8 \sum_{d \mid n, 4 \nmid d} d,$$

from which we can deduce that any positive integer can be written as a sum of 4 squares.

2.2.6 Rationality results.

As $M_8(1)$ and $M_{10}(1)$ are of dimension 1, we have

$$a_0(G_8)G_4^2 = a_0(G_4)^2G_8, \quad a_0(G_{10})G_4G_6 = a_0(G_4)a_0(G_6)G_{10}. \quad (*)$$

Let

$$\alpha = \frac{\Gamma(4)}{(-2\pi i)^2} \zeta(4), \quad \beta = \frac{\Gamma(8)}{(-2\pi i)^4} \zeta(8).$$

Substituting

$$G_4 = \alpha + q + 9q^2 + \cdots, \quad G_8 = \beta + q + 129q^2 + \cdots$$
in (\ast), compare the coefficients of \(q\) and \(q^2\), we have the following equations:

\[
\begin{align*}
2\alpha\beta &= \alpha^2 \\
\beta(1 + 18\alpha) &= 129\alpha^2
\end{align*}
\]

The solution is: \(\alpha = \frac{1}{240}\), \(\beta = \frac{1}{480}\). In particular, \(\alpha, \beta \in \mathbb{Q}\), which implies \(G_4\) and \(G_8\) have rational \(q\)-expansions, and \(\frac{\zeta(4)}{\pi^4} \in \mathbb{Q}\), \(\frac{\zeta(8)}{\pi^8} \in \mathbb{Q}\).

**Exercise.** \(a_0(G_6) = -\frac{1}{504}\), which implies \(\frac{\zeta(6)}{\pi^6} \in \mathbb{Q}\).

Let \(A\) be a subring of \(\mathbb{C}\), let

\[
M_k(\Gamma, A) = \{f \in M_k(\Gamma), a_n(f) \in A, \text{for all } n\},
\]

then \(M(\Gamma, A) = \sum_k M_k(\Gamma, A)\) is an \(A\)-algebra.

**Theorem 2.2.10.** (i) \(M(SL_2(\mathbb{Z}), \mathbb{Q}) \cong \mathbb{Q}[X, Y]\), where \(X = G_4, Y = G_6\).

(ii) \(M(SL_2(\mathbb{Z}), \mathbb{C}) = \mathbb{C} \otimes M(SL_2(\mathbb{Z}), \mathbb{Q})\).

**Proof.** If \(\sum_k f_k = 0\), where \(f_k \in M_k(SL_2(\mathbb{Z}), \mathbb{C})\), then for any \(z\), for any \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z})\), we have \(\sum_k cz^d f_k(z) = 0\). Therefore \(\sum_k (Xz + Y)^k f_k(z)\) is identically zero because it (as a polynomial in \(X\) and \(Y\)) has too many zeros. Hence \(f_k(z) = 0\), which implies that

\[
M(SL_2(\mathbb{Z}), \mathbb{C}) = \bigoplus_k M_k(SL_2(\mathbb{Z}), \mathbb{C}).
\]

Now if \(k = 12n\), \(G_4^{3n}, G_4^{3(n-1)} \Delta, \cdots, \Delta^n\) is a basis of \(M_k(\Gamma)\); if \(k = 12n + 2\), \(G_4^{3(n-1)+2} G_6, G_4^{3(n-2)+2} G_6 \Delta, \cdots, G_6^2 G_6 \Delta^{n-1}\) is a basis of \(M_k(\Gamma)\), and so on, \(\Delta = aG_4^3 + bG_6^2\), \(a, b \in \mathbb{Q}\). As \(G_4, G_6 \in M(SL_2(\mathbb{Z}), \mathbb{Q})\), this proves both results. \(\square\)

**Corollary 2.2.11.** Let \(f \in M_k(\Gamma)\), \(\sigma \in \text{Aut}(\mathbb{C})\), then \(f^\sigma = \sum a_n(f)^\sigma q^n \in M_k(\Gamma)\). Moreover, \(\frac{\zeta(k)}{(-2\pi i)^k} \in \mathbb{Q}\) if \(k\) is even and \(k \geq 4\).

**Proof.** The first assertion is a direct consequence of Theorem 2.2.10 (ii). For any \(\sigma \in \text{Aut}(\mathbb{C})\), we have

\[
G_k^\sigma - G_k = a_0(G_k)^\sigma - a_0(G_k) \in M_k(\Gamma).
\]

This implies \(a_0(G_k)^\sigma = a_0(G_k)\) for any \(\sigma \in \text{Aut}(\mathbb{C})\), therefore \(a_0(G_k) \in \mathbb{Q}\). \(\square\)
Remark. When \( k = 2 \), we can use
\[
4G_2^*(2z) - G_2^*(\frac{z}{2}) \in M_2(\Gamma_0, \mathbb{Q})
\]
to deduce \( \frac{\zeta(2)}{\pi^2} \in \mathbb{Q} \).

Remark. (i) The zeta function \( \zeta \) is a special case of \( L \)-functions, and \( \zeta(k) \) are special values of \( L \)-functions (i.e. values of \( L \)-functions at integers).

Siegel used the above method to prove rationality of special values of \( L \)-functions for totally real fields.

(ii) With a lot of extra work, we can prove integrality results. As
\[
G_k(z) = \frac{\Gamma(k)}{(-2\pi i)^k} \zeta(k) + \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n,
\]
and \( \sigma_{k-1}(n) = \int_{\mathbb{Z}_p} x^{k-1}(\sum_{d|n} \delta_d) \), we have all \( a_n(G_k) \) are given by measures on \( \mathbb{Z}_p \), therefore \( a_0(G_k) \) is also given by measures. From which we can deduce other constructions of Kubota-Leopoldt zeta functions (the work of Serre, Deligne, Ribet).

### 2.3 The algebra of all modular forms.

Let \( A \) be a subring of \( \mathbb{C} \), let
\[
\mathcal{M}_k(A) = \bigcup_{[\text{SL}_2(\mathbb{Z}):\Gamma]<+\infty} \mathcal{M}_k(\Gamma, A) = \left\{ \sum a_n q^n \in \mathcal{M}_k(\Gamma, \mathbb{C}), a_n \in A, n \in \mathbb{N} \right\}.
\]

Let \( \mathcal{M}(A) = \oplus \mathcal{M}_k(A) \), then it is an \( A \)-algebra. Let \( \mathcal{M}^{\text{cong}}(A) = \bigcup_{\Gamma \text{ congruence subgroup}} \mathcal{M}(\Gamma, A) \).

**Theorem 2.3.1.** (i) If \( f \in \mathcal{M}(\mathbb{C}) \), and \( \sigma \in \text{Aut}(\mathbb{C}) \), then \( f^\sigma \in \mathcal{M}(\mathbb{C}) \).

(ii) \( \mathcal{M}(\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{M}(\overline{\mathbb{Q}}) = \mathbb{C} \otimes_{\mathbb{Q}} \mathcal{M}(\mathbb{Q}) \).

(iii) Let \( \Pi_Q = \text{Aut}(\mathcal{M}(\overline{\mathbb{Q}})/\mathcal{M}(\text{SL}_2(\mathbb{Z}), \mathbb{Q})), G_Q = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), then we have an exact sequence:
\[
1 \longrightarrow \text{SL}_2(\mathbb{Z})^\wedge \longrightarrow \Pi_Q \longrightarrow G_Q \longrightarrow 1
\]
where $G^\wedge \triangleq \lim_{\overset{\Gamma \text{ normal}}{[G:\Gamma] \to \infty}} (G/\Gamma)$, and $G_Q \to \Pi_Q$ is induced by the action on Fourier coefficients.

(iv) $\mathcal{M}^\text{cong}(\mathbb{Q}^{ab})$ is stable by $\Pi_Q$, and

$$\text{Aut}(\mathcal{M}^\text{cong}(\mathbb{Q}^{ab})/\mathbb{M}(\text{SL}_2(\mathbb{Z}), \mathbb{Q})) \sim \text{GL}_2(\hat{\mathbb{Z}}).$$

Moreover, we have the following commutative diagram:

$$
\begin{array}{cccccc}
1 & \longrightarrow & \text{SL}_2(\mathbb{Z})^\wedge & \longrightarrow & G_Q & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \uparrow & & \downarrow \\
1 & \longrightarrow & \text{SL}_2(\hat{\mathbb{Z}}) & \longrightarrow & \text{GL}_2(\hat{\mathbb{Z}}) & \longrightarrow & \hat{\mathbb{Z}}^* & \longrightarrow & 1
\end{array}
$$

where $G_Q \to \hat{\mathbb{Z}}^*$ is the cyclotomic character, $\text{GL}_2(\hat{\mathbb{Z}}) \to \hat{\mathbb{Z}}^*$ is the determinant map, and $\hat{\mathbb{Z}}^* \to \text{GL}_2(\hat{\mathbb{Z}})$ maps $u$ to $\left( \begin{smallmatrix} 1 & 0 \\ 0 & u \end{smallmatrix} \right)$.

Remark. (i) $\text{SL}_2(\mathbb{Z})^\wedge$ is much bigger than $\text{SL}_2(\hat{\mathbb{Z}})$.

(ii) We can get an action of $G_Q$ on $\text{SL}_2(\mathbb{Z})^\wedge$ by inner conjugation in $\Pi_Q$. This is a powerful way to study $G_Q$ (Grothendieck, “esquisse d’un programme”).

(iii) There are $p$-adic representations of $G_Q$ attached to modular forms (by Deligne) for congruence subgroups. They come from the actions of $G_Q$ on $H^1(\text{SL}_2(\mathbb{Z})^\wedge, W)$, where $W = \text{Sym}^{k-2} V_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\text{SL}_2(\mathbb{Z})/\Gamma]$, $V_p$ is $\mathbb{Q}_p^2$ with actions of $\Pi_Q$ through $\text{GL}_2(\mathbb{Z}_p)$ and are cut out using Hecke operators on these spaces.

Proof of Theorem 2.3.1 (i). Let $N(\Gamma, A)$ denote the set of holomorphic functions $f : \mathcal{H} \to \mathbb{C}$ satisfying the following conditions:

(a) for any $\gamma \in \Gamma$, $f(\gamma z) = f(z),$

(b) for any $\gamma \in \Gamma \setminus \text{SL}_2(\mathbb{Z})$, $f \circ \gamma = \sum_{n \geq n_0(\gamma, f)} a_n q^n$, and $a_n \in A$ for any $n$.

As $\Delta \in S_{12}(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$ does not vanish on $\mathcal{H}$, $\Delta^\frac{1}{2} \in S_1(\text{SL}_2(\mathbb{Z}), \chi, \mathbb{Q})$, where $\chi : \text{SL}_2(\mathbb{Z}) \to \mu_{12}$. Let $\Gamma_0 = \text{Ker} \chi$. If $f \in M_k(\Gamma, A)$, $\Delta^\frac{1}{2} f \in \mathbb{M}(\Gamma \cap \Gamma_0, A)$. If $f \in N(\Gamma, A)$, $\Delta^k f \in \mathbb{M}_{12k}(\Gamma, A)$, where $k + n_0(\gamma, f) \geq 0$ for any $\gamma \in \Gamma$. 


\[ f = \sum_{n \geq n_0} a_n q^n \in \mathcal{N}(\mathbb{C}) = \bigcup_{\Gamma} \mathcal{N}(\Gamma, \mathbb{C}) \]

and \( \sigma \in \text{Aut} \mathbb{C} \), then \( f^\sigma \in \mathcal{N}(\mathbb{C}) \).

Let \( j = \frac{G_4^3}{a_0(G_4^3) \Delta} = q^{-1} + \cdots \in \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) \).

**Proposition 2.3.2.** (i) \( \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) = \mathbb{Q}[j] \), \( \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{C}) = \mathbb{C}[j] \).

(ii) \( j : \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H} \to \mathbb{C} \) is bijective.

(iii) \( j(z) - j(\alpha) \) has a zero at \( z = \alpha \) of order \( e(\alpha) = \begin{cases} 3 & \text{if } \alpha \in \text{SL}_2(\mathbb{Z}) \rho \\ 2 & \text{if } \alpha \in \text{SL}_2(\mathbb{Z}) i, \\ 1 & \text{otherwise} \end{cases} \)

(iv) \( j(i), j(\rho) \in \mathbb{Q} \).

**Proof.** (i) Note that \( G_3^2, G_4^2, \Delta, \cdots, \Delta^\alpha \) is a basis of \( \mathbb{M}_{12\alpha}(\text{SL}_2(\mathbb{Z}), \mathbb{Q}) \).

(ii) and (iii): For any \( \beta \in \mathbb{C} \), \( f = (j - \beta) \cdot \Delta \in \mathbb{M}_{12}(\text{SL}_2(\mathbb{Z}), \mathbb{C}) \), with \( v_\infty(f) = 0 \). As \( D = \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H} \), and

\[
\sum_{z \in D - \{\rho, i\}} \gamma_z(f) + \frac{1}{2} \gamma_i(f) + \frac{1}{3} f = 1,
\]

we can deduce the required results.

(iv) \( G_4(\rho) = 0, G_6(i) = 0 \). \( \square \)

Let \( f \in \mathbb{N}(\Gamma, \mathbb{C}) \),

\[
P_f(X) = \prod_{\delta \in \Gamma \setminus \text{SL}_2(\mathbb{Z})} (X - f \circ \delta) \in \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{C})[X] \subset \mathbb{C}((q))[X]
\]

\[
P_{f^\sigma}(X) = \prod_{\delta \in \Gamma \setminus \text{SL}_2(\mathbb{Z})} (X - (f \circ \delta)^\sigma) \in \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{C})[X] \subset \mathbb{C}((q))[X]
\]

Denote \( P_f(X) = \sum_{l=0}^{n} g_l X^l, P_{f^\sigma}(X) = \sum_{l=0}^{n} g_l^\sigma X^l \), where \( g_l \in \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{C}) \), and \( g_l^\sigma \in \mathbb{N}(\text{SL}_2(\mathbb{Z}), \mathbb{C}) \) thanks to the Corollary 2.2.11. We give the proof in two steps.
Step 1: Prove that $f^\sigma$ is holomorphic on $\mathcal{H}$, by the Proposition 2.3.2. We have

$$P_f(X) = \sum_{l=0}^{n} P_l(j)X^l, \quad P_{f^\sigma}(X) = \sum_{l=0}^{n} P^\sigma_l(j)X^l.$$ 

The roots of $P_f$ are the $f \circ \delta$'s, where $\delta \in \Gamma \setminus \text{SL}_2(\mathbb{Z})$. They are holomorphic on $\mathcal{H}$. The roots of $P_{f^\sigma}$ are multivalued holomorphic functions on $\mathcal{H}$. In order to prove that are single valued, it suffices to show there is no ramification. Let $\alpha$ be an arbitrary element in $\mathcal{H}$. we have, around $\alpha$,

$$\sum_{k=0}^{+\infty} a_{l,k}(\alpha)(j - j(\alpha))^k \quad (1 \leq l \leq n)$$

of $P_f(X) = 0$ as $(j - j(\alpha))^{\frac{k}{n(\alpha)}}$ is a local parameter around $\alpha$ by Proposition 2.3.2. Let $\beta^\sigma \in \mathcal{H}$ satisfies $j(\beta^\sigma) = j(\alpha)^\sigma$, then we have $e(\beta^\sigma) = e(\alpha)$. Therefore

$$\sum_{k=0}^{+\infty} a_{l,k}(\alpha)^\sigma(j - j(\beta^\sigma))^k \quad (1 \leq l \leq n)$$

are $n$ distinct formal solutions around $\beta^\sigma$. It follows that there is no ramification around $\beta^\sigma$, for any $\beta^\sigma$. Hence the roots of $P_{f^\sigma}$ are holomorphic on $\mathcal{H}$. In particular, $f^\sigma$ is holomorphic on $\mathcal{H}$. 

Step 2: Prove that there exists $\Gamma' \subset \text{SL}_2(\mathbb{Z})$ of finite index, such that $f^\sigma \circ \gamma = f^\sigma$ for any $\gamma \in \Gamma$. For any $\gamma \in \text{SL}_2(\mathbb{Z})$,

$$P_{f^\sigma}(f^\sigma \circ \gamma) = \sum_{l=0}^{n} g^\sigma_l(f^\sigma \circ \gamma)^l = \sum_{l=0}^{n} g^\sigma_l \circ \gamma(f^\sigma \circ \gamma)^l = P_{f^\sigma}(f^\sigma) \circ \gamma = 0$$

So $f^\sigma \circ \gamma$ belongs to the finite set of roots of $P_{f^\sigma}$, which leads to the required conclusion. 

\[\square\]

2.4 Hecke operators 

2.4.1 Preliminary. 

Let $\Gamma \subset G$ be groups (for example, $\Gamma = \text{SL}_2(\mathbb{Z})$, $G = \text{GL}_2(\mathbb{Q})^+$), let $x \in G$,

$$x\Gamma = \{x\gamma : \gamma \in \Gamma\}, \quad \Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$
Let $A$ be a ring, define $A[\Gamma \backslash G/\Gamma]$ to be the set of $\phi : G \to A$ satisfying the following two conditions:

(i) $\phi(\gamma x) = \phi(x \gamma) = \phi(x)$, for all $x \in G, \gamma \in \Gamma$.

(ii) There exists a finite set $I$ such that $\phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i}$.

**Remark.** (i) We impose $x_i$ to be distinct in $\Gamma \backslash G$, in this situation, the decomposition is unique, $\lambda_i$’s are unique.

(ii) For any $\gamma \in \Gamma$, $1_{\Gamma \gamma}x_i = 1_{\Gamma x_i}(x \gamma^{-1})$. So $\phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i} \in A[\Gamma \backslash G/\Gamma]$ implies

$$\sum_{i \in I} \lambda_i 1_{\Gamma x_i}(x) = \sum_{i \in I} \lambda_i 1_{\Gamma x_i}(x \gamma^{-1}) = \phi(x \gamma^{-1}) = \phi(x) = \sum_{i \in I} \lambda_i 1_{\Gamma x_i}(x)$$

Therefore there exists a permutation: $\sigma : I \to I$, and for any $i \in I$, there exists $\gamma_i \in \Gamma$, such that $\lambda_{\sigma(i)} = \lambda_i, x_i \gamma = \gamma_i x_{\sigma(i)}$.

**Proposition 2.4.1.** (i) If $\phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i}, \phi' = \sum_{j \in J} \mu_j 1_{\Gamma y_j} \in A[\Gamma \backslash G/\Gamma]$, then

$$\phi \ast \phi' = \sum_{(i,j) \in I \times J} \lambda_i \mu_j 1_{\Gamma x_i y_j} \in A[\Gamma \backslash G/\Gamma],$$

and it does not depend on the choices.

(ii) $(A[\Gamma \backslash G/\Gamma], +, \ast)$ is an associative $A$-algebra with $1_\Gamma$ as a unit.

(iii) If $M$ is a right $G$-module with $G$ action $m \mapsto m \ast g$, and $\phi = \sum_{i \in I} \lambda_i 1_{\Gamma x_i} \in A[\Gamma \backslash G/\Gamma]$, then for any $m \in M^\Gamma$, $m \ast \phi = \sum_{i \in I} \lambda_i m \ast x_i$ does not depend on the choices of $x_i$. Moreover, $m \ast \phi \in M^\Gamma$, $m \ast (\phi_1 \ast \phi_2) = (m \ast \phi_1) \ast \phi_2, m \ast (\phi_1 + \phi_2) = (m \ast \phi_1) + (m \ast \phi_2)$.

**Proof.** Exercise, using the previous remark.

**Remark.** If $\Gamma = 1$, then $A[\Gamma \backslash G/\Gamma] = A[G]$ is commutative if and only if $G$ is commutative.

### 2.4.2 Definition of Hecke operators: $R_n, T_n, n \geq 1$. 

Let $G = \text{GL}_2(\mathbb{Q})^+, \Gamma = \text{SL}_2(\mathbb{Z})$.

**Lemma 2.4.2.** Let $g \in G \cap M_2(\mathbb{Z})$, then there exists a unique pair $(a, d) \in \mathbb{N} - \{0\}$, and $b \in \mathbb{Z}$ unique mod $d\mathbb{Z}$, such that $\Gamma g = \Gamma \left( \begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right)$
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Proof. Let \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \), there exists \( \mu, \nu \in \mathbb{Z} \), such that \((\mu, \nu) = 1\), and \(\mu \alpha + \nu \gamma = 0\). And there exists \(x, y \in \mathbb{Z} \), such that \(x \nu - \mu y = 1\). Let

\[
\gamma_0 = \begin{pmatrix} x & y \\ \mu & \nu \end{pmatrix}
\]

if \(x \alpha + y \gamma \geq 0\); \(\gamma_0 = -\begin{pmatrix} x & y \\ \mu & \nu \end{pmatrix}\) if \(x \alpha + y \gamma < 0\). Then \(\gamma_0 g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\), where \(a > 0\). Thus completes the proof of existence.

If \(\gamma_1, \gamma_2 \in \Gamma\) satisfies

\[
\gamma_1 g = \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, \quad \gamma_2 g = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}
\]

then

\[
(\gamma_1 g)(\gamma_2 g)^{-1} = \begin{pmatrix} a_1 & \frac{a_2 b_1 - a_1 b_2}{a_2 d_2} \\ 0 & d_1 / d_2 \end{pmatrix} \in \text{SL}_2(\mathbb{Z})
\]

This implies \(a_1 = a_2\), \(d_1 = d_2\), \(b_1 - b_2\) divisible by \(d_1\).

Lemma-definition 2.4.3. For any \(n \geq 1\),

\[
R_n = 1_{\Gamma(\begin{smallmatrix} n & 0 \\ 0 & n \end{smallmatrix})} \in \mathbb{Z}[\Gamma \backslash G / \Gamma],
\]

\[
T_n = 1_{\{g \in M_2(\mathbb{Z}), \det g = n\}} \in \mathbb{Z}[\Gamma \backslash G / \Gamma].
\]

Proof. Left and right invariance come from \(\det gg' = \det g \det g'\). And Lemma 2.4.2 implies \(T_n = \sum_{a d = n, a \geq 1 \mod d} 1_{\Gamma(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix})}\), so get the finiteness needed.

Remark. If \(p\) is prime, then \(T_p = 1_{\Gamma(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix})}\) by elementary divisors for principle ideal domains.

Theorem 2.4.4. (i) For any \(n \geq 1\) and \(l \geq 1\), \(R_n R_l = R_{nl} = R_l R_n\), \(R_n T_l = T_l R_n\).

(ii) If \((l, n) = 1\), \(T_l T_n = T_{ln} = T_n T_l\).

(iii) If \(p\) is prime and \(r \geq 1\), \(T_{p^r} T_p = T_{p^{r+1}} + p R_p T_{p^{r-1}}\).

(iv) Let \(T_\mathbb{Z}\) be the subalgebra of \(\mathbb{Z}[\Gamma \backslash G / \Gamma]\) generated by \(R_n\) and \(T_n\) \((n \geq 1)\).

It is a commutative algebra.

Proof. (i) It is trivial.

(ii) We have

\[
T_n T_l = \sum_{a d = n, a \geq 1 \mod d} \sum_{a' d' = n, a' \geq 1 \mod d'} 1_{\Gamma(\begin{smallmatrix} a a' & b \mod d \\ 0 & d d' \end{smallmatrix})}.
\]
As \((n, l) = 1\), \((a, a') = 1\), \((a, d') = 1\). This implies \(\{a a' : a|n, a'|l\} = \{a'' : a''|nl\}\). Therefore in order to show \(T_n T_l = T_{nl}\), it suffices to verify that \(\{ab' + bd'\}\) is a set of representatives of \(\mathbb{Z}/(dd')\mathbb{Z}\), where \(b\) is a set of representatives of \(\mathbb{Z}/d\mathbb{Z}\), \(b'\) is a set of representatives of \(\mathbb{Z}/d'\mathbb{Z}\). It suffices to show the injectivity under the mod \(dd'\mathbb{Z}\) map. If

\[
ab_1' + b_1d' \equiv ab_2' + b_2d',
\]

then \(b_1' \equiv b_2'\mod d'\), so \(b_1' = b_2'\), which leads to the required conclusion.

(iii) We have

\[
T_{pr} = \sum_{i=0}^{r} \sum_{b \mod p^i} 1_{\Gamma \left( \begin{array}{cc} 0 & b \\ 0 & p^i \end{array} \right)}\), \(T_p = 1_{\Gamma \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right)} + \sum_{c \mod p} 1_{\Gamma \left( \begin{array}{cc} 1 & c \\ 0 & p \end{array} \right)}\)
\]

Then

\[
T_{pr} T_p = \sum_{i=0}^{r} \sum_{b \mod p^i} 1_{\Gamma \left( \begin{array}{cc} 0 & b \\ 0 & p^i \end{array} \right)} + \sum_{i=0}^{r} \sum_{b \mod p^i} \sum_{c \mod p} 1_{\Gamma \left( \begin{array}{cc} 0 & pb+p^{r-i}c \\ 0 & p^{i+1} \end{array} \right)}\)
\]

\[= T_{pr+1} + pR_p \sum_{i=0}^{r-1} \sum_{b \mod p^i} \sum_{c \mod p} 1_{\Gamma \left( \begin{array}{cc} 0 & p^{r-1-i}b+p^{r-1-i}c \\ 0 & p^i \end{array} \right)}\) = T_{pr+1} + pR_p T_{pr-1}.
\]

(iv) It follows from (i),(ii),(iii).

2.4.3 Action of Hecke operators on modular forms.

The following two propositions are exercises in group theory.

Proposition 2.4.5. Assume \(G \supset \Gamma\) are groups. Then

(i) If \([\Gamma : \Gamma'] < +\infty\), then \(\Gamma'\) contains some \(\Gamma''\) which is normal in \(\Gamma\), and \([\Gamma : \Gamma''] < +\infty\)\\

(ii) If \([\Gamma : \Gamma_1] < +\infty\), \([\Gamma : \Gamma_2] < +\infty\), then \([\Gamma : \Gamma_1 \cap \Gamma_2] < +\infty\).

(iii) If \(H' \subset H \subset G\), \([H : H'] < +\infty\), then \([H \cap \Gamma : H' \cap \Gamma] < +\infty\).

Proposition 2.4.6. (i) Suppose \(\alpha \in \text{GL}_2(\mathbb{Q})^+, \) and \(N \in \mathbb{N}\) such that \(N\alpha, N\alpha^{-1} \in \text{M}_2(\mathbb{Z})\), then

\[
\alpha^{-1} \text{SL}_2(\mathbb{Z})\alpha \cap \text{SL}_2(\mathbb{Z}) \supset \Gamma(N^2) := \text{SL}_2(\mathbb{Z}) \cap (1 + N^2 \text{M}_2(\mathbb{Z})).
\]

(ii) If \([\text{SL}_2(\mathbb{Z}) : \Gamma] < +\infty, \alpha \in \text{GL}_2(\mathbb{Q})^+, \) then

\[
[\text{SL}_2(\mathbb{Z}) : \text{SL}_2(\mathbb{Z}) \cap \alpha^{-1}\Gamma\alpha] < +\infty.
\]
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Proposition 2.4.7.

\[ \mathcal{M}_k(\mathbb{C}) = \bigcup_{[\text{SL}_2(\mathbb{Z})] \Gamma < +\infty} M_k(\Gamma, \mathbb{C}), \quad \mathcal{S}_k(\mathbb{C}) = \bigcup_{[\text{SL}_2(\mathbb{Z})] \Gamma < +\infty} S_k(\Gamma, \mathbb{C}) \]
are stable under \( \text{GL}_2(\mathbb{Q})^+ \).

Proof. For any \( \gamma \in \Gamma \), \( f|_k \gamma = f \). For \( \alpha \in \text{GL}_2(\mathbb{Q})^+ \), we have

\[ (f|_k \alpha)(\alpha^{-1} \gamma \alpha) = f|_k \alpha, \]
so \( f|_k \alpha \) is invariant for the group \( \alpha^{-1} \Gamma \alpha \cap \text{SL}_2(\mathbb{Z}) \).

To verify that \( f|_k \alpha \) is slowly increasing at \( \infty \), write \( \alpha = \gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \) for some \( \gamma \in \text{SL}_2(\mathbb{Z}) \), then

\[ (f|_k \alpha)(z) = (ad)^{k-1} d^{-k} (f|_k \gamma) \left( \frac{az + b}{d} \right), \]
then we get the result. \( \square \)

Let \( \Gamma = \text{SL}_2(\mathbb{Z}) \), \( G = \text{GL}_2(\mathbb{Q})^+ \), \( \varphi = \sum_{i \in I} \lambda_i 1_{\Gamma \gamma_i} \in \mathbb{Z}[\Gamma \backslash G/\Gamma] \), we define

\[ f|_k \varphi = \sum_{i \in I} \lambda_i f|_k \gamma_i, \quad \text{for } f \in M_k(1) = \mathcal{M}_k(\mathbb{C})^\Gamma. \]

The definition is independent of the choice of \( \gamma_i \). From the general theory, we have

\[ (f|_k \varphi)|_k \varphi'(z) = f|_k (\varphi * \varphi')(z). \]

If \( f \in M_k(1) \) (resp. \( S_k(1) \)), then \( f|_k \varphi \in M_k(1) \) (resp. \( S_k(1) \)).

Facts: \( f|_k R_n = n^{k-2} f \), and \( f|_k T_n = n^{k-1} \sum_{d|m, a \geq 1} d^{-k} f\left( \frac{az + b}{d} \right) \).

Proposition 2.4.8. If \( f = \sum_{m=0}^\infty a_m(f) q^m \), then \( a_m(f|_k T_n) = \sum_{a \mid mn, a \geq 1} a^{-1} a^{mn} a^n \).

Proof. For fixed \( d \mid n, d \geq 1 \),

\[ \sum_{b \mod d} d^{-k} f\left( \frac{az+b}{d} \right) = d^{-k} \sum_{b \mod d} \sum_{m=0}^\infty a_m(f) e^{2\pi im \frac{az+b}{d}} \]
\[ = d^{-k} \sum_{m=0}^\infty a_m(f) e^{2\pi imaz/d} \sum_{b \mod d} e^{2\pi imb/d} \]
\[ = d^{1-k} \sum_{m=0}^\infty a_m(f) e^{2\pi imaz/d} \]
\[ = d^{1-k} \sum_{l=0}^\infty a_{dl}(f) q^a. \]
So
\[ f |_{T_n} = n^{-k-1} \sum_{d|n, a \geq 1} d^{1-k} \sum_{l=0}^{\infty} a_{dl}(f) q^{al}, \]
summing the coefficients of \( q^m \), this gives:
\[ a_m(f |_{T_n}) = n^{-k-1} \sum_{a \geq 1} \frac{(n/a)^{1-k}a_{mn}}{a^2} (f) \]
\[ = \sum_{a \geq 1} \frac{a^{k-1}a_{mn}}{a^2} (f). \]

**Corollary 2.4.9.**

(i) \( M_k(\Gamma, \mathbb{Z}) \) and \( M_k(\Gamma, \mathbb{Q}) \) are stable under \( T_n \) and \( R_n \).

(ii) \( a_0(f |_{T_n}) = \sum_{a|n} a^{k-1}a_0(f) = \sigma_{k-1}(n)a_0(f). \)

(iii) \( a_1(f |_{T_n}) = a_n(f) \), therefore \( f \) is determined by \( T \mapsto a_1(f |_{T}). \)

### 2.5 Petersson scalar product.

**Lemma 2.5.1.**
\[
\int_{\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}} \frac{dx\,dy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\sqrt{1-x^2}}^{+\infty} \frac{dx\,dy}{y^2} = \frac{\pi}{3} < \infty.
\]

**Corollary 2.5.2.**

(i) If \( [\text{SL}_2(\mathbb{Z}) : \Gamma] < +\infty \), then
\[
\int_{\Gamma \setminus \mathcal{H}} \frac{dx\,dy}{y^2} = \frac{\pi}{3} C(\Gamma),
\]
where \( C(\Gamma) = [\text{PSL}_2(\mathbb{Z}) : \bar{\Gamma}] \), \( \bar{\Gamma} \) is the image of \( \Gamma \) in \( \text{PSL}_2(\mathbb{Z}) \).

(ii) If \( \alpha \in \text{GL}_2(\mathbb{Q})^+ \) such that \( \alpha^{-1}\Gamma\alpha \subset \text{SL}_2(\mathbb{Z}) \), then \( C(\alpha^{-1}\Gamma\alpha) = C(\Gamma) \).

**Proof.**

(i) Since \( \frac{dx\,dy}{y^2} \) is invariant under the action of \( \Gamma \), the integral is well defined. Put \( \{\gamma_i\} \) be a family of representatives of \( \Gamma \setminus \text{SL}_2(\mathbb{Z}) \), then \( \Gamma \setminus \mathcal{H} = \bigsqcup \gamma_i(D) \) up to sets of measure 0 (maybe have overlap in \( \text{SL}_2(\mathbb{Z})i \cup \text{SL}_2(\mathbb{Z})\rho \)).

(ii) Since \( \Gamma \setminus \mathcal{H} = \alpha(\alpha^{-1}\Gamma\alpha \setminus \mathcal{H}) \), the two integrals are the same by the invariance of \( \frac{dx\,dy}{y^2} \). \( \square \)
Let \( f, g \in S_k(\mathbb{C}) \), choose \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) of finite index such that \( f, g \in S_k(\Gamma, \mathbb{C}) \).

**Proposition 2.5.3.**

\[
\langle f, g \rangle := \frac{1}{C(\Gamma)} \int_{\Gamma \backslash \mathcal{H}} \overline{f(z)}g(z)\frac{dy\,dx}{y^2}
\]

converges and is independent of the choice of \( \Gamma \).

**Proof.** For \( \gamma \in \Gamma \), we have

\[
\overline{f(\gamma z)} = (cz + d)^k f(z), \quad g(\gamma z) = (cz + d)^k g(z),
\]

so \( \overline{f(z)}g(z)y^k \) is invariant under \( \Gamma \). Now \( \Gamma \backslash \mathcal{H} = \bigcup_{i \in I} \gamma_i D \) with \( |I| = C(\Gamma) \). So if \( \Gamma' \) also satisfy that \( f, g \in S_k(\Gamma', \mathbb{C}) \), then \( f, g \in S_k(\Gamma \cap \Gamma', \mathbb{C}) \), and

\[
\frac{1}{C(\Gamma)} \int_{\Gamma \backslash \mathcal{H}} \overline{f(z)}g(z)\frac{dy\,dx}{y^2} = \frac{1}{C(\Gamma \cap \Gamma')} \int_{(\Gamma \cap \Gamma') \backslash \mathcal{H}} \overline{f(z)}g(z)\frac{dy\,dx}{y^2} = \frac{1}{C(\Gamma')} \int_{\Gamma' \backslash \mathcal{H}} \overline{f(z)}g(z)\frac{dy\,dx}{y^2}.
\]

Because \( f_{|k}\gamma_i \) and \( g_{|k}\gamma_i \) are exponentially decreasing as \( y \to \infty \) on \( D \), \( \langle f, g \rangle \) converges. \( \square \)

**Remark.** In fact, we can choose one modular form and one cusp form, and the integral will still converge.

**Proposition 2.5.4.** For \( f \in S_k(1) \), we have \( \langle G_k, f \rangle = 0 \).

**Proof.** By definition,

\[
G_k(z) = \frac{1}{2} \, \Gamma(k) \sum_{m,n} \frac{1}{(mz + n)^k} \in M_k(1),
\]

and

\[
\sum_{m,n} \frac{1}{(mz + n)^k} = \sum_{a=1}^{\infty} \sum_{(m,n)=1} \frac{1}{(amz + an)^k} = \frac{\Gamma(k)}{(2\pi i)^k} \zeta(k) \sum_{\gamma \in \Gamma \backslash \text{SL}_2(\mathbb{Z})} \frac{1}{(cz + d)^k},
\]
where $\Gamma_\infty$ denotes the subgroup of $\text{SL}_2(\mathbb{Z})$ consisting of all upper triangular matrices. So we just compute $\langle \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \frac{1}{cz + d} f, f \rangle$. We have

$$\langle \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \frac{1}{cz + d} f, f \rangle = \int_{\text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}} \left( \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} \frac{1}{cz + d} \right) f(z) \frac{y^k \, dx \, dy}{y^2}$$

$$= \int_{\Gamma_\infty \setminus \mathcal{H}} f(z) \text{Im} (\gamma z) \frac{y^k \, dx \, dy}{y^2}$$

$$= \int_{\Gamma_\infty} \sum_{\gamma \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} f(z) y^k \, dx \, dy = 0,$$

where the last equality is because $a_0(f) = 0$ and $\int_0^1 e^{2\pi i nx} \, dx = 0$ for $n \geq 1$.

