THE KK-THEORY OF AMALGAMATED FREE PRODUCTS

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Abstract. In the presence of conditional expectations, we prove a long exact sequence in KK-theory for both the maximal and the vertex reduced amalgamated free product of unital C*-algebras that is valid even for non GNS-faithful conditional expectations. However, in the degenerated case, one has to introduce a new reduced amalgamated free product, that we call vertex-reduced. In the course of the proof we established the KK-equivalence between the full amalgamated free product and the vertex-reduced amalgamated free product. This results generalize and simplify the results obtained before by Germain and Thomsen. When the conditional expectations are extremely degenerated, i.e. when they are $*$-homomorphisms, our vertex-reduced amalgamated free product is isomorphic to the fiber direct sum. Hence our results also generalize a result of Cuntz.

1. Introduction

In 1982 J. Cuntz obtained a very elegant result about the full free product of unital C*-algebras with one-dimensional representations that leads to a conjectural long exact sequence for amalgamated free products in a general situation [Cu82]. At about the same time M. Pimsner's and D. Voiculescu's computation of the KK-theory for some group C*-algebras culminated in the computation of full and reduced crossed products by groups acting on trees [Pi86] (or by the fundamental group of a graph of groups in Serre's terminology). To go beyond the group situation has been difficult and it relied heavily on various generalizations of Voiculescu's absorption theorem (see [Th03] for the most general results in that direction). Note also that G. Kasparov and G. Skandalis had another proof of Pimsner long exact sequence when studying KK-theory for buildings [KS91].

Section 2 is a preliminary section in which we investigate the notion of reduced amalgamated free products of unital C*-algebras $A_1 *_B A_2$ in the presence of not necessarily GNS-faithful conditional expectations. The usual reduced version, due to D. Voiculecscu, which is obtained by looking at the module over B, is often too small. Indeed, when the conditional expectations onto B are both $*$-homomorphisms, the Voiculescu's reduced amalgamated free product is isomorphic to B and all the information about A_1 and A_2 is lost. This is why we consider another reduced amalgamated free product, that we call vertex-reduced, which is obtained by looking at the two modules over A_1 and A_2 and is an intermediate quotient between the full amalgamated free product and Voiculescu's reduced amalgamated free product. When the conditional expectations are GNS-faithful, these two reduced amalgamated free products coincide and when the conditional expectations are $*$-homomorphisms the vertex reduced amalgamated free product is isomorphic to the fiber sum $A_1 \oplus_B A_2$. Hence, even in the extreme degenerated case, the information on A_1

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and A_2 is still contained in the vertex-reduced amalgamated free product. As the vertex-reduced free product is a new construction, we devote some time to show some of its properties.

Before proving our long exact sequence in KK-theory we start with an auxiliary and easy result in Section 3. This result states that the full free product is always K-equivalent to the vertex-reduced free product. In particular, when the conditional expectations are morphisms, we get exactly Cuntz result [Cu82]. This result also generalizes and simplifies the previous result obtained by the second author [Ge96]. The proof is very natural, just a rotation trick. While finishing writing this paper, the authors have been made aware that K. Hasegawa just obtained the same result in the particular case of GNS-faithful conditional expectations. By a remark by Ueda ([Ue08]), this result also proves the K-equivalence between full and (vertex) reduced HNN extensions.

The main part and also the more difficult part of our paper comes in Section 4. Under the presence of conditional expectations, we show that the full amalgamated free product $A_1 \underset{B}{*} A_2$ is K-equivalent with the algebra D of continuous functions f from $]-1,1[$ to the full free product such that $f(]-1,0]) \subset A_1$, $f([0,1[) \subset A_2$ and $f(0) \in B$. This is done by generalizing a result in a paper by one of the authors ([Ge97]). Therefore the full amalgamated free product $A_1 \underset{B}{*} A_2$ sits inside a long exact sequence for the computation of its KK-groups. Of course the vertex reduced free product has the same long exact sequence. Explicitly, if C is any separable C*-algebra, then we have the two 6-terms exact sequences (see Corollary 4.12),

$$\begin{array}{ccccc} KK^0(C,B) & \longrightarrow & KK^0(C,A_1)\bigoplus KK^0(C,A_2) & \longrightarrow & KK^0(C,A_1 *_B A_2) \\ \uparrow & & & & \downarrow \\ KK^1(C,A_1 *_B A_2) & \longleftarrow & KK^1(C,A_1)\bigoplus KK^1(C,A_2) & \longleftarrow & KK^1(C,B) \end{array}$$

and

$$\begin{array}{ccccc} KK^0(B,C) & \longleftarrow & KK^0(A_1,C)\bigoplus KK^0(A_2,C) & \longleftarrow & KK^0(A_1 *_B A_2,C) \\ \downarrow & & & & \uparrow \\ KK^1(A_1 *_B A_2,C) & \longrightarrow & KK^1(A_1,C)\oplus KK^1(A_2,C) & \longrightarrow & KK^1(B,C) \end{array}$$

Again the HNN extension case follows using the isomorphism with an amalgamated free product. Note that this result greatly simplifies and generalizes the results of Thomsen [Th03] about KK-theory for amalgamated free products which are valid only when the amalgam is finite dimensional.

Let us mention some applications. As a direct corollary, we obtain that the amalgamated free product of discrete quantum groups is K-amenable if and only if the initial quantum groups are K-amenable. This generalizes the result of Vergnioux [Ve04] which was valid only for amenable discrete quantum groups and this also implies that a graph product of discrete quantum groups (see [CF14]) is K-amenable if and only if the initial quantum groups are K-amenable. Finally, let us mention that our results will be applied in a future paper to deduce a long exact sequence in KK-theory for fundamental C*-algebras of graph of C*-algebras, generalizing and simplifying the results of Pimsner [Pi86] and, as an application, the results of Fima-Freslon [FF13].

2. Preliminaries

2.1. **Notations and conventions.** All C*-algebras and Hilbert modules are supposed to be separable. For a C*-algebra A and a Hilbert A-module H we denote by $\mathcal{L}_A(H)$ the C*-algebra of A-linear adjointable operators from H to H and by $\mathcal{K}_A(H)$ the sub-C*-algebra of $\mathcal{L}_A(H)$ consisting of A-compact operators. For $a \in A$, we denote by $L_A(a) \in \mathcal{L}_A(A)$ the left multiplication operator by a. We refer the reader to [Bl86] for the basics on KK-theory. In general KK-theory is a bi-functor in the category of $\mathbb{Z}/2\mathbb{Z}$-graded C*-algebras. When the two C*-algebras are trivially graded, we end up with what is called $KK^0(A,B)$. It follows from the standard simplifications that any element in $KK^0(A,B)$ is the homotopy class of a A-B-Kasparov's module of the form (H,π,T), with H a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert B-module, i.e. $H = H_0 \oplus H_1$ is a direct sum of Hilbert B-modules, π a morphism of graded C*-algebras ($\mathcal{L}_B(H)$ is a naturally graded C*-algebra). As A is trivially graded, $\pi = \pi_0 \oplus \pi_1$, where $\pi_k : A \to \mathcal{L}_B(H_k)$ are $*$-homomorphisms. And T a self-adjoint 1-graded operator in $\mathcal{L}_B(H)$ with compact commutator any element of with $\pi(A)$. Therefore $T = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$ with $F \in \mathcal{L}_B(H_0,H_1)$ intertwines π_0 and π_1 up to compact operators. The operator T also have the additional property that $T^2 = 1$ modulo compact operator (A unital) and hence F is unitary up to compact operators in the case A is unital. In part 3 of this article, we refer to such a Kasparov module as (H,π,F) to simplify notation.

In part 4 of this article we must deal with KK^1 elements. Any element in $KK^1(A,B)$ has a simple description. It is the homotopy class of a triple (H,π_0,F), where H is a Hilbert B-module, $\pi_0 : A \to \mathcal{L}_B(H)$ is a $*$-homomorphism and $F \in \mathcal{L}_B(H)$ a selfadjoint operator which is unitary up to compact operators and commutes with π_0 up to compact operators. But it actually fits in the general description of Kasparov module but for the couple $(A, B \otimes \mathbb{C}_1)$ where $\mathbb{C}_1$ is the first non trivial Clifford algebra (see section 17.5.2 of [Bl86]) . As an $\mathbb{Z}/2\mathbb{Z}$-graded algebra, $\mathbb{C}_1 = \mathbb{C} \oplus \mathbb{C}$ where $(1,1)$ is 0 graded and $(1,-1)$ is 1-graded. If E is a Hilbert B-module then $E \otimes \mathbb{C}_1$ naturally becomes a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert module over $B \otimes \mathbb{C}_1$. If π is an action of the trivially graded C*-algebra A on this module, then compatibility with the grading as well as $\mathbb{C}_1$-linearity imply that π decomposes as $\pi_0 \oplus \pi_0$ with π_0 an action of A onto E. Now a self-adjoint 1-graded operator T in $\mathcal{L}_{B\otimes\mathbb{C}_1}(E \otimes \mathbb{C}_1)$ must be of the form $(F,-F)$ where F is a self-adjoint operator of $\mathcal{L}_B(E)$. So the simple description of a $KK^1(A,B)$ element gives a natural triple $(H \otimes \mathbb{C}_1, \pi_0 \oplus \pi_0, (F,-F))$ in $KK(A, B \otimes \mathbb{C}_1)$. It must be noted, although we don't use it, that by Kasparov stabilisation any element of $KK(A, B \otimes \mathbb{C}_1)$ is in the same class as an element of this simple form. For the most part of section 4, we use the first description except for proposition 4.6 where Connes-Skandalis characterization of the Kasparov product between a KK^0 and a KK^1 element forces us to use the general description.

2.2. **Conditional expectations.** Let A, B be unital C*-algebras and $\varphi : A \to B$ be a unital completely positive map (ucp). A *GNS construction* of φ is a triple (K,ρ,η), where K is a Hilbert B-module, $\eta \in K$ and $\rho : A \to \mathcal{L}_B(K)$ is a unital $*$-homomorphism such that $K = \overline{\rho(A)\eta \cdot B}$ and $\langle \eta, \rho(a)\eta \rangle = \varphi(a)$ for all $a \in A$. A GNS construction always exists and is unique, up to a canonical isomorphism. Note that, if $B \subset A$ and $E : A \to B$ is a conditional expectation, then the Hilbert B-submodule $\eta \cdot B$ of K, where (K,ρ,η) is a GNS construction of E, is complemented. Indeed, we have $K = \eta \cdot B \oplus K^\circ$, where $K^\circ = \overline{\text{Span}}\{\rho(a)\eta \cdot b : a \in A^\circ \text{ and } b \in B\}$ and $A^\circ = \text{Ker}(E)$. Since E is a conditional expectation onto B we have $bA^\circ \subset A^\circ$ for all $b \in B$. It follows that $\rho(b)K^\circ \subset K^\circ$ for all $b \in B$. Hence, the restriction of ρ to B (and to K°) gives a unital $*$-homomorphism $\rho : B \to \mathcal{L}_B(K^\circ)$.

A conditional expectation is called *GNS-faithful* (or *non-degenerate*) if for a given GNS construction (and hence for all GNS constructions) (K,ρ,η), the homomorphism ρ is faithful. In this paper we will consider reduced amalgamated free product with respect to non-necessary GNS-faithful conditional expectations. Actually, the degeneracy of the conditional expectations will naturally produce different types of reduced amalgamated free products. This is why we include the next proposition, which is well known to specialists but helps to understand the extreme degenerated case: when E is a homomorphism. We include a complete proof for the convenience of the reader.

Proposition 2.1. *Let $B \subset A$ be a unital inclusion of unital C*-algebras and $E : A \to B$ be a conditional expectation with GNS construction (K, ρ, η). The following are equivalent.*

(1) E is a homomorphism.
(2) $K = \eta \cdot B$.
(3) $K^\circ = \{0\}$.

Proof. Since $K = \eta \cdot B \oplus K^\circ$ the equivalence between (2) and (3) is obvious.

(1) $\Rightarrow$ (3). If E is a homomorphism from A to B then, since E is ucp, it is a unital $*$-homomorphism and we have for all $b \in B$ and all $a \in A^\circ$,

$$\langle \rho(a)\eta \cdot b, \rho(a)\eta \cdot b\rangle_K = b^*\langle \eta, \rho(a^*a)\eta\rangle_K b = b^*E(a^*a)b = b^*E(a)^*E(a)b = 0.$$

(3) $\Rightarrow$ (1). If $K^\circ = \{0\}$ then, for all $a \in A^\circ$, we have $E(a^*a) = \langle \rho(a)\eta, \rho(a)\eta\rangle_K = 0$. Hence

$$E((a - E(a))^*(a - E(a))) = 0 = E(a^*a) - E(a^*)E(a) - E(a)^*E(a) + E(a)^*E(a) \quad \text{for all } a \in A.$$

It follows that, for all $a \in A$, we have $E(a^*a) = E(a)^*E(a)$. Hence, the multiplicative domain of the ucp map E is equal to A which implies that E is a homomorphism. □

2.3. **The full and reduced amalgamated free products.** Let A_1, A_2 be two unital C*-algebras with a common C*-subalgebra $B \subset A_k$, $k = 1, 2$ and denote by A_f the full amalgamated free product. To be more precise, we sometimes write $A_f = A_1 \underset{B}{*} A_2$. It is well known that the canonical map from A_k to A_f is faithful for $k = 1, 2$. Hence, we will always view A_1 and A_2 as subalgebras of A_f.

We will now construct, in the presence of conditional expectations, two different reduced amalgamated free products. One of them, that we call the *edge-reduced amalgamated free product* has been extensively studied and it is called, in the literature, the reduced amalgamated free product. The other one, that we call the *vertex-reduced amalgamated free product*, does not seem to be known, even from specialists. As it will become gradually clear, the vertex-reduced amalgamated free product is actually much more natural than the edge-reduced amalgamated free product. It is an intermediate quotient of the full amalgamated free product and it is isomorphic to the edge-reduced amalgamated free product in the presence of GNS-faithful conditional expectations. This is the reason why it has not appeared before in the literature since many authors only consider amalgamated free product in the presence of GNS-faithful conditional expectations. Since the vertex-reduced and the edge-reduced amalgamated free product are the foundations of our proofs we will now explain in great detail their constructions.

In the sequel, we always assume that, for $k = 1, 2$, there exists a conditional expectation $E_k : A_k \to B$. We write $A_k^\circ = \{a \in A_k : E_k(a) = 0\}$, we denote by (K_k, ρ_k, η_k) a GNS construction of E_k and by K_k° the canonical orthogonal complement of $\eta_k \cdot B$ in K_k as explained in Section 2.2. Recall that the restriction of ρ_k to B (and to K_k°) gives a unital $*$-homomorphism $\rho_k : B \to \mathcal{L}_B(K_k^\circ)$.

We denote by I the subset of $\cup_{n\geq 1}\{1,2\}^n$ defined by

$$I=\{(i_1,\dots,i_n)\in\{1,2\}^n\,:\,n\geq 1\text{ and }i_k\neq i_{k+1}\text{ for all }1\leq k\leq n-1\},$$

Recall that an operator $x\in A_f$ is called *reduced* if $x\neq 0$ and x can be written as $x=a_1\dots a_n$ with $n\geq 1$ and $a_k\in A_{i_k}^\circ-\{0\}$ such that $\underline{i}=(i_1,\dots i_n)\in I$.

2.3.1. *The vertex-reduced amalgamated free products.* For $\underline{i}=(i_1,\dots,i_n)\in I$, we define a A_{i_1}-A_{i_n}-bimodule $H_{\underline{i}}$. As Hilbert A_{i_n}-module we have:

$$H_{\underline{i}}=\begin{cases}K_{i_1}\underset{B}{\otimes}K_{i_2}^\circ\underset{B}{\otimes}\dots\underset{B}{\otimes}K_{i_{n-1}}^\circ\underset{B}{\otimes}A_{i_n} & \text{if}\quad n\geq 3,\\ K_{i_1}\underset{B}{\otimes}A_{i_2} & \text{if}\quad n=2,\\ A_{i_1} & \text{if}\quad n=1.\end{cases}$$

The left action of A_{i_1} on $H_{\underline{i}}$ is given by the unital $*$-homomorphism defined by

$$\lambda_{\underline{i}}\,:\,A_{i_1}\to\mathcal{L}_{A_{i_n}}(H_{\underline{i}});\quad \lambda_{\underline{i}}=\begin{cases}\rho_{i_1}\underset{B}{\otimes}\mathrm{id} & \text{if}\quad n\geq 2,\\ L_{A_{i_1}} & \text{if}\quad n=1.\end{cases}$$

We consider, for $k,l\in\{1,2\}$, the subset $I_{k,l}=\{\underline{i}=(i_1,\dots,i_n)\in I\,:\,i_1=k\text{ and }i_n=l\}$ and the A_k-A_l-bimodule defined by

$$H_{k,l}=\bigoplus_{\underline{i}\in I_{k,l}}H_{\underline{i}}\quad\text{and}\quad\lambda_{k,l}=\bigoplus_{\underline{i}\in I_{k,l}}\lambda_{\underline{i}}\,:\,A_k\to\mathcal{L}_{A_l}(H_{k,l}).$$

For $k\in\{1,2\}$ we denote by $\overline{k}$ the unique element in $\{1,2\}\setminus\{k\}$.

Example 2.2. If, for $k\in\{1,2\}$, E_k is a homomorphism from A_k to B it follows from Proposition 2.1 that $K_k^\circ=\{0\}$. Hence, $H_{k,k}=A_k\oplus K_k\underset{B}{\otimes}K_{\overline{k}}^\circ\underset{B}{\otimes}A_k$ and $H_{\overline{k},k}=K_{\overline{k}}\underset{B}{\otimes}A_k$. Note that, since $K_k\simeq B$, we have $H_{k,k}\simeq A_k\oplus K_{\overline{k}}^\circ\underset{B}{\otimes}A_k\simeq K_{\overline{k}}\underset{B}{\otimes}A_k=H_{\overline{k},k}$. Also we have $H_{k,\overline{k}}=K_k\underset{B}{\otimes}A_{\overline{k}}$ and $H_{\overline{k},\overline{k}}=A_{\overline{k}}$. Again, $H_{k,\overline{k}}\simeq A_{\overline{k}}=H_{\overline{k},\overline{k}}$. Actually the isomorphism of Hilbert A_l-modules $H_{k,l}\simeq H_{\overline{k},l}$ is true in full generality as explained below.

