Amenable, transitive and faithful actions of groups acting on trees

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Abstract

We study under which condition an amalgamated free product or an HNN-extension over a finite subgroup admits an amenable, transitive and faithful action on an infinite countable set. We show that such an action exists if the initial groups admit an amenable and almost free action with infinite orbits (e.g. virtually free groups or infinite amenable groups). Our result relies on the Baire category Theorem. We extend the result to groups acting on trees.

1 Introduction

The notion of amenable groups or, more generally, amenable actions was first introduced by von Neumann [vN29] who proposed to study whether or not, given a group acting on a set X, there exists a mean on X invariant by the action (or equivalently a Følner sequence for the action).

Definition 1.1. An action $\Gamma \curvearrowright X$ of a countable group Γ on a countable set X is called *amenable* if there exists a sequence (C_n) of non-empty finite subsets of X such that

$$\frac{|C_n \Delta g C_n|}{|C_n|} \to 0 \quad \text{for all } \ g \in \Gamma.$$

Such a sequence (C_n) is called a *Følner* sequence.

This notion of amenability is different from the one introduced later by Zimmer [Zi84].

Obviously, every action of an amenable group (i.e. such that the left translation on itself is an amenable action) is amenable. Greenleaf [Gr69] asked for the converse: does the existence of an amenable action of Γ implies the amenability of Γ? To avoid any trivial negative answer, one should assume that the action is faithful and transitive. If the action is free then the converse holds. However van Douwen [vD90] gave a counter example: the free group $\mathbb{F}_2$ admits a faithful, transitive and amenable action.

This leads Glasner and Monod [GM05] to introduce the class $\mathcal{A}$ of countable groups admitting an amenable, transitive and faithful action. Grigorchuk and Nekrashevych [GN05] have constructed a class of amenable, transitive and faithful actions of finitely generated free groups using Schreier graphs. Simultaneously Glasner and Monod [GM05] gave a necessary and sufficient condition for a free product to be in the class $\mathcal{A}$. In particular, they showed that the class $\mathcal{A}$ is closed under free products. They also asked when free products with amalgamations and HNN-extensions are in $\mathcal{A}$.

S. Moon [Mo08], [Mo09] showed that a free product of finitely generated free groups amalgamated over a cyclic group is in $\mathcal{A}$. She also proved [Mo10] that an amalgamated free product of amenable groups over a finite group as well as an amalgamated free product of a residually finite group with an infinite amenable group over a finite group is in $\mathcal{A}$.

The initial motivation of the present work was to study the case of an HNN-extension $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$, where Σ is a subgroup of H and $\theta : \Sigma \to H$ is an injective group homomorphism. Few results are known: Monod and Popa [MP03] showed that $\Gamma \in \mathcal{A}$ whenever $\Sigma = H \in \mathcal{A}$ and S. Moon observed that the Baumslag-Solitar groups are in $\mathcal{A}$.

We say that an action has *infinite orbits* if every orbit is infinite. Our first result is as follows.

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Theorem 1.2. *Let Γ_1, Γ_2 and H be countable groups and Σ be a finite subgroup of Γ_1, Γ_2 and H. Let $\theta : \Sigma \to H$ be an injective group homomorphism.*

1. *If there exists an amenable and faithful action of H on a countable set with infinite orbits and free on Σ and $\theta(\Sigma)$ then* $\mathrm{HNN}(H, \Sigma, \theta) \in \mathcal{A}$.
2. *If, for $i = 1, 2$, there exists an amenable and faithful action of Γ_i on a countable set with infinite orbits and free on Σ then $\Gamma_1 *_\Sigma \Gamma_2 \in \mathcal{A}$.*

To prove Theorem 1.2 we use the Baire category Theorem. Such an approach has been used for example in [Ep70], [Di89], [GM05], [Mo08], [Mo09] and [Mo10].

An action is called *almost free* if every non-trivial group element acts with finitely many fixed points. During the investigation of the HNN-extension case we realized that the class $\mathcal{A}_\mathcal{F}$ of countable groups admitting an amenable and almost free action with infinite orbits on a countable set appears naturally. Observe that the class $\mathcal{A}_\mathcal{F}$ contains all infinite amenable groups. Moreover, the amenable, transitive and faithful action of $\mathbb{F}_2$ constructed in [vD90] is actually almost free and an obvious adaptation of his construction shows that $\mathbb{F}_n$ admits an amenable, transitive and almost free action on an infinite countable set for all $n \geq 2$. van Douwen also showed that the same conclusion holds for $\mathbb{F}_\infty$.

It is easy to check that if H has an amenable and almost free action with infinite orbits and if H is a finite index subgroup of Γ then, the induced action is still amenable and almost free with infinite orbits (and also transitive if the original action is). It follows that virtually free groups are in $\mathcal{A}_\mathcal{F}$. Moreover, the obstruction to be in $\mathcal{A}$ discovered in [GM05, Lemma 4.3] is also an obstruction to be in $\mathcal{A}_\mathcal{F}$. Namely, let $N \lhd H$ be a normal subgroup such that the pair (N, H) has the relative property (T). If $H \in \mathcal{A}_\mathcal{F}$ then N has finite exponent. In particular, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \notin \mathcal{A}_\mathcal{F}$. The proof of this assertion is an obvious adaptation of the proof of [GM05, Lemma 4.3].

We say that a graph is non-trivial if it has at least two edges, e and its inverse edge $\overline{e}$. Our second result is as follows.

Theorem 1.3. *Let Γ be a countable group acting without inversion on a non-trivial tree T with finite edge stabilizers and finite quotient graph T/Γ. If all the vertex stabilizers are in $\mathcal{A}_\mathcal{F}$ then $\Gamma \in \mathcal{A}$.*

Observe that Γ is virtually free and hence admits an amenable, transitive and almost free action when all the vertex stabilizers are finite.

We prove Theorem 1.3 by induction and by using a slightly stronger version of Theorem 1.2 (see Remarks 3.3 and 4.3). A particular case of Theorem 1.3 is the following. Let $\Gamma_1, \Gamma_2 \in \mathcal{A}_\mathcal{F}$. Then, for all finite common subgroup $\Sigma < \Gamma_1, \Gamma_2$ one has $\Gamma_1 *_\Sigma \Gamma_2 \in \mathcal{A}$. Also, if $H \in \mathcal{A}_\mathcal{F}$ then, for all finite subgroup $\Sigma < H$ and all injective group homomorphism $\theta : \Sigma \to H$, one has $\mathrm{HNN}(H, \Sigma, \theta) \in \mathcal{A}$.

The paper is organized as follows. The section 2 is a preliminary section in which we prove five basic general lemmas which will be used in the paper. In section 3 we prove the HNN-extension case of Theorem 1.2. The section 4 covers the case of an amalgamated free product. Finally, we prove Theorem 1.3 in section 5.

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2 Generalities

Our first lemma is certainly well known but we could not find any reference in the literature.