**Lemma 2.5.5.**

(i) For $\alpha \in \text{GL}_2(\mathbb{Q})^+$, we have

$$\langle f |_{k\alpha}, g |_{k\alpha} \rangle = (\det \alpha)^{k-2} \langle f, g \rangle.$$

(ii) Let $\alpha' = (\det \alpha)\alpha^{-1}$, then $\langle f |_{k\alpha}, g \rangle = \langle f, g |_{k\alpha} \rangle$.

**Proof.**

(i) Choose $\Gamma$ such that $f, g \in S_k(\Gamma)$ and $\alpha^{-1}\Gamma \subset \text{SL}_2(\mathbb{Z})$, then

$$C(\alpha^{-1}\Gamma) \langle f |_{k\alpha}, g |_{k\alpha} \rangle = (\det \alpha)^{2(k-1)} \int_{\alpha^{-1}\Gamma \setminus \mathcal{H}} \frac{f(\alpha z) g(\alpha z)}{|cz + d|^{2k}} y^k \, dx \, dy$$

$$= (\det \alpha)^{k-2} \int_{\Gamma \setminus \mathcal{H}} f(z) g(z) y^k \, dx \, dy$$

$$= (\det \alpha)^{k-2} C(\Gamma) \langle f, g \rangle.$$  

(ii) Replace $g$ by $g |_{k\alpha} \alpha^{-1}$, then we get

$$\langle f |_{k\alpha}, g \rangle = (\det \alpha)^{k-2} \langle f, g |_{k\alpha} \rangle$$

$$= (\det \alpha)^{k-2} \langle f, g |_{k \frac{1}{\det \alpha} \alpha' \rangle}$$

$$= \langle f, g |_{k\alpha'} \rangle.$$

\[\square\]

### 2.6 Primitive forms

**Theorem 2.6.1.**

(i) If $n \geq 1$, then $R_n$ and $T_n$ are hermitian.

(ii) The eigenvalues of $T_n$ are integers in a totally real field.

(iii) $S_k(1)$ has a basis of common eigenvectors for all $T_n$, $n \geq 1$. 
2.6. PRIMITIVE FORMS

Proof. (i) It is trivial for $R_n$. Since $T_Z$ is generated by $R_p$ and $T_p$ for $p$ prime, it suffices to consider $T_p$.

Let $\alpha \in M_2(\mathbb{Z})$, $\det \alpha = p$, then there exist $\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$ such that $\alpha = \gamma_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \gamma_2$, then

$$\langle f|k\alpha, g \rangle = \langle f|k\gamma_1 \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \gamma_2, g \rangle = \langle f|k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, g \rangle = \langle f, g|k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \rangle,$$

thus $\langle f|kT_p, g \rangle = (p + 1)\langle f|k\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, g \rangle = \langle f, g|kT_p \rangle$.

(ii) $S_k(SL_2(\mathbb{Z}), \mathbb{Z})$ is a lattice in $S_k(1)$ stable under $T_n$, so $\det(XI - T_n) \in \mathbb{Z}[X]$, so the roots are algebraic integers, and real since $T_n$ is hermitian.

(iii) $T_n$’s are hermitian, hence they are semisimple. Since the $T_n$ commute to each other, by linear algebra, there exists a common basis of eigenvectors for all $T_n$.

Theorem 2.6.2. Let $f = \sum_{n=0}^{\infty} a_n(f)q^n \in M_k(1) - \{0\}$. If for all $n$, $f|kT_n = \lambda_n f$, then

(i) $a_1(f) \neq 0$;
(ii) if $f$ is normalized, i.e. $a_1(f) = 1$, then $a_n(f) = \lambda_n$, for all $n$, and
(a) $a_{mn}(f) = a_m(f)a_n(f)$ when $(m, n) = 1$.
(b) $a_p(f)a_{p^r}(f) = a_{p^{r+1}}(f) + p^{k-1}a_{p^{r-1}}(f)$ for $p$ prime and $r \geq 1$.

Proof. (i) Since $a_n(f) = a_1(f|kT_n) = a_1(\lambda_n f) = \lambda_n a_1(f)$, if $a_1(f) = 0$, then $f = 0$.

(ii) The first assertion is obvious, and the other two follow by the same formulae for the $R_p, T_p$.

Definition 2.6.3. $f \in S_k(1)$ is called primitive if $a_1(f) = 1$ and $f$ is an eigenform for all Hecke operators.

Theorem 2.6.4. (i) If $f, g$ are primitive with the same set of eigenvalues, then $f = g$. (called “Multiplicity 1 theorem”).

(ii) The primitive forms are a basis of $S_k(1)$.

Proof. (i) Apply (i) of the previous theorem to $f - g$, since $a_1(f - g) = 0$, so $f = g$. 
(ii) By (iii) of Theorem 2.6.1, there exists a basis of primitive forms. For any two distinct such forms \( f \) and \( f' \), then there exist \( n \) and \( \lambda \neq \lambda' \) such that
\[
f|_k T_n = \lambda f, \quad f'|_k T_n = \lambda' f,
\]
then \( \lambda \langle f, f' \rangle = \langle f |_k T_n, f' \rangle = \lambda' \langle f, f' \rangle \), so \( \langle f, f' \rangle = 0 \). Therefore one has to take all the primitive forms to get a basis of \( S_k(1) \).

**Remark.** Since \((G_k)_k T_n = \sigma_{k-1}(n)G_k\), we get a basis of \( M_k(1) \) of eigenforms.

**Example 2.6.5.** Write
\[
\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n,
\]
where \( \tau(n) \) is Ramanujan’s \( \tau \)-function. Then
\[
\tau(mn) = \tau(m)\tau(n), \quad \text{if } (m, n) = 1,
\]
\[
\tau(p\tau(p^r)) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1}), \quad \text{if } p \text{ is a prime, } n \geq 1.
\]

**Proof.** Since \( S_{12}(1) = \mathbb{C} \cdot \Delta \), and is stable by the \( T_n \), \( \Delta \) is an eigenform of \( T_n \) with eigenvalue \( \tau(n) \).

**Remark.** In 1973, Deligne proved Ramanujan’s conjecture that
\[
|\tau(p)| \leq 2p^{11/2} \iff \Re(s) = 11/2, \quad \text{if } 1 - \tau(p)p^{-s} + p^{11-2s} = 0
\]
as a consequence of the proof of Riemann Hypothesis (Weil Conjecture) for zeta functions of varieties over finite fields.
Chapter 3

$p$-adic $L$-functions of modular forms

3.1 $L$-functions of modular forms.

3.1.1 Estimates for the fourier coefficients

**Proposition 3.1.1.** Let $\Gamma \subset \text{SL}_2(\mathbb{Z})$ be a subgroup of finite index, let $f = \sum_{n \in \mathbb{Z} \cap M} a_n(f) q^n \in M_k(\Gamma, \mathbb{C})$. Then

(i) $a_n(f) = \begin{cases} O(n^{k-1}), & \text{if } k \geq 3; \\ O(n \log n), & \text{if } k = 2; \\ O(\sqrt{n}), & \text{if } k = 1. \end{cases}$

(ii) $a_n(f) = O(n^{k/2})$, if $f \in S_k(\Gamma)$.

**Proof.** We have that

$$a_n(f) = e^{2\pi ny} y^{-\frac{k}{2}} \frac{1}{M} \int_0^M y^{\frac{k}{2}} f(x + iy) e^{-2\pi inx} dx, \quad \forall \ y.$$ 

Define

$$\varphi(z) = y^{\frac{k}{2}} \sup_{\delta \in \Gamma \setminus \text{SL}_2(\mathbb{Z})} |f_{\delta(z)}|.$$ 

It is finite since $[\text{SL}_2(\mathbb{Z}) : \Gamma] < +\infty$, and $\varphi(\gamma z) = \varphi(z)$ for $\gamma \in \text{SL}_2(\mathbb{Z})$. 

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Let $D$ be the fundamental domain of $\text{SL}_2(\mathbb{Z})$. For any $\delta \in \Gamma \setminus \text{SL}_2(\mathbb{Z})$, there exists $C_\delta$ such that, for all $z \in D$, 

$$|f_\delta(z) - a_0(f_\delta)| \leq C_\delta e^{-\frac{2\pi y}{M}}.$$ 

Let $C = \sup \{ C_\delta, \psi(z) = \sup_{(c,d) \neq (0,0)} \frac{y}{cz+d} \}, \text{ then } \varphi(z) \leq C \psi(z)^{k/2} + B \text{ for some } B.$

$$a_n(f) \leq e^{2\pi y \frac{k}{2} - \frac{1}{M}} \int_0^M \varphi(x + iy) dx \leq e^{2\pi y \frac{k}{2} - \frac{1}{M}} \int_0^M (C \psi(x + iy)^{k/2} + B) dx.$$ 

If $C = 0$, take $y = \frac{1}{Mn}$, then we get (ii).

We now need to evaluate

$$\int_0^M \psi(x + iy)^{\frac{k}{2}} dx.$$ 

Let $y \leq 1$ (in application, $y = \frac{1}{Mn}$), then $\psi(x + iy) \leq \frac{1}{y}$. Let $j \in \mathbb{N}$. If $\psi(x + iy) \geq \frac{1}{4^j y}$, there exists $(c, d)$ such that $c^2 y^2 + (cx + y)^2 \leq 4^j y^2$, hence there exist $c, d \in \mathbb{Z}$, such that

$$1 \leq |c| \leq 2^j, \quad |cx + d| \leq 2^j y.$$ 

Now

$$\text{Meas}\{x \in [0, M] : \exists d, s.t.|cx + d| \leq 2^j y\} \leq 2^{j+1} y M,$$

so

$$\text{Meas}\{x \in [0, M] : \psi(x + iy) \geq \frac{1}{4^j y}\} \leq 4^j 2y M,$$

and

$$\int_0^M \psi(x + iy)^{k/2} dx$$

$$\leq \sum_{j=1}^{[-\log_4 y]} \text{Meas}\{x \in [0, M] : \frac{1}{4^j y} \leq \psi(x + iy) \leq \frac{1}{4^{j-1} y}\} (\frac{1}{4^{j-1} y})^{k/2} + 4^{j/2} \text{Meas}\{x \in [0, M] : \psi(x + iy) \leq 4\}$$

$$\leq M 4^{j/2} + \sum_{j=1}^{[-\log_4 y]} 4^j 2y M (\frac{1}{4^{j-1} y})^{k/2}$$

$$= M 4^{j/2} (1 + 2 \sum_{j=1}^{[-\log_4 y]} y^{1-k/2} 4^{j(1-k/2)}).$$

When $k \geq 3$, let $y = 1/Mn$. As

$$\sum_{j=1}^{[-\log_4 y]} 4^{j(1-k/2)} \text{ converges, we get } a_n(f) = O(n^{k-1}).$$

When $k = 2$, it is obvious. For $k = 1$, 

$$\sum_{j=1}^{[-\log_4 y]} y^{1-k/2} 4^{j(1-k/2)} < 2 - y^{1/2} < 2,$$ 

then we get the result. □
Remark. (i) \( L(f, s) = \sum_{n \neq 0} a_n(f)n^{-s} \) converges for \( \text{Re}(s) \gg 0 \).

(ii) If \( \Gamma \) is a congruence subgroup, \( f \in S_k(\Gamma) \), Deligne showed that
\[
a_n(f) = O(n^{(k-1)/2+\varepsilon}), \quad \forall \varepsilon > 0
\]
in the same theorem mentioned above.

Question: What about the noncongruence subgroups?

### 3.1.2 Dirichlet series and Mellin transform

**Definition 3.1.2.** Let \( \{a_n\}_{n \geq 1} \) be a sequence in \( \mathbb{C} \), the *Dirichlet series* of \( (a_n) \) is \( D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \).

**Lemma 3.1.3.** If \( D(s_0) \) converges, then \( D(s) \) converges uniformly on compact subsets of \( \text{Re}(s) > \text{Re}(s_0) \).

**Proof.** One can assume \( s_0 = 0 \), then use Abel’s summation. \( \square \)

**Corollary 3.1.4.** There exists a maximal half plane of convergence (resp. absolute convergence).

**Remark.** (i) if \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), then the maximal open disc of convergence of \( f \) is the maximal open disc of absolute converge, and also is the maximal open disc of center 0 on which \( f \) can be extended analytically.

(ii) Let \( a_n = (-1)^{n-1} \), then \( D(s) = (1 - 2^{1-s}) \zeta(s) \), which converges for \( \text{Re}(s) > 0 \), absolutely converges \( \text{Re}(s) > 1 \) and can be extended analytically to \( \mathbb{C} \).

(iii) In general you can’t extend \( D(s) \) outside its half plane of absolute convergence, but for \( D(s) \) coming from number theory, it seems that you can always extend meromorphically to \( \mathbb{C} \) (Langlands program).

We review some basic facts about Mellin transform:

**Proposition 3.1.5.** (i) Let \( \varphi : \mathbb{R}^*_+ \to \mathbb{C} \) be in \( \mathcal{C}^r \), and suppose there exist \( A > B \) satisfying, for \( 0 \leq i \leq n \),
\[
\varphi^{(i)}(t) = \begin{cases} O(t^{A-i}) & \text{near 0} \\ O(t^{B-i}) & \text{near } \infty. \end{cases}
\]
Let
\[ \text{Mel}(\varphi, s) := \int_0^\infty \varphi(t)t^s \frac{dt}{t}. \]

Then it is holomorphic on \(-A < \text{Re}(s) < -B\), and \(O(|s|^{-r})\) on \(-A < a \leq \text{Re}(s) \leq b < -B\).

(ii) If \(r \geq 2\), \(\varphi(x) = \frac{1}{2\pi} \int_{C-i\infty}^{C+i\infty} \text{Mel}(\varphi, s) x^{-s} ds\), for any \(C\) with \(-a < C < -B\).

Proof. (i) The first assertion is clear. For the second, use
\[ \text{Mel}(\varphi, s) = (-1)^r \frac{1}{s(s+1)\cdots(s+r-1)} \text{Mel}(\varphi^{(r)}, s+r). \]

(ii) \(\text{Mel}(\varphi, C + it) = \hat{\psi}_C(t)\), where \(\psi_C(x) = \varphi(e^x)e^{Cx}\), and \(\hat{\psi}_C\) is the Fourier transform of \(\psi_C\). Then use Fourier inversion formula.

3.1.3 Modular forms and \(L\)-functions

For \(f = \sum_{n=0}^{\infty} a_n(f) q^n \in M_{2k}(1)\), define
\[ L(f, s) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s}, \quad \Lambda(f, s) = \frac{\Gamma(s)}{(2\pi)^s} L(f, s). \]

Example 3.1.6. Take \(f = G_{2k}\), we get
\[ L(G_{2k}, s) = \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^s} = \sum_{n=1}^{\infty} \left( \sum_{ad=n} d^{2k-1} \right) (ad)^{-s} = \left( \sum_{a=1}^{\infty} a^{-s} \right) \left( \sum_{d=1}^{\infty} d^{2k-1-s} \right) = \zeta(s) \zeta(s-2k+1). \]

Theorem 3.1.7. (i) \(L(f, s)\) absolutely converges for \(\text{Re}(s) > 2k\);

(ii) (a) \(\Lambda(f, s)\) has a meromorphic continuation to \(\mathbb{C}\);

(b) \(\Lambda(f, s)\) is holomorphic except for simple poles at \(s = 0\) of residue \(a_0(f)\) and \(2k\) of residue \((-1)^k a_0(f)\);

(c) \(\Lambda(f, 2k-s) = (-1)^k \Lambda(f, s)\);

(d) \(\Lambda(f, s)\) goes to zero at \(\infty\) in each vertical strip.
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\[ \text{Proof.} \]

(i) The result follows from \( a_n(f) = O(n^{2k-1}) \).

(ii) Let \( \varphi(t) = f(it) - a_0(f) \), then \( \varphi \) is \( C^\infty \) on \( \mathbb{R}_+ \), and \( \varphi(t) = O(e^{-2\pi t}) \) at \( \infty \). \( f \in M_{2k}(1) \) implies

\[ \varphi(t^{-1}) = (-1)^k t^{2k} \varphi(t) + (-1)^k a_0(f) t^{2k} - a_0(f). \]

For \( \text{Re}(s) > 0 \), we have

\[ \int_0^\infty e^{-2\pi n t} t^s dt = \frac{\Gamma(s)}{(2\pi n)^s}. \]

Then for \( \text{Re}(s) > k \),

\[ \Lambda(f, s) = \sum_{n=1}^\infty a_n(f) \frac{\Gamma(s)}{(2\pi n)^s}. \]

\[ = \int_1^\infty \varphi(t) t^s dt \]

\[ = \int_1^\infty \varphi(t) t^s dt + \int_1^\infty \varphi(t^{-1}) t^{-s} dt \]

\[ = \int_1^\infty \varphi(t)(t^s + (-1)^k t^{2k-s}) dt - a_0(f)(\frac{(-1)^k}{2k-s} + \frac{1}{s}), \]

since the first term is holomorphic for all \( s \in \mathbb{C} \), this gives (a) and (b). Replacing \( s \) by \( 2k - s \) in (\( * \)), we get (c). (d) follows from integration by part. \( \square \)

**Theorem 3.1.8 (Hecke’s converse theorem).** Let \( (c_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{C} \) such that \( L(s) = \sum_{n=1}^\infty \frac{c_n}{n^s} \) converges for \( \text{Re}(s) > A \), and \( \Lambda(s) = \frac{\Gamma(s)}{(2\pi)^s} L(s) \) satisfy (ii)(a) – (d) of previous theorem, then

\[ f(z) := \sum_{n=0}^\infty c_n q^n \in M_{2k}(1). \]

\[ g(z) = f(-\frac{1}{z}) - z^2 f(z) = 0 \quad \text{on} \quad \mathcal{H}. \]

It suffices to prove that \( g(it) = 0 \) for \( t > 0 \). Let

\[ \varphi(t) = f(it) - c_0 = \sum_{n=1}^\infty c_n e^{-2\pi n t}, \]

one can check that \( \Lambda(s) = \text{Mel}(\varphi, s) \). Take \( c > A \), then

\[ \varphi(t) = \frac{(-1)^k}{2\pi} \int_{c-i\infty}^{c+i\infty} \Lambda(s) t^{-s} ds - \frac{(-1)^k}{2\pi} \int_{c-i\infty}^{c+i\infty} \Lambda(s) t^{2k-s} ds \]

\[ = \frac{1}{2\pi i} \left( \int_{c-i\infty}^{c+i\infty} \Lambda(s) t^{-s} ds - \int_{c-i\infty}^{c+i\infty} \Lambda(2k-s) t^{s-2k} ds \right) \]

\[ = \frac{1}{2\pi i} \left( \int_{c-i\infty}^{c+i\infty} \Lambda(s) t^{-s} ds - \int_{2k-c-i\infty}^{2k-c+i\infty} \Lambda(s) t^{-s} ds \right) \]
Consider the integral of the function $\Lambda(s) t^{-s}$ around the closed path $\gamma$. Since $\Lambda(s) \to 0$ on vertical strips, by Cauchy formula,

$$\lim_{R \to +\infty} \int_{\gamma} R \Lambda(s) t^{-s} ds = 2\pi i \left( \text{res}_{s=0} (\Lambda(s) t^{-s}) + \text{res}_{s=-2k} (\Lambda(s) t^{-s}) \right) = 2\pi i \left( -c_0 + (-1)^k c_0 t^{-2k} \right).$$

So

$$\varphi(t) - \frac{(-1)^k}{t^{2k}} \varphi(t^{-1}) - (-c_0 + (-1)^k c_0 t^{-2k}) = 0,$$

by an easy computation, the left hand is just $\frac{(-1)^k}{t^{2k}} (-g(it))$, then we get $g(it) = 0$, which completes the proof. \qed

### 3.1.4 Euler products

**Theorem 3.1.9.** If $f = \sum_{n=0}^{\infty} a_n(f) q^n \in M_{2k}(1)$ is primitive, then

$$L(f, s) = \prod_p \frac{1}{1 - a_p(f)p^{-s} + p^{2k-1-2s}}.$$

**Proof.** By Theorem 2.6.2, $a_{nm}(f) = a_n(f)a_m(f)$ whenever $(n, m) = 1$, so

$$L(f, s) = \prod_p \left( \sum_{r=0}^{\infty} a_{p^r}(f)p^{-r} \right).$$

Since,

$$a_{p^r+1} - a_p a_{p^r} + p^{2k-1} a_{p^r-1} = 0,$$
multiplying by $p^{-(r+1)s}$, and summing over $r$ from 1 to $+\infty$, we get

$$
\sum_{r=2}^{\infty} a_{p'} p^{-rs} - a_p p^{-s} \sum_{r=1}^{\infty} a_{p'} p^{-rs} + p^{2k-1-2s} \sum_{r=0}^{\infty} a_{p'} p^{-rs} = 0.
$$

Using the fact that $a_1 = 1$, the result follows.

\[\square\]

### 3.2 Higher level modular forms

#### 3.2.1 Summary of the results

For $N \geq 2$, define

$$
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.
$$

and write $S_k(\Gamma_0(N)) = S_k(N)$.

**Exercise.** If $DM|N$, $f \in S_k(M)$, let $f_D(z) = f(Dz)$, then $f_D \in S_k(N)$. Such a form is said to be *old* if $M \neq N$.

**Definition 3.2.1.** $S^{\text{new}}_k(N) = \{ f \in S_k(N) : \langle f, g \rangle = 0, \forall \text{ } g \text{ } \text{“old”} \}$.

On $S_k(N)$, we have the Hecke operators $T_n$, $(n, N) = 1$,

$$
f_{|k} T_n = n^{k-1} \sum_{\substack{ad=n, a>1 \\ b \mod{d}}} d^{-k} f \left( \frac{az+b}{d} \right),
$$

and for $p \mid n$, the operator

$$
f_{|k} U_p = \frac{1}{p} \sum_{i=0}^{p-1} f \left( \frac{z+i}{p} \right).
$$

We also have a involution $w_N$ given by

$$
f_{|k} w_N = N^{-\frac{k}{2}} z^{-k} f \left( \frac{-1}{Nz} \right).
$$

**Definition 3.2.2.** $f \in S_k(N)$ is called *primitive* if $f \in S^{\text{new}}_k(N)$, $a_1(f) = 1$ and $f_{|k} T_n = a_n(f) f$, whenever $(n, N) = 1$. 

Theorem 3.2.3. (i) The primitive forms are a basis of $S_k^\text{new}(N)$.
(ii) If $f$ is primitive, then $\mathbb{Q}(\{a_n(f)\}, n \in \mathbb{N})$ is a totally real number field, $a_n(f)$ are integers, and $f^\sigma$ is primitive for all $\sigma \in \text{Aut}(\mathbb{C})$.
(iii) If $f$ is primitive, then
(a) $a_{nm}(f) = a_n(f)a_m(f)$ if $(n, m) = 1, (nm, N) = 1$;
(b) For $p \nmid N$, $a_{p^r+1} - a_p(f)a_{p^r}(f) + p^{k-1}a_{p^{r-1}}(f) = 0$.
(c) $f_Up = a_p(f)f$, and this implies $a_{p^r}(f) = (a_p(f))^r$ for $p|N$;
(d) There exists $\varepsilon_f = \pm 1$, such that $f_\varepsilon w_N = \varepsilon_ff$.

Theorem 3.2.4. Suppose $f = \sum_{n=1}^\infty a_nq^n \in S_k(N)$ is primitive. Define

$$L(f, s) = \sum_{n=1}^\infty \frac{a_n}{n^s}, \quad \Lambda(f, s) = \Gamma(s)(\frac{\sqrt{N}}{2\pi})^sL(f, s).$$

Then
(i) $L(f, s) = \prod_{p|N} \frac{1}{1-a_p^{-s}} \prod_{pr|N} \frac{1}{1-a_p^{-s}+p^{-1-s}}$;
(ii) $\Lambda(s)$ has an analytic continuation to $\mathbb{C}$. And

$$\Lambda(f, s) = i^{-k}\varepsilon_f\Lambda(f, k-s);$$

(iii) More generally, if $(D, N) = 1$, $\chi : (\mathbb{Z}/D\mathbb{Z})^* \to \mathbb{C}$ is a character of conductor $D$. Then
(a) $f \otimes \chi = \sum_{n=1}^\infty a_n\chi(n)q^n \in S_k(ND^2, \chi^2)$;
(b) $L(f \otimes \chi, s) = \prod_{p|N} \frac{1}{1-\chi(p)a_p^{-s}} \prod_{pr|N} \frac{1}{1-\chi(p)a_p^{-s}+\chi^2(p)p^{-1-s}}$;
(c) $\Lambda(f \otimes \chi, s) = \Gamma(s)(\frac{D\sqrt{N}}{2\pi})^sL(f \otimes \chi, s)$ has a analytic continuation to $\mathbb{C}$ and

$$\chi(-N)\frac{\Lambda(f \otimes \chi, s)}{G(x)} = i^{-k}\varepsilon_f\frac{\Lambda(f \otimes \chi^{-1}, s)}{G(\chi^{-1})}$$

where $G(\chi)$ is the Gauss sum

$$G(\chi) = \sum_{x \in (\mathbb{Z}/D\mathbb{Z})^*} \chi(x)e^{2\pi ix}. $$

Theorem 3.2.5 (Weil’s Converse Theorem). Conversely, if $(a_m)_{m \geq 1}$ satisfy (b) and (c) of condition (iii) of the above theorem for all $\chi$ of conductor $D$, $(D, N) = 1$, then $\sum_{m=1}^\infty a_mq^m \in S_k(N)$ and is primitive.
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3.2.2 Taniyama-Weil Conjecture

Let Λ be a finitely generated \( \mathbb{Z} \)-algebra. Define its Hasse-Weil zeta function \( \zeta_\Lambda(s) \) by

\[
\zeta_\Lambda(s) = \prod_{\wp \text{ prime in } \Lambda} \frac{1}{1 - |\Lambda/\wp|^{-s}}.
\]

Conjecture 3.2.6 (Hasse-Weil). \( \zeta_\Lambda \) has a meromorphic continuation to \( \mathbb{C} \).

Let \( E : y^2 = x^3 + ax^2 + bx + c, a, b, c \in \mathbb{Q} \) be an elliptic curve, \( \Lambda_E = \mathbb{Z}[x, y]/(y^2 - x^3 - ax^2 - bx - c) \) be its coordinate ring, which is a finitely generated algebra over \( \mathbb{Z} \).

Theorem 3.2.7 (Wiles, Breuil-Conrad-Diamond-Taylor). There exists a unique \( N_E \) and \( f_E \in S_2(N_E) \) which is primitive, such that

\[
\zeta_{\Lambda_E} \sim \frac{\xi(s - 1)}{L(f_E, s)}
\]

while \( \sim \) means up to multiplication by a finite numbers of Euler factors.

Remark. This proves Hasse-Weil conjecture in this case thanks to theorem 3.2.4.

Theorem 3.2.8 (Mordell-Weil). \( E(\mathbb{Q}) \cup \{\infty\} \simeq \mathbb{Z}^{r(E)} \oplus \text{finite group} \).

Conjecture 3.2.9 (Birch, Swinnerton-Dyer). \( \text{ord}_{s=1} L(f_E, s) = r(E) \).

3.3 Algebraicity of special values of L-functions

3.3.1 Modular symbols.

Let \( N \geq 1, f \in S_k(N), P \in A[x]_{k-2} \) (polynomials of degree \( \leq k - 2 \)) with \( A \subset \mathbb{C} \) a subring. For \( r \in \mathbb{Q} \), the integral \( \int_{i \infty}^{i \infty} f(z)P(z)dz \) converges because \( f \) is exponentially small around \( i \infty \) and \( r \). These integrals are called modular symbols.

For \( 0 \leq j \leq k - 2 \), define

\[
r_j(f) = \int_{i \infty} f(z)z^jdz = \frac{\Gamma(j + 1)}{(-2\pi i)^{j+1}}L(f, j + 1).
\]

Let \( L_f \) be the \( \mathbb{Z} \)-module generated by \( r_j(f_k \delta), 1 \leq j \leq k - 2 \) and \( \delta \in \Gamma_0(N) \setminus \text{SL}_2(\mathbb{Z}) \). Then \( L_f \) is finitely generated.
CHAPTER 3. P-ADIC L-FUNCTIONS OF MODULAR FORMS

Theorem 3.3.1. If \( P \in \mathbb{A}[x]^{(k-2)} \), \( r \in \mathbb{Q} \), then \( \int_{r}^{i \infty} f(z)P(z)dz \in \mathbb{A} \cdot L_f \subset \mathbb{C} \).

Proof. For \( \gamma \in \text{SL}_2(\mathbb{Z}) \),

\[
\int_{\gamma(0)}^{(i \infty)} f(z)P(z)dz = \int_{0}^{i \infty} f(\gamma z)P(z)dz = \int_{0}^{i \infty} f_{k}\gamma(z)P_{2-k}\gamma(z)dz,
\]
where \( P_{2-k}\gamma(z) = (cz + d)^{k-2}P \left( \frac{az + b}{cz + d} \right) \in \mathbb{A}[x]^{(k-2)} \). Take \( r = a/b, (a, b) = 1 \), then there exists \( \gamma = \left( \begin{array}{cc} a_{l-1} & a_l \\ b_{l-1} & b_l \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \) satisfying \( (a_0, b_0) = (1, 0) \), \( (a_n, b_n) = (a, b) \).

\[
\int_{r}^{i \infty} f(z)P(z)dz = \sum_{l=1}^{n} \int_{r}^{(i \infty)} f(z)P(z)dz = \sum_{l=1}^{n} \int_{0}^{i \infty} f_{k}\gamma_l(z)P_{2-k}\gamma_l(z)dz \in \mathbb{A} \cdot L_f.
\]

Exercise. For \( N = 1 \), let \( L^+_f \) (resp. \( L^-_f \)) be the \( \mathbb{Z} \)-module generated by \( r_j(f) \) for all odd (resp. even) \( j \). For \( P \in \mathbb{A}[X]^{(k-2)} \), \( r \in \mathbb{Q} \), \( \varepsilon = \pm \), then

\[
\int_{r}^{i \infty} f(z)P(z)dz = \varepsilon \int_{-r}^{(i \infty)} f(z)P(-z)dz \in \mathbb{A} \cdot L^\varepsilon_f.
\]

Corollary 3.3.2. (i) Suppose \( f \in \sum_{n=1}^{\infty} a_n q^n \), \( \phi : \mathbb{Z} \to \bar{\mathbb{Q}} \) is constant mod \( M\mathbb{Z} \) for some \( M \). Then \( L(f, \phi, s) = \sum_{n=1}^{\infty} \phi(n) \frac{a_n}{n^s} \) has an analytic continuation to \( \mathbb{C} \) and

\[
\Lambda(f, \phi, j) = \frac{\Gamma(j)}{(-2\pi i)^j} L(f, \phi, j) \in \bar{\mathbb{Q}} \cdot L_f,
\]
if \( 1 \leq j \leq k - 1 \).

(ii) If \( N = 1 \) and \( \phi(-x) = \varepsilon(-1)^j \phi(x) \), then \( \Lambda(f, \phi, j) \in \bar{\mathbb{Q}} \cdot L^\varepsilon_f \), if \( 1 \leq j \leq k - 1 \).
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Proof. we may assume \( \phi(n) = e^{2\pi i \frac{nu}{M}} \) for some \( 0 \leq u \leq M - 1 \) because such functions form a basis, then

\[
\frac{\Gamma(s)}{(2\pi)^s} L(f, \phi, s) = \int_0^\infty \sum_{n=1}^\infty a_n e^{2\pi i \frac{nu}{M}} e^{-2\pi ny} y^s dy\]

this proves the first assertion of (i) as \( f \) is exponentially small around \( i\infty \) and \( \frac{u}{M} \).

\[
\Lambda(f, \phi, j) = \int_{\frac{u}{M}}^{\infty} f(z)(z - \frac{u}{M})^{j-1}dz \in \bar{\mathbb{Q}} \cdot L_f.
\]

For (ii), we may assume \( \phi(n) = e^{2\pi i \frac{nu}{M}} + \varepsilon(-1)^j e^{-2\pi i \frac{nu}{M}} \), and similarly,

\[
\Lambda(f, \phi, j) = \int_{\frac{u}{M}}^{\infty} f(z)(z - \frac{u}{M})^{j-1}dz + \varepsilon(-1)^j \int_{-\frac{u}{M}}^{\infty} f(z)(z + \frac{u}{M})^{j-1}dz
\]

then one uses the exercise. \( \square \)

3.3.2 The results

Theorem 3.3.3. If \( f \) is primitive, then there exist \( \Omega_f^+ \) and \( \Omega_f^- \in \mathbb{C} \), if

\[
\phi : \mathbb{Z} \to \bar{\mathbb{Q}} \pmod{MZ}, \ 1 \leq j \leq k - 1, \ \phi(x) = \varepsilon(-1)^j \phi(-x),
\]

then \( \Lambda(f, \phi, j) \in \bar{\mathbb{Q}} \cdot \Omega_f \).

Proof. We prove the case \( N = 1, \ \varepsilon = 1 \).

We shall prove that

\[
r_{k-2}(f)r_l(f) \in \bar{\mathbb{Q}}\langle f, f \rangle, \text{ for } l \text{ odd}. \tag{3.1}
\]

This implies

\[
\Omega_f^+ \sim \frac{\langle f, f \rangle}{r_{k-2}(f)} \sim \frac{\langle f, f \rangle}{L(f, k-1)\pi^{k-2}}
\]

where \( \sim \) stands for equality up to multiplication by an algebraic number. The method to show (3.1) is the Rankin’s method in the following section. \( \square \)
3.3.3 Rankin’s method

Assume \( k = l + j \) for \( k, l, j \in \mathbb{N} \). Suppose \( \chi_1, \chi_2 : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) are multiplicative characters. Let

\[
    f = \sum_{n=1}^{+\infty} a_n q^n \in S_k(N, \chi_1^{-1}), \quad g = \sum_{n=0}^{+\infty} b_n q^n \in M_l(N, \chi_2).
\]

So

\[
    f(\gamma z) = \chi_1^{-1}(d)(cz + d)^k f(z), \quad g(\gamma z) = \chi_2(d)(cz + d)^l g(z).
\]

Let

\[
    G_{j, \chi_1 \chi_2, s}(z) = \frac{1}{2} \cdot \frac{\Gamma(j)}{(-2\pi i)^j} \cdot \sum_{N \mid m \mid (N,n) = 1} \frac{\chi_1 \chi_2(n) g^{s+1-k}}{(mz + n)^j | mz + n |^{2(s+1-k)}}
\]

\[
    = \frac{\Gamma(j)}{(-2\pi i)^j} L(\chi_1 \chi_2, j + 2(s + 1 - k)) \cdot \sum_{\gamma = (a \ b \, c \ d) \in \Gamma_\infty \backslash \Gamma_0(N)} \chi_1 \chi_2(d) (cz + d)^j \cdot \text{Im} (\gamma z)^{s+1-k}.
\]

We have

**Proposition 3.3.4.**

\[
    D(f,g,s) = L(\chi_1 \chi_2, j + 2(s + 1 - k)) \sum_{n=1}^{+\infty} \frac{a_n b_n}{n^s} = \frac{(4\pi)^s (-2\pi i)^j}{\Gamma(s)} \cdot \frac{\Gamma(s)}{\Gamma(j)} \cdot \langle f, g G_{j, \chi_1 \chi_2, s} \rangle \cdot [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)].
\]

**Proof.** Using the Fourier expansion, then

\[
    \sum_{n=1}^{+\infty} \frac{a_n b_n}{n^s} = \frac{\Gamma(s)}{(4\pi)^s} \int_0^{+\infty} \int_0^1 \frac{f(z)g(z)}{y} \, dx \cdot y^s \, dy
\]

\[
    = \frac{\Gamma(s)}{(4\pi)^s} \int_{\Gamma_\infty \backslash \mathcal{H}} \frac{f(z)g(z) y^{s+1}}{y^2} \, dx dy
\]

\[
    = \frac{\Gamma(s)}{(4\pi)^s} \int_{\Gamma_0(N) \backslash \mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (f(\gamma z)g(\gamma z) \text{Im} (\gamma z)^{s+1}) \, dx dy
\]

\[
    = \frac{\Gamma(s)}{(4\pi)^s} \int_{\Gamma_0(N) \backslash \mathcal{H}} \frac{f(z)}{(cz + d)^j} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \chi_1 \chi_2(d) \text{Im} (\gamma z)^{s+1-k} \left( cz + d \right)^j y^k \, dx dy,
\]

this implies the Proposition. \( \square \)
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**Theorem 3.3.5.** (i) $D(f, g, s)$ admits a meromorphic continuation to $\mathbb{C}$, which is holomorphic outside a simple pole at $s = k$ if $l = k$ and $\chi_1 \chi_2 = 1$.