For $k,l\in\{1,2\}$ we define a unitary $u_{k,l}\in\mathcal{L}_{A_l}(H_{k,l},H_{\overline{k},l})$, by the following formula. Let $\underline{i}=(i_1,\dots,i_n)\in I$, with $i_1=k$ and $i_n=l$. For $\xi\in H_{\underline{i}}$ we define $u_{k,l}\xi\in H_{\overline{k},l}$ in the following way.

- If $n\geq 2$, write $\underline{i}=(k,\underline{i}')$, where $\underline{i}'=(i_2,\dots,i_n)\in I_{\overline{k},l}$. For $\xi=\rho_k(a)\eta_k\otimes\xi'$, with $a\in A_k$ and $\xi'\in H_{\underline{i}'}$, we define $u_{k,l}\xi:=\begin{cases}\eta_{\overline{k}}\otimes\xi & \text{if}\quad E_k(a)=0,\\ \lambda_{\underline{i}'}(a)\xi' & \text{if}\quad a\in B.\end{cases}$
- If $n=1$ then $k=l$, $\underline{i}=(l)$ and $\xi\in A_l=H_{\underline{i}}$. We define $u_{k,l}\xi:=\eta_{\overline{k}}\otimes\xi$.

Since $\rho_k(b)\eta_k=\eta_k\cdot b$ for all $b\in B$, the operators $u_{k,l}$ are well defined and it is easy to check that, for all $k,l\in\{1,2\}$, the operators $u_{k,l}$ commute with the right actions of A_l on $H_{k,l}$ and $H_{\overline{k},l}$ and extend to a unitary operators, still denoted $u_{k,l}$, in $\mathcal{L}_{A_l}(H_{k,l},H_{\overline{k},l})$ such that $u_{k,l}^*=u_{\overline{k},l}$. Moreover, the definition of $u_{k,l}$ implies that,

$$u_{k,l}^*\lambda_{\overline{k},l}(b)u_{k,l}=\lambda_{k,l}(b)\quad\text{for all }b\in B.\tag{1}$$

Definition 2.3. Let $k \in \{1,2\}$. The *k-vertex-reduced amalgamated free product* is the C*-subalgebra $A_{v,k} \subset \mathcal{L}_{A_k}(H_{k,k})$ generated by $\lambda_{k,k}(A_k) \cup u_{k,k}^*\lambda_{\overline{k},k}(A_{\overline{k}})u_{k,k} \subset \mathcal{L}_{A_k}(H_{k,k})$. To be more precise, we use sometimes the notation $A_{v,k} = A_1 \underset{B}{\overset{k}{*}} A_2$.

For a fixed $k \in \{1,2\}$ the relations (1) imply the existence of a unique unital $*$-homomorphism $\pi_k : A_f \to A_{v,k}$ such that $\pi_k(a) = \begin{cases} \lambda_{k,k}(a) & \text{if } a \in A_k, \\ u_{k,k}^*\lambda_{\overline{k},k}(a)u_{k,k} & \text{if } a \in A_{\overline{k}}. \end{cases}$

In the sequel we will denote by ξ_k the vector $\xi_k := 1_{A_k} \in A_k \subset H_{k,k}$. We summarize the fundamental properties of $A_{v,k}$ in the following proposition.

Proposition 2.4. *Fix $k \in \{1,2\}$. The following holds.*

1. *The morphism π_k is faithful on A_k.*
2. *If $E_{\overline{k}}$ is GNS-faithful then π_k is faithful on $A_{\overline{k}}$.*
3. *There exists a unique ucp map $\mathbb{E}_k : A_{v,k} \to A_k$ such that $\mathbb{E}_k(\pi_k(a)) = a$ $\forall a \in A_k$ and*

$$\mathbb{E}_k(\pi_k(a_1 \dots a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced with } n \geq 2 \text{ or } n = 1 \text{ and } a = a_1 \in A_{\overline{k}}^\circ.$$

   Moreover, $\mathbb{E}_k$ is GNS-faithful.

4. *For any unital C*-algebra C with two unital $*$-homomorphisms $\nu_j : A_j \to C$, $j = 1,2$, such that*
   - *$\nu_1(b) = \nu_2(b)$ for all $b \in B$,*
   - *C is generated, as a C*-algebra, by $\nu_1(A_1) \cup \nu_2(A_2)$,*
   - *ν_k is faithful and there exists a GNS-faithful ucp map $E : C \to A_k$ such that $E(\nu_k(a)) = a$ for all $a \in A_k$ and*

$$E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced with } n \geq 2 \text{ or } n = 1 \text{ and } a = a_1 \in A_{\overline{k}}^\circ,$$

   there exists a unique unital $*$-isomorphism $\nu : A_{v,k} \to C$ such that $\nu \circ \pi_k(a) = \nu_k(a)$ for all $a \in A_1 \cup A_2$. Moreover, ν satisfies $E \circ \nu = \mathbb{E}_k$.

Proof. Fix $k \in \{1,2\}$. By definition of π_k we have, if $a \in A_k$, $\langle \xi_k, \pi_k(a)\xi_k \rangle = a$. It follows directly that π_k is faithful on A_k. Moreover, the map $\mathbb{E}_k : A_{v,k} \to A_k$, $x \mapsto \langle \xi_k, x\xi_k \rangle$ satisfies $\mathbb{E}_k(\pi_k(a)) = a$ $\forall a \in A_k$. By definition we have, for all reduced operators $x = a_1 \dots a_n$ with $\underline{i} = (i_1, \dots, i_n) \in I$ and $a_s \in A_{i_s}^\circ$ for all $s \in \{1, \dots, n\}$,

$$\pi_k(a_1 \dots a_n)\xi_k = \begin{cases} \rho_{i_1}(a_1)\eta_{i_1} \otimes \dots \otimes \rho_{i_{n-1}}(a_{n-1})\eta_{i_{n-1}} \otimes a_n & \text{if } i_1 = k \text{ and } i_n = k, \\ \eta_k \otimes \rho_{i_1}(a_1)\eta_{i_1} \otimes \dots \otimes \rho_{i_{n-1}}(a_{n-1})\eta_{i_{n-1}} \otimes a_n & \text{if } i_1 \neq k \text{ and } i_n = k, \\ \rho_{i_1}(a_1)\eta_{i_1} \otimes \dots \otimes \rho_{i_n}(a_n)\eta_{i_n} \otimes 1_{A_k} & \text{if } i_1 = k \text{ and } i_n \neq k, \\ \eta_k \otimes \rho_{i_1}(a_1)\eta_{i_1} \otimes \dots \otimes \rho_{i_n}(a_n)\eta_{i_n} \otimes 1_{A_k} & \text{if } i_1 \neq k \text{ and } i_n \neq k. \end{cases} \tag{2}$$

Hence we have $\mathbb{E}_k(\pi_k(a_1 \dots a_n)) = 0$ for all $a = a_1 \dots a_n \in A_f$ reduced with $n \geq 2$ or $n = 1$ and $a = a_1 \in A_{\overline{k}}^\circ$. It also follows easily from the previous set of equations that $\overline{\pi_k(A_f)\xi_k \cdot A_k} = H_{k,k}$. Hence the triple $(H_{k,k}, \mathrm{id}, \xi_k)$ is a GNS construction for $\mathbb{E}_k$. This shows that $\mathbb{E}_k$ is GNS-faithful. Note that the uniqueness statement of the third assertion is obvious since A_f is the linear span of B and the reduced operators. Also, the second statement becomes now obvious since, by the properties of $\mathbb{E}_k$ we have, for all $x \in A_{\overline{k}}$, $\mathbb{E}_k(\pi_k(x)) = \mathbb{E}_k(\pi_k(x - E_{\overline{k}}(x))) + \mathbb{E}_k(\pi_k(E_{\overline{k}}(x))) = \pi_k(E_{\overline{k}}(x))$. It follows easily from this equation that π_k is faithful on $A_{\overline{k}}$ whenever $E_{\overline{k}}$ is GNS-faithful. Indeed, let $x \in A_{\overline{k}}$ such that $\pi_k(x) = 0$. Then, for all $y \in A_{\overline{k}}$ we have $\pi_k(y^*x^*xy) = 0$.

Hence, $\pi_k \circ E_{\overline{k}}(y^*x^*xy) = \mathbb{E}_k \circ \pi_k(y^*x^*xy) = 0$ for all $y \in A_{\overline{k}}$. Since π_k is faithful on A_k we find $E_{\overline{k}}(y^*x^*xy) = 0$, for all $y \in A_{\overline{k}}$. Since $E_{\overline{k}}$ is GNS-faithful we conclude that $x = 0$.

(4). The proof is a routine. We write the argument for the convenience of the reader. Let (K, ρ, η) be the GNS construction of E. Since E is GNS-faithful we may and will assume that $\rho = \mathrm{id}$ and $C \subset \mathcal{L}_{A_k}(K)$. By the properties of $\mathbb{E}_k$ and E, the map $U : H_{k,k} \to K$ defined by, for $x = a_1 \dots a_n \in A_f$ reduced with $a_k \in A^{\circ}_{i_k}$, $U(\pi_k(x)\xi_k) := \nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)\eta$ and, for $x = b \in B$, $U(\pi_k(b)\xi_k) = \nu_1(b)\eta = \nu_2(b)\eta$, is well defined and extends to a unitary $U \in \mathcal{L}_{A_k}(H_{k,k}, K)$. By construction, the map $\nu(x) := UxU^*$, for $x \in A_{v,k}$, satisfies the claimed properties. The uniqueness is obvious. □

Remark 2.5. It is known that the canonical homomorphism from A_k to A_f is faithful for $k \in \{1, 2\}$ without assuming the existence of conditional expectations from A_k to B. However, assertion (1) of Proposition 2.4 gives a very simple proof of this fact, since it shows that the composition of the canonical homomorphism from A_k to A_f with the homomorphism π_k is faithful, which implies that the canonical homomorphism from A_k to A_f itself is faithful.

Example 2.6. Suppose that, for a given $k \in \{1, 2\}$, E_k is a homomorphism. Then, as observed in Example 2.2, we have $H_{\overline{k},\overline{k}} = A_{\overline{k}}$ (and $\lambda_{\overline{k},\overline{k}} = L_{A_{\overline{k}}}$). It follows from the definition of $\pi_{\overline{k}}$ that

$$\pi_{\overline{k}}(a) = \begin{cases} L_{A_{\overline{k}}}(a) & \text{if} \quad a \in A_{\overline{k}}, \\ 0 & \text{if} \quad a \in A^{\circ}_{k}. \end{cases}$$

Hence, since A_f the closed linear span of $A_{\overline{k}}$ and the reduced operators and $\pi_{\overline{k}} : A_f \to A_{v,\overline{k}}$ is surjective, we find that $A_{v,\overline{k}} = \pi_{\overline{k}}(A_{\overline{k}})$. Moreover, since $\pi_{\overline{k}}$ is faithful on $A_{\overline{k}}$ we conclude that the restriction of $\pi_{\overline{k}}$ to $A_{\overline{k}}$ gives an isomorphism $A_{\overline{k}} \simeq A_{v,\overline{k}}$.

Definition 2.7. The *vertex-reduced amalgamated free product* is the C*-algebra obtained by separation and completion of A_f with respect to the C*-semi-norm $\| \cdot \|_v$ on A_f defined by

$$\|x\|_v := \mathrm{Max}\{\|\pi_1(x)\|, \|\pi_2(x)\|\} \quad \text{for all } x \in A_f.$$

By *separation and completion* we mean the completion of the pre-C*-algebra obtained by considering the quotient by the null ideal of the C* semi-norm.

We will note it $A_1 \overset{v}{\underset{B}{*}} A_2$ or A_v for simplicity in the rest of this section and let $\pi : A_f \to A_v$ be the canonical surjective unital $*$-homomorphism. Note that, by construction of A_v, for all $k \in \{1, 2\}$, there exists a unique unital (surjective) $*$-homomorphism $\pi_{v,k} : A_v \to A_{v,k}$ such that $\pi_{v,k} \circ \pi = \pi_k$. We describe the fundamental properties of the vertex-reduced amalgamated free product in the following proposition. We call a family of ucp maps $\{\varphi_i\}_{i \in I}$, $\varphi_i : A \to B_i$ GNS-faithful if $\cap_{i \in I} \mathrm{Ker}(\pi_i) = \{0\}$, where (H_i, π_i, ξ_i) is a GNS-construction for φ_i. From Proposition 2.4 and the definition of A_v we deduce the following result.

Proposition 2.8. *The following holds.*

(1) *π is faithful on A_k for all $k \in \{1, 2\}$.*

(2) *For all $k \in \{1, 2\}$, there is a unique ucp map $\mathbb{E}_{A_k} : A_v \to A_k$ such that $\mathbb{E}_{A_k} \circ \pi(a) = a$ for all $a \in A_k$ and all $k \in \{1, 2\}$ and,*

$\mathbb{E}_{A_k}(\pi(a_1 \dots a_n)) = 0$ *for all* $a = a_1 \dots a_n \in A_f$ *reduced with* $n \geq 2$ *or* $n = 1$ *and* $a = a_1 \in A^{\circ}_{\overline{k}}$.

Moreover, the family $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$ is GNS-faithful.

(3) *Suppose that C is a unital C*-algebra with $*$-homomorphisms $\nu_k : A_k \to C$ such that*

- *$\nu_1(b) = \nu_2(b)$ for all $b \in B$,*
- *C is generated, as a C*-algebra, by $\nu_1(A_1) \cup \nu_2(A_2)$,*
- *ν_1 and ν_2 are faithful and, for all $k \in \{1,2\}$, there exists a ucp map $E_{A_k} : C \to A_k$ such that $E_{A_k} \circ \nu_k(a) = a$ for all $a \in A_k$ and all $k \in \{1,2\}$ and,*

$$E_{A_k}(\nu_{i_1}(a_1)\dots\nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1\dots a_n \in A_f \text{ reduced with } n \geq 2 \text{ or } n = 1 \text{ and } a = a_1 \in A_{\overline{k}}^{\circ},$$

and the family $\{E_{A_1}, E_{A_2}\}$ is GNS-faithful.

Then, there exists a unique unital $*$-isomorphism $\nu : A_v \to C$ such that $\nu \circ \pi(a) = \nu_k(a)$ for all $a \in A_k$ and all $k \in \{1,2\}$. Moreover, ν satisfies $E_{A_k} \circ \nu = \mathbb{E}_{A_k}$, $k \in \{1,2\}$.

Proof. (1). It is obvious since, by Proposition 2.4, π_k is faithful on A_k for $k = 1, 2$.

(2). By Proposition 2.4, the maps $\mathbb{E}_{A_k} = \mathbb{E}_k \circ \pi_{v,k}$ satisfy the desired properties and it suffices to check that the family $\{\mathbb{E}_{A_1}, \mathbb{E}_{A_2}\}$ is GNS-faithful. Let $x_0 \in A_f$ be such that $x = \pi(x_0) \in A_v$ satisfies $\mathbb{E}_{A_k}(y^*x^*xy) = 0$ for all $y \in A_v$ and all $k \in \{1,2\}$. Then, for all $k \in \{1,2\}$ we have $\mathbb{E}_k(y^*\pi_{v,k}(x^*x)y) = 0$ for all $y \in A_{v,k}$. Since $\mathbb{E}_k$ is GNS-faithful, this implies that $\pi_{v,k}(x) = \pi_k(x_0) = 0$ for all $k \in \{1,2\}$. Hence, $\|x\|_{A_v} = \text{Max}(\|\pi_1(x_0)\|, \|\pi_2(x_0)\|) = 0$.

(3). The proof is a routine. We include it for the convenience of the reader. Let (L_k, m_k, f_k) be the GNS construction of E_{A_k}. By the universal property of $A_{v,k}$, the C*-algebra $m_k(C) \subset \mathcal{L}_{A_k}(L_k)$ is canonically isomorphic to $A_{v,k}$. Hence, in the remainder of the proof we suppose that $m_k(C) = A_{v,k}$. By the universal property of A_f, we have a unital surjective $*$-homomorphism $\nu_f : A_f \to C$ such that $\nu_f|_{A_k} = \nu_k$. Note that, by the identification we made, $m_k \circ \nu_f = \pi_k$. Hence, by construction of A_v, there exists a unique unital (surjective) $*$-homomorphism $\nu_0 : C \to A_v$ such that $\pi_{v,k} \circ \nu_0 = m_k$ for all $k \in \{1,2\}$. Note that ν_0 is faithful since the identity $\pi_{v,k} \circ \nu_0 = m_k$, $k = 1, 2$, implies that $\text{Ker}(\nu_0) \subset \text{Ker}(m_1) \cap \text{Ker}(m_2) = \{0\}$ (because the pair (E_{A_1}, E_{A_2}) is GNS-faithful). Hence ν_0 is a unital $*$-isomorphism and $\nu := \nu_0^{-1}$ satisfies the required properties. □

Corollary 2.9. *If both E_1 and E_2 are homomorphisms then there is a canonical isomorphism $A_v \simeq A_1 \underset{B}{\oplus} A_2$, where $A_1 \underset{B}{\oplus} A_2 := \{(a_1, a_2) \in A_1 \oplus A_2 : E_1(a_1) = E_2(a_2)\}$.*

Proof. We use the universal property of A_v described in Proposition 2.8. Define $\nu_k : A_k \to A_1 \underset{B}{\oplus} A_2$ by $\nu_1(x) = (x, E_1(x))$ and $\nu_2(y) = (E_2(y), y)$. It is clear that ν_1 and ν_2 are both faithful unital $*$-homomorphisms such that $\nu_1(b) = \nu_2(b)$ for all $b \in B$. Define $E_{A_k} : A_1 \underset{B}{\oplus} A_2 \to A_k$ by $E_{A_1}(a_1, a_2) = a_1$ and $E_{A_2}(a_1, a_2) = a_2$. Then, for all $k \in \{1,2\}$, E_k is a unital $*$-homomorphism such that $E_{A_k} \circ \nu_k(a) = a$ for all $a \in A_k$. In particular, both E_1 and E_2 are conditional expectations and, since $\text{Ker}(E_{A_1}) \cap \text{Ker}(E_{A_2}) = \{0\}$, the family $\{E_{A_1}, E_{A_2}\}$ is GNS-faithful. Hence, it suffices to check the condition on the reduced operators. Since $\nu_1(A_1^{\circ}) = \{(x, 0) : x \in A_1^{\circ}\}$ and $\nu_2(A_2^{\circ}) = \{(0, y) : y \in A_2^{\circ}\}$, we have $\nu_1(A_1^{\circ})\nu_2(A_2^{\circ}) = \nu_2(A_2^{\circ})\nu_1(A_1^{\circ}) = \{0\}$. Hence, it suffices to check the condition on elements $(a_1, a_2) \in \nu_1(A_1^{\circ}) \cup \nu_2(A_2^{\circ})$ which is obvious. □

2.3.2. *The edge-reduced amalgamated free product.* In this section we show how the construction of the edge-reduced (or, in the literature, the reduced) amalgamated free product in full generality is related to the vertex-reduced free product we just defined.

