GRAPH PRODUCTS OF OPERATOR ALGEBRAS

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Abstract. Graph products for groups were defined by Green in her thesis [Gr90] as a generalization of both Cartesian and free products. In this paper we define the corresponding graph product for reduced and maximal C*-algebras, von Neumann algebras and quantum groups. We prove several stability properties including permanence properties of II_1-factors, the Haagerup property and under suitable conditions the property of Rapid Decay for quantum groups.

Introduction

A graph product is a group theoretic construction starting from a simplicial graph with a discrete group associated to each vertex. The graph product construction results in a new group and special cases depending on the graph are free products and Cartesian products. Important examples of graph products are Coxeter groups and right angled Artin groups.

Graph products preserve many important group theoretical properties. This yields important new examples of groups having such properties and gives (alternative) proofs of such properties for existing groups. For instance the graph product preserves soficity [CHR12], Haagerup property [AnDr13], residual finiteness [Gr90], rapid decay [CHR13], linearity [HsWi99] and many other properties, see e.g. [HeMei95], [AnMi11], [Chi12].

Whereas many of the stability properties above have important consequences for operator algebras, the actual operator algebras of graph products have been unexplored so far. The current paper develops the theory of reduced and universal/maximal C*-algebraic graph products as well as the graph product of von Neumann algebras and quantum groups. These objects generalize free products by adding commutation relations that are dictated by the graph.

We shall relate the basic properties of graph products of operator algebras/quantum groups to the ones of their vertices. This includes Tomita-Takesaki theory, commutants, GNS-representations, (co)representation theory, et cetera. We also show that any graph product of von Neumann algebras decomposes inductively into amalgamated free products of the von Neumann algebras at its edges. For notation we refer to Section 2.

Theorem 0.1. *Let Γ be a simplicial graph with von Neumann algebras $\mathsf{M}_v, v \in V\Gamma$ and graph product von Neumann algebra M. Fix $v \in V\Gamma$. Let M_1 be the graph product von Neumann algebra given by $\mathrm{Star}(v)$. Let M_2 be the graph product von Neumann algebra given by $\Gamma\backslash\{v\}$. Let N be the graph product von Neumann algebra given by $\mathrm{Link}(v)$. Then $\mathsf{M} \simeq \mathsf{M}_1 \star_{\mathsf{N}} \mathsf{M}_2$.*

There is a corresponding result of Theorem 0.1 for C*-algebras, see Section 2. Theorem 0.1 implies that any property of a von Neumann algebra that is being preserved by arbitrary amalgamated free products is automatically preserved by the graph product. However, there is a large number of properties which are not (or not known to be) preserved by amalgamated free products. For example, the Haagerup property is known not to be preserved by arbitrary amalgamated free products. But in fact we prove the following.

Theorem 0.2. *Let Γ be a simplicial graph with von Neumann algebras $\mathsf{M}_v, v \in V\Gamma$. Let M be the graph product von Neumann algebra. Then,*

(1) Suppose that every M_v is σ-finite. M has the Haagerup property if and only if for every $v \in V\Gamma$, M_v has the Haagerup property.

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(2) M *is a* II_1 *factor if for every* $v \in V\Gamma$, M_v *is a* II_1 *factor.*

And in the case of quantum groups:

Theorem 0.3. *Let* Γ *be a simplicial graph with compact quantum groups* $\mathbb{G}_v, v \in V\Gamma$. *Let* $\mathbb{G}$ *be its graph product and let* $\widehat{\mathbb{G}}_v, \widehat{\mathbb{G}}$ *be their duals. Then,*

(1) $\widehat{\mathbb{G}}$ *has the Haagerup property if and only if for every* $v \in V\Gamma$, $\widehat{\mathbb{G}}_v$ *has the Haagerup property.*
(2) Let Γ *be finite. If for every* $v \in V\Gamma$, $\widehat{\mathbb{G}}_v$ *is a classical group with the property of Rapid Decay (RD) or a quantum group with polynomial growth, then the graph product* $\widehat{\mathbb{G}}$ *has (RD).*
(3) Let Γ *be finite without edges. Then* $\mathbb{G} = \star_{v \in V\Gamma} \mathbb{G}_v$. *If for every* $v \in V\Gamma$, $\widehat{\mathbb{G}}_v$ *has (RD), then* $\widehat{\mathbb{G}}$ *has (RD). I.e. (RD) is preserved by finite free products.*

It must be emphasized that for compact quantum groups with tracial Haar state (i.e. of so-called Kac type) Theorem 0.3 (1) follows from Theorem 0.2 (1) by [DFSW13, Theorem 6.7]. However, it is unknown if the result of [DFSW13, Theorem 6.7] extends beyond Kac type quantum groups. In fact [CLR13] shows that the behaviour of approximation properties outside the Kac case can be quite different. In the group case our result gives an alternative proof of stability of the Haagerup property under graph products, which was first proved in [AnDr13].

Structure of this paper. Section 1 introduces the basic notions for graph products. In section 2 we develop the theory of graph products of operator algebras: graph products of Hilbert spaces, von Neumann algebras and maximal and reduced graph products of C*-algebras, study their representation theory and develop the unscrewing technique as explained in Theorem 0.1. We also prove some stability properties such as exactness for reduced graph product of C*-algebras and the Haagerup property for von Neumann algebras. In Section 3 we define graph products of quantum groups, study their representation theory and prove the stability of the Haagerup property. Section 4 proves stability of rapid decay for quantum groups under graph products.

General notation. We denote M_n for the $n \times n$ matrices over $\mathbb{C}$. We use bold face characters A and M for operator algebras and calligraphic characters $\mathcal{H}$ and $\mathcal{K}$ for Hilbert spaces. The symbol $\otimes$ denotes the tensor product of Hilbert spaces, C*-algebras (reduced one) or von Neumann algebras and it should be natural from the context which of these is meant. The symbol $\underset{\max}{\otimes}$ will denote the maximal tensor product of C*-algebras.

1. Preliminaries

Let Γ be a simplicial graph with vertex set $V\Gamma$ and edge set $E\Gamma \subseteq V\Gamma \times V\Gamma$. Simplicial means that the graph does not contain double edges and no loops, i.e. $(v, v) \notin E\Gamma$ for any $v \in V\Gamma$. We assume that the graph is non-oriented in the sense that if $(v, w) \in E\Gamma$ then also $(w, v) \in E\Gamma$. For $v \in V\Gamma$ we write $\mathrm{Link}(v)$ for the set of all $w \in V\Gamma$ such that $(v, w) \in E\Gamma$. We set $\mathrm{Star}(v) = \mathrm{Link}(v) \cup \{v\}$.

Definition 1.1. A *clique* in the graph Γ is a subgraph $\Gamma_0 \subseteq \Gamma$ such that for every $v, v' \in V\Gamma_0$ with $v \neq v'$ we have $(v, v') \in E\Gamma_0$ (so a complete subgraph of Γ). In particular every single vertex of Γ forms a clique (with no edges). By convention the empty graph is a clique. We denote $\mathrm{Cliq}(s)$ for all cliques in Γ with exactly s vertices.

Definition 1.2. For each $v \in V\Gamma$ let G_v be a discrete group. The graph product G_Γ is defined as the group obtained from the free product of $G_v, v \in V\Gamma$ by adding the relations

$$[s, t] = 1 \text{ for } s \in G_v, t \in G_w \text{ for every } v \neq w \text{ such that } (v, w) \in E\Gamma.$$

A *word* is a finite sequence $\mathbf{v} = (v_1, \ldots, v_n)$ of elements in $V\Gamma$. We shall commonly use bold face notation for words and write v_i for the entries of $\mathbf{v}$. The collection of words is denoted by $\mathcal{W}$ and by convention the empty word is not included in $\mathcal{W}$. We say that two words $\mathbf{v}$ and $\mathbf{w}$ are *equivalent* if they are equivalent modulo the equivalence relation generated by the two relations:

$$\begin{aligned} &\text{I} && (v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) \simeq (v_1, \ldots, v_i, v_{i+2}, \ldots, v_n) && \text{if} \quad v_i = v_{i+1}, \\ &\text{II} && (v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) \simeq (v_1, \ldots, v_{i+1}, v_i, \ldots, v_n) && \text{if} \quad v_i \in \mathrm{Link}(v_{i+1}). \end{aligned} \tag{1.1}$$

Moreover, we say that two words $\mathbf{v}$ and $\mathbf{w}$ are *type* II *equivalent* if they are equivalent modulo the sub-equivalence relation generated by the relation II. A word $\mathbf{v} \in \mathcal{W}$ is *reduced* if the following statement holds:

$$\begin{aligned} &\text{If there is a } v \in V\Gamma \text{ such that } v_k, v_l = v \text{ with } l > k, \\ &\text{then we do not have that all } v_{k+1}, \ldots, v_{l-1} \in \mathrm{Star}(v). \end{aligned} \tag{1.2}$$

We let $\mathcal{W}_{\mathrm{red}}$ be the set of all reduced words. Observe that if $\mathbf{v}$ is reduced and type II equivalent to $\mathbf{v}'$ then also $\mathbf{v}'$ is reduced.

Lemma 1.3. *We have,*

(1) Every word $\mathbf{v}$ is equivalent to a reduced word $\mathbf{w} = (w_1, \ldots, w_n)$.
(2) If $\mathbf{v}$ is also equivalent to a reduced word $\mathbf{w}'$, then the lengths of $\mathbf{w}$ and $\mathbf{w}'$ are equal.
(3) Moreover, there exists a permutation σ of $\{1, \ldots, n\}$ such that $\mathbf{w}' = (w_{\sigma(1)}, w_{\sigma(2)}, \ldots, w_{\sigma(n)})$.
(4) There is a unique such σ if we impose the condition that if $k > l$ and $w_k = w_l$, then $\sigma(k) > \sigma(l)$.

Proof. (1) This follows since if a word cannot be made shorter my means of permutations and cancellations (1.1) then it is reduced.

(2) This is essentially the normal form theorem [Gr90, Theorem 3.9]. It can be derived as follows. For each $v \in V\Gamma$ let G_v be the group $\mathbb{R}^+$ with multiplication. For $x \in \mathbb{R}^+$ we shall explicitly write x_v to identify it as an element of G_v. Associate to the word $\mathbf{w}$ of length n the group element $g_{\mathbf{w}} := 2_{w_1} 2_{w_2} \ldots 2_{w_n}$ in the graph product of the groups $G_v, v \in V\Gamma$, see Definition 1.2. Since $\mathbf{w}$ is reduced it follows that $g_{\mathbf{w}}$ is reduced in the sense of [Gr90]. Assume that $\mathbf{w}'$ has length m. Since $\mathbf{w}$ is equivalent to $\mathbf{w}'$, there exists elements $x_1, \ldots, x_m$ with $x_i \in G_{w_i'}$ and $x_i > 1$ such that $g_{\mathbf{w}}$ is equivalent to the graph product element $g_{\mathbf{w}'} = x_1 \ldots x_m$ (this can easily be seen by checking this on each step (1.1) to obtain this equivalence, in particular x_i is either a power or a root of 2). Since $\mathbf{w}'$ is reduced it also follows that $g_{\mathbf{w}'}$ is reduced. In all $g_{\mathbf{w}'}$ and $g_{\mathbf{w}}$ are reduced equivalent elements in the graph product of $G_v, v \in \Gamma$ and by the normal form Theorem [Gr90, Theorem 3.9], this implies that $m = n$. In fact [Gr90, Theorem 3.9] implies also that $x_i = 2$.

(3) Let m be the total number of times that a given v appears in $\mathbf{w}$. We need to show that v appears exactly m times in $\mathbf{w}'$. Suppose that this is not the case. Since $\mathbf{w}$ and $\mathbf{w}'$ have the same word length we may assume without loss of generality that it appears less than m times in $\mathbf{w}'$ since else we may change v to another vertex for which this is true. But since $\mathbf{w}'$ is obtained from $\mathbf{w}$ through the equivalences (1.1) this means that there exists some $l > k$ such that $w_l = w_k = v$ and $w_{k+1}, \ldots, w_{l-1} \in \mathrm{Star}(v)$ which contradicts the fact that $\mathbf{w}$ is reduced.

(4) The statement follows since it states that σ must for every w occuring in $\mathbf{w}$ be an increasing bijection between the sets $K_w := \{i \mid w_i = w\}$ and $K_w' = \{i \mid w_i' = w\}$. Such a bijection is unique. □

Let $\mathcal{W}_{\mathrm{min}}$ be a complete set of representatives of the reduced words under the equivalence relation described above. We call an element of $\mathcal{W}_{\mathrm{min}}$ a *minimal word*. It is then clear that every word $\mathbf{v}$ is equivalent to a unique minimal word $\mathbf{w}$. Note that $\mathcal{W}_{\mathrm{min}}$ excludes the empty word.

2. Graph products of operator algebras

In this section we construct graph products of operator algebras. In case the graph Γ does not have edges the graph product coincides with the free product for which we refer to [Vo85]. In addition it is important to emphasize that our constructions are different from [FiFr13]: indeed graph products impose *commutation relations* on the resulting algebra which in general cannot be written in terms of the *amalgamations* imposed by the constructions in [FiFr13].

2.1. **The graph product Hilbert space.** For all $v \in V\Gamma$ let $\mathcal{H}_v$ be a Hilbert space with a norm one vector $\xi_v \in \mathcal{H}_v$. Define $\mathcal{H}_v^\circ = \mathcal{H}_v \ominus \mathbb{C}\xi_v$ and let $\mathcal{P}_v$ be the orthogonal projection onto $\mathcal{H}_v^\circ$. For $\mathbf{v} \in \mathcal{W}_{\mathrm{red}}$ we let,

$$\mathcal{H}_{\mathbf{v}} = \mathcal{H}_{v_1}^\circ \otimes \ldots \otimes \mathcal{H}_{v_n}^\circ.$$

By Lemma 1.3 we see that if $\mathbf{v} \in \mathcal{W}_{\mathrm{red}}$ is equivalent to $\mathbf{w} \in \mathcal{W}_{\mathrm{red}}$ then there exists a uniquely determined unitary map,

$$\mathcal{Q}_{\mathbf{v},\mathbf{w}} : \mathcal{H}_{\mathbf{v}} \to \mathcal{H}_{\mathbf{w}} : \xi_{v_1} \otimes \ldots \otimes \xi_{v_n} \mapsto \xi_{v_{\sigma(1)}} \otimes \ldots \otimes \xi_{v_{\sigma(n)}}, \tag{2.1}$$

where σ is as in Lemma 1.3 (4). Since every $\mathbf{v} \in \mathcal{W}_{\mathrm{red}}$ has a unique minimal form $\mathbf{v}'$ we may simply write $\mathcal{Q}_{\mathbf{v}}$ for $\mathcal{Q}_{\mathbf{v},\mathbf{v}'}$.

Define the *graph product Hilbert space* $(\mathcal{H}, \Omega)$ by:

$$\mathcal{H} = \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w} \in \mathcal{W}_{\min}} \mathcal{H}_{\mathbf{w}}.$$

For $v \in V\Gamma$, let $\mathcal{W}_v$ be the set of minimal reduced words $\mathbf{w}$ such that the concatenation $v\mathbf{w}$ is still reduced and write $\mathcal{W}_v^c = \mathcal{W}_{\min} \setminus \mathcal{W}_v$. Define

$$\mathcal{H}(v) = \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w} \in \mathcal{W}_v} \mathcal{H}_{\mathbf{w}}.$$

We define the isometry $U_v : \mathcal{H}_v \otimes \mathcal{H}(v) \to \mathcal{H}$ in the following way:

$$\begin{array}{rccc} U_v : & \mathcal{H}_v \otimes \mathcal{H}(v) & \longrightarrow & \mathcal{H} \\ & \xi_v \otimes \Omega & \xrightarrow{\simeq} & \Omega \\ & \mathcal{H}_v^\circ \otimes \Omega & \xrightarrow{\simeq} & \mathcal{H}_v^\circ \\ & \xi_v \otimes \mathcal{H}_{\mathbf{w}} & \xrightarrow{\simeq} & \mathcal{H}_{\mathbf{w}} \\ & \mathcal{H}_v^\circ \otimes \mathcal{H}_{\mathbf{w}} & \xrightarrow{\simeq} & \mathcal{Q}_{v\mathbf{w}}(\mathcal{H}_v^\circ \otimes \mathcal{H}_{\mathbf{w}}) \end{array}$$

Here the actions are understood naturally. Observe that, for any reduced word $\mathbf{w}$, the word $v\mathbf{w}$ is not reduced if and only if $\mathbf{w}$ is equivalent to a reduced word that starts with v. It follows that U_v is surjective, hence unitary. Define, for $v \in V\Gamma$, the faithful unital normal $*$-homomorphism $\lambda_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{H})$ by

$$\lambda_v(x) = U_v(x \otimes 1)U_v^* \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}_v).$$

Observe the λ_v intertwines the vector states ω_{ξ_v} and ω_Ω.

Proposition 2.1. *For all $v \in V\Gamma$ and all $x \in \mathcal{B}(\mathcal{H}_v)$ one has:*

1. *$\lambda_v(x)\Omega = \mathcal{P}_v(x\xi_v) + \langle x\xi_v, \xi_v\rangle\Omega$.*
2. *$\lambda_v(x)\xi = \mathcal{P}_v(x\xi) + \langle x\xi, \xi_v\rangle\Omega$ for all $\xi \in \mathcal{H}_v^\circ$.*
3. *$\lambda_v(x)\xi = \mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v) \otimes \xi) + \langle x\xi_v, \xi_v\rangle\xi$ for all $\mathbf{w} \in \mathcal{W}_v$ and all $\xi \in \mathcal{H}_{\mathbf{w}}$.*
4. *Let $\mathbf{w} \in \mathcal{W}_v^c$ then there exists a unique $\mathbf{w}_v \in \mathcal{W}_v$ such that $\mathbf{w} \simeq v\mathbf{w}_v$ are II equivalent and, for all $\xi \in \mathcal{H}_{\mathbf{w}}$, one has*

$$\lambda_v(x)\xi = \mathcal{Q}_{v\mathbf{w}_v}(\mathcal{P}_v x \otimes id)\mathcal{Q}_{v\mathbf{w}_v}^*\xi + (\xi_v^* x \otimes id)\mathcal{Q}_{v\mathbf{w}_v}^*\xi.$$

Moreover, the images of λ_v and $\lambda_{v'}$ commute whenever $(v, v') \in E\Gamma$.

Proof. The first part of the proposition is an immediate consequence of the definition of U_v.
(1) One has $\lambda_v(x)\Omega = U_v(x\xi_v \otimes \Omega) = U_v(\mathcal{P}_v(x\xi_v) \otimes \Omega) + \langle x\xi_v, \xi_v\rangle U_v(\xi_v \otimes \Omega) = \mathcal{P}_v(x\xi_v) + \langle x\xi_v, \xi_v\rangle\Omega$.
(2) Let $\xi \in \mathcal{H}_v^\circ$, one has $\lambda_v(x)\xi = U_v(x\xi \otimes \Omega) = \mathcal{P}_v(x\xi) + \langle x\xi, \xi_v\rangle\Omega$.
(3) Let $\mathbf{w} \in \mathcal{W}_v$ and $\xi \in \mathcal{H}_{\mathbf{w}}$, one has $\lambda_v(x)\xi = U_v(x\xi_v \otimes \xi) = \mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v) \otimes \xi) + \langle x\xi_v, \xi_v\rangle\xi$.
(4) Let $\mathbf{w}_v \in \mathcal{W}_v$, $\xi \in \mathcal{H}_v^\circ$ and $\eta \in \mathcal{H}_{\mathbf{w}_v}$. We find $\lambda_v(x)\mathcal{Q}_{v\mathbf{w}_v}(\xi \otimes \eta) = \mathcal{Q}_{v\mathbf{w}_v}(\mathcal{P}_v(x\xi) \otimes \eta)) + \langle x\xi, \xi_v\rangle\eta$. Hence, for all $\xi \in \mathcal{H}_{v\mathbf{w}_v}$, one has $\lambda_v(x)\mathcal{Q}_{v\mathbf{w}_v}\xi = \mathcal{Q}_{v\mathbf{w}_v}(\mathcal{P}_v x \otimes \mathrm{id})\xi + (\xi_v^* x \otimes \mathrm{id})\xi$. Since $\mathcal{Q}_{v\mathbf{w}_v} : \mathcal{H}_{v\mathbf{w}_v} \to \mathcal{H}_{\mathbf{w}}$ is unitary, this gives the result.

We can now finish the proof of the proposition. Let $v, v' \in V\Gamma$ such that $(v, v') \in E\Gamma$. Let $x \in \mathcal{B}(\mathcal{H}_v)$ and $y \in \mathcal{B}(\mathcal{H}_{v'})$. Writing $\lambda_v(x) = \lambda_v(x - \langle x\xi_v, \xi_v\rangle 1) + \langle x\xi_v, \xi_v\rangle 1$ and $\lambda_{v'}(x) = \lambda_{v'}(y - \langle y\xi_{v'}, \xi_{v'}\rangle 1) + \langle y\xi_{v'}, \xi_{v'}\rangle 1$ we see that we may and will assume that $\langle x\xi_v, \xi_v\rangle = \langle y\xi_{v'}, \xi_{v'}\rangle = 0$.

Let $\mathbf{w}$ be the unique reduced minimal word equivalent to (vv'). Then $\mathbf{w} = (vv')$ or $\mathbf{w} = (v'v)$. In both cases we find $\mathcal{Q}_{vv'} = \mathcal{Q}_{v'v} \circ \Sigma$, where $\Sigma : \mathcal{H}_{vv'} = \mathcal{H}_v^\circ \otimes \mathcal{H}_{v'}^\circ \to \mathcal{H}_{v'}^\circ \otimes \mathcal{H}_v^\circ = \mathcal{H}_{v'v}$ is the flip map. Hence, we find:

$$\lambda_v(x)\lambda_{v'}(y)\Omega = \lambda_v(x)\mathcal{P}_{v'}(y\xi_{v'}) = \mathcal{Q}_{vv'}(\mathcal{P}_v(x\xi_v) \otimes \mathcal{P}_{v'}(y\xi_{v'})) = \mathcal{Q}_{v'v}(\mathcal{P}_{v'}(y\xi_{v'}) \otimes \mathcal{P}_v(x\xi_v)) = \lambda_{v'}(y)\lambda_v(x)\Omega.$$

Let $\xi \in \mathcal{H}_{v'}^{\circ}$. One has $\lambda_v(x)\lambda_{v'}(y)\xi = \lambda_v(x)\mathcal{P}_{v'}(y\xi) = \mathcal{Q}_{vv'}(\mathcal{P}_v(x\xi_v)\otimes\mathcal{P}_{v'}(y\xi))$ and,

$$\begin{aligned}\lambda_{v'}(y)\lambda_v(x)\xi &= \lambda_{v'}(y)(\mathcal{Q}_{vv'}(\mathcal{P}_v(x\xi_v)\otimes\xi)) = \lambda_{v'}(y)(\mathcal{Q}_{v'v}(\xi\otimes\mathcal{P}_v(x\xi_v))) = \mathcal{Q}_{v'v}(\mathcal{P}_{v'}(y\xi)\otimes\mathcal{P}_v(x\xi_v))\\ &= \mathcal{Q}_{vv'}(\mathcal{P}_v(x\xi_v)\otimes\mathcal{P}_{v'}(y\xi)) = \lambda_v(x)\lambda_{v'}(y)\xi.\end{aligned}$$

Claim 1. Let $(v,v')\in E\Gamma$ and $\mathbf{w}\in\mathcal{W}_{min}$.