(ii) If $f$ is primitive, $g \in M_l(N, \chi_2, \overline{\mathbb{Q}})$, then

$$D(f, g, k - 1) \in \overline{\mathbb{Q}} \cdot \pi^{j+k-1}(f, f).$$

**Proof.** As $D(f, g, s) = \langle f, gG_s \rangle$,

(i) we have to prove the same statement for $G_s$, which can be done by computing its Fourier extension. The pole comes from the constant Fourier coefficients.

(ii) For the case $N = 1$, $\chi_1 = \chi_2 = 1$ and $j \geq 3$, then $G_{j, \chi_1 \chi_2, k - 1} = G_j$, we are reduced to prove

$$\langle f, gG_j \rangle \in \overline{\mathbb{Q}} \langle f, f \rangle.$$

Let $f_i, i \in I$ be a basis of $S_k(1)$ of primitive forms, with $f_1 = f$. As $gG_j \in M_k(1, \overline{\mathbb{Q}})$, we can write $gG_j = \lambda_0 G_k + \sum \lambda_i f_i$, with $\lambda_i \in \overline{\mathbb{Q}}$. Since

$$\langle G_k, f \rangle = 0, \quad \langle f, f_j \rangle = 0, \text{ if } j \neq 1,$$

Then $\langle f, gG_j \rangle = \lambda_1 \langle f, f \rangle$.

**Remark.** The general case can be treated in the same way, once we prove that

$$G_{j, \chi_1 \chi_2, k - 1} \in M_j(N, \chi_1 \chi_2, \overline{\mathbb{Q}}) \text{ (if } j \neq 2 \text{ or } \chi_1 \chi_2 \neq 1).$$

**Proposition 3.3.6.** If

$$\sum_{n=1}^{+\infty} \bar{a}_n n^s = \left( \sum_{n \in \mathbb{Z}[\frac{1}{N}]^+} \bar{a}_n n^s \right) \prod_{p \nmid N} \frac{1}{(1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})}, \quad \alpha_p \beta_p = \chi_1(p)p^{k-1},$$

$$\sum_{n=1}^{+\infty} \bar{b}_n n^s = \left( \sum_{n \in \mathbb{Z}[\frac{1}{N}]^+} \bar{b}_n n^s \right) \prod_{p \nmid N} \frac{1}{(1 - \gamma_p p^{-s})(1 - \delta_p p^{-s})}, \quad \gamma_p \delta_p = \chi_2(p)p^{l-1},$$

then $D(f, g, s) =$

$$\left( \sum_{n \in \mathbb{Z}[\frac{1}{N}]^+} \bar{a}_n \bar{b}_n n^s \right) \prod_{p \nmid N} \frac{1}{(1 - \alpha_p \gamma_p p^{-s})(1 - \beta_p \gamma_p p^{-s})(1 - \alpha_p \delta_p p^{-s})(1 - \beta_p \delta_p p^{-s})}.$$
Proof. Exercise, noting that
\[ a_p^{r+1} = \frac{\alpha_p^{r+1} - \beta_p^{r+1}}{\alpha_p - \beta_p}, \quad b_p^{r+1} = \frac{\gamma_p^{r+1} - \delta_p^{r+1}}{\gamma_p - \delta_p}. \]

We give one application here:

**Corollary 3.3.7.** The claim (3.1) holds, i.e.
\[ r_{k-2}(f)r_{l}(f) \in \bar{\mathbb{Q}}(f, f), \quad \text{for } l \text{ odd.} \]

**Proof.** Let \( f \in S_k(1) \) be primitive, \( k \) given. For \( l \) even, let \( g = G_l \), then
\[
\sum_{n=1}^{+\infty} b_n \frac{1}{n^s} = \prod_p \frac{1}{(1 - p^{-s})(1 - p^{l-s-1})},
\]
hence \( D(f, G_l, s) = L(f, s)L(f, s - l + 1) \). Therefore
\[
L(f, k - 1)L(f, k - l) \in \bar{\mathbb{Q}} \cdot \pi^{j+k-1}\langle f, f \rangle
\]
which implies
\[
r_{k-2}(f)r_{k-l-1}(f) \in \bar{\mathbb{Q}}(f, f).
\]

**Remark.** In the general case,
\[
L(G_j, \chi_1\chi_2, k - 1, s) \sim \zeta(s)L(\chi_1\chi_2, s - l + 1).
\]

If \( f_1, f_2, \ldots, f_n \) are primitive forms \( \in S_k(N_i) \) for \( N_i \mid N \). Write
\[
L(f_i, s) = \prod_{p|N} \frac{1}{(1 - \alpha^{(i)}_{p,1}p^{-s})(1 - \alpha^{(i)}_{p,2}p^{-s})},
\]
then
\[
L(f_1 \otimes \cdots \otimes f_n, s) = \prod_{p|N} \prod_{j_1, j_2, \ldots, j_n \in \{1, 2\}} \frac{1}{(1 - \alpha^{(i)}_{p,j_1} \cdots \alpha^{(i)}_{p,j_n}p^{-s})}.
\]

One has the following conjecture:

**Conjecture 3.3.8 (Part of Langlands Program).** \( L(f_1 \otimes \cdots \otimes f_n, s) \) has a meromorphic continuation to \( \mathbb{C} \), and is holomorphic if \( f_i \neq \bar{f}_j \), for all \( i \neq j \).

**Remark.** Rankin’s method implies the above conjecture is OK for \( n = 2 \). The case for \( n = 3 \) is due to Paul Garrett. The case for \( n \geq 4 \) is still open.
3.4 \( p \)-adic \( L \)-functions of modular forms

In the following, we assume \( f \in S_k(N) \) is primitive.

**Definition 3.4.1.** \( \phi^+(x) = \frac{1}{2}(\phi(x) + \phi(-x)), \phi^-(x) = \frac{1}{2}(\phi(x) - \phi(-x)) \).

Then

\[
\tilde{\Lambda}(f, \phi, j) = \frac{\Lambda(f, \phi^+, j)}{\Omega_f^{j-1}} + \frac{\Lambda(f, \phi^-, j)}{\Omega_f^{j-1} + 1} \in \bar{Q}
\]

if \( \phi : \mathbb{Z} \to \bar{Q} \) and \( 1 \leq j \leq k - 1 \).

Fix an embedding \( \bar{Q} \to \bar{Q}_p \). The function \( L(f, s) \) has an Euler product

\[
L(f, s) = \prod_{\ell \text{ prime}} \frac{1}{E_\ell(s)}, \quad E_\ell(s) \in \bar{Q}[\ell^{-s}], \quad \deg E_\ell(s) \leq 2.
\]

Write \( E_\ell(s) = (1 - \alpha p^{-s})(1 - \beta p^{-s}) \) and assume \( \alpha \neq 0 \). Then \( \beta = 0 \) if and only if \( p \mid N \). Set

\[
f_\alpha(z) = f(z) - \beta f(pz).
\]

**Lemma 3.4.2.** \( f_\alpha \mid_k U_p = \alpha f_\alpha \) in all cases.

**Proof.** It is clear if \( p \mid N \) as in the case \( \beta = 0 \). If \( p \nmid N \), then

\[
\alpha + \beta = a_p, \quad \alpha \beta = p^{k-1}.
\]

and \( f \mid_k T_p = (\alpha + \beta)f \), thus

\[
f_\alpha \mid_k U_p - \alpha f_\alpha = \frac{1}{p} \sum_{i=0}^{p-1} f \left( \frac{z + i}{p} \right) - \beta f(z + i) - \alpha f(z) + \alpha \beta f(pz)
\]

\[
= - (\alpha + \beta)f(z) + f_\alpha \mid_k T_p = 0.
\]

If we write \( f_\alpha = \sum_{n=1}^{+\infty} b_n q^n \), the above lemma implies that \( b\alpha_p = \alpha b_n \) for all \( n \). Define \( b_n \) for \( n \in \mathbb{Z} \left[ \frac{1}{p} \right] \) as

\[
b_n = \alpha^{-r} b_{pn}, \quad r \gg 0.
\]
Take $\phi \in LC_c(\mathbb{Q}_p, \bar{\mathbb{Q}})$ a locally constant function with compact support and let
\[
L(f, \phi, s) = \sum_{n \in \mathbb{Z}[\frac{1}{p}]} \phi(n) \frac{\alpha_n}{n^s}.
\]
If $\phi$ has support in $p^{-r} \mathbb{Z}_p$, then $\phi(x) = \phi_0(p^r x)$ for $\phi_0 : \mathbb{Z} \to \bar{\mathbb{Q}}$ constant mod $p^m \mathbb{Z}$ for some $m$. Then
\[
L(f, \phi, s) = \alpha^{-r}p^r s L(f, \phi_0, s)
\]
which implies
\[
\tilde{\Lambda}(f, \phi, j) \in \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p, \text{ for all } \phi \in LC_c(\mathbb{Q}_p, \bar{\mathbb{Q}}).
\]

Definition 3.4.3. Assume $\phi \in LC_c(\mathbb{Q}_p, \bar{\mathbb{Q}})$ and $\phi$ is constant modulo $p^n \mathbb{Z}$. The discrete Fourier transform of $\phi$ is
\[
\hat{\phi}(x) = \frac{1}{p^{-m}} \sum_{\mathbf{y} \mod p^m} \phi(\mathbf{y}) e^{-2\pi i \mathbf{y} \cdot x},
\]
for $m \geq n - v_p(x)$, where $x y \in \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$. This definition does not depend on the choice of $m \geq n - v_p(x)$.

Exercise. (i) $\hat{\phi}$ is constant mod $p^m \mathbb{Z}_p$ if and only if $\phi$ has support in $p^{-m} \mathbb{Z}_p$.
(ii) $\hat{\phi}(x) = \phi(-x)$.
(iii) For $a \in \mathbb{Q}_p$, let $\phi_a(x) = \phi(ax)$, then $\hat{\phi}_a(x) = p^{v_p(a)} \hat{\phi}(\frac{x}{a})$.

Theorem 3.4.4. (i) There exists a unique $\mu_{f,\alpha} : LP^{[0,k-2]}(\mathbb{Z}_p, \bar{\mathbb{Q}}_p) \to \bar{\mathbb{Q}}_p$ such that for all $\phi \in LC(\mathbb{Z}_p, \bar{\mathbb{Q}})$,
\[
\int_{\mathbb{Z}_p} \phi(x) x^{j-1} \mu_{f,\alpha} = \tilde{\Lambda}(f_\alpha, \hat{\phi}, j), \ 1 \leq j \leq k - 1.
\]
Moreover, $\psi(\mu_{f,\alpha}) = \frac{1}{\alpha} \mu_{f,\alpha}$, or equivalently
\[
\int_{p \mathbb{Z}_p} \phi \left( \frac{x}{p} \right) \mu_{f,\alpha} = \frac{1}{\alpha} \int_{\mathbb{Z}_p} \phi \mu_{f,\alpha}.
\]
(ii) if $v_p(\alpha) < k - 1$, then $\mu_{f,\alpha}$ extends uniquely as an element of $D_{v_p(\alpha)}$. 
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**Proof.** (i) The existence of $\mu_{f,\alpha} : \mathbb{LP}^{[0,k-2]}(\mathbb{Z}_p, \mathbb{Q}_p) \to \overline{\mathbb{Q}}_p$ is just the linearity of $\phi \to \hat{\phi}$. The uniqueness is trivial. The second claim follows from

$$
\int_{\mathbb{Z}_p} \phi \left( \frac{x}{p} \right) \left( \frac{x}{p} \right)^{j-1} \mu_{f,\alpha} = \frac{1}{p^{j-1}} \tilde{\Lambda}(f, p^{-1} \hat{\phi}(px), j)
$$

$$
= \frac{1}{\alpha} \tilde{\Lambda}(f, \phi, j) = \frac{1}{\alpha} \int_{\mathbb{Z}_p} \phi(x)x^{j-1} \mu_{f,\alpha}.
$$

(ii) One needs to show there exists a constant $C$, such that

$$
v_p \left( \int_{a+p^n\mathbb{Z}_p} (x - a)^j \mu_{f,\alpha} \right) \geq C + (j - v_p(\alpha))n,
$$

for all $a \in \mathbb{Z}_p$, $n \in \mathbb{N}$, $j \leq k - 2$. Note that

$$
\tilde{1}_{a+p^n\mathbb{Z}_p}(x) = \begin{cases} 
  p^{-n}e^{-2\pi i ax}, & \text{if } x \in p^{-n}\mathbb{Z}_p, \\
  0, & \text{if not.}
\end{cases}
$$

for

$$
\phi_a(x) = \begin{cases} 
 e^{2\pi i \frac{ax}{p^n}}, & x \in \mathbb{Z}_p, \\
  0, & \text{otherwise.}
\end{cases}
$$

Then

$$
\int_{a+p^n\mathbb{Z}_p} (x - a)^j \mu_{f,\alpha} = \sum_{l=0}^{j} (-a)^l \binom{j}{l} p^{-n} \tilde{\Lambda}(f, \phi_a(p^n x), l + 1)
$$

$$
= \alpha^{-n} \sum_{l=0}^{j} (-1)^l \binom{j}{l} p^{nl} \tilde{\Lambda}(f, \phi_a, l + 1).
$$

Since

$$
p^{nl} \tilde{\Lambda}(f, \phi_a, l + 1) = p^{nl} \int_0^{i\infty} f_a(z - \frac{a}{p^n})z^l dz = \int_{-\frac{a}{p^n}}^{i\infty} f_a(z)(p^n z + a)^l dz,
$$

we get

$$
\sum_{l=0}^{j} (-1)^l \binom{j}{l} \int_{a+p^n\mathbb{Z}_p} (x - a)^j \mu_{f,\alpha} = \alpha^{-n} p^{nj} \int_{-\frac{a}{p^n}}^{i\infty} f_a(z)z^j dz \in \alpha^{-n} p^{nj} L_{f,\alpha}.
$$

We just pick $C = \min(v_p(\tilde{r}_j(f, \alpha|k\delta)))$. \qed
CHAPTER 3. P-ADIC L-FUNCTIONS OF MODULAR FORMS

Remark. (i) If \( p \mid N \), then \( \beta = 0 \), and \( \alpha \neq 0 \) implies \( v_p(\alpha) = \frac{k-2}{2} < k - 1 \), hence \( \mu_{f,\alpha} \) exists by the above Theorem.

(ii) If \( p \nmid N \), then \( v_p(\alpha), v_p(\beta) \geq 0 \). Since \( v_p(\alpha) + v_p(\beta) = k - 1 \), at least one of \( \mu_{f,\alpha} \) or \( \mu_{f,\beta} \) always exists.

In the case \( v_p(\alpha) = k - 1 \), then \( \alpha + \beta = a_p(f) \) is a unit. This case is called the ordinary case. The conditions are not strong enough for the uniqueness of \( \mu_{f,\alpha} \), as we can add the \((k-1)\)-th derivative of any \( \lambda \in \mathcal{D}_0 \).

(iii) In the case \( \alpha = \beta = 0 \), we do not understand what happens.

Definition 3.4.5. Let \( \chi : \mathbb{Z}_p^* \to \mathbb{C}_p^* \) be a continuous character. Set

\[
L_{p,\alpha}(f \otimes \chi, s) = \int_{\mathbb{Z}_p^*} x^{-1} \chi(x) \mu_{f,\alpha}.
\]

In particular, take \( \chi(x) = x^{\frac{k}{2}} \langle x \rangle^{s-\frac{k}{2}} \) where \( \langle x \rangle^t = \exp(t \log x) \). Set

\[
L_{p,\alpha}(f, s) = \int_{\mathbb{Z}_p^*} x^{\frac{k}{2} - 1} \langle x \rangle^{s-\frac{k}{2}} \mu_{f,\alpha}.
\]

Proposition 3.4.6. For \( 1 \leq j \leq k - 1 \),

\[
L_{p,\alpha}(f \otimes \chi^j) = (1 - \frac{p^{j-1}}{\alpha})(1 - \frac{\beta}{p^j}) \tilde{\Lambda}(f, j).
\]

Proof. Follows from

(i) \( \hat{1}_{\mathbb{Z}_p^*} = 1_{\mathbb{Z}_p} - p^{-1}1_{\mathbb{Z}_p} \),

(ii) \( \tilde{\Lambda}(f, 1_{\mathbb{Z}_p}, j) = (1 - \frac{\beta}{p^j}) \tilde{\Lambda}(f, j) \),

(iii) \( \psi(\mu_{f,\alpha}) = \frac{1}{\alpha} \mu_{f,\alpha} \).

Remark. (i) As \( \Lambda(f, s) = \Lambda(f, k - s) \) and \( \alpha \beta = p^{k-1} \) if \( p \nmid N \), then

\[
(1 - \frac{p^{j-1}}{\alpha}) = 1 - \frac{\beta}{p^{k-j}}.
\]

Note \( E_p(f, s) = (1 - \alpha p^{-s})(1 - \beta p^{-s}) \). Then the Euler factor of the \( p \)-adic \( L \)-function is actually the product of one part of the Euler factor for \( L(f, s) \) and one part of the Euler factor for \( L(f, k - s) \). This is a general phenomenon.

(ii) If \( p \mid N \), \( \alpha \neq 0 \), the \( v_p(\alpha) = \frac{k-2}{2} \). It can happen that \( \alpha = p^{\frac{k-2}{2}} \), which means \( L_{p,\alpha}(f, \frac{k}{2}) = 0 \). In this case
Conjecture 3.4.7 (Mazur-Tate-Teitelbaum Conjecture).

\[ L'_{p,\alpha}(f, \frac{k}{2}) = L_{\text{Font.}}(f, \frac{k}{2}). \]

Here the \( p \)-adic \( L \)-function is related to 2-dimensional \((\varphi, N)\)-filtered modules \( D \) with \( N \neq 0 \) and \( \text{Fil}^0 D = D \), \( \text{Fil}^1 D \neq D \). For the pair \((\lambda, \alpha)\) as in Fontaine’s course, where \( \lambda \) is the eigenvalue of \( \varphi \) and \( \alpha \) is the parameter associated to the filtration, \( \lambda \) is our \( \alpha \) and \( \alpha \) is our \( L_{\text{Font.}} \).

The conjecture is proved by Kato-Kurihara-Tsuji, Perrin-Riou, and Stevens, Orton, Emerton with other definitions of the \( L \)-invariant.

(iii) Mazur, Tate and Teitelbaum have also formulated a \( p \)-adic analog of the BSD conjecture. For \( E/\mathbb{Q} \) an elliptic curve, by Taniyama-Weil, it is associated to a primitive form \( f \in S_2(N) \). Set \( L_{p,\alpha}(E, s) = L_{p,\alpha}(f, s) \) if it exists, which is the case if \( E \) has either good reduction (hence \( p \nmid N \)) or multiplicative reduction (hence \( p \mid N, p^2 \nmid N \) mod \( p \)).

Conjecture 3.4.8 (\( p \)-adic BSD Conjecture).

\[ \text{ord}_{s=1} L_{p,\alpha}(E, s) = \begin{cases} \text{rank } E(\mathbb{Q}), & \text{if } p \nmid N \text{ or } \alpha \neq 1; \\ \text{rank } E(\mathbb{Q}) + 1, & \text{if } p \mid N \text{ and } \alpha = 1. \end{cases} \]

Kato showed that

\[ \text{ord}_{s=1} L_{p,\alpha}(E, s) \geq \begin{cases} \text{rank } E(\mathbb{Q}), & \text{if } p \nmid N \text{ or } \alpha \neq 1; \\ \text{rank } E(\mathbb{Q}) + 1, & \text{if } p \mid N \text{ and } \alpha = 1. \end{cases} \]

(iv) To prove Kato or Kato-Kurihara-Tsuji, we need another construction of \( p \)-adic \( L \)-functions via Iwasawa theory and \((\varphi, \Gamma)\)-modules; this construction is the subject of the next part of the course and is based on ideas of Perrin-Riou.
CHAPTER 3. P-ADIC L-FUNCTIONS OF MODULAR FORMS
Part II

Fontaine’s rings and Iwasawa theory
Chapter 4

Preliminaries

4.1 Some of Fontaine’s rings

This section is a review of notations and results from Fontaine’s course. For details, see Fontaine’s notes.

4.1.1 Rings of characteristic $p$

(1) $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}_p}$ for the valuation $v_p$ with $v_p(p) = 1$.

$$a = \{x \in \mathbb{C}_p, v_p(x) \geq \frac{1}{p}\}.$$ 

(2) $\tilde{E}^+$ is the ring $R$ in Fontaine’s course. By definition

$$\tilde{E}^+ := \{x = (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{C}_p/a, x_{n+1}^p = x_n, \forall n\}$$

is a ring of characteristic $p$ with an action of $G_{\mathbb{Q}_p}$. For $x = (x_n) \in \tilde{E}^+$, for every $x_n$, pick a lifting $\hat{x}_n \in \mathcal{O}_{\mathbb{C}_p}$, then

$$\lim_{k \to +\infty} (\hat{x}_{n+k})^p := x^{(n)} \in \mathcal{O}_{\mathbb{C}_p}$$

is a canonical lifting of $x_n$ such that

$$\tilde{E}^+ = \{x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in \mathcal{O}_{\mathbb{C}_p}, (x^{(n+1)})^p = x^{(n)}, \forall n\}$$

with the addition and multiplication by

$$(x + y)^{(n)} = \lim_{k \to +\infty} (x^{(n+k)} + y^{(n+k)})^p, \quad (xy)^{(n)} = x^{(n)} y^{(n)}.$$
\( \tilde{E}^+ \) is a valuation ring with valuation
\[
v_E(x) = v_p(x^{(0)})
\]
and maximal ideal
\[
\mathfrak{m}_{\tilde{E}^+} = \{x \in \tilde{E}^+, v_E(x) > 0\}.
\]

(3) Choose once for all
\[
\varepsilon = (1, \varepsilon^{(1)}, \ldots) \in \tilde{E}^+, \quad \varepsilon^{(1)} \neq 1.
\]
Then \( \varepsilon^{(n)} \) is a primitive \( p^n \)-th root of 1 for all \( n \). Set
\[
\bar{\pi} = \varepsilon - 1 \in \tilde{E}^+.
\]
We know that \( v_E(\bar{\pi}) = \frac{p^n}{p-1} > 0 \).

From now on, \( \chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^* \) will be the cyclotomic character. The action of \( G_{\mathbb{Q}_p} \) on \( \varepsilon \) is given by
\[
g(\varepsilon) = \varepsilon \chi(g) = \sum_{k=0}^{+\infty} \left( \chi(g) \right)_{\bar{\pi}}^k.
\]

(4) In the following, without further specification, \( K \subseteq \mathbb{Q}_p \) will be a finite extension of \( \mathbb{Q}_p \). Denote by \( k = k_K \) its residue field. Set
\[
K_n = K(\varepsilon^{(n)}), \quad K_{\infty} = \bigcup_{n \in \mathbb{N}} K_n.
\]
Set
\[
F \subseteq K = \text{the maximal unramified extension of } \mathbb{Q}_p \text{ inside } K,
\]
\[
F' \subseteq K_{\infty} = \text{the maximal unramified extension of } \mathbb{Q}_p \text{ inside } K_{\infty}.
\]
Set
\[
G_K = \text{Gal}(\mathbb{Q}_p/K), \quad H_K = \text{Ker } \chi = \text{Gal}(\mathbb{Q}_p/K_{\infty}),
\]
and
\[
\Gamma_K = G_K/H_K = \text{Gal}(K_{\infty}/K) \xrightarrow{\chi} \mathbb{Z}_p^*.
\]

(5) For every \( K \), let
\[
\tilde{E}_K^+ := \{x = (x_n) \in \tilde{E}^+, x_n \in \mathcal{O}_{K_{\infty}}/a, \forall n\} = (\tilde{E}^+)_{\mathcal{H}_K} \text{ (by Ax-Sen-Tate’s Theorem)},
\]
\[
E_K^+ := \{x = (x_n) \in \tilde{E}^+, x_n \in \mathcal{O}_{K_n}/a, \forall n \geq n(K)\}.
\]
4.1. SOME OF FONTAINE’S RINGS

Then
\[ \bar{\pi} \in E^+_K \subseteq \tilde{E}^+_K \subseteq \tilde{E}^+, \quad \forall K. \]

We set
\[ E_K := E^+_K[\bar{\pi}^{-1}] \subseteq \tilde{E}_K := \tilde{E}^+_K[\bar{\pi}^{-1}] \subseteq \tilde{E} = \tilde{E}^+[\bar{\pi}^{-1}] = \text{Fr } R \]
with valuation
\[ v_E(\bar{\pi}^{-k}x) = v_E(x) - kv_E(\bar{\pi}). \]

The following Theorem is the topics in the last section of Chapter 2 of Fontaine’s Notes.

**Theorem 4.1.1.** (i) \( \tilde{E} \) is a field complete for \( v_E \) with residue field \( \bar{F}_p \), ring of integers \( \tilde{E}^+ \) and \( G_{\bar{Q}_p} \) acts continuously with respect to \( v_E \).

(ii) \( E_F = k_F((\bar{\pi})) \) if \( F/\bar{Q}_p \) is unramified.

In general, \( E_K \) is a totally ramified extension of \( E_{F'} \) of degree \([K_\infty : F_{\infty}']\), thus a local field of characteristic \( p \), with ring of integers \( E^+_K \) and residue field \( k_{F'} \).

(iii) \( E = \bigcup_{[K : \bar{Q}_p] < +\infty} E_K \) is a separable closure of \( E_{\bar{Q}_p} \), is stable under \( G_{\bar{Q}_p} \) and \( \text{Gal}(E/E_K) = H_K \). So \( H_{\bar{Q}_p} \) acts continuously on \( E \) for the discrete topology.

(iv) \( \tilde{E} \) (resp. \( \tilde{E}_K \)) is the completion of the radical closure of \( E \) (resp. \( \tilde{E}_K \)), i.e., \( \bigcup_{n \in \mathbb{N}} E^{p-n} \) (resp. \( \bigcup_{n \in \mathbb{N}} E^{p-n}_K \)). In particular, \( \tilde{E} \) is algebraically closed.

### 4.1.2 Rings of characteristic 0

(6) Set
\[ \tilde{A}^+ := W(\tilde{E}^+) = W(R), \quad \tilde{A} := W(\tilde{E}) = W(\text{Fr } R). \]

Every element \( x \in \tilde{A} \) can be written as
\[ x = \sum_{k=0}^{+\infty} p^k[x_k] \]
while \( x_k \in \tilde{E} \) and \([x_k]\) is its Teichmüller representative.

As we know from the construction of Witt rings, there are bijections
\[ \tilde{A}^+ \cong (\tilde{E}^+)^\mathbb{N}, \quad \tilde{A} \cong (\tilde{E})^\mathbb{N}. \]
There are two topologies in $\tilde{A}^+$ and $\tilde{A}$:

(i) **Strong topology or $p$-adic topology:** topology by using the above bijection and the discrete topology on $\tilde{E}^+$ or $\tilde{E}$. A basis of neighborhoods of 0 are the $p^k \tilde{A}, \, k \in \mathbb{N}$.

(ii) **Weak topology:** topology defined by $v_E$. A basis of neighborhoods of 0 are the $p^k \tilde{A}^+ + [\tilde{\pi}^n] \tilde{A}^+, \, k, n \in \mathbb{N}$.

The commuting actions of $G_{\mathbb{Q}_p}$ and $\varphi$ on $\tilde{A}$ are given by

$$
g(\sum_{k=0}^{+\infty} p^k [x_k]) = \sum_{k=0}^{+\infty} p^k [g(x_k)], \quad \varphi(\sum_{k=0}^{+\infty} p^k [x_k]) = \sum_{k=0}^{+\infty} p^k [x_k^p].$$

(7) $\tilde{B} := \tilde{A}^{[\frac{1}{p}]}$ is the fraction field of $\tilde{A}$. $\tilde{B}$ is complete for the valuation $v_p$, its ring of integers is $\tilde{A}$ and its residue field is $\tilde{E}$.

For the $G_{\mathbb{Q}_p}$ and $\varphi$-actions,

$$\tilde{A}^{\varphi=1} = \mathbb{Z}_p, \quad \tilde{B}^{\varphi=1} = \mathbb{Q}_p,$$

$$\tilde{A}^{H_K} = W(\tilde{E}_K) := \tilde{A}_K, \quad \tilde{B}^{H_K} = \tilde{A}_K^{[\frac{1}{p}]} := \tilde{B}_K.$$

(8) Set

$$\pi = [\varepsilon] - 1, \quad t = \log[\varepsilon] = \log(1 + \pi).$$

The element $[\varepsilon]$ is the $p$-adic analogue of $e^{2\pi i}$. The $G_{\mathbb{Q}_p}$- and $\varphi$-actions are given by

$$\varphi(\pi + 1) = (\pi + 1)^p, \quad \varphi(\pi + 1) = (\pi + 1)^{\chi(g)}.$$

(9) Set

$$A_{\mathbb{Q}_p}^+ := \mathbb{Z}_p[[\pi]] \hookrightarrow \tilde{A}^+$$

which is stable under $\varphi$ and $G_{\mathbb{Q}_p}$. Set

$$A_{\mathbb{Q}_p} := \mathbb{Z}_p[[\pi]]^{[\frac{1}{p}]} \hookrightarrow \tilde{A}$$

while $\tilde{}$ stands for completion under the strong topology, thus

$$A_{\mathbb{Q}_p} = \{ \sum_{k \in \mathbb{Z}} a_k \pi^{-k} \mid a_k \in \mathbb{Z}_p, \lim_{k \to -\infty} v_p(a_k) = +\infty \}.$$

Set $B_{\mathbb{Q}_p} := A_{\mathbb{Q}_p}^{[\frac{1}{p}]}$, then $B_{\mathbb{Q}_p}$ is a field complete for the valuation $v_p$, with ring of integers $A_{\mathbb{Q}_p}$ and residue field $E_{\mathbb{Q}_p}$. 
Moreover, if $[K : \mathbb{Q}_p] < +\infty$, $\tilde{\mathcal{E}}$ contains a unique extension $B_K$ of $B_{\mathbb{Q}_p}$ whose residue field is $E_K$, and $A_K = B_K \cap \tilde{A}$ is the ring of integers. By uniqueness, $B_K$ is stable under $\varphi$ and $G_K$ acting through $\Gamma_K$.

The field $$\mathcal{E}_{ur} = \bigcup_{[K : \mathbb{Q}_p] < +\infty} B_K$$ is the maximal unramified extension of $B_{\mathbb{Q}_p} = \mathcal{E}$. Set $$B = \mathcal{E}_{ur}$$ be the closure of $\bigcup_{[K : \mathbb{Q}_p] < +\infty} B_K$ in $\tilde{B}$ for the strong topology. Then $A = B \cap \tilde{A}$ is the ring of integers $\mathcal{O}_{\mathcal{E}_{ur}}$ and the residue field of $B$ is $A/pA = E$. By Ax-Sen-Tate, $$B^{H_K} = B_K, \quad A^{H_K} = A_K.$$

**Remark.** If $\pi_K$ is a uniformising parameter of $E_K$, let $\pi_K \in A_K$ be any lifting. Then $$A_K = \{ \sum_{k \in \mathbb{Z}} a_k \pi_K^k | a_k \in \mathcal{O}_{F'}, \lim_{k \to -\infty} v_p(a_k) = +\infty \}.$$

**Remark.** In the above construction, the correspondence $\Lambda \rightarrow \tilde{\Lambda}$ is obtained by making $\varphi$ bijective and then complete, where $\Lambda = (E_K, E, A_K, A, B_K, B)$.

### 4.2 $(\varphi, \Gamma)$-modules and Galois representations.

Let $K$ be a fixed finite extension over $\mathbb{Q}_p$, let $\Gamma = \Gamma_K$.

**Definition 4.2.1.** (i) A $(\varphi, \Gamma)$-module over $A_K$ is a finitely generated $A_K$-module with semi-linear continuous (for the weak topology) and commuting actions of $\varphi$ and $\Gamma$.

A $(\varphi, \Gamma)$-module over $B_K$ is a finite dimensional $B_K$-vector space with semi-linear continuous (for the weak topology) and commuting actions of $\varphi$ and $\Gamma$.

(ii) A $(\varphi, \Gamma)$-module $D/A_K$ is étale (or of slope 0) if $\varphi(D)$ generates $D$ as an $A_K$-module.

A $(\varphi, \Gamma)$-module $D/B_K$ is étale (or of slope 0) if it has an $A_K$-lattice which is étale, equivalently, there exists a basis $\{e_1, \ldots, e_d\}$ over $B_K$, such that the matrix of $\varphi(e_1), \ldots, \varphi(e_d)$ in $e_1, \ldots, e_d$ is inside $\text{GL}_d(A_K)$. 

The following theorem is similar to Theorem 1.5.9 in §1.5.4 of Fontaine’s Notes.

**Theorem 4.2.2.** The correspondence

\[ V \mapsto D(V) := (A \otimes_{\mathbb{Z}_p} V)^{H_K} \]

is an equivalence of \( \otimes \) categories from the category of \( \mathbb{Z}_p \)-representations (resp. \( \mathbb{Q}_p \)-resp) of \( G_K \) to the category of étale \( (\varphi, \Gamma) \)-modules over \( A_K \) (resp. \( B_K \)), and the Inverse functor is

\[ D \mapsto V(D) = (A \otimes_{A_K} D)^{\varphi=1}. \]

**Remark.** (i) \( \Gamma_K \) is essentially pro-cyclic, so a \((\varphi, \Gamma)\)-module is given by two operators and commuting relations between them. For example, if \( D/A_K \) is free of rank \( d \), let \( U \) be the matrix of \( \gamma \) for \( \langle \gamma \rangle = \Gamma_K \), let \( P \) be the matrix of \( \varphi \), then

\[ U^\gamma(P) = P^\varphi(U), \quad U, \ P \in \text{GL}_d(A_K). \]

(ii) We want to recover from \( D(V) \) the known invariants of \( V \):

- \( H^i(G_K, V) \); we shall do so in the coming lectures. We will also recover the Iwasawa modules attached to \( V \) and thus give another construction of \( p \)-adic \( L \)-functions.

- \( D_{dR}(V), D_{cris}(V), D_{st}(V) \).
Chapter 5

$(\varphi, \Gamma)$-modules and Galois cohomology

5.1 Galois Cohomology

Let $M$ be a topological $\mathbb{Z}_p$-module (e.g. a finite module with discrete topology or a finitely generated $\mathbb{Z}_p$-module with $p$-adic topology, or a Fontaine’s ring $B_{dR}^+$), with a continuous action of $G_K$.