Lemma 2.1. *Let $\Gamma \curvearrowright X$ be an amenable action of a countable group. If the action has infinite orbits then there exists a Følner sequence (C_n) such that $|C_n| \to \infty$.*

Proof. Let (D_n) be a Følner sequence for $\Gamma \curvearrowright X$. Suppose that $|D_n| \nrightarrow \infty$. By taking a subsequence if necessary we may and will suppose that $(|D_n|)$ is bounded. It follows that, for all $g \in \Gamma$, $|D_n \Delta g D_n| \to 0$.

Define $K=\cup_{n\in\mathbb{N}}D_n$. Suppose that K is finite. Since $\mathbb{N}=\cup_{x\in K}\{n\in\mathbb{N}\,:\,x\in D_n\}$ there exists $x_0\in K$ such that $I=\{n\in\mathbb{N}\,:\,x_0\in D_n\}$ is infinite. Since the orbit of x_0 is infinite, there exists $h\in\Gamma$ such that $hx_0\notin K$. Hence, for all $n\in I$, $x_0\in D_n\Delta h^{-1}D_n$. Since I is infinite, this contradicts the fact that $|D_n\Delta h^{-1}D_n|\to 0$. It follows that K is infinite.

Write $\Gamma=\cup^{\uparrow}F_k$, where F_k are finite subsets. For all $k\in\mathbb{N}$, let $n_k,m_k\in\mathbb{N}$ such that $F_kD_n=D_n$ for all $n\geq n_k$ and $\left|\cup_{n=n_k}^{m_k}D_n\right|\geq k$. Define $C_k=\cup_{n=n_k}^{m_k}D_n$. Then (C_k) is a Følner sequence and $|C_k|\to\infty$. □

Given an action $\Gamma\curvearrowright X$ and $g\in\Gamma$, we define $\text{Fix}_X(g)=\{x\in X\,:\,gx=x\}$. If the context is clear, we omit the subscript X.

Lemma 2.2. *If $\Gamma\in\mathcal{A}_{\mathcal{F}}$ then, for every finite subset $F\subset H$ with $1\notin F$, there exists an amenable and almost free action with infinite orbits $\Gamma\curvearrowright X$ on a countable set X such that $\text{Fix}(g)=\emptyset$ for all $g\in F$.*

Proof. Let $\Gamma\curvearrowright Y$ be an amenable and almost free action with infinite orbits on a countable set Y. Let $m=\text{Max}\{|\text{Fix}_Y(g)|\,:\,g\in F\}+1$. The diagonal action $\Gamma\curvearrowright Y^m$ is still amenable, almost free and has infinite orbits. Define

$$X=\{(y_1,\dots,y_m)\,:\,y_i\neq y_j\text{ for all }i\neq j\}\subset Y^m.$$

Since X is globally invariant, we get an action of Γ on X. This action is almost free, every orbit is infinite and every element of F acts freely. Let us show that this action is also amenable. By Lemma 2.1, let (C_n) be a Følner sequence for the action $\Gamma\curvearrowright Y$ such that $|C_n|\to\infty$. One has

$$|C_n^m\cap X^c|\leq\sum_{1\leq i<j\leq m}|\{(y_1,\dots,y_m)\in C_n^m\,:\,y_i=y_j\}|=\frac{m(m-1)}{2}|C_n|^{m-1}.$$

Since $|C_n|\to\infty$ one has $\frac{|C_n^m\cap X^c|}{|C_n^m|}\to 0$ and $\frac{|C_n^m\cap X|}{|C_n^m|}\to 1$. Define $D_n=C_n^m\cap X$. Since $|C_n|\to\infty$ we have $|D_n|\to\infty$ and we may assume that $D_n\neq\emptyset$ for all $n\in\mathbb{N}$. It is easy to check that (C_n^m) is a Følner sequence for the action $\Gamma\curvearrowright Y^m$ hence, for all $g\in\Gamma$, one has

$$\begin{aligned}\frac{|D_n\Delta gD_n|}{|D_n|}&\leq\frac{|D_n\Delta C_n^m|}{|D_n|}+\frac{|C_n^m\Delta gC_n^m|}{|D_n|}+\frac{|gC_n^m\Delta gD_n|}{|D_n|}=2\frac{|D_n\Delta C_n^m|}{|D_n|}+\left(\frac{|C_n^m|}{|D_n|}\right)\frac{|C_n^m\Delta gC_n^m|}{|C_n^m|}\\&=2\frac{|C_n^m\cap X^c|}{|C_n^m\cap X|}+\left(\frac{|C_n^m|}{|C_n^m\cap X|}\right)\frac{|C_n^m\Delta gC_n^m|}{|C_n^m|}\to 0.\end{aligned}$$

□

The following Lemma is inspired by [Mo10, Lemma 6].

Lemma 2.3. *Let $\Gamma\curvearrowright Y$ be an action of a countable group Γ on a countable set Y. Define $X=Y\times\mathbb{N}$ and consider the action $\Gamma\curvearrowright X$ given by $g(y,n)=(gy,n)$ for all $g\in\Gamma$ and $(y,n)\in X$. If $\Gamma\curvearrowright Y$ is amenable then, for all sequence (a_n) of real numbers such that $a_n\to\infty$, there exists a Følner sequence (C_n) for $\Gamma\curvearrowright X$ and a subsequence $(a_{\varphi(n)})$ such that $\frac{a_{\varphi(n)}}{|C_n|}\to 1$.*

Proof. Write $\Gamma=\cup^{\uparrow}F_k$, where $(F_k)_k$ is an increasing sequence of finite subsets. It suffices to construct a strictly increasing sequence of integers $(n_k)_k$ and a sequence of non-empty finite subsets $(C_k)_k$ of X such that C_k is a $(\frac{1}{k},F_k)$-Følner set for all $k\geq 1$ and,

$$1\leq\frac{a_{n_k}}{|C_k|}<1+\frac{2}{k}\quad\text{for all}\quad k\geq 1.$$

In the sequel, given $x\in\mathbb{R}$, we denote by $[x]$ the unique integer such that $[x]\leq x<[x]+1$. For $k=1$, let D_1 be a $(1,F_1)$-Følner set for the action $\Gamma\curvearrowright Y$. Let $n_1\geq 1$ big enough such that $[a_{n_1}]\geq|D_1|$. By Euclidean division, we write $[a_{n_1}]=q_1|D_1|+r_1$, where $0\leq r_1<|D_1|$ and $q_1\geq 1$. Define $C_1=\sqcup_{n=1}^{q_1}D_1^{(n)}\subset X$, where $D_1^{(n)}=D_1\times\{n\}$. Then, $|C_1|=q_1|D_1|$ and, for all $g\in F_1$,

$$|gC_1\Delta C_1|\leq\sum_{n=1}^{q_1}|gD_1^{(n)}\Delta D_1^{(n)}|\leq q_1|D_1|=|C_1|.$$

Hence, C_1 is a $(1, F_1)$-Følner set. Moreover,

$$1 \leq \frac{[a_{n_1}]}{|C_1|} \leq \frac{a_{n_1}}{|C_1|} < \frac{[a_{n_1}]+1}{|C_1|} = \frac{q_1|D_1|+r_1+1}{|C_1|} = 1 + \frac{r_1+1}{q_1|D_1|} < 1 + \frac{1}{q_1} + \frac{1}{q_1|D_1|} \leq 1 + \frac{2}{q_1} \leq 1+2.$$