(1) Suppose that $\mathbf{w}\in\mathcal{W}_v\cap\mathcal{W}_{v'}$ and define $\mathbf{w}_1,\mathbf{w}_2\in\mathcal{W}_{min}$ such that $v'\mathbf{w}\simeq\mathbf{w}_1$ and $v\mathbf{w}\simeq\mathbf{w}_2$. Then $\mathbf{w}_1\in\mathcal{W}_v$, $\mathbf{w}_2\in\mathcal{W}_{v'}$, $v\mathbf{w}_1\simeq v'\mathbf{w}_2$ and, for all $\eta_v\in\mathcal{H}_v^{\circ}$, $\eta_{v'}\in\mathcal{H}_{v'}^{\circ}$ and $\xi\in\mathcal{H}_{\mathbf{w}}$ one has

$$\mathcal{Q}_{v\mathbf{w}_1}(\eta_v\otimes\mathcal{Q}_{v'\mathbf{w}}(\eta_{v'}\otimes\xi)) = \mathcal{Q}_{v'\mathbf{w}_2}(\eta_{v'}\otimes\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi)).$$

(2) Suppose that $\mathbf{w}\in\mathcal{W}_v\setminus\mathcal{W}_{v'}$ and define $\mathbf{w}_1\in\mathcal{W}_{v'}$, $\mathbf{w}_2\in\mathcal{W}_{min}$ such that $\mathbf{w}\simeq v'\mathbf{w}_1$ and $\mathbf{w}_2\simeq v\mathbf{w}$. Then, $\mathbf{w}_1\in\mathcal{W}_v$ and $\mathbf{w}_2\in\mathcal{W}_{v'}^c$. Let $\mathbf{w}_3\in\mathcal{W}_{v'}$ such that $\mathbf{w}_2\simeq v'\mathbf{w}_3$. For all $\xi\in\mathcal{H}_w$, $y\in\mathcal{B}(\mathcal{H}_{v'})$, $\eta\in\mathcal{H}_v^{\circ}$,

$$\eta_v\otimes\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^*\xi = \mathcal{Q}_{v\mathbf{w}}^*\mathcal{Q}_{v'\mathbf{w}_3}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_3}^*\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi) \quad \text{and,}$$

$$\mathcal{Q}_{v\mathbf{w}_1}\left(\eta_v\otimes(\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^*\xi\right) = (\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_3}^*\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi).$$

(3) Suppose that $\mathbf{w}\in\mathcal{W}_v^c\cap\mathcal{W}_{v'}^c$ and define $\mathbf{w}_1\in\mathcal{W}_{v'}$, $\mathbf{w}_2\in\mathcal{W}_v$ such that $\mathbf{w}\simeq v'\mathbf{w}_1$ and $\mathbf{w}\simeq v\mathbf{w}_2$. Then, $\mathbf{w}_1\in\mathcal{W}_v^c$ and $\mathbf{w}_2\in\mathcal{W}_{v'}^c$. Define $\mathbf{w}_1'\in\mathcal{W}_v$ and $\mathbf{w}_2'\in\mathcal{W}_{v'}$ such that $\mathbf{w}_1\simeq v\mathbf{w}_1'$ and $\mathbf{w}_2\simeq v'\mathbf{w}_2'$. One has

$$\begin{aligned}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^* &= \mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^*\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*,\\ (\xi_v^*x\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^* &= \mathcal{Q}_{v'\mathbf{w}_2'}(\mathcal{P}_{v'}y\otimes id)\mathcal{Q}_{v'\mathbf{w}_2'}^*(\xi_v^*x\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*,\\ \mathcal{Q}_{v\mathbf{w}_1'}(\mathcal{P}_vx\otimes id)\mathcal{Q}_{v\mathbf{w}_1'}^*(\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^* &= (\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^*\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*,\\ (\xi_v^*x\otimes id)\mathcal{Q}_{v\mathbf{w}_1'}^*(\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_1}^* &= (\xi_{v'}^*y\otimes id)\mathcal{Q}_{v'\mathbf{w}_2}^*(\xi_v^*x\otimes id)\mathcal{Q}_{v\mathbf{w}_2}^*.\end{aligned}$$

Proof of Claim 1. In each of the subclaims we let $u_1,\dots,u_n$ with $u_i\in V\Gamma$ denote (part of the) letters of $\mathbf{w}$.

(1) We may assume that $\xi=\xi_{u_1}\otimes\ldots\otimes\xi_{u_n}$ is a simple tensor product with $\xi_{u_i}\in\mathcal{H}_{u_i}$. Let l be such that $u_1,\dots,u_l\in\mathrm{Link}(v)$ and k be such that $u_1,\dots,u_k\in\mathrm{Link}(v')$. Moreover, we may choose l,k such that the words $(u_1,\dots,u_l,v,u_{l+1},\dots,u_n)$ and $(u_1,\dots,u_k,v',u_{k+1},\dots,u_n)$ are minimal. Without loss of generality we may assume that $L(v)<L(v')$. Then,

$$\begin{aligned}&\mathcal{Q}_{v\mathbf{w}_1}(\eta_v\otimes\mathcal{Q}_{v'\mathbf{w}}(\eta_{v'}\otimes\xi)) = \mathcal{Q}_{v\mathbf{w}_1}(\eta_v\otimes\xi_{u_1}\otimes\ldots\otimes\xi_{u_k}\otimes\eta_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n})\\ =&\xi_{u_1}\otimes\ldots\otimes\xi_{u_l}\otimes\eta_v\otimes\xi_{u_{l+1}}\otimes\ldots\otimes\xi_{u_k}\otimes\eta_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n},\end{aligned}$$

and using the fact that $(v,v')\in E\Gamma$ the same computation shows that this expression equals $\mathcal{Q}_{v'\mathbf{w}_2}(\eta_{v'}\otimes\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi))$.

(2) We may assume that $\xi=\xi_{u_1}\otimes\ldots\otimes\xi_{u_k}\otimes\eta_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n}$ and that $u_1\dots u_kv'u_{k+1}\dots u_n$ is reduced with $u_1,\dots,u_k\in\mathrm{Link}(v')$. Then, letting l be such that $u_1,\dots,u_l\in\mathrm{Link}(v)$ and $u_1\dots u_lvu_{l+1}\dots u_n$ minimal,

$$\begin{aligned}&\mathcal{Q}_{v\mathbf{w}}^*\mathcal{Q}_{v'\mathbf{w}_3}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}_{v'\mathbf{w}_3}^*\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi) = \mathcal{Q}_{v\mathbf{w}}^*\mathcal{Q}_{v'\mathbf{w}_3}((\mathcal{P}_{v'}y\eta_{v'})\otimes\xi_{u_1}\otimes\ldots\xi_{u_l}\otimes\eta_v\otimes\xi_{u_{l+1}}\otimes\ldots\otimes\xi_{u_n})\\ =&\eta_v\otimes\xi_{u_1}\otimes\ldots\otimes\xi_{u_k}\otimes\mathcal{P}_{v'}y\eta_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n} = \eta_v\otimes\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}_{v'\mathbf{w}_1}^*\xi \quad\text{and,}\\ &(\xi_{v'}^*y\otimes\mathrm{id})\mathcal{Q}_{v'\mathbf{w}_3}^*\mathcal{Q}_{v\mathbf{w}}(\eta_v\otimes\xi) = (\xi_{v'}^*y\otimes\mathrm{id})\eta_{v'}\otimes\xi_{u_1}\otimes\ldots\otimes\xi_{u_l}\otimes\eta_v\otimes\xi_{u_{l+1}}\otimes\ldots\otimes\xi_{u_n}\\ =&\langle y\eta_{v'},\xi_{v'}\rangle\xi_{u_1}\otimes\ldots\otimes\xi_{u_l}\otimes\eta_v\otimes\xi_{u_{l+1}}\otimes\ldots\otimes\xi_{u_n} = \mathcal{Q}_{v\mathbf{w}_1}(\eta_v\otimes\langle y\eta_{v'},\xi_{v'}\rangle\xi_{u_1}\otimes\ldots\otimes\xi_{u_n})\\ =&\mathcal{Q}_{v\mathbf{w}_1}(\eta_v\otimes(\xi_{v'}^*y\otimes\mathrm{id})\mathcal{Q}_{v'\mathbf{w}_1}^*\xi).\end{aligned}$$

(3) Assume that $\xi=\xi_{u_1}\otimes\ldots\otimes\xi_{u_l}\otimes\xi_v\otimes\xi_{u_{l+1}}\ldots\otimes\xi_{u_k}\otimes\xi_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n}$ with $u_1\dots u_lvu_{l+1}\dots u_kvu_{k+1}\dots u_n$ minimal and $u_1,\dots,u_l\in\mathrm{Link}(v)$, $u_1,\dots u_k\in\mathrm{Link}(v')$. Then, using $(v,v')\in E\Gamma$,

$$\begin{aligned}&\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}_{v'\mathbf{w}_1}^*\xi\\ =&\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi_{u_1}\otimes\ldots\otimes\xi_{u_k}\otimes\mathcal{P}_{v'}y\xi_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n}\\ =&\xi_{u_1}\otimes\ldots\otimes\xi_{u_l}\otimes\mathcal{P}_vx\xi_v\otimes\xi_{u_{l+1}}\otimes\ldots\otimes\xi_{u_k}\otimes\mathcal{P}_{v'}y\xi_{v'}\otimes\xi_{u_{k+1}}\otimes\ldots\otimes\xi_{u_n},\end{aligned}$$

which equals $\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}$ by a reverse computation. The other equalities follow in a similar way.

Remainder of the proof of the proposition. Let $\mathbf{w}\in\mathcal{W}_v\cap\mathcal{W}_{v'}$ and consider the minimal words $\mathbf{w}_1$, $\mathbf{w}_2$ introduced in the first assertion of Claim 1. One has,

$$\begin{aligned}\lambda_v(x)\lambda_{v'}(y)\xi &= \lambda_v(x)\mathcal{Q}_{v'\mathbf{w}}(\mathcal{P}_{v'}(y\xi_{v'})\otimes\xi) = \mathcal{Q}_{v\mathbf{w}_1}\left(\mathcal{P}_v(x\xi_v)\otimes\mathcal{Q}_{v'\mathbf{w}}(\mathcal{P}_{v'}(y\xi_{v'})\otimes\xi)\right)\\ &= \mathcal{Q}_{v\mathbf{w}_2}\left(\mathcal{P}_{v'}(y\xi_{v'})\otimes\mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v)\otimes\xi)\right) = \lambda_{v'}(y)\lambda_v(x)\xi.\end{aligned}$$

Let $\mathbf{w}\in\mathcal{W}_v\setminus\mathcal{W}_{v'}$ and consider the minimal words $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ introduced in the second assertion of Claim 1. One has,

$$\begin{aligned}\lambda_v(x)\lambda_{v'}(y)\xi &= \lambda_v(x)\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi + \lambda_v(x)(\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi\\ &= \mathcal{Q}_{v\mathbf{w}}\left(\mathcal{P}_v(x\xi_v)\otimes\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi\right) + \mathcal{Q}_{v\mathbf{w}_1}\left(\mathcal{P}_v(x\xi_v)\otimes(\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi\right)\text{ and,}\end{aligned}$$

$$\begin{aligned}\lambda_{v'}(y)\lambda_v(x)\xi &= \lambda_{v'}(y)\mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v)\otimes\xi)\\ &= \mathcal{Q}_{v'\mathbf{w}_3}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_3}\mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v)\otimes\xi) + (\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_3}\mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v(x\xi_v)\otimes\xi).\end{aligned}$$

These two expressions are equal by the second assertion of Claim 1. By symmetry, this is also true when $\mathbf{w}\in\mathcal{W}_{v'}\setminus\mathcal{W}_v$. Finally, let $\mathbf{w}\in\mathcal{W}_v^c\cap\mathcal{W}_{v'}^c$ and consider the minimal words $\mathbf{w}_1,\mathbf{w}_1',\mathbf{w}_2,\mathbf{w}_2'$ introduced in the third assertion of Claim 1. On has:

$$\begin{aligned}\lambda_v(x)\lambda_{v'}(y) &= \lambda_v(x)\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi + \lambda_v(x)(\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi.\\ &= \mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi + (\xi^*_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi\\ &\quad+\mathcal{Q}_{v\mathbf{w}_1'}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_1'}(\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi + (\xi^*_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_1'}(\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\xi\quad\text{and,}\\ \lambda_{v'}(y)\lambda_v(x)\xi &= \lambda_{v'}(y)\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi + \lambda_{v'}(y)(\xi^*_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi\\ &= \mathcal{Q}_{v'\mathbf{w}_1}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi + (\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_1}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi\\ &\quad+\mathcal{Q}_{v'\mathbf{w}_2'}(\mathcal{P}_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_2'}(\xi^*_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi + (\xi^*_{v'}y\otimes\mathrm{id})\mathcal{Q}^*_{v'\mathbf{w}_2}(\xi^*_vx\otimes\mathrm{id})\mathcal{Q}^*_{v\mathbf{w}_2}\xi.\end{aligned}$$

These two expressions are equal by the third assertion of Claim 1. It concludes the proof. □

We can also define the right versions of the unitaries U_v. For $v\in V\Gamma$, let $\mathcal{W}^v$ be the set of minimal reduced words $\mathbf{w}$ such that the concatenation $\mathbf{w}v$ is still reduced and write $(\mathcal{W}^v)^c=\mathcal{W}_{\min}\setminus\mathcal{W}^v$. Define

$$\mathcal{H}'(v)=\mathbb{C}\Omega\oplus\bigoplus_{\mathbf{w}\in\mathcal{W}^v}\mathcal{H}_{\mathbf{w}}.$$

We define the isometry $U_v':\mathcal{H}'(v)\otimes\mathcal{H}_v\to\mathcal{H}$ in the following way:

$$\begin{aligned}U_v':\quad \mathcal{H}'(v)\otimes\mathcal{H}_v &\longrightarrow \mathcal{H}\\ \Omega\otimes\xi_v &\xrightarrow{\simeq} \Omega\\ \Omega\otimes\mathcal{H}_v^\circ &\xrightarrow{\simeq} \mathcal{H}_v^\circ\\ \mathcal{H}_{\mathbf{w}}\otimes\xi_v &\xrightarrow{\simeq} \mathcal{H}_{\mathbf{w}}\\ \mathcal{H}_{\mathbf{w}}\otimes\mathcal{H}_v^\circ &\xrightarrow{\simeq} \mathcal{Q}_{\mathbf{w}v}(\mathcal{H}_{\mathbf{w}}\otimes\mathcal{H}_v^\circ)\end{aligned}$$

As before, U_v' is unitary. Define, for $v\in V\Gamma$, the faithful unital normal $*$-homomorphism $\rho_v:\mathcal{B}(\mathcal{H}_v)\to\mathcal{B}(\mathcal{H})$ by $\rho_v(x)=U_v'(1\otimes x)(U_v')^*$ for all $x\in\mathcal{B}(\mathcal{H}_v)$. Observe the ρ_v intertwines the vector states ω_{ξ_v} and ω_Ω. The analogue of Proposition 2.1 holds. We leave the details to the reader.

Proposition 2.2. *For all $v\in V\Gamma$ and all $x\in\mathcal{B}(\mathcal{H}_v)$ one has:*

(1) $\rho_v(x)\Omega=\mathcal{P}_v(x\xi_v)+\langle x\xi_v,\xi_v\rangle\Omega$.
(2) $\rho_v(x)\xi=\mathcal{P}_v(x\xi)+\langle x\xi,\xi_v\rangle\Omega$ for all $\xi\in\mathcal{H}_v^\circ$.
(3) $\rho_v(x)\xi=\mathcal{Q}_{\mathbf{w}v}(\xi\otimes\mathcal{P}_v(x\xi_v))+\langle x\xi_v,\xi_v\rangle\xi$ for all $\mathbf{w}\in\mathcal{W}^v$ and all $\xi\in\mathcal{H}_{\mathbf{w}}$.
(4) Let $\mathbf{w}\in(\mathcal{W}^v)^c$ then there exists a unique $\mathbf{w}_v'\in\mathcal{W}^v$ such that $\mathbf{w}\simeq\mathbf{w}_v'v$ and, for all $\xi\in\mathcal{H}_{\mathbf{w}}$, one has

$$\rho_v(x)\xi=\mathcal{Q}_{\mathbf{w}_v'v}(id\otimes\mathcal{P}_vx)\mathcal{Q}^*_{\mathbf{w}_v'v}\xi+(id\otimes\xi_v^*x)\mathcal{Q}^*_{\mathbf{w}_v'v}\xi.$$

Moreover, the images of ρ_v and $\rho_{v'}$ commute whenever $(v,v')\in E\Gamma$.

Proposition 2.3. *Let $v, v' \in V\Gamma$ and $x \in \mathcal{B}(\mathcal{H}_v)$, $y \in \mathcal{B}(\mathcal{H}_{v'})$. One has*

$$\lambda_v(x)\rho_{v'}(y) = \rho_{v'}(y)\lambda_v(x) \quad \text{whenever} \quad (v \neq v') \text{ or } (v = v' \text{ and } xy = yx).$$

Proof. We may and will assume that $\langle x\xi_v, \xi_v\rangle = 0 = \langle y\xi_{v'}, \xi_{v'}\rangle$. By Propositions 2.1 and 2.2, one has

$$\lambda_v(x)\rho_{v'}(y)\Omega = \lambda_v(x)(y\xi_{v'}) = \begin{cases} \mathcal{Q}_{vv'}(x\xi_v \otimes y\xi_{v'}) & \text{if} \quad v \neq v', \\ \mathcal{P}_v(xy\xi_v) + \langle xy\xi_v, \xi_v\rangle\Omega & \text{if} \quad v = v'. \end{cases}$$

Moreover, $\rho_{v'}(y)\lambda_v(x)\Omega = \rho_{v'}(y)(x\xi_v) = \begin{cases} \mathcal{Q}_{vv'}(x\xi_v \otimes y\xi_{v'}) & \text{if} \quad v \neq v', \\ \mathcal{P}_v(yx\xi_v) + \langle yx\xi_v, \xi_v\rangle\Omega & \text{if} \quad v = v'. \end{cases}$.

To finish the proof we need the following Claim.

Claim. Let $v, v' \in V\Gamma$, $\mathbf{w} \in \mathcal{W}_{min}$ and $\xi \in \mathcal{H}_{\mathbf{w}}$.

(1) If $\mathbf{w} \in \mathcal{W}_v \cap \mathcal{W}^{v'}$. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{min}$ such that $\mathbf{w}_1 \simeq \mathbf{w}v'$ and $\mathbf{w}_2 \simeq v\mathbf{w}$.

- *If $v\mathbf{w}v'$ is reduced then $\mathbf{w}_1 \in \mathcal{W}_v$, $\mathbf{w}_2 \in \mathcal{W}^{v'}$ and, for all $\eta_v \in \mathcal{H}_v$, $\eta_{v'} \in \mathcal{H}_{v'}$ one has*

$$\mathcal{Q}_{v\mathbf{w}_1}(\eta_v \otimes \mathcal{Q}_{\mathbf{w}v'}(\xi \otimes \eta_{v'})) = \mathcal{Q}_{\mathbf{w}_2v'}(\mathcal{Q}_{v\mathbf{w}}(\eta_v \otimes \xi) \otimes \eta_{v'}).$$

- *If $v\mathbf{w}v'$ is not reduced then $v = v'$, $\mathbf{w}_1 = \mathbf{w}_2 \simeq v\mathbf{w} \simeq \mathbf{w}v \in \mathcal{W}_v^c \cap (\mathcal{W}^v)^c$ and, for all $x, y \in \mathcal{B}(\mathcal{H}_v)$,*

$$\mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}}\mathcal{Q}_{\mathbf{w}v}(\xi \otimes y\xi_v) = \mathcal{Q}_{\mathbf{w}v}(id \otimes \mathcal{P}_v y)\mathcal{Q}^*_{\mathbf{w}v}\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi) \quad \text{and,}$$

$$(\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}}\mathcal{Q}_{\mathbf{w}v}(\xi \otimes y\xi_v) = (id \otimes \xi_v^* y)\mathcal{Q}^*_{\mathbf{w}v}\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi).$$

(2) If $\mathbf{w} \in \mathcal{W}_v^c \cap (\mathcal{W}^{v'})^c$. Let $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{min}$ such that $\mathbf{w} \simeq \mathbf{w}_1v' \simeq v\mathbf{w}_2$.