For $\underline{i} \in I$, we consider the B-B-module $K_{\underline{i}}^\circ = K_{i_1}^\circ \underset{B}{\otimes} \dots \underset{B}{\otimes} K_{i_n}^\circ$ as Hilbert B-module with the left action of B given by the unital $*$-homomorphism $\rho_{\underline{i}} : B \to \mathcal{L}_B(K_{\underline{i}}^\circ)$, $\rho_{\underline{i}}(b) = \rho_{i_1}(b) \underset{B}{\otimes} \mathrm{id}$ for all $b \in B$ and we define the Hilbert B-bimodule $K = B \oplus \left(\bigoplus_{\underline{i}\in I} K_{\underline{i}}^\circ\right)$.

Example 2.10. If, for some $k \in \{1,2\}$, E_k is a homomorphism then $K = B \oplus K_{\overline{k}}^\circ \simeq K_{\overline{k}}$. Hence, if both E_1 and E_2 are homomorphisms then $K = B$.

For $l \in \{1,2\}$ define $K(l) = B \oplus \left(\bigoplus_{\underline{i}\in I,\, i_1\neq l} K_{\underline{i}}^\circ\right)$ and note that we have a unital $*$-homomorphism $\rho_l : B \to \mathcal{L}_B(K(l))$ defined by $\rho_l = L_B \oplus \bigoplus_{\underline{i}\in I,\, i_1\neq l} \rho_{\underline{i}}$. Let $U_l \in \mathcal{L}_B(K_l \underset{\rho_l}{\otimes} K(l), K)$ be the unitary operator defined by

$$\begin{array}{rlcl} U_l : & K_l \underset{\rho_l}{\otimes} K(l) & \longrightarrow & K \\ & \eta_l \underset{\rho_l}{\otimes} B & \overset{\simeq}{\longrightarrow} & B \\ & K_l^\circ \underset{\rho_l}{\otimes} B & \overset{\simeq}{\longrightarrow} & K_l^\circ \\ & \eta_l \underset{\rho_l}{\otimes} H_{\underline{i}} & \overset{\simeq}{\longrightarrow} & K_{\underline{i}}^\circ \\ & K_l^\circ \underset{\rho_l}{\otimes} H_{\underline{i}} & \overset{\simeq}{\longrightarrow} & K_{(l,\underline{i})}^\circ \end{array}$$

where $(l,\underline{i}) = (l, i_1, \dots, i_n) \in I$ if $\underline{i} = (i_1, \dots, i_n) \in I$ with $i_1 \neq l$. We define the unital $*$-homomorphisms $\lambda_l : \mathcal{L}_B(K_l) \to \mathcal{L}_B(K)$ by $\lambda_l(x) = U_l(x \otimes 1)U_l^*$. By definition we have $\lambda_1(\rho_1(b)) = \lambda_2(\rho_2(b))$ for all $b \in B$. It follows that there exists a unique unital $*$-homomorphism $\rho : A_f \to \mathcal{L}_B(K)$ such that $\rho(a) = \lambda_k(\rho_k(a))$ for $a \in A_k$, for all $k \in \{1,2\}$.

Proposition 2.11. *There are canonical unitaries $V_k \in \mathcal{L}_B(H_{k,k} \underset{E_k}{\otimes} B, K)$ for $k = 1,2$ satisfying $V_k(\pi_k(a) \otimes 1)V_k^* = \rho(a)$ for all $a \in A_k$ and all $k \in \{1,2\}$.*

Proof. Note that, for $\underline{i} = (i_1, \dots, i_n) \in I$ with $i_1 = i_n = k$ (hence n is odd) we have, if $n = 1$, $H_{\underline{i}} \underset{E_k}{\otimes} B = A_k \underset{E_k}{\otimes} B \simeq K_k \simeq K_k^\circ \oplus B$, and, if $n \geq 3$, $H_{\underline{i}} \underset{E_k}{\otimes} B = K_k \underset{B}{\otimes} \left(K_{\overline{k}}^\circ \underset{B}{\otimes} \dots \underset{B}{\otimes} K_{\overline{k}}^\circ\right) \underset{B}{\otimes} K_k \simeq K_{\underline{i}}^\circ \oplus K_{\underline{i}'}^\circ \oplus K_{\underline{i}''}^\circ \oplus K_{\underline{i}'''}^\circ$, where $\underline{i}' = (i_2, \dots, i_n)$, $\underline{i}'' = (i_1, \dots, i_{n-1})$ and $\underline{i}''' = (i_2, \dots, i_{n-1})$. Hence the existence of $V_k : H_{k,k} \underset{E_k}{\otimes} B \to K$. It is easy to check that V_k satisfies $V_k(\pi_k(a) \otimes 1)V_k^* = \rho(a)$ for all $a \in A_k$ and all $k \in \{1,2\}$. ☐

Definition 2.12. The *edge-reduced* amalgamated free product is the C*-subalgebra $A_e \subset \mathcal{L}_B(K)$ generated by $\lambda_1(A_1) \cup \lambda_2(A_2) \subset \mathcal{L}_B(K)$. To be more precise, we use sometimes the notation $A_e = A_1 \overset{e}{\underset{B}{*}} A_2$.

The edge-reduced amalgamated free product has been constructed by Voiculescu in [Vo83] and is known in the literature as the Voiculescu's reduced amalgamated free product.

Example 2.13. If, for some $k \in \{1,2\}$, E_k is a homomorphism then A_e is the C*-algebra $\overline{\rho_{\overline{k}}(A_{\overline{k}})} \subset \mathcal{L}_B(K_{\overline{k}})$. If both E_1 and E_2 are homomorphisms then $A_e \simeq B$.

The preceding example shows that the edge reduced amalgamated free product may forget everything about the initial C*-algebras A_1 and A_2 in the extreme degenerated case: it only remembers B. This shows that, in general, one should consider instead the vertex-reduced amalgamated free product. Indeed, even in the extreme degenerated case, the vertex reduced amalgamated free product correctly remembers the C*-algebras A_1 and A_2, as shown in Corollary 2.9.

In the following proposition we recall the properties of A_e. The results below are well known when E_1 and E_2 are GNS-faithful. The proof is similar to the proof of Proposition 2.4 and we leave it to the reader.

Proposition 2.14. *The following holds.*

(1) ρ *is faithful on* B.
(2) *For any* $k \in \{1,2\}$, *if* E_k *is GNS-faithful then* ρ *is faithful on* A_k.
(3) *There exists a unique ucp map* $\mathbb{E} : A_e \to B$ *such that* $\mathbb{E} \circ \rho(b) = b$ *for all* $b \in B$ *and,*

$$\mathbb{E}(\rho(a_1, \dots a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced.}$$

Moreover, $\mathbb{E}$ *is GNS-faithful.*

(4) *For any unital C*-algebra* C *with two unital* $*$*-homomorphisms* $\nu_k : A_k \to C$, $k = 1, 2$, *such that*
- $\nu_1(b) = \nu_2(b)$ *for all* $b \in B$,
- C *is generated, as a C*-algebra, by* $\nu_1(A_1) \cup \nu_2(A_2)$,
- $\nu_1|_B = \nu_2|_B$ *is faithful and there exists a GNS-faithful ucp map* $E : C \to B$ *such that* $E \circ \nu_k(b) = b$ *for all* $b \in B$, $k = 1, 2$, *and,*

$$E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0 \text{ for all } a = a_1 \dots a_n \in A_f \text{ reduced,}$$

there exists a unique unital $*$*-isomorphism* $\nu : A_e \to C$ *such that* $\nu \circ \rho(a) = \nu_k(a)$ *for all* $a \in A_k$, $k \in \{1,2\}$. *Moreover,* ν *satisfies* $E \circ \nu = \mathbb{E}$.

Proposition 2.15. *For all* $k \in \{1,2\}$ *there exists a unique unital* $*$*-homomorphism*

$$\lambda_{v,k} : A_{v,k} \to A_e \quad \text{such that} \quad \lambda_{v,k} \circ \pi_k = \rho.$$

Moreover, $\lambda_{v,k}$ *is faithful on* $\pi_k(A_{\overline{k}})$ *and, if* E_k *is GNS-faithful,* $\lambda_{k,v}$ *is an isomorphism.*

Proof. The formulae $\lambda_{v,k}(x) = V_k(x \otimes 1)V_k^*$ defines a unital $*$-homomorphism $\lambda_{v,k} : A_{v,k} \to A_e$ satisfying $\lambda_{v,k} \circ \pi_k = \rho$. The uniqueness of $\lambda_{v,k}$ is obvious. Let us check that $\lambda_{v,k}$ is faithful on $\pi_k(A_{\overline{k}})$. Suppose that $x \in A_{\overline{k}}$ and $\lambda_{v,k}(\pi_k(x)) = 0$. Then, for all $y \in A_{\overline{k}}$, we have $\rho(y^*x^*xy) = \lambda_{v,k}(\pi_k(y^*x^*xy)) = 0$. Hence, $0 = \mathbb{E} \circ \rho(y^*x^*xy) = \mathbb{E} \circ \rho(E_{\overline{k}}(y^*x^*xy)) = E_{\overline{k}}(y^*x^*xy)$. It follows that $x \in \mathrm{Ker}(\rho_{\overline{k}})$ hence, $\lambda_{\overline{k},k}(x) = \oplus_{\underline{i} \in I_{\overline{k},k}} \rho_{\overline{k}}(x) \otimes 1 = 0$ which implies that $\pi_k(x) = u_{k,k}^* \lambda_{\overline{k},k}(x) u_{k,k} = 0$. The last statement follows from the universal property of A_e since the ucp map $E_k \circ \mathbb{E}_k : A_{v,k} \to B$ is GNS-faithful whenever E_k is GNS-faithful. □

In the next proposition, we study some associativity properties between the edge-reduced and the vertex-reduced amalgamated free product. The result is interesting in itself and it will be used to easily obtain ucp radial multipliers on the vertex-reduced amalgamated free product.

Proposition 2.16. *Let* A_1, A_2, A_3 *be unital C*-algebras with a common unital C*-subalgebra* B *and conditional expectations* $\mathrm{E}_k : A_k \to B$. *After identification of* A_1 *with a C*-subalgebra of both* $A_1 \overset{1}{\underset{B}{*}} A_2$ *and* $A_1 \overset{1}{\underset{B}{*}} A_3$, *the canonical GNS-faithful ucp maps* $A_1 \overset{1}{\underset{B}{*}} A_2 \to A_1$ *and* $A_1 \overset{1}{\underset{B}{*}} A_3 \to A_1$

become conditional expectations and, with respect to these GNS-faithful conditional expectations, we have canonical isomorphisms

- $\left(A_1 \overset{1}{\underset{B}{*}} A_2\right) \overset{e}{\underset{A_1}{*}} \left(A_1 \overset{1}{\underset{B}{*}} A_3\right) \simeq A_1 \overset{1}{\underset{B}{*}} \left(A_2 \overset{e}{\underset{B}{*}} A_3\right).$
- $\left(A_1 \overset{2}{\underset{B}{*}} A_2\right) \overset{e}{\underset{A_2}{*}} \left(A_3 \overset{2}{\underset{B}{*}} A_2\right) \simeq \left(A_1 \overset{e}{\underset{B}{*}} A_3\right) \overset{2}{\underset{B}{*}} A_2.$

Proof. We prove the first point. The proof of the second point is similar. We write $\widetilde{A} = A_1 \overset{1}{\underset{B}{*}} \left(A_2 \overset{e}{\underset{B}{*}} A_3\right)$. Let $\rho : A_2 \underset{B}{*} A_3 \to A_2 \overset{e}{\underset{B}{*}} A_3$ and $\widetilde{\pi} : A_1 \underset{B}{*} \left(A_2 \overset{e}{\underset{B}{*}} A_3\right) \to \widetilde{A}$ be the canonical surjections and $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ the canonical GNS-faithful ucp map. Define, for $k = 1, 2$, $\nu_k : A_k \to \widetilde{A}$ by $\nu_1 = \widetilde{\pi}|_{A_1}$ and $\nu_2 = \widetilde{\pi} \circ \rho|_{A_2}$. By definition, $\nu_1(b) = \nu_2(b)$ for all $b \in B$ and ν_1 is faithful. Let C be the C*-subalgebra of $\widetilde{A}$ generated by $\nu_1(A_1) \cup \nu_2(A_2)$. We claim that there exists a (unique) unital faithful $*$-homomorphism $\nu : A_1 \overset{1}{\underset{B}{*}} A_2 \to \widetilde{A}$ such that $\nu \circ \pi_1|_{A_k} = \nu_k$ for $k = 1, 2$, where $\pi_1 : A_1 \underset{B}{*} A_2 \to A_1 \overset{1}{\underset{B}{*}} A_2$ is the canonical surjection. By the universal property of the 1-vertex-reduced amalgamated free product, it suffices to show the following claim, where $E = \widetilde{\mathbb{E}}|_C : C \to A_1$.

Claim. *The ucp map E is GNS-faithful and satisfies $E \circ \nu_1 = id_{A_1}$ and, for all $a = a_1 \dots a_n \in A_f$ reduced with $a_k \in A_{i_k}^\circ$, $E(\nu_{i_1}(a_1) \dots \nu_{i_n}(a_n)) = 0$ whenever $n \geq 2$ or $n = 1$ and $a = a_1 \in A_2^\circ$.*

Proof of the Claim. The fact the E vanishes on the reduced operators (not in A_1°) is obvious, since $\widetilde{\mathbb{E}}$ satisfies the same property. The only non-trivial property to check is the fact that E is GNS-faithful: indeed, it is not true, in general, that the restriction of a GNS-faithful ucp map to a subalgebra is again GNS-faithful. So suppose that there exists $x \in C$ such that $E(y^*x^*xy) = 0$ for all $y \in C$ and let us show that x is equal to zero. Since $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ is GNS-faithful, it suffices to show that $\widetilde{\mathbb{E}}(y^*x^*xy) = 0$ for all $y \in \widetilde{A}$. By hypothesis, we know that it is true for all $y \in C$. Since $\widetilde{A}$ is the closed linear span of $\widetilde{\pi}(A_1)$ and $\widetilde{\pi}(z)$, for $z \in A_1 \underset{B}{*} \left(A_2 \overset{e}{\underset{B}{*}} A_3\right)$ a reduced operator not in A_1° and since $\widetilde{\pi}(A_1) \cup \widetilde{\pi} \circ \rho(A_2) \subset C$, it suffices to show that $\widetilde{\mathbb{E}}(y^*x^*xy) = 0$ for $y = \widetilde{\pi}(z)$ and $z = z_1 \dots z_n \in A_1 \underset{B}{*} \left(A_2 \overset{e}{\underset{B}{*}} A_3\right)$ a reduced operator with letters z_k alternating from A_1°, $\rho(A_2^\circ)$ and $\rho(A_3^\circ)$ and containing at least one letter in $\rho(A_3^\circ)$. Since one of the z_k is in $\rho(A_3^\circ)$ and $x \in C$ we have, by the property of $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{E}}(y^*(x^*x - \widetilde{\mathbb{E}}(x^*x))y) = 0$. Hence, $\widetilde{\mathbb{E}}(y^*x^*xy) = \widetilde{\mathbb{E}}(y^*\widetilde{\mathbb{E}}(x^*x)y) = \widetilde{\mathbb{E}}(y^*E(x^*x)y) = 0$, since $E(x^*x) = 0$.

End of the proof of the Proposition. Define, for $k = 1, 3$, the unital $*$-homomorphism $\eta_k : A_k \to \widetilde{A}$ by $\eta_1 = \widetilde{\pi}|_{A_1} = \nu_1$ and $\eta_3 = \widetilde{\pi} \circ \rho|_{A_3}$. Using the universal property of the 1-vertex-reduced amalgamated free product one can show, using exactly the same arguments we used to construct the homomorphism ν, that there exists a (necessarily unique) unital faithful $*$-homomorphism $\eta : A_1 \overset{1}{\underset{B}{*}} A_3 \to \widetilde{A}$ such that $\eta \circ \pi_1'|_{A_k} = \eta_k$ for $k = 1, 3$, where $\pi_1' : A_1 \underset{B}{*} A_3 \to A_1 \overset{1}{\underset{B}{*}} A_3$ is the canonical surjection. Note that $\nu(a) = \eta(a)$ for all $a \in A_1$ and $\widetilde{A}$ is generated, as a C*-algebra, by $\nu(A_1 \overset{1}{\underset{B}{*}} A_2) \cup \eta(A_1 \overset{1}{\underset{B}{*}} A_3)$. Since the GNS-faithful ucp map $\widetilde{\mathbb{E}} : \widetilde{A} \to A_1$ obviously satisfies the condition on the reduced operators we may use the universal property of the edge-reduced amalgamated free product to conclude that there exists a canonical $*$-isomorphism

$$\left(A_1 \overset{1}{\underset{B}{*}} A_2\right) \overset{e}{\underset{A_1}{*}} \left(A_1 \overset{1}{\underset{B}{*}} A_3\right) \to \widetilde{A}.$$

□

Using the previous identifications one can prove the following result about completely positive radial multipliers. For $\underline{i} = (i_1, \dots, i_n) \in I$ and $l \in \{1, 2\}$ we define the number

$$n(\underline{i}, l) = |\{s \in \{1, \dots, n\} : i_s = l\}|.$$

Proposition 2.17. *For all $k, l \in \{1, 2\}$ and all $0 < r \leq 1$ there exists a unique ucp map $\varphi_r : A_{v,k} \to A_{v,k}$ such that $\varphi_r(\pi_k(b)) = \pi_k(b)$ for all $b \in B$ and,*

$\varphi_r(\pi_k(a_1 \dots a_n)) = r^{n(\underline{i},l)}\pi_k(a_1 \dots a_n)$ for all $a_1 \dots a_n \in A_f$ reduced with $a_k \in A_{i_k}^{\circ}$ and $\underline{i} = (i_1, \dots, i_n)$.