Now, suppose that, for $k \geq 1$, we have constructed $n_1 < \ldots < n_k \in \mathbb{N}^*$ and $C_1, \ldots, C_k \subset X$ such that C_i is a $(\frac{1}{i}, F_i)$-Følner set and $1 \leq \frac{a_{n_i}}{|C_i|} < 1 + \frac{2}{i}$ for all $1 \leq i \leq k$. Let D_{k+1} be a $(\frac{1}{k+1}, F_{k+1})$-Følner set for $\Gamma \curvearrowright Y$ and $n_{k+1} > n_k$ big enough such that $[a_{n_{k+1}}] \geq (k+1)|D_{k+1}|$. Write $[a_{n_{k+1}}] = q_{k+1}|D_{k+1}| + r_{k+1}$, where $0 \leq r_{k+1} < |D_{k+1}|$ and $q_{k+1} \geq k+1$. Define $C_{k+1} = \sqcup_{n=1}^{q_{k+1}} D_{k+1}^{(n)} \subset X$, where $D_{k+1}^{(n)} = D_{k+1} \times \{n\}$. Then, $|C_{k+1}| = q_{k+1}|D_{k+1}|$ and, for all $g \in F_{k+1}$,

$$|gC_{k+1}\Delta C_{k+1}| \leq \sum_{n=1}^{q_{k+1}} |gD_{k+1}^{(n)}\Delta D_{k+1}^{(n)}| \leq q_{k+1}\frac{|D_{k+1}|}{k+1} = \frac{|C_{k+1}|}{k+1}.$$

Hence, C_{k+1} is a $(\frac{1}{k+1}, F_{k+1})$-Følner set. Moreover,

$$1 \leq \frac{[a_{n_{k+1}}]}{|C_{k+1}|} \leq \frac{a_{n_{k+1}}}{|C_{k+1}|} < \frac{[a_{n_{k+1}}]+1}{|C_{k+1}|} = \frac{q_{k+1}|D_{k+1}|+r_{k+1}+1}{|C_{k+1}|} < 1+\frac{1}{q_{k+1}}+\frac{1}{q_{k+1}|D_{k+1}|} \leq 1+\frac{2}{q_{k+1}} \leq 1+\frac{2}{k+1}.\square$$

Let $H < \Gamma$, $H \curvearrowright Y$ and consider the diagonal action $H \curvearrowright Y \times \Gamma$. Define $X = H \setminus (Y \times \Gamma)$. The *induced action* is the action $\Gamma \curvearrowright X$ given by right multiplication (by the inverse element) on the Γ part in X. The following proposition contains some standard observations on the induced action. We include a proof for the reader's convenience.

Lemma 2.4. *Let $H < \Gamma$, $H \curvearrowright Y$ and $\Gamma \curvearrowright X$ the induced action. The following holds.*

1. *If $H \curvearrowright Y$ is faithful then $\Gamma \curvearrowright X$ is faithful.*
2. *If $H \curvearrowright Y$ is amenable then $H \curvearrowright X$ is amenable.*
3. *If $H \curvearrowright Y$ has infinite orbits and H is malnormal [3] in Γ then $H \curvearrowright X$ has infinite orbits.*
4. *If $K < \Gamma$ is infinite and $gKg^{-1} \cap H$ is finite for all $g \in \Gamma$ then $K \curvearrowright X$ has infinite orbits.*
5. *Let $(\Sigma_\epsilon)_{\epsilon \in E}$ be a family of subgroups of H such that $\Sigma_\epsilon \curvearrowright Y$ is free for all $\epsilon \in E$. If, for all $g \in \Gamma \setminus H$, $gHg^{-1} \cap H \subset \cup_{\epsilon \in E,\, s \in H} s\Sigma_\epsilon s^{-1}$ then, for all $h \in H$ such that $\mathrm{Fix}_Y(h) = \emptyset$, one has $\mathrm{Fix}_X(h) = \emptyset$.*
6. *Let $\Sigma < H$ such that $\Sigma \curvearrowright Y$ is free. If $K < \Gamma$ is a subgroup and $\cup_{g \in \Gamma} gKg^{-1} \cap H \subset \cup_{h \in H} h\Sigma h^{-1}$ then $K \curvearrowright X$ is free.*

Proof. For $(y, g) \in Y \times \Gamma$ we denote by $[y, g]$ its class in X. Let $t \in \Gamma$. Observe that

$$\mathrm{Fix}_X(t) = \{[y, g] \in X \,:\, gtg^{-1} \in H \text{ and } y \in \mathrm{Fix}_Y(gtg^{-1})\}. \tag{2.1}$$

Suppose that $\mathrm{Fix}_X(t) = X$. Then, for all $y \in Y$, $[y, 1] \in \mathrm{Fix}_X(t)$ hence, $t \in H$ and $\mathrm{Fix}_Y(t) = Y$. Since $H \curvearrowright Y$ is faithful we have $t = 1$. Since the map $y \mapsto [y, 1]$ is H-equivariant (and injective), the action $H \curvearrowright X$ is amenable whenever that action $H \curvearrowright Y$ is. Let $y \in Y$ and $g \in H$. One has, for all $h \in H$, $h[y, g] = [y, gh^{-1}] = [hg^{-1}y, 1]$. Since the map $y \mapsto [y, 1]$ is injective, the H-orbit of $[y, g]$ is infinite for all $g \in H$ whenever $H \curvearrowright Y$ has infinite orbits. Let $y \in Y$ and $g \in \Gamma \setminus H$. If the H-orbit $H[y, g]$ is finite the stabilizer in H of $[y, g]$ must be infinite. However,

$$\{h \in H \,:\, h[y, g] = [y, g]\} \subset \{h \in H \,:\, ghg^{-1} \in H\} = g^{-1}Hg \cap H$$

which is finite since H is malnormal in Γ. The proof of 4 is similar: if the K-orbit $K[y, g]$ is finite and K is infinite then the stabilizer in K of $[y, g]$ must be infinite. However,

$$\{k \in K \,:\, k[y, g] = [y, g]\} \subset \{k \in K \,:\, gkg^{-1} \in H\} = K \cap g^{-1}Hg$$

[3] The subgroup H is called *malnormal* in Γ if, for all $g \in \Gamma \setminus H$, the set $g^{-1}Hg \cap H$ is finite.