- *If $\mathbf{w}_1 \in \mathcal{W}_v$ then $v = v'$, $\mathbf{w}_1 \in \mathcal{W}^v$, $w \simeq v\mathbf{w}_1 \simeq \mathbf{w}_1 v$ and $\mathbf{w}_1 = \mathbf{w}_2$ and, for all $x, y \in \mathcal{B}(\mathcal{H}_v)$,*

$$\begin{aligned} &\mathcal{Q}_{v\mathbf{w}_1}(x\xi_v \otimes (id \otimes \xi_v^* y)\mathcal{Q}^*_{\mathbf{w}_1 v}\xi) + \mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_1}\mathcal{Q}_{\mathbf{w}_1 v}(id \otimes \mathcal{P}_v y)\mathcal{Q}^*_{\mathbf{w}_1 v}\xi \\ =&\mathcal{Q}_{\mathbf{w}_1 v}((\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_1}\xi \otimes y\xi_v) + \mathcal{Q}_{\mathbf{w}_1 v}(id \otimes \mathcal{P}_v y)\mathcal{Q}^*_{\mathbf{w}_1 v}\mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_1}\xi, \\ &(\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_1}\mathcal{Q}_{\mathbf{w}_1 v}(id \otimes \mathcal{P}_v y)\mathcal{Q}^*_{\mathbf{w}_1 v}\xi = (id \otimes \xi_v^* y)\mathcal{Q}^*_{\mathbf{w}_1 v}\mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_1}\xi. \end{aligned}$$

- *If $\mathbf{w}_1 \in \mathcal{W}_v^c$ write $\mathbf{w}_1 \simeq v\mathbf{w}_3$, $\mathbf{w}_3 \in \mathcal{W}_{min}$ then $\mathbf{w}_2 \simeq \mathbf{w}_3v' \in (\mathcal{W}^{v'})^c$ and $\forall x \in \mathcal{B}(\mathcal{H}_v)$, $y \in \mathcal{B}(\mathcal{H}_{v'})$,*

$$\begin{aligned} \mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\mathcal{Q}_{\mathbf{w}_1 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi &= \mathcal{Q}_{\mathbf{w}_1 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi, \\ (\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\mathcal{Q}_{\mathbf{w}_1 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi &= \mathcal{Q}_{\mathbf{w}_3 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_3 v'}(\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi, \\ \mathcal{Q}_{v\mathbf{w}_3}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_3}(id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi &= (id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi, \\ (\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_3}(id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi &= (id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_3 v'}(\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi. \end{aligned}$$

(3) If $\mathbf{w} \in \mathcal{W}_v^c \cap \mathcal{W}^{v'}$ write $\mathbf{w}_1 \simeq \mathbf{w}v'$, $w \simeq v\mathbf{w}_2$, $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{min}$. Then, $\mathbf{w}_1 \in \mathcal{W}_v^c$, $\mathbf{w}_2 \in \mathcal{W}^{v'}$ and, if $\mathbf{w}_3 \in \mathcal{W}_{min}$ is such that $\mathbf{w}_1 \simeq v\mathbf{w}_3$ then we have $\mathbf{w}_2v' \simeq \mathbf{w}_3$ and, for all $x \in \mathcal{B}(\mathcal{H}_v)$, $y \in \mathcal{B}(\mathcal{H}_{v'})$,

$$\begin{aligned} \mathcal{Q}_{v\mathbf{w}_3}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_3}\mathcal{Q}_{\mathbf{w}v'}(\xi \otimes y\xi_{v'}) &= \mathcal{Q}_{\mathbf{w}v'}\left(\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi \otimes y\xi_{v'}\right), \\ (\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_3}\mathcal{Q}_{\mathbf{w}v'}(\xi \otimes y\xi_{v'}) &= \mathcal{Q}_{\mathbf{w}_2v'}(\xi_v^* x \otimes id)\mathcal{Q}^*_{v\mathbf{w}_2}\xi \otimes y\xi_{v'}. \end{aligned}$$

(4) If $\mathbf{w} \in \mathcal{W}_v \cap (\mathcal{W}^{v'})^c$ write $\mathbf{w} \simeq w_1v'$, $\mathbf{w}_2 \simeq v\mathbf{w}$, $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{W}_{min}$. Then, $\mathbf{w}_1 \in \mathcal{W}_v$, $\mathbf{w}_2 \in (\mathcal{W}^{v'})^c$ and, if $\mathbf{w}_3 \in \mathcal{W}_{min}$ is such that $\mathbf{w}_2 \simeq \mathbf{w}_3v'$ then we have $\mathbf{w}_3 \simeq v\mathbf{w}_1$ and, for all $x \in \mathcal{B}(\mathcal{H}_v)$, $y \in \mathcal{B}(\mathcal{H}_{v'})$,

$$\begin{aligned} \mathcal{Q}_{v\mathbf{w}}\left(x\xi_v \otimes \mathcal{Q}_{\mathbf{w}_1 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi\right) &= \mathcal{Q}_{\mathbf{w}_3 v'}(id \otimes \mathcal{P}_{v'} y)\mathcal{Q}^*_{\mathbf{w}_3 v'}\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi), \\ \mathcal{Q}_{v\mathbf{w}_1}(x\xi_v \otimes (id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_1 v'}\xi) &= (id \otimes \xi_{v'}^* y)\mathcal{Q}^*_{\mathbf{w}_3 v'}\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi). \end{aligned}$$

The proof of the Claim is analogous to the proof of Claim 1 in Proposition 2.1 and we shall leave the details to the reader.

Remainder of the proof of the proposition. Let $\mathbf{w} \in \mathcal{W}_{\min}$ and $\xi \in \mathcal{H}_{\mathbf{w}}$. We use freely the results and notations of the Claim and Propositions 2.1 and 2.2.

Case 1: $\mathbf{w} \in \mathcal{W}_v \cap \mathcal{W}^{v'}$. If moreover $v\mathbf{w}v'$ is reduced we have,

$$\begin{aligned} \lambda_v(x)\rho_{v'}(y)\xi &= \lambda_v(x)(\mathcal{Q}_{\mathbf{w}v'}(\xi \otimes y\xi_{v'})) = \mathcal{Q}_{v\mathbf{w}_1}(x\xi_v \otimes \mathcal{Q}_{\mathbf{w}v'}(\xi \otimes y\xi_{v'})), \\ \rho_{v'}(y)\lambda_v(x)\xi &= \rho_{v'}(y)(\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi)) = \mathcal{Q}_{\mathbf{w}_2v'}(\mathcal{Q}_{v\mathbf{w}}(x\xi_v \otimes \xi) \otimes y\xi_{v'}). \end{aligned}$$

These two expressions are equal by the Claim. Suppose now that $v\mathbf{w}v'$ is not reduced. Then $v = v'$ and,

$$\begin{aligned}\lambda_v(x)\rho_v(y)\xi &= \lambda_v(x)(\mathcal{Q}_{\mathbf{w}v}(\xi\otimes y\xi_v)) = \mathcal{Q}_{v\mathbf{w}}(\mathcal{P}_v x\otimes \mathrm{id})\mathcal{Q}_{v\mathbf{w}}^*\mathcal{Q}_{\mathbf{w}v}(\xi\otimes y\xi_v) + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}}^*\mathcal{Q}_{\mathbf{w}v}(\xi\otimes y\xi_v),\\ \rho_v(y)\lambda_v(x)\xi &= \rho_v(y)(\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi)) = \mathcal{Q}_{\mathbf{w}v}(\mathrm{id}\otimes\mathcal{P}_v y)\mathcal{Q}_{\mathbf{w}v}^*\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi) + (\mathrm{id}\otimes\xi_v^* y)\mathcal{Q}_{\mathbf{w}v}^*\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi).\end{aligned}$$

These two expressions are equal by the Claim.

Case 2: $\mathbf{w}\in\mathcal{W}_v^c\cap(\mathcal{W}^{v'})^c$. If moreover $\mathbf{w}_1\in\mathcal{W}_v$ then $v = v'$, $\mathbf{w}_1 = \mathbf{w}_2\in\mathcal{W}^v$, $\mathbf{w}\simeq v\mathbf{w}_1\simeq\mathbf{w}_1 v$ and,

$$\begin{aligned}\lambda_v(x)\rho_v(y)\xi &= \lambda_v(x)\left(\mathcal{Q}_{\mathbf{w}_1 v}(\mathrm{id}\otimes\mathcal{P}_v y)\mathcal{Q}_{\mathbf{w}_1 v}^*\xi + (\mathrm{id}\otimes\xi_v^* y)\mathcal{Q}_{\mathbf{w}_1 v}^*\xi\right)\\ &= \mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\mathcal{Q}_{\mathbf{w}_1 v}(\mathrm{id}\otimes\mathcal{P}_v y)\mathcal{Q}_{\mathbf{w}_1 v}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\mathcal{Q}_{\mathbf{w}_1 v}(\mathrm{id}\otimes\mathcal{P}_v y)\mathcal{Q}_{\mathbf{w}_1 v}^*\xi\\ &\quad +\mathcal{Q}_{v\mathbf{w}_1}(x\xi_v\otimes(\mathrm{id}\otimes\xi_v^* y)\mathcal{Q}_{\mathbf{w}_1 v}^*\xi),\\ \rho_v(y)\lambda_v(x)\xi &= \rho_v(y)\left(\mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\xi\right)\\ &= \mathcal{Q}_{\mathbf{w}_1 v}(\mathrm{id}\otimes\mathcal{P}_v y)\mathcal{Q}_{\mathbf{w}_1 v}^*\mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\xi + (\mathrm{id}\otimes\xi_v^* y)\mathcal{Q}_{\mathbf{w}_1 v}^*\mathcal{Q}_{v\mathbf{w}_1}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\xi\\ &\quad +\mathcal{Q}_{\mathbf{w}_1 v}((\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_1}^*\xi\otimes y\xi_v).\end{aligned}$$

These two expressions are equal by the Claim. Suppose now that $\mathbf{w}_1\in\mathcal{W}_v^c$, $\mathbf{w}_1\simeq v\mathbf{w}_3$, $\mathbf{w}_3\in\mathcal{W}_{\min}$. We have:

$$\begin{aligned}\lambda_v(x)\rho_{v'}(y)\xi &= \lambda_v(x)\left(\mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi + (\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi\right)\\ &= \mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi\\ &\quad +\mathcal{Q}_{v\mathbf{w}_3}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_3}^*(\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_3}^*(\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi.\end{aligned}$$

Moreover, since $\mathbf{w}_2\in(\mathcal{W}^{v'})^c$ and $\mathbf{w}_2\simeq\mathbf{w}_3 v'$, we find,

$$\begin{aligned}\rho_{v'}(y)\lambda_v(x)\xi &= \rho_{v'}(y)\left(\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi\right)\\ &= \mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi + (\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi\\ &\quad +\mathcal{Q}_{\mathbf{w}_3 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_3 v'}^*(\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi + (\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_3 v'}^*(\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi.\end{aligned}$$

These two expressions are equal by the Claim.

Case 3: $\mathbf{w}\in\mathcal{W}_v^c\cap\mathcal{W}^{v'}$. We have,

$$\begin{aligned}\lambda_v(x)\rho_{v'}(y)\xi &= \lambda_v(x)(\mathcal{Q}_{\mathbf{w}v'}(\xi\otimes y\xi_{v'})) = \mathcal{Q}_{v\mathbf{w}_3}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_3}^*\mathcal{Q}_{\mathbf{w}v'}(\xi\otimes y\xi_{v'}) + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_3}^*\mathcal{Q}_{\mathbf{w}v'}(\xi\otimes y\xi_{v'}),\\ \rho_{v'}(y)\lambda_v(x)\xi &= \rho_{v'}(y)\left(\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi + (\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi\right)\\ &= \mathcal{Q}_{\mathbf{w}v'}\left(\mathcal{Q}_{v\mathbf{w}_2}(\mathcal{P}_v x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi\otimes y\xi_{v'}\right) + \mathcal{Q}_{\mathbf{w}_2 v'}(\xi_v^* x\otimes\mathrm{id})\mathcal{Q}_{v\mathbf{w}_2}^*\xi\otimes y\xi_{v'}.\end{aligned}$$

These two expressions are equal by the Claim.

Case 4: $\mathbf{w}\in\mathcal{W}_v\cap(\mathcal{W}^{v'})^c$. We have,

$$\begin{aligned}\lambda_v(x)\rho_{v'}(y)\xi &= \lambda_v(x)\left(\mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi + (\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi\right)\\ &= \mathcal{Q}_{v\mathbf{w}}\left(x\xi_v\otimes\mathcal{Q}_{\mathbf{w}_1 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi\right) + \mathcal{Q}_{v\mathbf{w}_1}(x\xi_v\otimes(\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_1 v'}^*\xi),\\ \rho_{v'}(y)\lambda_v(x)\xi &= \rho_{v'}(y)(\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi)) = \mathcal{Q}_{\mathbf{w}_3 v'}(\mathrm{id}\otimes\mathcal{P}_{v'} y)\mathcal{Q}_{\mathbf{w}_3 v'}^*\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi) + (\mathrm{id}\otimes\xi_{v'}^* y)\mathcal{Q}_{\mathbf{w}_3 v'}^*\mathcal{Q}_{v\mathbf{w}}(x\xi_v\otimes\xi).\end{aligned}$$

These two expressions are equal by the Claim. □

2.2. **The graph product C*-algebra.** For all $v\in V\Gamma$, let A_v be a unital C*-algebra.

2.2.1. *The maximal graph product C*-algebra.*

Definition 2.4. The maximal graph product C*-algebra $\mathsf{A}_{\Gamma,m}$ is the universal unital C*-aglebra generated by the C*-algebras A_v, for $v\in V\Gamma$ and the relations

$$a_v a_{v'} = a_{v'} a_v \quad \text{for all} \quad a_v\in\mathsf{A}_v, a_{v'}\in\mathsf{A}_{v'} \quad \text{whenever} \quad (v,v')\in E\Gamma.$$

Remark 2.5. It is clear that $\mathsf{A}_{\Gamma,m}$ is not $\{0\}$ i.e. that the relations admit a non-trivial representation as bounded operators. Indeed, for any family of representations $\pi_v : \mathsf{A}_v \to \mathcal{B}(\mathcal{H}_v)$ and any family of norm one vectors $\xi_v \in \mathcal{H}_v$, the representations $\widetilde{\pi}_v = \lambda_v \circ \pi_v : \mathsf{A}_v \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the graph product Hilbert space of the family of pointed Hilbert spaces $(\mathcal{H}_v, \xi_v)_{v\in V\Gamma}$ and $\lambda_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{H})$ are the unital faithful morphisms defined in Section 2.1, satisfy the relations $\widetilde{\pi}_v(a_v)\widetilde{\pi}_{v'}(a_{v'}) = \widetilde{\pi}_{v'}(a_{v'})\widetilde{\pi}_v(a_v)$ for all $a_v \in \mathsf{A}_v, a_{v'} \in \mathsf{A}_{v'}$ and all $v, v' \in V\Gamma$ such that $(v, v') \in E\Gamma$ by Proposition 2.1. The associated representation $\pi : \mathsf{A}_{\Gamma,m} \to \mathcal{B}(\mathcal{H})$ such that $\pi|_{\mathsf{A}_v} = \widetilde{\pi}_v$ for all $v \in V\Gamma$ obtained by the universal property is called the *graph product representation*.

Example 2.6. Using the universal property of $\mathsf{A}_{\Gamma,m}$ we have:

- Let $\mathsf{A}_v = C^*_m(G_v)$ the maximal C*-algebra of a discrete group G_v, $v \in V\Gamma$. Then $\mathsf{A}_{\Gamma,m} = C^*_m(G_\Gamma)$.
- Let Γ be the graph with two connected vertices v_1 and v_2. Then $\mathsf{A}_\Gamma = \mathsf{A}_{v_1} \underset{\max}{\otimes} \mathsf{A}_{v_2}$.
- Let Γ be the graph with no edges. Then $\mathsf{A}_{\Gamma,m} = \underset{v\in V\Gamma}{*^m} \mathsf{A}_v$, where $*^m$ denotes the maximal free product.
- If $\Gamma = \mathrm{Star}(v)$ then $\mathsf{A}_{\Gamma,m}$ is a quotient of $\left(\underset{w\in \mathrm{Link}(v)}{*^m} \mathsf{A}_w\right) \underset{\max}{\otimes} \mathsf{A}_v$.

Remark 2.7. Let $\mathcal{A} \subset \mathsf{A}_{\Gamma,m}$ be the linear span of elements of the form $a_1 \dots a_n$ with $n \geq 1$ and $a_k \in \mathsf{A}_{v_k}$, where $\mathbf{v} = (v_1, \dots, v_n)$ is a reduced word. Observe that $\mathcal{A}$ is a dense $*$-subalgebra of $\mathsf{A}_{\Gamma,m}$. Indeed, the commutation relations defining $\mathsf{A}_{\Gamma,m}$ show that $\mathcal{A}$ is a $*$-subalgebra. It is dense since it contains all the A_v. Moreover, if $a = a_1 \dots a_n \in \mathsf{A}_{\Gamma,m}$ with $n \geq 1$, $a_k \in \mathsf{A}_{v_k}$ and $\mathbf{v} = (v_1, \dots, v_n)$ is a reduced word and if $\mathbf{w} = (w_1, \dots w_n)$ is a reduced word (type II) equivalent to $\mathbf{v}$ it follows from the commutation relations that $a = a_{\sigma(1)} \dots a_{\sigma(n)}$, where $\sigma \in S_n$ is the unique permutation such that $\mathbf{w} = \sigma(\mathbf{v})$ defined in Lemma 1.3 (4).

2.2.2. *The reduced graph product C*-algebra.* From this point we assume that each unital C*-algebra $\mathsf{A}_v, v \in V\Gamma$ is equipped with a GNS faithful state ω_v. Since the GNS representation is faithful we may assume that $\mathsf{A}_v \subset \mathcal{B}(\mathcal{H}_v)$, where $(\mathcal{H}_v, \mathrm{id}, \xi_v)$ is a GNS construction for ω_v. Let $(\mathcal{H}, \Omega)$ be the graph product of the pointed Hilbert spaces $(\mathcal{H}_v, \omega_v)$. Recall that $\mathcal{H}$ comes with faithful unital normal $*$-homomorphisms $\lambda_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{H})$.

Definition 2.8. The *reduced graph product* C*-algebra A_Γ is defined as the sub-C*-algebra of $\mathcal{B}(\mathcal{H})$ generated by $\bigcup_{v\in V\Gamma} \lambda_v(\mathsf{A}_v)$.

Since the λ_v are faithful, we may and will assume that $\mathsf{A}_v \subset \mathsf{A}_\Gamma$ and $\lambda_v|_{\mathsf{A}_v}$ is the inclusion for all $v \in V\Gamma$.

Remark 2.9. It follows from Proposition 2.1 that there exists a unique unital surjective $*$-homomorphism $\lambda_\Gamma : \mathsf{A}_{\Gamma,m} \to \mathsf{A}_\Gamma$ such that $\lambda_\Gamma(a) = a$ for all $a \in \mathsf{A}_v$ and all $v \in V\Gamma$. Moreover, if $a = a_1 \dots a_n \in \mathsf{A}$ with $n \geq 1$, $a_k \in \mathsf{A}_{v_k}$ and $\mathbf{v} = (v_1, \dots, v_n)$ is a reduced word and if $\mathbf{w} = (w_1, \dots w_n)$ is a reduced word (type II) equivalent to $\mathbf{v}$ it follows from the commutation relations that $a = a_{\sigma(1)} \dots a_{\sigma(n)}$, where $\sigma \in S_n$ is the unique permutation such that $\mathbf{w} = \sigma(\mathbf{v})$ defined in Lemma 1.3 (4).

Definition 2.10. An operator $a = a_1 \dots a_n \in \mathsf{A}_\Gamma$ is called *reduced* if $a_i \in \mathsf{A}^\circ_{v_i}$ with $\mathsf{A}^\circ_{v_i} = \{x \in \mathsf{A}_{v_i} \mid \omega_{v_i}(x) = 0\}$ and the word $\mathbf{v} = (v_1, \dots, v_n)$ is reduced. The word $\mathbf{v}$ is called the *associated word*.

Observe that the linear span of 1 and the reduced operators in a dense $*$-subalgebra of A_Γ.

Remark 2.11. For all $v \in V\Gamma$, let ω_v be a not necessarily GNS faithful state on A_v. The notion of reduced operators, relative to the family of states $(\omega_v)_{v\in V\Gamma}$, also makes sense in the maximal graph product C*-algebra and the linear span of 1 and the reduced operators in the maximal graph product C*-algebra is the $*$-algebra $\mathcal{A}$ introduced in Remark 2.7, which is dense.

It is clear from Proposition 2.1 that, whenever $a = a_1 \dots a_n \in \mathsf{A}_\Gamma$ is a reduced operator (with associated word in $\mathcal{W}_{\min}$) one has $a\Omega = \widehat{a}_1 \otimes \dots \otimes \widehat{a}_n$. Hence, the vector Ω is cyclic for A_Γ and $(\mathcal{H}, \mathrm{id}, \Omega)$ is a GNS construction for the (GNS faithful) state $\omega_\Gamma(\,\cdot\,) = \langle\, \cdot\, \Omega, \Omega\rangle$. We call ω_Γ the *graph product state*. It can be characterized as follows: it is the unique state on A_Γ satisfying $\omega_\Gamma(a) = 0$ for all reduced operators $a \in \mathsf{A}_\Gamma$. In particular, $\omega_\Gamma|_{\mathsf{A}_v} = \omega_v$ for all $v \in V\Gamma$. Actually the commutation relations and the properties of the graph product state determine the graph product C*-algebra.

Proposition 2.12. *Let B be a unital C*-algebra with a GNS faithful state ω and suppose that, for all $v \in V\Gamma$, there exists a unital faithful $*$-homomorphism $\pi_v : \mathsf{A}_v \to \mathsf{B}$ such that:*

- *B is generated, as a C*-algebra, by $\cup_{v\in V\Gamma}\pi_v(\mathsf{A}_v)$ and the images of π_v and $\pi_{v'}$ commute whenever $(v, v') \in E\Gamma$.*
- *For any operator $a = \pi_{v_1}(a_1)\dots\pi_{v_n}(a_n) \in \mathsf{B}$, where $\mathbf{v} = (v_1, \dots v_n)$ is a reduced word and $a_i \in \mathsf{A}_{v_i}^\circ$ one has $\omega(a) = 0$*

Then, there exists a unique $*$-isomorphism $\pi : \mathsf{A}_\Gamma \to \mathsf{B}$ such that $\pi|_{\mathsf{A}_v} = \pi_v$. Moreover, π intertwines the graph product state and ω.

Proof. The proof is a routine. We include it for the convenience of the reader. The uniqueness being obvious, let us show the existence. Since ω is GNS faithful we may and will assume that $B \subset \mathcal{B}(\mathcal{K})$ and $(\mathcal{K}, \mathrm{id}, \eta)$ is a GNS construction for ω. Define $V : \mathcal{H} \to \mathcal{K}$ by $V(\Omega) = \eta$ and,

$$V(a_1\dots a_n)\Omega = \pi_{v_1}(a_1)\dots\pi_{v_n}(a_n)\eta \quad \text{for all reduced } a = a_1\dots a_n \in \mathsf{A}_\Gamma \text{ with associated word } (v_1, \dots, v_n).$$

It is easy to check that V is well defined and isometric hence, it extends to an isometry. Since it also has a dense image, it is a unitary. Then, $\pi(\,\cdot\,) = V\,\cdot\,V^*$ does the job. □

Remark 2.13. Proposition 2.12 implies the following.

- Let $\mathsf{A}_v = C_r^*(G_v)$ be the reduced group C*-algebra of a discrete group $G_v, v \in V\Gamma$. Then $\mathsf{A}_\Gamma = C_r^*(G_\Gamma)$.
- Let Γ be the graph with two vertices v_1 and v_2 and no edges. Then $\mathsf{A}_\Gamma = \mathsf{A}_{v_1} \otimes \mathsf{A}_{v_2}$.
- Let Γ be the graph with no edges. Then $\mathsf{A}_{\Gamma,m} = \underset{v\in V\Gamma}{*}\mathsf{A}_v$.
- If $\Gamma_0 \subset \Gamma$ is a subgraph and, for all $v \in V_{\Gamma_0}$, $\mathsf{B}_v \subset \mathsf{A}_v$ is a unital C*-algebra then the sub-C*-algebra of A_Γ generated by $\cup_{v\in V\Gamma_0}\mathsf{B}_v$ is canonically isomorphic to graph product C*-algebras B_{Γ_0} obtained from B_v, $v \in V_{\Gamma_0}$.