Proof. We first prove the proposition for $k = 1$. We separate the proof in two cases.

Case 1: $l = 2$. Since π_1 is faithful on A_1, we may and will view $A_1 \subset A_{v,1}$. After this identification, the canonical GNS-faithful ucp map $\mathbb{E}_1 : A_{v,1} \to A_1$ becomes a conditional expectation. Consider the conditional expectation $\tau \otimes \mathrm{id} : C([0, 1]) \otimes B \to B$, where τ is the integral with respect to the normalized Lebesgue measure on $[0, 1]$. We will also view $A_1 \subset A_1 \overset{1}{\underset{B}{*}} (C([0, 1]) \otimes B)$ so that the canonical GNS-faithful ucp map $\widetilde{\mathbb{E}}_1 : A_1 \overset{1}{\underset{B}{*}} (C([0, 1]) \otimes B) \to A_1$ is a conditional expectation. Define $\widetilde{A} = A_{v,1} \overset{e}{\underset{A_1}{*}} \left(A_1 \overset{1}{\underset{B}{*}} (C([0, 1]) \otimes B)\right)$ with respect to the conditional expectations $\mathbb{E}_1$ and $\widetilde{\mathbb{E}}_1$. Since $\mathbb{E}_1$ and $\widetilde{\mathbb{E}}_1$ are GNS-faithful, the edge-reduced and the k-vertex-reduced amalgamated free products coincides for $k = 1, 2$. Hence, we may and will view $A_{v,1} \subset \widetilde{A}$ and we have a canonical GNS-faithful conditional expectation $\widetilde{\mathbb{E}} : \widetilde{A} \to A_{v,1}$. Also, by the first assertion of Proposition 2.16 we have a canonical identification $\widetilde{A} = A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2$, where $\widetilde{A}_2 = A_2 \overset{e}{\underset{B}{*}} (C([0, 1]) \otimes B)$. Let $\widetilde{\rho}_2 : A_2 \underset{B}{*} C([0, 1]) \otimes B \to \widetilde{A}_2$ be the canonical surjection from the full to the edge-reduced amalgamated free product and $\widetilde{\pi} : A_1 \underset{B}{*} \widetilde{A}_2 \to A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2 = \widetilde{A}$ be the canonical surjection from the full to the vertex-reduced amalgamated free product. Fix $t \in \mathbb{R}$ and define the unitary $v_t \in C([0, 1])$ by $v_t(x) = e^{2i\pi tx}$. Let $\rho_t = |\tau(v_t)|^2$ and $u_t = \widetilde{\pi} \circ \widetilde{\rho}_2(v_t \otimes 1_B) \in \widetilde{A}$. Define the unital $*$-homomorphisms $\nu_1 = \widetilde{\pi}|_{A_1} : A_1 \to \widetilde{A}$ and $\nu_2 : \widetilde{A}_2 \to \widetilde{A}$ by $\nu_2(x) = u_t\widetilde{\pi}(x)u_t^*$. Note that ν_1 is faithful. To simplify the notations we put $\widetilde{A}_1 := A_1$.

Claim. *For all $x = x_1 \dots x_n \in A_1 \underset{B}{*} \widetilde{A}_2$ reduced with $x_k \in \widetilde{A}_{i_k}^{\circ}$ and $\underline{i} = (i_1, \dots, i_n) \in I$ one has:*

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1) \dots \nu_{i_n}(x_n)) = \begin{cases} \rho_t^{n(\underline{i},l)}\widetilde{\pi}(x_1 \dots x_n) & \text{if } \widetilde{\pi}(x) \in A_{v,1}, \\ 0 & \text{if } \widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0. \end{cases}$$

Proof of the Claim. Note that $\widetilde{\pi}(x) \in A_{v,1}$ if and only if the letters x_k of x are alternating from A_1° and $\widetilde{\rho}_2(A_2^{\circ})$ and $\widetilde{\mathbb{E}}(\widetilde{\pi}(x)) = 0$ if and only if one of the letters of x comes from $\widetilde{\rho}_2((C([0, 1]) \otimes B)^{\circ})$.

We prove the formula by induction on n. If $n=1$ we have either $x\in A_1^\circ$ in that case $\widetilde{\mathbb{E}}(\nu_1(x)) = \widetilde{\mathbb{E}}(\widetilde{\pi}(x))=\widetilde{\pi}(x)$ or $x\in\widetilde{\rho}_2(\widetilde{A}_2^\circ)$ and

$$\begin{aligned}\widetilde{\mathbb{E}}(\nu_2(x)) &= \widetilde{\mathbb{E}}(u_t\widetilde{\pi}(x)u_t^*)\\ &= \widetilde{\mathbb{E}}((u_t-\tau(v_t))\widetilde{\pi}(x)(u_t^*-\overline{\tau(v_t)}))+\tau(v_t)\widetilde{\mathbb{E}}(\widetilde{\pi}(x)(u_t^*-\overline{\tau(v_t)}))\\ &\quad +\overline{\tau(v_t)}\widetilde{\mathbb{E}}((u_t-\tau(v_t))\widetilde{\pi}(x))+|\tau(v_t)|^2\widetilde{\mathbb{E}}(\widetilde{\pi}(x))\\ &= |\tau(v_t)|^2\widetilde{\mathbb{E}}(\widetilde{\pi}(x))=\rho_t\widetilde{\mathbb{E}}(\widetilde{\pi}(x)).\end{aligned}$$

Hence, $\widetilde{\mathbb{E}}(\nu_2(x))=\begin{cases}\rho_t\widetilde{\pi}(x) & \text{if } \ \widetilde{\pi}(x)\in A_{v,1},\\ 0 & \text{if } \ \widetilde{\mathbb{E}}(\widetilde{\pi}(x))=0.\end{cases}$

This proves the formula for $n=1$. Suppose that the formula holds for a given $n\geq 1$. Let $x=x_1\dots x_{n+1}$ be reduced with $x_k\in\widetilde{A}_{i_k}^\circ$ and define $x'=x_1\dots x_n$ and $z=\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)$. Let $\underline{i}=(i_1,\dots,i_{n+1})$ and $\underline{i}'=(i_1,\dots,i_n)$.

Suppose that $x_{n+1}\in A_1^\circ$. Then $n(\underline{i},2)=n(\underline{i}',2)$ and,

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1}))=\widetilde{\mathbb{E}}(\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)\widetilde{\pi}(x_{n+1}))=\widetilde{\mathbb{E}}(z)\widetilde{\pi}(x_{n+1}).$$

Hence, if $\widetilde{\pi}(x)\in A_{v,1}$ then also $\widetilde{\pi}(x')\in A_{v,1}$ and we have, by the induction hypothesis,

$$\widetilde{\mathbb{E}}(\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1}))=\rho_t^{n(\underline{i}',2)}\widetilde{\pi}(x')\widetilde{\pi}(x_{n+1})=\rho_t^{n(\underline{i},2)}\widetilde{\pi}(x).$$

If $\widetilde{\mathbb{E}}(\widetilde{\pi}(x))=0$ then also $\widetilde{\mathbb{E}}(\widetilde{\pi}(x'))=0$ and we have, by the induction hypothesis, $\widetilde{\mathbb{E}}(z)=0$ so $\widetilde{\mathbb{E}}(\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)\nu_{i_{n+1}}(x_{n+1}))=0$.

Suppose now that $x_{n+1}\in\widetilde{A}_2^\circ$ then $x_n\in A_1^\circ$ and we have,

$$\begin{aligned}\widetilde{\mathbb{E}}(z\nu_{i_{n+1}}(x_{n+1})) &= \widetilde{\mathbb{E}}(zu_t\widetilde{\pi}(x_{n+1})u_t^*)\\ &= \widetilde{\mathbb{E}}(z(u_t-\tau(v_t))\widetilde{\pi}(x_{n+1})(u_t^*-\overline{\tau(v_t)}))+\tau(v_t)\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})(u_t^*-\overline{\tau(v_t)}))\\ &\quad +\overline{\tau(v_t)}\widetilde{\mathbb{E}}(z(u_t-\tau(v_t))\widetilde{\pi}(x_{n+1}))+|\tau(v_t)|^2\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))\\ &= |\tau(v_t)|^2\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))=\rho_t\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1})).\end{aligned}$$

Hence, if $\widetilde{\pi}(x)\in A_{v,1}$ then also $\widetilde{\pi}(x')\in A_{v,1}$ and $x_{n+1}\in A_2^\circ$ so $\widetilde{\pi}(x_{n+1})\in A_{v,1}$ and $n(\underline{i},2)=n(\underline{i}',2)+1$. By the preceding computation and the induction hypothesis we find:

$$\widetilde{\mathbb{E}}(z\nu_{i_{n+1}}(x_{n+1}))=\rho_t\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))=\rho_t\widetilde{\mathbb{E}}(z)\widetilde{\pi}(x_{n+1})=\rho_t\rho_t^{n(\underline{i}',2)}\widetilde{\pi}(x')\widetilde{\pi}(x_{n+1})=\rho_t^{n(\underline{i},2)}\widetilde{\pi}(x).$$

Finally, if $\widetilde{\mathbb{E}}(\widetilde{\pi}(x))=0$, we need to prove that $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))=0$. Note that, since $x_n\in A_1^\circ$, we have $z=\nu_{i_1}(x_1)\dots\nu_{i_{n-1}}(x_{n-1})x_n$. Hence, if $\widetilde{\mathbb{E}}(\widetilde{\pi}(x'))=0$ so by the induction hypothesis we have $\widetilde{\mathbb{E}}(z)=0$, z may be written as a sum of reduced operators, containing at least one letter from $\widetilde{\rho}_2((C([0,1])\otimes B)^\circ)$ and ending with a letter from A_1°. It follows that $z\widetilde{\pi}(x_{n+1})$ may be written as a sum of reduced operators, containing at least one letter from $\widetilde{\rho}_2((C([0,1])\otimes B)^\circ)$. Hence, $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))=0$. If $\widetilde{\mathbb{E}}(\widetilde{\pi}(x))=0$ and $\widetilde{\mathbb{E}}(\widetilde{\pi}(x'))\in A_{v,1}$ then, $x_1,\dots x_n\in A_1^\circ\cup A_2^\circ$ but $\widetilde{\mathbb{E}}(\widetilde{\pi}(x_{n+1}))=0$. It follows that $z=\nu_{i_1}(x_1)\dots\nu_{i_{n-1}}(x_{n-1})x_n$ may be written as a sum of reduced operators ending with a letter from A_1°. Hence, $z\widetilde{\pi}(x_{n+1})$ be be written as a sum of reduced operators containing at least one letter from $\widetilde{\rho}_2((C([0,1])\otimes B)^\circ)$. Hence, $\widetilde{\mathbb{E}}(z\widetilde{\pi}(x_{n+1}))=0$.

End of the proof of the Proposition. By the Claim, $\mathbb{E}_1 \circ \widetilde{\mathbb{E}}(\nu_{i_1}(x_1)\dots\nu_{i_n}(x_n)) = 0$ for all reduced operators $x = x_1 \dots x_n \in A_1 \underset{B}{*} \widetilde{A}_2$ which are not in A_1 and, we obviously have, $\mathbb{E}_1 \circ \widetilde{\mathbb{E}} \circ \nu_1 = \mathrm{id}_{A_1}$. Viewing $\widetilde{A} = A_1 \overset{1}{\underset{B}{*}} \widetilde{A}_2$ and using the universal property of the vertex-reduced amalgamated free product, there exists, for all $t \in \mathbb{R}$, a unique unital $*$-isomorphism $\alpha_t : \widetilde{A} \to \widetilde{A}$ such that $\alpha_t(\widetilde{\pi}(a)) = \widetilde{\pi}(a)$ if $a \in A_1$ and $\alpha_t((\widetilde{\pi}(x)) = u_t\widetilde{\pi}(x)u_t^*$ if $x \in A_2 \overset{e}{\underset{B}{*}} (C([0,1]) \otimes B)$. In particular, it follows from the Claim that $\widetilde{\mathbb{E}} \circ \alpha_t|_{A_{v,1}} : A_{v,1} \to A_{v,1}$, which is a ucp map, satisfies the properties of the map φ_r described in the statement of the proposition, with $r = \rho_t = \left|\frac{\sin(\pi t)}{\pi t}\right|^2$. This concludes the proof.

Case 2: $l = 1$. The proof is similar. This time, the automorphism $\alpha_t : \widetilde{A} \to \widetilde{A}$ is defined, by the universal property, starting with the maps $\nu_1 : A_1 \to \widetilde{A}$ and $\nu_2 : \widetilde{A}_2 \to \widetilde{A}$ defined by $\nu_1(a) = u_t\widetilde{\pi}(a)u_t^*$ and $\nu_2(x) = \widetilde{\pi}(x)$. The remainder of the proof is the same.

The proof for $k = 2$ is the same, using the second assertion of Proposition 2.16. □

3. K-equivalence between the full and reduced amalgamated free products

Let A_1, A_2 be two unital C*-algebras with a common C*-subalgebra $B \subset A_k$, $k = 1, 2$ and denote by A_f the full amalgamated free product.

Let $A := A_1 \overset{v}{\underset{B}{*}} A_2$ be the vertex-reduced amalgamated free product. For $k = 1, 2$, let E_{A_k} (resp. E_B) be the canonical conditional expectation from A to A_k (resp. from A to B). We will denote by the same symbol $\mathcal{A}$ the set of reduced operators viewed in A or in A_f. Recall that the linear span of $\mathcal{A}$ and B is a dense unital $*$-subalgebra of A (resp. A_f).

We denote by $\lambda : A_f \to A$ the canonical surjective unital $*$-homomorphism which is the identity on $\mathcal{A}$. In this section we prove the following result.

Theorem 3.1. $[\lambda] \in \mathrm{KK}(A_f, A)$ *is invertible.*

The following lemma is well known (see [Ve04, Lemma 3.1]). We include a proof for the convenience of the reader.

Lemma 3.2. *Let $n \geq 1$, $a_k \in A_{l_k}^\circ$ for $1 \leq k \leq n$, and $a = a_1 \dots a_n \in A$ a reduced word. One has*

$$E_{A_k}(a^*a) = E_B(a^*a) \quad \text{whenever} \quad l_n \neq k.$$

Proof. We prove it for $k = 1$ by induction on n. The proof for $k = 2$ is the same.

It's obvious for $n = 1$. Suppose that $n \geq 2$, define $b = E_B(a_1^*a_1)^{\frac{1}{2}}$, $x = (ba_2)\dots a_n$. One has:

$$E_{A_1}(a^*a) = E_{A_1}(a_n^* \dots a_1^*a_1 \dots a_n) = E_{A_1}(a_n^* \dots a_2^* E_B(a_1^*a_1)a_2 \dots a_n) = E_{A_1}(x^*x) = E_B(x^*x),$$

where we applied the induction hypothesis to get the last equality. Since the same computation gives $E_B(a^*a) = E_B(x^*x)$, this concludes the proof. □

We denote by (H_k, π_k, ξ_k) (resp. (K, ρ, η)) the GNS construction of E_{A_k} (resp. E_B). We may and will assume that $A \subset \mathcal{L}_{A_k}(H_k)$ and $\pi_k = \mathrm{id}$.

Observe that the Hilbert A_k-module $\xi_k.A_k \subset H_k$ is orthogonally complemented i.e. $H_k = \xi_k.A_k \oplus H_k^\circ$, as Hilbert A_k-modules, where H_k° is the closure of $\{a\xi_k : a \in A,\ E_{A_k}(a) = 0\}$.

We now define a partial isometry $F_k \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ in the following way. First we put $F_k(\xi_k.a) = 0$ for all $a \in A_k$. Then, it follows from Lemma 3.2 that we can define an isometry $F_k : H_k^\circ \to K \underset{B}{\otimes} A_k$ by the following formula:

$$F_k(a_1 \dots a_n \xi_k) = \begin{cases} \rho(a_1 \dots a_n)\eta \underset{B}{\otimes} 1 & \text{if} \quad l_n \neq k \\ \rho(a_1 \dots a_{n-1})\eta \underset{B}{\otimes} a_n & \text{if} \quad l_n = k \end{cases} \quad \text{for all} \quad a_1 \dots a_n \in A \text{ a reduced operator.}$$

Let $q_k \in \mathcal{L}_B(K)$ is the orthogonal projection onto words which does no end with k i.e. onto the complemented B submodule $\bigoplus_{\underline{i}=(i_1,\dots,i_n)\in I, i_n\neq k} K_{\underline{i}}^\circ$ and note that F_k defines to a bounded linear map from H_k to $K \underset{B}{\otimes} A_k$ with image the complemented sub A_k-module $(q_k \otimes 1)K \underset{B}{\otimes} A_k$. Hence, $F_k \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ is a well defined partial isometry such that $1 - F_k^* F_k$ is the orthogonal projection onto $\xi_k.A_k$, and $F_k F_k^* = q_k \otimes_B 1$. Note also that the image of $1 - F_k F_k^*$ is

$$((1-q_k)\otimes 1)K \underset{B}{\otimes} A_k = (\eta \otimes 1).A_k \oplus \overline{\text{Span}}\{\rho(a_1 \dots a_n)\eta \otimes 1 \,:\, a = a_1 \dots a_n \in A \text{ reduced with } l_n = k\}.A_k.$$

We will denote in the sequel q_0 the orthogonal projection of K onto $\eta.B$. It is clear that $1 = q_1 + q_2 + q_0$ and that these projections are pairwise orthogonal. Define also $\overline{F}_k = F_k + \theta_{\eta\otimes_B 1, \xi_k}$. It is again clear that $\overline{F}_k$ is an isometry and $\overline{F}_k \overline{F}_k^* = q_k + q_0 = 1 - q_l$ for $k \neq l$.