which is finite. Let us prove 5. Let $h \in H \setminus \{1\}$, such that $\text{Fix}_X(h) \neq \emptyset$. Let $[y, g] \in \text{Fix}_X(h)$ then, $ghg^{-1} \in H$ and $y \in \text{Fix}_Y(ghg^{-1})$. If $g \in \Gamma \setminus H$ then, there exists $\epsilon \in E$, $\sigma \in \Sigma_\epsilon \setminus \{1\}$ and $s \in H$ such that $ghg^{-1} = s\sigma s^{-1}$. Hence, $y \in \text{Fix}_Y(ghg^{-1}) = \text{Fix}_Y(s\sigma s^{-1}) = s\text{Fix}_Y(\sigma)$. Hence, $\text{Fix}_Y(\sigma) \neq \emptyset$, a contradiction since $\Sigma_\epsilon \curvearrowright$ is free. Thus, $g \in H$ and $y \in \text{Fix}_Y(ghg^{-1}) = g\text{Fix}_Y(h)$ which implies that $\text{Fix}_Y(h) \neq \emptyset$. The proof of 6 is similar. Let $k \in K \setminus \{1\}$. If $[y, g] \in \text{Fix}_X(k)$ then, $gkg^{-1} \in H$ and $y \in \text{Fix}_Y(gkg^{-1})$. By the hypothesis on K, there exists $h \in H$ and $\sigma \in \Sigma \setminus \{1\}$ such that $gkg^{-1} = h\sigma h^{-1}$. Hence, $\text{Fix}_Y(gkg^{-1}) = \text{Fix}_Y(h\sigma h^{-1}) = h\text{Fix}_Y(\sigma) = \emptyset$, a contradiction. □

Example 2.5. The following holds.

1. Let $\Gamma = \text{HNN}(H, \Sigma, \theta) = \langle H, t \,|\, \theta(\sigma) = t\sigma t^{-1},\ \forall \sigma \in \Sigma \rangle$ be a non-trivial HNN-extension. It is not difficult to check, by induction on the length of g, that for all $g \in \Gamma \setminus H$ there exists $s \in H$ such that $gHg^{-1} \cap H \subset s\Sigma s^{-1}$ or $gHg^{-1} \cap H \subset s\theta(\Sigma)s^{-1}$. Hence, the hypothesis of 5 holds and H is malnormal in Γ whenever Σ is finite.

2. If $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is a non-trivial amalgamated free product. By induction on the length, for all $i = 1, 2$, for all $g \in \Gamma \setminus \Gamma_i$, there exists $s \in \Gamma_i$ such that $g\Gamma_i g^{-1} \cap \Gamma_i \subset s\Sigma s^{-1}$. Hence, the hypothesis of 5 holds (with $H = \Gamma_i$) and Γ_i is malnormal in Γ whenever Σ is finite. Moreover, it is not difficult to check that, for all $g \in \Gamma$, there exists $s \in \Gamma_1$ such that $g\Gamma_2 g^{-1} \cap \Gamma_1 \subset s\Sigma s^{-1}$. Hence, the hypothesis of 4 (if Σ is finite) and 6 hold with $H = \Gamma_1$ and $K = \Gamma_2$ and, by symmetry, with $H = \Gamma_2$ and $K = \Gamma_1$.

3 HNN-extensions in the class $\mathcal{A}$

This section is dedicated to the proof of Theorem 1.2.1.

Let H be a countable group, $\Sigma < H$ a subgroup and $\theta : \Sigma \to H$ an injective group homomorphism. Define the HNN-extension $\Gamma = \text{HNN}(H, \Sigma, \theta) = \langle H, t \,|\, \theta(\sigma) = t\sigma t^{-1},\ \forall \sigma \in \Sigma \rangle$.

Let X be an infinite countable set and denote by $S(X)$ the Polish[4] group of bijections of X.

For the rest of this section we suppose that $H < S(X)$ such that the actions of Σ and $\theta(\Sigma)$ on X are free. Define

$$Z = \{w \in S(X) \,:\, w\sigma w^{-1} = \theta(\sigma) \text{ for all } \sigma \in \Sigma\}.$$

It is clear that Z is a closed subset of $S(X)$. Moreover, since the actions of Σ and $\theta(\Sigma)$ are free, it is easy to see that Z is non-empty.

By the universal property, for all $w \in Z$, there exists a unique group homomorphism $\pi_w : \Gamma \to S(X)$ such that $\pi_w(t) = w$ and $\pi_w(h) = h$ for all $h \in H$. The strategy is to prove, under suitable assumptions on the action $H \curvearrowright X$, that the set of $w \in Z$ such that π_w is amenable, transitive and faithful is a dense G_δ in Z.

We first study the set of $w \in Z$ such that π_w is transitive.

Lemma 3.1. *If the action $H \curvearrowright X$ has infinite orbits then the set $U = \{w \in Z \,:\, \pi_w$ is transitive$\}$ is a dense G_δ in Z.*

Proof. Write $U = \cap_{x,y \in X} U_{x,y}$, where $U_{x,y} = \{w \in Z \,:\, \exists g \in \Gamma,\ \pi_w(g)x = y\}$. It suffices to show that $U_{x,y}$ is open and dense in Z for all $x, y \in X$. It is obvious that $U_{x,y}$ is open. Let us show that $U_{x,y}$ is dense. Let $w \in Z \setminus U_{x,y}$ and $F \subset X$ a finite subset. It suffices to construct $\gamma \in Z$ and $g \in \Gamma$ such that $\gamma|_F = w|_F$ and $\pi_\gamma(g)x = y$. Since the action has infinite orbits, there exists $h_0, h_1 \in H$ such that $h_0 y \notin w(\Sigma F)$ and $h_1 x \notin \Sigma F$. Observe that $\Sigma h_1 x \cap \Sigma w^{-1} h_0 y = \emptyset$. Indeed, if we have $\Sigma h_1 x = \Sigma w^{-1} h_0 y$ then, for some $\sigma \in \Sigma$, we have $\sigma h_1 x = w^{-1} h_0 y$. Hence, $\pi_w(g)x = y$ with $g = h_0^{-1} t\sigma h_1$, a contradiction. Define $Y = \Sigma h_1 x \sqcup \Sigma w^{-1} h_0 y$. Then, $F \subset Y^c$ and $w(Y) = \theta(\Sigma) w h_1 x \sqcup \theta(\Sigma) h_0 y$. Define a bijection $\gamma \in S(X)$ by $\gamma|_{Y^c} = w|_{Y^c}$ and

$$\gamma(\sigma h_1 x) = \theta(\sigma) h_0 y \quad \text{and} \quad \gamma(\sigma w^{-1} h_0 y) = \theta(\sigma) w h_1 x \quad \text{for all } \sigma \in \Sigma.$$

By construction, $\gamma \in Z$, $\gamma|_F = w|_F$ and $\pi_\gamma(h_0^{-1} t h_1)x = y$. □

[4] With the topology of pointwise convergence: $w_n \to w \Leftrightarrow$ for all finite subset $F \subset X$ $\exists n_0 \in \mathbb{N}$ such that $w_n|_F = w|_F$ $\forall n \geq n_0$.

Next, we give a sufficient condition for the set of $w \in Z$ for which π_w is amenable to be a dense G_δ.

Lemma 3.2. *If the action $H \curvearrowright X$ admits a Følner sequence (C_n) such that $|C_n| \to \infty$ then the set*

$$V = \{w \in Z \,:\, \pi_w \text{ is amenable}\}$$

is a dense G_δ in Z.

Proof. At first we prove the following claim.