Remark 2.14. Let $\Gamma_0 \subseteq \Gamma$ be a subgraph and consider the graph product C*-algebras A_{Γ_0} and A_Γ. By the universal property of Proposition 2.12, we may view $\mathsf{A}_{\Gamma_0} \subset \mathsf{A}_\Gamma$ canonically. Denote by $\mathcal{W}_{\min}^0 \subset \mathcal{W}_{\min}$ the subset of minimal reduced words in Γ_0 and let $\mathcal{H}_0 = \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w}\in\mathcal{W}_{\min}^0}\mathcal{H}_{\mathbf{w}} \subset \mathcal{H}$. Let P be the orthogonal projection onto $\mathcal{H}_0$. Then it is easy to check that $\mathcal{E}_{\Gamma_0} : x \mapsto PxP$ is a graph product state preserving conditional expectation from A_Γ onto A_{Γ_0}. Moreover, $\mathcal{E}_{\Gamma_0}$ is the unique conditional expectation from A_Γ to A_{Γ_0} such that $\mathcal{E}_{\Gamma_0}(a) = 0$ for all reduced operators $a \in \mathsf{A}_{\Gamma_0}$ with associated reduced word $\mathbf{v} = (v_1, \dots, v_n)$ satisfying the property that one of the v_i is not in Γ_0. In particular, for all $v \in V\Gamma$, there exists a unique conditional expectation $\mathcal{E}_v : \mathsf{A}_\Gamma \to \mathsf{A}_v$ such that $\mathcal{E}_v(a) = 0$ for all reduced operators $a \in \mathsf{A}_\Gamma \setminus \mathsf{A}_v$.

2.2.3. *Unscrewing technique.* Let $v \in \Gamma$, $\Gamma_1 = \mathrm{Star}(v)$, $\Gamma_2 = \Gamma \setminus \{v\}$ and set the following graph product C*-algebras: $\mathsf{A}_1 = \mathsf{A}_{\Gamma_1}$, $\mathsf{B} = \mathsf{A}_{\mathrm{Link}(v)}$, and $\mathsf{A}_2 = \mathsf{A}_{\Gamma_2}$. Recall that, by the universal property of Proposition 2.12, we may view $\mathsf{B} \subset \mathsf{A}_1 \subset \mathsf{A}_\Gamma$ and $\mathsf{B} \subset \mathsf{A}_2 \subset \mathsf{A}_\Gamma$ canonically. Moreover, by Remark 2.14, we have conditional expectations $\mathcal{E}_1 := \mathcal{E}_{\mathrm{Link}(v)}|_{\mathsf{A}_1} : \mathsf{A}_1 \to \mathsf{B}$ and $\mathcal{E}_2 := \mathcal{E}_{\mathrm{Link}(v)}|_{\mathsf{A}_2} : \mathsf{A}_2 \to \mathsf{B}$. Let us denote by $\mathsf{A}_1 \star_{\mathsf{B}} \mathsf{A}_2$ the reduced amalgamated free product with respect to these conditional expectations.

Theorem 2.15. *There exists a unique $*$-isomorphism $\pi : \mathsf{A}_1 \star_{\mathsf{B}} \mathsf{A}_2 \to \mathsf{A}_\Gamma$ such that $\pi|_{\mathsf{A}_1}$ (resp. $\pi|_{\mathsf{A}_2}$) is the canonical inclusion $\mathsf{A}_1 \subset \mathsf{A}_\Gamma$ (resp. $\mathsf{A}_2 \subset \mathsf{A}_\Gamma$). Moreover, π is state-preserving.*

Proof. Observe that A_Γ is generated by by A_1 and A_2. Let $\mathcal{E} = \mathcal{E}_{\mathrm{Link}(v)}$ be the canonical conditional expectation from A_Γ onto B. Define, for $k = 1, 2$, $\mathsf{A}_k^\circ = \ker(\mathcal{E}_k)$. By the universal property of amalgamated free products it suffices to show that for any $n \geq 2$, for any $a_1, \dots, a_n$ with $a_k \in \mathsf{A}_{l_k}^\circ$ and $l_k \neq l_{k+1}$, one has $\mathcal{E}(a_1\dots a_n) = 0$. Since A_k° is the closed linear span of reduced operators $a \in \mathsf{A}_k$ with associated reduced word $\mathbf{v} = (v_1, \dots, v_n)$, $v_i \in \Gamma_k$ satisfying the property that one of the v_i is not in $\mathrm{Link}(v)$ we may and will assume that each a_k is a reduced operator $a_k = x_1^k\dots x_{r_k}^k \in \mathsf{A}_{l_k}$ with associated reduced word $\mathbf{v}_k = (v_1^k, \dots v_{r_k}^k)$, $v_i^k \in \Gamma_{l_k}$ satisfying the property that one of the v_i^k is not in $\mathrm{Link}(v)$. One has $a := a_1\dots a_n = x_1^1\dots x_{r_1}^1 x_1^2\dots x_{r_2}^2\dots x_1^n\dots x_{r_n}^n$ with $x_{r_i}^k \in \mathsf{A}_{v_i^k}^\circ$. Let $\mathbf{v} = (v_1^1, \dots, v_{r_1}^1, v_1^2, \dots, v_{r_2}^2, \dots, v_1^n, \dots, v_{r_n}^n)$ be the associated, not necessarily reduced, word. Let $l = r_1 + \dots + r_n \geq n$.

Let us show, by induction on l, that $\mathcal{E}(a) = 0$. If $l = n$ then $a_k \in \mathsf{A}_{v_k}^\circ \subset \mathsf{A}_{l_k}$ and $v_k \in \Gamma_{l_k} \setminus \mathrm{Link}(v)$ for all k. Then $\mathbf{v}$ is reduced and since $v_k \notin \mathrm{Link}(v)$ we have $\mathcal{E}(a) = 0$. Indeed, if $\mathbf{v}$ is not reduced, there exists $i < j$ such that $v_i = v_j = w$ and $v_k \in \mathrm{Link}(v)$ for all $i < k < j$. Since $v_k \notin \mathrm{Link}(v)$ for all k, it follows that $j = i+1$. Hence, $w \in (\Gamma_{l_i} \setminus \mathrm{Link}(v)) \cap (\Gamma_{l_{i+1}} \setminus \mathrm{Link}(v)) = \{v\} \cap (\Gamma \setminus \{v\}) = \emptyset$, a contradiction.

Let $l \geq n$ and $a = a_1 \dots a_n$ is of the form described previously. We use the notations introduced at the beginning of the proof. If the word $\mathbf{v}$ associated to a is reduced then $\mathcal{E}(a) = 0$. Hence, we will assume that $\mathbf{v}$ is not reduced. Then there exists $i < j$ such that $v_{s_i}^i = w = v_{s_j}^j$ and $v_s^k \in \mathrm{Link}(w)$ whenever:

(1) $i < k < j$ and $1 \leq s \leq r_k$,
(2) $k = i$ and $s_i < s \leq r_i = r_k$,
(3) $k = j$ and $1 \leq s < s_j$.

Since we can replace $\mathbf{v}$ by a type II equivalent word and since any subword $\mathbf{v}_k$ is reduced, we may and will assume that $j = i + 1$ and $w = v_{r_i}^i = v_1^{i+1}$. Hence we have $w \in \Gamma_{l_i} \cap \Gamma_{l_j} = \Gamma_{l_i} \cap \Gamma_{l_{i+1}} = \Gamma_1 \cap \Gamma_2 = \mathrm{Star}(v) \cap \Gamma \setminus \{v\} = \mathrm{Link}(v)$. Write, for $x \in \mathsf{A}_w$, $\mathcal{P}_w(x) = x - \omega_w(x)$. One has

$$\begin{aligned}\mathcal{E}(a_1 \dots a_n) &= \mathcal{E}(a_1 \dots a_{i-1} x_1^i \dots x_{r_{i-1}}^i \mathcal{P}_w(x_{r_i}^i x_1^{i+1}) x_2^{i+1} \dots x_{r_{i+1}}^{i+1} \dots a_{i+2} \dots a_n) \\ &\quad + \omega_w(x_{r_i}^i x_1^{i+1}) \mathcal{E}(a_1 \dots a_{i-1} x_1^i \dots x_{r_{i-1}}^i x_2^{i+1} \dots x_{r_{i+1}}^{i+1} \dots a_{i+2} \dots a_n).\end{aligned}$$

The right hand side of this expression is zero by the induction hypothesis.

□

Remark 2.16. Theorem 2.15 is trivially true when we consider the maximal graph product and the maximal amalgamated free product.

Corollary 2.17. *A_Γ is exact if and only if A_v is exact for all $v \in V\Gamma$.*

Proof. By an inductive limit argument we may suppose the graph Γ finite and we conclude by induction using Theorem 2.15 and the results of [Dy04].

We explain now the inductive limit argument which will be used several time in this paper (even in the von Neumann algebra context). Let $\mathcal{F}(\Gamma)$ the set of finite subgraphs of Γ ordered by the inclusion. If $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{F}(\Gamma)$ and $\mathcal{G}_1 \subset \mathcal{G}_2$, we view $\mathsf{A}_{\mathcal{G}_1} \subset \mathsf{A}_{\mathcal{G}_2} \subset \mathsf{A}_\Gamma$. Hence, we get an inductive system of unital C*-algebras $(\mathsf{A}_\mathcal{G})_{\mathcal{G} \in \mathcal{F}(\Gamma)}$. Let $\mathsf{A}_\infty = \overline{\bigcup_{\mathcal{G} \in \mathcal{F}(\Gamma)} \mathsf{A}_\mathcal{G}} \subset \mathsf{A}_\Gamma$ be the inductive limit. We claim that actually $\mathsf{A}_\infty = \mathsf{A}_\Gamma$. Indeed, it is enough to show that every reduced operator $a = a_1 \dots a_n \in \mathsf{A}_\Gamma$, with associated word $\mathbf{v} = (v_1, \dots, v_n)$ lies in A_∞. In fact, such an operator a lies in $\mathsf{A}_\mathcal{G}$, where $\mathcal{G}$ is a finite subgrah of Γ containing the vertices $v_1, \dots v_n$. □

Remark 2.18. If Γ has n connected component $\Gamma_1, \dots, \Gamma_n$ then $(\mathsf{A}_\Gamma, \omega_\Gamma) \simeq (\mathsf{A}_{\Gamma_1} * \dots * \mathsf{A}_{\Gamma_n}, \omega_{\Gamma_1} * \dots * \omega_{\Gamma_n})$.

2.3. **The graph product of von Neumann algebras.** Suppose that for each $v \in V\Gamma$ we have a von Neumann algebra M_v with a *faithful* normal state ω_v. We may and will assume that $\mathsf{M}_v \subset \mathcal{B}(\mathcal{H}_v)$, where $(\mathcal{H}_v, \mathrm{id}, \xi_v)$ is a GNS construction for ω_v. Let $(\mathcal{H}, \Omega)$ by the graph product of the pointed Hilbert spaces $(\mathcal{H}_v, \omega_v)$. Recall that $\mathcal{H}$ comes with faithful unital normal $*$-homomorphisms $\lambda_v : \mathcal{B}(\mathcal{H}_v) \to \mathcal{B}(\mathcal{H})$.

Definition 2.19. The *graph product von Neumann algebra* is $\mathsf{M}_\Gamma := \big(\bigcup_{v \in V\Gamma} \lambda_v(\mathsf{M}_v)\big)'' \subset \mathcal{B}(\mathcal{H})$.

As before, we will assume that $\mathsf{M}_v \subset \mathsf{M}_\Gamma$ and $\lambda_v|_{\mathsf{M}_v}$ is the inclusion, for all $v \in V\Gamma$. We also have the same notion of reduced operators and the linear span of 1 and the reduced operators is a weakly dense $*$-subalgebra of M_Γ. The graph product state $\omega_\Gamma(\,\cdot\,) = \langle\, \cdot\, \Omega, \Omega\rangle$ is now a normal state on M_Γ. The graph product state is characterized as follows: it is the unique normal state on M_Γ satisfying $\omega_\Gamma(a) = 0$ for all reduced operators $a \in \mathsf{A}_\Gamma$. In particular, $\omega_\Gamma|_{\mathsf{M}_v} = \omega_v$ for all $v \in V\Gamma$.

Let us construct the right version of M_Γ. For $v \in V\Gamma$, we denote by $r_v(a)$ the right action of M_v on $\mathcal{H}_v$ i.e. $r_v(a) = J_v a^* J_v$ where J_v is the modular conjugation of ω_v. View r_v a faithful normal unital $*$-homomorphism from $\mathsf{M}_v^{\mathrm{op}}$ to $\mathcal{B}(\mathcal{H}_v)$. Denote by M_Γ^r the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\bigcup_{v \in V\Gamma} \rho_v \circ r_v(\mathsf{M}_v)$. Write $\rho_v^\Gamma = \rho_v \circ r_v$ and note that ρ_v^Γ is a faithful unital normal $*$-homomorphism from $\mathsf{M}_v^{\mathrm{op}}$ to M_Γ^r.

Observe that, by Proposition 2.3, $\mathsf{M}_\Gamma^r \subset \mathsf{M}_\Gamma'$.

As before, we call an operator $a = \rho_{v_1}^\Gamma(a_1)\dots\rho_{v_n}^\Gamma(a_n) \in \mathsf{M}_\Gamma^r$ *reduced* if $a_i \in \mathsf{M}_{v_i}^\circ$ and the word $\mathbf{v} = (v_1, \dots, v_n)$ is reduced. It is clear from the definitions that, whenever $a = \rho_{v_1}^\Gamma(a_1)\dots\rho_{v_n}^\Gamma(a_n) \in \mathsf{M}_\Gamma^r$ is a reduced operator (with associated word in $\mathcal{W}_{\min}$) one has $a\Omega = \widehat{a}_n \otimes \cdots \otimes \widehat{a}_1$. Hence, the vector Ω is cyclic for M_Γ^r so it is separating for M_Γ and the graph product state ω_Γ is faithful with GNS construction $(\mathcal{H}, \mathrm{id}, \Omega)$. It is now easy to compute the modular theory of ω_Γ. We denote by ∇_v, J_v and $(\sigma_t^v)_{t\in\mathbb{R}}$ the ingredients of the modular theory of ω_v, for $v \in V\Gamma$. For $\mathbf{w} \in \mathcal{W}$ a reduced word of the form $\mathbf{w} = (v_1, \dots, v_n)$, let $\overline{\mathbf{w}}$ be the unique minimal reduced word equivalent to the reduced word $\mathbf{w}^* = (v_n, \dots, v_1)$ and $\sigma_{\mathbf{w}}$ the unique bijection of $\{1, \dots, n\}$ such that $\overline{\mathbf{w}} = (v_{\sigma_{\mathbf{w}}(n)}, \dots, v_{\sigma_{\mathbf{w}}(1)})$. Define the unitary operator $\Sigma_{\mathbf{w}} : \mathcal{H}_{\mathbf{w}} \to \mathcal{H}_{\overline{\mathbf{w}}}$ by $\Sigma_{\mathbf{w}}(\xi_1 \otimes \dots \otimes \xi_n) = \mathcal{Q}_{\mathbf{w}^*,\bar{\mathbf{w}}}(\xi_n \otimes \dots \otimes \xi_1) = \xi_{\sigma_{\mathbf{w}}(n)} \otimes \dots \otimes \xi_{\sigma_{\mathbf{w}}(1)}$. Finally, denote by $J_{\mathbb{C}}$ the conjugation map on $\mathbb{C}$.

Proposition 2.20. *Let J, ∇ and $(\sigma_t)_{t\in\mathbb{R}}$ be the ingredients of the modular theory of ω_Γ. One has*

(1) $J = J_{\mathbb{C}} \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n)\in\mathcal{W}_{min}} (J_{v_{\sigma_{\mathbf{w}}(n)}} \otimes \dots \otimes J_{v_{\sigma_{\mathbf{w}}(n)}})\Sigma_{\mathbf{w}}$

(2) $\nabla = id_{\mathbb{C}\Omega} \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n)\in\mathcal{W}_{min}} \Sigma_{\mathbf{w}}^*(\nabla_{v_{\sigma_{\mathbf{w}}(n)}} \otimes \dots \otimes \nabla_{v_{\sigma_{\mathbf{w}}(n)}})\Sigma_{\mathbf{w}}$

(3) For any reduced operator $a = a_1 \dots a_n \in \mathsf{M}_\Gamma$ with associated word $\mathbf{v} = (v_1, \dots, v_n)$ one has

$$\sigma_t(a_1 \dots a_n) = \sigma_t^{v_1}(a_1) \dots \sigma_t^{v_n}(a_n) \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof. (3) follows easily from (2). Let S_v (resp. S) be the modular operator for ω_v (resp. ω_Γ). To get (1) and (2), it suffices to prove, by uniqueness of the polar decomposition, that

$$S = \mathrm{id}_{\mathbb{C}\Omega} \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n)\in\mathcal{W}_{\min}} (S_{v_{\sigma_{\mathbf{w}}(n)}} \otimes \dots \otimes S_{v_{\sigma_{\mathbf{w}}(n)}}) \circ \Sigma_{\mathbf{w}}.$$

Denote by T the right handside of the preceding equation. An easy computation gives that, for all reduced operators $a = a_1 \dots a_n \in \mathsf{A}_\Gamma$ or for $a \in \mathbb{C}1$, one has $S(a\Omega) = T(a\Omega)$. Hence, $S|_{\mathcal{M}_\Gamma\Omega} = T|_{\mathcal{M}_\Gamma\Omega}$, where $\mathcal{M}_\Gamma \subset \mathsf{M}_\Gamma$ is the linear span of 1 and the reduced operators and it suffices to show that $\mathcal{M}_\Gamma\Omega$ is a common core for S and T. By definition, $\mathsf{M}_\Gamma\Omega$ is a core for S. Since $\mathcal{M}_\Gamma$ is a weakly dense unital $*$-subalgebra of M_Γ, it follows from the Kaplanski's density Theorem that $\mathcal{M}_\Gamma\Omega$ is also a core for S. By definition of T, a core for T is given by the subspace

$$\mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n)\in\mathcal{W}_{\min}} \mathsf{M}_{v_1}^\circ\xi_{v_1} \otimes \cdots \otimes \mathsf{M}_{v_n}^\circ\xi_{v_n},$$

where the direct sums and tensor products are the algebraic ones. This subspace is exactly the linear span of Ω and vectors of the form $a\Omega$, where a is a reduced operator i.e. this is the space $\mathcal{M}_\Gamma\Omega$. □

Remark 2.21. It follows from the preceding proposition that, for all reduced operator $a = a_1 \dots a_n \in \mathsf{M}_\Gamma$, with $a_i \in \mathsf{M}_{v_i}$, one has $JaJ = \rho_{v_1}^\Gamma(a_1) \dots \rho_{v_n}^\Gamma(a_n)$. Hence we actually have $\mathsf{M}_\Gamma' = \mathsf{M}_\Gamma^r$.

The graph product von Neumann algebra also satisfies a universal property. The following Proposition 2.22 can be proved exactly as Proposition 2.1 since the isomorphism appearing in the proof of Proposition 2.12 is spacial.

Proposition 2.22. *Let N be a von Neumann algebra with a GNS faithful normal state ω and suppose that, for all $v \in V\Gamma$, there exists a unital normal faithful $*$-homomorphism $\pi_v : \mathsf{M}_v \to \mathsf{N}$ such that:*

- *N is generated, as a von Neumann algebra, by $\cup_{v\in V\Gamma}\pi_v(\mathsf{M}_v)$ and the images of π_v and $\pi_{v'}$ commute whenever $(v, v') \in E\Gamma$.*
- *For any operator $a = \pi_{v_1}(a_1) \dots \pi_{v_n}(a_n) \in \mathsf{N}$, where $\mathbf{v} = (v_1, \dots v_n)$ is a reduced word and $a_i \in \mathsf{M}_{v_i}^\circ$ one has $\omega(a) = 0$*

Then, there exists a unique normal $*$-isomorphism $\pi : \mathsf{M}_\Gamma \to \mathsf{N}$ such that $\pi|_{\mathsf{M}_v} = \pi_v$. Moreover, π intertwines the graph product state and ω. In particular, ω is faithful.

Remark 2.23. The preceding proposition implies the following.

- If $\mathsf{M}_v = \mathrm{L}(G_v)$ is the group von Neumann algebra of a discrete group $G_v, v \in V\Gamma$ then $\mathsf{M}_\Gamma = \mathrm{L}(G_\Gamma)$.

- Let Γ be the graph with two vertices v_1 and v_2 and one edge connecting them. Then $\mathsf{M}_\Gamma = \mathsf{M}_{v_1} \otimes \mathsf{M}_{v_2}$.
- Let Γ be the graph with no edges. Then $\mathsf{M}_\Gamma = \underset{v\in V\Gamma}{*}(\mathsf{M}_v, \omega_v)$.
- If $\Gamma_0 \subset \Gamma$ is a subgraph and, for all $v \in \Gamma_0$, $\mathsf{N}_v \subset \mathsf{M}_v$ is a unital von Neumann subalgebra then the graph product von Neumann N_{Γ_0} obtained from the N_v, $v \in \Gamma_0$, is canonically isomorphic to $\big(\bigcup_{v\in V\Gamma_0} \mathsf{N}_v\big)''$. In the sequel we will always do this identification without further explanations.
- There is a unique (state preserving) $*$-isomorphism $\mathsf{M}_{\mathrm{Star}(v)} \simeq \mathsf{M}_v \otimes \mathsf{M}_{\mathrm{Link}(v)}$ identifying $x \otimes y$ with xy, for all $x \in \mathsf{M}_v$ and all $y \in \mathsf{M}_{\mathrm{Link}(v)}$. In particular, $\mathsf{M}_v' \cap \mathsf{M}_{\mathrm{Star}(v)} = \mathsf{N}_{\mathrm{Star}(v)}$, where

$$\mathsf{N}_w = \begin{cases} \mathsf{M}_w & \text{if} \quad w \in \mathrm{Link}(v), \\ Z(\mathsf{M}_v) & \text{if} \quad w = v. \end{cases}$$

Remark 2.24. Let $\Gamma_0 \subseteq \Gamma$ be a subgraph and consider the graph product von Neumann algebras M_{Γ_0} and M_Γ. As in a C*-algebraic case, there exists a unique normal conditional expectation $\mathcal{E}_{\Gamma_0}$ from M_Γ to M_{Γ_0} preserving the graph product states and such that $\mathcal{E}_{\Gamma_0}(a) = 0$ for all reduced operator $a \in \mathsf{M}_{\Gamma_0}$ with associated reduced word $\mathbf{v} = (v_1, \dots v_n)$ satisfying the property that one of the v_i is not in Γ_0. In particular, for all $v \in V\Gamma$, there exists a unique state preserving normal conditional expectation $\mathcal{E}_v : \mathsf{M}_\Gamma \to \mathsf{M}_v$ such that $\mathcal{E}_v(a) = 0$ for all reduced operator $a \in \mathsf{M}_\Gamma \setminus \mathsf{M}_v$.