Let $H^i(G_K, M)$ be the $i$-th cohomology groups of $M$ of continuous cohomology. Then:

- $H^0(G_K, M) = M^{G_K} = \{x \in M : (g-1)x = 0 \ \forall g \in G_K\}$;
- $H^1(G_K, M) = \{c : G_K \to M \text{ continuous, } g_1c_{g_2} - g_{g_1g_2} + c_{g_1} = 0, \ \forall g_1, g_2 \in G_K\}$

To a 1-cocycle $c$, we associate a $G_K$ module $E_c$ such that

$$0 \to M \to E_c \to N \to 0$$

where $E_c \simeq \mathbb{Z}_p \times M$ as a $\mathbb{Z}_p$-module and $G_K$ acts on $E_c$ by

$$g(a, m) = (a, gm + c_g).$$

One can check easily

$$g_1(g_2(a, m)) = g_1(a, g_2m + c_{g_2}) = (a, g_1g_2m + g_1c_{g_2} + c_{g_1}) = g_1g_2(a, m).$$

$E_c$ is trivial if and only if there exists $\hat{1} \in E_c$, such that $g\hat{1} = \hat{1}$ for all $g$, i.e. $\hat{1} = (1, x), g\hat{1} = (0, gx - x + c_g) = 0$, that is, $c_g = (1 - g)x$ is a coboundary.
Chapter 5. \((\varphi, \Gamma)\)-modules and Galois cohomology

Theorem 5.1.1 (Tate’s Local Duality Theorem). Suppose \(K\) is a finite extension of \(\mathbb{Q}_p\). Let \(M\) be a \(\mathbb{Z}_p[\mathbb{G}_K]\)-module of finite length. Then:

(i) \(H^i(\mathbb{G}_K, M) = 0\) for \(i \geq 3\); \(H^i(\mathbb{G}_K, M)\) is finite if \(i \leq 2\).

(ii) \(\prod_{i=0}^{2} |H^i(\mathbb{G}_K, M)|(-1)^i = |M|^{-[K: \mathbb{Q}_p]}\).

(iii) \(H^{2-i}(\mathbb{G}_K, \text{Hom}(M, \mu_{p^\infty})) \simeq \text{Hom}(H^i(\mathbb{G}_K, M), \mathbb{Q}_p/\mathbb{Z}_p)\).

We will give a proof using \((\varphi, \Gamma)\)-module (Herr’s thesis).

Remark. (i) If \(M\) is a finitely generated \(\mathbb{Z}_p\)-module with \(p\)-adic topology, then \(M \simeq \varprojlim M/p^n M\), and \(H^i(\mathbb{G}_K, M) \simeq \varprojlim H^i(\mathbb{G}_K, M/p^n M)\).

Not tautological, the proof uses finiteness of (i) to ensure Mittag-Leffler conditions.

(ii) If \(V\) is a \(\mathbb{Q}_p\)-representation of \(\mathbb{G}_K\), let \(T \subset V\) be a \(\mathbb{Z}_p\)-lattice stable by \(\mathbb{G}_K\). Then \(H^i(\mathbb{G}_K, V) \simeq \mathbb{Q}_p \otimes H^i(\mathbb{G}_K, T)\).

Corollary 5.1.2. If \(V\) is a \(\mathbb{Q}_p\)-representation of \(\mathbb{G}_K\). Then:

(i) \(\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Q}_p} H^i(\mathbb{G}_K, V) = -[K : \mathbb{Q}_p] \dim_{\mathbb{Q}_p} V\);

(ii) \(H^2(\mathbb{G}_K, V) = H^0(\mathbb{G}_K, V^*(1))^*\).

5.2 The complex \(C_{\varphi, \gamma}(K, V)\)

Assume that \(\Gamma_K\) is pro-cyclic (\(\Gamma_{\mathbb{Q}_p} \simeq \mathbb{Z}_p^*\)), \(\gamma\) is a topological generator of \(\Gamma_K\). This assumption is automatic if \(p \geq 3\), or if \(K \supset \mathbb{Q}(\mu_4)\) when \(p = 2\). Let \(V\) be a \(\mathbb{Z}_p\)- or \(\mathbb{Q}_p\)-representation of \(\mathbb{G}_K\). Set

\[D(V) = (A \otimes_{\mathbb{Z}_p} V)^{H_K}\.\]

Definition 5.2.1. The complex \(C_{\varphi, \gamma}^*(K, V) = C_{\varphi, \gamma}(K, V)\) is

\[0 \to D(V) \xrightarrow{\varphi - 1, \gamma - 1} D(V) \oplus D(V) \xrightarrow{\gamma - 1, \varphi - 1} D(V) \to 0\.

It is easy to see \(C_{\varphi, \gamma}(K, V)\) is really a complex (as \(\varphi, \gamma\) commute to each other). We shall denote the complex by \(C^*(V)\) if no confusion is caused.
5.2. THE COMPLEX $C_{\varphi, \gamma}(K, V)$

have

$$H^0(C^\bullet(V)) = \{x \in D(V), \gamma(x) = x, \varphi(x) = x\},$$

$$H^1(C^\bullet(V)) = \{(x, y) : (\gamma - 1)x = (\varphi - 1)y\},$$

$$H^2(C^\bullet(V)) = \frac{D(V)}{(\gamma - 1, \varphi - 1)},$$

$$H^i(C^\bullet(V)) = 0, \text{ for } i \geq 3.$$

**Theorem 5.2.2.** $H^i(C_{\varphi, \gamma}(K, V)) \simeq H^i(G_K, V)$ for all $i$ in $\mathbb{N}$.

*Proof.* We have the following exact sequence (which can be proved by reducing mod $p$):

$$0 \rightarrow \mathbb{Z}_p \rightarrow A \xrightarrow{\varphi - 1} A \rightarrow 0,$$

here $A = \mathcal{O}_{\mathbb{Z}_p}$ in Fontaine’s course.

1. $i = 0$: For $x \in D(V)^{\varphi - 1}$, since $D(V) = (A \otimes_{\mathbb{Z}_p} V)^{H_K}$, we have

$$D(V)^{\varphi - 1} = (A^{\varphi - 1} \otimes_{\mathbb{Z}_p} V)^{H_K} = V^{H_K}, \text{ and } (V^{H_K})^{\gamma - 1} = V^{G_K}.$$

2. $i = 1$: Let $(x, y)$ satisfy the condition $(\gamma - 1)x = (\varphi - 1)y$. Choose $b \in (A \otimes_{\mathbb{Z}_p} V)^{H_K}, (\varphi - 1)b = x$. We define the map:

$$g \in G_K \rightarrow c_{x, y}(g) = \frac{g - 1}{\gamma - 1} y - (g - 1)b.$$

while the meaning of $\frac{g - 1}{\gamma - 1} y$ is: as $\chi(g) = \lim_{i \rightarrow +\infty} \chi(\gamma)^i$, $y$ is fixed by $H_K$, we let

$$\frac{g - 1}{\gamma - 1} y = \lim_{i \rightarrow +\infty} (1 + \gamma + \cdots + \gamma^{i - 1})y.$$

This is a cocycle with values in $V$, because $g \mapsto (g - 1)(\frac{y}{\gamma - 1} - b)$ is a cocycle, and $(\varphi - 1)c_{x, y}(g) = (g - 1)x - (\varphi - 1)(g - 1)b = 0$, which implies that $c_{x, y}(g) \in D(V)^{\varphi - 1} = V$.

Injectivity: If $c_{x, y} = 0$ in $H^1(G_K, V)$, then there exists $z \in V$, $c_{x, y}(g) = (g - 1)z$ for all $g \in G_K$, that is, $\frac{g - 1}{\gamma - 1} y = (g - 1)(b - z)$ for all $g$. Now $b - z \in D(V)$, because it is fixed by $g \in H_K$. Then we have: $y = (\gamma - 1)(b - z)$ and $x = (\varphi - 1)(b - z)$, hence $(x, y)$ equal to 0 in $H^1(C^\bullet(V))$.

Surjectivity: If $c \in H^1(G_K, V)$, we have:

$$0 \rightarrow V \rightarrow E_c \rightarrow \mathbb{Z}_p \rightarrow 0,$$
here \( E_c = \mathbb{Z}_p \times V, \ e \in E_c \mapsto 1 \in \mathbb{Z}_p \) and \( g \epsilon = e + c_g \) for \( g \mapsto c_g \) representing \( c \). We have:
\[
0 \to D(V) \longrightarrow D(E_c) \longrightarrow A_K \to 0,
\]
here \( D(E_c) \subset A \otimes E_c \) and \( \tilde{\epsilon} \in D(E_c) \mapsto 1 \in A_K \).

Let
\[
x = (\varphi - 1)\tilde{\epsilon}, \quad y = (\gamma - 1)\tilde{\epsilon},
\]
they are both in \( D(V) \) and satisfy \((\gamma - 1)x = (\varphi - 1)y\).

Let \( b = \tilde{\epsilon} - e \in A \otimes_{\mathbb{Z}_p} E_c \). Then \( c_{x,y}(g) = \frac{g-1}{\gamma-1}y - (g-1)b = c_g \) and \((\varphi - 1)(b) = x\).

(3) i general: from the exact sequence:
\[
0 \to Z_p \to A \varphi - 1 \longrightarrow A \longrightarrow 0,
\]
tensoring with \( V \) and taking the cohomology \( H^i(H_K, -) \), we get
\[
0 \to V^{H_K} \to D(V) \varphi - 1 \longrightarrow D(V) \longrightarrow H^1(H_K, V) \to 0,
\]
because \( A \otimes V \simeq \oplus(A/p^i) \) as \( H_K \)-modules and \( H^i(H_K, E) = 0 \), if \( i \geq 1 \), so \( H^i(H_K, A \otimes V) = 0 \) for all \( i \geq 1 \). Hence \( H^i(H_K, V) = 0 \) for all \( i \geq 1 \).

By the Hochschild-Serre Spectral Sequence for
\[
1 \to H_K \to G_K \to \Gamma_K \to 1,
\]
we have \( H^i(\Gamma_K, H^j(H_K, V)) \Rightarrow H^{i+j}(G_K, V) \). When \( j \) or \( i \geq 2 \), the cohomology vanishes. So we have:
\[
\begin{align*}
H^q(G_K, V) &= 0, \text{ if } q \geq 3 \\
H^2(G_K, V) &\simeq H^1(\Gamma_K, H^1(H_K, V)).
\end{align*}
\]
Since \( H^1(H_K, V) = \frac{D(V)}{\varphi - 1} \), we get
\[
H^2(G_K, V) \simeq \frac{D(V)}{\varphi - 1} \left/ \left( \gamma - 1 \right) \frac{D(V)}{\varphi - 1} \right. = \frac{D(V)}{(\varphi - 1, \gamma - 1)}.
\]

\[\square\]

\textbf{Remark.} (1) The inflation-restriction exact sequence becomes the commutative diagram
\[
\begin{array}{cccccc}
0 & \to & H^1(\Gamma_K, V^{H_K}) & \longrightarrow & H^1(G_K, V) & \longrightarrow & H^1(H_K, V^{\Gamma_K}) & \longrightarrow & 0 \\
\end{array}
\]
\[
\begin{array}{cccccc}
0 & \to & \frac{D(V)^{\varphi - 1}}{\gamma - 1} & \longrightarrow & H^1(C^{\varphi, \gamma}(K, V)) & \longrightarrow & (\frac{D(V)}{\varphi - 1})^{\Gamma_K} & \longrightarrow & 0
\end{array}
\]

where the map $H^1(C_{\varphi,\gamma}(K,V)) \to (\frac{D(V)}{\varphi-1})^{\Gamma_K}$ is given by sending $(x,y)$ to the image of $x$.

(2) Let $\gamma'$ be another generator of $\Gamma_K$, we have $\frac{\gamma - 1}{\gamma'} \in (\mathbb{Z}_p[[\Gamma_K]])^*$ and a commutative diagram:

$$
\begin{array}{cccccccc}
C_{\varphi,\gamma} : & 0 & \to & D(V) & \to & D(V) \oplus D(V) & \to & 0 \\
\downarrow & & & \downarrow & & \downarrow \text{Id} & & \\
C_{\varphi,\gamma'} : & 0 & \to & D(V) & \to & D(V) \oplus D(V) & \to & 0 \\
\end{array}
$$

It induces a commutative diagram

$$
\begin{array}{ccc}
H^1(C_{\varphi,\gamma}) & \xrightarrow{l_{K,\gamma'}} & H^1(C_{\varphi,\gamma'}) \\
\downarrow l_{K,\gamma} & & \downarrow l_{K,\gamma'} \\
H^1(G_K, V) & & H^1(G_K, V)
\end{array}
$$

where $l_{K,\gamma} = \frac{\log(\chi(\gamma))}{p^r(K)}$ for $\log \chi(\Gamma_K) \simeq p^r(K)\mathbb{Z}_p$. So $l_{K,\gamma}c_{\varphi,\gamma}$ "does not depend on the choice of $\gamma'". 

5.3 Tate’s Euler-Poincaré formula.

5.3.1 The operator $\psi$.

Lemma 5.3.1. (i) $\{1, \varepsilon, \ldots, \varepsilon^{p-1}\}$ is a basis of $E_{Q_p}$ over $\varphi(E_{Q_p})$;
(ii) $\{1, \varepsilon, \ldots, \varepsilon^{p-1}\}$ is a basis of $E_K$ over $\varphi(E_K)$, for all $[K : Q_p] < +\infty$;
(iii) $\{1, \varepsilon, \ldots, \varepsilon^{p-1}\}$ is a basis of $E$ over $\varphi(E)$;
(iv) $\{1, [\varepsilon], \ldots, [\varepsilon]^{p-1}\}$ is a basis of $A$ over $\varphi(A)$.

Proof. (i) Since $E_{Q_p} = \mathbb{F}_p((\bar{\varepsilon}))$ with $\bar{\varepsilon} = \varepsilon - 1$, we have $\varphi(E_{Q_p}) = \mathbb{F}_p((\bar{\varepsilon}^p))$;
(ii) Use the following diagram of fields, note that $E_{Q_p}/\varphi(E_{Q_p})$ is purely inseparable and $\varphi(E_K)/\varphi(E_{Q_p})$ is separable:

$$
\begin{array}{ccc}
E_K & \to & \varphi(E_K) \\
\downarrow & & \downarrow \\
E_{Q_p} & \to & \varphi(E_{Q_p})
\end{array}
$$

(iii) Because $E = \cup E_K$. 

(iv) To show that

\[(x_0, x_1, \cdots, x_{p-1}) \in A^p \mapsto \sum_{i=0}^{p-1} [\varepsilon]^i \varphi(x_i) \in A\]

is a bijection, it suffices to check it mod \(p\) and use (iii).

**Definition 5.3.2.** The operator \(\psi : A \rightarrow A\) is defined by

\[\psi(\sum_{i=0}^{p-1} [\varepsilon]^i \varphi(x_i)) = x_0.\]

**Proposition 5.3.3.** (i) \(\psi \varphi = \text{Id};\)

(ii) \(\psi\) commutes with \(G_{Q_p}\).

**Proof.** (i) The first statement is obvious.

(ii) Note that

\[g(\sum_{i=0}^{p-1} [\varepsilon]^i \varphi(x_i)) = \sum_{i=0}^{p-1} [\varepsilon]^{i\chi(g)} \varphi(g(x_i)).\]

If for \(1 \leq i \leq p-1\), write \(i \chi(g) = i_g + pj_g\) with \(1 \leq i_g \leq p - 1\), then

\[\psi(\sum_{i=0}^{p-1} [\varepsilon]^{i\chi(g)} \varphi(g(x_i)))) = \psi(\varphi(g(x_0)) + \sum_{i=1}^{p-1} [\varepsilon]^{i_g} \varphi([\varepsilon]^{j_g} g(x_i))) = g(x_0).\]

**Corollary 5.3.4.** (i) If \(V\) is a \(\mathbb{Z}_p\)-representation of \(G_K\), there exists a unique operator \(\psi : D(V) \rightarrow D(V)\) with

\[\psi(\varphi(a)x) = a\psi(x), \quad \psi(a\varphi(x)) = \psi(a)x\]

if \(a \in A_K, x \in D(V)\) and moreover \(\psi\) commute with \(\Gamma_K\).

(ii) If \(D\) is an étale \((\varphi, \Gamma)\)-module over \(A_K\) or \(B_K\), there exists a unique operator \(\psi : D \rightarrow D\) with as in (i). Moreover, for any \(x \in D\),

\[x = \sum_{i=0}^{p^n-1} [\varepsilon]^i \varphi^n(x_i)\]

where \(x_i = \psi^n([\varepsilon]^{-i} x)\).
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Proof. (i) The uniqueness follows from $A_K \otimes_{\varphi(A_K)} \varphi(D) = D$. For the existence, use $\psi$ on $A \otimes V \supset D(V)$. $D(V)$ is stable under $\psi$ because $\psi$ commutes with $H_K$, $\psi$ commutes with $\Gamma_K$ since $\psi$ commutes with $G_K$.

(ii) $D = D(V(D))$, thus we have existence and uniqueness of $\psi$. The rest is by induction on $n$.

Example 5.3.5. Let $D = A_{\mathbb{Q}_p} \supset A_{\mathbb{Q}_p}^+ = \mathbb{Z}_p[[\pi]]$ be the trivial $(\varphi, \Gamma)$-module, here $[\varepsilon] = (1 + \pi)$. Then for $x = F(\pi) \in A_{\mathbb{Q}_p}^+$, $\varphi(x) = F((1 + \pi)^p - 1)$. Write

$$F(\pi) = \sum_{i=0}^{p-1} (1 + \pi)^i F_i((1 + \pi)^p - 1),$$

then $\psi(F(\pi)) = F_0(\pi)$. It is easy to see if $F(\pi)$ belongs to $\mathbb{Z}_p[[\pi]]$, $F_i(\pi)$ belongs to $\mathbb{Z}_p[[\pi]]$ for all $i$. Then $\psi(E_{\mathbb{Q}_p}^+) \subset E_{\mathbb{Q}_p}^+ = F_p[[\pi]]$. Hence $\psi(A_{\mathbb{Q}_p}^+) \subset A_{\mathbb{Q}_p}^+$. Consequently, $\psi$ is continuous for the weak topology.

Moreover, we have:

$$\varphi(\psi(F)) = F_0((1 + \pi)^p - 1) = \frac{1}{p} \sum_{z} \sum_{i=0}^{p-1} (z(1 + \pi))^i F_i((z(1 + \pi))^{p} - 1)$$

$$= \frac{1}{p} \sum_{z} F(z(1 + \pi) - 1).$$

Recall $D_0(\mathbb{Z}_p, \mathbb{Q}_p) \simeq B_{\mathbb{Q}_p}^+ = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A_{\mathbb{Q}_p}^+$ by $\mu \mapsto A_{\mu}(\pi) = \int_{\mathbb{Z}_p} [\varepsilon]^x \mu$. Recall that $\psi(\mu)$ is defined by

$$\int_{\mathbb{Z}_p} \phi(x) \psi(\mu) = \int_{\mathbb{Z}_p} \phi\left(\frac{x}{p}\right) \mu.$$

From the above formula, we get, using formulas for Amice transforms,

$$A_{\psi(\mu)}(\pi) = \psi(A_{\mu})(\pi).$$

Proposition 5.3.6. If $D$ is an étale $\varphi$-module over $A_K$, then $\psi$ is continuous for the weak topology.

Proof. As $A_K$ is a free $A_{\mathbb{Q}_p}$-module of rank $[K_\infty : \mathbb{Q}_p(\mu_{p^\infty})]$, we can assume $K = \mathbb{Q}_p$. Choose $e_1, e_2, \cdots, e_d$ in $D$, such that

$$D = \oplus (A_{\mathbb{Q}_p}/p^{n_i}) e_i, \quad n_i \in \mathbb{N} \cup \{\infty\}.$$
Since $D$ is étale, we have $D = \bigoplus (A_{\mathbb{Q}_p}/p^n) \varphi(e_i)$. Then we have the following diagram:

\[
\begin{array}{ccc}
D & \xrightarrow{\psi} & D \\
\downarrow \iota & & \downarrow \iota \\
\bigoplus (A_{\mathbb{Q}_p}/p^n) \varphi(e_i) & \longrightarrow & \bigoplus (A_{\mathbb{Q}_p}/p^n)e_i \\
\end{array}
\]

Now $x \mapsto \psi(x)$ is continuous in $A_{\mathbb{Q}_p}$, hence $\psi$ is continuous in $D$.

### 5.3.2 $D^{\psi=1}$ and $D/(\psi - 1)$

**Lemma 5.3.7.** If $D$ is an étale $\varphi$-module over $E_{\mathbb{Q}_p}$, then:

(i) $D^{\psi=1}$ is compact;

(ii) $\dim_{E}(D/(\psi - 1)) < +\infty$.

**Proof.** (i) choose a basis $\{e_1, \ldots, e_d\}$, then $\{\varphi(e_1), \ldots, \varphi(e_d)\}$ is still a basis. Set $v_E(x) = \inf_i v_E(x_i)$ if $x = \sum_i x_i \varphi(e_i), x_i \in E_{\mathbb{Q}_p}^+$. We have

$$
\psi(x) = \sum_i \psi(x_i)e_i \text{ and } e_i = \sum_{i=1}^d a_{i,j} \varphi(e_j).
$$

Let $c = \inf_{i,j} v_E(a_{i,j})$, then we have

$$
v_E(\psi(x)) \geq c + \inf_i v_E(\psi(x_i)). \tag{5.1}
$$

From $\psi(E_{\mathbb{Q}_p}^+) \subset E_{\mathbb{Q}_p}^+$ and $\psi(\bar{\pi}^k x) = \bar{\pi}^k \psi(x)$, we get $v_E(\psi(x)) \geq \left\lfloor \frac{v_E(x)}{p} \right\rfloor$. So

$$
v_E(\psi(x)) \geq c + \inf_i \left\lfloor \frac{v_E(x_i)}{p} \right\rfloor \geq c + \left\lfloor \frac{v_E(x)}{p} \right\rfloor.
$$

If $v_E(x) < \frac{p(c-1)}{p-1}$, then $v_E(\psi(x)) > v_E(x)$. Now $D^{\psi=1}$ is closed since $\psi$ is continuous, and is a subset of the compact set

$$
M := \{x : v_E(x) \geq \frac{p(c-1)}{p-1}\} \subseteq \sum_{i=1}^d \bar{\pi}^k \mathbb{F}_p[[\bar{\pi}]] \cdot \varphi(e_i).
$$
Hence \( D^{\psi=1} \) is also compact.

(ii) \( \psi - 1 \) is bijective on \( D/M \) from the proof of (i). We only need to prove that \( M/((\psi - 1)D \cap M) \) is finite, equivalently, that \( (\psi - 1)D \) contains \( \{ x : v_E(x) \geq c' \} \) for some \( c' \).

\( \varphi(x_i) \) can be written uniquely as \( \varphi(x_i) = \sum_{j=1}^{d} b_{i,j} e_j \). Let \( c_0 = \inf_{i,j} v_E(b_{i,j}) \), then

\[
x = \sum_{i=1}^{d} x_i \varphi(e_i) = \sum_{i=1}^{d} x_i \sum_{j=1}^{d} b_{i,j} e_j = \sum_{j=1}^{d} (\sum_{i=1}^{d} x_i b_{i,j}) e_j.
\]

Let \( y_j = \sum_{i=1}^{d} x_i b_{i,j} \), then \( x = \sum_{j=1}^{d} y_j e_j \), and

\[
v_E(y_j) \geq c_0 + v_E(x).
\]

From \( \varphi(x) = \sum_{j=1}^{d} \varphi(y_j) \varphi(e_j) \), we get

\[
v_E(\varphi(x)) = \inf v_E(\varphi(y_j)) = p \inf v_E(y_j) \geq pv_E(x) + pc_0.
\]

So, if \( v_E(x) \geq \frac{pc_0}{p-1} + 1 \), then \( v_E(\varphi^n(x)) \geq p^n \). It implies \( y = \sum_{i=0}^{+\infty} \varphi^i(x) \) converges in \( D \). Now

\[
(\psi - 1)y = \sum_{i=0}^{+\infty} \varphi^i(x) - \sum_{i=1}^{+\infty} \varphi^i(x) = x
\]

implies that \( (\psi - 1)D \) contains \( \{ x|v_E(x) \geq \frac{pc_0}{p-1} + 1 \} \). \( \square \)

**Proposition 5.3.8.** If \( D \) is an étale \( \varphi \)-module over \( A_K \) (resp. over \( B_K \)), then:

(i) \( D^{\psi=1} \) is compact (resp. locally compact);

(ii) \( D/(\psi - 1) \) is finitely generated over \( \mathbb{Z}_p \) (resp. over \( \mathbb{Q}_p \)).

**Proof.** We can reduce to \( K = \mathbb{Q}_p \). \( B_K \) follows from \( A_K \) by \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} - \). So we consider \( D \) over \( A_{\mathbb{Q}_p} \).

(i) Note that \( D^{\psi=1} = \lim(D/p^nD)^{\psi=1} \). From the previous lemma we have \( D/p^nD \) is compact by easy induction on \( n \). So \( D^{\psi=1} \) is compact.

(ii) The quotient \( (D/(\psi - 1))/p \simeq (D/p)/\psi - 1 \) is finite dimensional over \( \mathbb{F}_p \). We have to check that
if \( x = (\psi - 1)y_n + p^n\mathbb{Z}_p \) for all \( n \), then \( x \in (\psi - 1)D \).

If \( m \geq n \), \( y_m - y_n \in (D/p^n)^{\psi=1} \), which is compact, we can extract a sequence converging mod \( p^n \). Thus we can diagonally extract a sequence converging mod \( p^n \) for all \( n \). Then \( y_n \) converges to \( y \) in \( D \) and \( x = (\psi - 1)y \).

\[ \square \]

### 5.3.3 The \( \Gamma \)-module \( D_{\psi=0} \).

If \( p \neq 2 \), we let \( \Gamma_0 = \Gamma_{\mathbb{Q}_p} \cong \mathbb{Z}_p^* \). Let \( \Gamma_n \subseteq \Gamma_0 \) and \( \Gamma_n \cong 1 + p^n\mathbb{Z}_p \) if \( n \geq 1 \). Then \( \Gamma_0 = \Delta \times \Gamma_1 \) where \( \Delta = \mu_{p-1} \), and \( \Gamma_n = \varprojlim_{\mathbb{Z}_p} \Gamma_n/\Gamma_{n+m} \). We define

\[ Z_p[[\Gamma_n]] = \varprojlim Z_p[\Gamma_n/\Gamma_{n+m}] = D(\Gamma_n, \mathbb{Z}_p). \]

If \( n \geq 1 \), let \( \gamma_n \) be a topological generator of \( \Gamma_n \). So \( \Gamma_n = \gamma_n\mathbb{Z}_p \). The correspondence

\[ Z_p[[\Gamma_n]] \xrightarrow{\sim} Z_p[[T]] \xrightarrow{\sim} A_{\mathbb{Q}_p}^+ \]

\[ \gamma_n - 1 \xleftarrow{\sim} T \xrightarrow{\sim} \pi \]

is just the Amice transform. Then

\[ Z_p[[\Gamma_0]] = Z_p[\Delta] \otimes Z_p[[\Gamma_1]]; \]
\[ Z_p\{\{\Gamma_n\}\} := (Z_p[[\Gamma_n]]/((\gamma_n - 1)^{-1}))^\wedge \cong A_{\mathbb{Q}_p} \text{ (as a ring)}, \]
\[ Z_p\{\{\Gamma_0\}\} = Z_p[\Delta] \otimes Z_p\{\{\Gamma_1\}\}. \]

Modulo \( p \), we get \( \mathbb{F}_p\{\{\Gamma_n\}\} \cong E_{\mathbb{Q}_p} \) as a ring.

**Remark.** \( Z_p[[\Gamma_0]] \cong D_0(\Gamma_0, \mathbb{Z}_p) \cong D_0(\mathbb{Z}_p^*, \mathbb{Z}_p) \cong (A_{\mathbb{Q}_p}^+)_{\psi=0} \psi = 0 \). So \( (A_{\mathbb{Q}_p}^+)_{\psi=0} \psi = 0 \) is a free \( Z_p[[\Gamma_0]] \)-module of rank 1. This a special case of a general theorem which will come up later on.

**Lemma 5.3.9.** (i) If \( M \) is a topological \( \mathbb{Z}_p \)-module \( (M = \varprojlim M/M_i) \) with a continuous action of \( \Gamma_n \) (i.e. for all \( i \), there exists \( k \), such that \( \Gamma_{n+k} \) acts trivially on \( M/M_i \)), then \( Z_p[[\Gamma_n]] \) acts continuously on \( M \);

(ii) If \( \gamma_n - 1 \) has a continuous inverse, then \( Z_p\{\{\Gamma_n\}\} \) also acts continuously on \( M \).
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**Lemma 5.3.10.** (i) If $n \geq 1$, $v_E(\gamma_n(\bar{\pi}) - \bar{\pi}) = p^n v_E(\bar{\pi})$;
(ii) For all $x$ in $E_{Q_p}$, we have $v_E(\gamma_n(x) - x) \geq v_E(x) + (p^n - 1)v_E(\bar{\pi})$.

**Proof.** Since $\chi(\gamma) = 1 + p^n u$, $u \in \mathbb{Z}_p^*$, we have

$$\gamma_n(\bar{\pi}) - \bar{\pi} = \gamma_n(1 + \bar{\pi}) - (1 + \bar{\pi}) = (1 + \bar{\pi})(1 + \bar{\pi})^u - 1 = (1 + \bar{\pi})((1 + \bar{\pi})^u - 1)^{p^n}.$$ Then we get (i).

In general, for $x = \sum_{k=k_0}^{+\infty} a_k \bar{\pi}^k$, then $v_E(x) = k_0 v_E(\bar{\pi})$. Now

$$\frac{\gamma_n(x) - x}{\gamma_n(\bar{\pi}) - \bar{\pi}} = \sum_{k=k_0}^{+\infty} a_k \frac{\gamma_n(\bar{\pi})^k - \bar{\pi}^k}{\gamma_n(\bar{\pi}) - \bar{\pi}},$$

and

$$v_E(\frac{\gamma_n(\bar{\pi})^k - \bar{\pi}^k}{\gamma_n(\bar{\pi}) - \bar{\pi}}) \geq (k - 1)v_E(\bar{\pi}).$$

**Proposition 5.3.11.** Let $D$ be an étale $(\varphi, \Gamma)$-module of dimension $d$ over $E_{Q_p}$. Assume $n \geq 1$, $(i, p) = 1$. Then
(i) $\gamma \in \Gamma$ induces $\varepsilon^i \varphi^n(D) \simeq \varepsilon^{\chi(\gamma)i} \varphi^n(D)$;
(ii) $\gamma - 1$ admits a continuous inverse on $\varepsilon^i \varphi^n(D)$. Moreover if $\{e_1, \cdots, e_d\}$ is a basis of $D$, then:

$$\mathbb{F}_p\{\{\Gamma_n\}\}^d \xrightarrow{\sim} \varphi^n(D)$$

$$(\lambda_1, \cdots, \lambda_d) \mapsto \lambda_1 * \varepsilon^i \varphi^n(e_1) + \cdots + \lambda_d * \varepsilon^i \varphi^n(e_d)$$

is a topological isomorphism.

**Proof.** (i) is obvious. Now, remark that (ii) is true for $n + 1$ implies (ii) is true for $n$, since

$$\varepsilon^i \varphi^n(D) = \varepsilon^i \varphi^n(\bigoplus_{j=0}^{p^n-1} \varepsilon^j \varphi(D)) = \bigoplus_{j=0}^{p^n-1} \varepsilon^{i + j} \varphi^{n+1}(D),$$

and for $n > 1$, $\gamma_{n+1} = \gamma_n^{p^n}$, so $\frac{1}{\gamma_{n+1}} = \frac{1}{\gamma_n^{p^n}}(1 + \gamma_n + \cdots + \gamma_n^{p^n-1})$, and

$$\mathbb{F}_p\{\{\Gamma_n\}\} = \mathbb{F}_p\{\{\Gamma_{n+1}\}\} + \cdots + \gamma_n^{p^n-1} \mathbb{F}_p\{\{\Gamma_{n+1}\}\}.$$
So we can assume \( n \) big enough.

Recall \( v_E(x) = \inf \{ v_E(x_i) \mid x = \sum_i x_i e_i \} \). We can, in particular, assume \( v_E(\gamma_n e_i - e_i) \geq 2v_E(\bar{\pi}) \), it implies \( v_E(\gamma_n(x) - x) \geq v_E(x) + 2v_E(\bar{\pi}) \) for all \( x \in D \) (as \( v_E(\gamma_n(x) - x) \geq v_E(x) + (p^n - 1)v_E(\bar{\pi}) \) for all \( x \in E_{\mathbb{Q}_p} \)). Now

\[
\chi(\gamma_n) = 1 + p^n u, \ u \in \mathbb{Z}_p^*,
\]
so

\[
\gamma_n(\varepsilon \varphi^n(x)) - \varepsilon \varphi^n(x) = \varepsilon(\varepsilon^{p^n u} \varphi^n(\gamma_n(x)) - \varphi^n(x)) = \varepsilon \varphi^n(\varepsilon^{iu} \gamma_n(x) - x).
\]

So we have to prove \( x \mapsto f(x) = \varepsilon^{iu} \gamma_n(x) - x \) has a continuous inverse on \( D \), and \( D \) is a \( \mathbb{F}_p \{ \{ f \} \} \)-module with basis \( \{ e_1, \cdots, e_d \} \). Let \( \alpha = \varepsilon^{iu} - 1; iu \in \mathbb{Z}_p^* \), so \( v_E(\alpha) = v_E(\bar{\pi}) \). Then \( v_E(\frac{f}{\alpha}(x) - x) \geq v_E(x) + v_E(\bar{\pi}) \). It implies \( \frac{f}{\alpha} \) has an inverse

\[
g = \sum_{n=0}^{+\infty} (1 - \frac{f}{\alpha})^n \text{ and } v_E(g(x) - x) \geq v_E(x) + v_E(\bar{\pi}).
\]

So \( f \) has an inverse \( f^{-1}(x) = g(\frac{x}{\alpha}) \) and \( v_E(f^{-1}(x) - \frac{x}{\alpha}) \geq v_E(x) \).

By induction, for all \( k \) in \( \mathbb{Z} \), we have

\[
v_E(f^k(x) - \alpha^k x) \geq v_E(x) + (k + 1)v_E(\bar{\pi}).
\]

Let \( M = E_{\mathbb{Q}_p}^+ e_1 \oplus \cdots \oplus E_{\mathbb{Q}_p}^+ e_d \), then \( f^k \) induces

\[
M/\pi^k M \simeq \alpha^k M/\alpha^{k+1} M \simeq \pi^k M/\pi^{k+1} M.
\]

So \( f^k \mathbb{F}_p[[f]] e_1 \oplus \cdots \oplus f^k \mathbb{F}_p[[f]] e_d \) is dense in \( \pi^k M \) and is equal by compactness.

\[\square\]

**Corollary 5.3.12.** \( \gamma - 1 \) has a continuous inverse on \( D^{\psi=0} \), and \( D^{\psi=0} \) is a free \( \mathbb{F}_p \{ \{ \Gamma_0 \} \} \)-module with basis \( \{ \varepsilon \varphi(e_1), \cdots, \varepsilon \varphi(e_d) \} \).

**Proof.** Copy the proof that (ii) for \( n + 1 \) implies (ii) for \( n \) in the previous proposition, using \( \gamma_1 = \gamma_0^{p-1} \).

\[\square\]

**Proposition 5.3.13.** If \( D \) is an étale \((\varphi, \Gamma)\)-module over \( A_K \) or \( B_K \), then \( \gamma - 1 \) has a continuous inverse on \( D^{\psi=0} \).
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Proof. \( B_K \) follows from \( A_K \) by \( \mathbb{Q}_p \otimes \mathbb{Z}_p \); and we can reduce \( A_K \) to \( A_{\mathbb{Q}_p} \).

Since \( D^{\psi=0} \rightarrow (D/p)^{\psi=0} \) is surjective, \( (\sum_{i=1}^{p-1} \varepsilon^i \varphi(x_i)) \) can be lifted to \( (\sum_{i=1}^{p-1} [\varepsilon]^i \varphi(\hat{x}_i)) \),

so we have the following exact sequence:

\[ 0 \rightarrow (pD)^{\psi=0} \rightarrow D^{\psi=0} \rightarrow (D/p)^{\psi=0} \rightarrow 0. \]

Everything is complete for the \( p \)-adic topology, so we just have to verify the result mod \( p \), which is in corollary 5.3.12. \( \square \)

5.3.4 Computation of Galois cohomology groups

Proposition 5.3.14. Let \( C_{\psi, \gamma} \) be the complex

\[ 0 \rightarrow D(V) \xrightarrow{(\psi-1, \gamma-1)} D(V) \oplus D(V) \xrightarrow{(\gamma-1)_{pr_1} - (\psi-1)_{pr_2}} D(V) \xrightarrow{} 0. \]

Then we have a commutative diagram of complexes

\[ C_{\psi, \gamma} : 0 \rightarrow D(V) \xrightarrow{} D(V) \xrightarrow{} D(V) \oplus D(V) \xrightarrow{} D(V) \xrightarrow{} 0 \]

\[ C_{\psi, \gamma} : 0 \rightarrow D(V) \xrightarrow{} D(V) \xrightarrow{} D(V) \oplus D(V) \xrightarrow{} D(V) \xrightarrow{} 0 \]

which induces an isomorphism on cohomology.