Claim. Let $Y, F \subset X$ be finite subsets and $w \in Z$ such that $\theta(\Sigma)Y \cap w(F) = \emptyset$. There exists $\gamma \in Z$ such that $\Sigma Y \cap \Sigma\gamma^{-1}Y = \emptyset$ and $\gamma|_F = w|_F$.

Proof of the Claim. Write $\theta(\Sigma)Y = \sqcup_{i=1}^n \theta(\Sigma)y_i$. Since the set $\Sigma Y \cup \Sigma F \cup \Sigma w^{-1}(Y)$ is finite and Σ-invariant, we can find n disjoint Σ-orbits $\Sigma z_1, \dots, \Sigma z_n$ in its complement.

Define $\widetilde{F} = \sqcup_{j=1}^n (\Sigma z_j \sqcup \Sigma w^{-1} y_j)$. Then $F \subset \widetilde{F}^c$ and $w(\widetilde{F}) = \sqcup_{j=1}^n (\theta(\Sigma) w z_j \sqcup \theta(\Sigma) y_j)$. By the freeness assumption, we can define a bijection $\gamma \in S(X)$ by $\gamma|_{\widetilde{F}^c} = w|_{\widetilde{F}^c}$ and,

$$\gamma(\sigma z_j) = \theta(\sigma) y_j \quad \text{and} \quad \gamma(\sigma w^{-1} y_j) = \theta(\sigma) w z_j \quad \sigma \in \Sigma,\ 1 \leq j \leq n.$$

By construction $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, $\Sigma Y \cap \Sigma\gamma^{-1}Y = \sqcup_{j=1}^n \Sigma Y \cap \Sigma\gamma^{-1}(y_j) = \sqcup_j \Sigma Y \cap \Sigma z_j = \emptyset$.□

End of the proof of Lemma 3.2. Write $H = \cup^{\uparrow} F_m$ where (F_m) is an increasing sequence of finite subsets. Since Γ is generated by H and t it follows that π_w is amenable if and only if, for all $m \geq 1$ there exists a non-empty finite set $C \subset X$ such that

$$\mathrm{Sup}_{g \in F_m \cup \{w\}} \frac{|gC \Delta C|}{|C|} < \frac{1}{m}.$$

Write $V = \cap_{m \geq 1} V_m$ where

$$V_m = \left\{ w \in Z \,:\, \exists C \subset X \text{ such that } 0 < |C| < \infty \text{ and } \mathrm{Sup}_{g \in F_m \cup \{w\}} \frac{|gC \Delta C|}{|C|} < \frac{1}{m} \right\}.$$

It is easy to see that V_m is open. Let us show that V_m is dense. Let $w \in Z$ and $F \subset X$ a finite subset. Observe that, for any Følner sequence (C_n) and any finite subset $K \subset H$, the sequence (KC_n) is again a Følner sequence. Also, if $|C_n| \to \infty$ then, for all finite subset $F \subset X$, there exists $n_0 \in \mathbb{N}$ such that the sequence $(C_n \setminus F)_{n \geq n_0}$ is a Følner sequence. Hence, up to a shifting of the indices, the sequence (D_n), where $D_n = \Sigma C_n \setminus (\Sigma F \cup \Sigma w(\Sigma F))$ is a Σ-globally invariant Følner sequence such that $D_n \cap F = \emptyset$ and $\theta(\Sigma) D_n \cap w(F) = \emptyset$ for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ large enough such that

$$|gD_N \Delta D_N| < \frac{|D_N|}{2m|\Sigma|} \quad \text{for all} \quad g \in F_m \cup \theta(\Sigma).$$

By the claim, there exists $\gamma_0 \in Z$ such that $\gamma_0|_F = w|_F$ and $D_N \cap \Sigma\gamma_0^{-1}(D_N) = \emptyset$. Write

$$D_N = \sqcup_{i=1}^L \Sigma x_i \quad \text{and} \quad \theta(\Sigma) D_N = \sqcup_{i=1}^K \theta(\Sigma) y_i,$$

where $K \geq L$. Observe that the sets Σx_i and $\Sigma\gamma_0^{-1} y_j$ are pairwise disjoint. Define $Y = \bigsqcup_{i=1}^L \Sigma x_i \sqcup \Sigma\gamma_0^{-1} y_i$. Observe that $F \subset Y^c$ and $\gamma_0(Y) = \bigsqcup_{i=1}^L \theta(\Sigma)\gamma_0 x_i \sqcup \theta(\Sigma) y_i$. Define a bijection $\gamma \in S(X)$ by $\gamma|_{Y^c} = \gamma_0|_{Y^c}$ and,

$$\gamma(\sigma x_i) = \theta(\sigma) y_i \quad \text{and} \quad \gamma(\sigma\gamma_0^{-1} y_i) = \theta(\sigma)\gamma_0 x_i \quad \sigma \in \Sigma,\ 1 \leq i \leq L.$$

By construction, $\gamma \in Z$ and $\gamma|_F = \gamma_0|_F = w|_F$. Observe that

$$|\theta(\Sigma) D_N \Delta \gamma(D_N)| = |\theta(\Sigma) D_N| - |\gamma(D_N)| = |\theta(\Sigma) D_N| - |D_N| = |\theta(\Sigma) D_N \Delta D_N|.$$

Moreover,

$$|D_N \Delta \theta(\Sigma) D_N| \leq \sum_{\sigma \in \Sigma} |D_N \Delta \theta(\sigma) D_N| < \frac{|D_N|}{2m}.$$

It follows that

$$|D_N \Delta \gamma(D_N)| \leq |D_N \Delta \theta(\Sigma) D_N| + |\theta(\Sigma) D_N \Delta \gamma(D_N)| < \frac{|D_N|}{m}.$$

We also have $|D_N \Delta g D_N| < \frac{|D_N|}{m}$ for all $g \in F_m$ hence, it suffices to define $C = D_N$ to see that $\gamma \in V_m$. □