Proposition 2.25. *Let $\Gamma_0, \Gamma_1 \subset \Gamma$ be subgraphs. One has $\mathsf{M}_{\Gamma_0} \cap \mathsf{M}_{\Gamma_1} = \mathsf{M}_{\Gamma_0\cap\Gamma_1}$.*

Proof. The inclusion $\mathsf{M}_{\Gamma_0\cap\Gamma_1} \subset \mathsf{M}_{\Gamma_0} \cap \mathsf{M}_{\Gamma_1}$ being obvious, let us show the other one. Let $\mathcal{M}_{\Gamma_0}$ be the linear span of 1 and the reduced operator in M_{Γ_0}. It suffices to show that $\mathcal{M}_{\Gamma_0} \cap \mathsf{M}_{\Gamma_1} \subset \mathsf{M}_{\Gamma_0\cap\Gamma_1}$. Indeed, if it is the case, then $\mathsf{M}_{\Gamma_0\cap\Gamma_1}$ contains $(\mathcal{M}_{\Gamma_0} \cap \mathsf{M}_{\Gamma_1})'' = \big(\mathcal{M}_{\Gamma_0}' \cup \mathsf{M}_{\Gamma_1}'\big)' = \mathcal{M}_{\Gamma_0}'' \cap \mathsf{M}_{\Gamma_1} = \mathsf{M}_{\Gamma_0} \cap \mathsf{M}_{\Gamma_1}$. Let $x \in \mathcal{M}_{\Gamma_0}$ and write $x = \omega_\Gamma(x)1 + \sum_i x_i$, where the sum is finite and the x_i are some reduced operators in M_{Γ_0}. If $x \in \mathsf{M}_{\Gamma_1}$ we have $x = \mathcal{E}_{\Gamma_1}(x) = \omega_\Gamma(x)1 + \sum_i \mathcal{E}_{\Gamma_1}(x_i)$. By definition of the conditional expectation, for all i, $\mathcal{E}_{\Gamma_1}(x_i)$ is either 0 or a reduced operator with associated vertices in $\Gamma_0 \cap \Gamma_1$. Hence, $x \in \mathsf{M}_{\Gamma_0\cap\Gamma_1}$. □

Let $v \in \Gamma$, $\Gamma_1 = \mathrm{Star}(v)$, $\Gamma_2 = \Gamma \setminus \{v\}$ and set the following graph product von Neumann algebras: $\mathsf{M}_1 = \mathsf{M}_{\Gamma_1}$, $\mathsf{N} = \mathsf{M}_{\mathrm{Link}(v)}$, and $\mathsf{M}_2 = \mathsf{M}_{\Gamma_2}$. By the universal property of Proposition 2.22, we may view $\mathsf{N} \subset \mathsf{M}_1 \subset \mathsf{M}_\Gamma$ and $\mathsf{N} \subset \mathsf{M}_2 \subset \mathsf{M}_\Gamma$ canonically. Let us denote by $\mathsf{M}_1 \star_{\mathsf{N}} \mathsf{M}_2$ the von Neumann algebraic amalgamated free product with respect to the graph product states. The following result can be proved exactly as Theorem 2.15, using the universal property of von Neumann algebraic amalgamated free products.

Theorem 2.26. *There exists a unique $*$-isomorphism $\pi : \mathsf{M}_1 \star_{\mathsf{N}} \mathsf{M}_2 \to \mathsf{M}_\Gamma$ such that $\pi|_{\mathsf{M}_1}$ (resp. $\pi|_{\mathsf{M}_2}$) is the canonical inclusion $\mathsf{M}_1 \subset \mathsf{M}_\Gamma$ (resp. $\mathsf{M}_2 \subset \mathsf{M}_\Gamma$). Moreover, π is state-preserving.*

Before the next Lemma, let us recall some standard notations. Let (M, τ) be a finite von Neumann algebra and $\mathsf{A}, \mathsf{B} \subset \mathsf{M}$ two unital von Neumann subalgebras. We write $\mathsf{A} \underset{\mathsf{M}}{\prec} \mathsf{B}$ if there exists a net $(u_i)_i$ of unitaries in A such that $\mathcal{E}_\mathsf{B}(xu_iy) \to 0$ for all $x, y \in \mathsf{M}$. We also write $\mathcal{N}_\mathsf{M}(\mathsf{A})$ the *normalizer* of A in M i.e.

$$\mathcal{N}_\mathsf{M}(\mathsf{A}) = \{u \in \mathcal{U}(\mathsf{M}) \,:\, u\mathsf{A}u^* = \mathsf{A}\}.$$

Lemma 2.27. *Suppose that ω_v is a trace for all $v \in V\Gamma$. Fix $v \in V\Gamma$. If $\mathsf{Q} \subset \mathsf{M}_v$ is a diffuse von Neumann subalgebra then*

$$\mathsf{Q} \underset{\mathsf{M}_{\mathrm{Star}(v)}}{\prec} \mathsf{M}_{\mathrm{Link}(v)}$$

and any Q-$\mathsf{M}_{\mathrm{Star}(v)}$-sub-bimodule of $\mathrm{L}^2(\mathsf{M}_{\mathrm{Star}(v)})$ which has finite dimension as right $\mathsf{M}_{\mathrm{Star}(v)}$-module is contained in $\mathsf{M}_{\mathrm{Star}(v)}$. In particular, $\mathsf{Q}' \cap \mathsf{M}_\Gamma \subset \mathcal{N}_{\mathsf{M}_\Gamma}(\mathsf{Q})'' \subset \mathsf{M}_{\mathrm{Star}(v)}$.

Proof. Since $\mathsf{M}_\Gamma = \mathsf{M}_{\mathrm{Star}(v)} \underset{\mathsf{M}_{\mathrm{Link}(v)}}{*} \mathsf{M}_{\Gamma\setminus\{v\}}$, we may apply [IPP08, Theorem1.1] and it suffices to show that that there exists a net $u_i \in \mathcal{U}(\mathsf{Q})$ such that $||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 \to 0$ for all $x, y \in \mathsf{M}_{\mathrm{Star}(v)}$.

Since Q is diffuse, let u_i be a net of unitaries in Q such that $\omega_v(au_ib) \to 0$ for all $a, b \in \mathsf{M}_v$.

Let us show that $||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 \to 0$ for all $x, y \in \mathsf{M}_{\mathrm{Star}(v)} \ominus \mathsf{M}_{\mathrm{Link}(v)}$. It suffices to show that

$$||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 \to 0$$

for all $x = a_1 \dots a_n, y = b_1 \dots b_l \in M_{\mathrm{Star}(v)}$ reduced operators with $a_k \in \mathsf{M}_{v_k}^\circ$, $b_k \in \mathsf{M}_{w_k}^\circ$, $v_k, w_k \in \mathrm{Star}(v)$ and such that $v_{i_0} \neq v$ and $w_{j_0} \neq v$ for some i_0 and j_0. Since x and y are reduced we may and will assume that:

- $(v_n = v$ and $v_k \in \mathrm{Link}(v)$ for all $k \neq n)$ or $(v_k \neq v$ for all $k)$,
- $(w_n = v$ and $w_k \in \mathrm{Link}(v)$ for all $k \neq n)$ or $(w_k \neq v$ for all $k)$.

In any case we can write $x = x'a$ and $y = by'$, where $x', y' \in \mathsf{M}_{\mathrm{Link}(v)}$ are reduced operators and $a, b \in \mathsf{M}_v$. Hence, for all $z \in \mathsf{M}_v^\circ$, the operator $x'zy' \in \mathsf{M}_{\mathrm{Star}(v)}$ is reduced and $\mathcal{E}_{\mathrm{Link}(v)}(x'zy') = 0$. It follows that:

$$\begin{aligned} ||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 &= ||\mathcal{E}_{\mathrm{Link}(v)}(x'(au_ib - \omega_v(au_ib)y')||_2 + |\omega_v(au_ib)|\,||\mathcal{E}_{\mathrm{Link}(v)}(x'y')||_2 \\ &\leq ||x'||\,||y'||_2|\omega_v(au_ib)| \to 0. \end{aligned}$$

Suppose now that $x, y \in \mathsf{M}_{\mathrm{Link}(v)}$. Then, since $u_i \in \mathsf{M}_v$,

$$||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 = ||x\mathcal{E}_{\mathrm{Link}(v)}(u_i)y||_2 \leq ||x||\,||y||\,||\mathcal{E}_{\mathrm{Link}(v)}(u_i)||_2 = ||x||\,||y||\,|\omega_v(u_i)| \to 0.$$

If $x \in \mathsf{M}_{\mathrm{Star}(v)} \ominus \mathsf{M}_{\mathrm{Link}(v)}$ and $y \in \mathsf{M}_{\mathrm{Link}(v)}$ we may assume, as before, that $x = x'a$, where $x' \in \mathsf{M}_{\mathrm{Link}(v)}$ is a reduced operator and $a \in M_v$ and we get

$$\begin{aligned} ||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 &= ||\mathcal{E}_{\mathrm{Link}(v)}(xu_i)y||_2 \leq ||y||\,||\mathcal{E}_{\mathrm{Link}(v)}(xu_i)||_2 = ||y||\,||E_{\mathrm{Link}(v)}(x'au_i)||_2 \\ &= ||y||\,|\omega_v(au_i)|\,||\mathcal{E}_{\mathrm{Link}(v)}(x')||_2 \leq ||x'||_2\,||y||\,|\omega_v(au_i)| \to 0. \end{aligned}$$

The proof that $||\mathcal{E}_{\mathrm{Link}(v)}(xu_iy)||_2 \to 0$ when $x \in \mathsf{M}_{\mathrm{Link}(v)}$ and $y \in \mathsf{M}_{\mathrm{Star}(v)} \ominus \mathsf{M}_{\mathrm{Link}(v)}$ is the same. □

Corollary 2.28. *Suppose that ω_v is a trace for all $v \in V\Gamma$. Fix $v \in V\Gamma$. For $w \in \mathrm{Star}(v)$ define*

$$\mathsf{N}_w = \begin{cases} \mathsf{M}_w & \text{if } \ w \in \mathrm{Link}(v), \\ Z(\mathsf{M}_v) & \text{if } \ w = v. \end{cases}$$

If M_v is diffuse then $\mathsf{M}_v' \cap \mathsf{M}_\Gamma = \mathsf{N}_{\mathrm{Star}(v)}$. In particular, $Z(\mathsf{M}_\Gamma) = \bigcap_{v \in V\Gamma} \mathsf{N}_{\mathrm{Star}(v)}$.

Proof. The inclusion $\mathsf{N}_{\mathrm{Star}(v)} \subset \mathsf{M}_v' \cap \mathsf{M}_\Gamma$ being obvious, let us prove the other inclusion. By Lemma 2.27 and the last assertion of Remark 2.23 we have, $\mathsf{M}_v' \cap \mathsf{M}_\Gamma \subset \mathsf{M}_v' \cap \mathsf{M}_{\mathrm{Star}(v)} = \mathsf{N}_{\mathrm{Star}(v)}$. □

Corollary 2.29. *If M_v is a II_1-factor for all $v \in V\Gamma$ then M_Γ is a II_1-factor.*

Proof. By the inductive limit argument we may and will assume that Γ is finite graph. By Corollary 2.28 we find $Z(\mathsf{M}_\Gamma) = \bigcap_{v \in V\Gamma} \mathsf{M}_{\mathrm{Link}(v)}$. It follows from Proposition 2.25 that $Z(\mathsf{M}_\Gamma) = \mathsf{M}_{\bigcap_{v \in V\Gamma} \mathrm{Link}(v)}$. Since $\bigcap_{v \in V\Gamma} \mathrm{Link}(v) \subset \bigcap_{v \in V\Gamma} \Gamma \setminus \{v\} = \emptyset$ we conclude that $Z(\mathsf{M}_\Gamma) = \mathbb{C}1$. □

2.3.1. *Completely positive maps of graph products.* Let $(\mathsf{M}_v, \omega_v)_{v \in V\Gamma}$ and $(\mathsf{N}_v, \mu_v)_{v \in V\Gamma}$ be two families of von Neumann algebras with faithful normal states.

Proposition 2.30. *For all $v \in V\Gamma$, let $\varphi_v : \mathsf{M}_v \to \mathsf{N}_v$ be a state-preserving normal ucp map. Then, there exists a unique normal ucp map $\varphi : \mathsf{M}_\Gamma \to \mathsf{N}_\Gamma$ such that, for all $a = a_1 \dots a_n \in \mathsf{M}_\Gamma$ reduced, with $a_k \in \mathsf{M}_{v_k}^\circ$,*

$$\varphi(a_1 \dots a_n) = \varphi_{v_1}(a_1) \dots \varphi_{v_n}(a_n).$$

Moreover, φ intertwines the graph product states and its L^2-extension is given by

$$T_\varphi : \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n) \in \mathcal{W}_{min}} \mathrm{L}^2(\mathsf{M}_{v_1})^\circ \otimes \dots \otimes \mathrm{L}^2(\mathsf{M}_{v_n})^\circ \to \mathbb{C}\Omega \oplus \bigoplus_{\mathbf{w}=(v_1,\dots,v_n) \in \mathcal{W}_{min}} \mathrm{L}^2(\mathsf{N}_{v_1})^\circ \otimes \dots \otimes \mathrm{L}^2(\mathsf{N}_{v_n})^\circ,$$

$$T_\varphi = id_{\mathbb{C}\Omega} \oplus \bigoplus T_{\varphi_{v_1}}|_{\mathrm{L}^2(\mathsf{M}_{v_1})^\circ} \otimes \dots \otimes T_{\varphi_{v_n}}|_{\mathrm{L}^2(\mathsf{M}_{v_n})^\circ}.$$

Proof. Let $(\mathcal{K}_v, \eta_v)$ be the pointed M_v-N_v bimodule obtained from the GNS construction of φ_v i.e. one has $\mathcal{K}_v = \overline{\mathsf{M}_v\eta_v\mathsf{N}_v}$ and $\langle a\eta_v b, \eta_v\rangle = \mu_v(\varphi_v(a)b)$. Denote by π_v^l (resp. π_v^r) the left (resp. right) action of M_v on $\mathcal{K}_v$. Observe that, since μ_v is faithful, the map π_v^r is faithful and, since ω_v is faithful and φ_v preserves the states, the maps π_v^l is also faithful. Let $(\mathcal{K}, \eta)$ be the graph product of the pointed Hilbert spaces $(\mathcal{K}_v, \eta_v)$ with the representations $\lambda_v, \rho_v : \mathcal{B}(\mathcal{K}_v) \to \mathcal{B}(\mathcal{K})$ and define $\widetilde{\pi}_v^l = \lambda_v \circ \pi_v^l$ and $\widetilde{\pi}_v^r = \rho_v \circ \pi_v^r$.

Let $\mathcal{M}$ (resp. $\mathcal{N}$) be the von Neumann algebra subalgebra of $\mathcal{B}(\mathcal{K})$ generated by $\cup_v \widetilde{\pi}_v^l(\mathsf{M}_v)$ (resp. $\cup_v \widetilde{\pi}_v^r(\mathsf{N}_v)$). Consider the vector state $\mu(x) = \langle x\eta, \eta\rangle$ on $\mathcal{M}$ and $\mathcal{N}$. Observe that, for all $a = a_1 \dots a_n \in \mathsf{M}_\Gamma$ reduced,

with associated word $\mathbf{v} = (v_1, \dots, v_n)$, Proposition 2.1 implies $\widetilde{\pi}^l_{v_1}(a_1)\dots\widetilde{\pi}^l_{v_n}(a_n)\eta = a_1\eta_{v_1} \otimes \cdots \otimes a_n\eta_{v_n}$. Also, for all $b = b_1 \dots b_n \in \mathsf{N}_\Gamma$ reduced, with associated word $\mathbf{v} = (v_1, \dots, v_n)$, Proposition 2.2 implies $\widetilde{\pi}^r_{v_1}(b_1)\dots\widetilde{\pi}^r_{v_n}(b_n)\eta = \eta_{v_n}b_n \otimes \cdots \otimes \eta_{v_1}b_1$. It follows that $\mu(a) = \mu(b) = 0$ for all $a \in \mathsf{M}_\Gamma$ and $b \in \mathsf{N}_\Gamma$ reduced. Moreover, $(\overline{\mathcal{M}\eta}, \pi, \eta)$ (resp. $(\overline{\mathcal{N}\eta}, \rho, \eta)$) is a GNS construction for μ on $\mathcal{M}$ (resp. on $\mathcal{N}$), where $\pi(x)$ is the restriction of x to the subspace $\overline{\mathcal{M}\eta}$ (resp. $\rho(y)$ is the restriction of y to the subspace $\overline{\mathcal{N}\eta}$). By Proposition 2.3 the images of $\widetilde{\pi}^l_v$ and $\widetilde{\pi}^r_{v'}$ commute for all $v, v' \in V\Gamma$. Hence, $\mathcal{N} \subset \mathcal{M}'$ and, by the preceding computations, we find $\overline{\mathcal{N}\mathcal{M}\eta} = \overline{\mathcal{M}\mathcal{N}\eta} = \mathcal{K}$. It follows that μ is GNS faithful on $\mathcal{M}$ (resp. on $\mathcal{N}$).

By Proposition 2.22, there exists two unital normal $*$-homomorphisms $\widetilde{\pi}_l : \mathsf{M}_\Gamma \to \mathcal{B}(\mathcal{K})$ and $\widetilde{\pi}_r : \mathsf{N}_\Gamma^{\mathrm{op}} \to \mathcal{B}(\mathcal{K})$ such that $\widetilde{\pi}_l|_{\mathsf{M}_v} = \widetilde{\pi}^l_v$ and $\widetilde{\pi}_r|_{\mathsf{N}_v} = \widetilde{\pi}^r_v$. It is easy to check that the images of $\widetilde{\pi}_l$ and $\widetilde{\pi}_r$ commute. Hence, $\mathcal{K}$ is a M_Γ-N_Γ bimodule and we will simply write $a\xi b$ for the element $\widetilde{\pi}_l(a)\widetilde{\pi}_r(b)\xi$, for $a \in \mathsf{M}_\Gamma$, $b \in \mathsf{N}_\Gamma$ and $\xi \in \mathcal{K}$. Define $V : \mathrm{L}^2(\mathsf{N}_\Gamma) \to \mathcal{K}$ by $V\widehat{x} = \eta.x$. One can easily check that, for all $a = a_1 \dots a_n \in \mathsf{M}_\Gamma$ reduced, with $a_k \in \mathsf{M}^\circ_{v_k}$,

$$V^*\widetilde{\pi}_l(a_1 \dots a_n)V = \varphi_{v_1}(a_1)\dots\varphi_{v_n}(a_n).$$

The fact the φ intertwines the graph product states and the L^2 extension formula are obvious from the formula defining φ. □

The ucp map obtained in Proposition 2.30 is called the *graph product ucp map*, it generalizes Boca's construction of free product of ucp maps [Bo93]. As a consequence we are able to show that the graph product preserves the Haagerup property (see [Boc93], [CaSk14] for free products of respectively finite and σ-finite von Neumann algebras). Recall the following definition from [CaSk13]. We refer to [OkTo13] and [COST14] for alternative (but equivalent) approaches to the Haagerup property and to [Cho83], [Jo02] for the case of a finite von Neumann algebra.

Definition 2.31. A pair (M, ω) of a von Neumann algebra M with normal, faithful state ω has the Haagerup property if there exists a net $\{\varphi_i\}_{i\in I}$ of cp maps $\varphi_i : \mathsf{M} \to \mathsf{M}$ such that $\omega \circ \varphi_i \leq \omega$ and such that the GNS-maps $T_i : x\Omega_\omega \mapsto \varphi_i(x)\Omega_\omega$ extend to compact operators converging to 1 strongly.

Remark 2.32. In [CaSk14] it was proved that if a pair (M, ω) has the Haagerup property, then the cp maps φ_i can be chosen unital and such that $\omega \circ \varphi_i = \omega$. Let $(\mathcal{H}_\omega, \Omega_\omega)$ be the GNS-space with cyclic vector Ω_ω and $\mathcal{H}^\circ_\omega$ the space orthogonal to Ω_ω. Define $\varphi'_i(x) = \frac{1}{1+\epsilon}(\varphi_i(x) + \epsilon\,\omega(x)), x \in \mathsf{M}$ and let T'_i be its GNS-map $\mathcal{H}_\omega \to \mathcal{H}_\omega$ determined by $x\Omega_\omega \mapsto \varphi'_i(x)\Omega_\omega$. The restriction of T'_i to the space $\mathcal{H}^\circ_\omega$ then has norm less than $\frac{1}{1+\epsilon}\|T_i\|$. This shows, letting $\epsilon \to 0$, that in Definition 2.31 we may assume that $\|T_i|_{\mathcal{H}^\circ_\omega}\| < 1$.

Corollary 2.33. *M_Γ has the Haagerup property if and only if M_v has the Haagerup property for all $v \in V\Gamma$.*

Proof. By considering inductive limits we may assume that the graph Γ is finite. Suppose that M_v has the Haagerup property for all $v \in V\Gamma$. Let $\varphi_{v,i_v} : \mathsf{M}_v \to \mathsf{M}_v$ be a net of state-preserving ucp maps with compact L^2-implementation T_{v,i_v} and such that $||\varphi_{v,i_v}(a) - a||_2 \to 0$ for all $a \in \mathsf{M}_v$ and all $v \in V\Gamma$. By Remark 2.32 we may assume that $\|T_{v,i_v}|_{\mathcal{H}^\circ_v}\| < 1$. Define the net of ucp map $\varphi_i : \mathsf{M}_\Gamma \to \mathsf{M}_\Gamma$, each φ_i is the graph product of the φ_{v,i_v}, $v \in V\Gamma$ and the net structure for φ_i is given by the product of the nets for φ_{v,i_v}. Since $\|T_{v,i_v}|_{\mathcal{H}^\circ_v}\| < 1$ and Γ is finite it follows that the L^2-implementation of φ_i is compact. Also, $||\varphi_i(a) - a||_2 \to 0$ for all reduced operators $a \in \mathsf{M}_\Gamma$. Since the linear span of 1 and the reduced operators is weakly dense in M_Γ, the convergence holds for all $a \in \mathsf{M}_\Gamma$, from which one easily deduces that M_Γ has the Haagerup property. The other implication is an obvious consequence of the fact that $\mathsf{M}_v \subseteq \mathsf{M}_\Gamma$ admits a normal state preserving conditional expectation value. □

3. Graph products of discrete quantum groups

In this paper we need compact and discrete quantum groups both in the C*-algebraic and von Neumann algebraic framework. We recall their preliminaries here. We define graph products of quantum groups and give their basic properties.

3.1. **C*-algebraic compact/discrete quantum groups.** We write $\overline{\text{Span}}$ for the closed linear span.

Definition 3.1 (Wornonowicz [Wo87])**.** A compact quantum group $\mathbb{G}$ is a pair (A, Δ) of a unital C*-algebra A together with a comultiplication $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ (minimal tensor product) which is a unital $*$-homomorphism such that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \otimes \Delta$ and such that the following cancellation laws hold:

$$\overline{\text{Span}}\, \Delta(\mathsf{A})(\mathsf{A} \otimes 1) = \overline{\text{Span}}\, (1 \otimes \mathsf{A})\Delta(\mathsf{A}) = \mathsf{A} \otimes \mathsf{A}.$$

Any compact quantum group $\mathbb{G}$ admits a unique state ω on A that satisfies $(\omega \otimes \mathrm{id}) \circ \Delta(x) = \omega(x)1_{\mathsf{A}} = (\mathrm{id} \otimes \omega) \circ \Delta(x)$. ω is called the Haar state. The GNS-space with respect to ω shall be denoted by $\mathcal{H}$.