Lemma 3.3. *For $k = 1, 2$ the following holds.*

1. *(1) $\rho(a)F_k = F_k a \in \mathcal{L}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all $a \in A_k$.*
2. *(2) $\text{Im}(\rho(a)F_k - F_k a) \subset (\rho(a)\eta \underset{B}{\otimes} 1).A_k \oplus (\eta \underset{B}{\otimes} 1).A_k$ for all $a \in A_l^\circ$ with $l \neq k$.*
3. *(3) $\rho(x)F_k - F_k x \in \mathcal{K}_{A_k}(H_k, K \underset{B}{\otimes} A_k)$ for all $x \in A$.*
4. *(4) $\rho(a)\overline{F}_k = \overline{F}_k a \ \forall a \in A_l$ with $l \neq k$ and $\rho(x)\overline{F}_k - \overline{F}_k x \in \mathcal{K}_{A_k}(H_k, K \underset{B}{\otimes} A_k) \ \forall x \in A$.*

Proof. We prove the lemma for $k = 1$. The proof for $k = 2$ is the same.

(1). When $a \in B$ the commutation is obvious hence we may and will assume that $a \in A_1^\circ$. One has $F_1 a \xi_1 = 0 = \rho(a)F_1\xi_1$. Let now $n \geq 1$ and $x = a_1 \dots a_n \in A$, $a_k \in A_{l_k}^\circ$, be a reduced operator with $E_{A_1}(x) = 0$. It suffices to show that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$. If $n = 1$ we must have $x \in A_2^\circ$ and $F_1 a x \xi_1 = \rho(ax)\eta \otimes 1 = \rho(a)F_1 x\xi_1$. Suppose that $n \geq 2$. If $l_1 = 2$ then ax is reduced and ends with a letter from $A_{l_n}^\circ$. It follows that $F_1 a x \xi_2 = \rho(a)F_1 x \xi_2$. If $l_1 = 1$ then we can write $ax = (aa_1)^\circ a_2 \dots a_n + E_B(aa_1)a_2 \dots a_n$. Since $a_2 \dots a_n$ is reduced and ends with l_n we find again that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$.

(2). Let $a \in A_2^\circ$ and put $X_a = (\rho(a)\eta \underset{B}{\otimes} 1).A_k \oplus (\eta \underset{B}{\otimes} 1).A_k$. We have $F_1 a \xi_1 = \rho(a)\eta \otimes 1$ and $\rho(a)F_1\xi_1 = 0$ hence, $(\rho(a)F_1 - F_1 a)\xi_1 = -\rho(a)\eta \otimes 1 \in X_a$. Let now $n \geq 1$ and $x = a_1 \dots a_n \in A$, $a_k \in A_{l_k}^\circ$, be a reduced operator with $E_{A_1}(x) = 0$. If $n = 1$ we must have $x \in A_2^\circ$. It follows that $F_1 a x \xi_1 = F_1(ax)^\circ \xi_1 + F_1 E_B(ax)\xi_1 = \rho((ax)^\circ)\eta \otimes 1$ and $\rho(a)F_1 x \xi_1 = \rho(ax)\eta \otimes 1$. Hence, $(\rho(a)F_1 - F_1 a)x\xi_1 = E_B(ax)\eta \otimes 1 = (\eta \otimes 1).E_B(ax) \in X_a$. If $n \geq 2$, arguing as in the proof of (1), we see that $F_1 a x \xi_1 = \rho(a)F_1 x \xi_1$. Hence, $\text{Im}(\rho(a)F_k - F_k a) \subset X_a$.

(3). Since A is generated, as a C*-algebra, by A_1 and A_2° and, by Assertion (1), $\rho(a)F_1 - F_1 a = 0$ if $a \in A_1$ we only have to consider an element $a \in A_2^\circ$. Define then θ_a and ψ_a in $\mathcal{K}(H_1, K \underset{B}{\otimes} A_1)$ as

$\theta_a(\zeta) = -\rho(a)\eta \otimes 1 < \xi_1, \zeta >_{H_1}$ and $\psi_a(\zeta) = \eta \otimes 1 < a^*\xi_1, \zeta >_{H_1}$. Using the computation of (2), we have that $\rho(a)F_1 - F_1 a$ is the compact operator $\theta_a + \psi_a$. Indeed if $\zeta = \xi_1$, $\theta_a(\zeta) = -\rho(a)\eta \otimes 1$ and $\psi_a(\zeta) = 0$, if $\zeta = x\xi_1$ with $x \in A_2^\circ$, then $\theta_a(\zeta) = 0$ and $\psi_a(\zeta) = \eta \otimes 1 E_{A_1}(ax) = \eta \otimes 1 E_B(ax)$ as ax is in A_2 and lastly if $\zeta = x\xi_1$ with x a reduced word of length at least 2 both $\theta_a(\zeta)$ and $\psi_a(\zeta)$ vanish.

(4). The second part is obvious in view of (3) as $\overline{F}_1$ is a compact perturbation of F_1, so let's concentrate on the exact commutation. Let $a \in A_2^\circ$. Clearly $\overline{F}_1 a\xi_1 = F_1 a\xi_1 = \rho(a)\eta \otimes 1$ and $\rho(a)\overline{F}_1\xi_1 = \rho(a)\eta \otimes 1$. Let now $n \geq 1$ and $x = a_1 \dots a_n \in A$, $a_k \in A_{l_k}^\circ$, be a reduced operator with $E_{A_1}(x) = 0$. If $n = 1$ we must have $x \in A_2^\circ$. It follows that $\overline{F}_1 ax\xi_1 = F_1(ax)^\circ\xi_1 + \theta_{\eta\otimes 1,\xi_1} E_B(ax)\xi_1 = \rho((ax)^\circ)\eta \otimes 1 + E_B(ax)\eta \otimes 1$ and $\rho(a)\overline{F}_1 x\xi_1 = \rho(a)F_1 x\xi_1 = \rho(ax)\eta \otimes 1$. If $n \geq 2$, arguing as in the proof of (1), we see that $\overline{F}_1 ax\xi_1 = F_1 ax\xi_1 = \rho(a)F_1 x\xi_1 = \rho(a)\overline{F}_1 x\xi_1$. □

We define the following Hilbert A_f-modules:

$$H_m = H_1 \underset{A_1}{\otimes} A_f \oplus H_2 \underset{A_2}{\otimes} A_f \quad \text{and} \quad K_m = K \underset{B}{\otimes} A_f = \left(K \underset{B}{\otimes} A_k\right) \underset{A_k}{\otimes} A_f,$$

with the canonical representations $\pi : A \to \mathcal{L}_{A_f}(H_m)$, $\pi(x) = x \underset{A_1}{\otimes} 1_{A_f} \oplus x \underset{A_2}{\otimes} 1_{A_f}$ and $\bar{\rho} : A \to \mathcal{L}_{A_f}(K_m)$, $\bar{\rho}(x) = \rho(x) \underset{B}{\otimes} 1_{A_f}$. We consider, for $k = 1, 2$, the partial isometry

$$F_k \underset{A_k}{\otimes} 1_{A_f} \in \mathcal{L}_{A_f}(H_k \underset{A_k}{\otimes} A_f, (K \underset{B}{\otimes} A_k) \underset{A_k}{\otimes} A_f).$$

Observe that $F_1 \underset{A_1}{\otimes} 1_{A_f}$ and $F_2 \underset{A_2}{\otimes} 1_{A_f}$ have orthogonal images. Indeed, the image of $F_k \underset{A_k}{\otimes} 1_{A_f}$ is the closed linear span of $\{\rho(a_1 \dots a_n)\eta \underset{B}{\otimes} y \,:\, y \in A_f$ and $a_1 \dots a_n \in A$ reduced with $a_n \notin A_k^\circ\}$. Hence the operator $F \in \mathcal{L}_{A_f}(H_m, K_m)$ defined by $F = F_1 \underset{A_1}{\otimes} 1_{A_f} \oplus F_2 \underset{A_2}{\otimes} 1_{A_f}$ is a partial isometry such that $1 - FF^*$ is the orthogonal projection onto $(\eta \underset{B}{\otimes} 1_{A_f}).A_f$ and $1 - F^*F$ is the orthogonal projection onto $(\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f$. In particular $1 - F^*F$ and $1 - FF^*$ belongs to $\mathcal{K}_{A_f}(H_m)$ and $\mathcal{K}_{A_f}(K_m)$ respectively. Moreover, it follows from lemma 3.3 that $F\pi(x) - \bar{\rho}(x)F \in \mathcal{K}_{A_f}(H_m, K_m)$ for all $x \in A$. Hence, we get an element $\alpha = [(H_m \oplus K_m, \pi \oplus \bar{\rho}, F)] \in \mathrm{KK}(A, A_f)$.

To prove Theorem 3.1 it suffices to prove that $\alpha \underset{A_f}{\otimes} [\lambda] = [\mathrm{id}_A]$ in $\mathrm{KK}(A, A)$ and $[\lambda] \underset{A}{\otimes} \alpha = [\mathrm{id}_{A_f}]$ in $\mathrm{KK}(A_f, A_f)$. We prove the easy part in the next proposition.

Proposition 3.4. *One has* $[\lambda] \underset{A}{\otimes} \alpha = [id_{A_f}]$ *in* $\mathrm{KK}(A_f, A_f)$.

Proof. Observe that $[\lambda] \underset{A}{\otimes} \alpha = [(H_m \oplus K_m, \pi_m \oplus \rho_m, F)]$ where $\pi_m = \pi \circ \lambda : A_f \to \mathcal{L}_{A_f}(H_m)$ and $\rho_m = \bar{\rho} \circ \lambda : A_f \to \mathcal{L}_{A_f}(K_m)$. Hence, by compact perturbation, $[\lambda] \underset{A}{\otimes} \alpha - [\mathrm{id}_{A_f}]$ is represented by the Kasparov triple $(H_m \oplus \widetilde{K}_m, \pi_m \oplus \widetilde{\rho}_m, \widetilde{F})$, where $\widetilde{K}_m = K_m \oplus A_f$ and $\widetilde{\rho}_m(x) = \rho_m(x) \oplus x$, where we view $A_f = \mathcal{L}_{A_f}(A_f)$ by left multiplication. Finally, $\widetilde{F} \in \mathcal{L}_{A_f}(H_m, \widetilde{K}_m)$ is the unitary defined by

$$\widetilde{F}(\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \eta \underset{B}{\otimes} 1_{A_f}, \quad \widetilde{F}(\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = 1_{A_f} \quad \text{and},$$

$$\widetilde{F}(\xi) = F(\xi) \text{ for all } \xi \in H_m \ominus \left((\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f \right).$$

We collect some computations in the following claim.

Claim. *Let $v \in \mathcal{L}_{A_f}(H_m)$ be the self-adjoint unitary defined by the identity on $H_m \ominus ((\xi_1 \underset{A_1}{\otimes} 1_{A_f}).A_f \oplus (\xi_2 \underset{A_2}{\otimes} 1_{A_f}).A_f)$ and $v(\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \xi_2 \underset{A_2}{\otimes} 1_{A_f}$, $v(\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = \xi_1 \underset{A_1}{\otimes} 1_{A_f}$. One has:*

1. *$\widetilde{F}^*\widetilde{\rho}_m(b)\widetilde{F} = \pi_m(b)$ and $v^*\pi_m(b)v = \pi_m(b)$ for all $b \in B$.*
2. *$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F} = v^*\pi_m(a)v$ for all $a \in A_1$.*
3. *$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F} = \pi_m(a)$ for all $a \in A_2$.*

*Proof of the claim.*The proof of (1) is obvious and we leave it to the reader.

(2). By (1), it suffices to prove (2) for $a \in A_1^\circ$. Let $a \in A_1^\circ$. On the one hand:

$$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}\xi_1 \underset{A_1}{\otimes} 1_{A_f} = \widetilde{F}^*(\rho(a)\eta \underset{B}{\otimes} 1_{A_f}) = a\xi_2 \underset{A_2}{\otimes} 1_{A_f} \quad \text{and} \quad \widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}\xi_2 \underset{A_2}{\otimes} 1_{A_f} = \widetilde{F}^*(a) = \xi_2 \underset{A_2}{\otimes} a.$$

On the other hand:

$$v^*\pi_m(a)v\xi_1 \underset{A_1}{\otimes} 1_{A_f} = v^*(a\xi_2 \underset{A_2}{\otimes} 1_{A_f}) = a\xi_2 \underset{A_2}{\otimes} 1_{A_f} \quad \text{and} \quad v^*\pi_m(a)v\xi_2 \underset{A_2}{\otimes} 1_{A_f} = v^*(a\xi_1 \underset{A_1}{\otimes} 1_{A_f}) = \xi_2 \underset{A_2}{\otimes} a.$$

Let now $x = a_1 \dots a_n \in A$ be a reduced operator with $a_k \in A_{l_k}^\circ$. We prove by induction on n that $\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} = v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f}$ for all $k \in \{1,2\}$. Suppose that $n = 1$ so $x \in A_1^\circ \cup A_2^\circ$ and let $k \in \{1,2\}$ such that $x \notin A_k^\circ$ (the case $x \in A_k^\circ$ has been done before). We have:

$$\widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} = \widetilde{F}^*(\rho(ax)\eta \underset{B}{\otimes} 1_{A_f}) = \begin{cases} (ax)^\circ\xi_2 \underset{A_2}{\otimes} 1_{A_f} + \xi_1 \underset{A_1}{\otimes} E_B(ax) & \text{if} \quad x \in A_1^\circ, \\ ax\xi_1 \underset{A_1}{\otimes} 1_{A_f} & \text{if} \quad x \in A_2^\circ. \end{cases}$$

On the other hand we have:

$$v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f} = v^*(ax\xi_k \underset{A_k}{\otimes} 1_{A_f}) = \begin{cases} (ax)^\circ\xi_2 \underset{A_2}{\otimes} 1_{A_f} + \xi_1 \underset{A_1}{\otimes} E_B(ax) & \text{if} \quad x \in A_1^\circ \ (k=2), \\ ax\xi_1 \underset{A_1}{\otimes} 1_{A_f} & \text{if} \quad x \in A_2^\circ \ (k=1). \end{cases}$$

Finally, suppose that $n \geq 2$ and the formula holds for $n-1$. Write $ax = y + z$, where, if $l_1 = 1$, $y = (aa_1)^\circ a_2 \dots a_n$ and $z = E_B(aa_1)a_2 \dots a_n$ and, if $l_1 = 2$, $y = ax$ and $z = 0$. Observe that, in both cases, y is a reduced operator ending with a letter from $A_{l_n}^\circ$ and z is either 0 or a reduced operator ending with a letter from $A_{l_n}^\circ$. By the induction hypothesis, we may and will assume that $k \neq l_n$. We have:

$$\begin{aligned} \widetilde{F}^*\widetilde{\rho}_m(a)\widetilde{F}x\xi_k \underset{A_k}{\otimes} 1_{A_f} &= \widetilde{F}^*(\rho(ax)\eta \underset{B}{\otimes} 1_{A_f}) = \widetilde{F}^*(\rho(y)\eta \underset{B}{\otimes} 1_{A_f}) + \widetilde{F}^*(\rho(z)\eta \underset{B}{\otimes} 1_{A_f}) \\ &= y\xi_k \underset{A_k}{\otimes} 1_{A_f} + z\xi_k \underset{A_k}{\otimes} 1_{A_f} = ax\xi_k \underset{A_k}{\otimes} 1_{A_f}. \end{aligned}$$

Moreover,

$$\begin{aligned} v^*\pi_m(a)vx\xi_k \underset{A_k}{\otimes} 1_{A_f} &= v^*(ax\xi_k \underset{A_k}{\otimes} 1_{A_f}) = v^*(y\xi_k \underset{A_k}{\otimes} 1_{A_f}) + v^*(z\xi_k \underset{A_k}{\otimes} 1_{A_f}) \\ &= y\xi_k \underset{A_k}{\otimes} 1_{A_f} + z\xi_k \underset{A_k}{\otimes} 1_{A_f} = ax\xi_k \underset{A_k}{\otimes} 1_{A_f}. \end{aligned}$$

The proof of (3) is similar. □

End of the proof of Proposition 3.4. Let $t \in \mathbb{R}$ and define $v_t = \cos(t) + iv\sin(t) \in \mathcal{L}_{A_f}(H_m)$. Since $v = v^*$ is unitary, v_t is a unitary for all $t \in \mathbb{R}$. Moreover, assertion (1) of the Claim implies that $v_t\pi_m(b)v_t^* = \pi_m(b)$ for all $b \in B$. It follows from the universal property of A_f that there exists a unique unital $*$-homomorphism $\pi_t : A_f \to \mathcal{L}_{A_f}(H_m)$ such that:

$$\pi_t(a) = \begin{cases} v_t^*\pi_m(a)v_t & \text{if} \quad a \in A_1, \\ \pi_m(a) & \text{if} \quad a \in A_2. \end{cases}$$

Then the triple $\alpha_t = (H_m \oplus \widetilde{K}_m, \pi_t \oplus \widetilde{\rho}_m, \widetilde{F})$ gives a homotopy between α_0 which represents $[\lambda] \underset{A}{\otimes} \alpha - [\mathrm{id}_{A_f}]$ and $\alpha_{\frac{\pi}{2}}$ which is degenerated by the claim. □

We finish the proof of Theorem 3.1 in the next proposition.

Proposition 3.5. *One has* $\alpha \underset{A_f}{\otimes} [\lambda] = [id_A]$ *in* $\mathrm{KK}(A, A)$.