Proof of Theorem 1.2.1. Suppose that H admits an amenable and faithful action with infinite orbits and free on Σ and $\theta(\Sigma)$. Define $\Gamma = \mathrm{HNN}(H, \Sigma, \theta) = \langle H, t \rangle$. By Lemma 2.4 and example 2.5, there exists a faithful action $\Gamma \curvearrowright Y$ with infinite H-orbits such that $\Sigma, \theta(\Sigma) \curvearrowright Y$ are free and the action $H \curvearrowright Y$ is amenable. Consider the faithful action $\Gamma \curvearrowright X$ with $X = Y \times \mathbb{N}$ given by $g(y,n) = (gy, n)$ for $g \in \Gamma$ and $(y,n) \in X$. We view $H < \Gamma < S(X)$. It is obvious that the actions $\Sigma, \theta(\Sigma) \curvearrowright X$ are free, the H-orbits are infinite and the action $H \curvearrowright X$ is amenable. Hence, by Lemma 2.1, there exists a Følner sequence (C_n) for the H-action whose size goes to infinity (one could also use Lemma 2.3). Thus, we can apply Lemmas 3.1 and 3.2 to the action $H \curvearrowright X$. Hence, it suffices to show that the set $O = \{w \in Z \,:\, \pi_w \text{ is faithful}\}$ is a dense G_δ in Z. Writing $O = \cap_{g \in \Gamma \setminus \{1\}} O_g$, where $O_g = \{w \in Z \,:\, \pi_w(g) \neq \mathrm{id}\}$ is obviously open, it suffices to show that O is dense. Write $X = \cup^{\uparrow} X_n$ where $X_n = \{(x,k) \in X \,:\, k \leq n\}$ is infinite and globally invariant under Γ. Let $w \in Z$ and $F \subset X$ a finite subset. Let $N \in \mathbb{N}$ large enough such that $\Sigma F \cup w(\Sigma F) \subset X_N$. The set $X_N \setminus \Sigma F$ (resp. $X_N \setminus w(\Sigma F)$) is infinite and globally invariant under Σ (resp. $\theta(\Sigma)$). Hence, there exists a bijection $\gamma_0 \,:\, X_N \setminus \Sigma F \to X_N \setminus w(\Sigma F)$ satisfying $\gamma_0 \sigma = \theta(\sigma)\gamma_0$ for all $\sigma \in \Sigma$. Define $\gamma \in S(X)$ by $\gamma|_{\Sigma F} = w|_{\Sigma F}$, $\gamma|_{X_N \setminus \Sigma F} = \gamma_0$ and $\gamma|_{X_N^c} = t|_{X_N^c}$. By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, since $\pi_\gamma(g)(y,n) = (gy,n)$ for all $n > N$ and since $\Gamma \curvearrowright Y$ is faithful, it follows that π_γ is faithful. $\square$

Remark 3.3. The following more general result is actually true.

For all amenable and faithful action on a countable set $H \curvearrowright Y$ with infinite orbits and free on Σ and $\theta(\Sigma)$, there exists an amenable, transitive and faithful action on a countable set $\Gamma \curvearrowright X$ such that, for all $h \in H$, $\mathrm{Fix}_Y(h) = \emptyset$ implies $\mathrm{Fix}_X(h) = \emptyset$.

Indeed, it follows from Lemma 2.4 and Example 2.5, that the replacement by the induced action preserves the property that elements in H have an empty fixed point set. Also, the replacement by the action on $X = Y \times \mathbb{N}$ preserves this property. Since $\pi_w(h) = h$ for all $h \in H$ and all $w \in Z$, this proves the remark.

4 Amalgamated free products in the class $\mathcal{A}$

In this section we prove Theorem 1.2.2. The notations are independent of the ones of section 3.

Let X be an infinite countable set. For the rest of this section we assume that $\Gamma_1, \Gamma_2 < S(X)$ are two countable subgroups of the Polish group of bijections of X with a common finite subgroup Σ such that $\Sigma \curvearrowright X$ is free. Define

$$Z = \{w \in S(X) \,:\, w\sigma = \sigma w \text{ for all } \sigma \in \Sigma\}.$$

Z is a non-trivial closed subgroup of $S(X)$. Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. By the universal property, for all $w \in Z$, there exists a unique group homorphism $\pi_w \,:\, \Gamma \to S(X)$ such that $\pi_w(g) = g$ and $\pi_w(h) = w^{-1}hw$ for all $g \in \Gamma_1, h \in \Gamma_2$.

Lemma 4.1. *If the actions $\Gamma_1 \curvearrowright X$ and $\Gamma_2 \curvearrowright X$ have infinite orbits then the set*

$$U = \{w \in Z \,:\, \pi_w \textit{ is transitive}\}$$

is a dense G_δ in Z.

Proof. Write $U = \cap_{x,y \in X} U_{x,y}$, where $U_{x,y} = \{w \in Z \,:\, \text{there exists } g \in \Gamma \text{ such that } \pi_w(g)x = y\}$. Since $U_{x,y}$ is open in Z for all $x, y \in X$, it suffices to show that it is dense in Z. Let $x, y \in X$, $w \in Z$ and $F \subset X$ a finite subset. Since $\Gamma_1 \curvearrowright X$ has infinite orbits, there exists $g_1 \in \Gamma_1$ such that $g_1 x \notin \Sigma F$. By the same argument, there exists also $g_2 \in \Gamma_1$ such that $g_2^{-1} y \notin \Sigma F \cup \Sigma g_1 x$. Take $z, t \in X$ in the same Γ_2-orbit and in the complement of the finite set $w(\Sigma F \cup \Sigma g_1 x \cup \Sigma g_2^{-1} y)$ such that $\Sigma z \cap \Sigma t = \emptyset$. Write $t = hz$ where $h \in \Gamma_2$ and define

$$Y = \Sigma g_1 x \sqcup \Sigma g_2^{-1} y \sqcup \Sigma w^{-1} z \sqcup \Sigma w^{-1} t.$$

One has $F \subset Y^c$ and $w(Y) = \Sigma w g_1 x \sqcup \Sigma w g_2^{-1} y \sqcup \Sigma z \sqcup \Sigma t$. Define $\gamma \in S(X)$ by $\gamma|_{Y^c} = w|_{Y^c}$ and,

$$\gamma(\sigma g_1 x) = \sigma z, \;\; \gamma(\sigma g_2^{-1} y) = \sigma t, \;\; \gamma(\sigma w^{-1} z) = \sigma w g_1 x, \;\; \gamma(\sigma w^{-1} t) = \sigma w g_2^{-1} y \quad \forall \sigma \in \Sigma.$$

By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, with $g = g_2 h g_1 \in \Gamma$, one has

$$\pi_\gamma(g)x = g_2 \gamma^{-1} h \gamma g_1 x = g_2 \gamma^{-1} h z = g_2 \gamma^{-1} t = g_2 g_2^{-1} y = y. \square$$

Lemma 4.2. *If there exist Følner sequences (C_n) and (D_n) for the actions $\Gamma_1 \curvearrowright X$ and $\Gamma_2 \curvearrowright X$ respectively such that $|C_n|, |D_n| \to \infty$ and $\frac{|D_n|}{|C_n|} \to 1$ then the set $V = \{w \in Z : \pi_w$ is amenable $\}$ is a dense G_δ in Z.*

Proof. We start the proof with the following simple claim.

Claim.

1. For all finite subsets $Y_1, Y_2 \subset X$ there exists Σ-invariant Følner sequences (C'_n) and (D'_n) for the actions $\Gamma_1 \curvearrowright X$ and $\Gamma_2 \curvearrowright X$ respectively such that $|C'_n|, |D'_n| \to \infty$, $\frac{|D'_n|}{|C'_n|} \to 1$ and $C'_n \cap Y_1 = \emptyset$, $D'_n \cap Y_2 = \emptyset$ for all $n \in \mathbb{N}$.

2. Let $F, Y_1 \subset X$ be finite subsets such that $\Sigma Y_1 \cap F = \emptyset$. For all finite subset $Y_2 \subset X$ and all $w \in Z$, there exists $\gamma \in Z$ such that $\gamma|_F = w|_F$ and $\Sigma Y_1 \cap \gamma^{-1}(\Sigma Y_2) = \emptyset$.