Let $\mathbb{G} = (\mathsf{A}, \Delta)$ be a compact quantum group. A (finite dimensional) unitary representation is a unitary operator $u \in \mathsf{A} \otimes M_n$ such that $(\Delta \otimes \mathrm{id})(u) = u_{13}u_{23}$ where $u_{23} = 1 \otimes u$ and $u_{13} = (\Sigma \otimes \mathrm{id})(u_{23})$ with $\Sigma : \mathsf{A} \otimes \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ the flip map. We denote $\mathrm{Irr}(\mathbb{G})$ for the equivalence classes of irreducible representations of $\mathbb{G}$ and for $\alpha \in \mathrm{Irr}(\mathbb{G})$ we let u^α be a concrete representation that is equivalent to α. Note that in the literature our notion of representation is often also called a corepresentation. We use n_α for the dimenstion of u^α, i.e. $u^\alpha \in \mathsf{A} \otimes M_{n_\alpha}$. We shall write $u^\alpha_{i,j}$ for the matrix coefficient $(\mathrm{id} \otimes \omega_{e_i,e_j})(u^\alpha) \in \mathsf{A}$ in case $e_i, 1 \leq i \leq n_\alpha$ is an orthonormal basis of $\mathbb{C}^{n_\alpha}$.

We set $\mathrm{Pol}(\mathbb{G}) \subseteq \mathsf{A}$ for the space of matrix coefficients of finite dimensional representations of $\mathbb{G}$. It is well-known that $\mathrm{Pol}(\mathbb{G})$ is a $*$-algebra. Let $\gamma \in \mathrm{Irr}(\mathbb{G})$ then we denote $p_\gamma \in \mathcal{B}(\mathcal{H})$ for the projection onto the closed linear span of the coefficients of u^γ identified within $\mathcal{H}$.

Let $\widehat{\mathbb{G}}$ denote the discrete dual quantum group of $\mathbb{G}$. Typically we will write $\widehat{\mathbb{G}} = (\widehat{\mathsf{A}}, \widehat{\Delta})$. We have $\widehat{\mathsf{A}} = \oplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{n_\alpha}$. We let $\widehat{\epsilon}$ be the counit of $\widehat{\mathbb{G}}$ which is the unique $*$-homomorphism $\widehat{\mathsf{A}} \to \mathbb{C}$ that satisfies $(\widehat{\epsilon} \otimes \mathrm{id}) \circ \Delta(x) = x = (\mathrm{id} \otimes \widehat{\epsilon}) \circ \Delta(x)$.

Every compact quantum group comes with a maximal (= universal) and a reduced version and we shall from this point fix a compact quantum group $\mathbb{G}$ and let (A, Δ) denote the associated reduced (compact) quantum group and let (A_m, Δ_m) denote the associated maximal (compact) quantum group. There exists a canonical surjection $\nu : \mathsf{A}_m \to \mathsf{A}$ that preserves the comultiplication. We refer to [Ku01] for the definition of maximal (= universal) quantum groups. There is no distinction between maximal and reduced versions of $\widehat{\mathbb{G}} = (\widehat{\mathsf{A}}, \widehat{\Delta})$ since for a discrete quantum group these always agree.

Remark 3.2. If $\mathbb{G}_1 = (\mathsf{A}_1, \Delta_1)$ and $\mathbb{G}_2 = (\mathsf{A}_2, \Delta_2)$ are compact quantum groups then $\mathbb{G}_1 \times \mathbb{G}_2$ is the quantum group whose C*-algebra is given by $\mathsf{A}_1 \otimes \mathsf{A}_2$ and with comultiplication $\Delta = (\mathrm{id} \otimes \Sigma \otimes \mathrm{id}) \circ \Delta_1 \otimes \Delta_2$, where $\Sigma : \mathsf{A}_1 \otimes \mathsf{A}_2 \to \mathsf{A}_2 \otimes \mathsf{A}_1$ is the flip map.

3.2. **Von Neumann algebraic quantum groups.** Let $\mathbb{G}$ be a compact quantum group. Let M be the von Neumann algebra given by the double commutant of A. $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ lifts uniquely to a unital, normal $*$-homorphism $\mathsf{M} \to \mathsf{M} \otimes \mathsf{M}$ which we keep denoting by Δ. Also the Haar state ω extends to a normal state ω on M. Then (M, Δ) forms a von Neumann algebraic quantum group in the sense of [KuVa03] with ω both the left and right Haar state.

We say that $\mathbb{G}$ is if Kac type if ω is tracial. If $\mathbb{G}$ is of Kac type then there exists a $*$-antihomomorphism $\kappa : \mathsf{M} \to \mathsf{M}$ called the antipode and which satisfies $\kappa(u^\alpha_{i,j}) = u^\alpha_{j,i}$. We let $\widehat{\kappa} : \widehat{\mathsf{M}} \to \widehat{\mathsf{M}}$ be the dual antipode. It may be characterized by $(\kappa \otimes \widehat{\kappa})(W) = W$ where W is the left multiplicative unitary from [KuVa03] (though this is not the definition, it suffices for our purposes).

3.3. **Graph products, their representation theory and Haar state.** For all $v \in V\Gamma$, let $\mathbb{G}_v$ be a compact quantum group with full C*-algebra $\mathsf{A}_{v,m}$, reduced C*-algebra A_v, von Neumann algebra M_v, Haar state ω_v and comultiplication Δ_v (on any of these algebras). Let $\mathsf{A}_m (= \mathsf{A}_{\Gamma,m})$ be the maximal graph product C*-algebra associated to the family of C*-algebras $(\mathsf{A}_{v,m})_{v \in V\Gamma}$. Since ω_v is faithful (resp. normal and faithful) on A_v (resp. on M_v), we can also consider the reduced graph product C*-algebra $\mathsf{A}\,(= \mathsf{A}_\Gamma)$ associated to the family $(\mathsf{A}_v, \omega_v)_{v \in V\Gamma}$ and the graph product von Neumann algebra $\mathsf{M}\,(= \mathsf{M}_\Gamma)$ associated to the family $(\mathsf{M}_v, \omega_v)_{v \in V\Gamma}$.

By the universal property of A_m, there exists a unique unital $*$-homomorphism $\Delta : \mathsf{A}_m \to \mathsf{A}_m \otimes \mathsf{A}_m$ such that $\Delta|_{\mathsf{A}_v} = \Delta_v$ for all $v \in V\Gamma$. From [Wa95, Definition 2.1'] we can show that $\mathbb{G} = (\mathsf{A}_m, \Delta)$ is a compact quantum group. Indeed, for all $v \in V\Gamma$, the inclusion $\mathsf{A}_{v,m} \subset \mathsf{A}_m$ intertwines the comultiplication, it induces an inclusion $\mathrm{Irr}(\mathbb{G}_v) \subset \mathrm{Irr}(\mathbb{G})$. Since the matrix coefficients of $\mathrm{Irr}(\mathbb{G}_v)$ generate A_m as a $*$-algebra this shows that the conditions of [Wa95, Definition 2.1'] are satisfied and hence $\mathbb{G}$ is a compact quantum group.

Note that it is at this point not clear that (A_m, Δ) is the underlying universal quantum group of $\mathbb{G}$ in the sense of [Ku01]. In fact this is true as follows from Theorem 3.4 below. We shall also prove that M and A are the algebras of the underlying von Neumann and reduced C*-algebraic quantum group. In order to distinguish notation we shall – only in this section – write $C_m(\mathbb{G}), C_r(\mathbb{G})$ and $\mathrm{L}^\infty(\mathbb{G})$ for the full and reduced C*-algebra associated with $\mathbb{G}$ as well as its von Neumann algebra. Also write $\nu_{\mathbb{G}} : C_m(\mathbb{G}) \to C_r(\mathbb{G})$ for the canonical surjection and $L^2(\mathbb{G})$ for the GNS space of $\mathbb{G}$.

Definition 3.3. A unitary representation u of $\mathbb{G}$ is said to be *reduced* if it is of the form $u = u^{\alpha_1} \otimes \ldots \otimes u^{\alpha_n}$, where $n \geq 1$, $\mathbf{v} = (v_1, \ldots, v_n)$ is a reduced word and $\alpha_k \in \mathrm{Irr}(\mathbb{G}_{v_k}) \setminus \{1\}$ for all $1 \leq k \leq n$.

Let $\nu_v : \mathsf{A}_{v,m} \to \mathsf{A}_v$ be the canonical surjection. By the universal property of A_m, we have a unique surjective and unital $*$-homomorphism $\nu : \mathsf{A}_m \to \mathsf{A}$ such that $\nu|_{\mathsf{A}_v} = \nu_v$.

Theorem 3.4. *We have,*

1. *The Haar state ω of $\mathbb{G}$ is given by $\omega = \omega_\Gamma \circ \nu$.*
2. *All the reduced representations are irreducible and any non-trivial irreducible representation of $\mathbb{G}$ is unitarily equivalent to a reduced one.*
3. *We have $C_m(\mathbb{G}) = \mathsf{A}_m$, $C_r(\mathbb{G}) = \mathsf{A}$, $\mathrm{L}^\infty(\mathbb{G}) = \mathsf{M}$ and $\nu = \nu_{\mathbb{G}}$.*

Proof. (1). Let $\mathcal{P} \subset \mathsf{A}_m$ be the linear span of the coefficients of the reduced representations (so $1 \notin \mathcal{P}$). Since $\mathcal{P}_m$ equals the linear span of the reduced operators $a \in \mathsf{A}_m$ relative to the family of states $(\omega_v)_{v \in V\Gamma}$ (see Remark 2.11) and of the form $a = a_1 \ldots a_n$, with $a_k \in \mathrm{Pol}(\mathbb{G}_{v_k})$ it follows that the linear span of 1 and $\mathcal{P}$ is dense in A_m. Hence, it suffices to show the invariance of ω on $\mathcal{P}$. Since $\Delta(\mathcal{P}) \subset \mathcal{P} \odot \mathcal{P}$ and $\nu(\mathcal{P})$ is again contained in $\mathcal{P}$ (viewed within A_Γ) of the reduced operators in A_Γ we have $(\mathrm{id} \otimes \omega)\Delta(\mathcal{P}) \subset (\mathrm{id} \otimes \omega)(\mathcal{P} \odot \mathcal{P}) = \{0\}$. In the same way we find $(\omega \otimes \mathrm{id})\Delta(\mathcal{P}) = \{0\}$. Hence, for all $a \in \mathcal{P}$, one has $(\mathrm{id} \otimes \omega)\Delta(a) = (\omega \otimes \mathrm{id})\Delta(a) = 0 = \omega(a)$.

(2). This assertion was already proved in (1). Indeed, since $\omega(\mathcal{P}) = \{0\}$, we have $(\omega \otimes \mathrm{id})(u) = 0$ for every reduced representation, which shows that the reduced representations are irreducibles. Moreover, since the linear span of $\mathcal{P}$ and 1 is dense in A_m, every non trivial irreducible representation is equivalent to a reduced one.

(3). Since ν is surjective and ω_Γ is faithful on A, it follows from (1) that $\mathsf{A} = C_r(\mathbb{G})$, $\mathcal{H} = \mathrm{L}^2(\mathbb{G})$ and $\mathsf{M} = \mathrm{L}^\infty(\mathbb{G})$. It follows from (2) that $\mathrm{Pol}(\mathbb{G})$ is the linear span of $\mathcal{P}$ and 1. Hence, $C_m(\mathbb{G})$ is generated, as a C*-algebra, by $\bigcup_{v \in V\Gamma} \mathrm{Pol}(\mathbb{G}_v)$ and the relations $a_v a_{v'} = a_{v'} a_v$ are satisfied in $C_m(\mathbb{G})$, for all $a_v \in \mathrm{Pol}(\mathbb{G}_v)$, $a_{v'} \in \mathrm{Pol}(\mathbb{G}_{v'})$ and all $v, v' \in V\Gamma$ such that $(v, v') \in E\Gamma$. From the inclusions $\mathrm{Pol}(\mathbb{G}_v) \subset C_m(\mathbb{G})$ and the universal property of $C_m(\mathbb{G}_v)$ we have, for all $v \in V\Gamma$, a unital $*$-homomorphism $\pi_v : C_m(\mathbb{G}_v) \to C_m(\mathbb{G})$ which is the identity on $\mathrm{Pol}(\mathbb{G}_v)$. The morphisms π_v are such that $\pi_v(a_v)\pi_{v'}(a_{v'}) = \pi_{v'}(a_{v'})\pi_v(a_v)$ for all $a_v \in \mathrm{Pol}(\mathbb{G}_v)$, $a_{v'} \in \mathrm{Pol}(\mathbb{G}_{v'})$ and all $v, v' \in V\Gamma$ and $C_m(\mathbb{G})$ is generated by $\bigcup_{v \in V\Gamma} \pi_v(C_m(\mathbb{G}_v))$. By universal property of A_m, we have a surjective unital $*$-homomorphism from A_m to $C_m(\mathbb{G})$ which is the identity on $\mathrm{Pol}(\mathbb{G})$. Hence, $\mathsf{A}_m = C_m(\mathbb{G})$. That $\nu = \nu_{\mathbb{G}}$ follows then since these maps are $*$-homomorphisms that agree on $\mathrm{Pol}(\mathbb{G})$. □

3.4. **Haagerup property of discrete quantum groups.** We show that the Haagerup property of discrete quantum groups is preserved by the graph product. In case the quantum group is of Kac type this follows from Corollary 2.33 and [DFSW13, Theorem 6.7]. Since it is unknown if the correspondence in [DFSW13, Theorem 6.7] holds beyond Kac type quantum groups the general case requires a proof. The special case of free products was proved in [DFSW13, Theorem 7.8] the special case of Cartesian products of quantum groups can be found in [Fr14, Proposition 3.4].

3.4.1. *General discrete quantum groups.* Firstly recall the following equivalent definition of the Haagerup property for discrete quantum groups, see [DFSW13, Proposition 6.2 and Lemma 6.24].

Proposition 3.5. *A discrete quantum group $\widehat{\mathbb{G}}$ has the Haagerup property if and only if there is a sequence of states $(\mu_k)_{k\in\mathbb{N}}$ on $\mathrm{Pol}(\mathbb{G})$ such that:*

(1) For each $k \in \mathbb{N}$ we have $((\mathcal{F}\mu_k)^\alpha)_{\alpha\in\mathrm{Irr}(\widehat{\mathbb{G}})} \in \prod_{\alpha\in\mathrm{Irr}(\widehat{\mathbb{G}})} M_{n_\alpha}$ is actually in $\oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} M_{n_\alpha}$.

(2) For each $\alpha \in \mathrm{Irr}(\widehat{\mathbb{G}})$ the net $((\mathcal{F}\mu_k)^\alpha)_{k\in\mathbb{N}}$ converges to the identity matrix.

If these conditions hold, then we may moreover impose the following conditions on the net $(\mu_k)_{k\in\mathbb{N}}$,

(3) For each $k \in \mathbb{N}$ and $\alpha \in \mathrm{Irr}(\widehat{\mathbb{G}})$ with $\alpha \neq 1$ we have that $\|(\mathcal{F}\mu_k)^\alpha\| \leq \exp(-\frac{1}{k})$.

Recall the following definition from [BLS96] and recall that $\mathrm{Pol}(\mathbb{G})$ is a unital $*$-algebra for every compact quantum group $\mathbb{G}$.

Definition 3.6. Let $\mathcal{A}$ be a unital $*$-algebra over $\mathbb{C}$. A linear map $\omega : \mathcal{A} \to \mathbb{C}$ is called a state if $\omega(1) = 1$, $\omega(a^*) = \overline{\omega(a)}$ and $\omega(a^*a) \geq 0$ for every $a \in A$.

Let $\mathcal{A}_v, v \in V\Gamma$ be unital $*$-algebras, each equipped with a state φ_i. Let $\mathcal{A}$ be its algebraic graph product which is defined as the unital $*$-algebra freely generated by $\mathcal{A}_v, v \in V\Gamma$ subject to the relation that $a_v a_w = a_w a_v$ for any $a_v \in \mathcal{A}_v, a_w \in \mathcal{A}_w$ such that $(v, w) \in E\Gamma$ (and the units of each $\mathcal{A}_v$ are identified). Using the decomposition $\mathcal{A}_i = \mathbb{C}1 \oplus \mathcal{A}_i^\circ$ with $\mathcal{A}_i = \ker\varphi_i$ we may identify $\mathcal{A}$ with the vector space $\mathbb{C}1 \oplus \bigoplus_{v_1\ldots v_n\in\mathcal{W}_{\min}} \mathcal{A}_{v_1}^\circ \otimes \mathcal{A}_{v_2}^\circ \otimes \ldots \otimes \mathcal{A}_{v_n}^\circ$. Suppose that there exists a state ψ_v on each $\mathcal{A}_v, v \in V\Gamma$, then the algebraic graph product functional $\psi = \diamond_{v\in V\Gamma}(\psi_v, \varphi_v)$ on $\mathcal{A}$ is defined as $\psi(a_1 \ldots a_n) = \psi_{v_1}(a_1)\ldots\psi_{v_n}(a_n)$ whenever $a_i \in A_{v_i}^\circ$ with $v_1 \ldots v_n \in \mathcal{W}_{\min}$. Note that ψ depends on the choice of φ_v.

Now let again $\mathbb{G}_v, v \in V\Gamma$ be a compact quantum group and $\mathbb{G}$ be its graph product.

Theorem 3.7. *The discrete quantum group $\widehat{\mathbb{G}}$ has the Haagerup property if and only if for every $v \in V\Gamma$ we have that $\widehat{\mathbb{G}}_v$ has the Haagerup property.*

Proof. Using [Fr14, Proposition 3.7] it suffices to prove the theorem under the condition that the graph Γ is finite. Firstly, suppose that for every $v \in V\Gamma$ the quantum group $\widehat{\mathbb{G}}_v$ has the Haagerup property. By Proposition 3.5 there exists a sequence $(\mu_{v,k})_{k\in\mathbb{N}}$ of states on $\mathrm{Pol}(\mathbb{G}_v)$ satisfying (1) - (3) of this proposition. Recall that ω_v is the Haar state of $\mathbb{G}_v$. Let $\mu_k = \diamond_{v\in V\Gamma}(\mu_{v,k}, \omega_v)$ denote the graph product functional as defined in the paragraph before this theorem.

We claim that $\mu_k, k \in \mathbb{N}$ is again a state. This follows from the following standard argument. For convenience of notation fix $k \in \mathbb{N}$. From the state $\mu_{v,k}$ on $\mathrm{Pol}(\mathbb{G}_v)$ we may follow the usual GNS-construction to find a pre-Hilbert space $\mathcal{H}_{v,0}$ with cyclic unit vector ξ_v and representation π_v such that $\mu_{v,k}(x) = \langle \pi_v(x)\xi_v, \xi_v\rangle$. Let $\mathsf{A}_{v,m}$ be the maximal C*-algebra associated with the quantum group $\mathbb{G}_v$. As in [DijKo93, Lemma 4.2] the map π_v extends to a $*$-homomorphism $\mathsf{A}_{v,m} \to B(\mathcal{H}_v)$ with $\mathcal{H}_v$ the completion of $\mathcal{H}_{v,0}$. Let B be the reduced graph product C*-algebra of $\pi_v(\mathsf{A}_{v,m}), v \in V\Gamma$ and let ξ denotes its cyclic vacuüm vector. Since $\pi_v(\mathsf{A}_{v,m})$ is included into B naturally we may regard π_v as a $*$-homomorphism $\mathsf{A}_{v,m} \to \mathsf{B}$. The universal property of the maximal graph product C*-algebra A_m then yields a $*$-homomorphism $\pi : \mathsf{A}_m \to \mathsf{B}$. Let μ_k be the state on A_m defined by $\mu_k(x) = \langle \pi(x)\xi, \xi\rangle$ and denote by μ_k the restriction to $\mathrm{Pol}(\mathbb{G})$. It follows from Theorem 3.4 that indeed $\mathrm{Pol}(\mathbb{G})$ is the algebraic graph product of $\mathrm{Pol}(\mathbb{G}_v), v \in V\Gamma$ and by construction it follows that μ_k is the graph product of the states $\mu_{k,v}, v \in V\Gamma$. In particular μ_k is again a state.

Let $\alpha_j, 1 \leq j \leq l$ be elements of $\mathrm{Irr}(\mathbb{G}_{v_j})$ with v_j such that $v_1 v_2 \ldots v_l \in \mathcal{W}_{\min}$. By definition of the graph product and the graph product representation $\alpha_1 \otimes \ldots \otimes \alpha_n$, see Theorem 3.4, we see that,

$$(\mathcal{F}\mu_k)^{\alpha_1\otimes\ldots\otimes\alpha_l} = \otimes_{j=1}^l (\mathcal{F}\mu_{v_j,k})^{\alpha_j}.$$

It is then straightforward to verify conditions (1) and (2) of Proposition 3.5. Note that the second condition follows from the fact that Γ is finite and that $\|(\mathcal{F}\mu_k)^{\alpha_1\otimes\ldots\otimes\alpha_l}\| \leq \exp(-\frac{l}{k})$. □

4. Rapid Decay

We prove that the property of Rapid Decay (RD) for discrete quantum groups is preserved by taking graph products of finite graphs under suitable conditions on the vertex quantum groups. In particular our result holds if every vertex quantum group is either a classical group or a quantum group with polynomial growth. This generalizes the result of [CHR13] which proves the corresponding result for discrete groups.

4.1. **Preliminaries on elements affiliated with a C*-algebra.** For unbounded operators affiliated with a C*-algebra we refer to [Wo91]. In case $\widehat{\mathsf{A}}$ is the C*-algebra of a discrete quantum group $\widehat{\mathbb{G}}$ the notion of affiliated elements simplifies. In that case the $*$-algebra $\widehat{\mathsf{A}}^\eta$ of affiliated elements with $\widehat{\mathsf{A}}$ can be identified with the algebraic product $\prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} M_{n_\alpha}$ and for each operator in $\widehat{\mathsf{A}}^\eta$ the vector space $\mathcal{H}_{\mathrm{Pol}}$ (the space of matrix coefficients of finite dimensional representations of $\mathbb{G}$ identified as subspace of $\mathcal{H}$) forms a core. For $L \in \widehat{\mathsf{A}}^\eta$ we will write $\prod_{\alpha\in\mathrm{Irr}(\mathbb{G})} L^{(\alpha)}$ for this representation. One can apply $*$-homomorphisms of $\widehat{\mathsf{A}}$ to $\widehat{\mathsf{A}}^\eta$ such as the counit $\widehat{\epsilon}$ and comultiplication $\widehat{\Delta}$ as well as proper maps such as the antipode $\hat{\kappa}$ of a Kac type discrete quantum group.