Proof. Observe that $\alpha \underset{A_f}{\otimes} [\lambda] = [(H_r \oplus K_r, \pi_r \oplus \rho_r, F_r)]$ where

$$H_r = H_m \underset{\lambda}{\otimes} A = H_1 \underset{A_1}{\otimes} A \oplus H_2 \underset{A_2}{\otimes} A \quad \text{and} \quad K_r = K_m \underset{\lambda}{\otimes} A = K \underset{B}{\otimes} A = \left(K \underset{B}{\otimes} A_k\right) \underset{A_k}{\otimes} A,$$

with the canonical representations $\pi_r : A \to \mathcal{L}_A(H_r)$, $\pi_r(x) = \pi(x) \underset{\lambda}{\otimes} 1 = x \underset{A_1}{\otimes} 1_A \oplus x \underset{A_2}{\otimes} 1_A$ and $\rho_r : A \to \mathcal{L}_A(K_r)$, $\rho_r(x) = \bar{\rho}(x) \underset{\lambda}{\otimes} 1 = \rho(x) \underset{B}{\otimes} 1_A$ and with the operator $F_r = F \underset{\lambda}{\otimes} 1 \in \mathcal{L}_A(H_r, K_r)$. Hence, $\alpha \underset{A_f}{\otimes} [\lambda] - [\mathrm{id}_A]$ is represented by the Kasparov triple $(H_r \oplus \widetilde{K}_r, \pi_r \oplus \widetilde{\rho}_r, \widetilde{F}_r)$, where $\widetilde{K}_r = K_r \oplus A$ and $\widetilde{\rho}_r(x) = \rho_r(x) \oplus x$, where we view $A = \mathcal{L}_A(A)$ by left multiplication. Finally, $\widetilde{F}_r \in \mathcal{L}_A(H_r, \widetilde{K}_r)$ is the unitary defined by

$$\widetilde{F}_r(\xi_1 \underset{A_1}{\otimes} 1_A) = \eta \underset{B}{\otimes} 1_A, \quad \widetilde{F}_r(\xi_2 \underset{A_2}{\otimes} 1_A) = 1_A \quad \text{and,}$$

$$\widetilde{F}(\xi) = F(\xi) \text{ for all } \xi \in H_r \ominus \left((\xi_1 \underset{A_1}{\otimes} 1_A).A \oplus (\xi_2 \underset{A_2}{\otimes} 1_A).A\right).$$

The claim in the proof of Proposition 3.4 implies the following claim.

Claim. *Let* $u \in \mathcal{L}_A(H_r)$ *be the self-adjoint unitary defined by the identity on* $H_r \ominus ((\xi_1 \underset{A_1}{\otimes} 1_A).A \oplus (\xi_2 \underset{A_2}{\otimes} 1_A).A)$ *and* $u(\xi_1 \underset{A_1}{\otimes} 1_A) = \xi_2 \underset{A_2}{\otimes} 1_A$, $u(\xi_2 \underset{A_2}{\otimes} 1_A) = \xi_1 \underset{A_1}{\otimes} 1_A$. *One has:*

1. $\widetilde{F}^*\widetilde{\rho}_r(b)\widetilde{F} = \pi_r(b)$ *and* $u^*\pi_r(b)u = \pi_r(b)$ *for all* $b \in B$.
2. $\widetilde{F}^*\widetilde{\rho}_r(a)\widetilde{F} = u^*\pi_r(a)u$ *for all* $a \in A_1$.
3. $\widetilde{F}^*\widetilde{\rho}_r(a)\widetilde{F} = \pi_r(a)$ *for all* $a \in A_2$.

Let $t \in \mathbb{R}$ and define the unitary $u_t = \cos(t) + iu\sin(t) \in \mathcal{L}_A(H_r)$. Assertion (1) of the Claim implies that $u_t^*\pi_r(b)u_t = \pi_r(b)$ for all $b \in B$. By the universal property of full amalgamated free products, for all $t \in \mathbb{R}$, there exists a unique unital $*$-homomorphism $\pi_t : A_f \to \mathcal{L}_A(H_r)$ such that:

$$\pi_t(a) = \begin{cases} u_t^*\pi_r(a)u_t & \text{if} \quad a \in A_1, \\ \pi_r(a) & \text{if} \quad a \in A_2. \end{cases}$$

Arguing as in the end of the proof of Proposition 3.4, we see that it suffices to show that, for all $t \in [0, \frac{\pi}{2}]$, π_t factorizes through A i.e. $\ker(\lambda) \subset \ker(\pi_t)$. Since it is obvious for $t = 0$, we only need to show that $\ker(\lambda) \subset \ker(\pi_t)$ for all $t \in]0, \frac{\pi}{2}]$. To do that, we need the following claim.

Claim. *For all $t \in \mathbb{R}$ and all $a = a_1 \dots a_n \in \mathcal{A}$ a reduced operator with $a_k \in A_{l_k}^\circ$ one has*

(1) $\pi_t(a)u_t^*(\xi_2 \underset{A_2}{\otimes} 1_A) = e^{-it}(a\xi_2 \underset{A_2}{\otimes} 1_A)$ *if* $l_n = 1$ *and* $\pi_t(a)(\xi_1 \underset{A_1}{\otimes} 1_A) = a\xi_1 \underset{A_1}{\otimes} 1_A$ *if* $l_n = 2$.

(2) $\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle = \sin^{2k}(t)a$ *where* $k = \begin{cases} \frac{n}{2} & \textit{if } n \textit{ is even,} \\ \frac{n-1}{2} & \textit{if } n \textit{ is odd and } l_n = 1, \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd and } l_n = 2. \end{cases}$

(3) $\langle \xi_2 \underset{A_2}{\otimes} 1_A, \pi_t(a)\xi_2 \underset{A_2}{\otimes} 1_A\rangle = \sin^{2k}(t)a$ *where* $k = \begin{cases} \frac{n}{2} & \textit{if } n \textit{ is even,} \\ \frac{n+1}{2} & \textit{if } n \textit{ is odd and } l_n = 1, \\ \frac{n-1}{2} & \textit{if } n \textit{ is odd and } l_n = 2. \end{cases}$

Proof of the Claim. (1) is obvious by induction on n once observed that $u_t\xi = e^{it}\xi$ (and $u_t^*\xi = e^{-it}\xi$) for all $\xi \in H_r \ominus (\xi_1 \underset{A_1}{\otimes} 1_A.A \oplus \xi_2 \underset{A_2}{\otimes} 1_A.A)$.

(2). Define, for $a_1 \dots a_n \in \mathcal{A}$, $F(a_1, \dots, a_n) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle$. First suppose that $a \in A_1^\circ$ then $F(a) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), u_t^*\pi_r(a)(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle = \langle \xi_1 \underset{A_1}{\otimes} 1_A, \xi_1 \underset{A_1}{\otimes} a\rangle = a$. Now, let $a = a_1 \dots a_n \in \mathcal{A}$ with $n \geq 2$ and $l_n = 1$. We have:

$$F(a_1, \dots, a_n) = \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(a_1 \dots a_{n-1})u_t^*(\xi_1 \underset{A_1}{\otimes} a_n)\rangle = F(a_1, \dots, a_{n-1})a_n.$$

Hence, it suffices to show the formula for $l_n = 2$. Suppose $a \in A_2^\circ$, we have:

$$\begin{aligned} F(a) &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_r(a)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle \cos(t)\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} 1_A, \cos(t)a\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a\rangle = \sin^2(t)a. \end{aligned}$$

Now suppose $a_1a_2 \in \mathcal{A}$, with $l_2 = 2$, $l_1 = 1$. We have:

$$\begin{aligned} F(a_1, a_2) &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \pi_r(a_1)u_t\pi_r(a_2)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \pi_r(a_1)u_t(\cos(t)a_2\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a_2)\rangle \\ &= \langle \xi_1 \underset{A_1}{\otimes} 1_A, \cos(t)e^{it}a_1a_2\xi_1 \underset{A_1}{\otimes} 1_A - i\cos(t)\sin(t)a_1\xi_2 \underset{A_2}{\otimes} a_2 + \sin^2(t)\xi_1 \underset{A_1}{\otimes} a_1a_2\rangle \\ &= \sin^2(t)a_1a_2. \end{aligned}$$

Finally, suppose that $n \geq 3$ and $a_1 \dots a_n \in \mathcal{A}$ with $l_n = 2$. Define $x = a_1 \dots a_{n-2}$. We have

$$\begin{aligned} F(a_1, \dots, a_n) &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t\pi_r(a_n)u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle \\ &= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\pi_r(a_{n-1})u_t(\cos(t)a_n\xi_1 \underset{A_1}{\otimes} 1_A - i\sin(t)\xi_2 \underset{A_2}{\otimes} a_n)\rangle \end{aligned}$$

$$= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*(\cos(t)e^{it}a_{n-1}a_n\xi_1 \underset{A_1}{\otimes} 1_A - i\cos(t)\sin(t)a_{n-1}\xi_2 \underset{A_2}{\otimes} a_n + \sin^2(t)\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n)\rangle$$
$$= \langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \cos(t)a_1\dots a_n\xi_1 \underset{A_1}{\otimes} 1_A - ie^{-it}\cos(t)\sin(t)a_1\dots a_{n-1}\xi_2 \underset{A_2}{\otimes} a_n\rangle$$
$$+\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \sin^2(t)\pi_t(x)u_t^*\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n)\rangle.$$

Hence we find:

$$F(a_1,\dots,a_n) = \sin^2(t)\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), \pi_t(x)u_t^*\xi_1 \underset{A_1}{\otimes} a_{n-1}a_n)\rangle = \sin^2(t)F(a_1,\dots,a_{n-2})a_{n-1}a_n.$$

The result now follows by an obvious induction on n. The proof of (3) is similar. □

End of the proof of Proposition 3.5. Fix $t \in]0, \frac{\pi}{2}]$ and let A_t be the C*-subalgebra of $\mathcal{L}_A(H_r)$ generated by $\pi_t(A_1) \cup \pi_t(A_2)$. Hence, $\pi_t : A_f \to A_t$ is surjective. Consider the ucp map $\varphi_t : A_t \to A$ defined by $\varphi_t(x) = \frac{1}{2}\left(\langle u_t^*(\xi_1 \underset{A_1}{\otimes} 1_A), xu_t^*(\xi_1 \underset{A_1}{\otimes} 1_A)\rangle + \langle \xi_2 \underset{A_2}{\otimes} 1_A, x\xi_2 \underset{A_2}{\otimes} 1_A\rangle\right)$ and note that φ_t is GNS faithful. Indeed, let $x \in A_t$ such that $\varphi_t(y^*x^*xy) = 0$ for all $y \in A_t$. Then $L \subset \ker(x)$ where,

$$\begin{aligned} L &= \overline{\text{Span}}\left(A_tu_t^*(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_t(\xi_2 \underset{A_2}{\otimes} 1_A).A\right) = \overline{\text{Span}}\left(A_t(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_t(\xi_2 \underset{A_2}{\otimes} 1_A).A\right) \\ &= \overline{\text{Span}}\left(A_t(\xi_1 \underset{A_1}{\otimes} 1_A).A \cup A_tu_t^*(\xi_2 \underset{A_2}{\otimes} 1_A).A\right) = H_r, \end{aligned}$$

where we used Assertion (1) of the Claim for the last equality. Hence $x = 0$. Let $A_{v,k}$ for $k = 1, 2$ be the k-vertex-reduced free product and call i_k the natural inclusion of A in $A_{v,k}$ and $\pi_k = i_k \circ \lambda$ the natural map from A_f to $A_{v,k}$. Clearly $||x||_A = \max(||i_1(x)||, ||, i_2(x)||)$ for any x in the vertex-reduced free product A. From the Assertions (2) and (3) of the Claim and Proposition 2.17 with $r = \sin^2(t) > 0$ we deduced that for any $k = 1, 2$ there exists two ucp maps ψ_1^k and ψ_2^k from $A_{v,k}$ to itself such that $i_k(\varphi_t(\pi_t(a))) = \frac{1}{2}(\psi_1^k(\pi_k(a)) + \psi_2^k(\pi_k(a)))$ for all $a \in A_f$. Therefore $||\varphi_t(\pi_t(a))||_A \le \max(||\pi_1(a)||, ||\pi_2(a)||) = ||\lambda(a)||$ for all $a \in A_f$. Let us show that $\ker(\lambda) \subset \ker(\pi_t)$. Let $x \in \ker(\lambda)$. Then, for all $y \in A_f$ we have $\lambda(y^*x^*xy) = 0$. Therefore $\varphi_t \circ \pi_t(y^*x^*xy) = 0$ for all $y \in A_f$. Since π_t is surjective we deduce that $\varphi_t(y^*\pi_t(x)^*\pi_t(x)y) = 0$ for all $y \in A_t$. Using that φ_t is GNS faithful we deduce that $\pi_t(x) = 0$. □

We obtain the following obvious Corollary of Theorem 3.1 and Corollary 2.9.

Corollary 3.6 ([Cu82]). *If we have conditional expectations $E_k : A_k \to B$ which are also unital $*$-homomorphism, then the canonical surjection $A_1 \underset{B}{*} A_2 \to A_1 \oplus_B A_2$ is K-invertible.*

4. A long exact sequence in KK-theory for full amalgamated free products

Let A_1 and A_2 two unital C*-algebras with a common unital C*-subalgebra B. We will denote by i_l the inclusion of B in A_l for $l = 1, 2$. The algebra A_f is the full amalgamated free product. To simplify notation we will denote by S the algebra $C_0(]-1, 1[)$.

Let D be the subalgebra of $S \otimes A_f$ consisting of functions f such that $f(]-1, 0[) \subset A_1$, $f(]0, 1[) \subset A_2$ and $f(0) \in B$. This algebra is of course isomorphic to the cone of $i_1 \oplus i_2$ from B to $A_1 \oplus A_2$. We call j the inclusion of D in the suspension of A_f.

Theorem 4.1. *Suppose that there exist unital conditional expectations from A_l to B for $l = 1, 2$, then the map j, seen as an element $[j]$ of $KK^0(D, S \otimes A_f)$, is invertible.*

The proof of this result will be done in several steps. We will start with the construction of an element x of $KK^1(A_f, D)$. As $KK^1(A_f, D)$ is isomorphic to $KK^0(S\otimes A_f, D)$ this will produce a candidate y for the inverse of j. The proof that $y \otimes_D [j]$ is the identity of the suspension of A_f in $KK^0(A_f, A_f)$ will use 3.4. Finally the proof that $[j] \otimes_{S\otimes A_f} y$ is the identity of D in $KK^0(D, D)$ will be done indirectly by using a short exact sequence for D.

4.1. **An inverse in KK-theory.** In order to present the inverse, we need some additional notations and preliminaries. Let κ_1 be the inclusion of $C_0(]-1, 0[; A_1)$ in D and κ_2 the inclusion of $C_0(]0, 1[; A_2)$ in D. There is also κ_0 the obvious map from $S \otimes B$ in D. As K of the preceding section is a B-module, we can define

$K_0 = (K\otimes S)\otimes_{\kappa_0} D$, $K_1 = (K\otimes_{i_1} A_1\otimes C_0(]-1, 0[))\otimes_{\kappa_1} D$ and $K_2 = (K\otimes_{i_2} A_2\otimes C_0(]0, 1[))\otimes_{\kappa_2} D$.

If one defines I_l as the images of κ_l in D for $l = 1, 2$, it is clear that these are ideals in D.

Lemma 4.2. *K_l is isomorphic to $\overline{K_0.I_l}$ for $l = 1, 2$ as D Hilbert module.*

Proof. We will show the statement for $l = 1$. Indeed as $I_1 = \overline{C_0(]-1, 0[).I_1}$ because an approximate unit for $C_0(]-1, 0[)$ is also one for I_1, it is easy to see that $\overline{K_0.I_1}$ is isomorphic to $\overline{(K \otimes S).C_0(]-1, 0[)} \otimes_{\kappa_0} D.I_1$, i.e. $(K \otimes C_0(]-1, 0[)) \otimes_{\kappa_0} D.I_1$. Considering that $C_0(]-1, 0[; A_1) \otimes_{\kappa_1} D$ is $D.I_1$, one gets that $\overline{K_0.I_1}$ is nothing but $(K \otimes C_0(]-1, 0[)) \otimes_{\tilde{\kappa}_0} C_0(]-1, 0[; A_1) \otimes_{\kappa_1} D$ where $\tilde{\kappa}_0$ is the natural inclusion of $C_0(]-1, 0[; B)$ in $C_0(]-1, 0[; A_1)$, i.e. $i_1 \otimes Id_{C_0(]-1,0[)}$. Therefore $(K \otimes_{i_1} A_1) \otimes C_0(]-1, 0[)$ is $(K \otimes C_0(]-1, 0[)) \otimes_{\tilde{\kappa}_0} C_0(]-1, 0[; A_1)$ and $\overline{K_0.I_1}$ is K_1. □

We will also need the following lemmas

Lemma 4.3.
1. *If $f \in C([-1, 1]; \mathbb{R})$, then f is a self-adjoint element in $Z(M(D))$ and more generally for any D-Hilbert module $\mathcal{E}$ the right multiplication by f induces an element $\hat{f} \in Z(\mathcal{L}_D(\mathcal{E}))$ such that the map $f \mapsto \hat{f}$ is an algebra homomorphism.*
2. *Let f in $C_0(]-1, 0[; \mathbb{R})$. Then $f \in I_1 \cap Z(D)$ and the right multiplication by f induces a morphism $\hat{f}$ of $\mathcal{L}_D(K_0, K_1)$ such that $\hat{f}^*\hat{f} = \hat{f}^2$ in $\mathcal{L}_D(K_0)$ and $\hat{f}\hat{f}^* = \hat{f}^2$ in $\mathcal{L}_D(K_1)$.*
3. *Let f in $C_0(]0, 1[; \mathbb{R})$. Then $f \in I_2 \cap Z(D)$ and the right multiplication by f induces a morphism $\hat{f}$ of $\mathcal{L}_D(K_0, K_2)$ such that $\hat{f}^*\hat{f} = \hat{f}^2$ in $\mathcal{L}_D(K_0)$ and $\hat{f}\hat{f}^* = \hat{f}^2$ in $\mathcal{L}_D(K_2)$.*

The first point is pretty obvious and (2) and (3) are also clear in view of Lemma 4.2.