Proof of the claim. 1. Take $C'_n = \Sigma C_n \setminus \Sigma Y_1$ and $D'_n = \Sigma D_n \setminus \Sigma Y_2$ and shift the indices if necessary. We leave the details to the reader.

2. Write $\Sigma Y_1 = \sqcup_{i=1}^l \Sigma x_i$. Since the set $\Sigma w Y_1 \cup \Sigma w F \cup \Sigma Y_2$ is finite and Σ invariant, we can find l disjoint Σ-orbits $\Sigma z_1, \dots \Sigma z_l$ in its complement. Define $Y = \sqcup_{i=1}^l \Sigma x_i \sqcup \Sigma w^{-1} z_i$. Then, $F \subset Y^c$ and $w(Y) = \sqcup_{i=1}^l \Sigma w x_i \sqcup \Sigma z_i$. Define $\gamma \in S(X)$ by $\gamma|_{Y^c} = w|_{Y^c}$ and,

$$\gamma(\sigma x_i) = \sigma z_i \quad \gamma(\sigma w^{-1} z_i) = \sigma w x_i \quad \text{for all } \sigma \in \Sigma, \ 1 \leq i \leq l.$$

By construction $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, $\gamma(\Sigma Y_1) \cap \Sigma Y_2 = \emptyset$. ☐

End of the proof of Lemma 4.2. Write $\Gamma_1 = \cup^\uparrow F_m$ and $\Gamma_2 = \cup^\uparrow G_m$, where $|F_m|, |G_m| < \infty$. Since Γ is generated by Γ_1 and Γ_2 it follows that π_w is amenable if and only if, for all $m \geq 1$ there exists a non-empty finite set $C \subset X$ such that

$$\frac{|\pi_w(g) C \Delta C|}{|C|} = \frac{|gC\Delta C|}{|C|} < \frac{1}{m} \quad \forall g \in F_m \quad \text{and} \quad \frac{|\pi_w(h) C \Delta C|}{|C|} = \frac{|hw(C)\Delta w(C)|}{|C|} < \frac{1}{m} \quad \forall h \in G_m.$$

Write $V = \cap_{m \geq 1} V_m$ where

$$V_m = \{w \in Z : \exists C \subset X, \, 0 < |C| < \infty, \text{ such that } \frac{|gC\Delta C|}{|C|} < \frac{1}{m} \text{ and } \frac{|hw(C)\Delta w(C)|}{|C|} < \frac{1}{m} \ \forall g \in F_m, h \in G_m\}.$$

Since V_m is open in Z, it suffices to show that V_m is dense in Z. Let $w \in Z$ and $F \subset X$ a finite subset. By the first assertion of the claim, we can assume that (C_n) and (D_n) are Σ-invariant Følner sequences such that $C_n \cap F = \emptyset$ and $D_n \cap w(F) = \emptyset$ for all $n \in \mathbb{N}$. Let $N \in \mathbb{N}$ large enough such that

$$\frac{|gC_N \Delta C_N|}{|C_N|} < \frac{1}{m}, \quad \frac{|hD_N \Delta D_N|}{|D_N|} < \frac{1}{4m}, \quad \left|1 - \frac{|D_N|}{|C_N|}\right| < \frac{1}{4m} \quad \text{for all } g \in F_m, \ h \in G_m.$$

By the second assertion of the claim we may and will assume that $C_N \cap w^{-1}(D_N) = \emptyset$. Write $C_N = \sqcup_{i=1}^l \Sigma x_i$ and $D_N = \sqcup_{j=1}^k \Sigma y_j$. Let $M = \mathrm{Min}(l, k)$ and define $Y = \sqcup_{i=1}^M \Sigma x_i \sqcup \Sigma w^{-1} y_i$. Then, one has $F \subset Y^c$ and $w(Y) = \sqcup_{i=1}^M \Sigma w x_i \sqcup \Sigma y_i$. Define $\gamma \in S(X)$ by $\gamma|_{Y^c} = w|_{Y^c}$ and,

$$\gamma(\sigma x_i) = \sigma y_i \quad \gamma(\sigma w^{-1} y_i) = \sigma w x_i \quad \text{for all } \sigma \in \Sigma, \ 1 \leq i \leq M.$$

By construction $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, $|\gamma(C_N)\Delta D_N| = \big|\,|D_N| - |C_N|\,\big| < \frac{|C_N|}{4m}$. Hence, for all $h \in G_m$, one has,

$$\begin{aligned} |h\gamma(C_N)\Delta\gamma(C_N)| &\leq |h\gamma(C_N)\Delta hD_N| + |hD_N \Delta D_N| + |D_N \Delta \gamma(C_N)| = 2|\gamma(C_N)\Delta D_N| + |hD_N\Delta D_N| \\ &< \frac{|C_N|}{2m} + \left(1 + \frac{1}{4m}\right)\frac{|hD_N\Delta D_N|}{|D_N|}|C_N| < \frac{|C_N|}{m}. \end{aligned}$$

Defining $C = C_N$, we see that $\gamma \in V_m$. ☐

Proof of Theorem 1.2.2. Suppose that the triple $(\Sigma, \Gamma_1, \Gamma_2)$ satisfies the hypothesis of Theorem 1.2.2 and define $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. By Lemma 2.4, for all $i \in \{1,2\}$, there exists a faithful action $\Gamma \curvearrowright Y_i$ with infinite Γ_i-orbits such that $\Sigma \curvearrowright Y_i$ is free and the action $\Gamma_i \curvearrowright Y_i$ is amenable. Moreover, by Example 2.5, $\Gamma \curvearrowright Y_i$ also has infinite Γ_j-orbits for $j \neq i$. Define $Y = Y_1 \sqcup Y_2$. Then the natural faithful action $\Gamma \curvearrowright Y$ has infinite Γ_i-orbits for $i = 1, 2$, $\Sigma \curvearrowright Y$ is free and $\Gamma_i \curvearrowright Y$ is amenable for $i = 1, 2$. Define $X = Y \times \mathbb{N}$ with the faithful Γ-action given by $g(y, n) = (gy, n)$ for $g \in \Gamma$ and $(y, n) \in X$. View $\Sigma < \Gamma_1, \Gamma_2 < \Gamma < S(X)$. It is clear that Σ acts freely on X. Moreover, by Lemma 2.3, we can find Følner sequences (C_n) and (D_n) for the actions of Γ_1 and Γ_2 on X respectively such that $|C_n| \to \infty$, $|D_n| \to \infty$ and $\frac{|D_n|}{|C_n|} \to 1$. Thus, we can apply Lemmas 4.1 and 4.2 to the actions $\Gamma_1, \Gamma_2 \curvearrowright X$. Hence, it suffices to show that the set $O = \{w \in Z : \pi_w \text{ is faithful}\}$ is a dense G_δ in Z. As in the proof of Theorem 1.2.1 it is easy to write O as a countable intersection of open sets. Hence, it suffices to show that O is dense in Z. Let $w \in Z$ and $F \subset X$ a finite subset. Write $X = \cup^{\uparrow} X_n$ where $X_n = \{(x, k) \in X : k \leq n\}$ is infinite globally invariant under Γ. Let $N \in \mathbb{N}$ large enough such that $\Sigma F \cup w(\Sigma F) \subset X_N$. Since the sets $X_N \setminus \Sigma F$ and $X_N \setminus w(\Sigma F)$ are infinite and Σ-invariant, there exists a bijection $\gamma_0 : X_N \setminus \Sigma F \to X_N \setminus w(\Sigma F)$ such that $\gamma_0 \sigma = \sigma \gamma_0$ for all $\sigma \in \Sigma$. Define $\gamma \in S(X)$ by $\gamma|_{\Sigma F} = w|_{\Sigma F}$, $\gamma|_{X_N \setminus \Sigma F} = \gamma_0$ and $\gamma|_{X_N^c} = \mathrm{id}|_{X_N^c}$. By construction, $\gamma \in Z$ and $\gamma|_F = w|_F$. Moreover, since $\pi_\gamma(g)(y, n) = (gy, n)$ for all $g \in \Gamma$ and all $(y, n) \in X$ with $n > N$ and because $\Gamma \curvearrowright Y$ is faithful, it follows that π_γ is faithful. □