4.2. **Definition of Rapid Decay.** Let $\mathbb{G} = (\mathsf{A}, \Delta)$ be a compact quantum group with discrete dual $\widehat{\mathbb{G}} = (\widehat{\mathsf{A}}, \widehat{\Delta})$. Then $\widehat{\mathsf{A}} = \oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} M_{n_\alpha}$ where n_α is the dimension of α. In case $\widehat{\mathbb{G}}$ is of Kac type its Haar weight $\widehat{\omega}$ is given by $\widehat{\omega} = \oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} n_\alpha \mathrm{Tr}_{M_{n_\alpha}}$ where $\mathrm{Tr}_{M_{n_\alpha}}$ is the normalized trace on M_{n_α}. For every $\alpha \in \mathrm{Irr}(\mathbb{G})$ let $u^\alpha \in \mathsf{A} \otimes M_{n_\alpha}$ be a corepresentation belonging to the equivalence class α. The Fourier transform $\mathcal{F}$ of $x = \oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} x_\alpha \in \widehat{\mathsf{A}}$ with finite direct sum, is defined as the element,

$$\sum_{\alpha\in\mathrm{Irr}(\mathbb{G})} (\mathrm{id}\otimes\widehat{\omega})(u^\alpha(1\otimes x_\alpha)).$$

Definition 4.1 (Lengths and central lengths)**.** A length on $\widehat{\mathbb{G}}$ is an (unbounded) operator affiliated with $\widehat{\mathsf{A}}$ that satisfies the following properties: $L \geq 0$, $\widehat{\epsilon}(L) = 0$, $\widehat{\kappa}(L)|_{\mathcal{H}_{\mathrm{Pol}}} = L|_{\mathcal{H}_{\mathrm{Pol}}}$ and $\widehat{\Delta}(L) \leq 1 \otimes L + L \otimes 1$. Given such a length we denote $q_n \in \mathcal{M}(\widehat{\mathsf{A}})$ (the multiplier algebra of $\widehat{\mathsf{A}}$) for the spectral projection of L associated to the interval $[n, n+1)$ with $n \in \mathbb{N}$. L is called central if each of its spectral projections are central in $\mathcal{M}(\widehat{\mathsf{A}})$.

Definition 4.2. Let L be a central length on the discrete quantum group $\widehat{\mathbb{G}} = (\widehat{\mathsf{A}}, \widehat{\Delta})$. We say that $(\widehat{\mathbb{G}}, L)$ has the property of Rapid Decay (RD) if the following condition is satisfied: there exists a polynomial $P \in \mathbb{R}[X]$ such that for every $k \in \mathbb{N}$ and $a \in q_k\widehat{\mathsf{A}}$ and for every $m, l \in \mathbb{N}$ we have $\|q_m\mathcal{F}(a)q_l\| \leq P(k)\|a\|_2$.

In fact there are other equivalent formulations of (RD), see [Ve07, Proposition and Definition 3.5] or [Jo90] for the group case.

4.3. **Permanence properties of (RD).** We prove permanence properties of (RD) under graph products. In particular we prove that (RD) is preserved by free products.

Lemma 4.3. *Let $\mathbb{G}$ be a compact quantum group of Kac type with discrete dual quantum group $\widehat{\mathbb{G}} = (\widehat{\mathsf{A}}, \widehat{\Delta})$. Let $\{u^\alpha \mid \alpha \in \mathrm{Irr}(\mathbb{G})\}$ denote a complete set of irreducible mutually non-equivalent corepresentations and let $u^\alpha_{i,j} = (id \otimes \omega_{e_i,e_j})(u^\alpha)$ denote its matrix coefficients with respect to some orthonormal basis e_i of the representation space $\mathcal{H}_\alpha$ for which,*

$$\omega((u^\alpha_{i,j})^* u^\alpha_{k,l}) = \delta_{i,k}\delta_{j,l} n_\alpha^{-1}, \tag{4.1}$$

(see [Da10, Proposition 2.1]*). The contragredient corepresentation $\overline{\alpha}$ is given by $u^{\overline{\alpha}}_{i,j} = u^\alpha_{j,i}$ (and this definition is consistent with* (4.1)*). Let $E^\alpha_{i,j} \in \widehat{\mathsf{A}}$ be the matrix with entry 1 on the i-th row and j-th column of the matrix block indexed by $\alpha \in \mathrm{Irr}(\mathbb{G})$ and zeros elsewhere. Then $\widehat{\kappa}(E^\alpha_{i,j}) = E^{\overline{\alpha}}_{j,i}$.*

Proof. The proof is a consequence of the relation $\kappa(u^\alpha_{i,j}) = (u^\alpha_{j,i})^*$ and using duality between $\mathbb{G}$ and $\widehat{\mathbb{G}}$. So let $\omega^\alpha_{i,j}(\,\cdot\,) = n_\alpha\,\omega((u^\alpha_{i,j})^*\,\cdot\,)$ so that by orthogonality (see [Da10, p. 1351]) we have,

$$(\omega^\alpha_{i,j}\otimes\mathrm{id})(W) = E^\alpha_{i,j}, \tag{4.2}$$

where $W = \oplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} u^{\alpha}$. Then we have using that for Kac algebras $\kappa^2 = \mathrm{id}$, κ is an anti-homomorphism, $\omega \circ \kappa = \omega$, traciality of the Haar state ω and the relation $\kappa(u^{\alpha}_{i,j}) = (u^{\alpha}_{j,i})^*$, see [Ti08],

$$\begin{aligned}\omega^{\alpha}_{i,j} \circ \kappa &= \omega(\kappa^2((u^{\alpha}_{i,j})^*)\kappa(\,\cdot\,)) = \omega(\kappa(\,\cdot\,\kappa((u^{\alpha}_{i,j})^*))) = \omega(\,\cdot\,\kappa((u^{\alpha}_{i,j})^*)) = \omega(\kappa((u^{\alpha}_{i,j})^*)\,\cdot\,) \\ &= \omega(\kappa(u^{\alpha}_{i,j})^*\,\cdot\,) = \omega(u^{\alpha}_{j,i}\,\cdot\,) = \omega((u^{\overline{\alpha}}_{j,i})^*\,\cdot\,),\end{aligned}$$

so that $(\omega^{\alpha}_{i,j} \circ \kappa \otimes \mathrm{id})(W) = E^{\overline{\alpha}}_{j,i}$ by (4.2). This means that using the relation $(\kappa \otimes \hat{\kappa})(W) = W$ [KuVa03],

$$\widehat{\kappa}(E^{\alpha}_{i,j}) = \hat{\kappa}(\omega^{\alpha}_{i,j} \otimes \mathrm{id})(W) = (\omega^{\alpha}_{i,j} \circ \kappa \otimes \mathrm{id})(W) = E^{\overline{\alpha}}_{j,i}.$$

□

Now let us return to graph products. So let Γ be again a simplicial graph and for each $v \in V\Gamma$ let $\mathbb{G}_v$ be a compact quantum group with discrete dual $\widehat{\mathbb{G}}_v$. Let $\mathbb{G}$ be the graph product of $\mathbb{G}_v, v \in V\Gamma$ and let $\widehat{\mathbb{G}}$ be its discrete dual. From Theorem 3.4 we see that the C*-algebra $\widehat{A}$ associated to $\widehat{\mathbb{G}}$ equals

$$\oplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} M_{n_{\alpha_1}} \otimes \ldots \otimes M_{n_{\alpha_l}},$$

in case $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_l$. For $k \in \mathbb{N}$ we shall use the notation $\mathsf{A}_{(k)}$ for the subspace defined by

$$\oplus_{\alpha \in \mathrm{Irr}(\mathbb{G}), \alpha = \alpha_1 \otimes \ldots \otimes \alpha_k} M_{n_{\alpha_1}} \otimes \ldots \otimes M_{n_{\alpha_k}},$$

so the subspace of exactly k-fold tensor products of matrices. Let $w \in V\Gamma$. We shall denote $P_w : \mathcal{H} \to \mathcal{H}$ for the projection onto the linear span of the Hilbert spaces $\mathcal{H}_{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{W}_{\min}$ a word that is equivalent to a word that starts with w. The following Lemma 4.4 is well defined now.

Lemma 4.4. *For $v \in V\Gamma$ suppose that $L_v = \oplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} L_v^{(\alpha)}$ is a central length for the discrete quantum group $\widehat{\mathbb{G}}_v$. Define,*

$$L = \prod_{\alpha \in \mathrm{Irr}(\mathbb{G})} \sum_{i=1}^{l(\alpha)} 1_{M_{n_{\alpha_1}}} \otimes \ldots \otimes 1_{M_{n_{\alpha_{i-1}}}} \otimes L^{(\alpha_i)}_{v_i} \otimes 1_{M_{n_{\alpha_{i+1}}}} \otimes \ldots \otimes 1_{M_{n_{\alpha_{l(\alpha)}}}}, \tag{4.3}$$

where each $\alpha \in \mathrm{Irr}(\mathbb{G})$ decomposes as the tensor product representation $\alpha_1 \otimes \ldots \otimes \alpha_{l(\alpha)}$ and $\alpha_i \in \mathrm{Irr}(\mathbb{G}_{v_i})$. Then L is a central length function for the discrete quantum group $\widehat{\mathbb{G}}$.

Proof. We first check that $\widehat{\Delta}(L) \leq L \otimes 1 + 1 \otimes L$. Recall from [Fi10, Eqn. (1) in Proposition 3] that,

$$\widehat{\Delta}(p_\gamma)(p_\alpha \otimes p_\beta) = \begin{cases} p_\gamma^{\alpha \otimes \beta} & \text{if } \gamma \subseteq \alpha \otimes \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

where $p_\gamma^{\alpha \otimes \beta} \in \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ is the projection onto the sum of all subrepresentations of $\alpha \otimes \beta$ that are equivalent to γ. Since the length functions $L_v, v \in V\Gamma$ are central, we know that $L_v = \oplus_{\alpha \in \mathrm{Irr}(\mathbb{G}_v)} f_v(\alpha) p_\alpha$ for some $f_v : \mathrm{Irr}(\mathbb{G}_v) \to [0, \infty)$ and similarly $L = \oplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} f(\alpha) p_\alpha$. In fact, by definition of L we have that $f(\alpha) = f_{v_1}(\alpha_1) + \ldots + f_{v_n}(\alpha_n)$ in case $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_n$. The condition $\widehat{\Delta}(L) \leq L \otimes 1 + 1 \otimes L$ now becomes equivalent to the property that for every $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$ we have $\widehat{\Delta}(L)(p_\alpha \otimes p_\beta) \leq (L \otimes 1 + 1 \otimes L) p_\alpha \otimes p_\beta$, which by (4.4) is equivalent to,

$$\sum_{\gamma \in \mathrm{Irr}(\mathbb{G}), \gamma \subseteq \alpha \otimes \beta} f(\gamma) p_\gamma^{\alpha \otimes \beta} \leq (f(\alpha) + f(\beta)) p_\alpha \otimes p_\beta. \tag{4.5}$$

Now fix $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_n \in \mathrm{Irr}(\mathbb{G})$ and $\beta = \beta_1 \otimes \ldots \otimes \beta_m \in \mathrm{Irr}(\mathbb{G})$. Let v_i and w_i be such that $\alpha_i \in \mathrm{Irr}(\mathbb{G}_{v_i})$ and $\beta_i \in \mathrm{Irr}(\mathbb{G}_{w_i})$. $\alpha \otimes \beta$ is not necessarily irreducible, c.f. Theorem 3.4. If $(v_i, v_{i+1}) \in E\Gamma$ then $\alpha_1 \otimes \ldots \otimes \alpha_n$ is unitarily equivalent to $\alpha_1 \otimes \ldots \otimes \alpha_{i-1} \otimes \alpha_{i+1} \otimes \alpha_i \otimes \alpha_{i+2} \otimes \ldots \alpha_n$ by intertwining with the flip map $\mathrm{id}^{\otimes i-1} \otimes \Sigma \otimes \mathrm{id}^{\otimes n-i-1}$. Therefore, without loss of generality we may assume that there exists r such that $v_1 \ldots v_r w_1 \ldots w_m$ is reduced and $v_1 \ldots v_r w_1 \ldots w_m \simeq v_1 \ldots v_n w_1 \ldots v_m$. Note that this implies that $w_1, \ldots, w_{n-r}$ commute and $\{w_1, \ldots, w_{n-r}\} = \{v_{r+1}, \ldots, v_m\}$. Therefore, without loss of generality we may

assume that $v_{r+1} = w_1, \ldots, v_n = w_{n-r}$ (since β is equivalent to a representation for which this is true, again by intertwining with flip maps). Then $\alpha \otimes \beta$ is equivalent to

$$\alpha_1 \otimes \ldots \otimes \alpha_r \otimes \alpha_{r+1} \otimes \beta_1 \otimes \alpha_{r+2} \otimes \beta_2 \otimes \ldots \otimes \alpha_n \otimes \beta_{n-r} \otimes \beta_{n-r+1} \otimes \ldots \otimes \beta_m. \tag{4.6}$$

Suppose that $\gamma \in \mathrm{Irr}(\mathbb{G})$ is contained in (4.6). Then by the Peter-Weyl decompositions of $\alpha_{r+1} \otimes \beta_1, \ldots, \alpha_n \otimes \beta_{n-r}$, there exist irreducible representations $\gamma_1, \ldots, \gamma_{n-r}$ with $\gamma_1 \subseteq \alpha_{r+1} \otimes \beta_1, \ldots, \gamma_{n-r} \subseteq \alpha_n \otimes \beta_{n-r}$ such that $\gamma \simeq \alpha_1 \otimes \ldots \otimes \alpha_r \otimes \gamma_1 \otimes \ldots \otimes \gamma_{n-r} \otimes \beta_{n-r+1} \otimes \ldots \otimes \beta_m$. This implies that $f(\gamma) = \sum_{i=1}^r f_{v_i}(\alpha_i) + \sum_{i=1}^{n-r} f_{v_{i+r}}(\gamma_i) + \sum_{i=n-r+1}^m f_{w_i}(\beta_i)$ and since $f_{v_{i+r}}$ is a length function, this implies that $f(\gamma) \leq \sum_{i=1}^r f_{v_i}(\alpha_i) + \sum_{i=1}^{n-r} \big(f_{v_{i+r}}(\alpha_{i+r}) + f_{u_i}(\beta_i)\big) + \sum_{i=n-r+1}^m f_{w_i}(\beta_i) = \sum_{i=1}^n f_{v_i}(\alpha_i) + \sum_{i=1}^m f_{w_i}(\beta_i)$ and so condition (4.5) holds.

Next we check the relation $\widehat{\kappa}(L) = L$. Let $\alpha \in \mathrm{Irr}(\mathbb{G})$ and assume that it decomposes as a reduced tensor product $\alpha_1 \otimes \ldots \otimes \alpha_n$. The contragredient representation (see Lemma 4.3) is then given by $\overline{\alpha_n} \otimes \ldots \otimes \overline{\alpha_1}$. This implies, using Lemma 4.3 and its notion, that $\widehat{\kappa}(E^{\alpha_1}_{i_1,j_1} \otimes \ldots \otimes E^{\alpha_n}_{i_n,j_n}) = E^{\overline{\alpha_n}}_{j_n,i_n} \otimes \ldots \otimes E^{\overline{\alpha_1}}_{j_1,i_1} = \widehat{\kappa}(E^{\alpha_n}_{i_n,j_n}) \otimes \ldots \otimes \widehat{\kappa}(E^{\alpha_1}_{i_1,j_1})$. Applying the latter observation to (4.3) yields that $\widehat{\kappa}(L) = L$.

Finally, we have $\widehat{\epsilon}(L) = f(\alpha_0)p_{\alpha_0}$, with $\alpha_0 \in \mathrm{Irr}(\mathbb{G})$ the trivial representation. Since $f(\alpha_0) = 0$ we have $\widehat{\epsilon}(L) = 0$.

□

The following lemma uses the notion of polynomial growth for which we refer to [Ve07]. Examples of discrete quantum groups with polynomial growth can be found in [BaVe09].

Lemma 4.5. *Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups such that $(\widehat{\mathbb{G}}_1, L_1)$ has (RD). If either $(\widehat{\mathbb{G}}_2, L_2)$ has polynomial growth or is a classical discrete group with (RD) then $\widehat{\mathbb{G}}_1 \times \widehat{\mathbb{G}}_2$ has (RD).*

Proof. Let $q_k^{(1)}, q_k^{(2)}$ and q_k be the spectral projections onto $[k, k+1)$ of respectively L_1, L_2 and L. Also write $q_{\leq k}^{(1)} = \sum_{j=0}^k q_j^{(1)}, q_{\leq k}^{(2)} = \sum_{j=0}^k q_j^{(2)}$. Let $\widehat{\mathsf{A}}_1, \widehat{\mathsf{A}}_2$ and $\widehat{\mathsf{A}}$ be the C*-algebras associated to the duals of respectively $\mathbb{G}_1, \mathbb{G}_2$ and $\mathbb{G}$. Let P_1, P_2 be the polynomials witnessing (RD) for $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ respectively.

In order to prove that for $x \in q_k\widehat{\mathsf{A}}$ we have $\|q_l \mathcal{F}(x) q_m\| \leq P(k)\|x\|_2$ for some polynomial P it suffices to prove for every k the following estimate holds,

$$\|\mathcal{F}(x)\| \leq P(k)\|x\|_2, \qquad \text{for all } x \in (q_{\leq k}^{(1)} \otimes q_{\leq k}^{(2)})\widehat{\mathsf{A}},$$

as follows from the fact that $q_{\leq k}^{(1)} \otimes q_{\leq k}^{(2)} \geq q_k$. Observe that, by definition, we have $\mathcal{F}(a \otimes b) = \mathcal{F}_1(a) \otimes \mathcal{F}_2(b)$ for all $a \in q_{\leq k}^{(1)}\widehat{\mathsf{A}}_1$ and all $b \in q_{\leq k}^{(2)}\widehat{\mathsf{A}}_2$.

First assume that $(\widehat{\mathbb{G}}_2, L_2)$ has polynomial growth (so, in particular it is an amenable discrete Kac algebra with property (RD) [Ve07]). Let $\widehat{\omega}_i$ be the Haar weight on $\widehat{\mathsf{A}}_i$, $i = 1, 2$. By polynomial growth we have $\widehat{\omega}_2\left(q_n^{(2)}\right) \leq P_3(n)$, where P_3 is a polynomial with $P_3(n) \geq 1$ for all n. Let $x \in (q_{\leq k}^{(1)} \otimes q_{\leq k}^{(2)})\widehat{\mathsf{A}}$ be a finite sum $x = \sum_i a_i \otimes b_i$, where $a_i \in q_{\leq k}^{(1)}\widehat{\mathsf{A}}_1$ with $\|a_i\|_2 < \infty$ and $b_i \in q_{\leq k}^{(2)}\widehat{\mathsf{A}}_2$ with $\|b_i\|_2 < \infty$ for all i. We may and will assume that (a_i) is an orthonormal system with respect to the scalar product given by $\widehat{\omega}_1$. Hence,

$\|x\|_2^2 = \sum_i \|b_i\|_2^2$ and,

$$\begin{aligned}
\|\mathcal{F}(x)\| &= \|\sum_i \mathcal{F}_1(a_i)\otimes\mathcal{F}_2(b_i)\| \leq \sum_i \|\mathcal{F}_1(a_i)\|\,\|\mathcal{F}_2(b_i)\| \leq P_1(k)P_2(k)\sum_i \|a_i\|_2\|b_i\|_2 \\
&= P_1(k)P_2(k)\sum_i \|b_i\|_2 \leq P_1(k)P_2(k)\sqrt{\dim(q^{(2)}_{\leq k}\widehat{\mathsf{A}}_2)}\sqrt{\sum_i \|b_i\|_2^2} = P_1(k)P_2(k)\sqrt{\dim(q^{(2)}_{\leq k}\widehat{\mathsf{A}}_2)}\|x\|_2 \\
&= P_1(k)P_2(k)\|x\|_2\sqrt{\widehat{\omega}_2\left(q^{(2)}_{\leq k}\right)} = P_1(k)P_2(k)\|x\|_2\sqrt{\sum_{j=0}^{k}\widehat{\omega}_2\left(q_j^{(2)}\right)} \leq P_1(k)P_2(k)\|x\|_2\sqrt{\sum_{j=0}^{k}P_3(j)} \\
&\leq P_1(k)P_2(k)\left(\sum_{j=0}^{k}P_3(j)\right)\|x\|_2 \leq Q(k)\|x\|_2,
\end{aligned}$$

where Q is a polynomial.

Now assume that $(\widehat{\mathbb{G}}_2, L_2)$ is a discrete group, denoted by G, with property (RD) so that we may take $x \in (q^{(1)}_{\leq k}\otimes q^{(2)}_{\leq k})\widehat{\mathsf{A}}$ to be a finite sum of the form $x = \sum_g a_g\otimes\delta_g$, where $a_g \in q^{(1)}_{\leq k}\widehat{\mathsf{A}}_1$ with $\|a_g\|_2 < \infty$ for all g, $\delta_g \in l^\infty(G)$ is the Dirac function at $g\in G$ and $a_g = 0$ for all $g\in G$ such that $L_2(g) > k$. Hence we have $\|x\|_2^2 = \sum_g \|a_g\|_2^2$ and $\mathcal{F}(x) = \sum_g \mathcal{F}_1(a_g)\otimes\lambda_g$, where $\lambda_g \in \mathcal{B}(l^2(G))$ is the left translation by $g\in G$. Let $\xi \in \mathrm{L}^2(\mathbb{G}_1)\otimes l^2(G)$ be a finite sum $\xi = \sum_h \xi_h\otimes\delta_h$. One has:

$$\begin{aligned}
\|\mathcal{F}(x)\xi\|_2^2 &= \|\sum_{g,h}\mathcal{F}_1(a_g)\xi_h\otimes\delta_{gh}\|_2^2 = \|\sum_{g,h}\mathcal{F}_1(a_g)\xi_{g^{-1}h}\otimes\delta_h\|_2^2 = \sum_h\|\sum_g\mathcal{F}_1(a_g)\xi_{g^{-1}h}\|_2^2 \\
&\leq \sum_h\left(\sum_g\|\mathcal{F}_1(a_g)\xi_{g^{-1}h}\|_2\right)^2 \leq P_1(k)^2\sum_h\left(\sum_g\|a_g\|_2\,\|\xi_{g^{-1}h}\|_2\right)^2 = P_1(k)^2\|\psi*\varphi\|_{l^2(G)}^2,
\end{aligned}$$

where $\psi,\varphi\in l^2(G)$ are defined by $\psi(g) = \|a_g\|_2$ and $\varphi(g) = \|\xi_g\|$. Observe that $\|\psi\|_{l^2(G)}^2 = \sum_g\|a_g\|_2^2 = \|x\|_2^2$ and $\|\varphi\|_{l^2(G)}^2 = \sum_g\|\xi_g\|^2 = \|\sum_g\xi_g\otimes\delta_g\|^2 = \|\xi\|^2$. Since ψ is supported on elements $g\in G$ of length less that k, we may use (RD) for G and we find:

$$\|\mathcal{F}(x)\xi\|^2 \leq P_1(k)^2P_2(k)^2\|\psi\|_{l^2(G)}^2\|\varphi\|_{l^2(G)}^2 = P_1(k)^2P_2(k)^2\|x\|_2^2\|\xi\|^2.$$

This finishes the proof. □

Let $P_m : \mathcal{H}\to\mathcal{H}$ be the projection onto the closure of the span of the spaces $\mathcal{H}_{\mathbf{w}}$ with $\mathbf{w}$ a minimal word of length $m\in\mathbb{N}$.