Proof. Since \( (-\psi)(\varphi-1) = \psi - 1 \) and \( \psi \) commutes with \( \gamma \) (i.e. \( \psi \gamma = \gamma \psi \)), the diagram commutes. \( \psi \) is surjective, hence the cokernel complex is 0. The kernel is nothing but

\[ 0 \rightarrow 0 \rightarrow D(V)^{\psi=0} \xrightarrow{\gamma-1} D(V)^{\psi=0} \rightarrow 0, \]

it has no cohomology by Proposition 5.3.13. \( \square \)

Theorem 5.3.15. If \( V \) is a \( \mathbb{Z}_p \) or a \( \mathbb{Q}_p \)-representation of \( G_K \), then \( C_{\psi, \gamma}(K, V) \) computes the Galois cohomology of \( V \):

(i) \( H^0(G_K, V) = D(V)^{\psi=1, \gamma=1} = D(V)^{\psi=1, \gamma=1} \).

(ii) \( H^2(G_K, V) \simeq \frac{D(V)}{[\psi-1, \gamma-1]} \).

(iii) One has an exact sequence

\[ 0 \rightarrow \frac{D(V)^{\psi=1}}{\gamma - 1} \rightarrow H^1(G_K, V) \rightarrow \left( \frac{D(V)}{\psi - 1} \right)^{\gamma=1} \rightarrow 0 \]

\[ (x, y) \mapsto x \]
Let $\mathcal{C}(V) = (\varphi - 1)D^{\psi = 1} \subset D^{\psi = 0}$, the exact sequence
\[
0 \longrightarrow D(V)^{\psi = 1} \longrightarrow D(V)^{\psi = 1} \longrightarrow \mathcal{C}(V) \longrightarrow 0
\]
induces an exact sequence
\[
0 \longrightarrow \frac{D(V)^{\psi = 1}}{\gamma - 1} \longrightarrow \frac{D(V)^{\psi = 1}}{\gamma - 1} \longrightarrow \mathcal{C}(V) \longrightarrow 0
\]
since $\mathcal{C}(V)^{\gamma = 1} \subset (D^{\psi = 0})^{\gamma = 1} = 0$.

**Proposition 5.3.16.** If $D$ is an étale $(\varphi, \Gamma)$-module of dimension $d$ over $E_{\mathbb{Q}_p}$, then $\mathcal{C} = (\varphi - 1)D^{\psi = 1}$ is a free $\mathbb{F}_p[[\Gamma_0]]$-module of rank $d$.

**Proof.** We know:

- $\mathcal{C} \subset D^{\psi = 0}$, it implies $\mathcal{C}$ is a $\mathbb{F}_p[[\Gamma_0]]$-module of rank less than $d$;
- $\mathcal{C}$ is compact, because $D^{\psi = 1}$ is compact;
- So we just have to prove (see proposition 5.3.11 and corollary 5.3.12) that $\mathcal{C}$ contains $\{\varepsilon \varphi(e_1), \ldots, \varepsilon \varphi(e_d)\}$, where $\{e_1, \ldots, e_d\}$ is any basis of $D$ over $E_{\mathbb{Q}_p}$.

Let $\{f_1, \ldots, f_d\}$ be any basis. Then $\varphi^n(\bar{\pi}^k f_i)$ goes to 0 when $n$ goes to $+\infty$ if $k \gg 0$. Let $g_i = \sum_{n=0}^{+\infty} \varphi^n(\varepsilon \varphi(\bar{\pi}^k f_i))$. Then we have:

- $\psi(g_i) = g_i$, because $\psi(\varepsilon \varphi(\bar{\pi}^k f_i)) = 0$;
- $(\varphi - 1)g_i = -\varepsilon \varphi(\bar{\pi}^k f_i) \in \mathcal{C}$.

We can take $e_i = \bar{\pi}^k f_i$. \hfill \Box

### 5.3.5 The Euler-Poincaré formula.

**Theorem 5.3.17.** If $V$ is a finite $\mathbb{Z}_p$-representation of $G_K$, then
\[
\chi(V) = \prod_{i=0}^{2} |H^i(G_K, V)|^{(-1)^i} = |V|^{-[K:Q_p]}.
\]
Proof. From Shapiro’s lemma, we have
\[ H^i(G_K, V) \simeq H^i(G_{\mathbb{Q}_p}, \text{Ind}_{G_K}^{G_{\mathbb{Q}_p}} V). \]
Since \( |\text{Ind}_{G_K}^{G_{\mathbb{Q}_p}} V| = |V|[K:\mathbb{Q}_p] \), we can assume \( K = \mathbb{Q}_p \). Given an exact sequence
\[ 0 \to V_1 \to V \to V_2 \to 0, \]
then \( \chi(V) = \chi(V_1)\chi(V_2) \) and \( |V| = |V_1||V_2| \) from the long exact sequence in Galois Cohomology, thus we can reduce to the case that \( V \) is a \( \mathbb{F}_p \)-representation of \( G_K \). Then we have:
\[ |H^0| = |D(V)^{\psi=1, \gamma=1}|; \]
\[ |H^1| = \left| \frac{D(V)^{\psi=1}}{\gamma - 1} \cdot |C(V)| \cdot \left( \frac{D(V)}{\psi - 1} \right)^{\gamma=1} \right|; \]
\[ |H^2| = \left| \frac{D(V)}{(\psi - 1, \gamma - 1)} \right|. \]
So \( |H^0||H^2||H^1|^{-1} = \frac{|C(V)|}{\gamma - 1} \), because \( D(V)^{\psi=1} \) and \( \frac{D(V)}{\psi - 1} \) are finite groups, and for a finite group \( M \), the exact sequence:
\[ 0 \to M^{\gamma=1} \to M \xrightarrow{\gamma-1} M \xrightarrow{\gamma-1} M \xrightarrow{\gamma-1} 0 \]
implies that \( |M^{\gamma=1}| = \left| \frac{M}{\gamma - 1} \right| \). Now \( \frac{C(V)}{\gamma - 1} \) is a \( (\mathbb{F}_p[[\Gamma_0]]/(\gamma - 1)) = \mathbb{F}_p \)-module of rank \( \dim_{\mathbb{F}_p} D(V) = \dim_{\mathbb{F}_p} V \). Hence \( |\frac{C(V)}{\gamma - 1}| = |V| \).

5.4 Tate’s duality and residues

Let \( M \) be a finite \( \mathbb{Z}_p \) module. We want to construct a perfect pairing
\[ H^i(G_K, M) \times H^{2-i}(G_K, M^\wedge(1)) \to \mathbb{Q}_p/\mathbb{Z}_p. \]

By using Shapiro’s lemma, we may assume \( K = \mathbb{Q}_p \).

Definition 5.4.1. Let \( x = \sum_{k \in \mathbb{Z}} a_k \pi^k \in B_{\mathbb{Q}_p} \), define
\[ \text{res}(x d\pi) = a_{-1}. \]
The residue of \( x \), denoted by \( \text{Res}(x) \) is defined as
\[ \text{Res}(x) = \text{res}(x \frac{d\pi}{1 + \pi}). \]
The map $\text{Res} : B_{\mathbb{Q}_p} \to \mathbb{Q}_p$ maps $A_{\mathbb{Q}_p}$ to $\mathbb{Z}_p$, thus it induced a natural map $B_{\mathbb{Q}_p}/A_{\mathbb{Q}_p} \to \mathbb{Q}_p/\mathbb{Z}_p$.

**Proposition 5.4.2.**

$$\text{Res}(\psi(x)) = \text{Res}(x);$$
$$\text{Res}(\gamma(x)) = \chi(\gamma)^{-1}\text{Res}(x)$$

*Proof.* Exercise. \qed

Let $D$ be an étale $(\varphi, \Gamma)$-module over $A_{\mathbb{Q}_p}$, denote $D^\vee = \text{Hom}_{A_{\mathbb{Q}_p}}(D, B_{\mathbb{Q}_p}/A_{\mathbb{Q}_p})$, let $x \in D^\vee$, $y \in D$, denote

$$\langle x, y \rangle = x(y) \in B_{\mathbb{Q}_p}/A_{\mathbb{Q}_p}.$$

Then

$$\langle \gamma(x), \gamma(y) \rangle = \gamma(\langle x, y \rangle),$$
$$\langle \varphi(x), \varphi(y) \rangle = \varphi(\langle x, y \rangle)$$

determines the $(\varphi, \Gamma)$-module structure on $D^\vee$. Set

$$[x, y] := \text{Res}(\langle x, y \rangle) \in \mathbb{Q}_p/\mathbb{Z}_p.$$

The main step is following proposition.

**Proposition 5.4.3.** (i) The map $x \mapsto (y \mapsto [x, y])$ gives an isomorphism from $D^\vee$ to $D^\vee(V) = \text{Hom}_{\text{cont}}(D, \mathbb{Q}_p/\mathbb{Z}_p)$.

(ii) The following formulas hold:

$$[x, \varphi(y)] = [\psi(x), y]$$
$$[\gamma(x), y] = \chi(\gamma)^{-1}[x, \gamma^{-1}(y)].$$

**Corollary 5.4.4.** Let $V^\vee(1) = \text{Hom}_{\mathbb{Z}_p}(V, (\mathbb{Q}_p/\mathbb{Z}_p)(1))$, then $D(V^\vee(1)) = D^\vee(1)$.

Now the two complexes

$$C_{\varphi, \gamma}(\mathbb{Q}_p, V) : D(V) \xrightarrow{d_1} D(V) \oplus D(V) \xrightarrow{d_2} D(V)$$

$$\xrightarrow{d'_2} D^\vee(1) \oplus D^\vee(1) \xrightarrow{d'_1} D^\vee(1) : C_{\psi, \gamma^{-1}}(\mathbb{Q}_p, V^\vee(1))$$
are in duality, where $d_1 z = ((\varphi - 1)z, (\gamma - 1)z)$, $d_2(x, y) = (\gamma - 1)x - (\varphi - 1)y$, $d_1'z' = ((\psi - 1)z', (\gamma^{-1} - 1)z')$, $d_2'(x', y') = (\gamma^{-1} - 1)x' - (\psi - 1)y'$, and the duality map in the middle given by $[(x, y), (x', y')] = [x', x] - [y', y]$.

One can check that the images are closed. Therefore their cohomology are in duality. For details, see Herr’s paper in Math Annalen (2001?).
Chapter 6

($\varphi, \Gamma$)-modules and Iwasawa theory

6.1 Iwasawa modules $H_{Iw}^i(K, V)$

6.1.1 Projective limits of cohomology groups

In this chapter we assume that $K$ is a finite extension of $\mathbb{Q}_p$ and $G_K$ is the Galois group of $\bar{K}/K$. Then $K_n = K(\mu_{p^n})$ and $\Gamma_n = \text{Gal}(K_\infty/K_n) = \gamma_n^{p^n}$ if $n \geq 1$ ($n \geq 2$ if $p = 2$) where $\gamma_n$ is a topological generator of $\Gamma_n$. We choose $\gamma_n$ such that $\gamma_n = \gamma_1^{p^{n-1}}$. The Iwasawa algebra $\mathbb{Z}_p[[\Gamma_K]]$ is isomorphic to $\mathbb{Z}_p[[T]]$ with the $(p, T)$-adic topology by sending $T$ to $\gamma - 1$. We have

$$\mathbb{Z}_p[[\Gamma_K]]/(\gamma_n - 1) = \mathbb{Z}_p[\text{Gal}(K_n/K)].$$

Furthermore $\mathbb{Z}_p[[\Gamma_K]]$ is a $G_K$-module: let $g \in G_K$ and $x \in \mathbb{Z}_p[[\Gamma_K]]$, then $gx = \bar{g}x$, where $\bar{g}$ is the image of $g$ in $\Gamma_K$. By the same way, $G_K$ acts on $\mathbb{Z}_p[\text{Gal}(K_n/K)]$.

Using Shapiro’s Lemma, we get, for $M$ a $\mathbb{Z}_p[G_K]$-module,

$$H^i(G_K, M) \xrightarrow{\sim} H^i(G_K, \mathbb{Z}_p[\text{Gal}(K_n/K)] \otimes M),$$

with the inverse map given by

$$((\sigma_1, ..., \sigma_i) \mapsto \sum_{g \in \text{Gal}(K_n/K)} g \otimes C_g(\sigma_1, ..., \sigma_i)) \mapsto ((\sigma_1, ..., \sigma_i) \mapsto C_{id}(\sigma_1, ..., \sigma_i)).$$

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Thus we have a commutative diagram:

\[
\begin{array}{ccc}
H^i(G_{K_{n+1}}, M) & \xrightarrow{\cong} & H^i(G_K, \mathbb{Z}_p[\text{Gal}(K_{n+1}/K)] \otimes M) \\
\downarrow \text{cor} & & \downarrow \\
H^i(G_K, M) & \xrightarrow{\cong} & H^i(G_K, \mathbb{Z}_p[\text{Gal}(K_n/K)] \otimes M)
\end{array}
\]

One can check that the second vertical arrow is just induced by the natural map \( \text{Gal}(K_{n+1}/K) \rightarrow \text{Gal}(K_n/K) \).

**Definition 6.1.1.** (i) If \( V \) is a \( \mathbb{Z}_p \)-representation of \( G_K \), define

\[
H^i_{Iw}(K, V) = \lim_{\leftarrow n} H^i(G_{K_n}, V)
\]

while the transition maps are the corestriction maps.

(ii) If \( V \) is a \( \mathbb{Q}_p \)-representation, choose \( T \) a stable \( \mathbb{Z}_p \)-lattice in \( V \), then define

\[
H^i_{Iw}(K, V) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{Iw}(K, T).
\]

**6.1.2 Reinterpretation in terms of measures**

**Proposition 6.1.2.** \( H^i(G_K, \mathbb{Z}_p[[\Gamma_K]] \otimes V) \cong H^i_{Iw}(K, V) \).

**Proof.** The case of \( \mathbb{Q}_p \) follows from the case of \( \mathbb{Z}_p \) by using \( \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \). Now assume that \( V \) is a \( \mathbb{Z}_p \)-representation of \( G_K \). By definition,

\[
\Lambda = \mathbb{Z}_p[[\Gamma_K]] = \lim_{\leftarrow n} \mathbb{Z}_p[[\Gamma_K]]/\gamma_n - 1,
\]

it induces the map \( \theta \):

\[
H^i(G_K, \Lambda \otimes V) \xrightarrow{\theta} \lim_{\leftarrow n} H^i(G_K, \Lambda/(\gamma_n - 1) \otimes V) = H^i_{Iw}(K, V)
\]

\[
\xrightarrow{\alpha} \lim_{\leftarrow n} H^i(G_K, \Lambda/(p^n, \gamma_n - 1) \otimes V)
\]

The surjectivity is general abstract nonsense.

The injectivity of \( \alpha \) implies the injectivity of \( \theta \); to prove that of \( \alpha \), it is enough to verify the Mittag-Leffler conditions of \( H^{i-1} \), which are automatic, because of the Finiteness Theorem: \( \Lambda/(p^n, \gamma_n - 1) \otimes V \) is a finite module, so \( H^{i-1}(G_K, \Lambda/(p^n, \gamma_n - 1) \otimes V) \) is a finite group. \( \square \)
6.1. IWASAWA MODULES $H^i_{IW}(K, V)$

**Remark.** (i) Recall that $\mathcal{D}_0(\Gamma_K, V)$ is the set of $p$-adic measures from $\Gamma_K$ to $V$: 

$$ Z_p[[\Gamma_K]] \otimes V \cong \mathcal{D}_0(\Gamma_K, V), \quad \gamma \otimes v \mapsto \delta_\gamma \otimes v, $$

where $\delta_\gamma$ is the Dirac measure at $\gamma$. Let $g \in G_K$, $\mu \in \mathcal{D}_0(\Gamma_K, V)$; the action of $G_K$ on $\mathcal{D}_0(\Gamma_K, V)$ is as follow:

$$ \int_{\Gamma_K} \phi(x)(g\mu) = g(\int_{\Gamma_K} \phi(gx)\mu). $$

Hence, for any $n \in \mathbb{N}$, the map $H^i(G_K, Z_p[[\Gamma_K]] \otimes V) \to H^i(G_K, V)$ (translation of Shapiro’s lemma) can be written in the following concrete way:

$$ ((\sigma_1, \ldots, \sigma_i) \mapsto \mu(\sigma_1, \ldots, \sigma_i)) \longmapsto ((\sigma_1, \ldots, \sigma_i) \mapsto \int_{\Gamma_K} 1_{\Gamma_K^n} \cdot \mu(\sigma_1, \ldots, \sigma_i) \in V)_{n \in \mathbb{N}}. $$

(ii) Let $g \in G_K$, $\lambda, \mu \in Z_p[[\Gamma_K]]$, $x \in V$, then

$$ g(\lambda \mu \otimes v) = \bar{g}\lambda \mu \otimes gv = \lambda \bar{g}\mu \otimes gv = \lambda g(\mu \otimes \mu). $$

So $\lambda$ and $g$ commutes, it implies that $H^i_{IW}(K, V)$ are $Z_p[[\Gamma_K]]$-modules.

### 6.1.3 Twist by a character (à la Soulé)

Let $\eta : \Gamma_K \rightarrow \mathbb{Q}_p^*$ be a continuous character. It induces a transform

$$ \mathcal{D}_0(\Gamma_K, V) \to \mathcal{D}_0(\Gamma_K, V), \quad \mu \mapsto \eta \cdot \mu. $$

For $\lambda \in Z_p[[\Gamma_K]]$, we have

$$ \eta \cdot (\lambda \mu) = (\eta \cdot \lambda)(\eta \cdot \mu). $$

Indeed, it is enough to check it on Dirac measures. In this case

$$ \eta \cdot (\delta_{\lambda_1}\delta_{\lambda_2} \otimes v) = \eta(\lambda_1 \lambda_2)\delta_{\lambda_1}\delta_{\lambda_2} \otimes v = (\eta \cdot \delta_{\lambda_1})(\eta \cdot \delta_{\lambda_2}) \otimes v. $$

Recall that $Z_p(\eta) = Z_p \cdot e_\eta$, where, if $g \in G_K$, then $ge_\eta = \eta(\bar{g})e_\eta$. Define $V(\eta) = V \otimes Z_p(\eta)$.

**Exercise.** The map $\mu \in \mathcal{D}_0(\Gamma_K, V) \mapsto (\eta \cdot \mu) \otimes e_\eta \in \mathcal{D}_0(\Gamma_K, V)$ is an isomorphism of $Z_p[G_K]$-modules.
By the above exercise, we have a commutative diagram:

\[
\begin{array}{ccc}
H^i_{Iw}(K, V) & \xrightarrow{i_\eta} & H^i_{Iw}(K, V(\eta)) \\
\uparrow & & \uparrow \\
H^i(G_K, D_0(\Gamma_K, V)) & \xrightarrow{\sim} & H^i(G_K, D_0(\Gamma_K, V(\eta)))
\end{array}
\]

So \(i_\eta\) is an isomorphism of cohomology groups. It can be written in a concrete way:

\[
i_\eta : (\sigma_1, \ldots, \sigma_i) \mapsto \mu(\sigma_1, \ldots, \sigma_i) \mapsto \int_{\Gamma_K} 1_{\Gamma_K} \eta \mu(\sigma_1, \ldots, \sigma_i) \otimes e_{\eta} \mid_{n \in \mathbb{N}}.
\]

It is an isomorphism of \(\mathbb{Z}_p\)-modules.

**Warning:** \(i_\eta\) is not an isomorphism of \(\mathbb{Z}_p[[\Gamma_K]]\)-modules, because \(i_\eta(\lambda x) = (\eta \cdot \lambda)i_\eta(x)\): there is a twist.

### 6.2 Description of \(H^i_{Iw}\) in terms of \(D(V)\)

**Remark.** \(H^i_{Iw}(K, V) = \lim_{\leftarrow n \geq n_0} H^i(G_{K_n}, V)\), so we can always assume \(n \gg 0\).

**Lemma 6.2.1.** Let \(\tau_n = \frac{\gamma_{n-1}}{\gamma_{n-1} - 1} = 1 + \gamma_{n-1} +, \ldots, + \gamma_{n-1}^{p-1} \in \mathbb{Z}_p[[\Gamma_K]]\), the diagram

\[
\begin{array}{ccc}
C_{\psi, \gamma_n}(K_n, V) : & 0 & \xrightarrow{\tau_n} & D(V) & \xrightarrow{\tau_n} & D(V) & \oplus & D(V) & \xrightarrow{\tau_n} & D(V) & \xrightarrow{\tau_n} & 0 \\
& & & \downarrow \tau_n & & \downarrow \tau_n & & \downarrow \text{Id} & & \downarrow \text{Id} & \\
C_{\psi, \gamma_{n-1}}(K_{n-1}, V) : & 0 & \xrightarrow{\tau_n} & D(V) & \xrightarrow{\tau_n} & D(V) & \oplus & D(V) & \xrightarrow{\tau_n} & D(V) & \xrightarrow{\tau_n} & 0
\end{array}
\]

is commutative and induces corestrictions on cohomology via

\[
H^i(C_{\psi, \gamma_n}(K_n, V)) \xrightarrow{\sim} H^i(G_{K_n}, V).
\]

**Proof.** \(\tau_n\) is a cohomological functor and induces \(\text{Tr}_{K_n/K_{n-1}}\) on \(H^0\), so it induces corestrictions on \(H^i\). \(\square\)

**Theorem 6.2.2.** If \(V\) is a \(\mathbb{Z}_p\) or \(\mathbb{Q}_p\) representation of \(G_K\), then we have:

\begin{enumerate}
  \item \(H^i_{Iw}(K, V) = 0\), if \(i \neq 1, 2\).
  \item \(H^1_{Iw}(K, V) \cong D(V)_{\psi=1}\), \(H^2_{Iw}(K, V) \cong \frac{D(V)}{\psi-1}\), and the isomorphisms are canonical.
\end{enumerate}
6.2. DESCRIPTION OF $H_{Iw}^1$ IN TERMS OF $D(V)$

Remark. (i) The isomorphism
\[ \text{Exp}^* : H_{Iw}^1(K, V) \to D(V)^{\psi=1} \]

is the map that will produce $p$-adic $L$-functions. Let’s describe $(\text{Exp}^*)^{-1}$. Let $y \in D(V)^{\psi-1}$, then $(\varphi - 1)y \in D(V)^{\psi=0}$. There exists unique $x_n \in D(V)^{\psi=0}$ satisfying that $(\gamma_n - 1)x_n = y_n$, then we can find $b_n \in A \otimes V$ such that $(\varphi - 1)b_n = x_n$. Then
\[ g \mapsto \frac{\log \chi(\gamma_n)}{p^n} \left( \frac{(g - 1)}{(\gamma_n - 1)}y - (g - 1)b_n \right) \]
gives a cocycle on $G_{K_n}$ with values in $V$, and $\frac{\log \chi(\gamma_n)}{p^n}$ does not depend on $n$. Denote by $t_{\psi,n}(y) \in H^1(G_{K_n}, V)$ the image of this cocycle, then
\[ (\text{Exp}^*)^{-1} : y \mapsto (\cdots, t_{\psi,n}(y), \cdots)_{n \in \mathbb{N}} \in H_{Iw}^1(K, V) \]
doesn’t depend on the choice of $\gamma_n$.

(ii) We see that $D(V)^{\psi-1}$ is dual to $D(V^\wedge(1))^{\psi=1} = V^\wedge(1)^{H_K}$, so $H_{Iw}^2(K, V) = \frac{D(V)}{\psi-1} = (V^\wedge(1)^{H_K})^\wedge$.

Before proving the theorem, we introduce a lemma.

Lemma 6.2.3. If $M$ is compact with continuous action of $\Gamma_K$, then
\[ M \simeq \lim \left( M/\gamma_n - 1 \right) \]

Proof. We have a natural map from $M$ to $\lim \left( M/\gamma_n - 1 \right)$.

Injectivity: let $V$ be an open neighborhood of 0. For all $x \in M$, there exists $n_x \in \mathbb{N}$ and $U_x \supseteq x$, an open neighborhood of $x$ such that $(\gamma - 1)x' \in V$ for $\gamma \in \Gamma_{K_{n_x}}$ and $x' \in U_x$. By compactness, $M = \bigcup_{i \in I} U_{x_i}$, where $I$ is a finite set. Let $n = \max_{i \in I} n_{x_i}$. It implies that $(\gamma - 1)M \subset V$, if $\gamma \in \Gamma_n$, then $\bigcap_{n \in \mathbb{N}} (\gamma_n - 1)M = 0$, this shows the injectivity.

Surjectivity: Let $(x_n)_{n \in \mathbb{N}} \in \lim \left( M/\gamma_n - 1 \right)$. From the proof of injectivity, we know that $x_n$ is a Cauchy-sequence. Because $M$ is compact, there exists $x = \lim x_n$. We have $x_{n+k} - x_n = (\gamma_n - 1)y_k$ for all $k \geq 0$, as $M$ is compact, there exists a subsequence of $y_k$ converging to $y$, passing to the limit, we get $x - x_n = (\gamma_n - 1)y$. This shows the surjectivity. \qed
Proof of Theorem 6.2.2. \( H^i_{Iw}(K, V) \) is trivial if \( i \geq 3 \) and the case of \( \mathbb{Q}_p \) follows from \( \mathbb{Z}_p \) by \( \mathbb{Q}_p \otimes \mathbb{Z}_p \).

For \( i = 0 \),
\[
H^0_{Iw}(K, V) = \lim_{\leftarrow \text{Tr}} V^{G_{K_n}}.
\]
\( V^{G_{K_n}} \) is increasing and \( \subset V \), as \( V \) is a finite dimensional \( \mathbb{Z}_p \)-module, the sequence is stationary for \( n \geq n_0 \). Then \( \text{Tr}_{K_{n+1}/K_n} \) is just multiplication by \( p \) for \( n \geq n_0 \), but \( V \) does not contain \( p \)-divisible elements. This shows that \( \lim_{\leftarrow \text{Tr}} V^{G_{K_n}} = 0 \).

For \( i = 2 \): \( H^2(G_{K_n}, V) = \frac{D(V)}{(\psi-1, \gamma_n-1)} \). The corestriction map is induced by \( \text{Id} \) on \( D(V) \), thus
\[
H^2_{Iw}(K, V) = \lim_{\leftarrow \text{Tr}} \frac{D(V)}{(\psi-1)} / (\gamma_n - 1) = \frac{D(V)}{\psi - 1}
\]
by Lemma 6.2.3, as \( \frac{D(V)}{\psi - 1} \) is compact (and even finitely generated over \( \mathbb{Z}_p \)).

For \( i = 1 \): we have commutative diagrams:

\[
\begin{array}{cccccc}
0 & \rightarrow & D(V)^{\psi=1}_{/\gamma_n - 1} & \rightarrow & H^1(G_{K_n}, V) & \rightarrow & 0 \\
& & p_1 & & \text{cor} & & \\
0 & \rightarrow & D(V)^{\psi=1}_{/\gamma_n - 1} & \rightarrow & H^1(G_{K_n-1}, V) & \rightarrow & 0
\end{array}
\]

where \( p_1(\bar{y}) = \bar{y} \), \( p_2((\bar{x}, \bar{y})) = \bar{x} \), for any \( x, y \in D(V) \). Using the functor \( \lim_{\leftarrow \text{Tr}} \), we get:
\[
0 \rightarrow \lim_{\leftarrow \text{Tr}} D(V)^{\psi=1}_{/\gamma_n - 1} \rightarrow \lim_{\leftarrow \text{Tr}} H^1(G_{K_n}, V) \rightarrow \lim_{\leftarrow \text{Tr}} \frac{(D(V))^{\psi=1}}{\gamma_n - 1}.
\]

Because \( D(V)^{\psi=1} \) is compact, by Lemma 6.2.3 we have \( D(V)^{\psi=1} \simeq \lim_{\leftarrow \text{Tr}} \frac{D(V)^{\psi=1}}{\gamma_n - 1} \).

By definition, \( H^1_{Iw}(K, V) = \lim_{\leftarrow \text{Tr}} H^1(G_{K_n}, V) \). The same argument for showing \( H^0_{Iw}(K, V) = 0 \) shows that \( \lim_{\leftarrow \text{Tr}} \frac{(D(V))^{\psi=1}}{\gamma_n - 1} = 0 \). So we get
\[
D(V)^{\psi=1} \sim H^1_{Iw}(K, V).
\]

\qed
6.3 Structure of $H^1_{Iw}(K, V)$

Recall that we proved that if $D$ is an étale $(\varphi, \Gamma)$-module of dim $d$ over $E_{Q_p}$, then $C = (\varphi - 1)D_{\psi=1}$ is a free $F_p[[\Gamma_{Q_p}]]$-module of rank $d$. The same proof shows that if $n \geq 1$, $i \in \mathbb{Z}^*_p$, $C \cap \varepsilon^{in}(D)$ is free of rank $d$ over $F_p[[\Gamma_n]]$.

**Corollary 6.3.1.** If $D$ is an étale $(\varphi, \Gamma)$-module of dimension $d$ over $E_K$, then $C$ is a free $F_p[[\Gamma_K]]$-module of rank $d$ over $E_{Q_p}$.

*Proof.* Exercise. Hint: $D$ is of dimension $d \cdot [H_{Q_p} : H_K]$ over $E_{Q_p}$ and $[K : Q_p] = [G_{Q_p} : G_K] = [\Gamma_{Q_p} : \Gamma_K][H_{Q_p} : H_K]$. \hfill $\square$

**Proposition 6.3.2.** If $V$ is a free $\mathbb{Z}_p$ or $\mathbb{Q}_p$ representation of rank $d$ of $G_K$, then

(i) $D(V)_{\psi=1}$ is the torsion sub-$\mathbb{Z}_p[[\Gamma_K \cap \Gamma_1]]$-module of $D(V)_{\psi=1}$.

(ii) We have exact sequences:

$$0 \rightarrow D(V)_{\psi=1} \rightarrow D(V)_{\psi=1} \rightarrow C(V) \rightarrow 0.$$ 

and $C(V)$ is free of rank $d \cdot [K : Q_p]$ over $\mathbb{Z}_p[[\Gamma_K]]$ (or over $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma_K]]$).

**Corollary 6.3.3.** If $V$ is a free $\mathbb{Z}_p$ representation of rank $d$ of $G_K$, then the torsion $\mathbb{Z}_p[[\Gamma_K \cap \Gamma_1]]$-module of $H^1_{Iw}(K, V)$ is $D(V)_{\psi=1} = V^{H_K}$, and $H^1_{Iw}(K, V)/V^{H_K}$ is free of rank $d \cdot [K : Q_p]$ over $\mathbb{Z}_p[[\Gamma_K]]$.

*Proof of Proposition 6.3.2.* $D(V)_{\psi=1} = V^{H_K}$ is torsion because it is finitely generated over $\mathbb{Z}_p$, so (ii) implies (i). To prove (ii), we have to prove $C(V)/pC(V)$ is free of rank $d \cdot [K : Q_p]$ over $F_p[[\Gamma_K]]$.

Consider the following commutative diagram with exact rows

$$
\begin{array}{c}
0 \rightarrow D(V)_{\psi=1} \rightarrow D(V)_{\psi=1} \rightarrow C(V) \rightarrow 0 \\
0 \rightarrow (D(V)/p)_{\phi=1} \rightarrow (D(V)/p)_{\psi=1} \rightarrow C(V/p) \rightarrow 0
\end{array}
$$

Using the exact sequence

$$0 \rightarrow pV \rightarrow V \rightarrow V/p \rightarrow 0$$
and apply the snake lemma to the vertical rows of the diagram above, we have the cokernel complex is

\[
p - \text{torsion of } \frac{D(V)}{(\varphi - 1)} \to p - \text{torsion of } \frac{D(V)}{(\psi - 1)} \to \frac{C(V/p)}{C(V)/pC(V)} \to 0.
\]

Note that the $p$-torsion of $\frac{D(V)}{(\psi - 1)}$ is a finite dimensional $\mathbb{F}_p$-vector space, thus $\frac{C(V/p)}{C(V)/pC(V)}$ is also a finite dimensional $\mathbb{F}_p$-vector space, hence $\frac{C(V/p)}{C(V)}$ is a $\mathbb{F}_p[[\Gamma_K]]$-module of rank $d \cdot [K : \mathbb{Q}_p]$ by Corollary 6.3.1.

\[\square\]

**Remark.** (i) The sequence

\[
0 \to D(V)^{\varphi=1} \to D(V)^{\psi=1} \to C(V) \to 0
\]

is just the inflation-restriction exact sequence

\[
0 \to H^1(\Gamma_K, \Lambda \otimes V^{H_K}) \to H^1(G_K, \Lambda \otimes V) \to H^1(H_K, \Lambda \otimes V)^{\Gamma_K} \to 0.
\]

(ii) Let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence, then the exact sequence

\[
0 \to D(V_1) \to D(V) \to D(V_2) \to 0
\]

and the snake lemma induces

\[
0 \to D(V_1)^{\psi=1} \to D(V)^{\psi=1} \to D(V_2)^{\psi=1} \to \frac{D(V_1)}{\psi - 1} \to \frac{D(V)}{\psi - 1} \to \frac{D(V_2)}{\psi - 1} \to 0.
\]

By Theorem 6.2.2, this is just

\[
0 \to H^1_{\text{Iw}}(K, V_1) \to H^1_{\text{Iw}}(K, V) \to H^1_{\text{Iw}}(K, V_2)
\]

\[
\to H^2_{\text{Iw}}(K, V_1) \to H^2_{\text{Iw}}(K, V) \to H^2_{\text{Iw}}(K, V_2) \to 0.
\]

It can also be obtained from the longer exact sequence in continuous cohomology from the exact sequence

\[
0 \to \mathbb{Z}_p[[\Gamma_K]] \otimes V_1 \to \mathbb{Z}_p[[\Gamma_K]] \otimes V \to \mathbb{Z}_p[[\Gamma_K]] \otimes V_2 \to 0.
\]
Chapter 7

$\mathbb{Z}_p(1)$ and Kubota-Leopoldt zeta function

7.1 The module $D(\mathbb{Z}_p(1))^{\psi=1}$

The module $\mathbb{Z}_p(1)$ is just $\mathbb{Z}_p$ with the action of $G_{\mathbb{Q}_p}$ by $g \in G_{\mathbb{Q}_p}$, $x \in \mathbb{Z}_p(1)$, $g(x) = \chi(g)x$. We shall study the exponential map

$$\text{Exp}^* : H^1_{Iw}(\mathbb{Q}_p, \mathbb{Z}_p(1)) \to D(\mathbb{Z}_p(1))^{\psi=1}.$$ 

Note that $D(\mathbb{Z}_p(1)) = (A \otimes \mathbb{Z}_p(1))^{H_{\mathbb{Q}_p}} = A_{\mathbb{Q}_p}(1)$, with usual actions of $\varphi$ and $\psi$, and for $\gamma \in \Gamma$, $\gamma(f(\pi)) = \chi(\gamma)f((1 + \pi)^{\chi(\gamma)} - 1)$, for all $f(\pi) \in A_{\mathbb{Q}_p}(1)$.

Proposition 7.1.1. (i) $A^{\psi=1}_{\mathbb{Q}_p} = \mathbb{Z}_p \cdot \frac{1}{\pi} \oplus (A^{\varphi=1}_{\mathbb{Q}_p})$.