Lemma 4.4.
1. *If $f \in C_0(]-1, 1[; \mathbb{R})$ then for any B-module $\mathcal{E}$ and $F \in \mathcal{K}_B(\mathcal{E})$, we have $(F \otimes 1_S) \otimes_{\kappa_0} 1_D\hat{f}$ is a compact operator of $(\mathcal{E} \otimes S) \otimes_{\kappa_0} D$.*
2. *If $f \in C_0(]-1, 0[; \mathbb{R})$ then for any A_1-module $\mathcal{E}$ and $F \in \mathcal{K}_{A_1}(\mathcal{E})$, we have $F \otimes 1_{C_0(]-1,0[;\mathbb{R})} \otimes_{\kappa_1} 1_D\hat{f}$ is a compact operator of $(\mathcal{E} \otimes C_0(]-1, 0[)) \otimes_{\kappa_1} D$.*
3. *Similarily for $f \in C_0(]0, 1[; \mathbb{R})$ and A_2-modules.*

Proof. Point (2) and (3) are similar to (1). To show (1), let F be the rank one operator $\theta_{\xi,\eta}$ for ξ and η vectors in $\mathcal{E}$ which is defined as $\theta_{\xi,\eta}(x) = \xi < \eta, x >$ for all x in $\mathcal{E}$. Then $(F \otimes 1_S) \otimes_{\kappa_0} 1_D\hat{f}$ is $\theta_{\xi\otimes f_2\otimes f_2,\eta\otimes f_2\otimes f_2}\hat{f}_1$ and therefore compact for any function $f = f_1 f_2^4$ with f_1 and f_2 in $C_0(]-1, 1[; \mathbb{R})$. As any function can be written like that, for example by polar decomposition, we get our result. □

Define now two functions in $C([-1,1];\mathbb{R})$: $C^+(t)$ is $\cos(\pi t)$ if $t \geq 0$ and 1 if $t \leq 0$, the function $C^-(t)$ is $\cos(\pi t)$ if $t \leq 0$ and 1 if $t \geq 0$. Similarly, we have two functions in S ; S^+ is $\sin(\pi t)$ if $t \geq 0$ and 0 if $t \leq 0$, the function $S^-(t)$ is $\sin(\pi t)$ if $t \leq 0$ and 0 if $t \geq 0$. And finally T is the identity function of $C([-1,1];\mathbb{R})$.
With the notation of the first part, we have a natural D-module

$$H = (H_1 \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D \oplus (H_2 \otimes C_0(]0,1[)) \otimes_{\kappa_2} D \oplus (K \otimes S) \otimes_{\kappa_0} D.$$

It is also clear that H is endowed with a natural (left) action of A_f as H_1, H_2 and K have it.

Let G be the operator of $\mathcal{L}_D(H)$ defined in matrix form by

$$G = \begin{pmatrix} \widehat{C^-} & 0 & -((F_1 \otimes 1_{C_0(]-1,0[)})^* \otimes_{\kappa_1} 1)\widehat{S^-} \\ 0 & -\widehat{C}^+ & ((F_2 \otimes 1_{C_0(]0,1[)})^* \otimes_{\kappa_2} 1)\widehat{S^+} \\ -\widehat{S^-}^*((F_1 \otimes 1_{C_0(]-1,0[)}) \otimes_{\kappa_1} 1) & \widehat{S^+}^*((F_2 \otimes 1_{C_0(]0,1[)}) \otimes_{\kappa_2} 1) & Z \end{pmatrix}$$

where $Z = -\widehat{C^-}(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}(q_2 \otimes 1_S) \otimes_{\kappa_0} 1 - \widehat{T}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$. Thanks to Lemma 4.3, G is well-defined. Moreover the following holds.

Proposition 4.5. *The operator G satisfies that $G^2 - 1$ is a compact operator of H and G commutes modulo compact operators with the action of A_f.*

Proof. Computing G^2 one gets as upper left 2×2 corner :

$$\begin{pmatrix} \widehat{C^-}^2 + F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{S^-}^* F_1 \otimes_{\kappa_1} 1 & -F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{S^+}^* F_2 \otimes_{\kappa_2} 1 \\ -F_2^* \otimes_{\kappa_2} 1\widehat{S^-}^*\widehat{S^+} F_1 \otimes_{\kappa_1} 1 & \widehat{C^+}^2 + F_2^* \otimes_{\kappa_1} 1\widehat{S^+}\widehat{S^+}^* F_2 \otimes_{\kappa_1} 1 \end{pmatrix}$$

As $F_1^* F_1$ is the identity modulo compact operator, using Lemma 4.4 (the function $(S^-)^2$ is in $C_0(]-1,1[)$) one has that $F_1^* \otimes_{\kappa_1} 1\widehat{(S^-)^2} F_1 \otimes_{\kappa_1} 1$ is $\widehat{(S^-)^2}$ modulo compact operators. Recalling also that $F_1^* F_2 = 0$, one gets that this matrix is then the identity modulo compact operators.

Let's focus now on the last row of G^2. We get first $-\widehat{C^-} F_1^* \otimes_{\kappa_1} 1\widehat{S^-} - F_1^* \otimes_{\kappa_1} 1\widehat{S^-} Z$. As $F_1^* q_1 \otimes_{i_1} 1 = F_1^*$ and $F_1^* q_2 \otimes_{i_1} 1 = 0$ along with $F_1^* q_0 \otimes_{i_1} 1 = 0$, $F_1^* \otimes_{\kappa_1} 1\widehat{S^-} Z$ is $-F_1^* \otimes_{\kappa_1} 1\widehat{S^-}\widehat{C^-}$. The second composant of that row is treated in the same way. Finally the last composant is $\widehat{S^-}^2(F_1 F_1^*) \otimes_{\kappa_1} 1 + \widehat{S^+}^2(F_2 F_2^*) \otimes_{\kappa_2} 1 + \widehat{C^-}^2(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}^2(q_2 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{T}^2(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$ as q_0, q_1, q_2 are commuting projections. But $F_l F_l^*$ is $q_l \otimes_{i_l} 1$ so $\widehat{S^-}^2(F_1 F_1^*) \otimes_{\kappa_1} 1$ is $\widehat{S^-}^2(q_1 \otimes 1_S) \otimes_{\kappa_0} 1$. Hence, as $q_1 + q_2 + q_0 = 1$, the last component is $1 + \widehat{T^2 - 1}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$. As $T^2 - 1$ is in $C_0(]-1,1[)$ and q_0 is compact, this composant is then 1 modulo compact operators.

Addressing now the compact commutation with the left action of A_f, it is very obvious using Lemma 4.4 and Lemma 3.3 (3) for every composant of G except Z as it contains multiplication with functions not in $C_0(]-1,1[)$. So let a be in A_1. We need to compute $[Z, \rho(a) \otimes_{\kappa_0} 1]$. But we know that $[q_1, \rho(a)] = 0$. As $q_2 = 1 - q_1 - q_0$ we get that $[Z, \rho(a) \otimes_{\kappa_0} 1] = -(\widehat{C^+ + T})[q_0, \rho(a)] \otimes_{\kappa_0} 1$ which is compact as $C^+ + T$ is a function that vanishes on -1 and 1. The case when a is in A_2 is treated in a similar way, hence the compact commutation property is proved for all a in A_f. □

As a consequence, the couple (H, G) defines an element of $KK^1(A_f, D)$ which we will call x in the sequel.

4.2. **K-equivalence.** In all the following proofs we will very often use the external tensor product of Kasparov elements. Instead of the traditional notation $\tau_C(x)$ for the tensorisation with the algebra C of an element x in $KK^*(A,B)$, we will write $1_C \otimes x$ for the element in $KK^*(C \otimes A, C \otimes B)$ or $x \otimes 1_C$ for the element in $KK^*(A \otimes C, B \otimes C)$. Of course $B \otimes C$ is (non canonically) isomorphic to $C \otimes B$, but as we will perform several times this operation, the order will matter. Note that we do not specify the tensor norm as the algebra C we will be using is always nuclear. Also when π is a morphism between A and B, we will write $[\pi]$ for the canonical element in $KK^0(A,B)$.

We will denote by b the element of $KK^1(\mathbb{C}, S)$ which is defined on the S-Hilbert module S itself by the operator $\widehat{T}$. It is well known that b is invertible. Indeed let's describe its form as an extension. The projection associated to the orthogonal symmetry $\widehat{T}$ is the multiplication by the function $p(t) = (1+t)/2$ on $C_b(]-1,1[)/C_0(]-1,1[)$. Now in $C_b(]-1,1[)$, the C*-algebra generated by $C_0(]-1,1[)$ and p is obviously $C_0(]-1,1])$. So the extension we have to consider is given by the map from $\mathbb{C}$ to $C_0(]-1,1])/C_0(]-1,1[)$ that sends λ to λp. Using the evaluation at 1, one gets the standard extension

$$0 \to C_0(]-1,1[) \to C_0(]-1,1]) \to \mathbb{C} \to 0.$$

Using UCT for example, as all K-groups appearing here are torsion-free, we deduce that b is invertible. The interested reader can also check section 19.2 of [Bl86].

Proposition 4.6. *With the hypothesis of Theorem 4.1, one has in $KK^1(A_f, A_f \otimes S)$ that $x \otimes_D [j]$ is homotopic to $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$.*

Proof. To prove that we will choose the representant of $[Id_{A_f}]$ that appear in 3.4 and show that its Kasparov product with b is homotopic to $x \otimes_D [j]$. Call j_l for $l = 1,2$ the inclusions of A_l in A_f and $j_0 = j_1 \circ i_1 = j_2 \circ i_2$ the inclusion of B in A_f. First it is obvious that $H \otimes_j (A_f \otimes S)$ is $H_1 \otimes_{j_1} A_f \otimes C_0(]-1,0[) \oplus H_2 \otimes_{j_2} A_f \otimes C_0(]0,1[) \oplus K \otimes_{j_0} A_f \otimes S$ which is not quite the same as $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S$. So we will realize now a homotopy to fix that.

Lemma 4.7. *Consider the following two spaces : $\Delta_1 = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le 1, -1 < t < s\}$ and $\Delta_2 = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le 1, -s < t < 1\}$. The Hilbert module $\overline{H} = H_1 \otimes_{j_1} A_f \otimes C_0(\Delta_1) \oplus H_2 \otimes_{j_2} A_f \otimes C_0(\Delta_2) \oplus K \otimes_{j_0} A_f \otimes S \otimes C([0,1])$ is endowed with a natural structure of $A_f \otimes S \otimes C([0,1])$ Hilbert module as $C_0(\Delta_1)$ and $C_0(\Delta_2)$ naturally embed in $C_0(]-1,1[\times[0,1])$ and A_f left action. Moreover the operator*

$$\overline{G} = \begin{pmatrix} \widehat{C^-} \otimes 1_{C([0,1])} & 0 & -F_1^* \otimes_{j_1} 1 \otimes 1_{\Delta_1} \widehat{S^-} \otimes 1_{C([0,1])} \\ 0 & -\widehat{C^+} \otimes 1_{C([0,1])} & F_2^* \otimes_{j_2} 1 \otimes 1_{\Delta_2} \widehat{S^+} \otimes 1_{C([0,1])} \\ -\widehat{S^-}^* \otimes 1_{C([0,1])} F_1 \otimes_{j_1} 1 \otimes 1_{\Delta_1} & \widehat{S^+}^* \otimes 1_{C([0,1])} F_2 \otimes_{j_2} 1 \otimes 1_{\Delta_2} & \overline{Z} \end{pmatrix}$$

with $\overline{Z} = \widetilde{Z} \otimes 1_{C([0,1])}$ where $\widetilde{Z} = -\widehat{C^-} q_1 \otimes_{j_0} 1 \otimes 1_S + \widehat{C^+} q_2 \otimes_{j_0} 1 \otimes 1_S - \widehat{T} q_0 \otimes_{j_0} 1 \otimes 1_S$ makes the pair $(\overline{H}, \overline{G})$ into an element of $KK^1(A_f, A_f \otimes S \otimes C([0,1]))$ for which the evaluation at $s = 0$ is $x \otimes_D [j]$ and the evaluation at $s = 1$ has $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S$ as module and $\widetilde{G} = \begin{pmatrix} \widehat{C^-} & 0 & -F_1^* \otimes_{j_1} 1 \otimes 1_S \widehat{S^-} \\ 0 & -\widehat{C^+} & F_2^* \otimes_{j_2} 1 \otimes 1_S \widehat{S^+} \\ -\widehat{S^-}^* F_1 \otimes_{j_1} 1 \otimes 1_S & \widehat{S^+}^* F_2 \otimes_{j_2} 1_S & \widetilde{Z} \end{pmatrix}$ *as operator.*

Proof. As it is a straightforward check, details will be omitted. □

Using Connes- Skandalis characterization of the Kasparov product, we now established that $\widetilde{G}$ is a representant of the Fredholm operator for the product $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$ by checking the connection and positivity properties (see [Bl86] Chap 18.4). But to do that we of course need to revert to the general presentation of KK^1 elements as graded KK elements (see preliminairies). Let's denote $e_0 = (1,1)$ and $e_1 = (1,-1)$ the basis of $\mathbb{C}_1$. The element of graded KK-theory that we have now for $x \otimes_D [j]$ is given by the module $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes \mathbb{C}_1$ and operator R such that if ξ is in $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f)$ and f in S, $R(\xi \otimes f \otimes e_0) = \widetilde{G}(\xi \otimes f) \otimes e_1$. As R is $\mathbb{C}_1$-linear, that completely characterizes R. There is a similar statement for b as an element of $KK(\mathbb{C}, S \otimes \mathbb{C}_1)$. We will call T the 1-graded operator that appears.

Looking first at the module for $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b)$, we obtain $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \hat{\otimes} (S \otimes \mathbb{C}_1)$. Note that we used the graded tensor product. Of course when one term is trivially graded the graded tensor product is the usual tensor product. At first look, it is the same as $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes \mathbb{C}_1$ except that the grading is not the same. But of course there is a $A_f \otimes S \otimes \mathbb{C}_1$-isomorphism U that corrects that, sending $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f) \otimes S \hat{\otimes} e_0 \oplus (K \otimes_{j_0} A_f) \otimes S \hat{\otimes} e_1$ to $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \otimes S \otimes e_0$. Throught this isomorphism, R becomes $\overline{R}$.

Let's look now at the connection condition (see [Bl86] Definition 18.3.1 p 170). As $\overline{R}$ and T are self-adjoint, there is only one condition to test. For ξ in $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f)$ and f in S, one should look at the $S \otimes \mathbb{C}_1$-linear map from $S \otimes \mathbb{C}_1$ to $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f \oplus K \otimes_{j_0} A_f) \hat{\otimes} (S \otimes \mathbb{C}_1)$ defined as $f \otimes e_0 \mapsto (-1)^{0 \times 1} \overline{R}(\xi \hat{\otimes} (f \otimes e_0)) - \xi \hat{\otimes} T(f \otimes e_0)$ and prove that it is compact. This is done by simply proving that at the evaluation at -1 and 1, the operator is 0. On both ends $\widetilde{G}$ is diagonal, equals to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ or the opposite matrix as $q_1 + q_2 + q_0 = 1$. So the evaluation at -1 of $(-1)^{0 \times 1} \overline{R}$ will send $(\xi \hat{\otimes} e_0)$ to $-\xi \hat{\otimes} e_1$ which is what the evaluation at -1 of T does. For the evaluation at 1, the two operators are also identical.

Now if ξ in $K \otimes_{j_0} A_f$, one looks at $(-1)^{1 \times 1} \overline{F}(\xi \hat{\otimes} (f \otimes e_0)) - \xi \hat{\otimes} T(f \otimes e_0)$. The evaluation at -1 of $(-1)^{1 \times 1} \overline{R}$ will send $(\xi \hat{\otimes} e_0)$ to $\xi \hat{\otimes} e_1$ which is again what the evaluation at -1 of T does and similarly for the evaluation at 1.

We now concentrates on the commutator condition (see [Bl86] Definition 18.4.1 p 172). One needs to compute the anti-commutator of $\overline{R}$ with $F \hat{\otimes} 1$, using the operator F that appeared before Proposition 3.4. We will call G_0 and G_1 the diagonal and anti-diagonal part of $\widetilde{G}$.

For ξ in $(H_1 \otimes_{j_1} A_f \oplus H_2 \otimes_{j_2} A_f)$ and f in S, one has that $F \hat{\otimes} 1(\xi \otimes f \otimes e_0) = F(\xi) \hat{\otimes} (f \otimes e_0)$. As $F(\xi)$ is then of degree 1, $\overline{R}(F(\xi) \hat{\otimes} (f \otimes e_0)) = R(F(\xi) \otimes f \otimes e_1) = U^* \widetilde{G}(F(\xi) \otimes f) \otimes e_0 = G_0(F(\xi) \otimes f) \hat{\otimes} e_1 + G_1(F(\xi) \otimes f) \hat{\otimes} e_0$. On the other hand $(F \hat{\otimes} 1).\overline{R}(\xi \hat{\otimes} (f \otimes e_0)) = (F \otimes 1_S).G_0(\xi \otimes f) \hat{\otimes} e_1 + (F \otimes 1_S).G_1(\xi \otimes f) \hat{\otimes} e_0$. As the same is true for ξ in $K \otimes_{j_0} A_f$, we will be done once the following Lemma is proved.

Lemma 4.8. *The anti-commutator of G_0 and $F \otimes 1_S$ is 0 modulo compact operators and the anti-commutator of G_1 and $F \otimes 1_S$ is positive modulo compact operators.*

Proof. It is clear that $\begin{pmatrix} \widehat{C^-} & 0 & 0 \\ 0 & -\widehat{C}^+ & 0 \\ 0 & 0 & \widetilde{Z} \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & F_1^* \otimes_{i_1} 1 \otimes 1_S \\ 0 & 0 & F_2^* \otimes_{i_2} 1 \otimes 1_S \\ F_1 \otimes_{i_1} 1 \otimes 1_S & F_2 \otimes_{i_2} 1 \otimes 1_S & 0 \end{pmatrix}$ anti-commutes modulo compact operator as we have (modulo compact operator) $q_1F_1 = F_1$ and $q_2F_1 = q_0F_1 = 0$. On the other hand the anti-commutator with the anti-diagonal part is

$$\begin{pmatrix} -2(F_1^*F_1) \otimes_{j_1} 1 \otimes 1_S\widehat{S^-} & 0 & 0 \\ 0 & 2(F_2^*F_2) \otimes_{j_2} 1 \otimes 1_S\widehat{S^+} & 0 \\ 0 & 0 & -2q_1 \otimes_{j_0} 1 \otimes 1_S\widehat{S^-} + 2q_2 \otimes_{j_2} 1 \otimes 1_S\widehat{S^+} \end{pmatrix}$$

As $-S^-$ and S^+ are positive functions and q_1 and q_2 are orthogonal projections, the previous matrix is a diagonal matrix of positive operators hence positive. □

End of the proof of Proposition 4.6. Having checked the two conditions that characterizes the Kasparov product we have our proposition. Note that as $[Id_{A_f}]$ is a Kasparov cycle given by a homomorphism, we obviously have $[Id_{A_f}] \otimes_{A_f} (1_{A_f} \otimes b) = (1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$. Hence $x \otimes_D [j]$ is also equal to $(1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$. This is the form we need in the final stage of our proof of the theorem. □

We need now the following two lemmas to get some information about $[j] \otimes_{A_f \otimes S} (x \otimes 1_S)$ as an element of $KK^1(D, D \otimes S)$.