Remark 4.3. The following more general result is actually true.

If, for $i = 1, 2$, there exists an amenable and faithful action on a countable set $\Gamma_i \curvearrowright Y_i$ with infinite orbits and free on Σ then, there exists an amenable, transitive and faithful action on a countable set $\Gamma \curvearrowright X$ with the property that, for all $i = 1, 2$, for all $h \in \Gamma_i$, $\mathrm{Fix}_{Y_i}(h) = \emptyset$ implies $\mathrm{Fix}_X(h) = \emptyset$.

Indeed, the first replacement by the induced action from Γ_i to Γ preserves the property that the elements in Γ_i have an empty fixed point set. Moreover, Lemma 2.4 and example 2.5 imply that $\Gamma_i \curvearrowright Y_j$ is free for $i \neq j$. Hence, the property to have an empty fixed point set for the actions $\Gamma_1, \Gamma_2 \curvearrowright Y = Y_1 \sqcup Y_2$ is preserved. The replacement by the action on $X = Y \times \mathbb{N}$ also preserves this property. Since, for all $w \in Z$, $\pi_w(g) = g$ for all $g \in \Gamma_1$ and $\mathrm{Fix}_X(\pi_w(h)) = w^{-1}(\mathrm{Fix}_X(h))$ for all $h \in \Gamma_2$, this proves the remark.

5 Groups acting on trees in the class $\mathcal{A}$

This section contains the proof of Theorem 1.3. Let $\mathcal{G}$ be a graph. We denote by $\mathrm{E}(\mathcal{G})$ its edge set and by $\mathrm{V}(\mathcal{G})$ its vertex set. For $e \in \mathrm{E}(\mathcal{G})$ we denote by $s(e)$ the source of e and $r(e)$ the range of e.

Let Γ be a countable group acting without inversion on a non-trivial tree T with finite quotient graph $\mathcal{G} = \mathrm{T}/\Gamma$ and finite edge stabilizers. By [Se83], the quotient graph $\mathcal{G}$ can be equipped with a structure of a graph of groups $(\mathcal{G}, \{\Gamma_p\}_{p \in \mathrm{V}(\mathcal{G})}, \{\Sigma_e\}_{e \in \mathrm{E}(\mathcal{G})})$ where each Σ_e is isomorphic to an edge stabilizer and each Γ_p is isomorphic to a vertex stabilizer and such that Γ is the fundamental group of this graph of groups i.e., given a fixed maximal subtree $\mathcal{T} \subset \mathcal{G}$, Γ is generated by the groups Γ_p for $p \in \mathrm{V}(\mathcal{G})$ and the edges $e \in \mathrm{E}(\mathcal{G})$ with the relations

$$\overline{e} = e^{-1}, \quad s_e(x) = e r_e(x) e^{-1}, \ \forall x \in \Sigma_e \quad \text{and} \quad e = 1 \ \ \forall e \in \mathrm{E}(\mathcal{T}),$$

where $s_e : \Sigma_e \to \Gamma_{s(e)}$ and $r_e : \Sigma_e \to \Gamma_{r(e)}$ are respectively the source and range group homomorphisms. We will prove the following stronger version of Theorem 1.3 by induction on $n = \frac{1}{2}|E(\mathcal{G})| \geq 1$.

Theorem 5.1. *If, for all $p \in \mathrm{V}(\mathcal{G})$, there exists an amenable and faithful action on a countable set $\Gamma_p \curvearrowright X_p$ with infinite orbits and free on $s_e(\Sigma_e)$ for all $e \in \mathrm{E}(\mathcal{G})$ such that $s(e) = p$. Then, there exists an amenable, faithful and transitive action on a countable set $\Gamma \curvearrowright X$ such that, for all $p \in \mathrm{V}(\mathcal{G})$ and all $h \in \Gamma_p$, $\mathrm{Fix}_{X_p}(h) = \emptyset$ implies $\mathrm{Fix}_X(h) = \emptyset$.*

Proof. If $n = 1$ then Γ is either an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ where $(\Sigma, \Gamma_1, \Gamma_2)$ satisfies the hypothesis of Theorem 1.2.2 or an HNN-extension $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ where (H, Σ, θ) satisfies the hypothesis of Theorem 1.2.1. In the amalgamated free product case we use Remark 4.3 and in the HNN-extension case we use Remark 3.3 to obtain that Γ satisfies the conclusion of the theorem. Let $n \geq 1$ and suppose that the conclusion holds for all $1 \leq k \leq n$. Suppose that $\frac{1}{2}|E(\mathcal{G})| = n+1$. Let $e \in \mathrm{E}(\mathcal{G})$ and let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by removing the edges e and $\overline{e}$.

Case 1: $\mathcal{G}'$ is connected. It follows from Bass-Serre theory that $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ where H is fundamental group of our graph of groups restricted to $\mathcal{G}'$, $\Sigma = r_e(\Sigma_e) < H$ and $\theta : \Sigma \to H$ is given by $\theta = s_e \circ r_e^{-1}$. By the induction hypothesis and Remark 3.3 it follows that Γ satisfies the conclusion of the theorem.

Case 2: $\mathcal{G}'$ is not connected. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the two connected components of $\mathcal{G}'$ such that $s(e) \in \mathrm{V}(\mathcal{G}_1)$ and $r(e) \in \mathrm{V}(\mathcal{G}_2)$. Bass-Serre theory implies that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$, where Γ_i is the fundamental group of our graph of groups restricted to $\mathcal{G}_i$, $i = 1, 2$, and $\Sigma = \Sigma_e$ is viewed as a subgroup of Γ_1 via the map s_e and as a subgroup of Γ_2 via the map r_e. By the induction hypothesis and Remark 4.3, Γ satisfies the conclusion of the theorem. □

The proof Theorem 1.3 follows from Theorem 5.1 and Lemma 2.2 since an almost free action on an infinite set is faithful.

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