(ii) We have an exact sequence:

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow (A_{\mathbb{Q}_p}^{\psi=1}) \varphi=1 \longrightarrow (\pi A_{\mathbb{Q}_p}^{\varphi=0}) \psi=0 \longrightarrow 0.$$

Remark. Under the map $\mu \mapsto \int_{\mathbb{Z}_p}[\varepsilon]^\mu$, $(\pi A_{\mathbb{Q}_p}^+)^{\psi=0}$ is the image of measures with support in $\mathbb{Z}_p^* (\psi = 0)$ and $\int_{\mathbb{Z}_p^*} \mu = 0$

$$(\pi A_{\mathbb{Q}_p}^+)^{\psi=0} = \mathcal{C}(\mathbb{Z}_p) = (\gamma - 1)\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]].$$

$\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]]$ can be viewed as measures on $\Gamma_{\mathbb{Q}_p} \cong \mathbb{Z}_p^*$, and $\mu \in (\gamma - 1)\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]]$ means $\int_{\mathbb{Z}_p} \mu = 0$. It implies that $\mathcal{C}(\mathbb{Z}_p)$ is free of rank 1 over $\mathbb{Z}_p[[\Gamma_{\mathbb{Q}_p}]]$ which is a special case of what we have proved.
CHAPTER 7. \(\mathbb{Z}_p(1)\) AND KUBOTA-LEOPOLDT ZETA FUNCTION

Proof. (i) We have proved

\[ \psi(A_{Q_p}^+) \subset A_{Q_p}^+, \quad \psi(\frac{1}{\pi}) = \frac{1}{\pi}, \]

\[ v_E(\psi(x)) \geq \left[ \frac{v_{E,x}}{p} \right], \quad \text{if} \ x \in E_{Q_p}. \]

These facts imply that \(\psi - 1\) is bijective on \(E_{Q_p}/\pi^{-1}E_{Q_p}^+\) and hence it is also bijective on \(A_{Q_p}/\pi^{-1}A_{Q_p}^+\). So

\[ \psi(x) = x \implies x \in \pi^{-1}A_{Q_p}^+. \]

(ii) We know that \((\varphi - 1)A_{Q_p}^+ \subset \pi A_{Q_p}^+\). For \(x \in (\pi A_{Q_p}^+)^{\psi = 0}\), then

\[ \varphi^n(x) \in \varphi^n(\pi)A_{Q_p}^+ \to 0 \text{ if } n \to \infty. \]

Hence \(y = \sum_{n=0}^{+\infty} \varphi^n(x)\) converges, and one check that \(\psi(y) = y, \ (\varphi - 1)y = -x\). This implies the surjectivity of \(\varphi - 1\). \(\square\)

7.2 Kummer theory

Recall that

\[ \varepsilon = (1, \varepsilon^{(1)}, \varepsilon^{(2)}, ..., \varepsilon^{(n)}, ...) \in E_{Q_p}^+ \subset \tilde{E}^+ = R, \ \varepsilon^{(1)} \neq 1. \]

Let \(\pi_n = \varepsilon^{(n)} - 1, F_n = Q_p(\pi_n)\) for \(n \geq 1\). Then \(\pi_n\) is a uniforming parameter of \(F_n\), and

\[ N_{F_{n+1}/F_n}(\pi_{n+1}) = \pi_n, \quad \mathcal{O}_{F_{n+1}} = \mathcal{O}_{F_n}[\pi_{n+1}]/((1 + \pi_{n+1})^p = 1 + \pi_n). \]

For an element \(a \in F_n^*\), choose \(x = (a, x^{(1)}, ...) \in \tilde{E}\). This \(x\) is unique up to \(\varepsilon^n\) with \(u \in \mathbb{Z}_p\). So if \(g \in G_{F_n}\), then

\[ \frac{g(x)}{x} = \varepsilon^c(g), \quad c(g) \in \mathbb{Z}_p \]

gives a 1-cocycle \(c\) on \(G_{F_n}\) with values in \(\mathbb{Z}_p(1)\). This defines the Kummer map:

\[ \kappa : F_n^* \longrightarrow H^1(G_{F_n}, \mathbb{Z}_p(1)) \]

\[ a \mapsto \kappa(a). \]
By Kummer theory, we have \( H^1(G_{F_n}, \mathbb{Z}_p(1)) = \mathbb{Z}_p \cdot \kappa(\pi_n) \oplus \kappa(\mathcal{O}_{F_n}^*) \). The diagram

\[
\begin{array}{ccc}
F_{n+1}^* & \xrightarrow{\kappa} & H^1(G_{F_{n+1}}, \mathbb{Z}_p(1)) \\
\downarrow \text{N}_{F_{n+1}/F_n} & & \downarrow \text{cor} \\
F_n^* & \xrightarrow{\kappa} & H^1(G_{F_n}, \mathbb{Z}_p(1))
\end{array}
\]

is commutative, we have a map:

\[
\kappa : \varprojlim F_n^* \rightarrow H^1_{Iw}(\mathbb{Q}_p, \mathbb{Z}_p(1))
\]

and

\[
H^1_{Iw}(\mathbb{Q}_p, \mathbb{Z}_p(1)) = \mathbb{Z}_p \cdot \kappa(\pi_n) \oplus \kappa(\varprojlim \mathcal{O}_{F_n}^*).
\]

### 7.3 Coleman’s power series

**Theorem 7.3.1 (Coleman’s power series).** Let \( u = (u_n)_{n \geq 1} \in \varprojlim(\mathcal{O}_{F_n}) - \{0\} \) (pour les applications \( \text{N}_{F_{n+1}/F_n} \)), then there exists a unique power series \( f_u \in \mathbb{Z}_p[[T]] \) such that \( f_u(\pi_n) = u_n \) for all \( n \geq 1 \).

**Lemma 7.3.2.** (i) If \( x \in \mathcal{O}_{F_n}, \gamma \in \text{Gal}(F_{n+1}/F_n) \), then \( \gamma(x) - x \in \pi_1\mathcal{O}_{F_{n+1}} \).

(ii) \( \text{N}_{F_{n+1}/F_n}x - x^p \in \pi_1\mathcal{O}_{F_{n+1}} \).

**Proof.** It is easy to see that (i) implies (ii) since \( [F_{n+1} : F_n] = p \). Write \( \chi(\gamma) = 1 + p^n u \) for \( u \in \mathbb{Z}_p \). Let \( x = \sum_{i=0}^{p-1} x_i (1 + \pi_{n+1})^i \), where \( x_i \in \mathcal{O}_{F_n} \). Then

\[
\gamma(x) - x = \sum_{i=0}^{p-1} x_i (1 + \pi_{n+1})^i ((1 + \pi_1)^{iu} - 1) \in \pi_1\mathcal{O}_{F_{n+1}}.
\]

\( \square \)

**Corollary 7.3.3.** \( \bar{u} = (\bar{u}_1^p, \bar{u}_1, ..., \bar{u}_n, ...) \in E_{\mathbb{Q}_p}^+, \) where \( \bar{u}_n \) is the image of \( u_n \mod \pi_1 \).

**Definition 7.3.4.** Let \( N : \mathcal{O}_{F_1}[[T]] \rightarrow \mathcal{O}_{F_1}[[T]] \) such that

\[
N(f)((1 + T)^p - 1) = \prod_{z^p=1} f((1 + T)z - 1).
\]
Lemma 7.3.5. (i) $N(f)(\pi_n) = N_{F_{n+1}/F_n}(f(\pi_{n+1}))$,
(ii) $N(Z_p[[T]]) \subset Z_p[[T]]$,
(iii) $N(f) - f \in \pi_1\mathcal{O}_{F_1}[[T]]$,
(iv) If $f \in \mathcal{O}_{F_1}[[T]]^*$, $k \geq 1$, if $(f - g) \in \pi_1^{k-1}\mathcal{O}_{F_1}[[T]]$, then

$$N(f) - N(g) \in \pi_1^{k+1}\mathcal{O}_{F_1}[[T]].$$

Proof. (i) The conjugates of $\pi_{n+1}$ under $\text{Gal}(F_{n+1}/F_n)$ are those $(1 + \pi_n)z - 1$ for $\varphi^p = 1$, this implies (i).

(ii) Obvious, is just Galois theory.

(iii) Look mod $\pi_1$, because $z = 1 \mod \pi_1$, we have $N(f)(T^p) = f(T)^p$.

(iv) We have $N(\bar{f}/g) = N(f)/N(g)$, so we can reduce to $f = 1$ and $g = 1 + \pi_1^k h$. Then

$$N(g)((1 + T)^p - 1) = 1 + \pi_1^k \sum_{z^p=1} h((1 + T)z - 1) \mod \pi_1^{k+1},$$

and $\sum_{z^p=1} h((1 + T)z - 1)$ is divisible by $p$.

Corollary 7.3.6. (i) If $\bar{u} \in E_{\mathbb{Q}_p}^+$ and $v_{E}(\bar{u}) = 0$, then there exists a unique $g_u \in Z_p[[T]]$ such that $N(g_u) = g_u$ and $g_u(\pi) = \bar{u}$.

(ii) If $x \in 1 + \pi_1^k\mathcal{O}_{F_{n+1}}$, then $N_{F_{n+1}/F_n}(x) \in 1 + \pi_1^{k+1}\mathcal{O}_{F_n}.$

Proof. (i) Take any $g \in Z_p[[T]]$ such that $g(\pi) = \bar{u}$, then $g \in Z_p[[T]]^*$, by (iv) of Lemma 7.3.5, $N^k(g)$ converges in $g + \pi_1\mathbb{Z}_p[[T]]$ and $g_u$ is the limit.

(ii) There exists $f \in 1 + \pi_1^k\mathcal{O}_{F_1}[T]$ such that $x = f(\pi_{n+1})$. Then use (i) and (iv) of Lemma 7.3.5.

Proof of Theorem 7.3.1. The uniqueness follows from the fact that $0 \neq f \in Z_p[[T]]$ has only many finitely zeros in $\mathfrak{m}_{C_p}^k$ (Newton polygons).

Existence: let $u = (u_n)$, write $u_n = \pi_n^k \alpha u'_n$, where $k \in \mathbb{Z}$ and $\alpha \in \mu_{p-1}$ do not depend on $n$, and $u'_n \in 1 + \mathfrak{m}_{F_n}$. Then $N_{F_{n+1}/F_n} u'_{n+1} = u'_n$. If for all $n$, $f_{u'}(\pi_n) = u'_n$, let $f_u = T^k \alpha f_{u'}$, then $f_u(\pi_n) = u_n$. Thus we are reduced to the case that $u_n \in 1 + \mathfrak{m}_{F_n}$ for all $n$.

By (i) of Corollary 7.3.6, we can find $g_u \in Z_p[[T]]$ for $\bar{u}$. We have to check that $g_u(\pi_n) = u_n$ for all $n \neq 1$. Write $v_n = g_u(\pi_n)$. Then $N(g_u) = g_u$, by (i) of Lemma 7.3.5, implies that $N_{F_{n+1}/F_n}(v_{n+1}) = v_n$; and $g_u(\pi) = \bar{u}$ implies that $v_n = u_n \mod \pi_1$ for all $n$. Let $w_n = \frac{v_n}{u_n}$, then we have

$$N_{F_{n+1}/F_n}(w_{n+1}) = w_n \quad \text{and} \quad w_n \in 1 + \pi_1\mathcal{O}_{F_n}.$$
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By (ii) of Corollary 7.3.6, we have

\[ w_n = \mathcal{N}_{F_{n+k}/F_n}(w_{n+k}) \in 1 + \pi_1^k \mathcal{O}_{F_n} \text{ for all } k, \]

then \( w_n = 1 \). This completes the proof.

\[ \square \]

Corollary 7.3.7.

\[ \mathcal{N}(f_u) = f_u, \quad \psi\left(\frac{\partial f_u}{f_u}\right) = \frac{\partial f_u}{f_u} \]

where \( \partial = (1 + T) \frac{d}{dT} \).

Proof. By (i) of Lemma 7.3.5, we have \( \mathcal{N}(f_u)(\pi_n) = \mathcal{N}_{F_{n+1}/F_n}(f_u(\pi_{n+1})) = f_u(\pi_n) \), for all \( n \), thus \( \mathcal{N}(f_u) = f_u \).

Using the formula \( \psi(\partial \log f) = \partial(\log \mathcal{N}(f)) \), we immediately get the result for \( \psi \). As for the proof of this last formula, we know that

\[ (1 + T) \frac{d}{dT} f_u((1 + T)z - 1) = \sum_{z^p = 1} (1 + T)zf_u'((1 + T)z - 1) = p\psi\left(\frac{\partial f_u}{f_u}\right)((1 + T)^p - 1), \]

hence the formula.
7.4 An explicit reciprocity law

Theorem 7.4.1. The diagram

\[
\xymatrix{
\lim_{\leftarrow} (\mathcal{O}_{F_n} - \{0\}) \ar[r]^\kappa \ar[dr]_{\sim} & H^1_{Iw}(\mathbb{Q}_p, \mathbb{Z}_p(1)) \ar[dr]_{\sim} \ar[l]^{\Exp^*} \\
& D(\mathbb{Z}_p(1))_{\psi=1} \ar[ur]_{\partial f_u} & \mathbb{F}_n = \mathbb{Q}_p(\mu_{p^n}) \ar[u]_{\sim}
}
\]

is commutative.

Remark. (i) The proof is typical of invariants defined via Fontaine’s rings: easy to define and hard to compute.

(ii) For another example, let \( X/K \) be a smooth and projective variety, then

\[
D_{dR}(H^i_{\et}(X \times \overline{K}, \mathbb{Q}_p)) = H^i_{dR}(X/K).
\]

The proof is very hard and is due to Faltings and Tsuji.

(iii) Let \( a \in \mathbb{Z} \) such that \( a \neq 1 \), \( (a, p) = 1 \). The element

\[
\gamma_{u_n} = \left( e^{-a \frac{2\pi i}{p^n}} - 1 \right) \in \mathbb{Q}(\mu_{p^n})
\]

is a cyclotomic unit in \( \mathcal{O}_{\mathbb{Q}(\mu_{p^n})} \) (whose units are called global units). Then

\[
\gamma_{u_n} \in F_n = \mathbb{Q}_p(\mu_{p^n}), \quad u_n = \frac{\gamma_{-a}(\pi_n)}{\gamma_{-1}(\pi_n)},
\]

where \( \gamma_b \in \Gamma_{\mathbb{Q}_p} \) such that \( \chi(\gamma_b) = b \). From \( N_{F_{n+1}/F_n}(\pi_{n+1}) = \pi_n \), one gets \( N_{F_{n+1}/F_n}(u_{n+1}) = u_n \) (\( \gamma \) commutes with norm), thus

\[
u = (u_n) \in \lim_{\leftarrow} \mathcal{O}_{F_n}.
\]

Obviously the Coleman power series

\[
f_u = \frac{(1 + T)^{-a} - 1}{(1 + T)^{-1} - 1}, \quad \frac{\partial f_u}{f_u} = \frac{a}{(1 + T)^a - 1} - \frac{1}{T}.
\]

So \( \frac{\partial f_u}{f_u} \) is nothing but the Amice transform of \( \mu_a \) that was used to construct \( p \)-adic zeta function. \( \Exp^* \) produces Kubota-Leopoldt zeta function from the system of cyclotomic units.
(iv) The example in (iii) is part of a big conjectural picture. For $V$ a fixed representation of $G_{\mathbb{Q}}$, then conjecturally

$$\{\text{compatible system of global elements of } V\} \longrightarrow H^1_{\text{Iw}}(\mathbb{Q}, V) \longrightarrow H^1_{\text{Iw}}(\mathbb{Q}_p, V)^{\text{Exp}^*} \longrightarrow D(V)^{\psi=1} \longrightarrow \text{Amice transform} \longrightarrow \text{p-adic } L\text{-functions}.$$ 

At present there are very few examples representation of $G_{\mathbb{Q}}$ for which this picture is known to work. The Amice transform works well for $\mathbb{Z}_p(1)$, because $\psi$ improves denominators in $\pi$, and $A^{\psi=1}_{\mathbb{Q}_p} \subset \frac{1}{\pi} A^+_\mathbb{Q}_p$ can be viewed as measures. In general, to use the properties of $\psi$, we will have to introduce overconvergent ($\varphi, \Gamma$)-modules.

### 7.5 Proof of the explicit reciprocity law

#### 7.5.1 Strategy of proof of Theorem 7.4.1

Let $u \in \varprojlim \mathcal{O}_{F_n}$, and $g \mapsto C_n(g)$ be the cocycle on $G_{F_n}$ by Kummer theory, i.e the image of $u$ under the composition of

$$\varprojlim \mathcal{O}_{F_n} - \{0\} \overset{\kappa}{\longrightarrow} H^1_{\text{Iw}}(\mathbb{Q}_p, \mathbb{Z}_p(1)) \longrightarrow H^1(G_{F_n}, \mathbb{Z}_p(1)).$$

Let $y \in D(\mathbb{Z}_p(1))^{\psi=1} = A^{\psi=1}_{\mathbb{Q}_p}(1)$, let $g \mapsto C'_n(g)$ be the image of $y$ under the composition of

$$D(\mathbb{Z}_p(1))^{\psi=1} = A^{\psi=1}_{\mathbb{Q}_p}(1) \overset{(\text{Exp}^*)^{-1}}{\longrightarrow} H^1_{\text{Iw}}(\mathbb{Q}_p, \mathbb{Z}_p(1)) \longrightarrow H^1(G_{F_n}, \mathbb{Z}_p(1)).$$

We need to prove that $C_n = C'_n$ for all $n$ implies $y = \frac{\partial f}{f_n}(\pi)$.

For $C'_n$, we have

$$C'_n(g) = \log \chi(\gamma_n) \frac{(\chi(g) - 1)}{\chi(\gamma_n) - 1} y - (\chi(g)g - 1)b_n,$$

where $b_n \in A$ is a solution of $(\varphi - 1)b_n = (\chi(\gamma_n)\gamma_n - 1)^{-1}(\varphi - 1)y$, we know that $(\varphi - 1)y \in A^{\psi=0}_{\mathbb{Q}_p}$. The exact value of $b_n$ is not important.

For $C_n$, choose $x_n = (x_n^{(0)}, \ldots, x_n^{(k)}, \ldots) \in \tilde{E}^+$ such that $x_n^{(0)} = u_n$. Let $\tilde{u}_n = [x_n]$, then

$$\frac{g(\tilde{u}_n)}{\tilde{u}_n} = [\varepsilon]C_n(g).$$
Proposition 7.5.1. Assume \( n \geq 1 \).

(i) There exists \( k \in \mathbb{Z}, b'_n \in \mathcal{O}_{C_p}/p^n \) such that

\[
p^2 C'_n(g) = p^2 \log \chi(\gamma_n) \frac{g - 1}{\gamma_n - 1} y(\pi_{n+k}) + (g - 1)b'_n \]

in \( \mathcal{O}_{C_p}/p^n \).

(ii) There exists \( k \in \mathbb{Z}, b''_n \in \mathcal{O}_{C_p}/p^n \) such that

\[
p^2 C_n(g) = p^2 \log \chi(g) \frac{\partial f_u}{f_u} y(\pi_n) + (g - 1)b''_n \]

in \( \mathcal{O}_{C_p}/p^n \).

Proposition 7.5.2. There exists a constant \( c \in \mathbb{N} \), such that for all \( n \) and for all \( k \), if \( x \in \mathcal{O}_{F_{n+k}}, b \in \mathcal{O}_{C_p} \) satisfy

\[
v_p \left( \frac{g - 1}{\gamma_n - 1} x + (g - 1)b \right) \geq n, \forall g \in G_{F_n}\]

then

\[
p^{-k} \text{Tr}_{F_{n+k}/F_n} x \in p^{n-c} \mathcal{O}_{F_n}.
\]

We shall prove Proposition 7.5.1 in the next \( n^0 \), and Proposition 7.5.2 in the third \( n^0 \). We first explain how the above two propositions imply the theorem:

If \( h(\pi) = \psi(h(\pi)) \), then \( h(\pi_n) = p^{-1} \text{Tr}_{F_{n+1}/F_n}(h(\pi_n + 1)) \). By hypothesis, \( \psi(y) = y \), we get

\[
p^{-k} \text{Tr}_{F_{n+k}/F_n}(y(\pi_{n+k})) = y(\pi_n), \forall n, \forall k \ (\ast).
\]

Let

\[
x = p^2 \log \chi(\gamma_n) \left( y(\pi_{n+k}) - \frac{\partial f_u}{f_u}(\pi_n) \right), \quad b = b'_n - b''_n.
\]

By Proposition 7.5.1, and the hypothesis \( C_n(g) = C'_n(g) \), we get

\[
\frac{g - 1}{\gamma_n - 1} x + (g - 1)b = p^2 (C'_n(g) - C_n(g)) = 0.
\]

The first equality is because for every \( x \in F_n, \frac{g - 1}{\gamma_n - 1} x = \frac{\log \chi(g)}{\log \chi(\gamma_n)} x \). Using Proposition 7.5.2, we get

\[
p^2 \frac{\log \chi(\gamma_n)}{p^n} \left( y(\pi_n) - \frac{\partial f_u}{f_u}(\pi_n) \right) \in p^{n-c} \mathcal{O}_{F_n},
\]
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then for every $n$,

$$y(\pi_n) - \frac{\partial f_u}{f_u}(\pi_n) \in p^{n-c-2}O_{F_n}.$$  

Let $h = y - \frac{\partial f_u}{f_u}$, then $\psi(h) = h$ and $h(\pi_n) \in p^{n-c-2}O_{F_n}$. Using the fact $p^{-k} \text{Tr}_{F_{n+k}/F_n}O_{F_{n+k}} \subset O_{F_n}$ and the formula (*), then for every $i \in \mathbb{N}, n \geq i$,

$$h(\pi_i) = p^{-(n-i)} \text{Tr}_{F_n/F_i}(h(\pi_n)) \in p^{n-c-2}O_{F_i},$$  

thus $h(\pi_i) = 0$ for every $i \in \mathbb{N}$, hence $h = 0$.

7.5.2 Explicit formulas for cocycles

This section is devoted to the proof of Proposition 7.5.1

(i) Recall that $\pi = [\varepsilon] - 1$, $\theta(\sum p^m x_m) = \sum p^m x_m(0)$ and $\theta(\pi) = 1 - 1 = 0$. Let $\tilde{\pi}_n = \varphi^{-n}(\pi) \in \tilde{A}^+$, then $\tilde{\pi}_n = [\varepsilon^{1/p^n}] - 1$, $\theta(\tilde{\pi}_n) = \pi_n$. Write $b_n = \sum_{l=0}^{\infty} p^l z_l$, where $z_l \in \tilde{E}$. As $C_n'(g) \in \mathbb{Z}_p$, we have

$$\varphi^{-(n+k)}C_n'(g) = C_n'(g), \quad \text{for all } n \text{ and } k.$$  

As

$$v_E(\varphi^{-k}(z_l)) = \frac{1}{p^k}v_E(z_l),$$

we can find $k$ such that

$$v_E(\varphi^{-(n+k)}(z_l)) \geq -1, \quad \text{for all } l \leq n - 1.$$  

Let $\tilde{p} = (p, \ldots) \in \tilde{E}^+$, then for every $l \leq n - 1$, $\tilde{p} \cdot \varphi^{-(n+k)}(z_l) \in \tilde{E}^+$. We have

$$[\tilde{p}]C_n'(g) = \frac{\log \chi(\gamma_n)}{p^n}[\tilde{p}] \cdot \frac{\chi(g)g - 1}{\chi(\gamma_n)\gamma_n - 1}y(\tilde{\pi}_{n+k}) + \frac{[\tilde{p}](\chi(g)g - 1)\varphi^{-(n+k)}(b_n)}{p^n}.$$  

Both sides live in $\tilde{A}^+ + p^n \tilde{A}$, reduce mod $p^n$ and use $\theta : \tilde{A}^+/p^n \to O_{C_p}/p^n$, then $[\tilde{p}] \mapsto p$ and

$$pC_n'(g) = p \frac{\log \chi(\gamma_n)}{p^n}p \cdot \frac{g - 1}{\gamma_n}y(\pi_{n+k}) + (g - 1)b_n'$$  

where $b_n' = \theta([\tilde{p}]\varphi^{-(n+k)}(b_n))$. 
(ii) Write \( u = (\pi^n_k)(v_n) \), where \( v_n \) are units in \( \mathcal{O}^*_E \). So we just have to prove the formula for \( (\pi^n_k) \) and \( (v_n) \). Thus we can assume \( v_p(u_n) \leq 1 \).

Let
\[
H : 1 + tB_{dR}^+ \rightarrow \mathbb{C}_p, \quad x \mapsto \theta\left(\frac{x-1}{\pi}\right) = \theta\left(\frac{x-1}{t}\right),
\]
recall that \( t = \log(1 + \pi) \). We have
\[
H((1 + \pi x')(1 + \pi y')) = H(1 + \pi (x' + y') + \pi^2 x'y') = \theta(x' + y') = H(1 + \pi x') + H(1 + \pi y'),
\]
thus \( H(xy) = H(x) + H(y) \).

Write \( \bar{u}_n = [(u_n, u_{n}^{\pi}, ...)] \), we have \( g(\bar{u}_n) = [\varepsilon]C_n(g) = 1 + C_n(g)\pi + \cdots \), thus
\[
C_n(g) = H\left(\frac{g(\bar{u}_n)}{\bar{u}_n}\right).
\]

We know \( u_n = f_u(\pi_n) \) and \( \theta(\bar{u}_n) = u_n \), then
\[
\theta(f_u(\pi_n)) = f_u(\theta(\pi_n)) = f_u(\pi_n) = u_n = \theta(\bar{u}_n).
\]

So, if we set \( a_n = f_u(\pi_n) \), then \( \theta(a_n) = 1 \).

We know that \( \bar{p}a_n \in \bar{A}^+ \) since \( v_p(u_n) \leq 1 \). Then we get \( H(a_n) \in \frac{1}{\pi_1} \mathcal{O}_{C_p} \) because of the following identity
\[
H(a_n) = \theta\left(\frac{[\bar{p}]a_n - [\bar{p}]}{[\bar{p}]\pi}\right) = \theta\left(\frac{[\bar{p}]a_n - [\bar{p}]}{\omega}\right) \cdot \theta\left(\frac{1}{[\bar{p}]\pi}\right),
\]
and because \( \omega = \frac{\pi_1}{\pi} \) is a generator of \( \text{Ker} \theta \) in \( \bar{A}^+ \) as \( \omega \in \text{Ker} \theta \), and
\[
\bar{\omega} = \frac{\varepsilon - 1}{\varepsilon^{1/p} - 1}, \quad \text{so} \quad v_E(\bar{\omega}) = (1 - \frac{1}{p})v_E(\varepsilon - 1) = 1.
\]

Then we have
\[
g\left(\frac{f_u(\pi_n)}{f_u(\pi_n)}\right) = \frac{f_u((1 + \pi_n)^{x(g)} - 1)}{f_u(\pi_n)}
\[
= \frac{f_u((1 + \pi_n)(1 + \pi) \frac{x(g)-1}{p^n} - 1)}{f_u(\pi_n)}
\[
= 1 + \frac{\partial f_u(\pi_n)}{f_u(\pi_n)} \cdot \frac{x(g)-1}{p^n} + \text{terms of higher degree in } \pi,
\]

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hence

\[ H\left(\frac{g(f_u(\pi_n))}{f_u(\pi_n)}\right) = \frac{\chi(g) - 1}{p^n} \cdot \frac{\partial f_u(\pi_n)}{f_u(\pi_n)}. \]

Using formula \( f_u(\pi_n) = \tilde{u}_n a_n \), we get

\[ C_n(g) = H\left(\frac{g(\tilde{u}_n)}{\tilde{u}_n}\right) = H\left(\frac{g(f_u(\pi_n))}{f_u(\pi_n)}\right) \]
\[ = \frac{\chi(g) - 1}{p^n} \cdot \frac{\partial f_u(\pi_n)}{f_u(\pi_n)} - (\chi(g) - 1)H(a_n). \]

We conclude the proof by multiplying \( p^2 \), noticing that \( \chi(g) = 1 \mod p^n \), so

\[ \frac{\chi(g) - 1}{p^n} = \exp(\log \chi(g)) - 1 \mod p^n; \]

set \( b''_n = -p^2 H(a_n) \), we get the result.

7.5.3 Tate’s normalized trace maps

Let \( \pi_n = \varepsilon^{(n)} - 1 \), \( F_n = \mathbb{Q}_p(\pi_n) \), \( F_\infty = \bigcup_n F_n \).

Lemma 7.5.3. If \( n \geq 1 \), \( x \in F_\infty \), then \( p^{-k} \text{Tr}_{F_{n+k}/F_n} x \) does not depend on \( k \) such that \( x \in F_{n+k} \).

Proof. Use the transitive properties of the trace map and the fact \([F_{n+k} : F_n] = p^k\). \( \square \)

Let \( R_n : F_\infty \to F_n \) be the above map. Denote

\[ Y_i = \{ x \in F_i, \text{Tr}_{F_i/F_{i-1}} x = 0 \}. \]

Lemma 7.5.4. (i) \( R_n \) commutes with \( \Gamma_{\mathbb{Q}_p} \), is \( F_n \) linear and \( R_n \circ R_{n+k} = R_n \).

(ii) Let \( x \in F_\infty \), then \( x = R_n(x) + \sum_{i=1}^{\infty} R_{n+i}^* (x) \), where \( R_{n+i}^*(x) = R_{n+i}(x) - R_{n+i-1}(x) \in Y_{n+i} \) and is 0 if \( i \gg 0 \).

(iii) Let \( k \in \mathbb{Z} \), then \( v_p(x) \geq kv_p(\pi_n) \) if and only if \( v_p(R_n(x)) \geq kv_p(\pi_n) \) and \( v_p(R_{n+i}^*(x)) \geq kv_p(\pi_n) \) for every \( i \in \mathbb{N} \).
Proof. (i) is obvious.

(ii) is also obvious, since $R_{n+i-1}(R^*_n(x)) = 0 \Rightarrow R^*_n(x) \in Y_{n+i}$.

(iii) $\Leftarrow$ is obvious. For $\Rightarrow$, let $x \in \mathcal{O}_{F_{n+k}}$, then

$$x = \sum_{j=0}^{p^{k-1}} a_j (1 + \pi_{n+k})^j, \quad a_j \in \mathcal{O}_{F_n}.$$ 

Write $j = p^{j-1}j'$ with $(j',p) = 1$, then

$$R_n(x) = a_0, \quad R^*_n(x) = \sum_{(j',p)=1} a_{p^{j-1}j'} (1 + \pi_{n+i})^{j'}$$

since

$$p^{-1} \text{Tr}_{F_{n+i}/F_{n+i-1}}(1 + \pi_{n+i})^j = \begin{cases} 
(1 + \pi_{n+i})^j, & \text{if } p \mid j \\
0, & \text{if } (j,p) = 1.
\end{cases}$$

Thus

$$v_p(x) \geq 0 \Rightarrow v_p(R_n(x)) \geq 0 \text{ and } v_p(R^*_n(x)) \geq 0.$$ 

By $F_n$-linearity we get the result. \hfill \square

Remark. In the whole theory, the following objects play similar roles:

$$\psi \longleftrightarrow p^{-1} \text{Tr}_{F_{n+i}/F_n}$$

$$\psi = 0 \longleftrightarrow Y_i.$$ 

Lemma 7.5.5. Assume that $j \leq i - 1$ and $j \geq 2$. and assume $\gamma_j$ is a generator of $\Gamma_j$. Let $u \in \mathbb{Q}_p^*$. If $v_p(u - 1) > v_p(\pi_1)$, then $u\gamma_j - 1$ is invertible on $Y_i$. Moreover if $x \in Y_i$, $v_p(x) \geq kv_p(\pi_n)$, then $v_p((u\gamma_j^{-1}x) \geq kv_p(\pi_n) - v_p(\pi_1)$.

Proof. If $\gamma_{i-1} = \gamma_j^{p^{i-j-1}}$, then

$$(u\gamma_j^{-1})^{-1} = (u^{p^{i-j-1}\gamma_{i-1}} - 1)^{-1}(1 + u\gamma_j + \cdots + (u\gamma_j)^{p^{i-j-1}-1}),$$

so it is enough to treat the case $j = i - 1$.

Let $x \in \mathcal{O}_{F_1} \cap Y_i$, write

$$x = \sum_{a=1}^{p-1} x_a (1 + \pi_i)^a, \quad x_a \in \mathcal{O}_{F_{i-1}},$$

where $x_a = \sum_{j=0}^{p^{a-1}} a_j (1 + \pi_{n+a})^j$, $a_j \in \mathcal{O}_{F_n}$.
write $\chi(\gamma_{i-1}) = 1 + p^{i-1}v$ with $v \in \mathbb{Z}_p^*$, then

$$(u\gamma_{i-1} - 1)x = \sum_{a=1}^{p-1} x_a (1 + \pi_i)^a (u(1 + \pi_1)^{av} - 1).$$

We can check directly

$$(u\gamma_{i-1} - 1)^{-1}x = \sum_{a=1}^{p-1} \frac{x_a}{(u(1 + \pi_1)^{av} - 1)} (1 + \pi_i)^a.$$ 

Moreover, if $v_p(x) \geq 0$, then $v_p((u\gamma_{i-1} - 1)^{-1}x) \geq -v_p(\pi_1)$. \hfill $\Box$

**Proposition 7.5.6.** Assume $n \geq 1$, $u \in \mathbb{Q}_p^*$ and $v_p(u - 1) > v_p(\pi_1)$, then

(i) $x \in F_\infty$ can be written uniquely as $x = R_n(x) + (u\gamma_n - 1)y$ with $R_n(y) = 0$, and we have

$$v_p(R_n(x)) > v_p(x) - v_p(\pi_n), \quad v_p(y) > v_p(x) - v_p(\pi_n) - v_p(\pi_1).$$

(ii) $R_n$ extends by continuity to $\hat{F}_\infty$, and let $X_n = \{ x \in \hat{F}_\infty, R_n(x) = 0 \}$. Then every $x \in \hat{F}_\infty$ can be written uniquely as $x = R_n(x) + (u\gamma_n - 1)y$ with $y \in X_n$ and $R_n(x) \in F_n$, and with the same inequalities

$$v_p(R_n(x)) \geq v_p(x) - v_p(\pi_n), \quad v_p(y) \geq v_p(x) - v_p(\pi_n) - v_p(\pi_1).$$

**Proof.** (i) As

$$x = R_n(x) + \sum_{i=1}^{+\infty} (u\gamma_n - 1)((u\gamma_n - 1)^{-1}R_{n+i}^*(x)).$$

we just let $y = \sum_{i=1}^{+\infty} (u\gamma_n - 1)^{-1}R_{n+i}^*(x)$.

(ii) By (i), we have $v_p(R_n(x)) \geq v_p(x) - C$, so $R_n$ extends by continuity to $\hat{F}_\infty$; the rest follows by continuity. \hfill $\Box$

**Remark.** (i) The maps $R_n : \hat{F}_\infty \longrightarrow F_n$ are Tate’s normalized trace maps.

(ii) they commutes with $\Gamma_{\mathbb{Q}_p}$ (or $G_{\mathbb{Q}_p}$).

(iii) $R_n(x) = x$ if $x \in F_\infty$ and $n \gg 0$, hence $R_n(x) \rightarrow x$ if $x \in \hat{F}_\infty$ and $n \rightarrow \infty$. 

7.5.4 Applications to Galois cohomology

**Proposition 7.5.7.** (i) The map

\[ x \in F_n \mapsto (\gamma \mapsto x \log \chi(\gamma)) \in H^1(\Gamma_{F_n}, F_n) \]

induces isomorphism

\[ F_n \sim H^1(\Gamma_{F_n}, F_n) \sim H^1(\hat{\Gamma}_{F_n}, \hat{F}_\infty). \]

(ii) If \( \eta : \Gamma_{F_n} \to \mathbb{Q}_p^* \) is of infinite order, then \( H^1(\Gamma_{F_n}, \hat{F}_\infty(\eta)) = 0 \).

**Proof.** If \( n \gg 0 \) so that \( v_p(\eta(\gamma_n) - 1) > v_p(\pi_1) \). Using the above proposition (let \( u = \eta(\gamma_n) \)), we get

\[ H^1(\Gamma_{F_n}, \hat{F}_\infty(\eta)) = \frac{\hat{F}_\infty}{(u\gamma_n - 1)} = \frac{F_n \bigoplus X_n}{(u\gamma_n - 1)} = \frac{F_n}{u\gamma_n - 1}. \]

If \( u = 1 \), we get \( (\gamma_n - 1)F_n = 0 \). If \( u \neq 1 \), we get \( F_n/(u - 1)F_n = 0 \).