Lemma 4.9. *Call ev_0 the morphism from D to B that evaluates a function at 0. Then we have in $KK^1(D, B \otimes S)$ that $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S) = -[ev_0] \otimes_B (1_B \otimes b)$.*

Proof. Let's first describe the left hand side. The Hilbert module is $K \otimes S$ as the module $(H_1 \otimes C_0(]-1,0[)) \otimes_{\kappa_1} D \otimes_{ev_0} B$ is 0. The left D action is given by $(\rho \otimes 1_S) \circ j$ and the operator is just $(-q_1 + q_2) \otimes 1_S$. We can replace this operator with $G_0 = (-q_1 + q_2) \otimes 1_S - \widehat{T}q_0 \otimes 1_S$ as for any f in D, $(\rho \otimes 1_S) \circ j(f)\widehat{T}q_0 \otimes 1_S$ is compact. Note now that the evaluation at -1 of G_0 is $-q_1 + q_2 + q_0 = (1 - 2q_1)$ and at 1 is $-q_1 + q_2 - q_0 = 2q_2 - 1$ as $q_1 + q_2 + q_0 = 1$. It then enables us to do a homotopy. Consider the pair $(K \otimes S \otimes C([0,1]), G_0 \otimes 1_{C([0,1])})$ where the left action of D is defined now for any f in D and $k \in C(]-1,1[\times[0,1]; K)$ as $\widehat{(f.k)}(t,s) = \rho(f(t(1-s)))k(t,s)$. This is still a Kasparov element as $(G_0^2 - 1) \otimes 1_{C([0,1])} = (\widehat{(T^2-1)}q_0 \otimes 1_S) \otimes 1_{C([0,1])}$ hence compact. Also the commutator of the left action with the operator $G_0 \otimes 1$ is compact. Indeed, as q_0 is compact, it is only necessary to check that the evaluation at -1 or 1 of any commutator is 0. But this is true as $[q_1, \rho(A_1)] = 0$ and $[q_2, \rho(A_2)] = 0$. Therefore $[j] \otimes_{A_f \otimes S} ((x \otimes_D [ev_0]) \otimes 1_S)$ is homotopic to an element of $KK^1(D, B \otimes S)$ which is described with the pair $(K \otimes S, G_0)$ where D acts on $K \otimes S$ as the constant morphism $\rho \circ ev_0$. So it is $[ev_0] \otimes_B z$ with z an element of $KK^1(B, B \otimes S)$ which is only non trivial on $q_0K \otimes S \simeq B \otimes S$ where G_0 acts as $-\widehat{T}$. Thus $z = -1_B \otimes b$. □

Recall that for $l = 1, 2$, κ_l is the inclusion of $A_l \otimes C(]-1,0[)$ in D. To be precise we will use $\bar{\kappa}_l$ for the induced map from $A_l \otimes S$ to D via the isomorphism of $C(]-1,0[)$ with S.

Lemma 4.10. *For all $l = 1, 2$, one has $[j_l] \otimes_{A_f} x = ([Id_{A_l}] \otimes b) \otimes_{A_l \otimes S} [\bar{\kappa}_l] \in KK^1(A_l, D)$.*

Proof. We will do the lemma for $l = 1$. The element $[j_1] \otimes_{A_f} x$ has the same module and operator as x, the only change is that we only consider a left action of A_1. We first perform a compact perturbation of the operator G. With the operators $\overline{F_l}$ defined before Lemma 3.3, consider

$$G_1 = \begin{pmatrix} \widehat{C^-} & 0 & -F_1^* \otimes_{\kappa_1} 1\widehat{S^-} \\ 0 & -\widehat{C^+} & \overline{F}_2^* \otimes_{\kappa_2} 1\widehat{S^+} \\ -\widehat{S^-}^* F_1 \otimes_{\kappa_1} 1 & \widehat{S^+}^* \overline{F}_2 \otimes_{\kappa_2} 1 & \overline{Z} \end{pmatrix},$$

where $\overline{Z} = -\widehat{C^-}(q_1 \otimes 1_S) \otimes_{\kappa_0} 1 + \widehat{C^+}(1 - q_1 \otimes 1_S) \otimes_{\kappa_0} 1$.
As $F_2 - \overline{F}_2$ is compact (see Lemma 3.3) and $\overline{Z} - Z = \widehat{C^+ + T}(q_0 \otimes 1_S) \otimes_{\kappa_0} 1$ is compact as $C^+ + T$ is in S, we get the same element of $KK^1(A_1, D)$. Observe now that when evaluating at any positive t, G_1^2 is the identity because $\overline{F}_2$ is an isometry and $\widehat{S^-} F_1 \otimes_{\kappa_1} 1$ vanishes and that for any t, G_1 commutes exactly with the left action of A_1 as F_1 and $\overline{F}_2$ does.

We will now construct a homotopy to remove the $[0, 1[$ part of our module. Consider the space $\Delta_3 = \{(t, s) \in \mathbb{R} : 0 \le s \le 1, 0 < t < s\}$ and $\Delta_4 = \{(t, s) \in \mathbb{R} : 0 \le s \le 1, -1 < t < s\}$ which are open in $] - 1, 1[\times[0, 1]$. Hence we also have a natural imbedding δ_4 of $C_0(\Delta_4; B)$ in $D \otimes C([0, 1])$ and δ_3 of $C_0(\Delta_3; A_2)$ in $D \otimes C([0, 1])$. Then $\widetilde{H} = (H_1 \otimes C_0(] - 1, 0[)) \otimes_{\kappa_1} D \otimes C([0, 1]) \oplus (H_2 \otimes C_0(\Delta_3)) \otimes_{\delta_3} D \otimes C([0, 1]) \oplus (K \otimes C_0(\Delta_4) \otimes_{\delta_4} D \otimes C([0, 1])$ is well defined and the pair $(\widetilde{H}, G_1 \otimes 1_{C([0,1])})$ is a Kasparov element in $KK^1(A_1, D \otimes C([0, 1]))$. Indeed the only thing to check is whether $G_1^2 \otimes 1_{C([0,1])}$ is the identity modulo compact operator as $G_1 \otimes 1_{C([0,1])}$ has exact commutation with the action of A_1. But this is true by the previous observation.

Therefore $[j_l] \otimes_{A_f} x$ can be represented by the evaluation at 0 of this Kasparov element. Let's describe it: the module part is $(H_1 \oplus K \otimes_{i_1} A_1) \otimes C_0(] - 1, 0[) \otimes_{\kappa_1} D$ with obvious left A_1 action as $(K \otimes C_0(] - 1, 0[)) \otimes_{\kappa_0} D$ is isomorphic to $(K \otimes_{i_1} A_1) \otimes C_0(] - 1, 0[) \otimes_{\kappa_1} D$. With this identification, the operator is

$$E_1 = \begin{pmatrix} \widehat{C^-} & -F_1^* \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1\widehat{S^-} \\ -\widehat{S^-}^* F_1 \otimes 1_{C_0(]-1,0[)} \otimes_{\kappa_1} 1 & -\widehat{C^-}(q_1 \otimes_{i_1} 1 \otimes 1_{C_0(]-1,0[)}) \otimes_{\kappa_1} 1 + (1 - q_1 \otimes_{i_1} 1 \otimes 1_{C_0(]-1,0[)}) \otimes_{\kappa_1} 1 \end{pmatrix}.$$

It is then clear, after identifying $C_0(] - 1, 0[)$ with S, that $[j_1] \otimes_{A_f} x$ is $z \otimes_{A_1 \otimes S} [\bar{\kappa}_1]$ with z in $KK^1(A_1, A_1 \otimes S)$. By recalling that $1 - q_1$ commutes with the left action of A_1, it is obvious that z is represented by the pair $((H_1 \oplus q_1 K \otimes_{i_1} A_1) \otimes S, \overline{E}_1)$ with $\overline{E}_1 = \begin{pmatrix} \widehat{C_1} & -F_1^* \otimes 1_S \widehat{S_1} \\ -\widehat{S_1}^* F_1 \otimes 1_S & -\widehat{C_1}(q_1 \otimes_{i_1} 1 \otimes 1_S) \end{pmatrix}$ where C_1 is the function $\cos(\pi(t/2 - 1/2))$ and S_1 the function $\sin(\pi(t/2 - 1/2))$.

Following the proof of Proposition 4.6, z is obviously the product $z' \otimes b$ where z' is the element of $KK^0(A_1, A_1)$ given by the module $H_1 \oplus q_1 K \otimes_{i_1} A_1$ with H_1 positively graded and the obvious left action of A_1 and the operator $\begin{pmatrix} 0 & F_1^* \\ F_1 & 0 \end{pmatrix}$. Now the action of A_1 stabilizes H_1° and commutes with F_1 by 3.3 (1) and moreover F_1 is a unitary between H_1° and $q_1 K \otimes_{i_1} A_1$. Hence this part is degenerated and can be removed from the Kasparov element. What remains is the graded module $\xi_1.A_1 \oplus 0$ with left action of A_1 by multiplication and 0 as operator. This is a description of $[Id_{A_1}]$.

□

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Call $a \in KK^1(S,\mathbb{C})$ the inverse of b. The element $y = (1_{A_f} \otimes a) \otimes_{A_f} x$ is an element of $KK^0(A_f \otimes S, D)$. We claim that this is the inverse of $[j]$. Indeed thanks to 4.6 we have that

$$y \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} x \otimes_D [j] = (1_{A_f} \otimes a) \otimes_{A_f} (1_{A_f} \otimes b) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S).$$

As $a \otimes_{\mathbb{C}} b = [Id_S]$ we get that $y \otimes_D [j] = (1_{A_f} \otimes [Id_S]) \otimes_{A_f \otimes S} ([Id_{A_f}] \otimes 1_S)$ is $[Id_{A_f \otimes S}]$. To prove the reverse equality, we will need a trick that can be found already in [Pi86]. Observe first that for any $l = 1,2$ and using Lemma 4.10,

$$\begin{aligned} [\bar\kappa_l] \otimes_D [j] \otimes_{A_f \otimes S} y &= [j \circ \bar\kappa_l] \otimes_{A_f \otimes S} y = ([j_l] \otimes 1_S) \otimes_{A_f \otimes S} (1_{A_f} \otimes a) \otimes_{A_f} x \\ &= (1_{A_l} \otimes a) \otimes_{A_l} [j_l] \otimes_{A_f} x \\ &= (1_{A_l} \otimes a) \otimes_{A_l} (1_{A_l} \otimes b) \otimes_{A_l} ([Id_{A_l}] \otimes 1_S) \otimes_{A_l \otimes S} [\bar\kappa_l] \\ &= [\bar\kappa_l]. \end{aligned}$$

Now we need to compute $[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]$. To do this we will use the following lemma.

Lemma 4.11. *In $KK^1(D \otimes S, A \otimes S)$, one has $([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = -(1_D \otimes a) \otimes_D [j]$.*

Proof. Indeed,

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S = [j] \otimes_{A_f \otimes S} (1_{A_f} \otimes (1_S \otimes b) \otimes_{S \otimes S} (a \otimes 1_S)).$$

If Σ is the flip automorphism of $S \otimes S$ then clearly $[\Sigma] = -[Id_{S \otimes S}]$ in $KK^0(S \otimes S, S \otimes S)$. As a consequence $(1_S \otimes b) \otimes_{S \otimes S} (a \otimes 1_S) = -1_S \otimes (b \otimes_{\mathbb{C}} a) = -[Id_S]$. Hence

$$(1_D \otimes b) \otimes_{D \otimes S} ([j] \otimes_{A_f \otimes S} (1_{A_f} \otimes a)) \otimes 1_S) = -[j].$$

Multiplying both side by $1_D \otimes a$ gives the result. ☐

In view of Lemmas 4.11 and 4.9 one has:

$$\begin{aligned} ([j] \otimes_{A_f \otimes S} y \otimes_D [ev_0]) \otimes 1_S &= -(1_D \otimes a) \otimes_D ([j] \otimes_{A_f \otimes S} (x \otimes_D [ev_0]) \otimes 1_S) \\ &= +(1_D \otimes a) \otimes_D [ev_0] \otimes_B (1_B \otimes b) \\ &= (1_D \otimes a) \otimes_D (1_D \otimes b) \otimes_{D \otimes S} ([ev_0] \otimes 1_S) \\ &= [ev_0] \otimes 1_S \end{aligned}$$

As $- \otimes 1_S$ from $KK(B_1, B_2)$ to $KK(B_1 \otimes S, B_2 \otimes S)$ is an isomorphism for any B_1 and B_2, we get $[j] \otimes_{A_f \otimes S} y \otimes_D [ev_0] = [ev_0]$. Denote now $q = [Id_D] - [j] \otimes_{A_f \otimes S} y$. As $y \otimes_D [j] = [Id_{A_f \otimes S}]$, q is an idempotent in the ring $KK^0(D,D)$. On the other hand, D fits into a short exact sequence

$$0 \to A_1 \otimes S \oplus A_2 \otimes S \overset{\bar\kappa_1 \oplus \bar\kappa_2}{\longrightarrow} D \overset{ev_0}{\longrightarrow} B \to 0.$$

The induced six term exact sequence for the functor $KK^0(D,-)$ then shows that, as $q \otimes_D [ev_0] = 0$, there exist q_l in $KK^0(D, A_l)$ for $l = 1,2$ such that $q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar\kappa_1] \oplus [\bar\kappa_2])$. So $q = q \otimes_D q = (q_1 \oplus q_2) \otimes_{A_1 \oplus A_2} ([\bar\kappa_1] \oplus [\bar\kappa_2]) \otimes_D q = 0$ because $[\bar\kappa_l] \otimes_D q = 0$ for $l = 1,2$ as observed before Lemma 4.11. Therefore $[Id_D] = [j] \otimes_{A_f \otimes S} y$ and the K-equivalence between A_f and D is established. ☐

We obtain the following immediate corollaries.

Corollary 4.12. *Let C be any separable C*-algebra. Recall that i_l is the inclusion of B in A_l and j_l is the inclusion of A_l in $A_1 *_B A_2$ for $l = 1$ or 2. Then we have the two 6-terms exact sequences,*

$$\begin{array}{ccccc} KK^0(C,B) & \overset{i_1^*\oplus i_2^*}{\longrightarrow} & KK^0(C,A_1)\bigoplus KK^0(C,A_2) & \overset{j_1^*+j_2^*}{\longrightarrow} & KK^0(C,A_1 *_B A_2) \\ \uparrow & & & & \downarrow \\ KK^1(C,A_1 *_B A_2) & \overset{j_1^*+j_2^*}{\longleftarrow} & KK^1(C,A_1)\bigoplus KK^1(C,A_2) & \overset{i_1^*\oplus i_2^*}{\longleftarrow} & KK^1(C,B) \end{array}$$

and

$$\begin{array}{ccccc} KK^0(B,C) & \overset{i_1^*+i_2^*}{\longleftarrow} & KK^0(A_1,C)\bigoplus KK^0(A_2,C) & \overset{j_1^*\oplus j_2^*}{\longleftarrow} & KK^0(A_1 *_B A_2,C) \\ \downarrow & & & & \uparrow \\ KK^1(A_1 *_B A_2,C) & \overset{j_1^*\oplus j_2^*}{\longrightarrow} & KK^1(A_1,C)\oplus KK^1(A_2,C) & \overset{i_1^*+i_2^*}{\longrightarrow} & KK^1(B,C) \end{array}$$

Proof. The proof can be found in [Ge97] or [Th03]. It is simply the application of the six-term exact sequence to the short exact sequence for D that has been used just above. Identification of the horizontal maps as well as the connecting maps can also be found there. □

The following is an generalization of a similar statement in [FF13].

Corollary 4.13. *Let G_1, G_2, H be compact quantum groups and suppose that $\widehat{H}$ is a common discrete quantum subgroup of both $\widehat{G_1}, \widehat{G_2}$ and $\widehat{G_k}$ is K-amenable for $k = 1, 2$. Then the amalgamated free product of the two discrete quantum groups is K-amenable.*

Proof. Write, for $k = 1, 2$, $C_m(G_k), C_m(H)$ the full C*-algebras and $C(G_k), C(H)$ the reduced C*-algebra and view $C_m(H) \subset C_m(G_k)$, $C(H) \subset C(G_k)$, for $k = 1, 2$. Let $\widehat{G}$ be the amalgamated free product discrete quantum group. One has $C_m(G) = C_m(G_1) \underset{C_m(H)}{*} C_m(G_2)$ and $C(G) = C(G_1) \overset{e}{\underset{C(H)}{*}} C(G_2)$, where the edge-reduced amalgamated free product is done with respect to the *faithful* Haar states on $C(G_k)$, for $k = 1, 2$. Let $\lambda_{G_k} : C_m(G_k) \to C(G_k)$ be the canonical surjection. By assumption, λ_{G_k} is K-invertible for $k = 1, 2$. Observe that the canonical surjection $\lambda_G : C_m(G) \to C(G)$ is given by $\lambda_G = \pi \circ \lambda$, where

$$\lambda : C_m(G_1) \underset{C_m(H)}{*} C_m(G_2) \to C(G_1) \underset{C(H)}{*} C(G_2)$$

is the free product of the maps λ_{G_1} and λ_{G_2} and $\pi : C(G_1) \underset{C(H)}{*} C(G_2) \to C(G_1) \overset{e}{\underset{C(H)}{*}} C(G_2)$ is the canonical quotient map. By Theorem 3.1 π is K-invertible and using the exact sequence of the full free product and the five Lemma, λ is K-invertible. □

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