Proposition 4.6. *Let Γ be a finite graph and for every $v\in V\Gamma$ let $\mathbb{G}_v$ be a compact quantum group such that $(\widehat{G}_v, L_v)$ has (RD). Moreover, assume that for every clique Γ_0 of Γ the graph product $\widehat{\mathbb{G}}_{\Gamma_0}$ has (RD). Let $\mathbb{G}$ be the graph product with respect to Γ and let $\widehat{\mathbb{G}} = (\widehat{A},\widehat{\Delta})$ be its discrete dual. There exist a polynomial $P\in\mathbb{R}[X]$ such that for every $k,l,m\in\mathbb{N}$ such that $|k-l|\leq m\leq k+l$ and $a\in\widehat{\mathsf{A}}_{(k)}$ we have $\|P_m\mathcal{F}(a)P_l\|\leq P(k)\|a\|_2$.*

Proof. For each $v\in V\Gamma$ we let $a_{v,j}, j\in J_v$ be elements of A_v° such that $\widehat{a}_{v,j} := \widehat{a}_{v,j}\Omega_v, j\in J_v$ is an orthonormal basis of $\mathcal{H}_v^\circ$. Set $\xi_{v,j} = \widehat{a}_{v,j}^* = a_{v,j}^*\Omega_v, j\in J_v$ which also is an orthonormal basis of $\mathcal{H}_v^\circ$ since $\mathbb{G}_v$ has a tracial Haar weight [Ve07, Proposition 4.7]. In particular, $\|\xi_{v,j}\| = \|\widehat{a}_{v,j}\|$. Throughout the proof we shall use the convention that a summation $\sum_j a_{v,j}$ in fact is the summation over $j\in J_v$. To prove the proposition it suffices to assume that,

$$\begin{aligned}
\mathcal{F}(a) &= \sum_{\mathbf{w}\in\mathcal{W}_{\min},l(\mathbf{w})=k}\sum_{j_1\cdots j_k}\lambda_{\mathbf{w},j_1,\ldots,j_k}a_{w_1,j_1}\ldots a_{w_k,j_k}, \\
\xi &= \sum_{\mathbf{v}\in\mathcal{W}_{\min},l(\mathbf{v})=l}\sum_{i_1\ldots i_l}\mu_{\mathbf{v},i_1,\ldots,i_l}\xi_{v_1,i_1}\otimes\ldots\otimes\xi_{v_l,i_l}.
\end{aligned}$$

Firstly, using the notation introduced before Lemma 4.4,

(4.7)
$$\begin{aligned} &P_m\mathcal{F}(a)\xi \\ =&P_m\left(\sum_{\mathbf{w}\in\mathcal{W}_{\min},l(\mathbf{w})=k}\sum_{j_1,\dots,j_k}\lambda_{\mathbf{w},j_1,\dots,j_k}(P_{w_1}+P_{w_1}^\perp)a_{w_1,j_1}(P_{w_1}+P_{w_1}^\perp)\dots(P_{w_k}+P_{w_k}^\perp)a_{w_k,j_k}(P_{w_k}+P_{w_k}^\perp)\right) \\ &\times\left(\sum_{\mathbf{v}\in\mathcal{W}_{\min},l(\mathbf{v})=l}\sum_{i_1,\dots,i_l}\mu_{\mathbf{v},i_1,\dots,i_l}\xi_{v_1,i_1}\otimes\dots\otimes\xi_{v_l,i_l}\right)\end{aligned}$$

A large part of the terms in the product of these sums vanishes in fact as follows from the following observations.

Reduction of the operator part. Firstly, consider an expression:

$$Q_{w_1}^{(2)}a_{w_1,j_1}Q_{w_1}^{(1)}\dots Q_{w_k}^{(2)}a_{w_k,j_k}Q_{w_k}^{(1)}, \tag{4.8}$$

with $Q_{w_i}^{(1)}$ and $Q_{w_i}^{(2)}$ equal to either P_{w_i} or $P_{w_i}^\perp$. Assume that (4.8) is non-zero, then this implies the following:

(1) If $Q_{w_i}^{(1)}=P_{w_i}^\perp$ then $Q_{w_i}^{(2)}=P_{w_i}$.
(2) If $Q_{w_i}^{(2)}=P_{w_i}$, then it must be true that $Q_{w_{i-1}}^{(1)}=P_{w_{i-1}}^\perp$ or $(w_{i-1},w_i)\in E\Gamma$.

These observations yield that without loss of generality (4.8) can assumed to be of a specific form.

- We claim that (4.8) can be assumed to be of the form:

$$Q_{w_1}^{(2)}a_{w_1,j_1}Q_{w_1}^{(1)}\dots Q_{w_s}^{(2)}a_{w_s,j_s}Q_{w_s}^{(1)}P_{w_{s+1}}^\perp a_{w_{s+1},j_{s+1}}P_{w_{s+1}}\dots P_{w_k}^\perp a_{w_k,j_k}P_{w_k}, \tag{4.9}$$

  where for every $1\leq i\leq s$ we do not have that $Q_{w_i}^{(2)}=P_{w_i}^\perp$ and $Q_{w_i}^{(1)}=P_{w_i}$. In order to prove this claim first note that if $Q_{w_i}^{(2)}=P_{w_i}^\perp$ and $Q_{w_i}^{(1)}=P_{w_i}$ then it follows from (2) that either $Q_{w_{i+1}}^{(2)}=P_{w_{i+1}}^\perp$ and $Q_{w_{i+1}}^{(1)}=P_{w_{i+1}}$ or $(w_i,w_{i+1})\in E\Gamma$. It then suffices to show that in the latter case the operators $P_{w_i}^\perp a_{w_i,j_i}P_{w_i}$ and $Q_{w_{i+1}}^{(2)}a_{w_{i+1},j_{i+1}}Q_{w_{i_1}}^{(1)}$ commute. So firstly observe that $P_{w_i}P_{w_{i+1}}$ is a projection and hence P_{w_i} and $P_{w_{i+1}}$ commute. By taking complements any of the projections $P_{w_i},P_{w_i}^\perp,P_{w_{i+1}}$ and $P_{w_{i+1}}^\perp$ commute. It follows from Lemma 2.3 that $P_{w_i}^\perp a_{w_i,j_i}P_{w_i}$ and $Q_{w_{i+1}}^{(2)}a_{w_{i+1},j_{i+1}}Q_{w_{i+1}}^{(1)}$ commute. This concludes (4.9).
- An analogous argument as in the previous bullet point yields that without loss of generality we may assume that (4.8) has the form,

$$Q_{w_1}^{(2)}a_{w_1,j_1}Q_{w_1}^{(1)}\dots Q_{w_r}^{(2)}a_{w_r,j_r}Q_{w_r}^{(1)}P_{w_{r+1}}^{(2)}a_{w_{r+1},j_{r+1}}P_{w_{r+1}}^{(1)}\dots P_{w_s}^{(2)}a_{w_s,j_s}P_{w_s}^{(1)}P_{w_{s+1}}^\perp a_{w_{s+1},j_{s+1}}P_{w_{s+1}}\dots P_{w_k}^\perp a_{w_k,j_k}P_{w_k}, \tag{4.10}$$

  and that for every $1\leq i\leq r$ we do not have that $Q_{w_i}^{(1)}=P_{w_i}$.
- If $Q_{w_i}^{(1)}=P_{w_i}^\perp$ then this implies that $Q_{w_i}^{(2)}=P_{w_i}$ by (1). So (4.10) shows that the expression (4.8) can be written as,

$$P_{w_1}a_{w_1,j_1}P_{w_1}^\perp\dots P_{w_r}a_{w_r,j_r}P_{w_r}^\perp P_{w_{r+1}}a_{w_{r+1},j_{r+1}}P_{w_{r+1}}\dots P_{w_s}a_{w_s,j_s}P_{w_s}P_{w_{s+1}}^\perp a_{w_{s+1},j_{s+1}}P_{w_{s+1}}\dots P_{w_k}^\perp a_{w_k,j_k}P_k, \tag{4.11}$$

  for some $0\leq r\leq s\leq k$ (the cases $r=0$ and $s=k$ should be understood naturally).
- Moreover, suppose that $s>r+1$. Then it follows from (1) that $(w_{r+1},w_{r+2})\in E\Gamma$. As in the first bullet point this implies that $P_{w_{r+1}}a_{w_{r+1},j_{r+1}}P_{w_{r+1}}$ and $P_{w_{r+2}}a_{w_{r+2},j_{r+2}}P_{w_{r+2}}$ commute. Hence it follows from (2) that $(w_{r+1},w_{r+3})\in E\Gamma$ (provided that $s>r_2$) and inductively we find that $(w_{r+1},w_i)\in E\Gamma$ for every $r+1\leq i\leq s$. The same argument yields that actually $(w_i,w_j)\in E\Gamma$ for every $r+1\leq i,j\leq s$. We conclude that $w_{r+1},\dots,w_s$ are in clique of Γ.

Reduction of the vector part. Now suppose that a vector $\xi_{v_1,i_1}\otimes\dots\otimes\xi_{v_l,i_l}$ is not in the kernel of (4.11). Then this implies that we may assume (using the commutation relations given by $E\Gamma$ to permute

terms in (4.11)) that $v_1 = w_k, \ldots, v_{k-r} = w_{r+1}$, that $w_s \ldots w_{r+1}$ is contained in a clique and furthermore that $v_{k-r+1} \neq w_r$. And in that case,

(4.12)
$$\Big(P_{w_1} a_{w_1,j_1} P_{w_1}^{\perp} \ldots P_{w_r} a_{w_r,j_r} P_{w_r}^{\perp} P_{w_{r+1}} a_{w_{r+1},j_{r+1}} P_{w_{r+1}} \ldots P_{w_s} a_{w_s,j_s} P_{w_s} P_{w_{s+1}}^{\perp} a_{w_{s+1},j_{s+1}} P_{w_{s+1}} \ldots P_{w_k}^{\perp} a_{w_k,j_k} P_k\Big)$$
$$(\xi_{v_1,i_1} \otimes \ldots \otimes \xi_{v_l,i_l}) =$$
$$\widehat{a}_{w_1,j_1} \otimes \ldots \otimes \widehat{a}_{w_r,j_r} \otimes P_{w_s} a_{w_s,j_s} \xi_{v_{k-s+1},i_{k-s+1}} \otimes \ldots \otimes P_{w_{r+1}} a_{w_{r+1},j_{r+1}} \xi_{v_{k-r},i_{k-r}}$$
$$\otimes \xi_{v_{k-r+1},i_{k-r+1}} \otimes \ldots \otimes \xi_{v_l,i_l} \times \langle a_{w_k,j_k} \xi_{v_1,i_1}, \Omega\rangle \ldots \langle a_{w_{s+1},j_{s+1}} \xi_{v_{k-s},i_{k-s}}, \Omega\rangle,$$

where we explicitly mention that some of the indices in the triple dots of the right hand side of this expression either increase or decrease by steps of 1. Looking at the length of tensor products shows that (4.12) is in the kernel of P_m unless $m + k - l = s + r$.

Remainder of the proof. Now we conclude from (4.7) and (4.12) that,

(4.13)
$$\begin{aligned}
&\|P_m \mathcal{F}(a)\xi\|_2^2 \\
\leq &\sum_{\substack{m+k-l=s+r \\ 0 \leq s,r \leq k}} \sum_{(u_1,\ldots,u_{s-r}) \in \mathrm{Cliq}_\Gamma(s-r)} \sum_{\substack{\mathbf{w},\mathbf{v} \in \mathcal{W}_{\min}, \\ l(\mathbf{w})=k, l(\mathbf{v})=l, \\ v_1 = w_k \ldots v_{k-r} = w_{r+1}, \\ (v_{k-s+1},\ldots,v_{k-r}) = (w_s,\ldots,w_{r+1}) = (u_1,\ldots,u_{s-r})}} \sum_{\substack{j_1,\ldots,j_r, \\ j_{s+1},\ldots,j_k}} \sum_{\substack{i_1,\ldots,i_{k-s}, \\ i_{k-r+1},\ldots,i_l}} \\
&\|\widehat{a}_{w_1,j_1}\|_2^2 \ldots \|\widehat{a}_{w_r,j_r}\|_2^2 \times \delta_{j_k,i_1} \|\widehat{a}_{w_k,j_k}\|_2^4 \ldots \delta_{j_{s+1},i_{k-s}} \|\widehat{a}_{w_{s+1},j_{s+1}}\|_2^4 \\
&\left\| \sum_{j_{r+1},\ldots,j_s} a_{w_{r+1},j_{r+1}} \ldots a_{w_s,j_s} \sum_{i_{k-r},\ldots,i_{k-s+1}} \xi_{v_{k-r},i_{k-r}} \otimes \ldots \otimes \xi_{v_{k-s+1},i_{k-s+1}} \right\|_2^2
\end{aligned}$$

We have, since $\widehat{G}_{\Gamma_0}$ has (RD) by assumption for every clique Γ_0 in Γ,

(4.14)
$$\begin{aligned}
&\left\| \sum_{j_{r+1},\ldots,j_s} a_{w_{r+1},j_{r+1}} \ldots a_{w_s,j_s} \sum_{i_{k-r},\ldots,i_{k-s+1}} \xi_{v_{k-r},i_{k-r}} \otimes \ldots \otimes \xi_{v_{k-s+1},i_{k-s+1}} \right\|_2^2 \\
\leq &P(s-r) \left\| \sum_{j_{r+1},\ldots,j_s} a_{w_{r+1},j_{r+1}} \ldots a_{w_s,j_s} \right\|_2^2 \left\| \sum_{i_{k-r},\ldots,i_{k-s+1}} \xi_{v_{k-r},i_{k-r}} \otimes \ldots \otimes \xi_{v_{k-s+1},i_{k-s+1}} \right\|_2^2,
\end{aligned}$$

for some polynomial P. Let Q be a polynomial such that $P(s-r) \leq Q(k)$ for any choice of $s, r \in \mathbb{N}$ with $0 \leq s, r \leq k$. Combining (4.13) and (4.14) we see that,

$$\begin{aligned}
&\|P_m \mathcal{F}(a)\xi\|_2^2 \\
\leq &\sum_{\substack{m+k-l=s+r \\ 0 \leq s,r \leq k}} \sum_{(u_1,\ldots,u_{s-r}) \in \mathrm{Cliq}_\Gamma(s-r)} \sum_{\substack{\mathbf{w},\mathbf{v} \in \mathcal{W}_{\min}, \\ l(\mathbf{w})=k, l(\mathbf{v})=l, \\ v_1 = w_k \ldots v_{k-r} = w_{r+1}, \\ (v_{k-s+1},\ldots,v_{k-r}) = (w_s,\ldots,w_{r+1}) = (u_1,\ldots,u_{s-r})}} \sum_{j_1,\ldots,j_k} \sum_{i_1,\ldots,i_l} \\
&Q(k) \|\widehat{a}_{w_1,j_1}\|_2^2 \ldots \|\widehat{a}_{w_k,j_k}\|_2^2 \|\xi_{v_1,i_1}\|_2^2 \ldots \|\xi_{v_l,i_l}\|_2^2 \\
\leq &M(k+1)^2 Q(k) \|a\|_2^2 \|\xi\|_2^2.
\end{aligned}$$

where M is the number of cliques in Γ, which is finite since Γ is finite. □

Lemma 4.7. *Let $\mathbb{G}$ be a compact quantum group and let L be a central length associated with $\widehat{\mathbb{G}}$. Then there exists a central length $L' \geq L$ associated with $\widehat{\mathbb{G}}$ such that $L'p_\alpha \geq 1$ for every $\alpha \in \mathrm{Irr}(\mathbb{G})$ nontrivial.*

Proof. Since L is a central length we may write $L = \oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} f(\alpha)p_\alpha$. We define the central length $L' = \oplus_{\alpha\in\mathrm{Irr}(\mathbb{G})} f'(\alpha)p_\alpha$, where $f'(\alpha) = f(\alpha)+1$ if α is nontrivial and $f'(\alpha) = f(\alpha)$ in case α is trivial. As in the proof of Lemma 4.4 the condition $\widehat{\Delta}(L') \leq L'\otimes 1 + 1\otimes L'$ is equivalent to checking that $\sum_{\gamma\in\mathrm{Irr}(G),\gamma\subseteq\alpha\otimes\beta} f'(\gamma)p_\gamma^{\alpha\otimes\beta} \leq (f'(\alpha)+f'(\beta))p_\alpha\otimes p_\beta$. However, this condition easily follows from the fact that if both α and β are trivial then $\alpha\otimes\beta$ is trivial and so γ is trivial whenever $\gamma\subseteq\alpha\otimes\beta$. The condition $\widehat{\kappa}(L') = L'$ follows as in the proof of Lemma 4.4, see also Lemma 4.3. And finally by definition of the counit we have $\widehat{\epsilon}(L') = f'(\alpha_0) = f(\alpha_0) = 0$ with $\alpha_0 \in \mathrm{Irr}(\mathbb{G})$ trivial. □

Theorem 4.8. *Let Γ be a finite graph and let for every $v \in V\Gamma$, $\widehat{\mathbb{G}}_v$ be a compact quantum group such that $(\widehat{\mathbb{G}}_v, L_v)$ has (RD). Assume that for every clique Γ_0 the graph product $\widehat{\mathbb{G}}_{\Gamma_0}$ has (RD). Then the graph product $\widehat{\mathbb{G}} := \widehat{G}_\Gamma$ has (RD).*

Proof. Let $L_v, v \in V\Gamma$ be the lengths for $\widehat{\mathbb{G}}_v$ and let L be the length defined in Lemma 4.4. Assume by Lemma 4.7 and [Ve07, Remark 3.6] that,

$$L_v p_\alpha \geq 1, \qquad \forall v \in V\Gamma, \alpha \in \mathrm{Irr}(\mathbb{G}_v). \tag{4.15}$$

This implies that $Lp_\alpha \geq l(\alpha)$ where $l(\alpha)$ the length of the reduced expression $\alpha = \alpha_1 \otimes \ldots \otimes \alpha_{l(\alpha_0)}$ with $\alpha \in \mathrm{Irr}(\mathbb{G})$.

By Proposition 4.6 there exist a polynomial P such that for every $k, l, m \in \mathbb{N}$ such that $|k-l| \leq m \leq k+l$ and $a \in \widehat{\mathsf{A}}_{(k)}$ we have $\|P_m\mathcal{F}(a)P_l\| \leq P(k)\|a\|_2$.

Now, let $a \in q_k\widehat{\mathsf{A}}$ and write $a = \sum_{j=0}^k a_{(j)}$ with $a_{(j)} \in \widehat{\mathsf{A}}_{(j)}$, which is possible by the first paragraph. Take a vector $v \in q_l\mathcal{H}$ and write $v = \sum_{i=0}^l v_{(i)}$ with $v_{(i)} = P_i v$. Since $\sum_{r=0}^m P_r \geq q_m$ and the projections P_r are orthogonal, it follows that $\|q_m\mathcal{F}(a_{(j)})q_l v\|_2^2 \leq \sum_{r=0}^m \|P_r\mathcal{F}(a_{(j)})q_l v\|_2^2$. Next, we have an elementary equality that follows by considering word lengths and an inequality which follows from Cauchy-Schwarz and the triangle inequality,

$$\begin{aligned}\sum_{r=0}^m \|P_r\mathcal{F}(a_{(j)})q_l v\|_2^2 =& \sum_{r=0}^m \|\sum_{i=|j-r|}^{j+r} P_r\mathcal{F}(a_{(j)})v_{(i)}\|_2^2 \\ \leq& (2j+1)\sum_{r=0}^m \sum_{i=|j-r|}^{j+r} \|P_r\mathcal{F}(a_{(j)})v_{(i)}\|_2^2\end{aligned}$$

Now, for $|j-i| \leq r \leq j+i$ we have,

$$\|P_r\mathcal{F}(a_{(j)})v_{(i)}\|_2^2 \leq P(j)^2\|a_{(j)}\|_2^2\|v_{(i)}\|_2^2$$

For other values of r we have $\|P_r\mathcal{F}(a_{(j)})v_{(i)}\|_2^2 = 0$. Since as we observed $|i-j| \leq r \leq j+i$ for any given value of i this shows that we can estimate,

$$\begin{aligned}\sum_{r=0}^m \|P_r\mathcal{F}(a_{(j)})q_l v\|_2^2 \leq& P(j)^2(2j+1)\sum_{i=0}^{j+m}\sum_{r=0}^m \|a_{(j)}\|_2^2\|v_{(i)}\|_2^2 \\ \leq& P(j)^2(2j+1)^2\sum_{i=0}^{j+m} \|a_{(j)}\|_2^2\|v_{(i)}\|_2^2 \\ \leq& P(j)^2(2j+1)^2\|a_{(j)}\|_2^2\|v\|_2^2.\end{aligned}$$

Now, using the triangle inequality and the Cauchy-Schwarz inequality we have

$$\|q_m\mathcal{F}(a)q_l v\|_2^2 \leq (\sum_{j=0}^{k}\|q_m\mathcal{F}(a_{(j)})q_l v\|_2)^2 \leq (k+1)\sum_{j=0}^{k}\|q_m\mathcal{F}(a_{(j)})q_l v\|_2^2$$

$$\leq (k+1)\sum_{j=0}^{k}P(j)^2(2j+1)^2\|a_{(j)}\|_2^2\|v\|_2^2 \leq (k+1)(2k+1)^2P'(k)\|a\|_2^2\|v\|_2^2 = P''(k)\|a\|_2^2\|v\|_2^2,$$

for some polynomials P, P', P'' that satisfy the property that for every $0 \leq j \leq k$ we have $P(j)^2 \leq P'(k)$ and $P''(k) = (k+1)(2k+1)^2P'(k)$. □

Corollary 4.9. *Let Γ be a finite graph. For every v in$V\Gamma$ let $\mathbb{G}_v$ be a compact quantum group and assume that $\mathbb{G}_v$ has either polynomial growth or is a classical compact group with (RD). Then the discrete dual of the graph product, i.e. $\widehat{\mathbb{G}}$, has (RD).*

Proof. This is a consequence of Theorem 4.8 and Lemma 4.5. □

Corollary 4.10. *Let Γ be finite and without edges. Let $\mathbb{G} = \star_{v\in V\Gamma}\mathbb{G}_v$. If for every $v \in V\Gamma$, $\widehat{\mathbb{G}}_v$ has (RD), then $\widehat{\mathbb{G}}$ has (RD). I.e. (RD) is preserved by finite free products.*

Proof. This is a consequence of Theorem 4.8 and Lemma 4.5. □

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