For \( n \) small, using inflation and restriction sequence, as \( \text{Gal}(F_{n+k}/F_n) \) is finite, and \( \hat{F}_\infty(\eta) \) is a \( \mathbb{Q}_p \)-vector space, we have

\[ H^1(\Gamma_{F_n}, \hat{F}_\infty(\eta)) = 0, \quad H^2 = 0, \]

then we get an isomorphism

\[ H^1(\Gamma_{F_n}, \hat{F}_\infty(\eta)) \sim H^1(\Gamma_{F_{n+k}}, \hat{F}_\infty(\eta))^{\text{Gal}(F_{n+k}/F_n)}. \]

From the case of \( n \gg 0 \), we immediately get the result. \( \square \)

Recall that the following result is the main step in Ax’s proof of the Ax-Sen-Tate theorem (cf. Fontaine’s course).

**Proposition 7.5.8.** There exists a constant \( C \in \mathbb{N} \), such that if \( x \in \mathbb{C}_p \), if \( H \subset G_{\mathbb{Q}_p} \) is a closed subgroup, if for all \( g \in H \), \( v_p((g - 1)x) \geq a \) for some \( a \), then there exists \( y \in \mathbb{C}_p^H \) such that \( v_p(x - y) \geq a - C \).

The following corollary is Proposition 7.5.2 in the previous section.

**Corollary 7.5.9.** For \( x \in \mathcal{O}_{\hat{F}_\infty} \), if there exists \( c \in \mathcal{O}_{\mathbb{C}_p} \) such that

\[ v_p\left( \frac{g - 1}{\gamma_n - 1} x - (g - 1)c \right) \geq n, \quad \text{for all} \quad g \in G_{F_n}. \]

Then we have

\[ v_p(R_n(x)) \geq n - C - 1(\text{or } 2). \]
Proof. By assumption, we get
\[ v_p((g-1)c) \geq n, \ \forall \ g \in H_{Q_p} = \text{Ker} \chi, \]
then by Ax, there exists \( c' \in \hat{F}_\infty \) such that
\[ v_p(c - c') \geq n - C. \]
Take \( g = \gamma_n \), then \( v_p(x - (\gamma_n - 1)c') \geq n - C. \) As \( R_n\gamma_n = \gamma_n R_n = R_n \), we get
\[ v_p(R_n(x)) = v_p(R_n(x - (\gamma_n - 1)c')) \geq n - C - v_p(\pi_1) - v_p(\pi_n), \]
hence the result. \( \square \)

7.5.5 No 2\pi i in \( C_p \! \)!

Theorem 7.5.10. (i) \( C_p \) does not contain \( \log 2\pi i \), i.e. there exists no \( x \in C_p \) satisfies that \( g(x) = x + \log \chi(g) \) for all \( g \in G_K \) where \( K \) is a finite extension of \( Q_p \).

(ii) \( C_p(k) = 0 \), if \( k \neq 0 \).

Proof. (i) If \( K = Q_p \), if there exists such an \( x \), by Ax-Sen-Tate, we get \( x \in \hat{F}_\infty = C_p^{H_{Q_p}} \). Then we have:
\[ R_n(g(x)) = g(R_n(x)) = R_n(x) + \log \chi(g). \]
Because \( R_n(x) \in F_n \), it has only finite number of conjugates but \( \log \chi(g) \) has infinitely many values, contradiction!

Now for \( K \) general, we can assume \( K/Q_p \) is Galois, let
\[ y = \frac{1}{[K : Q_p]} \sum_{\sigma \in S} \sigma(x) \]
where \( S \) are representatives of \( G_{Q_p}/G_K \). For \( g \in G_{Q_p} \), we can write \( g\sigma = \sigma'_\sigma h_\sigma \) for \( h_\sigma \in G_K \) and \( \sigma'_\sigma \in S \). From this we get
\[ \sum_{\sigma \in S} \log \chi(h_\sigma) = [K : Q_p] \log \chi(g). \]
Then we have
\[ g(y) = \frac{1}{[K : \mathbb{Q}_p]} \sum_{\sigma \in S} g(\sigma(x)) = \frac{1}{[K : \mathbb{Q}_p]} \sum_{\sigma \in S} \sigma' \delta_{\sigma} x \]
\[ = \frac{1}{[K : \mathbb{Q}_p]} \sum_{\sigma \in S} \sigma'(x + \log \chi(h_{\sigma})) \]
\[ = \frac{1}{[K : \mathbb{Q}_p]} \sum_{\sigma \in S} \sigma(x) + \frac{1}{[K : \mathbb{Q}_p]} \sum_{\sigma \in S} \log \chi(h_{\sigma}) \]
\[ = y + \log \chi(g). \]

Then by the case \( K = \mathbb{Q}_p \), we get the result.

(ii) If \( 0 \neq x \in \mathbb{C}_p(k) \), then \( g(x) = \chi(g)^{-k}x \). Let \( y = \frac{\log x}{-k} \), then we have \( g(y) = y + \log \chi(g) \), which is a contradiction by (i). \( \square \)
Chapter 8

$(\varphi, \Gamma)$-modules and $p$-adic $L$-functions

8.1 Tate-Sen’s conditions

8.1.1 The conditions (TS1), (TS2) and (TS3)

Let $G_0$ be a profinite group and $\chi : G_0 \to \mathbb{Z}_p^*$ be a continuous group homomorphism with open image. Set $v(g) = v_p(\log \chi(g))$ and $H_0 = \text{Ker} \chi$.

Suppose $\tilde{\Lambda}$ is a $\mathbb{Z}_p$-algebra and

$$v : \tilde{\Lambda} \longrightarrow \mathbb{R} \cup \{+\infty\}$$

satisfies the following conditions:

(i) $v(x) = +\infty$ if and only if $x = 0$;
(ii) $v(xy) \geq v(x) + v(y)$;
(iii) $v(x + y) \geq \inf(v(x), v(y))$;
(iv) $v(p) > 0, \ v(px) = v(p) + v(x)$.

Assume $\tilde{\Lambda}$ is complete for $v$, and $G_0$ acts continuously on $\tilde{\Lambda}$ such that $v(g(x)) = v(x)$ for all $g \in G_0$ and $x \in \tilde{\Lambda}$.

Definition 8.1.1. The Tate-Sen’s conditions for the quadruple $(G_0, \chi, \tilde{\Lambda}, v)$ are the following three conditions TS1–TS3.

(TS1). For all $C_1 > 0$, for all $H_1 \subset H_2 \subset H_0$ open subgroups, there exists an $\alpha \in \tilde{\Lambda}^{H_1}$ with

$$v(\alpha) > -C_1 \text{ and } \sum_{\tau \in H_2/H_1} \tau(\alpha) = 1.$$
(In Faltings' terminology, \( \tilde{\Lambda}/\tilde{\Lambda}^{H_0} \) is called \textit{almost étale}.)

(TS2). \textit{Tate's normalized trace maps}: there exists \( C_2 > 0 \) such that for all open subgroups \( H \subset H_0 \), there exist \( n(H) \in \mathbb{N} \) and \( (\Lambda_{H,n})_{n \geq n(H)} \), an increasing sequence of sub \( \mathbb{Z}_p \)-algebras of \( \tilde{\Lambda}^H \) and maps

\[
R_{H,n} : \tilde{\Lambda}^H \to \Lambda_{H,n}
\]
satisfying the following conditions:

(a) if \( H_1 \subset H_2 \), then \( \Lambda_{H_2,n} = (\Lambda_{H_1,n})^{H_2} \), and \( R_{H_1,n} = R_{H_2,n} \) on \( \tilde{\Lambda}^{H_2} \);

(b) for all \( g \in G_0 \),

\[
g(\Lambda_{H,n}) = \Lambda_{gHg^{-1},n} \quad g \circ R_{H,n} = R_{gHg^{-1},n} \circ g;
\]

(c) \( R_{H,n} \) is \( \Lambda_{H,n} \)-linear and is equal to \( \text{Id} \) on \( \Lambda_{H,n} \);

(d) \( v(R_{H,n}(x)) \geq v(x) - C_2 \) if \( n \geq n(H) \) and \( x \in \tilde{\Lambda}^H \);

(e) \( \lim_{n \to +\infty} R_{H,n}(x) = x \).

(TS3). There exists \( C_3 \), such that for all open subgroups \( G \subset G_0 \), \( H = G \cap H_0 \), there exists \( n(G) \geq n(H) \) such that if \( n \geq n(G) \), \( \gamma \in G/H \) and \( v(\gamma) = v_p(\log \chi(\gamma)) \leq n \), then \( \gamma - 1 \) is invertible on \( X_{H,n} = (R_{H,n} - 1)\tilde{\Lambda} \) and

\[
v((\gamma - 1)^{-1}x) \geq v(x) - C_3
\]
for \( x \in X_{H,n} \).

\textbf{Remark}. \( R_{H,n} \circ R_{H,n} = R_{H,n} \), so \( \tilde{\Lambda}^H = \Lambda_{H,n} \oplus X_{H,n} \).

\textbf{8.1.2 Example : the field} \( \mathbb{C}_p \)

\textbf{Theorem 8.1.2}. The quadruple \( (\tilde{\Lambda} = \mathbb{C}_p, v = v_p, G_0 = G_{\mathbb{Q}_p} \) and \( \chi=\text{the cyclotomic character}) \) satisfies (TS1), (TS2), (TS3).

\textit{Proof}. (TS1): In Fontaine's course, we know that for any \( \mathbb{Q}_p \subset K \subset L \) such that \( [L : \mathbb{Q}_p] < +\infty \), then

\[
v_p(\mathfrak{a}_{L_n/K_n}) \to 0 \quad \text{as} \quad n \to +\infty.
\]

The proof showed that \( v_p(\gamma(\pi_n) - \pi_n) \to 0 \) as \( n \to +\infty \), where \( \pi_n \) is a uniformizer of \( L_n \) and \( \gamma \in \text{Gal}(L_n/K_n) = \text{Gal}(L_{\infty}/K_{\infty}) \) when \( n \gg 0 \). We also have

\[
\text{Tr}_{L_{\infty}/K_{\infty}} = \text{Tr}_{L_n/K_n}
\]
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on $L_n$ if $n \gg 0$ and

$$\text{Tr}_{L_n/K_n}(\mathcal{O}_{L_n}) \supset \mathfrak{o}_{L_n/K_n} \bigcap \mathcal{O}_{K_n},$$

thus $\text{Tr}_{L_\infty/K_\infty}(\mathcal{O}_{L_\infty})$ contains elements with $v_p$ as small as we want. Take $x \in \mathcal{O}_{L_\infty}$ and let $\alpha = \frac{x}{\text{Tr}_{L_\infty/K_\infty}(x)}$, then

$$\sum_{\tau \in H_{K/K_L}} \tau(\alpha) = \text{Tr}_{L_\infty/K_\infty}(\alpha) = 1.$$ 

Then for all $C_1 > 0$, we can find $x \in \mathcal{O}_{L_\infty}$ such that $v_p(\text{Tr}_{L_\infty/K_\infty}(x))$ is small enough, thus $v_p(\alpha) > -C_1$.

(TS2) and (TS3): By Ax-Sen-Tate, $\mathbb{C}_p^{H_K} = \hat{K}_\infty$, let $\Lambda_{H_K,n} = K_n$, and $R_{H_K,n} = p^{-k} \text{Tr}_{K_{n+k}/K_n}$ on $K_{n+k}$.

If $K = \mathbb{Q}_p$, $R_{H_K,n} = R_n$, that’s what we did in last chapter. We are going to use what we know about $R_n$.

For $G = G_K$, then $H = H_K$, choose $m$ big enough such that for any $n \geq m$, $v_p(\mathfrak{o}_{K_n/F_n})$ is small and $[K_\infty : F_\infty] = [K_n : F_n] = d$. Let $\{e_1, \ldots, e_d\}$ be a basis of $\mathcal{O}_{K_n}$ over $\mathcal{O}_{F_n}$ and $\{e_1^*, \ldots, e_d^*\}$ be the dual basis of $K_n$ over $F_n$ for the trace map $(x, y) \mapsto \text{Tr}_{K_n/F_n}(xy)$. This implies that $\{e_1^*, \ldots, e_d^*\}$ is a basis of $\mathfrak{o}_{K_n/F_n}^{-1}$ and $v_p(e_i^*) > v_p(\mathfrak{o}_{K_n/F_n})$ are small. Any $x \in K_\infty$ can be written as

$$x = \sum_{i=1}^d \text{Tr}_{K_\infty/K}(xe_i)e_i^*,$$

then

$$\inf_i v_p(\text{Tr}_{K_\infty/F_\infty}(xe_i)) \geq v_p(x) \geq \inf_i v_p(\text{Tr}_{K_\infty/F_\infty}(xe_i)) - v_p(\mathfrak{o}_{K_n/F_n}),$$

and

$$R_{H_K,n}(x) = \sum_{i=1}^d R_n(\text{Tr}_{K_\infty/F_\infty}(xe_i))e_i^*, \quad n \geq m.$$ 

So use what we know about $R_n$ to conclude. \qed

Remark. By the same method as Corollary 7.5.7, we get

(i) $H^1(\Gamma, \hat{K}_\infty) \cong K$, where the isomorphism is given by $x \in K \mapsto (\gamma \mapsto x \log \chi(\gamma))$.

(ii) $H^1(\Gamma, \hat{K}_\infty(\eta)) = 0$ if $\eta$ is of infinite order.
8.2 Sen’s method

Proposition 8.2.1. Assume $\tilde{\Lambda}$ verifying (TS1), (TS2) and (TS3). Let $\sigma \mapsto U_\sigma$ be a continuous cocycle from $G_0$ to $\mathrm{GL}_d(\tilde{\Lambda})$. If $G \subset G_0$ is an open normal subgroup of $G_0$ such that $v(U_\sigma - 1) > 2C_2 + 2C_3$ for any $\sigma \in G$. Set $H = G \cap H_0$, then there exists $M \in \mathrm{GL}_d(\tilde{\Lambda})$ with $v(M - 1) > C_2 + C_3$ such that

$$\sigma \mapsto V_\sigma = M^{-1}U_\sigma\sigma(M)$$

satisfies $V_\sigma \in \mathrm{GL}_d(\Lambda_{H,n}(G))$ and $V_\sigma = 1$ if $\sigma \in H$.

Example 8.2.2. Example of Sen: For the case $\tilde{\Lambda} = \mathbb{C}_p$, for $U_\sigma$ a 1-cocycle on $G_K$ with values in $\mathrm{GL}_d(\mathbb{C}_p)$, there exists $[L : K] < \infty$, such that $U_\sigma$ is cohomologous to a cocycle that which is trivial on $H_L$ and with values in $\mathrm{GL}_d(L_n)$ for some $n$.

The proof of Proposition 8.2.1 needs three Lemmas below. It is technical, but the techniques come over again and again.

8.2.1 Almost étale descent

Lemma 8.2.3. If $\tilde{\Lambda}$ satisfies (TS1), $a > 0$, and $\sigma \mapsto U_\sigma$ is a 1-cocycle on $H$ open in $H_0$ and

$$v(U_\sigma - 1) \geq a$$

for any $\sigma \in H$,

then there exists $M \in \mathrm{GL}_d(\tilde{\Lambda})$ such that

$$v(M - 1) \geq \frac{a}{2}, \quad v(M^{-1}U_\sigma\sigma(M) - 1) \geq a + 1.$$

Proof. The proof is approximating Hilbert’s Theorem 90.

Fix $H_1 \subset H$ open and normal such that $v(U_\sigma - 1) \geq a + 1 + a/2$ for $\sigma \in H_1$, which is possible by continuity. Because $\tilde{\Lambda}$ satisfies (TS1), we can find $\alpha \in \tilde{\Lambda}^{H_1}$ such that

$$v(\alpha) \geq -a/2, \quad \sum_{\tau \in H/H_1} \tau(\alpha) = 1.$$

Let $S \subset H$ be a set of representatives of $H/H_1$, denote $M_S = \sum_{\sigma \in S} \sigma(\alpha)U_\sigma$, we have $M_S - 1 = \sum_{\sigma \in S} \sigma(\alpha)(U_\sigma - 1)$, this implies $v(M_S - 1) \geq a/2$ and moreover

$$M_S^{-1} = \sum_{n=0}^{+\infty} (1 - M_S)^n,$$
so we have \( v(M^{-1}_S) \geq 0 \) and \( M_S \in \text{GL}_d(\bar{\Lambda}) \).

If \( \tau \in H_1 \), then \( U_{\sigma} - U_{\sigma}U(\sigma(U_\tau) - 1) \). Let \( S' \subset H \) be another set of representatives of \( H/H_1 \), so for any \( \sigma' \in S' \), there exists \( \tau \in H_1 \) and \( \sigma \in S \) such that \( \sigma' = \sigma \tau \), so we get

\[
M_S - M_{S'} = \sum_{\sigma \in S} \sigma(\alpha)(U_{\sigma} - U_{\sigma \tau}) = \sum_{\sigma \in S} \sigma(\alpha)U_{\sigma}(1 - \sigma(U_\tau)),
\]

thus

\[
v(M_S - M_{S'}) \geq a + 1 + a/2 - a/2 = a + 1.
\]

For any \( \tau \in H \),

\[
U_{\tau}(M_S) = \sum_{\sigma \in S} \sigma(\alpha)U_{\tau}(U_{\sigma}) = M_{\tau S}.
\]

Then

\[
M_{\tau S} = \sum_{\sigma \in S} \sigma(\alpha)U_{\tau}(U_{\sigma}) = M_{\tau S},
\]

with \( v(M_S^{-1}(M_{\tau S} - M_S)) \geq a + 1 \). Take \( M = M_S \) for any \( S \), we get the result.

**Corollary 8.2.4.** Under the same hypotheses as the above lemma, there exists \( M \in \text{GL}_d(\bar{\Lambda}) \) such that

\[
v(M - 1) \geq a/2, \quad M^{-1}U_{\sigma} \sigma(M) = 1, \forall \sigma \in H.
\]

**Proof.** Repeat the lemma \( (a \mapsto a + 1 \mapsto a + 2 \mapsto \cdots) \), and take the limits. \( \square \)

**Exercise.** Assume \( \bar{\Lambda} \) satisfies (TS1), denote by \( \bar{\Lambda}^+ = \{ x \in \bar{\Lambda} | v(x) \geq 0 \} \). Let \( M \) be a finitely generated \( \bar{\Lambda}^+ \)-module with semi-linear action of \( H \), an open subgroup of \( H_0 \). Then \( H^i(H, M) \) is killed by any \( x \in \bar{\Lambda}^H \) with \( v(x) > 0 \).

**Hint:** Adapt the proof that if \( L/K \) is finite Galois and \( M \) is a \( L \)-module with semi-linear action of \( \text{Gal}(L/K) \), then \( H^i(\text{Gal}(L/K), L) = 0 \) for all \( i \geq 1 \). Let \( \alpha \in L \) such that \( \text{Tr}_{L/K}(\alpha) = 1 \). For any \( c(g_1, \cdots, g_n) \) an \( n \)-cocycle, let

\[
c'(g_1, \cdots, g_{n-1}) = \sum_{h \in \text{Gal}(L/K)} g_1 \cdots g_{n-1} h(\alpha)c(g_1, \cdots, g_{n-1}, h),
\]

then \( dc' = c \).
CHAPTER 8. \((\varphi, \Gamma)\)-MODULES AND P-ADIC L-FUNCTIONS

Theorem 8.2.5.  
(i) The map \(x \mapsto (g \mapsto x \log \chi(g))\) gives an isomorphism \(K \cong H^1(G_K, \mathbb{C}_p)\).

(ii) If \(\eta : G_K \to \Gamma_K \to \mathbb{Q}_p^*\) is of infinite order, then \(H^1(G_K, \mathbb{C}_p(\eta)) = 0\).

Proof. Using the inflation and restriction exact sequence

\[
0 \longrightarrow H^1(\Gamma_K, \mathbb{C}_p(\eta))^{H_K} \overset{\text{inf}}{\longrightarrow} H^1(G_K, \mathbb{C}_p(\eta)) \overset{\text{res}}{\longrightarrow} H^1(H_K, \mathbb{C}_p(\eta))^{\Gamma_K}.
\]

by the above exercise, \(H^1(H_K, \mathbb{C}_p(\eta))^{\Gamma_K} = 0\), then the inflation map is actually an isomorphism. We have \(\mathbb{C}_p(\eta)^{H_K} = \widehat{K}_\infty(\eta)\), and use Corollary 7.5.7.

In fact \(K = H^1(\Gamma_K, \widehat{K}_\infty) = H^1(\Gamma_K, K) = \text{Hom}(\Gamma, K) = K \cdot \log \chi\),

the last equality is because \(\Gamma_K\) is pro-cyclic. \(\square\)

8.2.2 Decompletion

Now recall that we have the continuous character: \(G_0 \xrightarrow{\chi} \mathbb{Z}_p^*, H_0 = \text{Ker} \chi\). \(\tilde{\Lambda}\) is complete for \(v\), with continuous action of \(G_0\). \(H\) is an open subgroup of \(H_0\), and we have the maps: \(R_{H,n} : \tilde{\Lambda}^H \to \Lambda_{H,n}\). By (TS2), \(v(R_{H,n}(x)) \geq v(x) - C_2\); and by (TS3), \(v((\gamma - 1)^{-1}x) \geq v(x) - C_3\), if \(R_{H,n}(x) = 0\) and \(v_p(\log \chi(\gamma)) \leq n\).

We can use these properties to reduce to something reasonable.

Lemma 8.2.6. Assume given \(\delta > 0\), \(b \geq 2C_2 + 2C_3 + \delta\), and \(H \subset H_0\) is open. Suppose \(n \geq n(H)\), \(\gamma \in G/H\) with \(n(\gamma) \leq n\), \(U = 1 + U_1 + U_2\) with

\[
U_1 \in M_d(\Lambda_{H,n}), \ v(U_1) \geq b - C_2 - C_3 \\
U_2 \in M_d(\tilde{\Lambda}^H), \ v(U_2) \geq b.
\]

Then, there exists \(M \in GL_d(\tilde{\Lambda}^H), v(M - 1) \geq b - C_2 - C_3\) such that

\[
M^{-1}U \gamma(M) = 1 + V_1 + V_2,
\]

with

\[
V_1 \in M_d(\Lambda_{H,n}), \ v(V_1) \geq b - C_2 - C_3), \\
V_2 \in M_d(\tilde{\Lambda}^H), \ v(V_2) \geq b + \delta.
\]
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Proof. Using (TS2) and (TS3), one gets $U_2 = R_{H,n}(U_2) + (1 - \gamma)V$, with

$$v(R_{H,n}(U_2)) \geq v(U_2) - C_2, \quad v(V) \geq v(U_2) - C_2 - C_3.$$  

Thus,

$$(1 + V)^{-1} U_\gamma (1 + V) = (1 - V + V^2 - \cdots)(1 + U_1 + U_2)(1 + \gamma(V))$$

$$= 1 + U_1 + (\gamma - 1)V + U_2 + (\text{terms of degree } \geq 2)$$

Let $V_1 = U_1 + R_{H,n}(U_2) \in M_d(\Lambda_{H,n})$ and $W$ be the terms of degree $\geq 2$. Thus $v(W) \geq 2(b - C_2 - C_3) \geq b + \delta$. So we can take $M = 1 + V$, $V_2 = W$. □

Corollary 8.2.7. Keep the same hypotheses as in Lemma 8.2.6. Then there exists $M \in \text{GL}_d(\tilde{\Lambda}^H)$, $v(M - 1) \geq b - C_2 - C_3$ such that $M^{-1} U_\gamma (M) \in \text{GL}_d(\Lambda_{H,n})$.

Proof. Repeat the lemma ($b \mapsto b + \delta \mapsto b + 2\delta \mapsto \cdots$), and take the limit. □

Lemma 8.2.8. Suppose $H \subset H_0$ is an open subgroup, $i \geq n(H)$, $\gamma \in G/H$, $n(\gamma) \geq i$ and $B \in \text{GL}_d(\tilde{\Lambda}^H)$. If there exist $V_1, V_2 \in \text{GL}_d(\Lambda_{H,i})$ such that

$$v(V_1 - 1) > C_3, \quad v(V_2 - 1) > C_3, \quad \gamma(B) = V_1 BV_2,$$

then $B \in \text{GL}_d(\Lambda_{H,i})$.

Proof. Take $C = B - R_{H,i}(B)$. We have to prove $C = 0$. Note that $C$ has coefficients in $X_{H,i} = (1 - R_{H,i})\tilde{\Lambda}^H$, and $R_{H,i}$ is $\Lambda_{H,i}$-linear and commutes with $\gamma$. Thus,

$$\gamma(C) - C = V_1 CV_2 - C = (V_1 - 1)CV_2 + V_1 C(V_2 - 1) - (V_1 - 1)C(V_2 - 1)$$

Hence, $v(\gamma(C) - C) > v(C) + C_3$. By (TS3), this implies $v(C) = +\infty$, i.e. $C = 0$. □

Proof of Proposition 8.2.1. Let $\sigma \mapsto U_\sigma$ be a continuous 1-cocycle on $G_0$ with values in $\text{GL}_d(\tilde{\Lambda})$. Choose an open normal subgroup $G$ of $G_0$ such that

$$\inf_{\sigma \in G} v(U_\sigma - 1) > 2(C_2 + C_3).$$

By Lemma 8.2.3, there exists $M_1 \in \text{GL}_d(\tilde{\Lambda})$, $v(M_1 - 1) > 2(C_2 + C_3)$ such that $\sigma \mapsto U'_\sigma = M_1^{-1}U_\sigma \sigma(M_1)$ is trivial in $H = G \cap H_0$ (In particular, it has values in $\text{GL}_d(\tilde{\Lambda}^H)$).
Now we pick $\gamma \in G/H$ with $n(\gamma) = n(G)$. In particular, we want $n(G)$ big enough so that $\gamma$ is in the center of $G_0/H$. Indeed, the center is open, since in the exact sequence:

$$1 \to H_0/H \to G_0/H \to G/H \to 1,$$

$G/H \simeq \mathbb{Z}_p \times \text{finite}$, and $H_0/H$ is finite. So we are able to choose such a $n(G)$.

Then we have $v(U'_\gamma) > 2(C_2 + C_3)$, and by Corollary 8.2.7, there exists $M_2 \in \text{GL}_d(\tilde{\Lambda}^H)$ satisfying

$$v(M_2 - 1) > C_2 + C_3 \text{ and } M_2^{-1}U'_\gamma \gamma(M_2) \in \text{GL}_d(\Lambda_{H,n(G)}).$$

Take $M = M_1 \cdot M_2$, then the cocycle

$$\sigma \mapsto V_{\sigma} = M^{-1}U_{\sigma}\sigma(M)$$

a cocycle trivial on $H$ with values in $\text{GL}_d(\tilde{\Lambda}^H)$, and we have

$$v(V_{\gamma} - 1) > C_2 + C_3 \text{ and } V_{\gamma} \in \text{GL}_d(\Lambda_{H,n(G)}).$$

This implies $V_{\sigma}$ comes by inflation from a cocycle on $G_0/H$.

The last thing we want to prove is $V_\tau \in \text{GL}_d(\Lambda_{H,n(G)})$ for any $\tau \in G_0/H$. Note that $\gamma \tau = \tau \gamma$ as $\gamma$ is in the center, so

$$V_\tau \tau' V_\gamma = V_{\tau' \gamma} = V_{\tau' \gamma}(V_\gamma)$$

which implies $\gamma(V_\gamma) = V_{\gamma}^{-1}V_\tau V_\gamma(V_\gamma)$. Apply Lemma 8.2.8 with $V_1 = V_{\gamma}^{-1}, V_2 = \tau(V_{\gamma})$, then we obtain what we want. \hfill \square

### 8.2.3 Applications to $p$-adic representations

**Proposition 8.2.9.** Let $T$ be a free $\mathbb{Z}_p$-representation of $G_0$, $k \in \mathbb{N}$, $v(p^k) > 2C_2 + 2C_3$, and suppose $G \subset G_0$ is an open normal subgroup acting trivially on $T/p^kT$, and $H = G \cap H_0$. Let $n \in \mathbb{N}, n \geq n(G)$. Then there exists a unique $D_{H,n}(T) \subset \tilde{\Lambda} \otimes T$, a free $\Lambda_{H,n}$-module of rank $d$, such that:

(i) $D_{H,n}(T)$ is fixed by $H$, and stable by $G$;

(ii) $\tilde{\Lambda} \otimes_{\Lambda_{H,n}} D_{H,n}(T) \sim \tilde{\Lambda} \otimes T$;

(iii) there exists a basis $\{e_1, \ldots, e_d\}$ of $D_{H,n}$ over $\Lambda_{H,n}$ such that if $\gamma \in G/H$, then $v(V_{\gamma} - 1) > C_3$, $V_{\gamma}$ being the matrix of $\gamma$. 


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Proof. Translation of Proposition 8.2.1, by the correspondence

\[ \tilde{\Lambda}\text{-representations of } G_0 \overset{\sim}{\longleftrightarrow} H^1(G_0, \text{GL}_d(\tilde{\Lambda})). \]

For the uniqueness, one uses Lemma 8.2.8.

Remark. \(H_0\) acts through \(H_0/H\) (which is finite) on \(D_{H,n}(T)\). If \(\Lambda_{H,n}\) is étale over \(\Lambda_{H_0,n}\) (the case in applications), and then \(D_{H_0,n}(T) = D_{H,n}(T)\overset{(H_0/H)}{\sim}\), \(\Lambda_{H,n}\) locally free over \(\Lambda_{H_0,n}\) (in most cases it is free), and

\[ \Lambda_{H,n} \boxtimes_{\Lambda_{H_0,n}} D_{H_0,n}(T) \overset{\sim}{\rightarrow} D_{H,n}(T). \]

Example 8.2.10. For \(\tilde{\Lambda} = \mathbb{C}_p\), let \(V\) be a \(\mathbb{Q}_p\)-representation of \(G_K\) for \([K : \mathbb{Q}_p] < +\infty\), \(T \subset V\) be a stable lattice. Then

\[ D_{\text{Sen},n}(V) := D_{H,K,n}(T) \]

is a \(K_n\)-vector space of dimension \(d = \dim_{\mathbb{Q}_p} V\) with a linear action of \(\Gamma_{K_n}\). Sen’s operator is defined as follows:

\[ \Theta_{\text{Sen}} = \frac{\log \gamma}{\log \chi(\gamma)}, \text{ where } \gamma \in \Gamma_{K_n}, \log \chi(\gamma) \neq 0. \]

It is easy to see:

**Proposition 8.2.11.** \(V\) is Hodge-Tate if and only if \(\Theta_{\text{Sen}}\) is semi-simple, and the eigenvalues lie in \(\mathbb{Z}\). These eigenvalues are the Hodge-Tate weights of \(V\).

**Remark.** For general \(V\), the eigenvalues of \(\Theta_{\text{Sen}}\) are the generalized Hodge-Tate weights of \(V\).

8.3 Overconvergent \((\varphi, \Gamma)\)-modules

8.3.1 Overconvergent elements

**Definition 8.3.1.** (i) For \(x = \sum_{i=0}^{+\infty} p^i[x_i] \in \tilde{A}, x_i \in \tilde{E} = \text{Fr} R, k \in \mathbb{N}\), define

\[ w_k(x) := \inf_{x_i \in \tilde{E}} v_E(x_i) \] (One checks easily that \(w_k(x) \geq v_E(\alpha), \alpha \in \tilde{E}, \text{ if and only if } [\alpha]x \in \tilde{A}^+ + p^{k+1}\tilde{A}).\)
(ii) For a real number \( r > 0 \), define
\[
v^{(0,r)}(x) := \inf_{k \in \mathbb{N}} w_k(x) + \frac{k}{r} = \inf_{k \in \mathbb{N}} v_E(x_k) + \frac{k}{r} \in \mathbb{R} \cup \{\pm \infty\}.
\]

(iii) \( \widetilde{A}^{(0,r)} := \{ x \in \widetilde{A} : \lim_{k \to +\infty} (v_E(x_k) + \frac{k}{r}) = \lim_{k \to +\infty} (w_E(x_k) + \frac{k}{r}) = +\infty \} \).

**Proposition 8.3.2.** \( \widetilde{A}^{(0,r)} \) is a ring and \( v = v^{(0,r)} \) satisfies the following properties:

(i) \( v(x) = +\infty \iff x = 0 \);

(ii) \( v(xy) \geq v(x) + v(y) \);

(iii) \( v(x+y) \geq \inf(v(x), v(y)) \);

(iv) \( v(px) = v(x) + \frac{1}{r} \);

(v) \( v([\alpha]x) = v_E(\alpha) + v(x) \) if \( \alpha \in \tilde{E} \);

(vi) \( v(g(x)) = v(x) \) if \( g \in G_{\mathbb{Q}_p} \);

(vii) \( v^{(0,r^{-1}r)}(\varphi(x)) = pv^{(0,r)}(x) \).

**Proof.** Exercise. \( \square \)

**Lemma 8.3.3.** Given \( x \in \sum_{k=0}^{+\infty} p^k [x_k] \in \widetilde{A} \), the following conditions are equivalent:

(i) \( \sum_{k=0}^{+\infty} p^k [x_k] \) converges in \( B_{dR}^+ \);

(ii) \( \sum_{k=0}^{+\infty} p^k x_k^{(0)} \) converges in \( \mathbb{C}_p \);

(iii) \( \lim_{k \to +\infty} (k + v_E(x_k)) = +\infty \);

(iv) \( x \in \widetilde{A}^{(0,1)} \).

**Proof.** (iii) \( \iff \) (iv) is by definition of \( \widetilde{A}^{(0,r)} \). (ii) \( \iff \) (iii) is by definition of \( v_E \). (i) \( \Rightarrow \) (ii) is by the continuity of \( \theta : B_{dR}^+ \to \mathbb{C}_p \). So it remains to show (ii) \( \Rightarrow \) (i).

Write \( \tilde{p} = (p, p^{1/p}, \ldots) \in \tilde{E}^+ \), then \( \xi = [\tilde{p}] - p \) is a generator of \( \text{Ker} \theta \cap \tilde{A}^+ \).

We know
\[
a_k = k + [v_E(x_k)] \to +\infty \text{ if } k \to +\infty.
\]

Write \( x_k = \tilde{p}^{ak} y_k \), then \( y_k \in \tilde{E}^+ \). We have
\[
p^k [x_k] = \left( \frac{p}{\tilde{p}} \right)^k [\tilde{p}]^{ak} [y_k] = p^{ak} (1 + \frac{\xi}{p})^{ak-k} [y_k].
\]
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Note that \(p^k(1 + \frac{\xi}{p})^{a_k-1} \in p^{a_k-m}\tilde{A} + (\text{Ker } \theta)^{m+1}\) for all \(m\). Thus, \(a_k \to +\infty\) implies that \(p^k[x_k] \to 0 \in B_{dR}^+/(\text{Ker } \theta)^{m+1}\) for every \(m\), and therefore also in \(B_{dR}^+\) by the definition of the topology of \(B_{dR}^+\). □

**Remark.** We just proved \(\tilde{A}^{[0,1]} := B_{dR}^+ \cap \tilde{A}\), and we can use

\[
\varphi^{-n} : \tilde{A}^{[0,p^{-n}]} \xrightarrow{\sim} \tilde{A}^{[0,1]}
\]

to embed \(\tilde{A}^{[0,r]}\) in \(B_{dR}^+\), for \(r \geq p^{-n}\).

Define

\[
\tilde{A}^\dagger := \bigcup_{r>0} \tilde{A}^{[0,r]} = \{x \in \tilde{A} : \varphi^{-n}(x) \text{ converges in } B_{dR}^+ \text{ for } n \gg 0\}.
\]

**Lemma 8.3.4.** \(x \in \sum_{k=0}^{+\infty} p^k[x_k] \text{ is a unit in } \tilde{A}^{[0,r]} \text{ if and only if } x_0 \neq 0 \text{ and } v_E\left(\frac{x_k}{p^k}\right) > -\frac{k}{r} \text{ for all } k \geq 1\).

**Proof.** Exercise. Just adapt the proof of Gauss Lemma. □

Set

\[
\tilde{B}^{[0,r]} = \tilde{A}^{[0,r]}[\frac{1}{p}] = \bigcup_{n \in \mathbb{N}} p^{-n}\tilde{A}^{[0,r]},
\]

endowed with the topology of inductive limit, and

\[
\tilde{B}^\dagger = \bigcup_{r>0} \tilde{B}^{[0,r]},
\]

again with the topology of inductive limit.

**Theorem 8.3.5.** \(\tilde{B}^\dagger\) is a subfield of \(\tilde{B}\), stable by \(\varphi\) and \(G_{\mathbb{Q}_p}\), both acting continuously.

\(\tilde{B}^\dagger\) is called the field of overconvergent elements. We are going to prove elements of \(D(V)_{\psi=1}\) are overconvergent.

**Definition 8.3.6.** (i) \(B^\dagger = \tilde{B}^\dagger \cap B\), \(A^\dagger = \tilde{A}^\dagger \cap B\) (so \(B^\dagger\) is a subfield of \(B\) stable by \(\varphi\) and \(G_{\mathbb{Q}_p}\)), \(A^{[0,r]} = \tilde{A}^{[0,r]} \cap B\).

(ii) If \(K/\mathbb{Q}_p\) is a finite extension and \(\Lambda \in \{\tilde{A}^\dagger, \tilde{B}^\dagger, A^\dagger, B^\dagger, A^{[0,r]}, B^{[0,r]}\}\), define \(\Lambda_K = \Lambda^{H_K}\). For example \(A^{[0,r]}_K = \tilde{A}^{[0,r]} \cap A_K\).

(iii) If \(\Lambda \in \{A, B, A^\dagger, B^\dagger, A^{[0,r]}, B^{[0,r]}\}\), and \(n \in \mathbb{N}\), define \(\Lambda_{K,n} = \varphi^{-n}(\Lambda_K) \subset \tilde{B}\).
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We now want to make $A_{K}^{(0,r)}$ more concrete. Let $F' \subset K_\infty$ be the maximal unramified extension of $\mathbb{Q}_p$, $\pi_K$ be a uniformizer of $E_K = k_{F'}((\pi_K))$, $\bar{P}_K \in E_{F'}[X]$ be a minimal polynomial of $\bar{\pi}_K$. Let $P_K \in A_{F'}^+[X]$ (note that $A_{F'}^+ = \mathcal{O}_{F'}[[\pi]]$) be a lifting of $\bar{P}_K$. By Hensel’s lemma, there exists a unique $\pi_K \in A_K$ such that $P_K(\pi_K) = 0$ and $\bar{\pi}_K = \pi_K \mod p$. If $K = F'$, we take $\pi_K = \pi$.

Lemma 8.3.7. If we define
\[
    r_K = \begin{cases} 
    1, & \text{if } E_K/E_{Q_p} \text{ is unramified}, \\
    (2v_E(\theta_{E_K/E_{Q_p}}))^{-1}, & \text{otherwise}.
\end{cases}
\]
then $\pi_K$ and $P_K'(\pi_K)$ are units in $A_{K}^{(0,r)}$ for all $0 < r < r_K$.

Proof. The proof is technical but not difficult and is left to the readers. $\Box$

Proposition 8.3.8. (i) $A_K = \{ \sum_{n \in \mathbb{N}} a_n \pi_K^n : a_n \in \mathcal{O}_{F'}, \lim_{n \to -\infty} v_p(a_n) = +\infty \};$

(ii) $A_{K}^{(0,r)} = \{ \sum_{n \in \mathbb{N}} a_n^{(0,r)} \pi_K^n : a_n \in \mathcal{O}_{F'}, \lim_{n \to -\infty} (v_p(a_n) + n rv_E(\bar{\pi}_K)) = +\infty \}.$
So $f \mapsto f(\pi_K)$ is an isomorphism from bounded analytic functions on the annulus $0 < v_p(T) \le rv_E(\bar{\pi}_K)$ to the ring $B_{K}^{(0,r)}$.

Proof. The technical but not difficult proof is again left as an exercise. See Cherbonnier-Colmez Invent. Math. 1998. $\Box$

Corollary 8.3.9. (i) $A_{K}^{(0,r)}$ is a principal ideal domain;

(ii) If $L/K$ is a finite Galois extension, then $A_L^{(0,r)}$ is an étale extension of $A_K^{(0,r)}$ if $r < r_L$, and the Galois group is nothing but $H_K/H_L$.

Define $\bar{\pi}_n = \varphi^{-n}(\pi), \bar{\pi}_{K,n} = \varphi^{-n}(\pi_{K,n}).$

Proposition 8.3.10. (i) If $p^n r_K \ge 1$, $\theta(\bar{\pi}_{K,n})$ is a uniformizer of $K_n$;

(ii) $\bar{\pi}_{K,n} \in K_n[[t]] \subset B_{dR}^+$.

Proof. First by definition
\[
    \bar{\pi}_n = [\varepsilon^{1/p^n}] - 1 = \varepsilon^{(n)} e^{t/p} - 1 \in F_n[[t]] \subset B_{dR}^+
\]
(for $[\varepsilon^{1/p^n}] = \varepsilon^{(n)} e^{t/p^n}$ : the $\theta$ value of both sides is $\varepsilon^{(n)}$, and the $p^n$-th power of both side is $\varepsilon$ (recall $t = \log[\varepsilon]$)). This implies the proposition in the unramified case.
For the ramified case, we proceed as follows.

By the definition of $E_K$, $\pi_{K,n} = \theta(\tilde{\pi}_{K,n})$ is a uniformizer of $K_n \mod a = \{x : v_p(x) \geq \frac{1}{p}\}$. Write $\omega_n$ be the image of $\pi_{K,n}$ in $K_n \mod a$. So we just have to prove $\pi_{K,n} \in K_n$.

Write

$$P_K(x) = \sum_{i=0}^{d} a_i(\pi)x^i, \ a_i(\pi) \in \mathcal{O}_{F'}[[\pi]].$$

Define

$$P_{K,n}(x) = \sum_{i=0}^{d} a_i^{\varphi^{-n}}(\pi_n)x_i,$$

then $P_{K,n}(\pi_{K,n}) = \theta(\varphi(P_K(\pi_n))) = 0$. Then we have $v_p(P_{K,n}(\omega_n)) \geq \frac{1}{p}$ and

$$v_p(P_{K,n}'(\omega_n)) = \frac{1}{p^n}v_E(P_K'(\tilde{\pi}_n)) = \frac{1}{p^n}v_E(\mathfrak{o}_{E_K/E_{Q_p}}) < \frac{1}{2p}$$

if $p^n r_K > 1$.

Then one concludes by Hensel’s Lemma that $\pi_{K,n} \in K_n$.

For (ii), one uses Hensel’s Lemma in $K_n[[t]]$ to conclude $\tilde{\pi}_{K,n} \in K_n[[t]]$. □

**Corollary 8.3.11.** If $0 < r < r_K$ and $p^n r \geq 1$, $\varphi^{-n}(A_K^{(0,r)}) \subset K_n[[t]] \subset B_{dR}^+$. 

### 8.3.2 Overconvergent representations

Suppose $V$ is a free $\mathbb{Z}_p$ representation of rank $d$ of $G_K$. Let

$$D^{(0,r)} := (A^{(0,r)} \otimes_{\mathbb{Z}_p} V)^{H_K} \subset D(V).$$

This is a $A_K^{(0,r)}$-module stable by $\Gamma_K$. As for $\varphi$, we have

$$\varphi : D^{(0,r)}(V) \longrightarrow D^{(0,p^{-1}r)}(V).$$

**Definition 8.3.12.** $V$ is **overconvergent** if there exists an $r_V > 0, r_V \leq r_K$ such that

$$A_K \bigotimes_{A_K^{(0,rv)}} D^{(0,rv)}(V) \iso D(V).$$

By definition, it is easy to see for all $0 < r < r_V$,

$$D^{(0,r)}(V) = A_K^{(0,r)} \bigotimes_{A_K^{(0,rv)}} D^{(0,rv)}(V).$$
Proposition 8.3.13. If $V$ is overconvergent, then there exists a $C_V$ such that if $\gamma \in \Gamma_K$, $n(\gamma) = v_p(\log(\chi(\gamma)))$ and $r < \inf\{p^{-1}r_V, p^{-n(\gamma)}\}$, then $\gamma - 1$ is invertible in $D^{(0, r)}(V)_{\psi=0}$ and

$$v^{(0, r)}((\gamma - 1)^{-1}x) \geq v^{(0, r)}(x) - C_V - p^n(\gamma)\nu_E(\bar{\pi}).$$

**Proof.** Write $x = \sum_{i=1}^{p-1} [\varepsilon]^i \varphi(x_i)$ and adapt the proof of the same statement as in the characteristic $p$ case. One has to use the fact that $[\varepsilon]^{p^n - 1}$ is a unit in $A^{(0, r)}_K$ if $r < p^{-n}$ and $i \in \mathbb{Z}_p^*$. \qed

**Remark.** This applies to $(A^{(0, r)}_K)_{\psi=0}$.

**Theorem 8.3.14 (Main Theorem).** (i) All (free $\mathbb{Z}_p$ or $\mathbb{Q}_p$) representations of $G_K$ are overconvergent.

(ii) $D(V)_{\psi=1} \subset D^{(0, r_V)}(V)$.

**Sketch of Proof.** (ii) is just because $\psi$ improves convergence.

(i) follows from Sen’s method applied to

$$\tilde{\Lambda} = \tilde{A}^{(0, 1)}, v = v^{(0, 1)}, G_0 = G_K, \Lambda_{H, K, n} = \varphi^{-n}(A^{(0, 1)}_K).$$

Now we show how to check the three conditions.

(TS1). Let $L \supset K \supset \mathbb{Q}_p$ be finite extensions, for $\alpha = [\bar{\pi}_L](\sum_{\tau \in H_K/H_L} \tau([\bar{\pi}_L]))^{-1}$, then for all $n$,

$$\sum_{\tau \in H_K/H_L} \tau(\varphi^{-n}(\alpha)) = 1,$$

and

$$\lim_{n \to +\infty} v^{(0, 1)}(\varphi^{-n}(\alpha)) = 0.$$

(TS2). First $\Lambda_{H, K, n} = A^{(0, 1)}_{K, n}$. Suppose $p^n r_K \geq 1$. We can define $R_{K, n}$ by the following commutative diagram:

$$R_{K, n}: \tilde{A}^{(0, 1)}_K \to A^{(0, 1)}_{K, n} \to A^{(0, 1)}_{K, n+k} \to A^{(0, 1)}_{K, n+k} \to \varphi^n \circ \psi^{n+k} \circ \varphi^{n+k}.$$
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One verifies that \(\varphi^{-n} \circ \psi^{n+k} \circ \varphi^{n+k}\) does not depend on the choice of \(k\), using the fact \(\psi \varphi = \text{Id}\). Then the proof is entirely parallel to that for \(C_p\) with \(\psi\) in the role of \(p^{-1} \text{Tr}_{F_{n+1}/F_n}\) and \(\tilde{\pi}_{n+k}\) in the role of \(\pi_{n+k}\).

(TS3). For an element \(x\) such that \(R_{K,n}(x) = 0\), write

\[ x = \sum_{i=0}^{+\infty} R^*_{K,n}(x), \quad \text{where } R^*_{K,n}(x) \in \varphi^{-(n+i+1)}((A_K^{(0,p^{-(n+i+1)})})^\psi = 0). \]

Then just apply Proposition 8.3.13 on \((A_K^{(0,p^{-(n+i+1)})})^\psi = 0).

Now Sen’s method implies that there exists an \(n\) and a \(A_{K,n}^{[0,1]}\)-module \(D_{K,n}^{[0,1]} \subset \tilde{A}^{[0,1]} \otimes V\) such that

\[ \tilde{A}^{[0,1]} \otimes A_{K,n}^{[0,1]} D_{K,n}^{[0,1]} \cong \tilde{A}^{[0,1]} \otimes V. \]

Play with (TS3) just like Lemma 8.2.8, one concludes that \(D_{K,n}^{[0,1]} \subset \varphi^{-n}(D(V))\) and \(\varphi^n(D_{K,n}^{[0,1]}) \subset D^{(0,p^{-n})}(V)\). We can just take \(r_V = n\).

8.3.3 \(p\)-adic Hodge theory and \((\varphi, \Gamma)\)-modules

Suppose we are given a representation \(V\), \(0 < r < r_V\) and \(n\) such that \(p^n r > 1\). Then we have

\[ \varphi^{-n}(D^{(0,r]}(V)) \hookrightarrow B_{dR}^+ \otimes V \xrightarrow{\theta} C_p \otimes V \]

and

\[ \varphi^{-n}(A_K^{(0,r]}) \hookrightarrow K_n[[t]] \xrightarrow{\theta} K_n. \]

So we get the maps

\[ \theta \circ \varphi^{-n} : K_n \otimes A_K^{(0,r]} D^{(0,r]}(V) \longrightarrow C_p \otimes V \quad (8.1) \]

and

\[ \varphi^{-n} : t^i K_n[[t]] \otimes A_K^{(0,r]} D^{(0,r]}(V) \longrightarrow t^i B^+_{dR} \otimes V, \forall i \in \mathbb{Z}. \quad (8.2) \]

**Theorem 8.3.15.** There exists an \(n(V) \in \mathbb{N}\) such that if \(n \geq n(V)\), then we have

(i) the image of \(\theta \circ \varphi^{-n}\) in (8.1) is exactly \(D_{\text{Sen}, n}(V)\);

(ii) \(\text{Fil}^i D_{dR}(V) = (\text{Im } \varphi^{-n})^\Gamma_K\) in (8.2) for all \(i\);

(iii) \(D_{dR}(V) = (K_n((t)) \otimes A_K^{(0,r]} D^{(0,r]}(V))^\Gamma_K\).
Let $K/\mathbb{Q}_p$ be a finite extension, and define

$$B_K^\dagger = \{F(\pi_K) : F \text{ is a bounded analytic function on } 0 < v_p(t) \leq r(F), \ r(F) > 0\},$$

$$B_{\text{rig.}, K}^\dagger = \{F(\pi_K) : F \text{ is an analytic function on } 0 < v_p(t) \leq r(F), \ r(F) > 0\}$$

(this last ring is the Robba ring in the variable $\pi_K$), and

$$B_{\log, K}^\dagger = B_{\text{rig.}, K}^\dagger [\log \pi_K].$$

Extend $\varphi, \Gamma_K$ by continuity on $B_{\text{rig.}, K}^\dagger$, and set

$$\varphi(\log \pi_K) = p \log \pi_K + \log \frac{\varphi(\pi_K)}{\pi_K^p},$$

$$\gamma(\log \pi_K) = \log \pi_K + \log \frac{\gamma(\pi_K)}{\pi_K}$$

where $\log \frac{\varphi(\pi_K)}{\pi_K} \in B_K^\dagger$ and $\log \frac{\gamma(\pi_K)}{\pi_K} \in B_{\text{rig.}, K}^\dagger$. Let

$$N = -\frac{1}{v_E(\pi_K)} \cdot \frac{d}{d \log \pi_K}.$$

**Theorem 8.3.16 (Berger).** For

$$D^\dagger(V) = (B^\dagger \otimes V)^{H_K} = \bigcup_{r > 0} D^{(0, r]}(V),$$

if $V$ is semi-stable, then

$$B_{\log, K}^\dagger \left[ \frac{1}{t} \right] \otimes_{K_0} D_{st}(V) = B_{\log, K}^\dagger \left[ \frac{1}{t} \right] \otimes_{B_K^\dagger} D^\dagger(V)$$

is an isomorphism of $(\varphi, N, \Gamma_K)$-modules. This implies that $D_{st}(V)$ is the invariant under $\Gamma_K$.

### 8.3.4 A map of the land of the rings

The following nice picture outlines most of the objects that we have discussed till now and that we shall have to discover more about in the future.
8.4. Explicit reciprocity laws and \( p \)-adic \( L \)-functions

8.4.1 Galois cohomology of \( B_{dR} \)

Suppose \( K \) is a finite extension of \( \mathbb{Q}_p \). Recall that we have the following:

**Proposition 8.4.1.** For \( k \in \mathbb{Z} \), then

(i) if \( k \neq 0 \), then \( H^i(G_K, \mathbb{C}_p(k)) = 0 \) for all \( i \)

(ii) if \( k = 0 \), then \( H^i(G_K, \mathbb{C}_p) = 0 \) for \( i \geq 2 \), \( H^0(G_K, \mathbb{C}_p) = K \),
and \( H^1(G_K, \mathbb{C}_p) \) is a 1-dimensional \( K \)-vector space generated by \( \log \chi \in H^1(G_K, \mathbb{Q}_p) \). (i.e., the cup product \( x \mapsto x \cup \log \chi \) gives an isomorphism \( H^0(G_K, \mathbb{C}_p) \cong H^1(G_K, \mathbb{C}_p) \)).
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Remark. This has been proved for \(i \leq 1\). For \(i \geq 2\), \(H^i(H_K, \mathbb{C}_p(k)) = 0\) by using the same method as for \(H^1\). Then just use the exact sequence

\[
1 \longrightarrow H_K \longrightarrow G_K \longrightarrow \Gamma_K \longrightarrow 1
\]

and Hochschild-Serre spectral sequence to conclude.

Proposition 8.4.2. Suppose \(i < j \in \mathbb{Z} \cup \{\pm \infty\}\), then if \(i \geq 1\) or \(j \leq 0\),

\[
H^1(G_K, t^i B^{+}_{dR}/t^j B^{+}_{dR}) = 0;
\]

if \(i \leq 0\) and \(j > 0\), then \(x \mapsto x \cup \log \chi\) gives an isomorphism

\[
H^0(G_K, t^i B^{+}_{dR}/t^j B^{+}_{dR}) \simeq H^1(G_K, t^i B^{+}_{dR}/t^j B^{+}_{dR}).
\]

Proof. Use the long exact sequence in continuous cohomology attached to the exact sequence

\[
0 \longrightarrow t^{i+n} \mathbb{C}_p(\simeq \mathbb{C}_p(i + n)) \longrightarrow t^i B^{+}_{dR}/t^{n+i+1} B^{+}_{dR} \longrightarrow t^i B^{+}_{dR}/t^{i+n} B^{+}_{dR} \longrightarrow 0,
\]

and use induction on \(j - i\) (note that in the base step \(j = i+1\), \(t^i B^{+}_{dR}/t^j B^{+}_{dR} \simeq \mathbb{C}_p(i)\)), and Proposition 8.4.1 to do the computation. This concludes for the case where \(i, j\) are finite. For the general case, one proves it by taking limits. \(\square\)

8.4.2 Bloch-Kato’s dual exponential maps

Let \(V\) be a de Rham representation of \(G_K\), we have

\[
B_{dR} \otimes_{\mathbb{Q}_p} V \cong B_{dR} \otimes_K D_{dR}(V) \cong H^0(G_K, B_{dR} \otimes V)
\]

and

\[
H^1(G_K, B_{dR} \otimes V) = H^1(G_K, B_{dR} \otimes_K D_{dR}(V)) = H^1(G_K, B_{dR}) \otimes_K D_{dR}(V).
\]

So we get an isomorphism

\[
D_{dR}(V) \cong H^1(G_K, B_{dR} \otimes V); \quad x \mapsto x \cup \log \chi.
\]
8.4. EXPLICIT RECIPROCITY LAWS AND $p$-ADIC $L$-FUNCTIONS

**Definition 8.4.3.** The exponential map $\exp^*$ is defined through the commutative diagram:

\[
\begin{array}{c}
\exp^* : H^1(G_K, V) \\
\sim \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
H^1(G_K, B_{dR} \otimes V) \\
\end{array} \xrightarrow{\sim} D_{dR}(V)
\]

**Proposition 8.4.4.** (i) The image of $\exp^*$ lies in $\text{Fil}^0 D_{dR}(V)$.

(ii) For $c \in H^1(G_K, V)$, $\exp^*(c) = 0$ if and only if the extension $E_c$

\[
0 \longrightarrow V \longrightarrow E_c \longrightarrow \mathbb{Q}_p \longrightarrow 0,
\]

is de Rham as a representation of $G_K$.

**Proof.** (ii) is just by the definition of de Rham. For (i), $c \in H^1(G_K, V)$ implies $c = 0 \in H^1(G_K, (B_{dR}/B_{dR}^+ \otimes V)$. But $x \mapsto x \cup \log \chi$ gives an isomorphism

\[
D_{dR}(V)/\text{Fil}^0(D_{dR}(V)) = H^0(G_K, (B_{dR}/B_{dR}^+ \otimes V)) \longrightarrow H^1(G_K, (B_{dR}/B_{dR}^+ \otimes V)).
\]

So $\exp^*(c) = 0 \pmod{\text{Fil}^0}$ \hfill $\square$

**Remark.** $\exp^*$ is a very useful tool to prove the non-triviality of cohomology classes.

Now suppose $k \in \mathbb{Z}$, $L$ is a finite extension of $K$. Then $V(k)$ is still de Rham as a representation of $G_L$. Define

\[
D_{dR, L}(V(k)) := H^0(G_L, B_{dR} \otimes V(k)) = t^{-k}L \otimes_K D_{dR}(V)
\]

by an easy computation. Thus,

\[
\text{Fil}^0(D_{dR, L}(V(k))) = t^{-k} \otimes_K \text{Fil}^k D_{dR}(V)
\]

and this is 0 if $k \gg 0$. So for every $k \in \mathbb{Z}$, for $L/K$ finite,

\[
\exp^* : H^1(G_L, V(k)) \longrightarrow t^{-k}L \otimes_K D_{dR}(V)
\]

is identically 0 for $k \gg 0$. 

8.4.3 The explicit reciprocity law

Recall that

\[ H^1_{\text{Iw}}(K, V) \xrightarrow{\sim} H^1(G_K, \mathbb{Z}_p[[\Gamma_K]] \otimes V) = H^1(G_K, \mathcal{D}_0(\Gamma_K, V)). \]

If \( \eta : \Gamma_K \to \mathbb{Q}_p^* \) is a continuous character, for \( n \in \mathbb{N} \),

\[ \mu \in H^1(G_K, \mathcal{D}_0(\Gamma_K, V)) \mapsto \int_{\Gamma_K^n} \eta \mu \in H^1(G_{K_n}, V \otimes \eta). \]

where we write \( V \otimes \eta \), not as \( V(\eta) \) to distinguish from \( V(k) = V \otimes \chi^k \). Then

\[ \exp^*(\int_{\Gamma_K^n} \chi^k \mu) \in t^{-k} K_n \otimes_K D_{dR}(V) \]

and is 0 if \( k \gg 0 \).

Recall also that we have the isomorphism \( \text{Exp}^* : H^1(K, V) \xrightarrow{\sim} D(V)^{\psi=1} \)

that \( D(V)^{\psi=1} \subset D^{(0,rv]}(V) \) and that there exists \( n(V) \) such that

\[ \varphi^{-n}(D^{(0,rv]}(V)) \subset K_n((t)) \otimes_K D_{dR}(V), \text{ for all } n \geq n(V). \]

Now denote by

\[ \text{Tr}_{K_{n+i}/K_n} = \text{Tr}_{K_{n+i}(t)/K_n(t)} \otimes \text{Id} : K_{n+i}(t) \otimes D_{dR}(V) \to K_n((t)) \otimes D_{dR}(V). \]

**Theorem 8.4.5 (Explicit Reciprocity Law).** Let \( V \) be a de Rham representation of \( G_K \) and \( \mu \in H^1_{\text{Iw}}(K, V) \).

(i) If \( n \geq n(V) \), then

\[ p^{-n} \varphi^{-n}(\text{Exp}^*(\mu)) = \sum_{k \in \mathbb{Z}} \exp^*(\int_{\Gamma_K^n} \chi^k \mu). \]

(ii) For \( n \in \mathbb{N}, n + i \geq n(V) \), then

\[ \text{Exp}^*_{K_n}(\mu) := \text{Tr}_{K_{n+i}/K_n}(p^{-(n+i)}\varphi^{-(n+i)}(\text{Exp}^*(\mu))) \]

does not depend on \( i \), and \( \text{Exp}^*_{K_n}(\mu) = \sum_{k \in \mathbb{Z}} \exp^*(\int_{\Gamma_K^n} \chi^k \mu). \)
8.4. EXPLICIT RECIPROCITY LAWS AND P-ADIC L-FUNCTIONS

Proof. (ii) follows from (i) and from the commutative diagram:

\[
\begin{array}{ccc}
H^1(G_{L_2}, V) & \xrightarrow{\exp^*} & L_2 \otimes_K D_{dR}(V) \\
\text{cor} & \downarrow & \text{Tr}_{L_2/L_1} \otimes_K \text{Id} \\
H^1(G_{L_1}, V) & \xrightarrow{\exp^*} & L_1 \otimes_K D_{dR}(V)
\end{array}
\]

where \( L_1 \subset L_2 \) are two finite extensions of \( K \).

For (i), suppose \( y = \text{Exp}^*(\mu), x \in D(V), \) and \( x(k) \) is the image of \( x \) in \( D(V(k)) = D(V)(k) \) (Thus, \( \varphi(x(k)) = \varphi(x)(k) \) and \( \gamma(x(k)) = \chi(\gamma)(\gamma(x)(k)) \)).

The integral \( \int_{\Gamma_n} \chi^k \mu \) is represented by the cocycle:

\[
g \mapsto c_g = \frac{\log \chi(\gamma_n)}{p^n} \cdot \left( \frac{g - 1}{\gamma_n - 1} y(k) - (g - 1)b \right)
\]

where \( b \in A \otimes V \) is the solution of

\[
(\varphi - 1)b = (\gamma_n - 1)^{-1}((\varphi - 1)(y)(k)).
\]

From \( y \in D^{(0,rv)}(V)_{\psi=1} \) one gets

\[
(\varphi - 1)y \in D^{(0,p^{-1}rv)}(V)_{\psi=0}
\]

and then

\[
(\gamma_n - 1)^{-1}(\varphi - 1)y \in D^{(0,p^{-n})}(V)_{\psi=0}.
\]

Thus \( b \in A^{(0,p^{1-n})} \otimes V \). This implies that \( \varphi^{-n}(b) \) and \( \varphi^{-n}(y) \) both converge in \( B_{dR}^+ \otimes V \). Then \( c_g = \varphi^{-n}(c_g) \) differs from

\[
c'_g = \frac{\log \chi(\gamma_n)}{p^n} \cdot \frac{g - 1}{\gamma_n - 1} \cdot \varphi^{-n}(y)(k)
\]

by the coboundary \( (g - 1)(\varphi^{-n}(b)) \). Therefore, they have the same image in \( H^1(G_{K_n} B_{dR}^+ \otimes V(k)) \). Write

\[
p^{-n} \varphi^{-n}(y) = \sum_{i \geq i_0} y_i t^i, \quad y_i \in K_n \otimes_K D_{dR}(V),
\]

then

\[
c'_g = \log \chi(g)y_{-k} t^{-k} + \sum_{i \neq -k} \frac{\chi(g)^{i+k} - 1}{\chi(\gamma_n)^{i+k} - 1} \cdot y_i t^i
\]

\[
= \log \chi(g)y_{-k} t^{-k} + (g - 1) \sum_{i \neq -k} \frac{y_i t^i}{(\chi(\gamma_n)^{i+k} - 1)}.
\]

So we get \( \exp^*(\int_{\Gamma_n} \chi^k \mu) = y_{-k} t^{-k} \). \( \square \)
8.4.4 Cyclotomic elements and Coates-Wiles morphisms.

Let $K = \mathbb{Q}_p$, $V = \mathbb{Q}_p(1)$, $u = \left(\pi_n^{\frac{1}{1+\pi_n}}\right)_{n \geq 1} \in \varprojlim \mathcal{O}_{F_n}$, $\kappa(u) \in H^1_{Iw}(\mathbb{Q}_p, \mathbb{Q}_p(1))$, the Coleman power series $f_u = \frac{T}{1+T}$. Then we have
\[
\text{Exp}^*(\kappa(u)) = (1 + T) \frac{df_u}{dT}(\pi) = \frac{1}{\pi}.
\]
Note that \[
\varphi^{-1}(\pi)^{-1} = \left([e^{1/p}] - 1\right)^{-1} = \frac{1}{(1 + \pi_1) e^{t/p} - 1},
\]
then
\[
\text{Exp}^*_q(\kappa(u)) = \frac{1}{p} \text{Tr}_{\mathbb{Q}_p(\pi_1)/\mathbb{Q}_p} \varphi^{-1}(\pi)^{-1} = \frac{1}{p} \sum_{2^p = 1, z \neq 1} \frac{1}{e^{t/p} - 1} = \frac{1}{e^t - 1} - \frac{1}{p} \cdot \frac{1}{e^{t/p} - 1} = \frac{1}{t} \cdot \left(\frac{t}{e^t - 1} - \frac{t/p}{e^{t/p} - 1}\right)
\]
\[
= \sum_{n=1}^{+\infty} (1 - p^{-n}) \zeta(1 - n) \frac{(-t)^{n-1}}{(n-1)!}.
\]
So
\[
\exp^*\left(\int_{\Gamma_{\mathbb{Q}_p}} \chi^k \kappa(\mu)\right) = \begin{cases} 
0, & \text{if } k \geq 0; \\
(1 - p^k) \zeta(1 + k) \frac{(-t)^{-k-1}}{(-k-1)!}, & \text{if } k \leq -1.
\end{cases}
\]

Remark. (i) The map
\[
\varprojlim \mathcal{O}_{F_n} - \{0\} \longrightarrow H^1_{Iw}(\mathbb{Q}_p, \mathbb{Q}_p(1)) \longrightarrow \mathbb{Q}_p, \quad u \mapsto t^{k+1} \exp^*\left(\int_{\Gamma_{\mathbb{Q}_p}} \chi^k \kappa(u)\right)
\]
is the Coates-Wiles homomorphism.

(ii) Since $\zeta(1 + k) \neq 0$ if $k \leq -1$ is even, the above formula implies that the extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(k+1)$ constructed via cyclotomic elements are non-trivial and are even not de Rham.

(iii) $\dim_{\mathbb{Q}_p} H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(k)) = 1$ if $k \neq 0, 1$.

Corollary 8.4.6. Non-trivial extensions of $\mathbb{Q}_p$ by $\mathbb{Q}_p(k)$ are not de Rham if $k \leq 0$ is odd.

Exercise. (i) Prove that this is also true for $k \leq -1$ even by taking a general element of $D(\mathbb{Q}_p(1))^{\psi=1}$.

(ii) For $[K : \mathbb{Q}_p] < \infty$, prove the same statement.
8.4.5 Kato’s elements and $p$-adic $L$-functions of modular forms.

Now we come to see the relations with modular forms. Suppose

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N), \ k \geq 2$$

is primitive. So $Q(f) = Q(a_1, \ldots, a_n, \ldots)$ is a finite extension of $Q$, and $Q_p(f) = Q_p(a_1, \ldots, a_n, \ldots)$ is a finite extension of $Q_p$.

**Theorem 8.4.7 (Deligne).** There exists a representation $V_f$ of $G_{\mathbb{Q}}$ of dimension 2 over $Q_p(f)$, non-ramified outside $Np$, such that if $\ell \nmid Np$, for $\varphi_\ell$ the arithmetic Frobenius at $\ell$ ($(\varphi_\ell(e^{2\pi i \frac{p^n}{p}})) = e^{2\pi i \frac{p^n}{p}}$), then

$$\det(1 - X \varphi_\ell^{-1}) = 1 - a_\ell X + \ell^{k-1}X^2.$$

**Remark.** A $Q_p(f)$-representation $V$ of dimension $d$ is equivalent to a $Q_p$ representation of dimension $d \cdot [Q_p(f) : Q_p]$ endowed with a homomorphism $Q_p(f) \hookrightarrow \text{End}(V)$ commuting with $G_{\mathbb{Q}}$. Therefore, $D_{\text{cris}}(V_f), D_{\text{st}}(V), D_{\text{dR}}(V)$ are all $Q_p(f)$-vector spaces.

**Theorem 8.4.8 (Faltings-Tsuji-Saito).** (i) $V_f$ is a de Rham representation of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights 0 and 1 − $k$, the 2-dimensional $Q_p(f)$-vector space $D_{\text{dR}}(V_f)$ contains naturally $f$, and

$$D_{\text{dR}}^0(V_f) = D_{\text{dR}}(V_f), \ D_{\text{dR}}^i(V_f) = 0, \ D_{\text{dR}}^i(V_f) = Q_p(f)f \ i \leq 1 \leq k - 1.$$

(ii) If $p \nmid N$, then $V_f$ is crystalline and

$$\det(X - \varphi) = X^2 - a_p X + p^{k-1}.$$

If $p | N$ but $a_p \neq 0$, then $V_f$ is semi-stable but not crystalline and $a_p$ is the eigenvalue of $\varphi$ on $D_{\text{cris}}(V)$; if $a_p = 0$, then $V_f$ is potentially crystalline.

**Remark.** If $V$ is a representation of $G_K, \mu \in H^1_{\text{Iw}}(K, V)$,

$$\int_{\Gamma_K} \chi^k \mu \in H^1(G_{K_n}, V(k)),$$

then this is also true for $\int_{a \Gamma_K} \chi^k \mu$ for all $a \in \Gamma_K$ and for $\int_{\Gamma_K} \phi(x) \chi^k \mu$, with $\phi : \Gamma_K \to \mathbb{Z}_p$ being constant modulo $\Gamma_{K_n}$. 
Theorem 8.4.9 (Kato). There exists a unique element $z_{Kato} \in H^1_{Iw}(\mathbb{Q}_p, V_f)$ (obtained by global methods using Siegel units on modular curves), such that if $0 \leq j \leq k-2$, $\phi$ is locally constant on $\mathbb{Z}_p^* \cong \Gamma_{\mathbb{Q}_p}$ with values in $\mathbb{Q}(f)$, then

$$\exp^*(\int_{\mathbb{Z}_p^*} \phi(x)x^{k-1-j} \cdot z_{Kato}) = \frac{1}{j!} \tilde{\Lambda}(f, \phi, j + 1) \cdot \frac{f}{t^{k-1-j}}$$

where

$$\tilde{\Lambda}(f, \phi, j + 1) \in \mathbb{Q}(f, \mu_{p^n}), \quad \frac{f}{t^{k-1-j}} \in \text{Fil}^0(D_{dR}(V_f(k - 1 - j))).$$

Our goal is to recover $L_{p, \alpha}(f, s)$ from $z_{Kato}$ (recall $L_{p, \alpha}$ is obtained from $\mu_{f, \alpha} \in D_{v_p(\alpha)}(\mathbb{Z}_p)$ before). We have $\text{Exp}^*(z_{Kato}) \in D(V_f)^{\psi=1}$, but the question is how to relate this to $D_{\text{cris}}(V_f), D_{\text{st}}(V_f)$.

If $p \mid N$, let $\alpha$ be a root of $X^2 - a_p X + p^{k-1}$ with $v_p(\alpha) < k - 1$; if $p \nmid N$, let $\alpha = a_p \neq 0$ (in this case $pa^2 = p^{k-1}$). In both cases, take $\beta = p^{k-1-\alpha^{-1}}$. Thus, $\alpha, \beta$ are eigenvalues of $\varphi$ on $D_{\text{st}}(V_f)$.

Assume $\alpha \neq \beta$ (which should be the case for modular forms by a conjecture). Define $\Pi_\beta = \frac{\varphi - \alpha}{\beta - \alpha}$ to be the projection on the $\beta$-eigenspace in $D_{\text{st}}(V_f)$ and extend it by $B_{\log, K}^\dagger$-linearity to $B_{\log, K}^\dagger \boxtimes K_0 D_{\text{st}}(V_f) \longrightarrow B_{\log, K}^\dagger \boxtimes B_{K}^\dagger D^\dagger(V_f)$.

Theorem 8.4.10. (i) $\Pi_\beta(f) \neq 0$;
(ii) $\Pi_\beta(\text{Exp}^*(z_{Kato})) = \left(\int_{\mathbb{Z}_p} [\varepsilon]^x \mu_{f, \alpha} \frac{\Pi_\beta(f)}{t^{k-1}}\right)$.

Remark. $\mu_{f, \alpha}$ exists up to now only in the semi-stable case, but $z_{Kato}$ exists all the time. So a big question is:

How to use it for $p$-adic $L$-function?